Math. Program., Ser. A (2014) 144:307–314
DOI 10.1007/s10107-013-0634-3

FULL LENGTH PAPER

Disjunctive programming and relaxations of polyhedra

Michele Conforti · Alberto Del Pia

Received: 27 April 2012 / Accepted: 16 January 2013 / Published online: 7 February 2013

Abstract Given a polyhedron L with h facets, whose interior contains no integral points, and a polyhedron P, recent work in integer programming has focused on characterizing the convex hull of P minus the interior of L. We show that to obtain such a characterization it suffices to consider all relaxations of P defined by at most $n(h-1)$ among the inequalities defining P. This extends a result by Andersen, Cornuéjols, and Li.

Keywords Mixed integer programming · Disjunctive programming · Polyhedral relaxations

Mathematics Subject Classification (2000) 90C10 · 90C11 · 90C57 · 52B11

1 Introduction

Given polyhedra $P, L \subseteq \mathbb{R}^n$, we denote with

$$P \backslash L := \overline{\text{conv}}(P - \text{int} L), \tag{1}$$

Supported by the Progetto di Eccellenza 2008-2009 of the Fondazione Cassa Risparmio di Padova e Rovigo.

M. Conforti
Dipartimento di Matematica Pura ed Applicata, Universitá degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy
e-mail: conforti@math.unipd.it

A. Del Pia (✉)
IFOR, Department of Mathematics, ETH Zürich, 8092 Zürich, Switzerland
e-mail: alberto.delpia@math.ethz.ch

where "$\overline{\text{conv}}$" indicates the closed convex hull, "−" the set difference, and "int" the topological interior. Let $Ax \leq b$ be a system of inequalities defining P. We denote by $\mathcal{R}^q(A, b)$ the family of the polyhedral relaxations of P that consist of the intersection of the half-spaces corresponding to at most q inequalities of the system $Ax \leq b$. In this note we prove the following theorem:

Theorem 1 *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and L be polyhedra in $\mathbb{R}^n$ and let $h \geq 2$ be the number of facets of L. Then*

$$P \setminus L = \bigcap_{R \in \mathcal{R}^{n(h-1)}(A,b)} R \setminus L.$$

In the next section we provide a proof of this theorem, and we sketch a construction showing that the result does not hold if one considers polyhedra in $\mathcal{R}^{n(h-1)-1}(A, b)$. We now motivate it by providing an application to mixed integer programming.

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and let $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$, for some p, $1 \leq p \leq n$. A mixed-integer set $\mathcal{F}$ is a set of the form $\{x \in P \cap S\}$. Most of the research has focused on obtaining inequalities that are valid for $\mathcal{F}$, or equivalently, for conv$\mathcal{F}$, where "conv" indicates the convex hull. The operator defined in (1) was first considered in the mixed integer programming community by Andersen et al. [2], and it may be viewed as a special case of the disjunctive programming approach invented by Balas [3]. A convex set L is *S-free* if int$L$ does not contain any point in S. Given a mixed-integer set $\mathcal{F}$ in the form described above and an S-free polyhedron L, $\mathcal{F}$ is obviously contained in $P \setminus L$. It follows that any valid inequality for $P \setminus L$ is also valid for $\mathcal{F}$. The converse is also true: If P is a rational polyhedron and $ax \leq \beta$ is a valid inequality for $\mathcal{F}$, then $ax \leq \beta$ is valid for $P \setminus L$, for some S-free polyhedron L [13,7]. This provides a motivation for the study of valid inequalities for $P \setminus L$ when L is a polyhedron, a setting that is receiving extensive interest from the community (see for example [4,6,10–13]).

Theorem 1 shows that in order to derive the inequalities that are essential in a description of $P \setminus L$, it is necessary and sufficient to consider inequalities that are valid for a relaxation of P comprising a number of inequalities that is a function of the dimension of the ambient space and of the number of facets of L.

Let $S = \mathbb{Z}^p \times \mathbb{R}^{n-p}$, for some p, $1 \leq p \leq n$. A *split* is a set L such that $L = \{x \in \mathbb{R}^n : \pi_0 \leq (\pi, 0)\, x \leq \pi_0 + 1\}$, for some $\pi \in \mathbb{Z}^p$, $\pi_0 \in \mathbb{Z}$. Clearly a split is an S-free convex set. Balas and Perregaard [5] prove Theorem 1 when P is contained in the unit cube and L is a split of the form $\{x \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$, $1 \leq i \leq p$. Andersen et al. [1] prove Theorem 1 when L is a split, and they pose as an open question if their result generalizes to other polyhedra L. A shorter proof of the same result has been recently provided by Dash et al. [9], and uses the equivalence between split cuts and mixed-integer rounding (MIR) cuts. All these proofs do not seem to be extendable to a more general case.

Andersen et al. [1] also prove that, if L is a split in $\mathbb{R}^n$, in Theorem 1 it is enough to consider polyhedra in $\mathcal{R}^n(A, b)$ defined by linearly independent inequalities. Furthermore they show that if L is defined by only two inequalities, one cannot generally restrict to polyhedra in $\mathcal{R}^n(A, b)$ defined by linearly independent inequalities.

2 Proof of main result

The following lemma is well-known, as it is an equivalent formulation of Carathéodory's theorem (see for example [14]).

Lemma 1 *Let G be a matrix of size $m \times d$ and let $\bar{r}$ be an extreme ray of the cone $\{r \in \mathbb{R}^m : r \geq 0,\ rG = 0\}$. Then $\bar{r}$ has at most $d+1$ positive components.*

Corollary 1 *Let $A^i, i = 1, \dots, k$ be $m^i \times n$ matrices and let $b^i, i = 1, \dots, k$ be vectors of dimension m^i. Let $(\bar{r}^i \in \mathbb{R}^{m^i},\ \bar{s}^i \in \mathbb{R} : i = 1, \dots, k)$ be an extreme ray of the cone defined by the system*

$$
\begin{aligned}
-r^1 A^1 + r^i A^i &= 0 && i = 2, \dots, k \\
r^1 b^1 - r^i b^i + s^1 - s^i &= 0 && i = 2, \dots, k \\
r^i &\geq 0 && i = 1, \dots, k \\
s^i &\geq 0 && i = 1, \dots, k.
\end{aligned}
$$

Then $(\bar{r}^i,\ \bar{s}^i : i = 1, \dots, k)$ has at most $n(k-1) + k$ positive components.

Proof The system

$$
\begin{aligned}
-r^1 A^1 + r^i A^i &= 0 && i = 2, \dots, k \\
r^1 b^1 - r^i b^i + s^1 - s^i &= 0 && i = 2, \dots, k
\end{aligned}
$$

comprises of $(n+1)(k-1)$ equations. By Lemma 1, $(\bar{r}^i,\ \bar{s}^i : i = 1, \dots, k)$ has at most $(n+1)(k-1) + 1 = n(k-1) + k$ positive components. □

(In the above proof, if $k = 1$ we intend the set of indices $i = 2, \dots, k$ to be empty.)

For $i = 1, \dots, k$ consider polyhedra $P^i = \{x \in \mathbb{R}^n : A^i x \leq b^i\}$ and cones $C^i := \{x \in \mathbb{R}^n : A^i x \leq 0\}$. So C^i is the recession cone of P^i if P^i is nonempty. By Minkowski-Weil's theorem (see for example [14]) there exist polytopes Q^i, for $i = 1, \dots, k$, such that

$$P^i = Q^i + C^i, \quad i = 1, \dots, k,$$

where $P^i = \emptyset$ if and only if $Q^i = \emptyset$. Let

$$\tilde{P} := \operatorname{conv} \bigcup_{i=1}^{k} Q^i + \operatorname{cone} \bigcup_{i=1}^{k} C^i, \tag{2}$$

where "cone" denotes the conic hull. Again, $\tilde{P} = \emptyset$ if and only if $\bigcup_{i=1}^{k} Q^i = \emptyset$.

Let S' be the following system of inequalities:

$$A^i x^i - b^i \lambda^i \leq 0 \quad i = 1, \dots, k \tag{3}$$

$$x - \sum_{i=1}^{k} x^i = 0 \tag{4}$$

$$\sum_{i=1}^{k} \lambda^i = 1 \tag{5}$$

$$\lambda^i \geq 0 \quad i = 1, \dots, k. \tag{6}$$

Given a polyhedron $P = \{(x, y) \in \mathbb{R}^{n+d} : Ax + Gy \leq b\}$, we denote with $\text{proj}_x P \subseteq \mathbb{R}^n$ the orthogonal projection of P onto the space of the x-variables. More precisely $\text{proj}_x P := \{x \in \mathbb{R}^n, \exists y \in \mathbb{R}^d : Ax + Gy \leq b\}$. The following theorem is similar to Balas' theorem on union of polyhedra [3].

Theorem 2 [8] *Given k polyhedra $P^i = \{x \in \mathbb{R}^n : A^i x \leq b^i\} = Q^i + C^i$, let $\tilde{P}$ defined as in (2), and let $Y' \subset \mathbb{R}^{n+(n+1)k}$ be the polyhedron defined by the system (3)–(6). Then $\tilde{P} = \text{proj}_x Y'$.*
Furthermore, if either $P^i = \emptyset, i = 1, \dots, k$, or if $P^i \neq \emptyset, i = 1, \dots, k$, then $\tilde{P} = \overline{\text{conv}} \bigcup_{i=1}^{k} P^i$.

We now prove Theorem 1.

Proof Clearly $P \backslash L \subseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A,b)} R \backslash L$, thus we need to show the reverse inclusion.

Every inequality in the system $Ax \leq b$ is valid for some $R \in \mathcal{R}^1(A, b)$. Since $h \geq 2$, $R \in \mathcal{R}^{n(h-1)}(A, b)$ and therefore $P \supseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A,b)} R \backslash L$.

If L is not full-dimensional, $\text{int} L = \emptyset$, $P \backslash L = P \supseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A,b)} R \backslash L$, and the theorem follows. So we assume that L is a full-dimensional polyhedron with h facets. Hence $L = \{x \in \mathbb{R}^n : c^i x \leq \delta^i, \ i = 1, \dots, h\}$, where each inequality $c^i x \leq \delta^i$ defines a facet of L.

For $i = 1, \dots, h$, let $A^i x \leq b^i$ be the system obtained from $Ax \leq b$ by adding inequality $-c^i x \leq -\delta^i$ and let $P^i := \{x \in \mathbb{R}^n : A^i x \leq b^i\}$. Let k be defined as follows. If $P^i = \emptyset$ for every $i = 1, \dots, h$, let $k = h$. Otherwise let $k \geq 1$ be the number of nonempty polyhedra among $P^i, i = 1, \dots, h$, and we assume that the nonempty polyhedra are $P^1, \dots, P^k$. It follows from the definition of $P \backslash L$ that

$$P \backslash L = \overline{\text{conv}} \bigcup_{i=1}^{k} P^i.$$

Let S be the following system, obtained from (3)–(6) by using Eqs. (4) and (5) to eliminate vector x^1 and scalar λ^1:

$$
\begin{aligned}
A^1x - A^1\sum_{i=2}^{k} x^i + b^1 \sum_{i=2}^{k} \lambda^i &\le b^1 \\
A^i x^i - b^i \lambda^i &\le 0 \quad i = 2,\dots,k \\
\sum_{i=2}^{k} \lambda^i &\le 1 \\
\lambda^i &\ge 0 \quad i = 2,\dots,k.
\end{aligned}
$$

Let Y be the polyhedron defined by S. Note that Y is a polyhedron in $\mathbb{R}^{n+(n+1)(k-1)}$ involving vectors $x, x^2, \dots, x^k$ and scalars $\lambda^2, \dots, \lambda^k$. Furthermore Theorem 2 implies that

$$P \backslash L = \text{proj}_x Y.$$

Let U be the set of the extreme rays $(r^i, s^i \;:\; i = 1,\dots,k)$ of the cone defined by the system

$$
\begin{aligned}
-r^1A^1 + r^iA^i &= 0 \quad i = 2,\dots,k & (7)\\
r^1b^1 - r^ib^i + s^1 - s^i &= 0 \quad i = 2,\dots,k & (8)\\
r^i &\ge 0 \quad i = 1,\dots,k & (9)\\
s^i &\ge 0 \quad i = 1,\dots,k. & (10)
\end{aligned}
$$

Since $P \backslash L = \text{proj}_x Y$, it is well-known that

$$P \backslash L = \{x \in \mathbb{R}^n \;:\; r^1A^1x \le r^1b^1 + s^1,\ \forall (r^i, s^i \;:\; i = 1,\dots,k) \in U\}. \quad (11)$$

Let $(\bar{r}^i, \bar{s}^i \;:\; i = 1,\dots,k)$ be a ray in U, and let $ax \le \beta$ be the corresponding valid inequality for $P \backslash L$, where $a = \bar{r}^1A^1, \beta = \bar{r}^1b^1 + \bar{s}^1$. To prove $P \backslash L \supseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A,b)} R \backslash L$, it suffices to show that there exists a polyhedron $\bar{R} \in \mathcal{R}^{n(h-1)}(A,b)$ such that $ax \le \beta$ is valid for $\bar{R} \backslash L$. Since $P \supseteq \bigcap_{R \in \mathcal{R}^{n(h-1)}(A,b)} R \backslash L$, we assume that the inequality $ax \le \beta$ is not valid for P. We now construct a polyhedron $\bar{R} \in \mathcal{R}^{n(h-1)}(A,b)$ such that $ax \le \beta$ is valid for $\bar{R} \backslash L$.

For $i = 1,\dots,k$, let R^i be the polyhedron defined by the inequalities in $Ax \le b$ corresponding to positive components of $\bar{r}^i$.

Note that when $k < h$, by definition of k, $P \ne \emptyset$ and for $i = k+1,\dots,h$, $P^i = P \cap \{x \in \mathbb{R}^n \;:\; c^ix \ge \delta^i\} = \emptyset$. Since $P \ne \emptyset$, it follows by Carathéodory's theorem (see for example [14]) that, for $i = k+1,\dots,h$, there exist a polyhedron R^i defined by at most n linearly independent inequalities in $Ax \le b$ such that $R^i \cap \{x \in \mathbb{R}^n \;:\; c^ix \ge \delta^i\} = \emptyset$.

We now show that for $i = 1,\dots,h$, inequality $ax \le \beta$ is valid for $R^i \cap \{x \in \mathbb{R}^n \;:\; c^ix \ge \delta^i\}$. For $i = 1,\dots,k$, by (7)–(11) we have that $a = \bar{r}^iA^i, \beta = \bar{r}^ib^i + \bar{s}^i$, and $\bar{r}^i, \bar{s}^i \ge 0$, thus $ax \le \beta$ is valid for $R^i \cap \{x \in \mathbb{R}^n \;:\; c^ix \ge \delta^i\}$. Moreover for $i = k+1,\dots,h$, $ax \le \beta$ is valid for $R^i \cap \{x \in \mathbb{R}^n \;:\; c^ix \ge \delta^i\} = \emptyset$. Now let $\bar{R} = \bigcap_{i=1}^{h} R^i$. Hence $ax \le \beta$ is valid for $\bar{R} \cap \{x \in \mathbb{R}^n : c^i x \ge \delta^i\}$ for every $i = 1, \dots, h$. This shows that $ax \le \beta$ is valid for $\bar{R} \setminus L$.

We finally show $\bar{R} \in \mathcal{R}^{n(h-1)}(A, b)$. For $i = 1, \dots, k$, since $ax \le \beta$ is not valid for P and $P \subseteq R^i$, $ax \le \beta$ is not valid for R^i. Since by (7)–(11) we have that $a = \bar{r}^i A^i$, $\beta = \bar{r}^i b^i + \bar{s}^i$, and $\bar{r}^i, \bar{s}^i \ge 0$, it follows that the component of $\bar{r}^i$ corresponding to $c^i x \ge \delta^i$ must be positive. By Corollary 1 the positive components of the vector $(\bar{r}^i : i = 1, \dots, k)$ are at most $n(k-1)+k$, and by the previous argument, the k components of $(\bar{r}^i : i = 1, \dots, k)$ corresponding to the inequalities $c^i x \ge \delta^i, i = 1, \dots, k$, are all positive. This shows that $\bigcap_{i=1}^{k} R^i$ is defined by at most $n(k-1)$ inequalities of $Ax \le b$. Moreover for $i = k+1, \dots, h$, R^i is defined by at most n inequalities of $Ax \le b$. It follows that $\bar{R}$ is defined by at most $n(k-1) + n(h-k) = n(h-1)$ inequalities of $Ax \le b$, hence $\bar{R} \in \mathcal{R}^{n(h-1)}(A, b)$. □

We conclude this paper showing that the bound given in Theorem 1 is tight. For $n = 1$ the result is trivial since L has at most 2 facets, so assume $n \ge 2$. For every $n \ge 2$ and $h \ge 2$, we sketch the construction of a polyhedron P in $\mathbb{R}^n$ and a polyhedron L with h facets such that

$$P \setminus L \subset \bigcap_{R \in \mathcal{R}^{n(h-1)-1}(A,b)} R \setminus L.$$

Figure 1 illustrates the construction for $n = 2, h = 3$.

Let $L' = \{x \in \mathbb{R}^n : c^i x \le \delta^i,\ i = 1, \dots, h\}$ be a full dimensional polyhedron, where inequalities $c^i x \le \delta^i$ are in one to one correspondence with the $h \ge 2$ facets F^i of L'. For every $i = 1, \dots, h$, let f^i be a point in the relative interior of F^i. Let $\epsilon > 0$ be such that for every $i = 1, \dots, h$

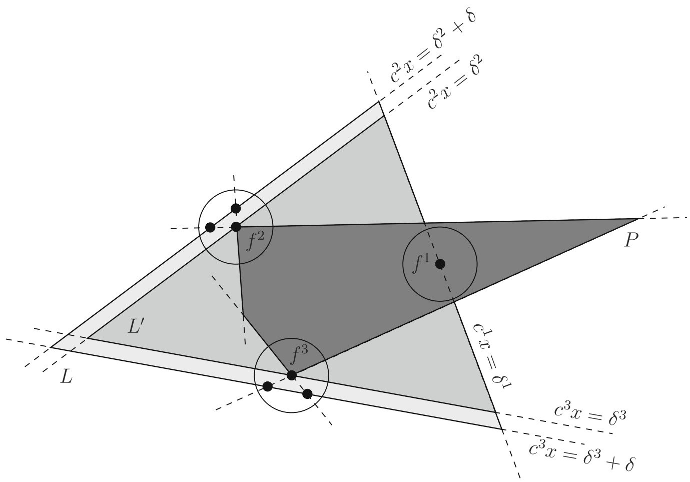

Fig. 1 Construction for $n = 2, h = 3$

i) the strict inequalities $c^j x < \delta^j$ are valid for $f^i + \epsilon B$, for $j = 1, \ldots, h$ with $j \neq i$, where B is the unit ball in $\mathbb{R}^n$.

For every $i = 2, \ldots, h$, let $A^i x \leq b^i$ be a system of n linearly independent inequalities, such that:

ii) $A^i f^i = b^i$,
iii) $c^i x \leq \delta^i$ is valid for $R^i := \{x \in \mathbb{R}^n : A^i x \leq b^i\}$, and $R^i \cap \{x \in \mathbb{R}^n : c^i x = \delta^i\} = f^i$,
iv) $f^j + \epsilon B \subseteq R^i$, for $j = 1, \ldots, h$ with $j \neq i$.

(The existence of such systems follows from the definition of f^i, $i = 1, \ldots, h$, and by i)). For $i = 2, \ldots, h$ and $j = 1, \ldots, n$, let $a^{i_j} x \leq \beta^{i_j}$ be the jth inequality of the system $A^i x \leq b^i$, and let $A^{i_j} x \leq b^{i_j}$ be the system obtained from $A^i x \leq b^i$ by removing $a^{i_j} x \leq \beta^{i_j}$.

Since for $i = 2, \ldots, h$, the polyhedra R^i are translate of polyhedral cones and by ii) R^i has apex f^i, it follows from iii) that for every $i = 2, \ldots, h$, $j = 1, \ldots, n$, and $\delta > 0$, there exists a unique point x^{i_j} that satisfies

v) $A^{i_j} x^{i_j} = b^{i_j}$ and $c^i x^{i_j} = \delta^i + \delta$.

Let $\delta > 0$ be small enough such that $x^{i_j} \in f^i + \epsilon B$ for every $i = 2, \ldots, h$ and $j = 1, \ldots, n$.

Let $L := \{x \in \mathbb{R}^n : c^1 x \leq \delta^1,\ c^i x \leq \delta^i + \delta,\ i = 2, \ldots, h\}$ and let $P = \bigcap_{i=2}^h R^i$. Note that P is defined by the system $Ax \leq b$ consisting of all inequalities in systems $A^i x \leq b^i$, $i = 2, \ldots, h$. Since by iii), for $i = 2, \ldots, h$, inequalities $c^i x \leq \delta^i$ are valid for P and $\delta > 0$, then $P \cap \{x \in \mathbb{R}^n : c^i x \geq \delta^i + \delta\} = \emptyset$ for every $i = 2, \ldots, h$. This shows that $P \backslash L = P \cap \{x \in \mathbb{R}^n : c^1 x \geq \delta^1\}$. Since by i), $c^1 f^2 < \delta^1$ and by ii), iv), $f^2 \in P$, the inequality $c^1 x \geq \delta^1$ is not valid for P, and so $c^1 x \geq \delta^1$ is irredundant for the system defining $P \backslash L$.

We now show that for every $R \in \mathcal{R}^{n(h-1)-1}(A, b)$, the inequality $c^1 x \geq \delta^1$ is not valid for $R \backslash L$.

Let $R \in \mathcal{R}^{n(h-1)-1}(A, b)$. Since the system $Ax \leq b$ contains $n(h-1)$ inequalities, R contains the polyhedron defined by the system $Ax \leq b$ deprived of a single inequality. We assume without loss of generality that this inequality is $a^{2_1} x \leq \beta^{2_1}$, and so is the first inequality of the system $A^2 x \leq b^2$. By v), the point x^{2_1} is such that $A^{2_1} x^{2_1} = b^{2_1}$ and $c^2 x^{2_1} = \delta^2 + \delta$. By the choice of δ, $x^{2_1} \in f^2 + \epsilon B$, so it follows by iv) that $x^{2_1} \in R^i$ for every $i = 3, \ldots, h$. Hence $x^{2_1} \in R$.

Since $c^2 x^{2_1} = \delta^2 + \delta$, and $c^2 x \leq \delta^2 + \delta$ is valid for L, x^{2_1} does not belong to the interior of L. This shows that x^{2_1} belongs to $R \backslash L$. Since x^{2_1} belongs to $f^2 + \epsilon B$, then by i), $c^1 x^{2_1} < \delta^1$. Hence $c^1 x \geq \delta^1$ is not valid for $R \backslash L$.

References

1. Andersen, K., Cornuéjols, G., Li, Y.: Split closure and intersection cuts. Math. Program. A **102**(3), 457–493 (2005)
2. Andersen, K., Louveaux, Q., Weismantel, R.: An analysis of mixed integer linear sets based on lattice point free convex sets. Math. Oper. Res. **35**(1), 233–256 (2010)
3. Balas, E.: Disjunctive programming: properties of the convex hull of feasible points. Discret. Appl. Math. **89**(1–3), 3–44 (1998)

4. Balas, E., Margot, F.: Generalized intersection cuts and a new cut generating paradigm. Math. Program. A **137**(1–2), 19–35 (2013)
5. Balas, E., Perregaard, M.: A precise correspondence between lift-and-project cuts, simple disjunctive cuts, and mixed integer gomory cuts for 0-1 programming. Math. Program. **94**(2–3), 221–245 (2003)
6. Basu, A., Cornuéjols, G., Margot, F.: Intersection cuts with infinite split rank. Math. Oper. Res. **37**(1), 21–40 (2012)
7. Conforti, M., Cornuéjols, G., Zambelli, G.: Eqivalence between intersection cuts and the corner polyhedron. Oper. Res. Lett. **38**(3), 153–155 (2010)
8. Conforti, M., Cornuéjols, G., Zambelli, G.: Integer programming. In preparation, (2012)
9. Dash, S., Günlük, O., Raack, C.: A note on the MIR closure and basic relaxations of polyhedra. Oper. Res. Lett. **39**(3), 198–199 (2011)
10. Del Pia, A.: On the rank of disjunctive cuts. Math. Oper. Res. **37**(2), 372–378 (2012)
11. Del Pia, A., Weismantel, R.: On convergence in mixed integer programming. Math. Program. A **135**(1), 397–412 (2012)
12. Dey, S.S.: A note on the split rank of intersection cuts. Math. Program. A **130**(1), 107–124 (2011)
13. Jörg, M.: k-disjunctive cuts and cutting plane algorithms for general mixed integer linear programs. PhD thesis, Technische Universität München, München, (2008)
14. Schrijver, A.: Theory of Linear and Integer Programming. Wiley, Chichester (1986)