Synthesis of Real Weyl-Heisenberg Signal Frames with Desired Frequency-Time Localization

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***Abstract*—An algebraic approach to the synthesis of optimal real Weyl-Heisenberg frames with the best frequency-time localization oriented to the processing of discrete signals is developed. The chosen optimality criterion ensures the construction of a tight signal frame with the lowest standard deviation of frame functions from the desired standard. In addition, a special algebraic structure of the synthesis algorithm in the form of a product of sparse matrices allows for efficient computational implementation and flexible adjustment of the frequency-time resolution of the signal functions of the frame. The results of the experiment confirming the effective computational implementation of the algorithm and a desired time-frequency localization of frame functions are presented.**

I. Introduction

One of the most important tasks of modern intelligent information processing systems is the development of effective methods and algorithms for spectral-time analysis of the processes observed at the output of various recording devices. Such devices, for example, can be biomedical sensors, echo signal receivers for radar (or sonar) various purpose systems, seismic sensors, earth's surface monitoring systems devices, etc. All the information obtained in these cases is usually digitized, so it can be processed either in real time or stored and processed later using specialized algorithms.

An important specific of most of the observed signals is that they are not the stationary processes, which greatly complicates or limits the use of classical algorithms for digital spectral analysis. In addition, the useful information that you want to extract from the received signal is usually multifactorial by nature and to identify all its features flexible multi-level algorithms of frequency-time analysis required, able to quickly adapt to specific tasks. Signals transmitted through telecommunications communication systems of 4-5G generations have the same features, but in this case there are problems not only of their optimal reception, but also the development of the most suitable signal design with dense time-frequency multiplexing and at the same time a good separation of signal constellation points.

For these purposes the paper proposes to use tight discrete Weyl-Heisenberg frames (WH-frames), defined on a finite time interval and focused on batch processing of discrete signals. This is in good agreement with the presentation of observed processes at the output of most digital recording devices and broadband signals that can be used in future 5G telecommunication systems. The general theory of WH-frames (including discrete) and the related theory of extensions of Gabor are described in [1-8].

Discrete functions included in the WH-frame are obtained by uniform shifts in time and frequency of the same forming pulse with desired frequency-time localization.

The shape of this pulse determines the frequency-time resolution of the WH-frame, and the number of shifts in time and frequency – its frequency-time range. In this case, nonstationarity of the observed process will appear as different behavior of the frame decomposition coefficients in the time-shifted "Windows" of the forming pulse. Therefore, the observed signal can be well approximated by a finite linear combination of frame functions, choosing a suitable forming pulse, the number and structure of time/frequency shifts (factor parameters of the WH-frame). This means that the WH-decomposition of the signal can be considered as a discrete multifactorial time-frequency model of the observed nonstationary process, and the adjustment of these factors to obtain the best approximation can be considered as a procedure for identification of the WH-frame.

Thus, the possibility of fast optimal adjustment of the WH-frame for a specific observed process is practically important for the approximation and time-frequency analysis of signals. The paper proposes an algebraic approach to the synthesis of the WH-frame identification algorithm based on its optimal adjustment to the desired reference system. As such a reference system we choose a real system of pulses uniformly shifted in time and frequency with the desired symmetry properties, frequency-time resolution and range. Following the terminology [1-8], the reference system with the specified shift structure represents a certain system of Gabor functions forming a WH-frame. Note that the desired Gabor system is usually not a tight frame, so an attempt to decompose the signal into a linear combination of standard functions is a complex problem that does not always have a stable solution and does not necessarily lead to a good approximation.

The paper deals with the problem of synthesis of a tight signal WH-frame, which has the desired properties, while the deviation of the frame system from the reference system is minimized by the standard deviation criterion. In this sense, the resulting tight WH frame is optimal. It is shown that the optimal adjustment algorithm is a linear matrix operator that is factorized into the product of sparse matrices. This allows us to provide a fast computational implementation of the algorithm of WH-frame identification and flexible configuration of the time-frequency resolution characteristics by changing the corresponding parameters of the desired reference system.

Note, the obtained results can be considered as a generalization and further development of the results on optimization of complex WH-bases described in [9-12] for the case of tight real frames. In particular, for the oversampling coefficient equal to two after a structural adjustment of the optimal WH-frame real matrix, we obtain a complex orthogonal WH-basis. However, the advantage of the proposed WH-frames versus basis is that, increasing the oversampling coefficient of the samples, it is possible to achieve much better frequency-temporal localization of frame function and, as a result, to improve/increase the time-frequency diversity/multiplexing in the signal structures of 5G telecommunication systems. In addition, discrete real WH-frames and a fast matrix-vector algorithm for their identification are of separate scientific and practical interest, since most of the above observed processes are real, non-stationary and supposed to be digitally processed at different intervals of observation.

The results of the experiment confirming the effective computational implementation of this algorithm and a desired time-frequency localization of frame functions are presented.

II. MATHEMATICAL FORMALIZATION OF THE PROBLEM AND ITS RESOLUTION

Let the continuous forming pulse $g(t), t \in \mathrm{R}$ of the desired frame system be an even function, with effective duration T and bandwidth $F = 1/T$, the number of basic time and frequency shifts are L and M, respectively. Then the product $N = M \cdot L \geq M$ determines the total number of basic elements of time-frequency resolution, the overlapping frequency range is $\Delta_f = [0, W], W = M/T$, and the time range $\Lambda_t = [0, T_-]$, $T_- = LT$. In this article, we assume that the analyzed signals are bandpass, localized in the low frequency band. This is not a significant limitation, and, if necessary, will allow the transfer of the observed signal spectrum to the low-frequency band at the stage of its sampling. For such bandpass signals, the minimum sampling rate is equal $f_d = W$, and the total number of discrete time samples within the range Λ_t is equal $N = T_- f_d = ML$.

With this in mind, after sampling at the interval Λ_t, the forming pulse will be

$$g[n] = g(n / f_d) \,, n \in J_N = \{0,1,..., N-1\} .$$

For an adequate definition of time shifts and parity properties at the finite discrete interval, we perform an additional cyclic reduction

$$g_{ep}[n] = (g[(n)_{[n]}] + g[(-n)_{[n]}]\,)/2 \quad , n \in J_N, \quad (1)$$

where $(n)_{[N]} = n \,(\mathrm{mod}\, N)$, $n \in J_N$. As a result, $g_{ep}[n]$ will satisfy the condition of N-periodicity and N-symmetry with respect to point 0

$$g_{ep}[n] = g_{ep}[qN+n]\,,\; g_{ep}[n] = g_{ep}[-n]\,,\; \forall n \in J_N, q \in \mathbf{Z} \quad (2)$$

Let the oversampling coefficient $P \geq 2$ to be an even natural number, M a multiple of P, $L_o = M/P$, $K = LP$, $K_o = K/2$, then, the desired reference system of discrete real Gabor functions at a finite discrete time interval $J_N = \{0,1,..., N-1\}$ is described by expressions:

$$\mathbf{G}_\alpha = \left(\mathbf{G}_\alpha^{(0)}, \ldots, \mathbf{G}_\alpha^{(PN-1)}\right) = \left(\mathbf{\Phi}^{ч}_{m,l}[n], \mathbf{\Phi}^{нч}_{m,l}[n]\right), \quad (3)$$

$$\mathbf{\Phi}^{ч}_{m,l}[n] = \begin{pmatrix} g_{ep}[n-2lL_0]\cos[\omega m(n-\alpha/2)] \\ g_{ep}[n-2lL_0]\sin[\omega m(n-\alpha/2)] \end{pmatrix},$$

$$\mathbf{\Phi}^{нч}_{m,l}[n] = \begin{pmatrix} -g_{ep}[n-(2l+1)L_0]\sin[\omega m(n-\alpha/2)] \\ g_{ep}[n-(2l+1)L_0]\cos[\omega m(n-\alpha/2)] \end{pmatrix}, \quad (4)$$

$$m \in J_M,\; l \in J_{K_o},\; n \in J_N,\; \omega = 2\pi/M,\; N = ML,$$

where $\mathbf{G}_\alpha = \left(G_\alpha(i_o, j_o)\right)$, $i_o \in J_{2N}, j_o \in J_{PN}$ is a real rectangular matrix of dimension $2N \times PN$, in which the vectors-columns

$$\mathbf{G}_\alpha^{(j_o)} = [G_\alpha^{(j_o)}(0), \ldots, G_\alpha^{(j_o)}(2N-1)]^T$$

at constant $m \in J_M = \{0, \ldots M-1\}$ and even values $2l$ are vector-functions $\mathbf{\Phi}^{ч}_{m,l} = \{\mathbf{\Phi}^{ч}_{m,l}[n], n \in J_N\}$, and at odd values $(2l+1)$ – are vector-functions $\mathbf{\Phi}^{нч}_{m,l} = \{\mathbf{\Phi}^{нч}_{m,l}[n], n \in J_N\}$ with dimension $2N$.

From (4), (2) we can find out that the functions $\mathbf{\Phi}^{ч}_{m,l}$ and $\mathbf{\Phi}^{нч}_{m,l}$ are pairwise orthogonal and consist of quadrature components cyclically shifted in time by a value L_o. Indexes $m \in J_M$ determine the basic frequency shifts, and the indexes $2l,\ (2l+1),\ l \in J_{K_o}$ determines odd and even frame shifts in time, taking into account the oversampling coefficient P. Their total number is equal $K = LP > L$, i.e. P times exceeds the number of basic time shifts L. The dimension of the signal space "stretched" on the system of reference functions $\mathbf{G}_\alpha = \left\{\left(\mathbf{\Phi}^{ч}_{m,l}, \mathbf{\Phi}^{нч}_{m,l}\right), m \in J_M,\ l \in J_{K_o}\right\}$ is equal to $N_o = 2N$. The total number of frame functions is equal to $M_o = PN \geq N_o$. The phase parameter $\alpha \in \mathrm{R}$ is used for additional adjustment of the reference system. From(3)-(4) it follows that the elements indexes (i_o, j_o) of the matrix structure $\mathbf{G}_\alpha = \left(G_\alpha(i_o, j_o)\right)$ are associated with frame variables $(m,\ l,\ n)$ by expressions:

$$i_o = \begin{cases} 2n, & i_o - even \\ 2n+1, & i_o - odd \end{cases},$$

$$j_o = \begin{cases} 2(l+mK_o), & j_o - even \\ 2(l+mK_o)+1, & j_o - odd \end{cases}$$

$$i_o \in J_{N_o} = \{0, \ldots N_o - 1\},\; j_o \in J_{M_o} = \{0, \ldots M_o - 1\},$$

$$m \in J_M,\; l \in J_{K_o},\; n \in J_N .$$

For a better understanding of the problem, the terminology and notations used, we recall a number of definitions from the frame theory applied to discrete real finite-dimensional spaces.

We denote $\mathrm{M}_{m,n}$ by the set of all real size $m \times n$ matrices . If $m = n$ so, the abbreviated M_n will be used. According to [2-8], a system of discrete functions

$$\mathbf{G} = \left(\mathbf{G}^{(0)},\ldots,\mathbf{G}^{(M_o-1)}\right) \in \mathrm{M}_{N_o,M_o}$$

with the structure of the frequency-time shifts described above is called a WH-frame if an inequality

$$A\|\mathbf{s}\|^2 \le \|\mathbf{G}^T\mathbf{s}\|^2 \le B\|\mathbf{s}\|^2$$

is satisfied for any signal vector $\mathbf{s} = (\dot{s}[0],\ldots,\dot{s}[N_o-1])^T$, where $0 < A \le B < \infty$ are called bounds of the frame $\|\cdot\|$ - is Euclidean norm. If $A = B$, then the frame is called tight, if $A = B = 1$ and $\|\mathbf{G}^{(\mu)}\| = 1, \forall\mu$, then $\mathbf{G}$ is an orthonormal signal WH-basis. The rectangular matrix $\mathbf{G}^T \in \mathrm{M}_{M_o,N_o}$ describes the frame operator, which application to the signal $\mathbf{s} \in \mathrm{R}^{N_o}$ leads to the vector

$$\mathbf{x} = \mathbf{G}^T\mathbf{s} \in \mathrm{R}^{M_o},$$

that in the engineering interpretation describes a frame-based time-frequency spectrum.

For most practical applications, frame functions $\mathbf{G}^{(\mu)} \in \mathrm{R}^{N_o}, \mu \in J_{M_o}$ must have a suitable time-frequency localization to allow the corresponding frame spectrum to distinguish its specific features in the signal. These additional WH frame requirements we will define as desired or reference requirements.

Note that in the algebraic interpretation the signal $\mathbf{s} \in \mathrm{R}^{N_o}$ reconstruction procedure by its frame spectrum is described by the expression

$$\mathbf{s} = \mathbf{G}\mathbf{x} = \mathbf{G}\mathbf{G}^T\mathbf{s}$$

and such reconstruction to be possible if the square symmetric matrix $\mathbf{B} = \mathbf{G}\mathbf{G}^T \in \mathrm{C}^{N_o \times N_o}$ is an identity. This means that for accurate restoration, the rectangular matrix of the reference system $\mathbf{G}$ should consist of orthonormal rows and form a tight frame with borders $A = B = 1$, which, however, taking into account the above construction (3-4) is not guaranteed. Therefore, all further construction will be carried to finding a tight WH-frame with singular boundaries, which best approximates the desired reference system $\mathbf{G}$. In this case, we assume that the matrix $\mathbf{B} = \mathbf{G}\mathbf{G}^T$ is positively determined, which in practice is always the case.

Let's define a set of real rectangular matrices with orthonormal rows

$$\mathcal{A}_{N_o,M_o} = \left\{\mathbf{U} \in \mathrm{M}_{N_o,M_o} : \mathbf{U}^T\mathbf{U} = \mathbf{I}_{N_o}, M_o \ge N_o\right\}$$

and a set of real square matrices

$$\mathcal{A}_{N_o} = \left\{\mathbf{U} \in \mathrm{M}_{N_o} : \mathbf{U}^T\mathbf{U} = \mathbf{I}_{N_o}\right\}.$$

Then the problem of synthesis of an optimal tight WH-frame matrix $\mathbf{U}_{opt} \in \mathcal{A}_{N_o,M_o}$ according to a given model (3-4) can be carried out to solving a two-stage extremum problem:

$$\mathbf{U}_{opt} : \min_{\alpha\in\mathrm{R}}\left(\min_{\mathbf{U}\in\mathcal{A}_{N_o,M_o}} \|\mathbf{G}_\alpha - \mathbf{U}\|_E^2\right), \quad (5)$$

where, first, for an arbitrary phase parameter $\alpha \in \mathrm{R}$, we look among all matrices $\mathbf{U} \in \mathcal{A}_{N_o,M_o}$ a matrix

$$\mathbf{U}_\alpha : \min_{\mathbf{U}\in\mathcal{A}_{N_o,M_o}} \|\mathbf{G}_\alpha - \mathbf{U}\|_E^2 \quad (6)$$

closed to the reference system $\mathbf{G}_\alpha$ in the matrix norm $\|\mathbf{A}\|_E^2 = \mathrm{tr}(\mathbf{A}\mathbf{A}^T)$, then corrects extremum problem.

$$\alpha_{opt} : \min_{\alpha\in\mathrm{R}} \|\mathbf{G}_\alpha - \mathbf{U}_\alpha\|_E^2 \quad (7)$$

minimizing deviation from the reference system by parameter $\alpha \in \mathrm{R}$. The solution of the first problem (6) is formulated in the form of theorem 1, the proof of which, valid for an arbitrary rectangular matrix, is based on the methodology described in [10,11].

Theorem 1. The optimal matrix which ensure the minimum in the extremum problem (6) is defined by the expression

$$\mathbf{U}_\alpha = \mathbf{S}_\alpha\mathbf{W}_\alpha^T \quad (8)$$

where $\mathbf{S}_\alpha \in \mathcal{A}_{N_o}, \mathbf{W}_\alpha \in \mathcal{A}_{N_o,M_o}$ – matrices, included in the singular value decomposition of the matrix

$$\mathbf{G}_\alpha = \mathbf{S}_\alpha\mathbf{\Sigma}_\alpha\mathbf{W}_\alpha^T, \ \mathbf{G}_\alpha \in \mathrm{M}_{N_o,M_o}$$

with a diagonal matrix $\mathbf{\Sigma}_\alpha = \mathrm{diag}\{\sigma_{\alpha 1}, \sigma_{\alpha 2}, \ldots, \sigma_{\alpha N_o}\}$ consisting of the N_o singular numbers $\sigma_i > 0$ of the matrix $\mathbf{G}_\alpha$. The value of the achieved minimum in the problem (6) is equal to

$$\varepsilon_\alpha = \|\mathbf{G}_\alpha - \mathbf{U}_\alpha\|_E^2 = \sum_{n=1}^{N_o}(\sigma_{\alpha n} - 1)^2 = \mathrm{Tr}(\mathbf{\Sigma}_\alpha - \mathbf{I}_{N_o}), \quad (9)$$

where $\mathrm{Tr}(\cdot)$ is the matrix trace operator

It follows from (9) that the smaller the standard deviation of singular numbers $\sigma_{\alpha n}$ from 1, the better the approximation of the reference system $\mathbf{G}_\alpha$ will be.

The solution of an additional extremum problem (7) performed by the method described in [13] leads to the result, which we formulate in the form of the following theorem.

Theorem 2. The optimal value α_{opt} which delivers the minimum to extremum problem (7) and the corresponding optimal solution $\mathbf{U}_{opt} \in \mathcal{A}_{N_o,M_o}$ for a tight WH-frame in the extremum problem (5) is described by the expressions

$$\alpha_{opt} = \pm kM/P, \ k \in \mathbf{Z} \quad (10)$$

$$\mathbf{U}_{opt} = \mathbf{U}_{\alpha_{opt}} = \mathbf{S}_{\alpha_{opt}}\mathbf{W}_{\alpha_{opt}}^T \quad (11)$$

Note that the specific choice of value and sign of α_{opt} can be important in the filter implementation of the WH-frame in the form of a filter Bank. Usually in this case it is enough to choose $\alpha_{opt} = M/P$.

The resulting matrix

$$\mathbf{U}_{opt} = (U_{opt}(i_o, j_o)) = \left(\mathbf{u}_0, \ldots, \mathbf{u}_{M_o-1}\right) \in \mathcal{A}_{N_o, M_o}$$

with vector-functions

$$\mathbf{u}_{j_o} = [U_{opt}(0, j_o), \ldots, U_{opt}(N_o - 1, j_o)]^T$$

describes the desired tight WH-frame with singular boundaries, which best approximates the desired reference system (3)-(4). Therefore, the frame time-frequency spectrum

$$\mathbf{s}_* = (s_*[0], \ldots, s_*[M_o - 1])^T$$

of the signal $\mathbf{s} = (s[0], \ldots, s[N_o - 1])^T$ and its spectral decomposition by WH-frame functions are described, respectively, by expressions

$$\mathbf{s}_* = \mathbf{U}_{opt}^T \mathbf{s} = (\langle \mathbf{u}_0, \mathbf{s} \rangle, \langle \mathbf{u}_1, \mathbf{s} \rangle, \ldots \langle \mathbf{u}_{M_o-1}, \mathbf{s} \rangle)^T,$$

$$\mathbf{s} = \mathbf{U}_{opt}\, \mathbf{s}_* = \sum_{j_o=0}^{M_o-1} s_*[j_o]\, \mathbf{u}_{j_o},$$

where $\langle \cdot, \cdot \rangle$ is the function of the scalar product of vectors. The last expression actually represents the approximation of the signal by the frame WH-model.

We further show that matrices $\mathbf{S}_{\alpha_{opt}} \in \mathcal{A}_{N_o}, \mathbf{W}_{\alpha_{opt}} \in \mathcal{A}_{N_o, M_o}$ can be found explicitly without the use of a singular value decomposition procedure, and the algorithm for finding them admits an efficient computational implementation. Let us first consider the construction of the matrix $\mathbf{S}_{\alpha_{opt}}$.

According to theorem 1, a matrix $\mathbf{S}_\alpha \in \mathcal{A}_{N_o}$ consists of eigenvectors of a symmetric positive-definite matrix , i.e. true an equality $\mathbf{S}_\alpha^T \mathbf{B}_\alpha \mathbf{S}_\alpha = \mathbf{\Lambda}_\alpha$ should be valid, where – $\mathbf{\Lambda}_\alpha = \mathbf{\Sigma}_\alpha^2 = \mathrm{diag}\{\lambda_{\alpha 1}, \lambda_{\alpha 2}, \ldots, \lambda_{\alpha N_o}\}$ is diagonal matrix of positive eigenvalues $\lambda_{\alpha i} = \sigma_{\alpha i}^2$ of the matrix $\mathbf{B}_\alpha$. Moreover, the matrices $\mathbf{S}_\alpha$ and $\mathbf{\Lambda}_\alpha$ are determined uniquely up to the permutation of columns and diagonal elements, respectively.

Using (3)-(4), we present the matrix $\mathbf{B}_\alpha$ explicitly through the matrix elements $\mathbf{G}_\alpha$

$$\begin{gathered}
\mathbf{B}_\alpha = \left(\mathbf{B}_\alpha(i,j)\right)_{i,j \in J_N} = \sum_{m=0}^{M-1} \mathbf{B}_{\alpha\, m}, \\
\mathbf{B}_{\alpha\, m} = \left(\mathbf{B}_\alpha^{(m)}(i,j)\right)_{i,j \in J_N}, \; \mathbf{B}_\alpha^{(m)}(i,j) = \mathbf{V}_{\alpha\, i}^m \mathbf{A}_{i,j} \mathbf{V}_{\alpha\, j}^{-m}, \\
\mathbf{A}_{i,j} = \begin{pmatrix} a^{(ч)}(i,j) & 0 \\ 0 & a^{(нч)}(i,j) \end{pmatrix}, \quad i, j \in J_N, \\
\mathbf{V}_{\alpha\, i}^m = \begin{pmatrix} \cos[\omega(i - \alpha/2)] & -\sin[\omega(i - \alpha/2)] \\ \sin[\omega(i - \alpha/2)] & \cos[\omega(i - \alpha/2)] \end{pmatrix}^m,
\end{gathered} \tag{12}$$

$$\begin{gathered}
a^{(ч)}(i,j) = \sum_{l=0}^{L-1} G_l^{(ч)}(i) G_l^{(ч)}(j), \\
a^{(нч)}(i,j) = \sum_{l=0}^{L-1} G_l^{(нч)}(i) G_l^{(нч)}(j),
\end{gathered} \tag{13}$$

$$G_l^{(ч)}(i) = g_{ep}[n - 2l\, L_0], \quad G_l^{(нч)}(j) = g_{ep}[n - (2l+1)\, L_0],$$

where $\mathbf{B}_\alpha, \mathbf{B}_{\alpha\, m} \in \mathrm{M}_{N_o}$ block matrices consisting of two-dimensional blocks, $\mathbf{B}_\alpha(i,j), \mathbf{B}_\alpha^{(m)}(i,j) \in \mathrm{M}_2$, respectively. Taking into account (12), we transform $\mathbf{B}_\alpha$ to a more simple form

$$\begin{gathered}
\mathbf{B}_\alpha = \left(\mathbf{B}_\alpha(i,j)\right)_{i,j \in J_N} = \left(\sum_{m=0}^{M-1} \mathbf{B}_\alpha^{(m)}(i,j)\right)_{i,j \in J_N}, \\
\mathbf{B}_\alpha(i,j) = \sum_{m=0}^{M-1} \mathbf{V}_{\alpha\, i}^m \mathbf{A}_{i,j} \mathbf{V}_{\alpha\, j}^{-m} = \\
= \begin{pmatrix} \sum_{m=0}^{M-1} b_{\alpha 1}^{(m)}(i,j) & \sum_{m=0}^{M-1} b_{\alpha 3}^{(m)}(i,j) \\ \sum_{m=0}^{M-1} b_{\alpha 3}^{(m)}(i,j) & \sum_{m=0}^{M-1} b_{\alpha 2}^{(m)}(i,j) \end{pmatrix} = \\
= \begin{pmatrix} b_{\alpha 1}(i,j) & b_{\alpha 3}(i,j) \\ b_{\alpha 3}(i,j) & b_{\alpha 2}(i,j) \end{pmatrix},
\end{gathered} \tag{14}$$

where elements of the last matrix are described by expressions:

$$\begin{gathered}
b_{\alpha 1}(i,j) = \gamma^+(i,j) \sum_{m=0}^{M-1} \cos[\omega m(i-j)] + \\
+ \gamma^-(i,j) \sum_{m=0}^{M-1} \cos[\omega m(i+j-\alpha)], \\
b_{\alpha 2}(i,j) = \gamma^+(i,j) \sum_{m=0}^{M-1} \cos[\omega m(i-j)] + \\
- \gamma^-(i,j) \sum_{m=0}^{M-1} \cos[\omega m(i+j-\alpha)], \\
b_{\alpha 3}(i,j) = -\gamma^+(i,j) \sum_{m=0}^{M-1} \sin[\omega m(i-j)] + \\
+ \gamma^-(i,j) \sum_{m=0}^{M-1} \sin[\omega m(i+j-\alpha)],
\end{gathered} \tag{15}$$

$$\begin{gathered}
\gamma^+(i,j) = [a^{(ч)}(i,j) + a^{(нч)}(i,j)]/2, \\
\gamma^-(i,j) = [a^{(ч)}(i,j) - a^{(нч)}(i,j)]/2
\end{gathered} \tag{16}$$

Using (13), it is easy to verify the validity of the following equations

$$\begin{gathered}
\gamma^-(i,j) \sum_{m=0}^{M-1} \cos[\omega m(i+j-\alpha_{opt})] = 0, \\
\sum_{m=0}^{M-1} \sin[\omega m(i-j)] = 0, \quad \sum_{m=0}^{M-1} \sin[\omega m(i+j-\alpha)] = 0
\end{gathered}. \tag{17}$$

Therefore, after substitution (17) in (15), the expression (14) for the matrix $\mathbf{B}_{\alpha_{opt}}$ is significantly simplified and takes the following canonical form

$$\mathbf{B}_{\alpha_{opt}} = \mathbf{B}_o \otimes \mathbf{I}_2, \quad \mathbf{B}_o = \mathbf{\Gamma}^+ \circ \mathbf{U}_c^-, \tag{18}$$

$$\mathbf{\Gamma}^+ = (\gamma^+(i,j))_{i,j \in J_N}, \quad \mathbf{U}_c^- = \left(\sum_{m=0}^{M-1} \cos[\omega(i-j)]\right)_{i,j \in J_N}, \tag{19}$$

where $\otimes$, $\circ$ – are operators of direct and element product of matrices, respectively, $\mathbf{\Gamma}^+, \mathbf{U}_c^- \in \mathrm{M}_N$ – are symmetric matrices. Note that when

$$\alpha \neq \alpha_{opt}, \; \gamma^-(i,j) \sum_{m=0}^{M-1} \cos[\omega m(i+j-\alpha)] \neq 0,$$

the expression for the matrix $\mathbf{B}_\alpha$ is significantly complicated, and all subsequent arguments associated with its transformations and the calculation of eigenvalues are unfair.

For this reason , we further consider only the case, $\alpha = \alpha_{opt}$, keeping the corresponding matrices index α_{opt}.

Let's define a block-diagonal singular matrix of dimension $(N_o \times N_o)$

$$\mathbf{F}_{-} = \mathbf{I}_{2L_o} \otimes \mathbf{F}_o \,, \quad \mathbf{F}_o = K^{-1/2}\left(\exp(j2\pi pq / K)\right)_{p,q\in J_K}, \quad (20)$$

in which there are diagonally unitary Fourier matrices $\mathbf{F}_o$ of dimension $(K \times K)$, where $K = PL$, and a symmetric orthogonal matrix similar in structure

$$\mathbf{Q}_{-} = \mathbf{I}_{2L_o} \otimes \mathbf{Q}_o, \quad \mathbf{Q}_o = \mathrm{Re}(\mathbf{F}_o) + \mathrm{Im}(\mathbf{F}_o) \quad (21)$$

Let's define an orthogonal permutation matrix $\mathbf{P} \in \mathcal{A}_{N_o}$:

$$\mathbf{P}_{-} = \left(P_{-}(i_o, j_o)\right)_{i_o, j_o \in J_{N_o}},$$

$$P_{-}(i_o, j_o) = \begin{cases} 1, & j_o = 2L_o (i_o)_{[K]} + 2\left\lfloor i_{o[N]} / K \right\rfloor + \left\lfloor i_o \,/\, N \right\rfloor \\ 0, & else \end{cases} \quad (22)$$

where $a_{[n]} = a \,(\mathrm{mod}\, n)$ – is the value of number a modulo n, $\lfloor \cdot \rfloor$ – is the operator of taking integer part of the number with rounding down, and using (20)-(21) we define two multiplicative compositions of matrices

$$\mathbf{S}'_{-} = \mathbf{P}_{-}^T \mathbf{F}_{-}\,, \qquad \mathbf{S}_{-} = \mathbf{P}_{-}^T \mathbf{Q}_{-}\,, \quad (23)$$

here $\mathbf{S}'_{-}$ – a complex singular and $\mathbf{S}_{-}$ – real orthogonal matrices. Then, if you perform the similarity transformation over matrix $\mathbf{B}_{\alpha_{opt}}$

$$\tilde{\mathbf{B}}_{\alpha_{opt}} = \mathbf{P}_{-} \mathbf{B}_{\alpha_{opt}} \mathbf{P}_{-}^T, \quad (24)$$

$$\begin{aligned} \mathbf{\Lambda}'_{\alpha_{opt}} &= \mathbf{F}_{-}^* \tilde{\mathbf{B}}_{\alpha_{opt}} \mathbf{F}_{-} = \mathbf{S}'^{*}_{-} \mathbf{B}_{\alpha_{opt}} \mathbf{S}'_{-}, \\ \mathbf{\Lambda}_{\alpha_{opt}} &= \mathbf{Q}_{-}^T \tilde{\mathbf{B}}_{\alpha_{opt}} \mathbf{Q}_{-} = \mathbf{S}_{-}^T \mathbf{B}_{\alpha_{opt}} \mathbf{S}_{-} \end{aligned}, \quad (25)$$

(where $(\cdot)^*$ – operator of Hermitian conjugation of matrices) then, using the found canonical representation (18), it is possible to prove the following two theorems.

Theorem 3. The matrix $\tilde{\mathbf{B}}_{\alpha_{opt}} \in \mathrm{M}_{N_o}$ has a block-diagonal circulant structure

$$\tilde{\mathbf{B}}_{\alpha_{opt}} = \left(\tilde{\mathbf{B}}_{p',q'}\right)_{p',q' \in J_{2L_o}},$$

$$\tilde{\mathbf{B}}_{p',q'} = \begin{cases} \tilde{\mathbf{B}}_{p',p'} & p' = q' \\ \mathbf{0} & p' \neq q' \end{cases}, \quad \tilde{\mathbf{B}}_{p',p'} = \left(\tilde{b}_{p'}[(p-q)_{[K]}]\right)_{p,q \in J_K}, \quad (26)$$

where $\mathbf{0}_K \in \mathrm{M}_K$ – zero matrix, $\tilde{\mathbf{B}}_{p',p'}$ – located on diagonal circulant matrix blocks with forming elements $\tilde{b}_{p'}[i], i \in J_K$:

$$\tilde{b}_{p'}[i] = \begin{cases} \dfrac{M}{2} \displaystyle\sum_{l=0}^{K-1} g_{ep}[lL_o + p'] g_{ep}[(i+l)_{[K]} L_o + p'], & i_{[P]} = 0 \\ 0\,, & i_{[P]} \neq 0 \end{cases} \quad (27)$$

Moreover, the forming vector

$$(\tilde{b}_{p'}[0], \ldots, \tilde{b}_{p'}[K-1])^T = \tilde{\mathbf{b}}_{p'} \in \mathrm{R}^K$$

of circulant matrix $\tilde{\mathbf{B}}_{p',p'}$, $p' \in J_{2L_o}$ consists of $K/P = L$ nonzero elements, i.e. is a P multiple decimated vector.

Theorem 4. Similarity transformations (25) with matrices (23) are diagonalizing and lead to identical real matrices $\mathbf{\Lambda}'_{\alpha_{opt}} = \mathbf{\Lambda}_{\alpha_{opt}} = \mathrm{diag}(\boldsymbol{\lambda}_{-}) \in \mathrm{M}_{N_o}$ in which the eigenvalues $[\lambda_1, \lambda_2, \ldots, \lambda_{N_o}]^T = \boldsymbol{\lambda}_{-}$ of the matrix $\mathbf{B}_{\alpha_{opt}}$ are on the diagonal . Moreover, all eigenvalues $\lambda_{i_o} > 0$ are positive, have multiplicity not less then $2P$ and can be obtained from the composite forming vector $\tilde{\mathbf{b}}_{-} = (\tilde{\mathbf{b}}_0^T, \ldots, \tilde{\mathbf{b}}_{2L_o-1}^T)^T \in \mathrm{R}^{N_o}$ by means of orthogonal transformations (20-21) by any of the following formulas

$$\boldsymbol{\lambda}_{-} = \sqrt{K}\, \mathbf{F}_{-}^* \tilde{\mathbf{b}}_{-}\,, \qquad \boldsymbol{\lambda}_{-} = \sqrt{K}\, \mathbf{Q}_{-}^T \tilde{\mathbf{b}}_{-}\,. \quad (28)$$

Thus, we found an explicit form of the real orthogonal transformation

$$\mathbf{S}_{\alpha_{opt}} = \mathbf{S}_{-} = \mathbf{P}_{-}{}^T \mathbf{Q}_{-}\,, \quad (29)$$

of the optimal solution (11), which diagonalizes the matrix $\mathbf{B}_{\alpha_{opt}} \in \mathrm{M}_{N_o}$. To find the orthogonal transformation $\mathbf{W}_{\alpha_{opt}}$, we use the statement proved in [10], according to which

$$\mathbf{W}_{\alpha_{opt}} = \mathbf{G}_{\alpha_{opt}}^T \mathbf{S}_{\alpha_{opt}} \mathbf{\Lambda}_{\alpha_{opt}}^{-1/2}\,. \quad (30)$$

Substituting (29),(30) into (11), we obtain the WH-frame matrix $\mathbf{U}_{opt}$ after the transformations

$$\mathbf{U}_{opt} = \mathbf{H}_{\alpha_{opt}} \mathbf{G}_{\alpha_{opt}}\,, \quad \mathbf{H}_{\alpha_{opt}} = \mathbf{S}_{\alpha_{opt}} \mathbf{\Lambda}_{\alpha_{opt}}^{-1/2} \mathbf{S}_{\alpha_{opt}}^T\,, \quad (31)$$

where $\mathbf{H}_{\alpha_{opt}}$ determines the desired matrix operator of the optimal adjustment (identification of the WH-frame) to the desired reference (reference system) system of functions $\mathbf{G}_{\alpha_{opt}}$, which was mentioned in the problem description part of the article. Moreover, taking into account (31), (29) the matrix $\mathbf{H}_{\alpha_{opt}}$ is factorized into the product of sparse matrices.

$$\mathbf{H}_{\alpha_{opt}} = \mathbf{P}_{-}^T \mathbf{Q}_{-} \mathbf{\Lambda}_{\alpha_{opt}}^{-1/2} \mathbf{Q}_{-}^T \mathbf{P}_{-}\,. \quad (32)$$

We further will show that finding the diagonal matrix $\mathbf{\Lambda}_{\alpha_{opt}}$, and hence the adjusting procedure (31), can be greatly simplified by using the canonical representation of the matrix (24).

Theorem 5. Matrix $\tilde{\mathbf{B}}_{\alpha_{opt}} = \mathbf{P}_{-} \mathbf{B}_{\alpha_{opt}} \mathbf{P}_{-}^T \in \mathrm{M}_{N_o}$ admits the following canonical representation

$$\tilde{\mathbf{B}}_{\alpha_{opt}} = \mathbf{I}_2 \otimes \tilde{\mathbf{B}}_{o\alpha_{opt}}\,, \quad \tilde{\mathbf{B}}_{o\alpha_{opt}} = \mathbf{P}\, \mathbf{B}_o \mathbf{P}^T\,, \quad (33)$$

where $\mathbf{B}_o \in \mathrm{M}_N$ – is matrix from (18), $\mathbf{P} \in \mathcal{A}_N$ – the orthogonal matrix of the permutation

$$\mathbf{P} = \left(P(i,j)\right)_{i,j \in J_N}, \quad P(i,j) = \begin{cases} 1, & j = L_o\, i_{[K]} + \lfloor i / K \rfloor \\ 0, & else \end{cases} \quad (34)$$

$\tilde{\mathbf{B}}_{o\alpha_{opt}} \in \mathrm{M}_N$ – is a block-diagonal circulant matrix

$$\tilde{\mathbf{B}}_{o\alpha_{opt}} = \left(\tilde{\mathbf{B}}_{p',q'}\right)_{p',q'\in J_{L_o}},$$

$$\tilde{\mathbf{B}}_{p',q'} = \begin{cases} \tilde{\mathbf{B}}_{p',p'} & p'=q' \\ \mathbf{0} & p' \neq q' \end{cases}, \quad \tilde{\mathbf{B}}_{p',p'} = \left(\tilde{b}_{p'}[(p-q)_{[K]}]\right)_{p,q\in J_K}, \qquad (35)$$

In which forming elements $(\tilde{b}_{p'}[0],\dots,\tilde{b}_{p'}[K-1])^T = \tilde{\mathbf{b}}_{p'} \in \mathrm{R}^K$ of the circulant blocks $\tilde{\mathbf{B}}_{p',p'} \in \mathrm{M}_K$ are described by the expression (27).

Thus, the structure of the matrix (35) is similar to the structure (26), but its dimension is two times smaller. By analogy with (20), (21), (23) we will determine the orthogonal dimension matrix $(N \times N)$

$$\mathbf{F} = \mathbf{I}_{L_o} \otimes \mathbf{F}_o, \quad \mathbf{Q} = \mathbf{I}_{L_o} \otimes \mathbf{Q}_o, \qquad (36)$$

$$\mathbf{S}' = \mathbf{P}^T\mathbf{F}, \qquad \mathbf{S} = \mathbf{P}^T\mathbf{Q}, \qquad (37)$$

The following Theorem will be valid.

Theorem 6. Similarity transformations with matrices (37) are diagonalizing for the symmetric matrix (35):

$$\begin{aligned} \mathbf{\Lambda}'_{o\alpha_{opt}} &= \mathbf{F}^*\tilde{\mathbf{B}}_{o\alpha_{opt}}\mathbf{F} = \mathbf{S}'^*\mathbf{B}_o\mathbf{S}', \\ \mathbf{\Lambda}_{o\alpha_{opt}} &= \mathbf{Q}^T\tilde{\mathbf{B}}_{o\alpha_{opt}}\mathbf{Q} = \mathbf{S}^T\mathbf{B}_o\mathbf{S} \end{aligned} \qquad (38)$$

and lead to the same real diagonal matrices $\mathbf{\Lambda}'_{o\alpha_{opt}} = \mathbf{\Lambda}_{o\alpha_{opt}} \in \mathrm{M}_N$ in which the diagonals are the eigenvalues $[\lambda_1,\lambda_2,\dots,\lambda_N]^T = \boldsymbol{\lambda}$ of the matrix $\mathbf{B}_o$. All eigenvalues a $\lambda_i > 0$ are positive, have multiplicity not less than P and can be obtained from the composite forming vector $\tilde{\mathbf{b}} = (\tilde{\mathbf{b}}_0^T,\dots,\tilde{\mathbf{b}}_{L_o-1}^T)^T \in \mathrm{R}^N$ by orthogonal transformations (36) using any of the following formulas

$$\boldsymbol{\lambda} = \sqrt{K}\,\mathbf{F}^*\tilde{\mathbf{b}}, \qquad \boldsymbol{\lambda} = \sqrt{K}\,\mathbf{Q}^T\tilde{\mathbf{b}} \qquad (45)$$

From Theorems 4-6 it follows that

$$\tilde{\mathbf{b}}_- = (\tilde{\mathbf{b}}^T,\tilde{\mathbf{b}}^T)^T, \quad \boldsymbol{\lambda}_- = (\boldsymbol{\lambda}^T,\boldsymbol{\lambda}^T)^T, \qquad (46)$$

$$\mathbf{\Lambda}_{\alpha_{opt}} = \mathbf{I}_2 \otimes \mathbf{\Lambda}_{o\alpha_{opt}}, \quad \mathbf{\Lambda}_{o\alpha_{opt}} = \mathrm{diag}\,(\boldsymbol{\lambda}), \qquad (47)$$

It twice simplifies the process of calculating diagonal matrix $\mathbf{\Lambda}_{\alpha_{opt}}$, which is part of the operator of the optimal adjustment $\mathbf{H}_{\alpha_{opt}}$ (32).

III. Experimental result

Fig. 1 shows the results of a computational experiment in which the matrices $\mathbf{P}_-$, $\mathbf{Q}_-$, $\mathbf{H}_{\alpha_{opt}}$ included in (32) were calculated for the selected parameters $P=4, L=3, M=8$, $N_o = 48$ and the forming pulse (1) of the desired reference system $\mathbf{G}_{\alpha_{opt}}$.

The structure of these matrices is displayed as points characterizing non-zero elements, and at the bottom of each image of the matrix the total number of its non-zero elements is indicated (the indices of the matrices in the figure are omitted).

The analysis of the structure shows that not only each of the cofactors $\mathbf{P}_-$, $\mathbf{Q}_-$, $\mathbf{\Lambda}_{\alpha_{opt}}$ in (32) is a sparse matrix, but their product – matrix $\mathbf{H}_{\alpha_{opt}}$ is also strongly sparse. This fact is of separate special interest, since in general the product of sparse matrices does not have to be a sparse matrix.

All this allows for fast computational implementation of the WH-frame identification algorithm, and therefore flexible adjustment of the frequency-time resolution in the process of adjusting the parameters of the reference system $\mathbf{G}_{\alpha_{opt}}$.

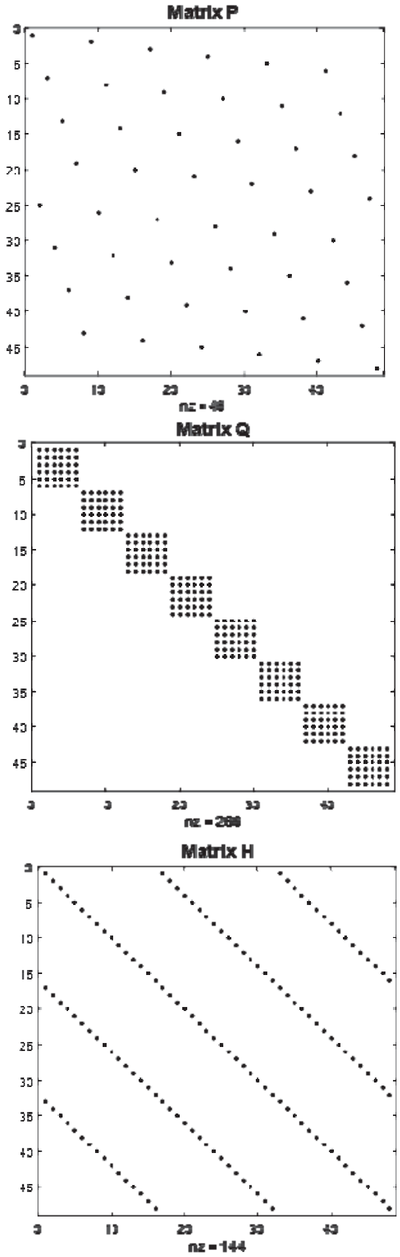

Fig. 1. . Results of computational experiment

Note that the matrices $\mathbf{P}_{-}$, $\mathbf{Q}_{-}$ included in the right part of the equation (32) do not depend of the forming pulse (1), and their structure is determined only by the number of basic shifts in time L, frequency M and oversampling coefficient P. The diagonal matrix $\mathbf{\Lambda}_{\alpha_{opt}} \in \mathrm{M}_{N_o}$ from (32) in the selected structure L, M, P depends only on the shape of the pulse (1),

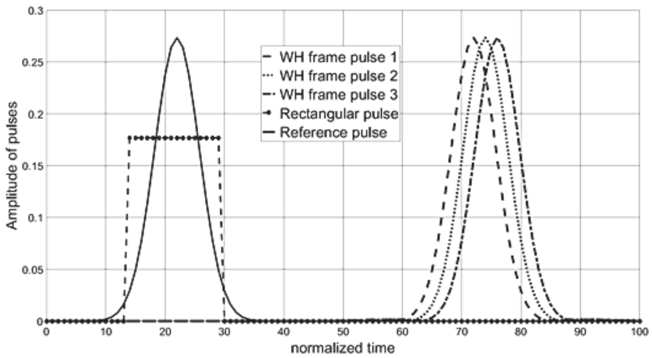

Fig. 2. Pulses of the optimal WH-frame

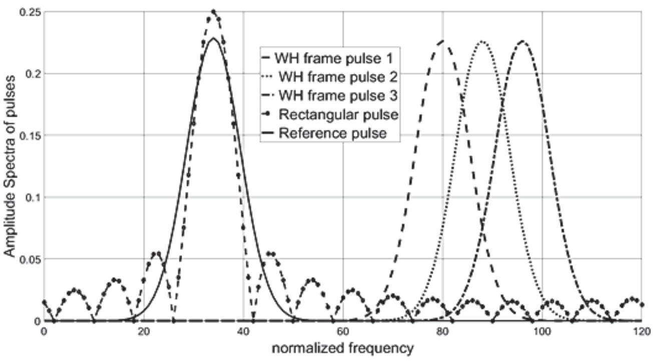

Fig. 3. . Spectra pulses of the optimal WH-frame

Figures 2-3 show the results of the computational experiment of the optimal WH-frame (31) for the values $L = 8, M = 16, P = 8$, which allow to estimate the time-frequency localization of the three adjacent in time and frequency functions included in the matrix $\mathbf{U}_{opt}$, as well as to compare them with one of the pulses of the reference system $\mathbf{G}_{\alpha_{opt}}$ and a rectangular pulse having a similar effective duration.

At Fig. 2 the pulses of the optimal WH-frame for any of its columns $\mathbf{U}_{\alpha_{opt}}^{(j_o)}$ are constructed as an envelopes of quadrature components formed from even and odd samples of this column. Similarly, the pulses corresponding to any column $\mathbf{G}_{\alpha}^{(j_o)}$ of the reference model are constructed. Spectra pulses of the optimal WH-frame $\mathbf{U}_{opt}$ and reference system $\mathbf{G}_{\alpha_{opt}}$ are defined as discrete Fourier transform from these pulses, and their image in Fig. 3 represents the module of these spectra.

The analysis of the presented graphs shows that the optimal WH-frame pulses are very close in shape to the desired pulses of the system $\mathbf{G}_{\alpha_{opt}}$ and are well localized in time and frequency, allowing to provide the required characteristics of the frequency-time resolution.

IV. Conclusion

1) The problem of synthesis of the optimal WH-frame with the desired properties is solved. The selected quality criterion minimizes the value of its deviation from the desired standard by the standard criterion and greatly simplifies the structure of subsequent decisions.

2) On the basis of the algebraic approach, the WH-frame identification algorithm is synthesized, based on its optimal adjustment to the desired reference model, in the form of a real system of pulses uniformly shifted in time and frequency with the desired symmetry property, frequency-time resolution and range.

3) It is shown that the developed vector-matrix WH-frame identification algorithm is represented as a product of sparse matrices, which allows for its fast computational implementation in object-oriented programming using the sparse matrices algebra approach.

4) The analysis of the presented graphs shows that the optimal W-frame pulses are very close in shape to the desired pulses of the standard and are well localized in time and frequency, allowing to provide the required characteristics of the frequency-time resolution.

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