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GRADIENT ESTIMATES FOR A NONLINEAR ELLIPTIC EQUATION ON COMPLETE NONCOMPACT RIEMANNIAN MANIFOLD

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Abstract. Let (M,g) be an n-dimensional complete noncompact Riemannian manifold (with possibly empty boundary). We derive local and global gradient estimates on positive solutions $u(x)$ to the following nonlinear elliptic equation

$$\Delta u(x)+au^s(x)+\lambda(x)u(x)=0, \qquad x\in M,$$

where a and s are constants, $a\in\mathbb{R}\setminus\{0\}$, $s>1$ and $\lambda(x)$ is bounded on M. Our gradient estimates yield differential Harnack inequalities as an application. This paper extends results of Y. Yang [17] and J. Li [11, Theorem 3.1].

1. Introduction

Let M be a complete noncompact Riemannian manifold (with possibly empty boundary) of dimension n. In this paper we study gradient estimates and Harnack inequalities on positive solutions $u(x)$ to the nonlinear elliptic PDE

$$\Delta u(x)+au^s(x)+\lambda(x)u(x)=0 \tag{1}$$

on M, where $s>1$, $\lambda(x)>0$ and $a\in\mathbb{R}\setminus\{0\}$. By integrating along a geodesic between two points in M our gradient estimates yield classical Harnack inequalities, which compare solutions at different points. The importance of gradient estimates and Harnack inequalities cannot be overemphasized in geometric analysis and mathematical physics in general, they have been used to find Hölders continuity of solutions, sharp estimate on the fundamental solution, estimate on the principal eigenvalue to mention but a few. For examples, see the following references [6, 7, 8, 10, 11, 12, 13, 15, 17]. In particular, Ma [13] studies gradient estimate on positive solutions to the elliptic equation

$$\Delta u(x)+au(x)\log u(x)=0 \quad \text{on } M, \tag{2}$$

while its parabolic counterpart is considered by Yang [16]. For $\lambda(x)\equiv 0$, physical applications of (1) are found in the theory of stellar structure in Astrophysics $(n=3)$

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and Yang Mills' problem for $n=4$ and $s=(n+2)/(n-3)$ in Physics (see [8, 4]). See also [8, Appendix B] for abstract examples.

In [17] Yang obtained interesting gradient estimates for positive solutions to the following elliptic equation with singular nonlinearity

$$\Delta u(x)+cu^{-\alpha}(x)=0, \tag{3}$$

on M whose $Ric(M)\geqslant -Kg$, $K\geqslant 0$, where α, c are two real constants and $\alpha>0$. Precisely, he obtained

$$\frac{|\nabla u|^2}{u^2}+cu^{-(\alpha+1)}\leqslant \frac{n(2n+1)\varepsilon^2}{R^2}+\frac{n(n-1)\varepsilon^2}{R}\sqrt{K}+\frac{n\nu}{R^2}+2nK \tag{4}$$

for $c>0$ and

$$\begin{aligned}\frac{|\nabla u|^2}{u^2}+cu^{-(\alpha+1)}\leqslant{}& \Big(n(\alpha+1)(\alpha+1)+\sqrt{n}(\alpha+1)\Big)|c|\Big(\inf_{B_{2R}(p)} u\Big)^{-(\alpha+1)}\\ &+\frac{n\nu}{R^2}+\Big(2n+\frac{\sqrt{n}}{\alpha+1}\Big)K\\ &+\frac{n\varepsilon^2}{R^2}\Big(2n+1+\frac{\sqrt{n}}{2(\alpha+1)}+(n-1)\sqrt{K}R\Big)\end{aligned} \tag{5}$$

for $c<0$ on metric ball of radius $2R$, $R>0$ around a point p in M with $\varepsilon>0$ and $\nu>0$ being some universal constants independent of the geometry of M. An immediate application of (4) is the following

COROLLARY 1. ([17, Corollary 1.2]) *Let M be a noncompact complete Riemannian manifold of dimension n without boundary. Suppose that the Ricci curvature of M is nonnegative and that $\alpha, c>0$ are two positive constants. Then (3) does not have a smooth positive solution on M.*

Zhang and Ma [19] have extended Yang's result to the followng elliptic equation

$$\Delta_f u(x)+cu^{-\alpha}(x)=0, \tag{6}$$

where $\Delta_f=\Delta-\nabla f\nabla$ and $f\in C^\infty(M)$ on the condition that N-Bakry-Emery Ricci tensor is bounded from below. In [11, Theorem 3.1] Li derived a gradient estimate and Harnack inequality on solutions $u(x)>0$ of

$$\Delta u(x)+b(x)\nabla u(x)+h(x)u^\alpha(x)=0 \tag{7}$$

on M with $Ric(g)\geqslant -k$, $h(x)\in C^2(M)$, $b(x)\in C^\infty(M)$ and $1<\alpha<n/(n-2)$, $n\geqslant 4$.

Our results extend the above mentioned results in [17] and [11]. Estimates on solutions to the parabolic counterpart of (1) are of interest too. We are prompted by this to study the following nonlinear equation

$$\frac{\partial u(t,x)}{\partial t}=\Delta u(t,x)+au^s(t,x)+\lambda(x)u(t,x) \tag{8}$$

where $(t,x) \in ([0,\infty) \times M)$. The results [2] for the case the metrics $g = g(t)$ evolve by the abstract geometric flow will appear somewhere else. Interested readers can also see [1, 3, 9] for similar results in this direction.

The rest of the paper is organized as follows. In Section 2 we give some notations and state our main results. Section 3 is devoted to the proofs of the main theorems. We first state and prove a technical lemma that would be applied to prove the results. We also give a description of the cut-off function needed in the proof. Lastly in this section, we derive a Harnack inequality by integrating along the space path.

2. Notation and main results

Let (M,g) be an n-dimensional complete noncompact Riemannian manifold (possibly with empty boundary) and the Ricci curvature bounded below by a negative constant. In local coordinates, we denote the Laplace Beltrami operator by

$$\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i}\Big(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x_j}\Big),$$

the gradient of a smooth function f by

$$\nabla f = (\nabla f)^i = g^{ij} \frac{\partial f}{\partial x_j} \quad \text{and} \quad |\nabla f|^2 = g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$$

and the Ricci curvature tensor of M by

$$Ric_{ij} = \frac{\partial \Gamma^k_{ij}}{\partial x_k} - \frac{\partial \Gamma^k_{ij}}{\partial x_i} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{il}\Gamma^l_{kj},$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\Big(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\Big)$$

are the Christoffel's symbols and g_{ij}, $|g|$ and $g^{ij} = (g_{ij})^{-1}$ are respectively the components, determinant and the inverse of metric matrix g. Repeated indices are summed up. For convinience sake we avoid working in orthonormal frames.

A natural function that will be defined on M is the distance function from a given point, namely, let $y \in M$ and define $d(x,y)$ for all $x \in M$, where $d(\cdot,\cdot)$ is the geodesic distance. Note that d is everywhere continuous except on the cut locus of y and on the point where x and y coincide. It is then easy to see that $|\nabla d| = 1$ on $M \setminus \{\{y\} \cup cut(y)\}$. Let $d(x,y)$ be the geodesic distance between x and y, we shall define a smooth cut-off function $\varphi(x)$ with support in the geodesic ball around p

$$\mathscr{B}_{2R}(p) := \{p \in M : d(x,p) \leqslant 2R,\ R > 0\}.$$

In a situation where M has an empty boundary we shall choose $\mathscr{B}_{2R}(p) \subset M$ such that it does not touch the boundary of M.

Let $u \in C^2(M)$ be a positive solution to the following nonlinear elliptic equation

$$\Delta u(x) + au^s(x) + \lambda(x)u(x) = 0 \tag{9}$$

on M, where $s > 1$, $\lambda(x) > 0$ and $a \in \mathbb{R} \setminus \{0\}$. Note that a simple calculation shows that if we take $w(x) = \log u(x)$, then, $w(x)$ solves

$$\Delta w(x) + |\nabla w(x)|^2 + ae^{(s-1)w(x)} + \lambda(x) = 0, \qquad s > 1 \tag{10}$$

another nonlinear elliptic equation.

Our main results are stated as follows.

2.1. Main results

THEOREM 1. (Local gradient estimate) *Let (M,g) be an n-dimensional complete noncompact Riemannian manifold. Suppose there exists a nonnegative constant $k := k(2R)$ such that the Ricci curvatute of M is bounded below by $-k$, i.e., $Ric(g) \geqslant -k(2R)$ in the geodesic ball $\mathscr{B}_{2R}(p)$. For any smooth positive solution $u(x)$ to equation (1) in the metric ball $\mathscr{B}_{2R}(p)$, where $u \leqslant \mathbb{M}$, $\lambda > 0$ and $\Delta\lambda \geqslant 0$, then*

$$\frac{|\nabla u|^2}{u^2} + au^{s-1} + \lambda \leqslant \frac{n\Big[(n-1)(1+\sqrt{k}R)+2\Big]C_1^2 + C_2)}{R^2} + \frac{n^2C_1^2}{R^2} + 2nk \tag{11}$$

for $a < 0$ and

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} + 2au^{s-1} + \lambda \leqslant{}& \frac{n\Big[(n-1)(1+\sqrt{k}R)+2\Big]C_1^2 + C_2)}{R^2} + \frac{n^2C_1^2}{R^2} + 2nk \\ & +2a^2n(s-1)\mathbb{M}^{2(s-1)}\end{aligned} \tag{12}$$

for $a > 0$. Here C_1, C_2 are absolute constants independent of the geometry of M.

Letting $R \to \infty$, we obtain the following global gradient estimates for the solutions of (1).

THEOREM 2. (Global gradient estimates) *Assume the conditions of Theorem 1. Then, the following global gradient estimate holds*

$$\frac{|\nabla u|^2}{u^2} + au^{s-1} + \lambda \leqslant 2nk, \qquad a < 0 \tag{13}$$

and

$$\frac{|\nabla u|^2}{u^2} + au^{s-1} + \lambda \leqslant 2nk + +2a^2n(s-1)\mathbb{M}^{2(s-1)}, \qquad a > 0 \tag{14}$$

for all $x \in M$.

As an application of the above theorem we derive the following classical Harnack inequality. This is a comparison of values of positive solution at different points.

COROLLARY 2. *Suppose that $a, s, \lambda(x), R$ satisfy the hypotheses of Theorem 1, if $u(x)$ is a positive solution of (1) and satisfies $u \leqslant \mathbb{M}$, then*

$$\sup_{\mathscr{B}_{2R}} u(x) \leqslant e^{B(n,\, k,\, |a|,\, s,\, \|\lambda\|_\infty,\, R,\, \mathbb{M})} \inf_{\mathscr{B}_{2R}} u(x). \tag{15}$$

where $B(n,\, k,\, |a|,\, s,\, \|\lambda\|_\infty,\, R,\, \mathbb{M})$ is a quantity depending on $n,\, k,\, |a|,\, s,\, R$ and

$$\|\lambda\|_\infty = \|\lambda\|_{L^\infty} =: \sup_{\mathscr{B}_{2R}} |\lambda(x)|.$$

Our method follows from the method of gradient estimates which originated first in Yau [18] and Cheng–Yau [7]. Meanwhile a fundamental formula in differential geometry which is very crucial to proving our results is the following:

LEMMA 1. (Bochner-Weitzenböck's formula) *Given any smooth function f on (M,g), it holds that*

$$\Delta(|\nabla f|^2) = 2|\nabla^2 f|^2 + 2\nabla f \nabla(\Delta f) + 2Ric(g)(\nabla f, \nabla f), \tag{16}$$

where $\nabla^2 f$ is the Hessian of f and $Ric(g)$ is the Ricci tensor of the metric g.

3. Proof of main results

3.1. Technical lemma

The following technical lemma is very crucial to derivation of our gradient estimate.

LEMMA 2. *Let (M,g) be a complete noncompact Riemannian manifold. Suppose there exists a nonnegative constant $k := k(2R)$ such that the Ricci curvatute of M is bounded below by $-k$, i.e., $Ric(g) \geqslant -k(2R)$. For any smooth positive solution $u(x)$ to equation (1) in the metric ball $\mathscr{B}_{2R}(p)$, where $\lambda > 0$, $\Delta\lambda > 0$, it then holds that*

$$\begin{aligned}\Delta G \geqslant & -2\nabla w \nabla G + \frac{2}{n}\Big(-G + (\beta - a)e^{(s-1)w}\Big)^2 \\ & + \Big(\Big[\beta(s-1)^2 + 2(\beta - a)(s-1)\Big]e^{(s-1)w} - 2k\Big)\Big(G - \beta e^{(s-1)w} - \lambda\Big) \\ & + \beta(s-1)\Big(-G + (\beta - a)e^{(s-1)w}\Big)e^{(s-1)w},\end{aligned} \tag{17}$$

where $w = \log u$, $G = |\nabla w|^2 + \beta e^{(s-1)w} + \lambda$ and β is a nonzero constant to be determined.

Proof. Define a Harnack quantity

$$G(x) = |\nabla w|^2 + \beta e^{(s-1)w} + \lambda.$$

Notice that

$$G = -\Delta w + (\beta - a)e^{(s-1)w}, \tag{18}$$

$$\Delta w = -G + (\beta - a)e^{(s-1)w} \tag{19}$$

and

$$|\nabla w|^2 = G - \beta e^{(s-1)w} - \lambda. \tag{20}$$

Also by the Bochner-Weitzenböck's formula (16) and the assumption on the Ricci curvature tensor we have

$$\Delta|\nabla w|^2 \geqslant 2|\nabla^2 w|^2 + 2\nabla w \nabla(\Delta w) - 2k|\nabla w|^2. \tag{21}$$

Now

$$\Delta G = \Delta|\nabla w|^2 + \Delta(\beta e^{(s-1)w}) + \Delta\lambda. \tag{22}$$

We now compute using (18)-(22) (with the condition that $\Delta\lambda > 0$)

$$\begin{aligned}
\Delta G \geqslant\ & 2|\nabla^2 w|^2 + 2\nabla w \nabla(\Delta w) - 2k|\nabla w|^2 + \beta(s-1)\Delta w e^{(s-1)w} \\
& + \beta(s-1)^2|\nabla w|^2 e^{(s-1)w}. \\
=\ & 2|\nabla^2 w|^2 + 2\nabla w \nabla\Big(-G + (\beta - a)e^{(s-1)w}\Big) - 2k|\nabla w|^2 \\
& + \beta(s-1)\Big(-G + (\beta - a)e^{(s-1)w}\Big)e^{(s-1)w} + \beta(s-1)^2|\nabla w|^2 e^{(s-1)w} \\
\geqslant\ & \frac{2}{n}(\Delta w)^2 - 2\nabla w \nabla G + 2(\beta - a)(s-1)|\nabla w|^2 e^{(s-1)w} - 2k|\nabla w|^2 \\
& + \beta(s-1)\Big(-G + (\beta - a)e^{(s-1)w}\Big)e^{(s-1)w} + \beta(s-1)^2|\nabla w|^2 e^{(s-1)w} \\
=\ & \frac{2}{n}(\Delta w)^2 - 2\nabla w \nabla G + \beta(s-1)\Big(-G + (\beta - a)e^{(s-1)w}\Big)e^{(s-1)w} \\
& + \Big(\Big[\beta(s-1)^2 + 2(\beta - a)(s-1)\Big]e^{(s-1)w} - 2k\Big)|\nabla w|^2
\end{aligned}$$

where we have also used the identity

$$|\nabla^2 w|^2 \geqslant \frac{1}{n}(\Delta w)^2 \tag{23}$$

derived by Cauchy-Schwarz inequality.

Rearranging, we arrive at the desired result. Our calculation is valid in the metric ball $\mathscr{B}_{2R}(x)$. □

3.2. Estimating the cut-off function

Define a smooth cut-off function $\varphi(x)$ with support in the metric ball

$$\mathscr{B}_{2R}(p) := \{x \in M : d(x,p) \leqslant 2R\}.$$

Let $\psi(r)$ on $[0,+\infty)$ be a C^2 cut-off function such that $\psi(r) = 1$ for $0 \leqslant r \leqslant 1$, $\psi(r) = 0$ for $2 \leqslant r \leqslant +\infty$ and $0 \leqslant \psi(r) \leqslant 1$. Furthermore, let $\psi(r)$ satisfy the following estimates

$$0 \geqslant \psi'(r) \geqslant -C_1 \psi^{\frac{1}{2}}(r) \quad \text{and} \quad \psi''(r) \geqslant -C_2$$

for some absolute constants C_1, C_2. Fix a point $p \in M$, denote by $d(x,p)$ the geodesic distance between points p and x in M. Let $R \geqslant 0$ and define a smooth function

$$\varphi(x) = \psi\Big(\frac{d(x,p)}{R}\Big) \quad \text{and} \quad \varphi\Big|_{\mathscr{B}_{2R}} = 1.$$

Using Calabi's trick [5] (see also Cheng ang Yau [7]) we assume without loss of generality that the function $\varphi(x)$ is everywhere smooth with support in $\mathscr{B}_{2R}(p)$ since $\psi(r)$ is in general Lipschitz. Then by direct calculation we have on $\mathscr{B}_{2R}$

$$\frac{|\nabla\varphi|^2}{\varphi} = \frac{|\psi'|^2 \cdot |\nabla d|^2}{R^2\psi} \leqslant \frac{C_1^2}{R^2} \tag{24}$$

and by the Laplacian comparison theorem [14] we have

$$\Delta\varphi = \frac{\psi'\Delta d}{R} + \frac{\psi''|\nabla d|^2}{R^2} \geqslant -\frac{(n-1)(1+\sqrt{k}R)C_1^2 + C_2}{R^2}. \tag{25}$$

3.3. Proof of gradient estimates

Proof of Theorem 1. The next argument is to consider the point $x_0 \in B_{2R}(p)$ at which the following function φG attains its maximum value P. We assume P to be positve, i.e.,

$$P(y) = \sup_{\mathscr{B}_{2R}(p)} \varphi G(x_0) > 0$$

otherwise the proof becomes trivially true. At $x_0 \in \mathscr{B}_{2R}(p)$ we have by the maximum principle that

$$\nabla(\varphi G)(x_0) = 0, \quad \text{and} \quad \Delta(\varphi G)(x_0) \leqslant 0, \tag{26}$$

which respectively imply

$$\varphi\nabla G = -G\nabla\varphi$$

and

$$\varphi\Delta G + G\Delta\varphi - 2G\frac{|\nabla\varphi|^2}{\varphi} \leqslant 0, \tag{27}$$

Using estimates (24), (25) and (27) we have

$$AG \geqslant \varphi\Delta G, \tag{28}$$

where

$$A = \frac{\Big((n-1)(1+\sqrt{k}R)+2\Big)C_1^2 + C_2}{R^2}$$

Using Lemma 2 in (28) at the maximum point x_0 we have

$$\begin{aligned}
AG \geqslant & -2\varphi\nabla w\nabla G + \frac{2}{n}\varphi\Big(-G+(\beta-a)e^{(s-1)w}\Big)^2 \\
& + \Big(\Big[\beta(s-1)^2+2(\beta-a)(s-1)\Big]e^{(s-1)w}-2k\Big)\varphi\Big(G-\beta e^{(s-1)w}-\lambda\Big) \\
& + \beta(s-1)\varphi\Big(-G+(\beta-a)e^{(s-1)w}\Big)e^{(s-1)w} \\
\geqslant & -\frac{2C_1}{R}\varphi^{\frac{1}{2}}G\Big(G-\beta e^{(s-1)w}-\lambda\Big)^{\frac{1}{2}} + \frac{2}{n}\varphi\Big(-G+(\beta-a)e^{(s-1)w}\Big)^2 \\
& + \Big(\Big[\beta(s-1)^2+2(\beta-a)(s-1)\Big]e^{(s-1)w}-2k\Big)\varphi\Big(G-\beta e^{(s-1)w}-\lambda\Big) \\
& + \beta(s-1)\varphi\Big(-G+(\beta-a)e^{(s-1)w}\Big)e^{(s-1)w}
\end{aligned}$$

where we have used the following relation

$$\begin{aligned}
-2\varphi\nabla w\nabla G = 2G\nabla\varphi\nabla w = 2G|\nabla w||\nabla\varphi| &\geqslant -2\frac{C_1}{R}\varphi^{\frac{1}{2}}G|\nabla w| \\
&= -\frac{2C_1}{R}\varphi^{\frac{1}{2}}G\Big(G-\beta e^{(s-1)w}-\lambda\Big)^{\frac{1}{2}}.
\end{aligned}$$

Letting $\mu = e^{(s-1)w}/G$ i.e., $\mu G = e^{(s-1)w}$ we have (since $\lambda > 0$)

$$\begin{aligned}
AG \geqslant & -\frac{2C_1}{R}(1-\beta\mu)^{\frac{1}{2}}\varphi^{\frac{1}{2}}G^{\frac{3}{2}} + \frac{2}{n}(-1+(\beta-a)\mu)^2\varphi G^2 \\
& + \Big(\Big[\beta(s-1)^2+2(\beta-a)(s-1)\Big]\mu G-2k\Big)\varphi(1-\beta\mu)G \\
& + [\beta(s-1)(-1+(\beta-a)\mu)\varphi\mu G^2.
\end{aligned}$$

Multiplying the last inequality through by φ, we get

$$\begin{aligned}
AP \geqslant & -\frac{2C_1}{R}(1-\beta\mu)^{\frac{1}{2}}P^{\frac{3}{2}} + \frac{2}{n}(1+(a-\beta)\mu)^2P^2 + [\beta(s-1)((\beta-a)\mu-1)]\mu P^2. \\
& + \Big(\Big[\beta(s-1)^2+2(\beta-a)(s-1)\Big]\mu G-2k\Big)(1-\beta\mu)P,
\end{aligned}$$

which implies

$$\begin{aligned}
A \geqslant & -\frac{2C_1}{R}(1-\beta\mu)^{\frac{1}{2}}P^{\frac{1}{2}} + \Big[\frac{2}{n}(1+(a-\beta)\mu)^2 + \beta(s-1)((\beta-a)\mu-1)\mu\Big]P. \\
& + \Big(\Big[\beta(s-1)^2+2(\beta-a)(s-1)\Big]\mu G-2k\Big)(1-\beta\mu).
\end{aligned} \tag{29}$$

Applying Young inequality

$$-\frac{2C_1}{R}(1-\beta\mu)^{\frac{1}{2}}P^{\frac{1}{2}} \geqslant -\frac{1}{n}(1+(a-\beta)\mu)^2P - \frac{nC_1^2(1-\beta\mu)}{R^2(1+(a-\beta)\mu)^2}. \tag{30}$$

Substituting inequality (30) into (29) we have

$$
\begin{aligned}
A \geqslant \frac{1}{n}(1+(a-\beta)\mu)^2+\beta(s-1)((\beta-a)\mu-1)\mu\Big]P-\frac{nC_1^2(1-\beta\mu)}{R^2(1+(a-\beta)\mu)^2} \\
+\Big(\Big[\beta(s-1)^2+2(\beta-a)(s-1)\Big]\mu G-2k\Big)(1-\beta\mu).
\end{aligned}
\tag{31}
$$

Next we consider the following two cases: (1) Case $a<0$ and (2) $a>0$.

(1) *Case* $a<0$. In the case $a<0$, we have two possibilites, namely; (i) $\mu>0$ ($\Longrightarrow G>0$) and (ii) $\mu<0$ ($\Longrightarrow G<0$). Notice that $\mu G = e^{(s-1)w}>0$ (by definition) in both possibilities, then the term involving μG can be excluded in (31). In this case we choose $\beta=a$ and we obtain

$$
A \geqslant \frac{1}{n}P-\frac{nC_1^2(1-a\mu)}{R^2}-2k(1-a\mu)
$$

which implies

$$
P \leqslant n\Big(A+\frac{nC_1^2(1-a\mu)}{R^2}+2k(1-a\mu)\Big).
$$

(i) For the possibility $\mu>0$ ($\Longrightarrow G>0$), note that

$$
G=|\nabla w|^2+ae^{(s-1)w}+\lambda > ae^{(s-1)w}
$$

implies $1>a\mu$ i.e, $1-a\mu>0$. Then we have

$$
\sup_{\mathscr{B}_{2R}} G \leqslant P \leqslant n\Big(A+\frac{nC_1^2}{R^2}+2k\Big). \tag{32}
$$

(ii) For the possiblity $\mu<0$ ($\Longrightarrow G<0$) ($a<0$) the estimate in (11) of Theorem 1 is trivial. Hence we assume $G>0$ and the analysis will also yield (32).

Therefore the estimate in (11) of Theorem 1 follows from (32).

(2) *Case* $a>0$. For the case $a>0$ we have $G=|\nabla w|^2+ae^{(s-1)w}+\lambda > ae^{(s-1)w} > 0$ and $1-a\mu>0$.

Note that (17) of Lemma 2 can be written as

$$
\begin{aligned}
\Delta G \geqslant & -2\nabla w\nabla G+\frac{2}{n}\Big(-G+(\beta-a)e^{(s-1)w}\Big)^2 \\
& +\Big(\beta(s-1)^2+(\beta-2a)(s-1)\Big)e^{(s-1)w}G \\
& -\Big(\beta(\beta-a)(s-1)+\beta(s-1)^2\Big)e^{(s-1)w}e^{(s-1)w}-2k(G-\beta e^{(s-1)w}) \\
& +\Big[\beta(s-1)^2+2(\beta-a)(s-1)\Big]e^{(s-1)w}\lambda+2k\lambda.
\end{aligned}
\tag{33}
$$

Follow a similar computation as before

$$
\begin{aligned}
\varphi\Delta G \geqslant & -\frac{2C_1}{R}(1-\beta\mu)^{\frac{1}{2}}\varphi^{\frac{1}{2}}G^{\frac{3}{2}}+\frac{2}{n}(-1+(\beta-a)\mu)^2\varphi G^2 \\
& +\Big(\beta(s-1)^2+(\beta-2a)(s-1)\Big)\varphi\mu G^2+\beta\Big((a-\beta)(s-1)-\beta(s-1)^2\Big)\varphi\mu^2G^2 \\
& -2k(1-\mu\beta)\varphi G+\Big[\beta(s-1)^2+2(\beta-a)(s-1)\Big]\mu G\lambda+2k\lambda.
\end{aligned}
$$

Then

$$
\begin{aligned}
AP \geqslant & -\frac{2C_1}{R}(1-\beta\mu)^{\frac{1}{2}}P^{\frac{3}{2}}+\frac{2}{n}(1+(a-\beta)\mu)^2P^2+\Big(\beta(s-1)^2+(\beta-2a)(s-1)\Big)\mu P^2. \\
& +\beta\Big((a-\beta)(s-1)-\beta(s-1)^2\Big)\mu^2P^2-2k(1-\beta\mu)P \\
& +\Big[\beta(s-1)^2+2(\beta-a)(s-1)\Big]\mu G\lambda+2k\lambda.
\end{aligned}
$$

In this case $a>0$, note that $\mu>0$, $\mu G>0$ and $2k\lambda\geqslant 0$. Choosing $\beta=2a$ we can exlude the last two terms. We then have by using the inequality in (30)

$$
\begin{aligned}
A \geqslant & \frac{1}{n}(1+(a-\beta)\mu)^2P-\frac{nC_1^2(1-\beta\mu)}{R^2(1+(a-\beta)\mu)^2}+\Big([\beta(s-1)^2+2(\beta-2a)(s-1)]\mu \\
& -\beta[(\beta-a)(s-1)+\beta(s-1)^2]\mu^2\Big)P-2k(1-\beta\mu) \\
= & \Big(\frac{1}{n}(1-a\mu)^2+2a(s-1)^2(1-2a\mu)\mu-2a^2(s-1)\mu^2\Big)P \\
& -\frac{nC_1^2(1-2a\mu)}{R^2(1-a\mu)^2}-2k(1-2a\mu).
\end{aligned}
$$

Notice that

$$
\frac{(1-2a\mu)}{(1-a\mu)^2}\leqslant 1\,,\quad |1+|a|\mu|^2\geqslant 1\ \text{ and }\ \mu^2P\leqslant e^{2(s-1)w}\leqslant \mathbb{M}^{2(s-1)}.
$$

Hence, we arrive at

$$
A\geqslant \frac{1}{n}P-2a^2(s-1)\mathbb{M}^{2(s-1)}-\frac{nC_1^2}{R^2}-2k
$$

which implies

$$
\sup_{\mathscr{B}_{2R}} G\leqslant P\leqslant n\Big(A+\frac{nC_1^2}{R^2}+2k+2a^2(s-1)\mathbb{M}^{2(s-1)}\Big). \tag{34}
$$

Therefore the estimate (12) of Theorem 1 follows from (34). $\square$

3.4. Harnack inequalities

PROPOSITION 1. *Suppose the conditions of Theorem 1 hold. If in addition* $u^{s-1} \leqslant \mathbb{M}$ *for all* $x \in \mathscr{B}_{2R}(p)$, *then*

$$\frac{|\nabla u|^2}{u^2} \leqslant B_1 + |a|\mathbb{M} + \lambda, \tag{35}$$

where

$$B_1 = An + 2nk \quad \textit{for } \ a < 0$$

$$B_1 = An + 2nk + 2a^2 n(s-1)\mathbb{M}^{2(s-1)} \quad \textit{for } \ a > 0.$$

and

$$A = \frac{((n-1)(1+\sqrt{k}R)+2)C_1^2 + C_2}{R^2}$$

Proof of Corollary 2. Let γ be a shortest curve in $\mathscr{B}_R(p)$ joinning y to x. Clearly, the length is at most $2R$. Integrating the quantity $|\nabla u|/u$ along γ and applying the gradient etimate of Proposition 1 yields

$$\begin{aligned} \log u(x) - \log u(y) &\leqslant \int_\gamma \frac{|\nabla u|^2}{u^2} \\ &\leqslant \int_\gamma (B_1 + |a|\mathbb{M} + \lambda)^{\frac{1}{2}} \\ &\leqslant (B_1 + |a|\mathbb{M} + \lambda)^{\frac{1}{2}} \cdot 2R. \end{aligned}$$

The Corollary follows at once by exponentiation. □

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