

Interpolation of Nonlinear Maps*

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Abstract—Let (X_0, X_1) and (Y_0, Y_1) be complex Banach couples and assume that $X_1 \subseteq X_0$ with norms satisfying $\|x\|_{X_0} \leq c\|x\|_{X_1}$ for some $c > 0$. For any $0 < \theta < 1$, denote by $X_\theta = [X_0, X_1]_\theta$ and $Y_\theta = [Y_0, Y_1]_\theta$ the complex interpolation spaces and by $B(r, X_\theta)$, $0 \leq \theta \leq 1$, the open ball of radius $r > 0$ in X_θ centered at zero. Then, for any analytic map $\Phi: B(r, X_0) \rightarrow Y_0 + Y_1$ such that $\Phi: B(r, X_0) \rightarrow Y_0$ and $\Phi: B(c^{-1}r, X_1) \rightarrow Y_1$ are continuous and bounded by constants M_0 and M_1 , respectively, the restriction of Φ to $B(c^{-\theta}r, X_\theta)$, $0 < \theta < 1$, is shown to be a map with values in Y_θ which is analytic and bounded by $M_0^{1-\theta}M_1^\theta$.

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1. INTRODUCTION

Let us first recall some basic notations and definitions of the interpolation theory for Banach spaces. Following [1], we say that two complex Banach spaces X_0 and X_1 are a *complex Banach couple* or *Banach couple* for short, (X_0, X_1) , if they are both linearly and continuously embedded into a linear complex Hausdorff space \mathcal{X} , i.e., $X_0 \subseteq \mathcal{X}$ and $X_1 \subseteq \mathcal{X}$. The spaces $X_\cap := X_0 \cap X_1$ and

$$X_+ := X_0 + X_1 = \{x \in \mathcal{X} \mid x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

are also Banach spaces when endowed with the norms

$$\|x\|_\cap := J(1, x), \quad \text{respectively,} \quad \|x\|_+ := K(1, x),$$

where, for any $t \geq 0$,

$$J(t, x) := \max\{\|x\|_{X_0}, t\|x\|_{X_1}\} \quad \forall x \in X_\cap$$

and for any $x \in X_+$

$$K(t, x) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} \mid x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}. \quad (1)$$

Without loss of generality, we will assume in what follows that $\mathcal{X} = X_+$. We say that (X_0, X_1) is a *regular Banach couple* if X_1 is continuously embedded into X_0 , i.e., $X_1 \subseteq X_0$ and there exists $c > 0$ so that

$$\|x\|_{X_0} \leq c\|x\|_{X_1} \quad \forall x \in X_1.$$

There are several methods of constructing, for any given Banach couple (X_0, X_1) , interpolation spaces X , $X_\cap \subseteq X \subseteq X_+$, that satisfy the interpolation property discussed below. The most familiar

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interpolation methods are the real and the complex ones. The real method comprises among others the K -method, the L -method, the J -method, the mean-methods, and the trace method (see, e.g., [2], [1]). Up to equivalent norms, these methods all lead to the same interpolation spaces $X_{\theta,p}$, where $0 < \theta < 1$ and $1 \leq p \leq \infty$. In particular, for any fixed $1 \leq p < \infty$ and $0 < \theta < 1$, the K -method defines the interpolation spaces $X_{\theta,p} \equiv (X_0, X_1)_{\theta,p}$ as follows

$$X_{\theta,p} := \left\{ x \in X_+ \mid \|x\|_{\theta,p} = \left(\int_0^\infty (t^{-\theta} K(t,x))^p \frac{dt}{t} \right)^{1/p} < \infty \right\}, \tag{2}$$

where $K(t,x)$ is defined by (1). If $p = \infty$, the space $(X_0, X_1)_{\theta,\infty}$ consists of the elements $x \in X_+$ with $\|x\|_{\theta,\infty} = \sup_{0 < t < \infty} t^{-\theta} K(t,x) < \infty$.

Now let us turn to the complex method of interpolation. For the notion of an analytic map between complex Banach spaces, we refer the reader, e.g., to [3] or [4, Appendix A]. Following [1], denote by S the vertical strip in the complex plane given by

$$S := \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\},$$

and by \bar{S} its closure. For any given Banach couple (X_0, X_1) , we then introduce the complex vector space $\mathcal{H}(X_0, X_1)$ of maps $f: \bar{S} \rightarrow X_+$ with the following properties:

- (H0) $f: \bar{S} \rightarrow X_+$ is continuous and bounded;
- (H1) $f|_S: S \rightarrow X_+$ is analytic;
- (H2) for any $t \in \mathbb{R}$ $f(it) \in X_0$, $f(1+it) \in X_1$, and the maps $\mathbb{R} \rightarrow X_0$, $t \mapsto f(it)$, and $\mathbb{R} \rightarrow X_1$, $t \mapsto f(1+it)$, are bounded and continuous.

Then

$$\|f\|_{\mathcal{H}} := \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{X_0}, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{X_1} \right\}$$

defines the norm on $\mathcal{H}(X_0, X_1)$. For any $0 < \theta < 1$, the complex interpolation space $X_\theta := [X_0, X_1]_\theta$ is the space

$$X_\theta := \{x \in X_+ \mid \exists f(z) \in \mathcal{H}(X_0, X_1) \text{ with } f(\theta) = x\}$$

endowed with the norm

$$\|x\|_{X_\theta} := \inf_{f \in \mathcal{H}} \{\|f\|_{\mathcal{H}} \mid f(\theta) = x\}. \tag{3}$$

It is well known that the spaces X_θ (and $X_{\theta,p}$) have the *interpolation property*: given any two Banach couples (X_0, X_1) and (Y_0, Y_1) and any bounded linear operator $T: X_+ \rightarrow Y_+$ such that

$$\|Tx\|_{Y_0} \leq M_0 \|x\|_{X_0} \quad \forall x \in X_0 \quad \text{and} \quad \|Tx\|_{Y_1} \leq M_1 \|x\|_{X_1} \quad \forall x \in X_1$$

for some $M_0 > 0$ and $M_1 > 0$, it follows that, for any $0 < \theta < 1$, T maps X_θ into Y_θ and

$$\|Tx\|_{Y_\theta} \leq M_0^{1-\theta} M_1^\theta \|x\|_{X_\theta} \quad \forall x \in X_\theta.$$

An important problem is to identify classes of nonlinear maps for which (a version of) the above interpolation property holds. It was investigated by many authors, see, e.g., [5]–[12], as well as the books [10], [1] and references therein for results on nonlinear maps defined on the entire spaces X_0 and X_1 . In applications, such an assumption is often too restrictive. The first results for nonlinear maps which are defined only locally were obtained by Tartar [13, Theorem 1]. Using the real interpolation method (constructed with the help of the K -functional (see (2)) and a setup where $X_0 = X_+$ and $Y_0 = Y_+$, Tartar proved the interpolation property for a class of nonlinear maps $\Phi: U \rightarrow Y_0$ defined on an open, nonempty subset $U \subset X_0$ with the following properties:

- (T1) Φ is locally α -Hölder continuous for some $0 < \alpha \leq 1$;

(T2) for any $x \in U$, there exist a neighborhood $V \subset U$ of x and a constant $c > 0$ such that

$$\forall y \in V \cap X_1 \quad \|\Phi(y)\|_{Y_1} \leq c(\|y\|_{X_1}^\beta + 1) \quad \text{for some } \beta > 0.$$

Note that the set $V \cap X_1$ may be of infinite diameter in X_1 even if V is a (small) ball in X_0 . Hence in applications, property (T2) is often not satisfied or difficult to verify. It is this fact that motivated our study of nonlinear interpolation in [14], [15], and [16], where applications to inverse problems of spectral theory were considered. Within the setup of the complex method of interpolation, in [14] and [15], we established the interpolation property for a class of nonlinear maps, defined on balls with center at the origin, satisfying assumptions that are rather easy to prove in many applications. In this paper, our aim is to extend the results of [14] and [16] to a more general setup. Results in this direction have been obtained previously by Böhm [17] and Cwikel [18]. For a discussion of these results, we refer the reader to Remark 1 below. To the best of our knowledge, the type of result obtained in Theorem 2 below, is new.

2. MAIN RESULTS

To state our results, we first need to introduce some additional notation and establish some auxiliary results. For any given complex Banach space X , denote by $B(r, X)$ the open ball in X of radius r centered at the origin. Throughout this section, let (X_0, X_1) be a complex Banach couple with $X_1 \subseteq X_0$, and let $X_\theta = [X_0, X_1]_\theta$, $0 < \theta < 1$, be the complex Banach spaces constructed by the complex method of interpolation. We have the following auxiliary lemma.

Lemma 1. *Assume that the norms of the Banach spaces $X_1 \subseteq X_0$ satisfy*

$$\|x\|_{X_0} \leq c\|x\|_{X_1} \quad \forall x \in X_1 \tag{4}$$

for some positive constant $c > 0$. Then, for any $0 < \theta < 1$,

$$\begin{aligned} \text{(i)} \quad & \|x\|_{X_0} \leq c^\theta \|x\|_{X_\theta} \quad \forall x \in X_\theta, \\ \text{(ii)} \quad & \|x\|_{X_\theta} \leq c^{1-\theta} \|x\|_{X_1} \quad \forall x \in X_1. \end{aligned} \tag{5}$$

In particular,

$$B(c^{-1}r, X_1) \subseteq B(c^{-\theta}r, X_\theta) \subseteq B(r, X_0). \tag{6}$$

Proof. Inequality (i) in (5) follows from the interpolation property of linear operators. Indeed, by (4) the identity operators

$$I: X_1 \rightarrow X_0 \quad \text{and} \quad I: X_0 \rightarrow X_0$$

are bounded by c and 1 , respectively. Hence,

$$I: X_\theta = [X_0, X_1]_\theta \rightarrow [X_0, X_0]_\theta = X_0$$

is bounded by c^θ . To prove inequality (ii) in (5), consider for any given $x \in X_1$, the analytic function $f: \mathbb{C} \rightarrow X_1$, $f(z) := c^{z-\theta}x$. Clearly, $f \in \mathcal{H}(X_0, X_1)$. Furthermore, in view of the definition of the norms $\|\cdot\|_{X_\theta}$, $\|f\|_{\mathcal{H}}$ and of (4), we have

$$\|x\|_{X_\theta} \leq \|f\|_{\mathcal{H}} = \max\{c^{-\theta}\|x\|_{X_0}, c^{1-\theta}\|x\|_{X_1}\} \leq c^{1-\theta}\|x\|_{X_1}. \quad \square$$

The first main result of this paper is the following.

Theorem 1. *Assume that the norms of the Banach spaces $X_1 \subseteq X_0$ satisfy*

$$\|x\|_{X_0} \leq c\|x\|_{X_1} \quad \forall x \in X_1 \tag{7}$$

for some positive constant $c > 0$. Let (Y_0, Y_1) be an arbitrary Banach couple and, for some $r > 0$, let

$$\Phi: B(r, X_0) \rightarrow Y_+ \tag{8}$$

be an analytic map with $\Phi(B(r, X_0)) \subseteq Y_0$ and $\Phi(B(c^{-1}r, X_1)) \subseteq Y_1$, so that

$$\Phi: B(r, X_0) \rightarrow Y_0 \tag{9}$$

and

$$\Phi|_{B(c^{-1}r, X_1)}: B(c^{-1}r, X_1) \rightarrow Y_1 \tag{10}$$

are continuous and bounded by the constants M_0 and M_1 , respectively,

$$\sup_{x \in B(r, X_0)} \|\Phi(x)\|_{Y_0} \leq M_0 \quad \text{and} \quad \sup_{x \in B(c^{-1}r, X_1)} \|\Phi(x)\|_{Y_1} \leq M_1. \tag{11}$$

Then, for any $0 < \theta < 1$, the inclusions $B(c^{-\theta}r, X_\theta) \subseteq B(r, X_0)$ and $\Phi(B(c^{-\theta}r, X_\theta)) \subseteq Y_\theta$ hold. Furthermore,

$$\Phi|_{B(c^{-\theta}r, X_\theta)}: B(c^{-\theta}r, X_\theta) \rightarrow Y_\theta \tag{12}$$

is bounded. More precisely, for any $x \in B(c^{-\theta}r, X_\theta)$

$$\|\Phi(x)\|_{Y_\theta} \leq M_0^{1-\theta} M_1^\theta. \tag{13}$$

Remark 1. Theorem 1 is related to [18, Theorem 1.1], where sufficient conditions are provided for non-linear maps defined on balls to have the interpolation property. Our proof of Theorem 1 is self-contained, making no reference to Theorem 1.1 in [18]. At the same time, it implicitly verifies some conditions which appear in [18]. Theorem 1 is also related to [17, Theorem 1] stating a result comparable to ours. However, condition (P2), assumed in Theorem 1 in [17], is too strong for our purposes: expressed in the setup of our paper, this condition says that the map Φ continuously maps $B(r, X_0) \cap X_1$ into the space Y_1 . As already pointed out in our discussion of Tartar’s result of [13], $B(r, X_0) \cap X_1$ may be of infinite diameter in X_1 . In comparison, in Theorem 1, it is only assumed that Φ continuously maps $B(c^{-1}r, X_1)$ into Y_1 .

Proof of Theorem 1. By Lemma 1, for any $0 < \theta < 1$,

$$B(c^{-\theta}r, X_\theta) \subseteq B(r, X_0).$$

Hence the map Φ is well defined on $B(c^{-\theta}r, X_\theta)$. It remains to prove that $\Phi(B(c^{-\theta}r, X_\theta)) \subseteq Y_\theta$ and that (13) holds. Take an arbitrary $x \in B(c^{-\theta}r, X_\theta)$. By the definition of the norm $\|x\|_\theta$, there exists a function $f \in \mathcal{H}(X_0, X_1)$ such that

$$f(\theta) = x \quad \text{and} \quad \|f\|_{\mathcal{H}} < c^{-\theta}r.$$

In particular, in view of the definition of the norm $\|f\|_{\mathcal{H}}$, one has, for any $t \in \mathbb{R}$,

$$\|f(it)\|_{X_0} < c^{-\theta}r \quad \text{and} \quad \|f(1+it)\|_{X_1} < c^{-\theta}r. \tag{14}$$

Consider the function

$$g: \overline{S} \rightarrow X_+, \quad z \mapsto g(z) := c^{\theta-z} f(z).$$

Clearly, $g \in \mathcal{H}(X_0, X_1)$ and $x = g(\theta)$. Since X_0 and X_+ coincide and the norms of X_0 and X_+ are equivalent, it follows that $g: \overline{S} \rightarrow X_0$ is continuous and bounded, and $f|_S: S \rightarrow X_0$ is analytic. Moreover, in view of (7) and (14), for any $t \in \mathbb{R}$, we have

$$\|g(1+it)\|_{X_0} = c^{\theta-1} \|f(1+it)\|_{X_0} \leq c^\theta \|f(1+it)\|_{X_1} < r \tag{15}$$

$$\|g(it)\|_{X_0} \leq c^\theta \|f(it)\|_{X_0} < r. \tag{16}$$

Applying Hadamard’s three line theorem (see, e.g., [2, Lemma 1.1.2]), then yields

$$\|g(z)\|_{X_0} < r \quad \forall z \in \overline{S}. \tag{17}$$

Further, it follows from (14) that

$$\|g(1+it)\|_{X_1} = c^{\theta-1} \|f(1+it)\|_{X_1} < c^{-1}r. \tag{18}$$

Using inequality (17), we can define $F: \overline{S} \rightarrow Y_+$ by setting

$$F(z) := M_0^{z-1} M_1^{-z} \Phi(g(z)).$$

Since $g: \overline{S} \rightarrow B(r, X_0) \subseteq X_0$ is continuous and, by assumption, $\Phi: B(r, X_0) \rightarrow Y_0$ is bounded and continuous, we conclude from the continuity of the embedding $Y_0 \subseteq Y_+$ that $F: \overline{S} \rightarrow Y_+$ is also continuous and bounded. Similarly, since $g|_S: S \rightarrow B(r, X_0) \subseteq X_0$ and, by our assumption, $\Phi: B(r, X_0) \rightarrow Y_+$ are analytic, the function $F|_S: S \rightarrow Y_+$ is analytic as well. The inequalities (16) and (18), together with the bounds (11), as well as the continuity and the boundedness of the maps $\Phi: B(r, X_0) \rightarrow Y_0$ and $\Phi|_{B(c^{-1}r, X_1)}: B(c^{-1}r, X_1) \rightarrow Y_1$, then imply that $F \in \mathcal{H}(Y_0, Y_1)$ and $\|F\|_{\mathcal{H}} \leq 1$. Further, in view of the equality $F(\theta) = M_0^{\theta-1} M_1^{-\theta} \Phi(x)$, we conclude that

$$\|\Phi(x)\|_{Y_\theta} = M_0^{1-\theta} M_1^\theta \|F(\theta)\|_{Y_\theta} \leq M_0^{1-\theta} M_1^\theta \|F(z)\|_{\mathcal{H}} \leq M_0^{1-\theta} M_1^\theta.$$

This completes the proof of Theorem 1. □

If the nonlinear maps $\Phi: B(r, X_0) \rightarrow Y_0$ and $\Phi|_{B(c^{-1}r, X_1)}: B(c^{-1}r, X_1) \rightarrow Y_1$ in Theorem 1 admit polynomial bounds of order n for some $n \geq 1$, then it follows from the next statement that the same is true for the maps $\Phi|_{B(c^{-\theta}r, X_\theta)}: B(c^{-\theta}r, X_\theta) \rightarrow Y_\theta$ with arbitrary $0 < \theta < 1$. Actually, the following statement generalizes the corresponding result of [16].

Corollary. *Assume that all assumptions of Theorem 1 are satisfied except that the boundedness condition (11) is replaced for some integer $n \geq 0$ by*

$$\|\Phi(x)\|_{Y_0} \leq M_0 \|x\|_{X_0}^n \quad \forall x \in B(r, X_0), \tag{19}$$

$$\|\Phi(x)\|_{Y_1} \leq M_1 \|x\|_{X_1}^n \quad \forall x \in B(c^{-1}r, X_1). \tag{20}$$

Then, for any $0 \leq \theta \leq 1$ and any $x \in B(c^{-\theta}r, X_\theta)$,

$$\|\Phi(x)\|_{Y_\theta} \leq M_0^{1-\theta} M_1^\theta \|x\|_{X_\theta}^n. \tag{21}$$

Proof of the corollary. Let $x \in B(c^{-\theta}r, X_\theta)$, and set $\rho := \|x\|_{X_\theta}$. Choose $\varepsilon > 0$ so that

$$r_1 := c^\theta(\rho + \varepsilon) < r.$$

Then, by (19),

$$\|\Phi(y)\|_{Y_0} \leq M_0 r_1^n \quad \forall y \in B(r_1, X_0)$$

and, by (20),

$$\|\Phi(y)\|_{Y_1} \leq M_1 (c^{-1}r_1)^n \quad \forall y \in B(c^{-1}r_1, X_1).$$

Theorem 1 then implies that, for all $y \in B(c^{-\theta}r_1, X_\theta)$,

$$\|\Phi(y)\|_{Y_\theta} \leq M_0^{1-\theta} M_1^\theta (r_1)^{n(1-\theta)} (c^{-1}r_1)^{n\theta} = M_0^{1-\theta} M_1^\theta (\rho + \varepsilon)^n.$$

In particular, the above inequality holds for $y = x$. Since $\varepsilon > 0$ can be chosen arbitrarily small, the corollary follows. □

Our second main result says that the map

$$\Phi|_{B(c^{-\theta}r, X_\theta)}: B(c^{-\theta}r, X_\theta) \rightarrow Y_\theta,$$

in Theorem 1 is analytic.

Theorem 2. *The map $\Phi|_{B(c^{-\theta}r, X_\theta)}: B(c^{-\theta}r, X_\theta) \rightarrow Y_\theta$, in Theorem 1 is analytic.*

Proof. Inspired by arguments used in [16], we prove the claim by showing that $\Phi|_{B(c^{-\theta}r, X_\theta)}$ is represented by a series of analytic maps and the series converges absolutely and uniformly in Y_θ on any ball $B(c^{-\theta}\rho, X_\theta)$ with $0 < \rho < r$ (see [4, Theorem 2, Appendix A]). First, note that Y_0 and Y_1 are continuously embedded in Y_+ , so, by Lemma 2 in the Appendix, the maps

$$\Phi: B(r, X_0) \rightarrow Y_0 \quad \text{and} \quad \Phi|_{B(c^{-1}r, X_1)}: B(c^{-1}r, X_1) \rightarrow Y_1$$

of (9) and (10) are analytic. Being analytic, the map $\Phi: B(r, X_0) \rightarrow Y_0$ is represented by its Taylor’s series at 0 with values in Y_0 ,

$$\Phi(h) = \Phi(0) + \sum_{n=1}^{\infty} \Phi_n(h), \quad h \in B(r, X_0). \tag{22}$$

Here

$$\Phi(0) \in Y_1 \quad \text{and} \quad \Phi_n(h) = \frac{1}{n!} d_0^n \Phi(h, \dots, h), \quad n \geq 1,$$

with $d_0^n \Phi$ denoting the n th derivative of Φ at 0 (cf. (A4) in Appendix). We remark that, for any $n \geq 1$, $\Phi_n(h)$ is a bounded homogeneous polynomial of degree n in $h \in X_0$ with values in Y_0 and hence is analytic, and that the series in (22) converges absolutely and uniformly in h on any ball $B(\rho, X_0)$ with $0 < \rho < r$. Moreover, in view of Cauchy’s formula (cf. the Appendix), for any $h \in X_0$ with $h \neq 0$,

$$\Phi_n(h) = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{\Phi(zh)}{z^{n+1}} dz, \tag{23}$$

where ρ is chosen arbitrarily so that $0 < \rho < r/\|h\|_{X_0}$. Using (23) and the first inequality in (11), we see that, for any $h \in X_0$ with $h \neq 0$,

$$\|\Phi_n(h)\|_{Y_0} \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\|\Phi(\rho e^{it}h)\|_{X_0}}{|\rho e^{it}|^{n+1}} |\rho i e^{it}| dt \leq \frac{M_0}{\rho^n}.$$

Since this inequality holds for any $0 < \rho < r/\|h\|_{X_0}$, it follows that, for any $h \in X_0$ with $h \neq 0$,

$$\|\Phi_n(h)\|_{Y_0} \leq \frac{M_0}{r^n} \|h\|_{X_0}^n. \tag{24}$$

Note that the latter estimate holds trivially for $h = 0$. Applying the arguments above to the analytic map $\Phi|_{B(c^{-1}r, X_1)}: B(c^{-1}r, X_1) \rightarrow Y_1$, we see that, for any $h \in B(c^{-1}r, X_1)$, $\Phi_n(h) \in Y_1$ and the series (22) converges in Y_1 . Further, by (23), for any $h \in X_1$,

$$\|\Phi_n(h)\|_{Y_1} \leq \frac{M_1}{(c^{-1}r)^n} \|h\|_{X_1}^n. \tag{25}$$

Applying the corollary to the the map $\Phi_n: X_0 \rightarrow Y_0 \subseteq Y_+$, we conclude from (24) and (25) that, for any $h \in X_\theta$, $\Phi_n(h) \in Y_\theta$ and

$$\|\Phi_n(h)\|_{Y_\theta} \leq \left(\frac{M_0}{r^n}\right)^{1-\theta} \left(\frac{M_1}{(c^{-1}r)^n}\right)^\theta \|h\|_{X_\theta}^n \leq M_0^{1-\theta} M_1^\theta \left(\frac{\|h\|_{X_\theta}}{c^{-\theta}r}\right)^n.$$

This inequality shows that, for any $h \in B(c^{-\theta}r, X_\theta)$, the series in (22) converges in Y_θ and that it converges absolutely and uniformly on any ball $B(c^{-\theta}\rho, X_\theta)$ with $0 < \rho < r$. \square

Finally, we remark that Theorem 1, combined with Theorem 2, generalizes Theorem 1.1 in [14] in the context of the setup chosen in this paper.

3. APPENDIX

In this appendix, we review the notion of an analytic map between complex Banach spaces and discuss properties of such maps used in Sec. 2. For more details we refer the reader, e.g., to [3] or [4, Appendix A].

Let X and Y be complex Banach spaces and let $U \subseteq X$ be an open set in X . A map $F: U \rightarrow Y$ is said to be *analytic* if it is Fréchet differentiable over \mathbb{C} at any point $x \in U$. The map $F: U \rightarrow Y$ is said to be *weakly analytic* if for any $x \in U$, $h \in X$, and $f \in Y^*$, the complex-valued function $f(F(x + zh))$ is holomorphic in a small disk in \mathbb{C} centered at zero.

Let us recall the following analyticity criteria (see, e.g., [4, Theorem 1.1, Appendix A]). Assume that X and Y are complex Banach spaces and that $\Phi: U \rightarrow Y$ is a map defined on an open subset U of X , $U \subseteq X$. Then the following statements are equivalent:

(A1) $\Phi: U \rightarrow Y$ is analytic;

(A2) $\Phi: U \rightarrow Y$ is weakly analytic and locally bounded;

(A3) $\Phi: U \rightarrow Y$ is continuous and for any $x \in U$ and $h \in X$, there exists a disk $D_r = \{z \in \mathbb{C} \mid |z| < r\}$ such that, for any $0 < \rho < r$, Cauchy's formula holds

$$F(z) = \frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{\Phi(\xi)}{\xi - z} d\xi \quad \forall |z| < \rho; \tag{26}$$

(A4) $\Phi: U \rightarrow Y$ is infinitely differentiable on U and is represented by its Taylor series in a neighborhood of each point of U , i.e., for any $x \in U$,

$$\Phi(x + h) = \Phi(x) + \sum_{n=1}^{\infty} d_x^n \Phi(h, \dots, h),$$

where $d_x^n \Phi$ denotes the n th derivative of Φ at x and the series converges absolutely and uniformly for any $h \in B(\rho, X)$, with $0 < \rho < r$ so that $B(r, X) \subset U$.

Now let us prove the following lemma.

Lemma 2. *Let X , Y , and Y_+ be Banach spaces such that $Y \subset Y_+$. Let $U \subseteq X$ be an open set in X and let $\Phi: U \rightarrow Y_+$ be analytic, $\Phi(U) \subseteq Y$, and $\Phi: U \rightarrow Y$ be continuous. Then $\Phi: U \rightarrow Y$ is analytic.*

Proof. For any $x \in U$ and $h \in X$, consider the map

$$F: D_r \rightarrow Y_+, \quad F(z) := \Phi(x + zh),$$

where the disk $D_r \subseteq \mathbb{C}$ is centered at zero and chosen so that $x + zh \in U$ for any $z \in D_r$. Since, by assumption, $F: D_r \rightarrow Y_+$ is analytic in the Y_+ -norm, it follows that Cauchy's formula (26) holds in Y_+ for any given $z \in D$. Furthermore, it follows from the assumptions of the lemma that $F: D_r \rightarrow Y$ is continuous with respect to the Y -norm; then the integral on the right-hand side of (26) defines a continuous map $G: D_r \rightarrow Y$. In view of the embedding $Y \subseteq Y_+$, one has $G = F$. \square

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