Center-of-mass angular momentum and memory effect in asymptotically flat spacetimes

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Gravitational-wave (GW) memory effects are constant changes in the GW strain and its time integrals, which are closely connected to changes in the charges that characterize asymptotically flat spacetimes. The first GW memory effect discovered was a lasting change in the GW strain. It can occur when GWs or massless fields carry away 4-momentum from an isolated source. Subsequently, it was shown that fluxes of intrinsic angular momentum can generate a new type of memory effect called the spin memory, which is an enduring change in a portion of the time integral of the GW strain. In this paper, we note that there is another new type of memory effect. We call it the "center-of-mass (CM) memory effect," because it is related to changes in the CM part of the angular momentum of a spacetime. We first examine a few properties of the CM angular momentum. Specifically, we describe how it transforms under the supertranslation symmetry transformations of the Bondi-Metzner-Sachs group, and we compute a new expression for the flux of CM angular momentum carried by GWs in terms of a set of radiative multipole moments of the GW strain. We then turn to the CM memory effect. The CM memory effect appears in a quantity which has the units of the time integral of the GW strain. We define the effect in asymptotically flat spacetimes that start in a stationary state, radiate, and settle to a different stationary state. We show that it is invariant under infinitesimal supertranslation symmetries in this context. To determine the magnitude of the flux of CM angular momentum and the CM memory effect, we compute these quantities for nonspinning, quasicircular compact binaries in the post-Newtonian approximation. The CM memory effect arises from terms in the gravitational waveform for such binaries beginning at third and fourth post-Newtonian order for unequal- and equal-mass binaries, respectively. Finally, we estimate the amplitude of the CM memory effect for these binaries. We anticipate that it will be unlikely for current or upcoming GW detectors to measure the effect.

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I. INTRODUCTION

Far from an isolated gravitating source, spacetime can be described as asymptotically flat if it satisfies the conditions set forth by Bondi *et al.* [1] and Sachs [2,3] (see also, e.g., the review [4]). These spacetimes encompass the asymptotic region of a wide range of interesting astrophysical systems. The gravitational waveforms used for detecting the five binary-black-hole mergers by the LIGO-Virgo Collaboration [5–9] and the one binary-neutron-star merger [10], for example, are determined from numerical simulations with asymptotically flat boundary conditions. The symmetry group of asymptotically flat spacetimes is the Bondi-Metzner-Sachs (BMS) group, which consists of the Lorentz transformations and an infinite-dimensional, Abelian group called the supertranslations. The supertranslations include the four spacetime translations, but they predate and are not related to supersymmetry. Related to all the infinitesimal BMS symmetries are corresponding charges (see, e.g., [11]). For the Lorentz symmetries, the conjugate charges are the angular momenta [which can be split into the spin and center-of-mass (CM) parts]; for the supertranslations, the charges are called "supermomenta" (by analogy with how the charges related to the four spacetime translations are called 4-momenta).

More recently, the symmetries of asymptotically flat spacetimes have been reexamined, and larger symmetry algebras than the BMS algebra have been proposed (see, e.g., [12–15]). The extensions of the BMS algebra involve enlargements of the Lorentz part of the algebra, and the conjugate charges can be thought of as generalizations of relativistic angular momentum. These charges were called superspin and super CM in [16], by analogy with nomenclatures used to describe the Lorentz charges and the supermomentum charges. Collectively, we will call these charges "super angular momentum" (though they have also

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been called super-rotation charges [17] after the name given to the extended symmetry vector fields). Thus, there may be an infinite number of additional charges that characterize an asymptotically flat spacetime. These charges have garnered much attention recently, because they, and related quantities on black-hole horizons, were proposed as a type of "soft hair" on black holes that could be a part of the resolution to the black-hole-information paradox [18].

The super angular momentum and the supermomentum are also of interest because of their relation to gravitational-wave (GW) memory effects. The first GW memory effect discovered—which in this paper we will simply refer to as *the* GW memory effect[1]—is characterized by a nonzero change in the GW strain between early and late times. In an idealized detector composed of freely falling test masses, the GW memory causes the proper distance of the masses before and after the GWs have passed through the detector to differ. The GW memory was initially computed within the context of linearized gravity by Zel'dovich and Polnarev [20], and it was subsequently computed in full (nonlinear) general relativity by Christodoulou [21]. Note, however, that the idea of a nonlinear GW memory effect dates back (at least) to Payne [22] (including the notion that the memory is related to supertranslation symmetries and to Weinberg's soft theorem [23]) as well as to an unpublished habilitation thesis, of which certain results were later published in [24] (see [25] for more detail).[2] The sources of the GW memory are changes in the supermomentum charges and in the quadrupole and higher-multipole moments of the flux of 4-momentum radiated in massless fields and GWs (see, e.g., [16,27,28]).

Pasterski *et al.* [29] also realized that there can be a new kind of GW memory effect, which they called the "spin memory effect." The spin memory is characterized by a change in the time integral of the magnetic-parity part of the GW strain.[3] It can be measured by a Sagnac detector following a particular accelerating trajectory [29] or by a family of freely falling observers surrounding a source of GWs [31]. The sources of the spin memory are changes in the superspin charges or in the quadrupole and higher-multipole moments of the flux of the intrinsic part of the angular momentum carried by massless fields and GWs [16]. The spin memory also has a signature in the gravitational waveform from compact binaries that could be detected by third-generation GW observatories [19], such as the Einstein Telescope [32] and Cosmic Explorer [33].

There has not yet been any discussion of a memory effect related to changes in the quadrupole and higher multipole moments of the flux of the CM portion of the angular momentum or in the super-CM charges. We find that there can be such an effect, which we call the "center-of-mass memory effect".[4] Defining this effect; understanding its properties; and computing the effect from nonspinning, quasicircular compact binaries are all goals of this paper. To help reach these goals, we will also need to discuss the properties of the flux of CM angular momentum and the context in which the CM memory effect is defined. We organize the discussion of these topics as follows.

In Sec. II, we review some properties of the flux of (super) angular momentum in asymptotically flat spacetimes. We first provide some background on the Bondi-Sachs framework, the space of stationary and nonradiative solutions of Einstein's equations in asymptotically flat spacetimes, and BMS symmetries and their corresponding charges and fluxes. We then discuss how changes in the (super) angular momentum transform under supertranslations and how they can be interpreted physically. Even for spacetimes that start in a stationary state, radiate, and then settle to a different stationary state (a stationary-to-stationary transition), the changes in the charges can transform nontrivially. We also give an expression for the flux of CM angular momentum carried by GWs, when the GW strain is expanded in a set of radiative multipole moments.

In Sec. III, we introduce the CM memory effect, we discuss the context in which it is defined, and we show that it is invariant under infinitesimal BMS supertranslation symmetry transformations. We also give an expression for the CM memory effect in terms of multipole moments of the GW strain in this part.

The results of Secs. II and III are then used in Sec. IV to compute the leading-order expressions for the CM memory effect and flux of CM angular momentum for nonspinning, quasicircular compact binaries in the post-Newtonian (PN) approximation. We find that both equal- and

[1] There seem to be two competing naming systems for GW memory effects: one is based on the type of physical effect that could be measured as a consequence of the GW memory; the other employs the name of the flux of the "conserved" quantity which can act as a source of the corresponding memory effect. Thus, the two nomenclatures would suggest calling it the displacement memory (as in [19]) or the 4-momentum (or supermomentum) memory (a name that, as far as we can tell, has never been used). However, because this is the first memory effect discovered, we will opt against adding cumbersome modifiers and simply refer to it as *the* GW memory effect (and we will typically drop the emphasis on the word "the" hereafter).

[2] It seems plausible to argue that the GW memory in full general relativity was previously realized as a possibility by Newman and Penrose (see the discussion in [26]); however, we will not attempt to settle the question of the first reference to the GW memory effect here.

[3] By "magnetic-parity part," we mean the part that can be decomposed into magnetic-parity tensor spherical harmonics (see, e.g., [30] for a review of these harmonics). This turns out to be equivalent to the part parametrized by the scalar function Ψ in Eq. (3.9). It is also sometimes called just the "magnetic part," for short.

[4] Note that we follow the convention of naming the memory effect after the type of charge that can generate the effect when it varies in time. The primary reason for this is to maintain a parallel with the naming of the spin memory effect. A secondary reason is that the measurable effect related to the CM memory is somewhat involved (as we discuss later), and it does not lend itself to a simple name.

unequal-mass binaries have a CM memory effect, but the leading-PN-order sources of these memory effects come from the ordinary and null parts of the memory, respectively (using the terminology of Bieri and Garfinkle [27]). When we estimate the amplitude of the part of the gravitational waveform responsible for the CM memory, we find that both the null and the ordinary parts will be unlikely to be observed (though for different reasons), even for the next generation of ground-based GW detectors, such as the Einstein Telescope or Cosmic Explorer. We conclude in Sec. V.

Throughout this paper we use units in which $G = c = 1$, and we use the conventions for spacetime indices and metric and curvature tensors given in [34].

II. PROPERTIES OF THE FLUX OF (SUPER) ANGULAR MOMENTUM

Before discussing the properties of the flux of (super) angular momentum and the interpretation of the CM part of the flux, we briefly review a few features of the Bondi-Sachs framework that will be needed throughout this paper.

A. Aspects of the Bondi-Sachs framework

The metric of asymptotically flat spacetimes can be expressed in Bondi coordinates, (u, r, θ^A) (where $A = 1, 2$). These coordinates are a retarded time (u), an affine parameter along outgoing null rays (as well as an areal radius r), and coordinates on a 2-sphere (θ^A). The general form of the metric and the corresponding Einstein equations were derived assuming axisymmetry in [1]. Subsequently, in Ref. [2], Einstein's equations without imposing axisymmetry were given in part; the full expressions for the hypersurface and evolution equations in vacuum (and with respect to a particular parametrization of Bondi-Sachs coordinates) were given in [35]. The hypersurface and evolution equations with matter sources (and in a covariant notation with respect to the 2-sphere cross sections) were written later in [36]. We will not give these (somewhat lengthy) expressions here; however, we will briefly discuss the structure of Einstein's equations as elaborated in these references.

Of the ten components of Einstein's equations, four take the form of "hypersurface" equations, which do not involve u derivatives, and which constrain different metric functions on hypersurfaces of constant u. Two other components are evolution-type equations for the transverse-traceless parts of the metric. The final four components are sometimes called the "conservation" equations, though one component is trivially satisfied. The remaining three components have the property that if they are satisfied at a fixed value of r on an outgoing null cone in a Bondi coordinate chart, then they are satisfied for all such values of r. This follows from the contracted Bianchi identities (which are equivalent to local stress-energy conservation for spacetimes with matter sources).

We next briefly review the procedure involved in the derivation of the components of Einstein's equations that we will need in the discussion below. We start from the Bondi-Sachs metric, which we write as

$$ds^2 = -Ue^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}(d\theta^A - U^A du)(d\theta^B - U^B du). \tag{2.1}$$

We then assume that the functions U, β, U^A and γ_{AB} can be expanded in a series in $1/r$ with the asymptotic fall-off conditions given in [1]. When the spacetime contains matter sources, it is also necessary to assume fall-off conditions on the stress-energy tensor T_{ab}. We use those discussed in [16], which are based on the stress-energy tensor of a radiating scalar field in flat spacetime:

$$T_{uu} = r^{-2}\hat{T}_{uu}(u, \theta^A) + O(r^{-3}), \tag{2.2a}$$

$$T_{uA} = r^{-2}\hat{T}_{uA}(u, \theta^B) + O(r^{-3}), \tag{2.2b}$$

$$T_{rA} = r^{-3}\hat{T}_{rA}(u, \theta^B) + O(r^{-4}), \tag{2.2c}$$

$$T_{rr} = r^{-4}\hat{T}_{rr}(u, \theta^A) + O(r^{-5}), \tag{2.2d}$$

$$(T_{AB})^{\mathrm{TF}} = r^{-2}\hat{T}_{AB}(u, \theta^C) + O(r^{-3}). \tag{2.2e}$$

The superscript TF means to take the trace-free part of the expression on the left-hand side of the equation with respect to the metric on the 2-sphere, h_{AB}. Note that local stress-energy conservation requires that the functions $\hat{T}_{rr}$ and $\hat{T}_{rA}$ be related by

$$\hat{T}_{rA}(u, \theta^B) = \check{T}_{rA}(\theta^B) - \frac{1}{2}D_A T_{rr}(u, \theta^B) \tag{2.3}$$

(see, e.g., [16]). The derivative operator D_A is the Levi-Civita connection compatible with the metric h_{AB}.

The hypersurface-type components of Einstein's equations can then be applied to determine the precise form of the expansion of the functions U, β, U^A and γ_{AB} in a series in $1/r$. At the accuracy in $1/r$ needed for the discussion of Einstein's equations below, these functions are given by

$$\gamma_{AB} = h_{AB}\left(1 + \frac{1}{4r^2}C_{CD}C^{CD} + \frac{1}{2r^3}\mathcal{D}_{CD}C^{CD}\right) + \frac{1}{r}C_{AB} + \frac{1}{r^2}\mathcal{D}_{AB} + \frac{1}{r^3}\mathcal{E}_{AB} + O(r^{-4}), \tag{2.4a}$$

$$U^A = -\frac{1}{2r^2}D_B C^{AB} + \frac{1}{r^3}\left[-\frac{2}{3}N^A + \frac{1}{16}D^A(C_{BC}C^{BC}) + \frac{1}{2}C^{AB}D^C C_{BC}\right] + O(r^{-4}), \tag{2.4b}$$

$$U = 1 - \frac{2m}{r} + O(r^{-2}), \tag{2.4c}$$

$$\beta = -\frac{1}{r^2}\left(\pi\hat{T}_{rr} + \frac{1}{32}C_{AB}C^{AB}\right) + O(r^{-3}). \tag{2.4d}$$

In the expressions above, all the scalars and tensors on the right-hand side are functions of the coordinates (u, θ^A), which have been omitted to make the notation more compact; also all capital latin indices are raised and lowered with the metric h^{AB} and its inverse. The tensors C_{AB}, $\mathcal{D}_{AB}$, and $\mathcal{E}_{AB}$ in the expansion of γ_{AB} are symmetric and trace free. This, as well as the form of the term proportional to h_{AB}, is required to satisfy the determinant condition of Bondi gauge: $\partial_r \det(\gamma_{AB}) = 0$. Two of the hypersurface-type components of Einstein's equations also require that $D^B\mathcal{D}_{BA} = -8\pi\check{T}_{rA}$. The two additional functions $m(u, \theta^A)$ and $N_A(u, \theta^B)$ are often called the Bondi mass and angular momentum aspects, respectively. We use a convention for the angular momentum aspect like that used by Sachs [2], in which it is proportional to the $1/r^4$ parts of certain components of the Riemann tensor.

The three nontrivial conservation components of Einstein's equations require that the Bondi mass and angular momentum aspects satisfy the following equations:

$$\dot{m} = -4\pi\hat{T}_{uu} - \frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}D_A D_B N^{AB}, \tag{2.5a}$$

$$\begin{aligned}\dot{N}_A = {} & -8\pi\hat{T}_{uA} + \pi D_A\partial_u\hat{T}_{rr} + D_A m \\ & + \frac{1}{4}D_B D_A D_C C^{BC} - \frac{1}{4}D_B D^B D^C C_{CA} \\ & + \frac{1}{4}D_B(N^{BC}C_{CA}) + \frac{1}{2}D_B N^{BC}C_{CA}.\end{aligned} \tag{2.5b}$$

The dot over the variables on the left-hand side is a shorthand notation for ∂_u. We define the news tensor as $N_{AB} = \partial_u C_{AB}$ (twice that defined in [1]). The news tensor is a quantity that arises from solving the evolution equations for the traverse-traceless components of Einstein's equations at leading order in $1/r$. The news tensor is unconstrained, but it can be shown that it vanishes when the spacetime is not radiating GWs [37].

Expanding the evolution-type components of Einstein's equations at higher order in $1/r$ leads first to the equation $\dot{\mathcal{D}}_{AB} = 0$, which is consistent with the hypersurface-type equations $D^B\mathcal{D}_{BA} = -8\pi\check{T}_{rA}$. When the spacetime is vacuum, it follows that $\mathcal{D}_{AB} = 0$. The tensor $\mathcal{E}_{AB}$ satisfies a nontrivial evolution equation:

$$\begin{aligned}\dot{\mathcal{E}}_{AB} = {} & -4\pi\hat{T}_{AB} - 2\pi(\partial_u\hat{T}_{rr})C_{AB} - \frac{1}{2}\mathcal{D}_{AB} + \frac{1}{2}mC_{AB} \\ & + \pi\left(D_A D_B - \frac{1}{2}h_{AB}D^2\right)\hat{T}_{rr} + \frac{1}{3}D_{(A}N_{B)} \\ & - \frac{1}{6}h_{AB}(D_C N^C) + \frac{1}{4}C_{AB}(N_{CD}C^{CD}) \\ & - \frac{1}{8}\epsilon_A{}^C C_{CB}(\epsilon_{\mathrm{DE}}D^E D_C C^{CD}).\end{aligned} \tag{2.6}$$

A closely related equation in axisymmetry and in vacuum appears in the paper [1]. Restricting Eq. (2.6) to vacuum, it is equivalent to an equation derived by van der Burg [35] after taking into account differences in notation and convention used (Sachs [2] also derives a related equation, but does not present all the nonlinear terms). The linearized limit of Eq. (2.6) also agrees with the nonvacuum expression given in, e.g., [38]. Because the tensor $\mathcal{E}_{AB}$ is closely related to the Newman-Penrose scalar ψ_0 [39] (discussed in [2]), it is also closely related to evolution equations for this scalar (see, e.g., the review [40]).

B. Stationary and nonradiative regions and transitions between these regions

For computing memory effects, we specialize to asymptotically flat spacetimes that begin in a stationary or a nonradiative ($N_{AB} = 0$) state, radiate GWs and massless fields, and then settle into a different nonradiative or stationary state. We will often make the further assumptions that the initial stationary or nonradiative region is in vacuum ($T_{ab} = 0$), the radiative region of the spacetime is not in vacuum [and the stress-energy tensor satisfies the conditions in Eq. (2.2)], and the final stationary or nonradiative region also is in vacuum. Einstein's equations in (2.5) and (2.6) constrain the form of the Bondi-metric functions m, N_A, and $\mathcal{E}_{AB}$ in stationary or nonradiative regions, which restricts the space of solutions to Einstein's equations therein. However, it does not imply that a given set of astrophysical sources will necessarily realize the full space of solutions consistent with the vacuum and stationary or nonradiative conditions.

This type of issue (as it relates to the GW memory effect) was discussed by Frauendiener [41]. From the perspective of Einstein's equations, the news tensor can be an arbitrary function $N_{AB}(u, \theta^C)$, and in a nonradiative-to-nonradiative transition, the memory can have any amplitude and angular dependence (this should hold for the changes in m, N_A, and $\mathcal{E}_{AB}$, too). However, from the perspective of solving a specific initial-value problem for a certain astrophysical source, the news tensor cannot be specified freely; rather, it follows from the dynamics of the source. The Bondi news tensor can be determined through some sort of matching procedure analytically (e.g., through post-Newtonian-expanded, multipolar-post-Minkowski calculations [42]) or numerically (e.g., through Cauchy-characteristic extraction [43]). For the specific systems treated in this paper (inspiraling compact binaries, and particularly binary-black-hole mergers), the allowed values of the memory, and the changes in m, N_A, and $\mathcal{E}_{AB}$ are more restricted than those allowed by the general solutions of Einstein's equations in a stationary or nonradiative region.[5] In this paper, we will focus on this latter perspective, because we are ultimately interested in GW memory effects arising from the inspiral and merger of nonspinning compact binaries. Nevertheless, we will first describe the general solutions of Einstein's equations in nonradiative and stationary regions, to make clear the types of restrictions we are making in specializing to particular sources.

In a nonradiative and vacuum region, the first line of Eq. (2.5) requires that the Bondi mass aspect be independent of u, so that it is just a function of angular coordinates, $m(\theta^A)$. From the other lines of Eq. (2.5), it then follows that N_A can have a piece that depends linearly on u. The electric-parity part of N_A depends on $D_A m$, while the magnetic-parity part depends on the magnetic-parity part of C_{AB}. Although the magnetic-parity part vanishes in stationary regions (see [35,45]), it need not vanish in nonradiative regions. N_A can also have a part that is independent of u (with both electric and magnetic parities). Finally, from Eq. (2.6), it then implies that $\mathcal{E}_{AB}$ can have terms proportional to u^2, u, and independent of u with both electric and magnetic parities, in a nonradiative, vacuum region. Summarizing these results by explicitly solving Eqs. (2.5) and (2.6) in such a region, we find that

$$m = m(\theta^A), \tag{2.7a}$$

$$N_A = uD_A m + \frac{u}{4}(D_B D_A D_C C^{BC} - D^2 D^B C_{AB}) + N_A^{(0)}(\theta^B), \tag{2.7b}$$

$$\mathcal{E}_{AB} = \frac{u^2}{24}[4D_A D_B m - 2D^2 m h_{AB} + D_B D_A D_C C^{BC} - D^2 D^B C_{AB}] + \frac{u}{2} m C_{AB} + \frac{u}{6}(2D_{(A} N^{(0)}_{B)} - D^C N_C^{(0)} h_{AB}) - \frac{u}{8}\epsilon_A{}^C C_{CB}(\epsilon_{\mathrm{DE}} D^E D_C C^{CD}) + \mathcal{E}_{AB}^{(0)}(\theta^C). \tag{2.7c}$$

Recall that while Eq. (2.7) is the most general solution for m, N_A, and $\mathcal{E}_{AB}$ consistent with a nonradiative and vacuum region of future null infinity, it is not clear if the nonradiative regions of a specific astrophysical system, such as a merging compact binary, will realize this level of generality.

Stationary vacuum regions, for example, have frames in which the Bondi metric functions are independent of u [35,45]. Applying this condition to Eq. (2.7), we find that m is a constant, the magnetic-parity part of C_{AB} is zero, and $N_A^{(0)}$ is composed of both $l = 1$ vector spherical harmonics and $l > 1$ harmonics that satisfy $2D_{(A}N^{(0)}_{B)} - (D^C N_C^{(0)})h_{AB} = -3mC_{AB}$. These frames can then be transformed to the "canonical" frame described in [16], in which m is constant, $C_{AB} = 0$, and $N_A^{(0)}$ is composed of $l = 1$ magnetic-parity vector harmonics.

Because our primary focus in this paper is on merging compact binaries composed of black holes, we will need to know the properties of the nonradiative regions for these binaries at early and late times in their evolution. At early times, the binaries can be approximated well by PN theory. One assumption in this approximation is that there is a (finite) time before which the system was stationary in the past (see, e.g., [42]). This could correspond to a time early in the evolution of the binary, when the binary's components are sufficiently widely separated and slowly moving that the system can be treated as stationary. The outcome of a binary-black-hole merger is a stationary black hole. For studying binary-black-hole mergers, therefore, it should be sufficient to consider stationary-to-stationary transitions. It is also important to briefly describe the types of restrictions assuming a stationary-to-stationary transition will cause, so as to better understand the generality of our results.

For simplicity, in most of the subsequent calculations and discussion, we will assume that the initial stationary frame is the canonical frame of the system. At late times, the stationary frame will generally not be the canonical frame, but one that differs from the canonical frame by a BMS transformation (which can be decomposed into a rotation, followed by a boost, and then a supertranslation). From these properties of the initial and final frames, we anticipate that there will be two different types of restrictions from assuming a stationary-to-stationary transition.

The first is that the magnetic-parity part of the shear will vanish in both stationary regions (although, in general m will not be constant and N_A will not consist of just $l = 1$

[5]This issue can be recast in terms of how the conservation-type components of Einstein's equations are treated. These equations are automatically satisfied for all r on an outgoing null cone in a Bondi coordinate patch, so long as they are satisfied on some 2-sphere of fixed r. One choice for this 2-sphere is at infinite radius (i.e., at future null infinity). At this boundary of an asymptotically flat spacetime, it is possible to allow for any value of the news tensor N_{AB}, because quantities at null infinity can be defined without reference to the interior of the spacetime. From this perspective, however, it is not clear if these values of the Bondi news tensor correspond to any astrophysical solution of Einstein's equations in the interior of the spacetime. The other viewpoint, which fits more with the aims of this paper, is to allow the Bondi functions to satisfy the conservation-type components of Einstein's equations at finite r and to determine their evolution by matching to a specific initial-value (Cauchy) solution for a given system (as described, e.g., in [44]).

magnetic-parity vector harmonics in the final stationary region). This does not seem to be a very strong restriction, because Mädler and Winicour [38] have shown that there is no magnetic-parity memory effect in the absence of incoming radiation or time-dependent, anisotropic, magnetic-parity material stresses near null infinity (which compact binaries, for example, are generally not expected to have). Several common classes of stress-energy tensors also do not give rise to magnetic-parity memory [38]. The second type of restriction relates to the ordinary part of the GW memory (using the terminology of [27]). Assuming a stationary-to-stationary transition makes the ordinary part of the GW memory a function of just the change in the 4-momentum radiated by the spacetime. It, therefore, would exclude certain physically relevant systems, like the gravitational scattering of astrophysical objects considered in [20]. Note, however, that the assumption of a stationary-to-stationary transition does not have a significant effect on the null part of the GW memory (neither the linear nor the nonlinear parts).

Finally, because the set of stationary-to-stationary transitions is contained within the larger set of nonradiative-to-nonradiative transitions, imposing the former assumption will generally restrict the types of possible memory effects. Because stationary-to-stationary transitions contain an interesting set of physical systems (compact-binary mergers), it has sufficient generality to allow for some nontrivial memory effects (even if they are not the most general effects possible). Having elaborated our assumptions and their consequences, we next discuss BMS symmetries and their conjugate charges and fluxes.

C. Symmetries, charges, and fluxes

The vector fields at future null infinity that define the (extended) BMS algebra, $\vec{\zeta}$, are parametrized by a scalar function $\alpha(\theta^A)$ and a vector on the 2-sphere $Y^A(\theta^B)$ as follows:

$$\vec{\zeta} = [\alpha(\theta^A) + uD_A Y^A(\theta^B)/2]\vec{\partial}_u + Y^A(\theta^B)\vec{\partial}_A. \tag{2.8}$$

The quantity $\alpha(\theta^A)$ is a smooth function that corresponds to a supertranslation, and $Y^A(\theta^B)$ are $l = 1$ vector spherical harmonics, for the standard BMS group. For the extended BMS algebra [17], Y^A are elements of a Virasoro algebra, or for the generalized BMS group [14], they are smooth vector fields on the 2-sphere. The standard and extended BMS symmetries at null infinity can be defined at finite r in Bondi coordinates by requiring that the spacetime metric continue to satisfy the Bondi gauge conditions and the same scaling with r under pullback along the symmetry vector fields. The vector fields in (2.8) have a series expansion in $1/r$ in the interior of the spacetime, and the Bondi metric functions (C_{AB}, m, and N_A) transform nontrivially under these (extended) BMS symmetries. The formulas for the BMS vector fields and the transformations of the Bondi functions are given, e.g., in [46].

For most of the computations in this paper, we are interested in how the Bondi functions transform under supertranslations in vacuum and in stationary or nonradiative regions of the types described in the previous subsection. Specializing the results in [16], for example, we find that m is invariant under supertranslations and

$$\delta C_{AB} = (-2D_A D_B + h_{AB} D^2)\alpha \equiv -2C^{(\alpha)}_{AB} \tag{2.9a}$$

$$\begin{aligned}\delta N_A = {} & \alpha D_A m + 3m D_A \alpha + \frac{1}{4} C_{AB} D^B D^2 \alpha \\ & - \frac{3}{4} D_B \alpha (D^B D^C C_{CA} - D_A D_C C^{BC}) \\ & + \frac{3}{8} D_A (C^{BC} C^{(\alpha)}_{BC}) + \frac{1}{2} C^{(\alpha)}_{AB} D_C C^{BC}\end{aligned} \tag{2.9b}$$

(the first line can be found from the results in [2] or [26]). In the equation above, we have introduced the notation $C^{(\alpha)}_{AB}$ to denote the electric-parity part of the shear generated by a scalar "potential" $\alpha(\theta^A)$.

The (super) angular momentum in a vacuum, nonradiative region of null infinity, on a cut $\mathcal{C}$ of constant $u = u_0$, is given by

$$\begin{aligned}Q[\vec{\zeta}_Y; \mathcal{C}] = {} & \frac{1}{128\pi} \int d^2\Omega Y^A [16(N_A - u_0 D_A m) \\ & - D_A (C_{BC} C^{BC}) - 4 C_{AB} D_C C^{BC}],\end{aligned} \tag{2.10}$$

where by $\vec{\zeta}_Y$, we mean a BMS vector field with $\alpha = 0$, and which is thus parametrized by the vector on a 2-sphere, Y^A. The prescription to compute this charge corresponding to a vector field $Y^A(\theta^B)$ is outlined in [16], which is based on the procedure in [11] (and which gives equivalent results to those defined via a different procedure in [17], in the nonradiative and vacuum regions treated here).

The integral of the flux of (super) angular momentum between two cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ in vacuum, nonradiative regions given by $u = u_1$ and $u = u_2$, respectively, is

$$\begin{aligned}\Delta\tilde{Q}[\vec{\zeta}_Y; \mathcal{C}_2, \mathcal{C}_1] = {} & -\frac{1}{64\pi} \int_{u_1}^{u_2} du \int d^2\Omega [u D_A (2 D_B D_C N^{BC} \\ & - N_{BC} N^{BC} - 32\pi \hat{T}_{uu}) + D_A (C_{BC} N^{BC}) \\ & + 2 N^{BC} D_A C_{BC} - 4 D_B (N^{BC} C_{AC}) \\ & + 64\pi \hat{T}_{uA}] Y^A.\end{aligned} \tag{2.11}$$

It was shown in [16] that the changes in the charges between the two cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ do not equal the integral of the flux in Eq. (2.11) for the meromorphic super-rotation vector fields Y^A (i.e., when Y^A is not one of the six generators of the Lorentz group). To restore equality for these extended BMS symmetries, an additional term of the form

$$\Delta\mathcal{F}[\vec{\zeta}_Y;\mathcal{C}_2,\mathcal{C}_1] \equiv \frac{1}{32\pi}\int_{u_1}^{u_2} du \int d^2\Omega Y^A \epsilon_{AB}\epsilon^{CD} \times D^B D_D D^E C_{CE} \tag{2.12}$$

must be added. The change in the charges is then given by

$$Q[\vec{\zeta}_Y;\mathcal{C}_2] - Q[\vec{\zeta}_Y;\mathcal{C}_1] = \Delta\tilde{Q}[\vec{\zeta}_Y;\mathcal{C}_2,\mathcal{C}_1] - \Delta\mathcal{F}[\vec{\zeta}_Y;\mathcal{C}_2,\mathcal{C}_1]. \tag{2.13}$$

We reiterate that the term $\Delta\mathcal{F}[\vec{\zeta}_Y;\mathcal{C}_2,\mathcal{C}_1]$ vanishes for the standard BMS group; it is only needed for the additional elements of the extended BMS algebra. The term $\Delta\mathcal{F}[\vec{\zeta}_Y;\mathcal{C}_2,\mathcal{C}_1]$ is also closely related to the spin memory effect of [29], as discussed in [16].

It will be convenient to define a quantity that is equal to the change in the charges:

$$\Delta Q[\vec{\zeta}_Y;\mathcal{C}_2,\mathcal{C}_1] \equiv Q[\vec{\zeta}_Y;\mathcal{C}_2] - Q[\vec{\zeta}_Y;\mathcal{C}_1]. \tag{2.14}$$

After some algebra (described in [16]), it was shown that the change in the charges can be written as

$$\begin{aligned}\Delta Q[\vec{\zeta}_Y;\mathcal{C}_2,\mathcal{C}_1] = -\frac{1}{64\pi}\int_{u_1}^{u_2} du \int d^2\Omega [& u D_A(2D_B D_C N^{BC} \\ & - N_{BC}N^{BC} - 32\pi\hat{T}_{uu}) + C^{BC}D_B N_{AC} \\ & - N^{BC}D_B C_{AC} + 3(N_{AB}D_C C^{BC} \\ & - C_{AB}D_C C^{BC}) + 64\pi\hat{T}_{uA} + 16\pi\partial_u\hat{T}_{rA} \\ & + 2\epsilon_{AB}\epsilon^{CD}D^B D_D D^E C_{CE}]Y^A. \end{aligned} \tag{2.15}$$

D. Transformation properties of (super) angular momentum under supertranslations

We now point out a few features of the (super) angular momentum charges and fluxes that we have not seen discussed explicitly elsewhere, but which may be related to two other aspects of the charges and fluxes that have been previously noted. The first is that nonlinear terms involving the shear in the super angular momentum can make it behave nontrivially: for example, it can be nonvanishing in spacetimes that are flat aside from a defect at the origin [47]. The second is the observation that the flux of angular momentum will depend upon nonradiative (or "Coulombic") parts of the Bondi metric functions and stress-energy tensor [48].

To illustrate the transformation properties of the (super) angular momentum, we will examine the same stationary-to-stationary transition from the perspective of two different Bondi frames. For the first frame, we use the canonical frame associated with the initial stationary region. Constructing this frame fixes all the degrees of freedom in the BMS group except for a global SO(3) rotation and a time translation (a BMS transformation with $\vec{\zeta} = u_0\vec{\partial}_u$, for a constant, u_0). We denote by $\mathcal{C}_1$ a cut corresponding to a retarded time $u = u_1$ in the initial stationary region and by $\mathcal{C}_2$ a cut of constant $u = u_2$ in the latter stationary region. For the second Bondi frame, we will consider one that is supertranslated from the canonical Bondi frame of the initial stationary region by an amount α. Because $u' = u + \alpha$, we will denote the cuts by $\mathcal{C}_1{}'$ and $\mathcal{C}_2{}'$, which correspond to 2-sphere cross sections of constant $u' = u'_1$ and $u' = u'_2$, respectively. Finally, we also assume that the spacetime has GW memory, which is determined by a potential $\Delta\Phi(\theta^A)$ and which is given by

$$\begin{aligned}\Delta C_{AB} &= C_{AB}(u_2) - C_{AB}(u_1) \\ &= \frac{1}{2}(2D_A D_B - h_{AB}D^2)\Delta\Phi. \end{aligned} \tag{2.16}$$

The memory is invariant under supertranslations (i.e., is equivalent to the related quantity measured at the times u'_2 and u'_1). We will treat the supertranslation, α, as small, and we will compute the transformation of the charges to linear order in α. We will not linearize with respect to the potential $\Delta\Phi$ that determines the GW memory.

We are particularly interested in comparing the changes in the charges between the cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ with those between the cuts $\mathcal{C}_1{}'$ and $\mathcal{C}_2{}'$. Performing such a comparison is somewhat subtle, because the (extended) BMS vector fields corresponding to (super) Lorentz transformations on cuts of constant u and u' are different. Namely, the quantity

$$\vec{\zeta}_Y = \frac{1}{2}u D_A Y^A \vec{\partial}_u + Y^A\vec{\partial}_A \tag{2.17}$$

and the equivalent vector fields adapted to the cuts of constant u' differ by a supertranslation (e.g., [3] and [17]). The charges associated with these two vector fields will therefore include different amounts of supermomentum. While this is to be expected, the difference in the charges arising from the dependence of the charges on the cut will mix with the difference that comes from the dependence of the charges on the generators adapted to those cuts. Instead, we will compute the change in the charges between the cuts of constant u' with the generators adapted to cuts of constant u. The vector field $\vec{\zeta}_Y$ expressed in terms of the primed coordinates is given by

$$\vec{\zeta}_Y = \left[\frac{1}{2}(u'-\alpha)(D_A Y^A) + Y^A D_A\alpha\right]\vec{\partial}_{u'} + Y^A\vec{\partial}_A \tag{2.18}$$

(see, e.g., [49]).

We will now show that in stationary vacuum regions, the (super) angular momentum transforms nontrivially under supertranslations (unlike the supermomentum, which is supertranslation invariant in this context). The reason for this is as follows. Although we use the same BMS vector field, $\vec{\zeta}_Y$, to compute the charges in the cuts defined by u and u', because the cuts of constant u' are supertranslated from the cuts of constant u, the values of the Bondi metric functions C_{AB} and N_A differ between the two sets of cuts (even in the vacuum and stationary regions). In addition, the split of the vector field $\vec{\zeta}_Y$ into parts tangent and orthogonal to cuts of u and u' will differ, which will also influence the value of the charges. Finally, because the (super) angular momentum charge depends on C_{AB} and its derivatives quadratically, it follows from Eqs. (2.9) and (2.10) that the (super) angular momentum charges that are supertranslated from a stationary region in which $C_{AB} = 0$ differ from the charges that are supertranslated from a frame with a nonzero C_{AB}. While the physical reason for this is not immediately obvious, we speculate that these nonlinear terms capture a difference in the "origin" about which the (super) angular momentum is computed in these two cases.

Let us explicitly compute how this change in the charges produced by a supertranslation (which we will denote by $\delta Q[\vec{\zeta}_Y; \mathcal{C}', \mathcal{C}]$) will affect the change in the (super) angular momentum between the two stationary regions (i.e., $\Delta Q[\vec{\zeta}_Y; \mathcal{C}_2, \mathcal{C}_1]$ versus $\Delta Q[\vec{\zeta}_Y; \mathcal{C}_2{}', \mathcal{C}_1{}']$). In the stationary region including u_1 and u_1', because we are working to linear order in the supertranslation α from the canonical frame, then the change is similar to a result for the (super) angular momentum charges in [16]. To linear order in α, we find that

$$\delta Q[\vec{\zeta}_Y; \mathcal{C}_1', \mathcal{C}_1] = \frac{1}{8\pi} \int d^2\Omega (5Y^A D_A \alpha - \alpha D_A Y^A) m_1, \tag{2.19}$$

where we have used the notation $m_1 = m(u_1) = m(u_1')$.

Next, let us compute $\delta Q[\vec{\zeta}_Y; \mathcal{C}_2', \mathcal{C}_2]$. The expression for this quantity is somewhat lengthier, because we are allowing C_{AB} to be nonzero at late times (and equal to the GW memory, ΔC_{AB}, in the cut $u = u_2$). Using Eqs. (2.9) and (2.10), we find that to linear order in α

$$\begin{aligned}\delta Q[\vec{\zeta}_Y; \mathcal{C}_2', \mathcal{C}_2] = {}& \frac{1}{8\pi} \int d^2\Omega (5Y^A D_A \alpha - \alpha D_A Y^A) m_2 \\ & + \frac{1}{64\pi} \int d^2\Omega Y^A [2\Delta C_{AB} D^B D^2 \alpha \\ & - 6 D_B \alpha (D^B D^C \Delta C_{CA} - D_A D_C \Delta C^{BC}) \\ & + 5 D_A (\Delta C^{BC} C^{(\alpha)}_{BC}) + 8 C^{(\alpha)}_{AB} D_C \Delta C^{BC} \\ & + 4 \Delta C_{AB} D_C C^{BC}_{(\alpha)}.\end{aligned} \tag{2.20}$$

Given the relationship in Eq. (2.14), then by construction, the changes in the charges between the cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ and the cuts $\mathcal{C}_1{}'$ and $\mathcal{C}_2{}'$ are related by

$$\begin{aligned}\Delta Q[\vec{\zeta}_Y; \mathcal{C}_2', \mathcal{C}_1'] = {}& \Delta Q[\vec{\zeta}_Y; \mathcal{C}_2, \mathcal{C}_1] + \delta Q[\vec{\zeta}_Y; \mathcal{C}_2{}', \mathcal{C}_2] \\ & - \delta Q[\vec{\zeta}_Y; \mathcal{C}_1{}', \mathcal{C}_1].\end{aligned} \tag{2.21}$$

Using Eqs. (2.19) and (2.20), we can compute a difference in the changes of the charges, $\Delta Q[\vec{\zeta}_Y; \mathcal{C}_2', \mathcal{C}_1'] - \Delta Q[\vec{\zeta}_Y; \mathcal{C}_2, \mathcal{C}_1]$, which we find is

$$\begin{aligned}&\Delta Q[\vec{\zeta}_Y; \mathcal{C}_2', \mathcal{C}_1'] - \Delta Q[\vec{\zeta}_Y; \mathcal{C}_2, \mathcal{C}_1] \\ &\quad = \frac{1}{8\pi} \int d^2\Omega (5Y^A D_A \alpha - \alpha D_A Y^A) \Delta m + \frac{1}{64\pi} \int d^2\Omega Y^A \\ &\qquad \times [2\Delta C_{AB} D^B D^2 \alpha - 6 D_B \alpha (D^B D^C \Delta C_{CA} - D_A D_C \Delta C^{BC}) \\ &\qquad + 5 D_A (\Delta C^{BC} C^{(\alpha)}_{BC}) + 8 C^{(\alpha)}_{AB} D_C \Delta C^{BC} \\ &\qquad + 4 \Delta C_{AB} D_C C^{BC}_{(\alpha)}].\end{aligned} \tag{2.22}$$

We defined $\Delta m = m_2 - m_1$ in the expression above. This result is interesting, because the change in the charges is related to the integral of the flux (plus the additional term $\Delta \mathcal{F}[\vec{\zeta}_Y; \mathcal{C}_2', \mathcal{C}_1']$). Thus, while it was not very surprising that the (super) angular momentum charges transform under supertranslations, it is more surprising that this change arising from a BMS transformation does not cancel between early and late times in a stationary-to-stationary transition (i.e., the flux transforms nontrivially under supertranslations).

From Eq. (2.22), it is clear that this lack of cancellation occurs when the system radiates supermomentum or when there is GW memory. Thus, the result in Eq. (2.22) is a combined effect of the GW memory, changes in the supermomentum, and the transformation properties of the (super) angular momentum under supertranslations. This is an interesting feature of the change in the (super) angular momentum that will be relevant when we discuss the flux of the CM angular momentum in the next subsections. We do not anticipate that it will play an important role for the CM memory effect: in Sec. III, we show that the CM memory is invariant under infinitesimal supertranslations α. It may also be possible to modify this transformation property of the change in the charges by an appropriate redefinition of the charges. Investigating this issue, however, goes beyond the scope of this work.

E. Center-of-mass part of (super) angular momentum and its flux

In this part, we focus on a few issues that apply specifically to the (super) CM part of the angular momentum. CM angular momentum is the conserved quantity conjugate to Lorentz boost symmetries. In special relativity, it is usually denoted by K^i, and it is closely related to the mass-weighted CM position, G^i. When there are no external forces, these two quantities satisfy the relationships

$$K^i = G^i - tP^i, \qquad \frac{dG^i}{dt} = P^i, \qquad \frac{dK^i}{dt} = 0 \tag{2.23}$$

(see, e.g., [50]). Thus, we see that K^i is the conserved quantity in this context, and that it represents the mass times the CM position in the center-of-momentum frame. It is also a trivial quantity in this context, because by translating the origin of coordinates around which the CM is computed, the CM part of the angular momentum can be set to zero.

In stationary regions of asymptotically flat spacetimes, the (super) CM angular momentum [defined by the integral of the electric-parity part of the integrand in Eq. (2.10) against a vector field Y^A] is again trivial; by performing the BMS transformations needed to reach the canonical frame, we can make the (super) CM angular momentum vanish (see [16]). We argue below that the change in the (super) CM angular momentum can be nontrivial in a stationary-to-stationary transition from the canonical frame of the first stationary region (in the sense that the CM angular momentum contains additional information not contained in the changes of other BMS charges or in the GW memory or spin memory effects, in this context). We provide further evidence for this by computing the flux of CM angular momentum in the PN approximation in Sec. IV. It could be of interest to compare this result to a related calculation of the flux of CM angular momentum in numerical relativity simulations in [51], though we will not attempt to do this in this paper. Instead, we will first point out a few more general features about the CM angular momentum and its flux, before we investigate these quantities for compact-binary sources in Sec. IV.

Because the CM angular momentum, K^i, is the mass times the CM position in the rest frame of the system (in the context of special relativity, with no external forces), it is worth briefly discussing the physical interpretation of this quantity when the CM of the system is changing because of radiated linear momentum. To do so, let us recast Eq. (2.15) for the change in the charges in terms of the instantaneous flux on a cut of constant u:

$$\begin{aligned}\dot{K}_{\vec{\zeta}_Y} = -\frac{1}{64\pi}\int d^2\Omega Y^A[&uD_A(2D_BD_CN^{BC} - N_{BC}N^{BC} \\ &- 32\pi\hat{T}_{uu}) + C^{BC}D_BN_{AC} - N^{BC}D_BC_{AC} \\ &+ 3(N_{AB}D_CC^{BC} - C_{AB}D_CC^{BC}) \\ &+ 64\pi\hat{T}_{uA} + 16\pi\partial_u\hat{T}_{rA}].\end{aligned} \tag{2.24}$$

Note that we have denoted this flux by $\dot{K}_{\vec{\zeta}_Y}$ to parallel the notation commonly used for the CM angular momentum in special relativity. Integrating the first three terms in Eq. (2.24) by parts, we find that these terms have exactly the same form as the flux of supermomentum; however, instead of a scalar $\alpha(\theta^A)$ appearing in the charge integral, it is $uD_AY^A(\theta^B)/2$. This is to be expected given that the BMS vector field (2.8) contains a sum of both α and $uD_AY^A/2$ in the $\vec{\partial}_u$ direction. The remaining terms in the integrand [which are related to the part $\vec{Y} = Y^A\vec{\partial}_A$ of the vector field $\vec{\zeta}_Y$ in Eq. (2.17)] have a similar form to the flux of the (super) spin; however, they are now the electric-parity part of the integrand, rather than the magnetic-parity part. Because it is the electric-parity part, the term related to the spin memory in Eq. (2.12) does not contribute. To emphasize the contributions from the two types of terms, we will write the instantaneous flux as the sum of two terms as follows:

$$\dot{K}_{\vec{\zeta}_Y} = \dot{k}_{\vec{Y}} + \frac{u}{2}\dot{P}_{(D_AY^A)}. \tag{2.25}$$

The second term involving $\dot{P}_{(D_AY^A)}$ has the same form as the supermomentum flux (for a scalar function D_AY^A rather than α), and the quantity $\dot{k}_{\vec{Y}}$ contains the remaining terms, which are related to the part of $\vec{\zeta}_Y$ not proportional to $\vec{\partial}_u$.

Consider now the change in the (super) CM angular momentum in a stationary-to-stationary transition. Given the splitting in Eq. (2.25), this change can be written as

$$\Delta K_{\vec{\zeta}_Y}(u_2, u_1) = \Delta k_{\vec{Y}} + \int_{u_1}^{u_2} du\frac{u}{2}\dot{P}_{(D_AY^A)}. \tag{2.26}$$

Note that this is a specialization of and rewriting of Eq. (2.15); we have used the notation $\Delta K_{\vec{\zeta}_Y}(u_2, u_1)$ rather than $\Delta Q[\vec{\zeta}_Y; \mathcal{C}_2, \mathcal{C}_1]$ to emphasize that it applies specifically to the change of the CM angular momentum. We also write the vector field as a subscript and use u_2 and u_1 rather than $\mathcal{C}_2$ and $\mathcal{C}_1$ to make the notation more compact (which will be particularly helpful for when we derive the multipolar expansion of the change in the CM angular momentum, which we do in the next subsection). Integrating the second term in Eq. (2.26) by parts, the change has the form

$$\begin{aligned}\Delta K_{\vec{\zeta}_Y}(u_2, u_1) = \Delta k_{\vec{Y}} &+ \frac{1}{2}[uP_{(D_AY^A)}]\big|_{u_1}^{u_2} \\ &- \frac{1}{2}\int_{u_1}^{u_2} duP_{(D_AY^A)}.\end{aligned} \tag{2.27}$$

Thus, we can now better understand the physical interpretation of the change in the (super) CM part of the angular momentum in a stationary-to-stationary transition. The first term $\Delta k_{\vec{Y}}$ represents a change in the (super) CM angular momentum, which is similar to the integral of the flux of the intrinsic angular momentum (but involves the electric-parity part of the integrand, rather than the magnetic-parity part). The last term in Eq. (2.27) represents the change in the (super) CM part of the angular momentum that arises from integrating the time dependence of a term like the supermomentum associated with the quantity $D_A Y^A$. This term would typically grow linearly with u when there is a net change in the supermomentum; however, the middle term in Eq. (2.27) also grows linearly with u, and will cancel this growth from the last term. The quantity $\Delta K_{\vec{\zeta}_Y}(u_1, u_2)$, therefore, is finite for spacetimes that radiate supermomentum over finite retarded-time intervals $u \in [u_1, u_2]$, and it contains information about the time dependence of the supermomentum beyond what is given by the net change in the supermomentum.[6] Thus, although the (super) CM part of the angular momentum can be made to vanish in a stationary region, its change in a stationary-to-stationary transition does not necessarily vanish. In addition, it contains additional information that is not captured in the net changes in the supermomenta or in the other BMS charges.

There is a interesting feature specific to the flux of (super) CM angular momentum that we now point out. Suppose we specialize Eq. (2.22) to the case in which $\alpha = u_0$ is a constant shift in retarded time. All terms except the first vanish, and we find that

$$\Delta K_{\vec{\zeta}_Y}(u_2', u_1') - \Delta K_{\vec{\zeta}_Y}(u_2, u_1) = -\frac{u_0}{2}\Delta P_{(D_A Y^A)}. \tag{2.28}$$

Thus, when there is a net change in the linear momentum or the supermomentum, the change in the (super) CM angular momentum is not invariant under shifts in the cuts by constant values of u_0. This transformation property of the CM angular momentum could be useful for defining a specific BMS frame in an asymptotically flat spacetime. As we noted in the previous subsection, the canonical frame associated with the initial stationary region fixes all the BMS transformations except for a time translation $\vec{\zeta} = u_0 \vec{\partial}_u$ and a global SO(3) rotation. In these stationary regions, the charges are invariant under time translations; thus, they cannot be used to determine a "preferred" retarded time in a stationary region. When there is a flux of 4-momentum [i.e., when $\Delta P_{(D_A Y^A)} \neq 0$], its time dependence allows for a preferred reference time (i.e., an "origin" of the time coordinate) to be picked out. One natural choice comes from requiring that the magnitude of the change in the CM angular momentum be minimized. This is satisfied by a value of u_0 given by

$$u_0 = \frac{2\Delta K_{\vec{\zeta}_Y}\Delta P_{(D_A Y^A)}}{\Delta P^2_{(D_A Y^A)}}. \tag{2.29}$$

This value of u_0 can be computed from changes in the BMS charges at infinity, and it is a geometrically motivated method of determining a reference time for spacetimes that radiate 4-momentum. A possible application of this property of the CM angular momentum is defining a reference time for comparing gravitational waveforms from numerical relativity simulations of compact binaries that radiate linear momentum. While we will not investigate this point in greater detail in this paper, we will make use of this reference time for computing the change in the CM angular momentum in Sec. IV.

F. Multipole expansion of the flux of CM angular momentum carried by GWs

To compute an expression for the flux of the CM angular momentum carried by GWs in terms of a set of multipole moments of the GW strain, we will closely follow the methods used to calculate the GW memory and spin memory effects given in [19]. We will expand C_{AB} in terms of electric- and magnetic-parity tensor spherical harmonics as

$$C_{AB} = \sum_{l,m}(U_{lm}T^{(e),lm}_{AB} + V_{lm}T^{(b),lm}_{AB}), \tag{2.30}$$

where the conventions we use for the second-rank tensor spherical harmonics are given in an Appendix of [19]. Because the tensor C_{AB} is real, and because the tensor spherical harmonics satisfy the relationships

$$T^{(e),l-m}_{AB} = (-1)^m \bar{T}^{(e),lm}_{AB}, \qquad T^{(b),l-m}_{AB} = (-1)^m \bar{T}^{(b),lm}_{AB} \tag{2.31}$$

(where the overline denotes complex conjugation), the coefficients of this expansion in spherical harmonics obey the related properties

$$U_{l-m} = (-1)^m \bar{U}_{lm}, \qquad V_{l-m} = (-1)^m \bar{V}_{lm}. \tag{2.32}$$

These tensor spherical harmonics are also related to spin-weighted spherical harmonics, a complex vector

$$\vec{m} = \frac{1}{\sqrt{2}}(\vec{\partial}_\theta + i\csc\theta\vec{\partial}_\phi), \tag{2.33}$$

and its complex conjugate. The GW flux in Eq. (2.24) (which is a product of the shear, the news tensor, and the derivatives of both quantities) can be expressed as a product of vector and second- and third-rank tensor spherical harmonics (see [19] for more detail). For simplicity, we will assume that the stress-energy tensor of the matter fields

[6] If we take the limits $u_1 \to -\infty$ and $u_2 \to +\infty$, then we must make additional assumptions about the rate at which the supermomentum approaches a constant in the limits $u \to \pm\infty$ to ensure that the change in CM angular momentum is finite. For example, if we assume the leading-order time dependence goes as $P_{(D_A Y^A)} \sim P_0(1 + |u/u_0|^{-n})$ as $u \to \pm\infty$ (u_0 is a reference time), then it is clear that we would need to require $n > 1$. A detailed study of these types of asymptotics is beyond the scope of this work, and it will not be necessary for spacetimes that radiate for a finite interval of retarded time, $u \in [u_1, u_2]$.

vanishes, although this could be included trivially, because the flux is linear in the material stress-energy tensor.

To compute the $l = 1$ moments of the flux, we will integrate minus[7] the flux in Eq. (2.24) against vector fields of the form $Y_A = D_A \bar{Y}_{1,m}$, where $Y_{l,m}(\theta, \phi)$ are scalar spherical harmonics with the Condon-Shortley phase convention. It is then possible to express the multipole moments of the flux in terms of integrals of products of three spin-weighted spherical harmonics (with the conventions for the harmonics given in [19]). Before evaluating these integrals, it will again be useful to perform integration by parts on the set of terms in Eq. (2.24) that are the divergence of a scalar quantity. Once this is done, the flux splits naturally into two types of terms,

$$\frac{dK^{(\mathrm{GW})}_{1,m}}{du} = \frac{dk^{(\mathrm{GW})}_{1,m}}{du} - u \frac{dP^{(\mathrm{GW})}_{1,m}}{du}, \tag{2.34}$$

as in Eq. (2.25). Note that the apparent factor of -2 difference between the second terms on the right-hand sides of Eqs. (2.25) and (2.34) comes from a difference in convention for the supermomentum associated with a scalar function $D_A Y^A$ and the convention commonly used for the $l = 1$ moments of the flux of linear momentum. The multipolar expansion of the first term on the right-hand side of Eq. (2.34) has not been computed before (as far as we are aware). The second term is the same as the flux of linear momentum multiplied by minus the retarded time u. The multipolar expansion of the linear-momentum flux has been computed before (e.g., in [30]).

The integrals of products of three spin-weighted spherical harmonics that arise in the flux of the CM angular momentum are relatively simple functions of l and m. It will be helpful to define a few coefficients, so as to express the multipolar expansion for the CM angular momentum flux produced by GWs more concisely. These coefficients are

$$a_l = \sqrt{\frac{(l-1)(l+3)}{(2l+1)(2l+3)}}, \tag{2.35a}$$

$$b^{(\pm)}_{lm} = \sqrt{(l \pm m + 1)(l \pm m + 2)}, \tag{2.35b}$$

$$c_{lm} = \sqrt{(l - m + 1)(l + m + 1)}, \tag{2.35c}$$

$$d^{(\pm)}_{lm} = \sqrt{(l \pm m + 1)(l \mp m)}. \tag{2.35d}$$

After a lengthy calculation, it is possible to write the first term on the right-hand side of Eq. (2.34) as

$$\begin{aligned}\frac{dk^{(\mathrm{GW})}_{1,0}}{du} = -\frac{1}{64\pi}\sqrt{\frac{3}{\pi}}\sum_{l,m} a_l c_{lm} [\bar{U}_{lm}\dot{U}_{(l+1)m} \\ - \bar{U}_{(l+1)m}\dot{U}_{lm} + \bar{V}_{lm}\dot{V}_{(l+1)m} \\ - \bar{V}_{(l+1)m}\dot{V}_{lm}],\end{aligned} \tag{2.36a}$$

$$\begin{aligned}\frac{dk^{(\mathrm{GW})}_{1,\pm1}}{du} = -\frac{1}{64\pi}\sqrt{\frac{3}{2\pi}}\sum_{l,m} a_l [b^{(\pm)}_{lm}(\bar{U}_{lm}\dot{U}_{(l+1)m\pm1} \\ + \bar{V}_{lm}\dot{V}_{(l+1)m\pm1}) - b^{(\mp)}_{lm}(\bar{U}_{(l+1)m\mp1}\dot{U}_{lm} \\ + \bar{V}_{(l+1)m\mp1}\dot{V}_{lm})],\end{aligned} \tag{2.36b}$$

and the second term as

$$\begin{aligned}\frac{dP^{(\mathrm{GW})}_{1,0}}{du} = \frac{1}{32\pi}\sqrt{\frac{3}{\pi}}\sum_{l,m}\frac{1}{l+1}\Big[a_l c_{lm}(\dot{\bar{U}}_{lm}\dot{U}_{(l+1)m} \\ + \dot{\bar{V}}_{lm}\dot{V}_{(l+1)m}) - \frac{2im}{l}\dot{\bar{U}}_{lm}\dot{V}_{lm}\Big],\end{aligned} \tag{2.37a}$$

$$\begin{aligned}\frac{dP^{(\mathrm{GW})}_{1,\pm1}}{du} = \frac{1}{32\pi}\sqrt{\frac{3}{2\pi}}\sum_{l,m}\frac{1}{l+1}\Big[a_l b^{(\pm)}_{lm}(\dot{\bar{U}}_{lm}\dot{U}_{(l+1)m\pm1} \\ + \dot{\bar{V}}_{lm}\dot{V}_{(l+1)m\pm1}) \pm \frac{2i}{l} d^{(\pm)}_{lm}\dot{\bar{U}}_{lm}\dot{V}_{l(m\pm1)}\Big].\end{aligned} \tag{2.37b}$$

All the sums in Eqs. (2.36) and (2.37) run over $l \geq 2$, and $-l \leq m \leq l$.

The translation subgroup of the BMS group is four-dimensional, and it can be treated as a manifold with a flat Minkowski metric (see, e.g., [37]). We can then express the $l = 1$ moments of the flux of CM angular momentum in terms of vectors on this flat Minkowski manifold. Here, we will give the components in a set of Cartesian-type coordinates, (x, y, z), which we define from the spherical-polar coordinates (θ, ϕ) commonly used with the spherical harmonics employed in this paper. A method to transform from the $l = 1$ moments to the Cartesian components is described in [16], which we now summarize.

First, define the unit vector n^i and its gradient with respect to the derivative operator D_A via

$$n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \qquad e^i_A = D_A n^i. \tag{2.38}$$

The 1-form $D_A \bar{Y}_{1,m}$ can then be expressed in terms of a linear combination of the Cartesian components of e^i_A as

[7]Because GWs carry away energy from an isolated system with no incoming radiation, the flux is always negative. Thus, it has become a common convention (e.g., [30]) to define the energy carried away by GWs as a positive number, with it being implicit that this positive change in the energy causes the Bondi mass of the system to decrease. Similar sign conventions are used for the linear momentum and intrinsic part of the angular momentum. We also follow this convention with the flux of CM angular momentum, but we add the superscript "(GW)" to this flux to make this convention explicit.

$$D_A \bar{Y}_{1,m} = \omega_{0i}^{1,m} e_A^i, \tag{2.39}$$

where the coefficients $\omega_{0i}^{1,m}$ are given by

$$\omega_{0x}^{1,0} = 0 = \omega_{0y}^{1,0}, \qquad \omega_{0z}^{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}} \tag{2.40a}$$

$$\omega_{0x}^{1,\pm 1} = \mp \frac{1}{2}\sqrt{\frac{3}{2\pi}}, \qquad \omega_{0y}^{1,\pm 1} = \frac{i}{2}\sqrt{\frac{3}{2\pi}}, \qquad \omega_{0z}^{1,\pm 1} = 0. \tag{2.40b}$$

Then, from the fact that the flux of CM angular momentum satisfies

$$\frac{dK_{1,m}^{(\mathrm{GW})}}{du} = \omega_{0i} \frac{dK^i_{(\mathrm{GW})}}{du}, \tag{2.41}$$

we find that the Cartesian components are

$$\frac{dK^x_{(\mathrm{GW})}}{du} = -\sqrt{\frac{2\pi}{3}}\left[\left(\frac{dk_{1,1}^{(\mathrm{GW})}}{du} - \frac{dk_{1,-1}^{(\mathrm{GW})}}{du}\right) - u\left(\frac{dP_{1,1}^{(\mathrm{GW})}}{du} - \frac{dP_{1,-1}^{(\mathrm{GW})}}{du}\right)\right], \tag{2.42a}$$

$$\frac{dK^y_{(\mathrm{GW})}}{du} = -i\sqrt{\frac{2\pi}{3}}\left[\left(\frac{dk_{1,1}^{(\mathrm{GW})}}{du} + \frac{dk_{1,-1}^{(\mathrm{GW})}}{du}\right) - u\left(\frac{dP_{1,1}^{(\mathrm{GW})}}{du} + \frac{dP_{1,-1}^{(\mathrm{GW})}}{du}\right)\right], \tag{2.42b}$$

$$\frac{dK^z_{(\mathrm{GW})}}{du} = 2\sqrt{\frac{\pi}{3}}\left(\frac{dk_{1,0}^{(\mathrm{GW})}}{du} - u\frac{dP_{1,0}^{(\mathrm{GW})}}{du}\right). \tag{2.42c}$$

As was explained in more detail in the previous subsection, the flux in Eq. (2.42) represents the change in the CM part of the angular momentum, for which the origin of the retarded time coordinate is chosen to be $u = 0$. It contains nontrivial information about the flux of CM angular momentum from the system that is not contained in the fluxes of the other BMS charges.

III. CENTER-OF-MASS GRAVITATIONAL-WAVE MEMORY EFFECT

In this section, after giving an argument for why the CM memory effect should exist, we define the effect, describe some of its basic properties, and derive an expansion for the CM memory effect in terms of multipole moments of the GW strain.

A. Rationale for the existence of the CM memory effect

Consider, for simplicity, an asymptotically flat spacetime undergoing a stationary-to-stationary transition as it radiates GWs for a finite time. In each stationary region, there is a canonical reference frame in which the Bondi mass aspect is constant, the shear vanishes, and the Bondi angular-momentum aspect is a linear combination of $l = 1$ magnetic-parity vector spherical harmonics (though the values of the mass and angular-momentum aspects will generally be different in the canonical frames of the two stationary regions). The two canonical frames typically will not be the same, but there will be a BMS transformation (a Lorentz transformation and supertranslation) that relates the two. The supertranslation between the two canonical frames is equivalent to the GW memory (e.g., [16]), and the Lorentz transformation is related to the change in the 4-momentum between and the relative rotation of a set of fiducial observers in each of the two stationary regions.

Next, we integrate Eq. (2.24) with respect to u to relate the change in the charges to the net flux between the cuts:

$$\begin{aligned}\Delta K_{\zeta_Y} = -\frac{1}{64\pi}\int_{u_1}^{u_2} du \int d^2\Omega Y^A [& u D_A (2 D_B D_C N^{BC} \\ & - N_{BC}N^{BC} - 32\pi \hat{T}_{uu}) + C^{BC} D_B N_{AC} \\ & - N^{BC} D_B C_{AC} + 3 N_{AB} D_C C^{BC} \\ & - 3 C_{AB} D_C C^{BC} + 64\pi \hat{T}_{uA} + 16\pi \partial_u \hat{T}_{rA}].\end{aligned} \tag{3.1}$$

The left-hand side of Eq. (3.1), the change in the charges, depends on just the values of the 4-momentum and angular momentum in the canonical frames in the stationary regions and the BMS transformation that contains information about the net rotation, boost, and supertranslation between the two canonical frames. We argued in Sec. II, however, that the net change in the (super) CM angular momentum, as computed using the right-hand side of Eq. (3.1), contains additional information besides the change in the super-momentum, angular momentum, and GW memory. Thus, there appears to be an inconsistency. It could be resolved, if there is a cancellation between certain terms in the flux, for example.

Such a cancellation occurs with the GW memory, which we will now review. First, recall that the potential $\Delta\Phi$ that determines the memory [see Eq. (2.16)] can be found by integrating the conservation-type equation for the Bondi mass aspect (2.5):

$$\mathcal{D}\Delta\Phi = \mathcal{P}\left[8\Delta m + \int_{u_1}^{u_2} du (32\pi \hat{T}_{uu} + N_{AB}N^{AB})\right] \tag{3.2}$$

(see, e.g., [16]). We have defined a differential operator

$$\mathcal{D} \equiv D^2(D^2 + 2) \tag{3.3}$$

and a projector $\mathcal{P}$, which removes the $l = 0$ and $l = 1$ spherical harmonics from the right-hand side of Eq. (3.2). The projector is needed to invert the operator $\mathcal{D}$ and solve for $\Delta\Phi$, because the $l = 0$ and $l = 1$ harmonics are in its kernel of the operator $\mathcal{D}$. In the terminology of [27], the first term on the right-hand side in Eq. (3.2) is the ordinary part of the GW memory, and the remaining terms in the integral are collectively the null part of the memory. We will, therefore, express $\Delta\Phi$ as a sum of two parts

$$\Delta\Phi = \Delta\Phi_{(\mathrm{o})} + \Delta\Phi_{(\mathrm{n})}, \tag{3.4}$$

which correspond to the parts of the solution to Eq. (3.2) for the ordinary and null parts, respectively (and which is possible because the equation is linear in $\Delta\Phi$).

Now, let us return to the cancellation that occurs for the GW memory effect. The ordinary part of the memory (the change in the supermomentum charges, up to a normalization factor) depends on just the net change in the rest mass of the system and the relative boost of the observers that determine the canonical frames. The null memory, however, can be arbitrary. Thus, the integral of $D_A D_B N^{AB}$ with respect to u must be nonzero, so that Einstein's equations (and, equivalently, charge conservation) are satisfied. For the spin memory, the values of the change in the superspin charges are also restricted in a stationary-to-stationary transition, but the null part of the spin memory is not limited in this way. The additional term in the flux, Eq. (2.12), therefore, is necessary to ensure that charge conservation holds.

Finally, let us revisit the "inconsistency" discussed below Eq. (3.1) about the change in the charges in light of the discussion above. The related inconsistencies for supermomentum and superspin charge conservation were resolved by the GW memory and spin memory effects, respectively. Thus, it seems natural to suggest a similar resolution for super-CM charge conservation: namely, that there must be a CM memory effect. Because the GW memory and spin memory effects come about from terms in the fluxes that are linear in the Bondi news and shear tensors, respectively, we expect that the CM memory will arise from a similar type of term in the flux of CM angular momentum. The term linear in the Bondi news tensor in Eq. (3.1) (proportional to $u D_A D_B D_C N^{BC}$), therefore, is the most obvious term that could give rise to the CM memory. As we discuss in the next subsection, it turns out to be the u integral of a quantity related to this term that will be the CM memory effect.

B. Definition and properties of the CM memory effect

Let us then define a quantity

$$\Delta\mathcal{C}_{(D_A Y^A)} \equiv -\int_{u_1}^{u_2} du \int d^2\Omega\, u\left(D_B D_C N^{BC} - \frac{1}{2}\mathcal{D}\dot{\Phi}_{(\mathrm{n})}\right) \times (D_A Y^A), \tag{3.5}$$

which should be interpreted as a part of u times the u integral of a quantity proportional to a portion of the Bondi news tensor, with the part of the news tensor responsible for the null GW memory removed [this latter part of the news tensor is denoted by the potential $\dot{\Phi}_{(\mathrm{n})}$].[8] This quantity has the units of the time integral of the GW strain (like the spin memory effect), and it will be our definition of the CM memory effect. We now investigate some of its properties.

Integrating Eq. (3.5) by parts with respect to u, we find that

$$\begin{aligned}\Delta\mathcal{C}_{(D_A Y^A)} = &\int_{u_1}^{u_2} du \int d^2\Omega\left(D_B D_C C^{BC} - \frac{1}{2}\mathcal{D}\Phi_{(\mathrm{n})}\right)\\ &\times (D_A Y^A) - u\int d^2\Omega (D_A Y^A)\\ &\times \left(D_B D_C C^{BC} - \frac{1}{2}\mathcal{D}\Phi_{(\mathrm{n})}\right)\Big|_{u_1}^{u_2}.\end{aligned} \tag{3.6}$$

Thus, we see that $\Delta\mathcal{C}_{(D_A Y^A)}$ contains information about the time integral of C_{AB}, but it removes the part that grows linearly with u, which arises when there is ordinary GW memory.[9] It is, therefore, the part of the time integral of the electric-parity part of C_{AB} that becomes constant in a stationary-to-stationary transition.

Next, we will consider how the CM memory effect behaves in a set of cuts that are supertranslated from the cuts u used to compute the effect above. Under a supertranslation, α, the news tensor transforms as

$$\delta N_{AB} = \alpha \dot{N}_{AB}, \tag{3.7}$$

to linear order in α. Using this relationship, integration by parts, and the facts that $u' = u + \alpha$ and the news tensor and $\hat{T}_{uu}$ vanish in a nonradiative region, it is then straightforward to show from Eq. (3.5) that

$$\begin{aligned}\Delta\mathcal{C}'_{(D_A Y^A)} = &-\int_{u'_1}^{u'_2} du' \int d^2\Omega\, u'\Big(D^B D^C N'_{BC}\\ &-\frac{1}{2}\mathcal{D}\dot{\Phi}'_{(\mathrm{n})}\Big)(D_A Y^A) = \Delta\mathcal{C}_{(D_A Y^A)}.\end{aligned} \tag{3.8}$$

In the equation above, we computed $\Delta\mathcal{C}'_{(D_A Y^A)}$ with respect to the generators adapted to cuts of constant u'.

[8] Using Einstein's equations (2.5), we could have written this as a term proportional to u times the u derivative of the Bondi mass aspect. We deliberately avoided writing it in this form, so as to reinforce the notion that this is an observable with the units of the time integral of the GW strain, rather than a quantity with the units of the time integral of the supermomentum.

[9] A similar caveat to that elaborated in footnote 6 holds: namely, for finite values of u_1 and u_2, we do not need to suppose that the ordinary GW memory approaches a constant at a given rate; however, in the limit that $u_1 \to -\infty$ and $u_2 \to +\infty$, we would need to assume similar fall-off rates to those given for the supermomentum in footnote 6.

Thus, $\Delta\mathcal{C}_{(D_A Y^A)}$ is invariant under infinitesimal supertranslations, for stationary-to-stationary transitions.

When computing memory effects within the Bondi framework, it can be useful to define a scalar potential as the memory observable (see, e.g., [16]). We now define this quantity. The shear tensor, C_{AB}, can be expressed in terms of two potentials that encompass its two degrees of freedom as follows:

$$C_{AB} = \frac{1}{2}(2D_A D_B - h_{AB}D^2)\Phi + \epsilon_{C(A}D_{B)}D^C\Psi. \tag{3.9}$$

Using this decomposition and integrating by parts with respect to u, we find that Eq. (3.5) can be written as

$$\Delta\mathcal{C}_{(D_A Y^A)} = \frac{1}{2}\int d^2\Omega(D_A Y^A)\mathcal{D}\left[\int_{u_1}^{u_2} du(\Phi - \Phi_{(\mathrm{n})}) - u(\Phi - \Phi_{(\mathrm{n})})|_{u_1}^{u_2}\right]. \tag{3.10}$$

The CM memory observable that we define is

$$\Delta\mathcal{K} \equiv \int_{u_1}^{u_2} du(\Phi - \Phi_{(\mathrm{n})}) - u(\Phi - \Phi_{(\mathrm{n})})|_{u_1}^{u_2}, \tag{3.11}$$

which is a potential for the time integral of the electric-parity part of the shear with the part that grows linearly with u from the ordinary part of the GW memory removed.[10] The quantity $\Delta\mathcal{C}_{(D_A Y^A)}$ can be expressed in terms of $\Delta\mathcal{K}$ by

$$\Delta\mathcal{C}_{(D_A Y^A)} = \frac{1}{2}\int d^2\Omega(D_A Y^A)\mathcal{D}\Delta\mathcal{K}. \tag{3.12}$$

Using Eq. (3.1), we can also solve for $\Delta\mathcal{C}_{(D_A Y^A)}$ from the change in the super-CM angular momentum and the quadrupole and higher multipole moments of the flux of CM angular momentum carried by GWs and matter fields:

$$\begin{aligned}\Delta\mathcal{C}_{(D_A Y^A)} = {} & -32\pi\mathcal{P}\Delta K_{\vec{\zeta}_Y} - \frac{1}{2}\mathcal{P}\int_{u_1}^{u_2} du \int d^2\Omega Y^A \\ & \times (C^{BC}D_B N_{AC} - N^{BC}D_B C_{AC} \\ & + 3N_{AB}D_C C^{BC} - 3C_{AB}D_C C^{BC} \\ & + 64\pi\hat{T}_{uA} + 16\pi\partial_u \hat{T}_{rA}).\end{aligned} \tag{3.13}$$

As with the GW memory and spin memory, the CM memory has two parts: the first given by $\Delta K_{\vec{\zeta}_Y}$ is the ordinary part, whereas the portion involving the retarded-time integral is the null part. Because the CM memory is invariant under infinitesimal supertranslations, but the changes in the CM part of the super angular momentum transform in the way given in Eq. (2.22), then the ordinary and null parts of the CM memory must transform in opposite ways.

To more easily compute the amplitude of the CM memory effect produced by astrophysical sources, we expand Eq. (3.13) in a set of multipole moments of the GW strain in the next subsection.

C. Multipolar expansion of the CM memory effect

We find it simplest to compute the multipole moments of the CM memory effect by integrating the right-hand side of Eq. (3.13) with respect to the smooth vector fields. Specifically, we use the electric-parity vector spherical harmonics, $D_A\bar{Y}_{lm}/\sqrt{l(l+1)}$ (analogously to what was done in the calculation of the spin memory in [19]). These functions are a useful basis for smooth vector fields, like those used by Campiglia and Laddha [14,15]. Because the integral of a meromorphic super-rotation vector field [12] with a smooth vector field is finite (see, e.g., [16]), then it could also represent the part of the super-rotation symmetry that has overlap with these vector spherical harmonics (although this decomposition may not be unique [47]).

The method for calculating the multipole moments of the CM memory observable, which we will denote by $\Delta\mathcal{K}_{lm}$, is very similar to the procedure to compute similar moments of the spin memory described in [19] (as well as that described in Sec. II for computing the multipole moments of the CM angular momentum flux). The basic strategy of the calculation is to change the tensorial expression for the multipole moments of the shear and its derivatives into a sum of products of three spin-weighted spherical harmonics. The conventions for the vector, tensor, and spin-weighted harmonics are given in detail in [19]. Using these conventions, we define a set of coefficients

$$\mathcal{B}_l(s', l', m'; s'', l'', m'') \equiv \int d^2\Omega({}_{s'}Y_{l'm'})({}_{s''}Y_{l''m''})({}_{s'+s''}\bar{Y}_{l(m'+m'')}), \tag{3.14}$$

[10] Given the somewhat complicated nature of the CM memory observable, the reader might be concerned about whether this quantity is measurable by freely falling observers, in principle. Because the GW strain, the GW memory, and their time integrals can be measured by freely falling observers, the basic ingredients needed to construct the CM memory observable are measurable. The CM memory effect corresponds to the electric-parity part of the time-integrated GW strain, and this part could be separated from the magnetic-parity part by having many observers surrounding an isolated source measuring the GW strain. Thus, the one remaining potential subtlety relates to extracting just the null part of the memory. This could be performed by directly measuring the flux of GWs and massless fields with appropriate detectors or by determining the time dependence of the ordinary part of the memory (i.e., the flux of the supermomentum charges) by measuring components of the asymptotic Riemann tensor with a generalization of the procedure described in [52,53], for example. Thus, we see no obstacle for observing the CM memory, in principle, but a more detailed analysis of its measurability would be beneficial.

as in [19].[11] We have restricted to integrals in which the complex-conjugated spin-weighted spherical harmonic has spin weight $s = s' + s''$ and has azimuthal number $m = m' + m''$, because the integrals are zero for all other values of s and m. Furthermore, the only values of l for which the integral is nonvanishing are those with $l \in \{\max(|l' - l''|, |m' + m''|, |s' + s''|), \ldots, l' + l'' - 1, l' + l''\}$. The reason for these "selection rules" comes from the fact that the coefficients $\mathcal{B}_l(s', l', m'; s'', l'', m'')$ can be expressed in terms of products of Clebsch-Gordan coefficients $\langle l', m'; l'', m''|l, m' + m''\rangle$ via the relationship

$$\mathcal{B}_l(s', l', m'; s'', l'', m'') = (-1)^{l+l'+l''}\sqrt{\frac{(2l'+1)(2l''+1)}{4\pi(2l+1)}} \times \langle l', s'; l'', s''|l, s' + s''\rangle\langle l', m'; l'', m''|l, m' + m''\rangle. \tag{3.15}$$

These coefficients, therefore, satisfy similar identities to those of the Clebsch-Gordan coefficients when the signs of the spin weight or the azimuthal numbers are changed (see, e.g., [19]).

Next, we specialize to vacuum spacetimes, and we compute the multipole moments $\Delta\mathcal{K}_{lm}$ of the CM memory produced by GWs. Nonvacuum cases can be treated by simply adding the appropriate multipole moments of the relevant components of the stress-energy tensor given in Eq. (3.13). To make the expression more compact, we write the result as

$$\Delta\mathcal{K}_{lm} = \frac{(l-2)!}{(l+2)!}\frac{1}{\sqrt{l(l+1)}}\mathcal{P}\left(\int_{u_1}^{u_2} du \frac{dk_{lm}^{(\mathrm{CM})}}{du} + 64\pi\Delta K_{lm}\right). \tag{3.16}$$

The first term in the integral $dk_{lm}^{(\mathrm{CM})}/du$ comes from the higher multipole moments of the quantity that gives rise to the term $dk_{1m}^{(\mathrm{GW})}/du$ in the flux of CM angular momentum (though with a different overall normalization). The second term, ΔK_{lm}, is a spherical harmonic moment of the change in the super-CM charges $\Delta K_{\vec{\zeta}_Y}$. Before giving the explicit form of the term $dk_{lm}^{(\mathrm{CM})}/du$, we make a few additional definitions of coefficients so as to write the result more compactly:

$$s_{l';l''}^{l,(\pm)} = 1 \pm (-1)^{l+l'+l''}, \tag{3.17a}$$

$$c_{l',m';l'',m''}^{l} = 3\sqrt{(l'-1)(l'+2)}\mathcal{B}_l(-1, l', m'; 2, l'', m'') + \sqrt{(l''-2)(l''+3)}\mathcal{B}_l(-2, l', m'; 3, l'', m''). \tag{3.17b}$$

After a lengthy calculation, it is possible to show that

$$\frac{dk_{lm}^{(\mathrm{CM})}}{du} = \frac{1}{4}\sum_{l',l'',m',m''} c_{l',m';l'',m''}^{l}[s_{l';l''}^{l,(+)}(U_{l'm'}\dot{U}_{l''m''} - \dot{U}_{l'm'}U_{l''m''} + V_{l'm'}\dot{V}_{l''m''} - \dot{V}_{l'm'}V_{l''m''}) + is_{l';l''}^{l,(-)}(U_{l'm'}\dot{V}_{l''m''} + \dot{V}_{l'm'}U_{l''m''} - \dot{U}_{l'm'}V_{l''m''} - V_{l'm'}\dot{U}_{l''m''})]. \tag{3.18}$$

The sum runs over l', $l'' \geq 2$, and for $-l' \leq m' \leq l'$ and $-l'' \leq m'' \leq l''$. For an arbitrary source, an infinite number of products of multipoles will be needed to compute the CM memory effect. For compact binaries in the PN approximation, the number of multipole moments that contribute at leading order is a small number, as we discuss next.

IV. FLUX OF CM ANGULAR MOMENTUM AND CM MEMORY IN THE PN APPROXIMATION

In this part, we introduce a few essential elements of the PN formalism for compact binaries that we need for the calculations in this section. Our summary is based on the much more comprehensive review [42]. We then present the main results of this section: expressions for the leading-PN-order flux of CM angular momentum and CM memory effect for nonspinning, quasicircular compact binaries. We also comment on the terms in the gravitational waveform responsible for producing the CM memory effect and on the prospects for detecting these features in the waveform with future GW detectors.

A. Summary of selected results from PN theory

In PN theory, the gravitational waveform is typically described by a transverse-traceless tensor h_{ij}^{TT}. It can be expanded in second-rank electric- and magnetic-parity tensor spherical harmonics as

$$h_{ij}^{\mathrm{TT}} = \frac{1}{r}\sum_{l,m}(U_{lm}T_{ij}^{(e),lm} + V_{lm}T_{ij}^{(b),lm}), \tag{4.1}$$

where the sum runs over $l \geq 2$ and $-l \leq m \leq l$. It was argued in [19] that the coefficients U_{lm} and V_{lm} that appear in both Eqs. (2.30) and (4.1) are the same in linearized theory (though care would need to be taken to properly include any nondynamical terms in h_{ij}^{TT}, as noted by [48]). It is often convenient to work with the complex GW strain $h = h_+ - ih_\times$, which is related to the tensorial strains by

[11]The coefficients $\mathcal{B}_l(s', l', m'; s'', l'', m'')$ are identical to the coefficients denoted $\mathcal{C}_l(s', l', m'; s'', l'', m'')$ in [19]; however, we have renamed them here, so as to avoid confusion with the quantity $\Delta\mathcal{C}_{(D_A Y^A)}$ that is related to the CM memory effect.

$$h = r^{-1} C_{AB} \bar{m}^A \bar{m}^B = h^{\mathrm{TT}}_{ij} e^i_A e^j_B \bar{m}^A \bar{m}^B, \tag{4.2}$$

where m^A is defined in Eq. (2.33) and e^i_A is given in Eq. (2.38). When h is expanded in spin-weighted spherical harmonics, a short calculation shows that

$$h = \sum_{l,m} h_{lm} ({}_{-2}Y_{lm}), \qquad h_{lm} = \frac{1}{r\sqrt{2}} (U_{lm} - iV_{lm}). \tag{4.3}$$

The convention used here for h differs from that in [42] by an overall minus sign, but the multipole moments U_{lm} and V_{lm} as well as the tensorial GW strain agree (as they must).

Through a matching procedure, summarized in the review [42], it is possible to relate the radiative moments U_{lm} and V_{lm} to source multipole moments I_{lm} and J_{lm} that, as their name suggests, describe the multipole moments of the source in the near zone. To simplify the matching, we will choose our coordinates such that the orbital angular momentum of our nonspinning, compact-binary source points in the z coordinate direction. The matching procedure takes place through a third set of intermediate "canonical" multipole moments,[12] M_{lm} and S_{lm}, as well as a set of multipole moments that parametrize a coordinate transformation between two solutions of the linearized Einstein's equations. For multipoles with $m \neq 0$, the relationship between the radiative and canonical moments is given by

$$U_{lm} = M^{(l)}_{lm} + O(c^{-3}), \qquad V_{lm} = S^{(l)}_{lm} + O(c^{-3}), \tag{4.4}$$

where here—and everywhere else hereafter—the remainder means there are *relative* PN corrections (where PN corrections conventionally scale as the power of c to the minus one-half), and where the superscript (l) means to take l derivatives with respect to u. These corrections consist of terms that get called "tails" (including higher PN-order generalizations, such as "tails of tails"), "instantaneous" nonlinear terms, and "hereditary" (or "memory") terms. For understanding the terms in the GWs that give rise to the CM memory effect, the instantaneous, nonlinear terms will play the most important role. The canonical moments are related to the source moments by

$$M_{lm} = I_{lm} + O(c^{-5}), \qquad S_{lm} = J_{lm} + O(c^{-5}). \tag{4.5}$$

The 2.5PN remainder here means that there are additional nonlinear terms entering at this order in the PN expansion that are not captured by the PN expansion of the source multipoles I_{lm} and J_{lm} to that PN order.

[12]While the term "canonical" is used to describe both these moments and a specific Bondi frame associated with a stationary region in asymptotically flat spacetimes, this repeated usage is just an unfortunate repetition of the term "canonical"; there is no obvious connection between the two concepts.

For computing the leading-order flux of CM angular momentum and the CM memory effect, it turns out that we will be able to use the leading Newtonian expressions for the radiative multipole moments in terms of the source moments (for $m \neq 0$),

$$U_{lm} = I^{(l)}_{lm} + O(c^{-3}), \qquad V_{lm} = J^{(l)}_{lm} + O(c^{-3}). \tag{4.6}$$

The $U_{2,0}$ mode below comes from the GW memory, which does not satisfy Eq. (4.6), even though it is a leading, Newtonian-order effect in the waveform. We will also need to use one higher-PN-order calculation for the flux of linear momentum carried by GWs. For comparing the parts of the GWs responsible for the CM memory with the expressions for the multipole moments of the waveform in PN theory, however, we will need to be aware of the higher-order PN corrections to the radiative multipole moments. Given that the corrections have distinct mathematical forms (tail, instantaneous, and hereditary terms), we will be able to identify the relevant nonlinear terms to make this comparison analytically. Identifying these terms observationally in the GWs from compact-binary mergers will be much more challenging.

For nonspinning compact binary sources in quasicircular orbits, the radiative multipole moments can be expressed conveniently in terms of just a few parameters, most of which involve the masses of the two bodies, $m_{(A)}$ and $m_{(B)}$: the total mass $M = m_{(A)} + m_{(B)}$, the mass difference $\delta m = m_{(A)} - m_{(B)}$, the symmetric mass ratio $\eta = m_{(A)} m_{(B)}/M^2$, the orbital frequency ω, the PN parameter $x = (M\omega)^{2/3}$, and the orbital phase φ (see, e.g., [42]). In terms of these quantities, the radiative moments that we will need for our calculations are

$$U_{2,2} = -8\sqrt{\frac{2\pi}{5}} M\eta x e^{-i2\varphi} + O(c^{-2}), \tag{4.7a}$$

$$U_{2,0} = \frac{4}{7}\sqrt{\frac{5\pi}{3}} M\eta x + O(c^{-2}), \tag{4.7b}$$

$$U_{3,1} = -\frac{2i}{3}\sqrt{\frac{\pi}{35}} \delta m \eta x^{3/2} e^{-i\varphi} + O(c^{-2}), \tag{4.7c}$$

$$U_{3,3} = 6i\sqrt{\frac{3\pi}{7}} \delta m \eta x^{3/2} e^{-i3\varphi} + O(c^{-2}), \tag{4.7d}$$

$$V_{2,1} = \frac{8}{3}\sqrt{\frac{2\pi}{5}} \delta m \eta x^{3/2} e^{-i\varphi} + O(c^{-2}), \tag{4.7e}$$

where the orbital phase is given by

$$\varphi = -\frac{x^{-5/2}}{32\eta} + O(c^{-2}). \tag{4.8}$$

The modes with negative azimuthal number can be obtained by using the relationships given in Eq. (2.32). The u derivatives of these multipole moments can be expressed in terms of x by using the chain rule and the fact that

$$\dot{x} = \frac{64\eta}{5M} x^5 + O(c^{-2}). \tag{4.9}$$

The results are as follows:

$$\dot{U}_{2,2} = 16i\sqrt{\frac{2\pi}{5}}\eta x^{5/2} e^{-i2\varphi} + O(c^{-2}), \tag{4.10a}$$

$$\dot{U}_{2,0} = \frac{256}{7}\sqrt{\frac{\pi}{15}}\eta^2 x^5 + O(c^{-2}), \tag{4.10b}$$

$$\dot{U}_{3,1} = -\frac{2}{3}\sqrt{\frac{\pi}{35}}\frac{\delta m}{M}\eta x^3 e^{-i\varphi} + O(c^{-2}), \tag{4.10c}$$

$$\dot{U}_{3,3} = 18\sqrt{\frac{3\pi}{7}}\frac{\delta m}{M}\eta x^3 e^{-i3\varphi} + O(c^{-2}), \tag{4.10d}$$

$$\dot{V}_{2,1} = -\frac{8i}{3}\sqrt{\frac{2\pi}{5}}\frac{\delta m}{M}\eta x^3 e^{-i\varphi} + O(c^{-2}). \tag{4.10e}$$

Because the quantity $\dot{U}_{2,0}$ is several PN orders higher than the other derivatives, it will not appear in most of the calculations below.

B. Flux of CM angular momentum

In this part, we give the leading-PN order expression for the flux of the CM part of the angular momentum. We begin by computing the term $dk_{1,1}^{(\mathrm{GW})}/du$ in Eq. (2.34). It is given by

$$\frac{dk_{1,1}^{(\mathrm{GW})}}{du} = -\frac{1}{64\pi}\sqrt{\frac{3}{7\pi}}[\sqrt{15}(U_{2,-2}\dot{U}_{3,3} + U_{3,3}\dot{U}_{2,-2}) + (U_{2,2}\dot{U}_{3,-1} + U_{3,-1}\dot{U}_{2,2}) + \sqrt{6}U_{2,0}\dot{U}_{3,1}] + O(c^{-2}). \tag{4.11}$$

Substituting the relevant components in Eqs. (4.7) and (4.10) into Eq. (4.11), we find that it can be written as a function of x as follows:

$$\frac{dk_{1,1}^{(\mathrm{GW})}}{du} = \frac{627}{980}\sqrt{\frac{3}{2\pi}}\delta m\eta^2 x^4 e^{-i\varphi} + O(c^{-2}). \tag{4.12}$$

The $l = 1$, $m = 0$ term vanishes for nonspinning, quasicircular compact binaries. This can be shown using arguments based on parity, like those given in [54].

The second term on the right-hand side of Eq. (2.34) requires computing the flux of linear momentum. This has been computed before (it can be inferred from [55], for example) and is given by

$$\frac{dP_{1,1}^{(\mathrm{GW})}}{du} = \frac{1}{96\pi}\sqrt{\frac{3}{7\pi}}(\sqrt{15}\dot{U}_{2,-2}\dot{U}_{3,3} + \dot{U}_{2,2}\dot{U}_{3,-1} + i\sqrt{14}\dot{U}_{2,2}\dot{V}_{2,-1}) + O(c^{-2}), \tag{4.13}$$

at leading PN order. Inserting the appropriate values of the radiative moments given in Eq. (4.10) into Eq. (4.13), we find

$$\frac{dP_{1,1}^{(\mathrm{GW})}}{du} = -i\frac{232}{105}\sqrt{\frac{3}{2\pi}}\frac{\delta m}{M}\eta^2 x^{11/2} e^{-i\varphi} + O(c^{-2}), \tag{4.14}$$

a result that traces back to [56]. The $l = 1$, $m = 0$ mode of the flux of linear momentum also vanishes, which follows from the arguments based on parity in [54].

To compute the net change in the CM angular momentum, we must evaluate

$$\Delta K_{1,1}^{(\mathrm{GW})} = \int du\left(\frac{dk_{1,1}^{(\mathrm{GW})}}{du} - u\frac{dP_{1,1}^{(\mathrm{GW})}}{du}\right). \tag{4.15}$$

Because at leading order, the retarded time u goes as

$$(u_c - u) = \frac{5M}{256\eta}x^{-4} + O(c^{-2}) \tag{4.16}$$

(where u_c is the retarded time of coalescence of the binary in PN theory), then by comparing powers of x, we see that the first term on the right-hand side of Eq. (4.15) is 2.5 PN orders higher than the second term is. While this might make the reader wonder why we do not neglect this term and focus just on the second term, we now revisit some of the discussion around Eqs. (2.28) and (2.29) in the context of nonspinning PN compact binaries.

In PN theory, the BMS supertranslations are fixed by the fiducial Minkowski spacetime that is the background about which the PN expansion is computed. This leaves the Poincaré group as the remaining symmetries. There is a relatively natural way to fix the boost transformations (by moving to the rest frame of the source in the initial stationary region, for example). The rotations in the Lorentz group can be specified by aligning the orbital angular momentum to fall along the z axis and the separation to be along the x axis (at some fiducial time) in the initial stationary region. Finally, one way to constrain the spatial translations is to require that the CM of the system coincide with the origin of the coordinates initially. This will make the CM part of the angular momentum equal to zero in this region. A translation in time will not affect the values of the 4-momentum, supermomentum, and (super) angular momentum in the initial stationary region in this frame. Thus, there is no obvious prescription for using the (extended) BMS charges in a stationary region to constrain this remaining degree of freedom in the BMS group. However, the flux of (super) CM angular momentum is not invariant under such transformations in a stationary-to-stationary transition, as was highlighted in Eq. (2.28). To compute this flux, therefore, it is necessary to specify a reference time u_0 about which it is computed. We will use the prescription defined in Eq. (2.29) that minimizes the flux of the CM angular momentum in our computation below. This then fixes the previously unconstrained time-translation freedom in the BMS group.

As was shown in [57], through 2PN order, the flux of linear momentum is parallel to the orbital velocity of the reduced mass of the system (and thus the change in the linear momentum is directed radially outward). In terms of the multipole moment $dP_{1,1}^{(\mathrm{GW})}/du$, this is related to the fact that the coefficient multiplying $e^{-i\varphi}$ is a strictly imaginary quantity (i.e., has vanishing real part) through 2PN order. With the one real degree of freedom in u_0, we can choose this reference time to make the flux of the CM part of the angular momentum arising from the second term in Eq. (4.15) vanish through 2PN order.

The first term on the right-hand side of Eq. (4.15), however, leads to a change in the CM angular momentum that is $\pi/2$ out of phase with that from the second term at 2PN order [i.e., the coefficient multiplying $e^{-i\varphi}$ for $dk_{1,1}^{(\mathrm{GW})}/du$ is real]. In addition, the 2.5PN corrections to $dP_{1,1}^{(\mathrm{GW})}/du$ have terms that are in phase with $dk_{1,1}^{(\mathrm{GW})}/du$. It is possible to continue canceling the imaginary part of the coefficient of $e^{-i\varphi}$ of the second term in Eq. (4.15) through 2.5PN order by appropriately choosing the reference time u_0; however, it is not possible also to cancel the real part of this coefficient in this manner. This implies that to compute the leading-PN-order expression for the flux of the CM angular momentum, we need the leading-order expression for $dk_{1,1}^{(\mathrm{GW})}/du$ in Eq. (4.12), the leading-order expression for the time to coalescence in Eq. (4.16), and a 2.5PN order correction to the leading expression for $dP_{1,1}^{(\mathrm{GW})}/du$ in Eq. (4.14) [specifically, the part that is in phase with $dk_{1,1}^{(\mathrm{GW})}/du$]. Thus, with this choice of reference time, the two terms on the right-hand side of Eq. (4.15) contribute at the same PN order.

The relevant 2.5PN corrections to the linear momentum flux have been computed in [58] for nonspinning, quasicircular compact binaries. We express their result in terms of the $l=1$, $m=1$ moment of the flux by using the fact that

$$\frac{dP_{1,1}}{du} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\left(\frac{dP_x}{du} - i\frac{dP_y}{du}\right), \tag{4.17}$$

which can be obtained by inverting a relation like the one given in Eq. (2.42). It then follows from the results of [58] that

$$\frac{dP_{1,1}^{(2.5\mathrm{PN})}}{du} = ix^{5/2}\left(\frac{dP_{1,1}^{(\mathrm{GW})}}{du}\right)(p_{(0)} + p_{(1)}\eta), \tag{4.18}$$

where we have defined the coefficients

$$p_{(0)} = -\frac{106187}{50460} + \frac{32835}{841}\log 2 - \frac{77625}{3364}\log 3, \tag{4.19a}$$

$$p_{(1)} = \frac{10126}{4205} - \frac{109740}{841}\log 2 + \frac{66645}{841}\log 3. \tag{4.19b}$$

We can then integrate the flux with respect to u. To evaluate the integral, we change variables to write it as an integral with respect to x by using Eq. (4.9). We find that the result can be expressed in terms of the fluxes in Eqs. (4.12) and (4.18) as

$$\Delta K_{1,1}^{(\mathrm{GW})} = iMx^{-3/2}\left(\frac{dk_{1,1}^{(\mathrm{GW})}}{du} + \frac{5M}{256\eta}x^{-4}\frac{dP_{1,1}^{(2.5\mathrm{PN})}}{du}\right) + O(c^{-2}). \tag{4.20}$$

The quantity $\Delta K_{1,1}^{(\mathrm{GW})}$ scales with the PN parameter x as $x^{5/2}e^{-i\varphi}$. We have not seen an expression for the change in the CM part of the angular momentum before, although it is may be related to a part of the time-dependent mass-dipole moment computed in [59], for example.

It is relatively straightforward to understand the physics underlying Eq. (4.20). For nonspinning compact binaries in quasicircular orbits, the orbital velocity is tangent to the circular orbit up to 2PN order. At 2.5PN order, however, radiation reaction causes the system to inspiral, thereby producing a small radial velocity. It is not possible to remove the effects of this radial velocity on the CM angular momentum while preserving the properties of the canonical frame associated with the initial stationary region. This implies that there is a change in the CM part of the angular momentum, given by Eq. (4.20).

To conclude this subsection, we briefly discuss radiation reaction and balance equations in PN theory (in the sense of [60]). The fluxes of energy and intrinsic angular momentum cause the corresponding conserved Poincaré charges in the near zone to change at 2.5PN order, and the flux of linear momentum also causes such a change in the corresponding charge, though at higher (3.5PN) order. In general, the flux of CM angular momentum also produces a change in the near-zone CM angular momentum that begins at 3.5PN order. For nonspinning, quasicircular compact binary sources, however, we showed that through an appropriate choice of reference time, the flux of CM angular momentum begins at 2.5PN orders higher than the leading effect (which, therefore, corresponds to a 6PN-order effect in the near zone). Because the conserved quantities for nonspinning compact binaries in PN theory are currently computed to 4PN order [61], this flux leads to a change in the CM angular momentum that is two PN orders higher than the accuracy of the CM angular momentum computed in [61]. Thus, we do not anticipate that the flux of CM angular momentum will have a significant impact on computations of the dynamics of nonspinning, quasicircular compact binaries in the PN approximation.

C. Center-of-mass GW memory effect

In the two parts of this section, we compute first the null and then the ordinary parts of the CM memory effect for nonspinning, quasicircular compact binaries in the PN approximation.

1. Nonlinear and null part of the CM memory

We compute the nonlinear part of the null CM memory from the multipolar expressions given in Eqs. (3.16) and (3.18) and the relevant definitions of the coefficients that appear in the latter equation. There are five (independent) nonzero spherical-harmonic modes of this null nonlinear CM memory at leading PN order, which are given by

$$\Delta\mathcal{K}_{3,1} = \frac{1}{2880\sqrt{\pi}}\int_{u_1}^{u_2} du[\sqrt{10}(U_{3,3}\dot{U}_{2,-2} - \dot{U}_{3,3}U_{2,-2}) + 2\sqrt{6}(U_{3,-1}\dot{U}_{2,2} - \dot{U}_{3,-1}U_{2,2}) + 3U_{2,0}\dot{U}_{3,1}] + O(c^{-2}), \tag{4.21a}$$

$$\Delta\mathcal{K}_{3,3} = \frac{1}{2880\sqrt{\pi}}\int_{u_1}^{u_2} du[\sqrt{10}(U_{3,1}\dot{U}_{2,2} - \dot{U}_{3,1}U_{2,2}) - 5U_{2,0}\dot{U}_{3,3}] + O(c^{-2}), \tag{4.21b}$$

for the $l = 3$ modes and

$$\Delta\mathcal{K}_{5,1} = \frac{1}{50400\sqrt{77\pi}}\int_{u_1}^{u_2} du[(U_{3,3}\dot{U}_{2,-2} - \dot{U}_{3,3}U_{2,-2}) + \sqrt{15}(U_{3,-1}\dot{U}_{2,2} - \dot{U}_{3,-1}U_{2,2}) - 3\sqrt{10}U_{2,0}\dot{U}_{3,1}] + O(c^{-2}), \tag{4.21c}$$

$$\Delta\mathcal{K}_{5,3} = \frac{1}{50400\sqrt{11\pi}}\int_{u_1}^{u_2} du[\sqrt{10}(U_{3,1}\dot{U}_{2,2} - \dot{U}_{3,1}U_{2,2}) - 2U_{2,0}\dot{U}_{3,3}] + O(c^{-2}), \tag{4.21d}$$

$$\Delta\mathcal{K}_{5,5} = \frac{1}{1680\sqrt{330\pi}}\int_{u_1}^{u_2} du(U_{3,3}\dot{U}_{2,2} - \dot{U}_{3,3}U_{2,2}) + O(c^{-2}), \tag{4.21e}$$

for the $l = 5$ modes. We can then substitute the expressions for the multipole moments in Eqs. (4.7) and (4.10) to find that

$$\Delta\mathcal{K}_{3,1} = i\frac{6463}{12600}\sqrt{\frac{\pi}{21}}M\delta m\eta^2 x^{5/2}e^{-i\varphi}|_{x_1}^{x_2} + O(c^{-2}), \tag{4.22a}$$

$$\Delta\mathcal{K}_{3,3} = -i\frac{647}{22680}\sqrt{\frac{\pi}{35}}M\delta m\eta^2 x^{5/2}e^{-3i\varphi}|_{x_1}^{x_2} + O(c^{-2}), \tag{4.22b}$$

for the $l = 3$ modes and

$$\Delta\mathcal{K}_{5,1} = i\frac{677}{154350}\sqrt{\frac{\pi}{330}}M\delta m\eta^2 x^{5/2}e^{-i\varphi}|_{x_1}^{x_2} + O(c^{-2}), \tag{4.22c}$$

$$\Delta\mathcal{K}_{5,3} = -i\frac{11}{198450}\sqrt{\frac{11\pi}{35}}M\delta m\eta^2 x^{5/2}e^{-3i\varphi}|_{x_1}^{x_2} + O(c^{-2}), \tag{4.22d}$$

$$\Delta\mathcal{K}_{5,5} = i\frac{1}{875}\sqrt{\frac{\pi}{77}}M\delta m\eta^2 x^{5/2}e^{-5i\varphi}|_{x_1}^{x_2} + O(c^{-2}), \tag{4.22e}$$

for the $l = 5$ modes. We have used the notation x_2 and x_1 to denote the values of the PN parameter at retarded times u_2 and u_1, respectively.[13] Note that unlike the leading-PN part of the GW memory or the spin memory effects, the leading nonlinear, null part of the CM memory effect appears in the $m \neq 0$ modes of the multipolar expansion of the effect (specifically modes with odd m and l). While there are higher-order PN corrections to the GW and the spin memory effects that appear in the modes with nonzero m, it is a distinctive feature of the CM memory that the leading-order nonlinear, null CM memory effect appears in modes with nonzero m. However, it is also not too surprising, because the flux of CM angular momentum for nonspinning, quasicircular compact binaries has no $m = 0$ mode (only $m = \pm 1$ modes).

2. Ordinary part of the CM memory

A second interesting difference between the GW memory and spin memory effects and the CM memory

[13]Because x_1 and x_2 are related to the orbital frequency of the binary at times u_1 and u_2, respectively, it is clear that the spacetime is not stationary at either time (which breaks one of the assumptions we made in deriving the CM memory effect). Thus, the results presented in Eq. (4.22) should be taken as suggestive of how the CM memory effect would grow with x, in the PN context (namely, that it grows in amplitude like $x^{5/2}$, like the change in the CM angular momentum does). The full effect will depend on the details of the merger of the compact binary and would need to be computed by numerical relativity simulations of merging compact objects.

effect is the role of the ordinary part of the memory. In the PN approximation for nonspinning, quasicircular compact binaries, the nonlinear null GW memory appears at leading Newtonian order in the waveform, whereas the ordinary part of the memory is typically ignored, because it will appear at a PN order that is much higher than that at which the PN-expanded gravitational waveform has been computed. For the spin memory, the ordinary part of the memory is again of a very high PN order.

Let us now consider the ordinary part of the CM memory effect. It was shown in [16] that the change in the super-CM charges is nonzero when there is GW memory. To linear order in the GW memory, this change is given by

$$\Delta K_{lm} = -\frac{3M}{16\pi}\sqrt{l(l+1)}\Delta\Phi_{lm}, \tag{4.23}$$

where $\Delta\Phi_{lm}$ are the moments of the scalar function $\Delta\Phi$ in Eq. (2.16) with respect to scalar spherical harmonics (recall that ΔK_{lm} was computed with respect to electric-parity vector spherical harmonics). The leading GW memory appears in the $m = 0$ modes with $l = 2$ and $l = 4$, and the values of the potential $\Delta\Phi_{lm}$ are given, e.g., in [19]. Combining the results of [19] with the expressions in Eqs. (3.16) and (4.23), we can then compute the leading-PN-order prediction for the ordinary part of the CM memory. The result is

$$\Delta\mathcal{K}_{2,0} = -\frac{M}{168}\sqrt{\frac{5}{\pi}}\int_{u_1}^{u_2} du|\dot{U}_{2,2}|^2 + O(c^{-2}), \tag{4.24a}$$

$$\Delta\mathcal{K}_{4,0} = -\frac{M}{453600\sqrt{\pi}}\int_{u_1}^{u_2} du|\dot{U}_{2,2}|^2 + O(c^{-2}). \tag{4.24b}$$

We can then use Eqs. (4.9) and (4.10) to show that in terms of the PN parameter x, Eq. (4.24) can be expressed as

$$\Delta\mathcal{K}_{2,0} = -\frac{\sqrt{5\pi}}{21}M^2\eta(x_2 - x_1) + O(c^{-2}), \tag{4.25a}$$

$$\Delta\mathcal{K}_{4,0} = -\frac{\sqrt{\pi}}{56700}M^2\eta(x_2 - x_1) + O(c^{-2}). \tag{4.25b}$$

Thus, the nonlinear null part of the CM memory enters at 1.5 PN orders higher than the ordinary part of the CM memory. Moreover, the ordinary part of the CM memory is nonoscillatory ($m = 0$) at leading order, whereas the null part is oscillatory ($m \neq 0$).

The reader might then wonder why we compute the nonlinear null part of the CM memory, when it is weaker than the ordinary part, for quasicircular, nonspinning compact binaries. We do so because, for the CM memory, it will be useful to understand which terms in the gravitational waveform are responsible for generating the effect. For the spin memory, there is an easily identifiable term in the GW strain that produces the effect, when it is integrated in time. It is also possible, in principle, to measure the terms in the GWs that produce the spin memory effect with the next generation of ground-based interferometers [19] (and likely space-based interferometers, too). To see if the terms in the GWs responsible for the CM memory effect might also be measured, we must first identify the pertinent terms. The nonlinear, null part of the CM memory turns out to be the leading PN-order effect in the GW strain, as we discuss in more detail in the next subsection.

D. GW modes that produce the CM memory effect

Because the CM memory observable $\Delta\mathcal{K}$ is a potential for a portion of the time integral of the electric-parity part of the GW strain (with the terms that grow linearly with u in nonradiative regions removed), then there must be terms in the GWs that, when integrated in time, give rise to the CM memory effect. Because the moments $\Delta\mathcal{K}_{lm}$ are the spherical harmonic modes of the potential $\Delta\mathcal{K}$ expanded in scalar harmonics, whereas the radiative multipoles U_{lm} correspond to an expansion of the GW strain in second-rank, symmetric-trace-free tensor harmonics, there is the following relationship between these quantities:

$$U_{lm}^{(\mathrm{CM})} = \frac{1}{\sqrt{2}}\sqrt{\frac{(l+2)!}{(l-2)!}}\dot{\mathcal{K}}_{lm}. \tag{4.26}$$

We used the notation $U_{lm}^{(\mathrm{CM})}$ to denote just the part of U_{lm} that is related to the CM memory (U_{lm} will generally have other contributions) and $\dot{\mathcal{K}}_{lm}$ to denote the quantity that when integrated in time gives rise to $\Delta\mathcal{K}_{lm}$.

1. Nonlinear and null part of the CM memory

We find that the nonlinear, null part of the CM memory is a consequence of terms in the gravitational waveform of the form

$$\begin{aligned} U_{3,1}^{(\mathrm{CM})} = {} & \frac{1}{96\sqrt{30\pi}}[2\sqrt{5}(U_{3,3}\dot{U}_{2,-2} - \dot{U}_{3,3}U_{2,-2}) \\ & + 4\sqrt{3}(U_{3,-1}\dot{U}_{2,2} - \dot{U}_{3,-1}U_{2,2}) + 3\sqrt{2}U_{2,0}\dot{U}_{3,1}] \\ & + O(c^{-2}), \end{aligned} \tag{4.27a}$$

$$\begin{aligned} U_{3,3}^{(\mathrm{CM})} = {} & \frac{1}{96\sqrt{30\pi}}[2\sqrt{5}(U_{3,1}\dot{U}_{2,2} - \dot{U}_{3,1}U_{2,2}) \\ & - 5\sqrt{2}U_{2,0}\dot{U}_{3,3}] + O(c^{-2}), \end{aligned} \tag{4.27b}$$

for the $l = 3$ modes and

$$U_{5,1}^{(\mathrm{CM})} = \frac{1}{1680\sqrt{165\pi}}[(U_{3,3}\dot{U}_{2,-2} - \dot{U}_{3,3}U_{2,-2}) + \sqrt{15}(U_{3,-1}\dot{U}_{2,2} - \dot{U}_{3,-1}U_{2,2}) - 3\sqrt{10}U_{2,0}\dot{U}_{3,1}] + O(c^{-2}), \tag{4.27c}$$

$$U_{5,3}^{(\mathrm{CM})} = \frac{1}{240\sqrt{1155\pi}}[\sqrt{10}(U_{3,1}\dot{U}_{2,2} - \dot{U}_{3,1}U_{2,2}) - 2U_{2,0}\dot{U}_{3,3}] + O(c^{-2}), \tag{4.27d}$$

$$U_{5,5}^{(\mathrm{CM})} = \frac{1}{120\sqrt{154\pi}}(U_{3,3}\dot{U}_{2,2} - \dot{U}_{3,3}U_{2,2}) + O(c^{-2}), \tag{4.27e}$$

for the $l = 5$ modes.

Instead of directly substituting the expressions for the multipole moments given in Eqs. (4.7) and (4.10) into Eq. IV D 1 to compute the analog of Eq. (4.27) for the quantities $U_{lm}^{(\mathrm{CM})}$, we note that for the $m \neq 0$ modes, there is the simple relationship

$$\dot{\mathcal{K}}_{lm} = -i\frac{m}{M}x^{3/2}\Delta\mathcal{K}_{lm} \tag{4.28}$$

at the PN order at which we are calculating. By combining Eqs. (4.22), (4.26), and (4.28), we can easily determine the results for $U_{lm}^{(\mathrm{CM})}$ in terms of x. It follows that all the moments scale as $x^4 e^{-im\varphi}$, which means that they are 3PN contributions to the gravitational waveform (for the $l = 3$ modes they are relative 2.5PN-order corrections, and for the $l = 5$ modes, they are relative 1.5PN-order corrections). Because the 3PN waveform from compact binaries has been computed to this order [62], it is possible to compare the expressions for $U_{lm}^{(\mathrm{CM})}$ with the equivalent modes in the PN waveform. There are a few subtleties about making this comparison that we will discuss further after computing the terms in the GWs that produce the ordinary part of the CM memory effect.

2. Ordinary part of the CM memory

Using Eq. (4.26) to convert the expressions for $\Delta\mathcal{K}_{l,0}$ in Eq. (2.24) into expressions for $U_{l,0}^{(\mathrm{CM})}$, we find that

$$U_{2,0}^{(\mathrm{CM})} = -\frac{M}{84}\sqrt{\frac{15}{\pi}}|\dot{U}_{2,2}|^2 + O(c^{-2}), \tag{4.29a}$$

$$U_{4,0}^{(\mathrm{CM})} = -\frac{M}{75600}\sqrt{\frac{5}{\pi}}|\dot{U}_{2,2}|^2 + O(c^{-2}). \tag{4.29b}$$

For these $m = 0$ modes, they can be expressed in terms of x as

$$U_{2,0}^{(\mathrm{CM})} = -\frac{128}{7}\sqrt{\frac{\pi}{15}}M\eta^2 x^5 + O(c^{-2}), \tag{4.30a}$$

$$U_{4,0}^{(\mathrm{CM})} = -\frac{32}{4725}\sqrt{\frac{\pi}{5}}M\eta^2 x^5 + O(c^{-2}). \tag{4.30b}$$

Because the $U_{l,0}^{(\mathrm{CM})}$ modes scale as x^5, then they are a 4PN effect in the gravitational waveform. The gravitational waveform at 4PN order has not yet been computed, which prohibits us from making a comparison with existing PN results. However, we anticipate that future PN calculations will find evidence for such terms.

E. Comparison with existing PN results

Because the null part of the CM memory arises from a 3PN effect in the gravitational waveform, and because the PN waveform has been computed to this accuracy, it would be a useful consistency check of the CM memory effect to identify certain terms in the PN expansion of the gravitational waveform that are responsible for the CM memory effect. We find that we can make such an identification in a certain approximation, which we will describe in more detail.

Before we do so, however, we must clarify a few notational differences between the PN results given in, e.g., [42] and those in this paper. The expressions for the PN radiative (as well as canonical and source) multipole moments in [42] are expressed in terms of symmetric-trace-free, spatial, rank-l tensors $\mathcal{U}_L$ and $\mathcal{V}_L$ rather than the multipole moments U_{lm} and V_{lm} (which are scalar functions of u). There are well-known prescriptions for converting between the two types of moments, which are described in [30] (or more recently in [63], for example). The relationships for the radiative mass moments are given by

$$U_{lm} = \frac{16\pi}{(2l+1)!!}\sqrt{\frac{(l+1)(l+2)}{2l(l-1)}}\mathcal{U}_L\bar{\mathcal{Y}}^L_{lm}, \tag{4.31a}$$

$$\mathcal{U}_L = \frac{l!}{4}\sqrt{\frac{2l(l-1)}{(l+1)(l+2)}}\sum_m U^{lm}\mathcal{Y}^{lm}_L, \tag{4.31b}$$

where $\mathcal{Y}^{lm}_L$ are a set of basis functions for the rank-l, symmetric-trace-free tensors [30], and the double factorial means a product of all odd integers less than or equal to $(2l+1)$. Similar relationships exist for the current multipole moments $\mathcal{V}_L$ and V_{lm}, though we will not need them in the subsequent discussion.

Having addressed the differences in notation, we must identify the relevant terms in the PN waveform. Because the CM memory effect comes from the integral of the product of radiative moments, then the corresponding terms in the PN waveform must be able to be expressed as an instantaneous product of radiative moments (at the relevant

PN order). Thus, the other effects at 3PN order in the waveform (contributions from time derivatives of 3PN accurate near-zone multipole moments, from components of a gauge transformation needed to relate the near-zone moments to the intermediate canonical moments, and from tail and hereditary terms) will not be needed here. The instantaneous and nonlinear terms in the 3PN-accurate radiative moments, however, are expressed in terms of the canonical moments. We now reproduce the expressions for these parts of the $l = 3$ and $l = 5$ radiative moments, which can be found, e.g., in Eqs. (95a) and (95e) of [42]:

$$\begin{aligned}\mathcal{U}^{(\mathrm{IN})}_{ijk} = &-\frac{4}{3}M^{(3)}_{a\langle i}M^{(3)}_{jk\rangle a} - \frac{9}{4}M^{(4)}_{a\langle i}M^{(2)}_{jk\rangle a} + \frac{1}{4}M^{(2)}_{a\langle i}M^{(4)}_{jk\rangle a} \\ &- \frac{3}{4}M^{(5)}_{a\langle i}M^{(1)}_{jk\rangle a} + \frac{1}{4}M^{(1)}_{a\langle i}M^{(5)}_{jk\rangle a} + \frac{1}{12}M^{(6)}_{a\langle i}M_{jk\rangle a} \\ &+ \frac{1}{4}M_{a\langle i}M^{(6)}_{jk\rangle a},\end{aligned} \tag{4.32a}$$

$$\begin{aligned}\mathcal{U}^{(\mathrm{IN})}_{ijkpq} = &-\frac{710}{21}M^{(3)}_{\langle ij}M^{(3)}_{kpq\rangle} - \frac{265}{7}M^{(4)}_{\langle ij}M^{(2)}_{kpq\rangle} \\ &- \frac{120}{7}M^{(2)}_{\langle ij}M^{(4)}_{kpq\rangle} - \frac{155}{7}M^{(5)}_{\langle ij}M^{(1)}_{kpq\rangle} \\ &- \frac{41}{7}M^{(1)}_{\langle ij}M^{(5)}_{kpq\rangle} - \frac{34}{7}M^{(6)}_{\langle ij}M_{kpq\rangle} - \frac{14}{7}M_{\langle ij}M^{(6)}_{kpq\rangle}.\end{aligned} \tag{4.32b}$$

The superscript "(IN)" is short for "instantaneous and nonlinear," the repeated index a is being summed over in the first three lines, and the angled brackets mean to take the symmetric trace-free part of the tensor.

It is not immediately obvious how to relate the products of canonical moments that appear in Eq. (4.32) to the products of radiative moments that appear in the GW modes that produce the CM memory effect. The reason is that the canonical moments that appear in Eq. (4.32) have fewer than l derivatives with respect to time (where l is the multipole order of the different canonical moments that appear in the products of the moments). Thus, we cannot directly use the analog of the relationships in Eq. (4.4) for the rank-l symmetric-trace-free tensors to express the radiative moments in terms of the canonical moments. Instead, we would have to express these derivatives of the canonical moments in terms of integrals of the radiative moments by integrating an expression like Eq. (4.4). In performing this procedure, we would need to introduce new constants of integration, but we do not have a prescription for determining the values of these constants.

Because the CM memory effect involves a time integral of the GW strain, however, it is equally relevant to know whether the time integral of the PN expressions in Eq. (4.32) agree with the time integral of the modes in Eqs. (4.27) [after using the relationships in Eq. (4.31)]. In making this comparison, we can integrate by parts to obtain an equivalent expression that involves a different linear combination of derivatives of the canonical moments (and boundary terms from integrating by parts). If the boundary terms vanish, then the integrand is a new expression for the relevant parts of the radiative moments that give rise to the same CM memory effect.[14]

We will use this procedure of integrating in time, integrating by parts, and differentiating the expression to get a new PN expression for the instantaneous, nonlinear terms. In this procedure, we will integrate by parts so that we can write the result in terms of products of the radiative moments and their derivatives (but not their integrals). This will avoid issues with unknown constants of integration, which were mentioned above. The result of this process is that Eq. (4.32) can be written as

$$\mathcal{U}'^{(\mathrm{IN})}_{ijk} = \frac{1}{12}(\mathcal{U}_{a\langle i}\dot{\mathcal{U}}_{jk\rangle a} - \dot{\mathcal{U}}_{a\langle i}\mathcal{U}_{jk\rangle a}), \tag{4.33a}$$

$$\mathcal{U}'^{(\mathrm{IN})}_{ijkpq} = \frac{2}{21}(\mathcal{U}_{\langle ij}\dot{\mathcal{U}}_{kpq\rangle} - \dot{\mathcal{U}}_{\langle ij}\mathcal{U}_{kpq\rangle}). \tag{4.33b}$$

We have added an apostrophe to the modes $\mathcal{U}'^{(\mathrm{IN})}_L$ to indicate that they were obtained from the expressions for $\mathcal{U}^{(\mathrm{IN})}_L$ in Eq. (4.32) by integrating by parts and differentiating the resulting expression [as well as using the analog of Eq. (4.4) for the symmetric-trace-free tensors]. With the relationships in Eq. (4.31), we can then recover the $l = 3$ and $l = 5$ modes given in Eq. (4.27), namely,

$$U'^{(\mathrm{IN})}_{3m} = U^{(\mathrm{CM})}_{3m}, \qquad U'^{(\mathrm{IN})}_{5m} = U^{(\mathrm{CM})}_{5m}, \tag{4.34}$$

for odd integers m. Therefore, there are terms in the already computed 3PN waveform that give rise to the same CM memory effect, under the prescription described above for rewriting the instantaneous and nonlinear terms in the 3PN waveform.

F. Discussion of PN results

Because the CM memory effect arises from 3PN and 4PN terms in the GWs from a compact binary, it is of interest to determine whether these terms in the

[14]A subtle issue will be whether the boundary terms vanish. This clearly will not be true if we consider the binary as it evolves between two nonzero frequencies x_1 and x_2, but this will also break the assumption of a stationary-to-stationary transition (as was further discussed in footnote 13). Thus, we will consider the full evolution of the binary as it makes a stationary-to-stationary transition; this will eliminate the majority of the boundary terms. However, there are some additional boundary terms that will not vanish in stationary regions, which occur because of the GW memory effect. These terms are of a sufficiently high PN order that we will not need to treat them at the PN accuracy at which we perform the calculation. As a result, we are able to ignore boundary terms when integrating by parts when we make this comparison.

gravitational waveform could be detected by any current or upcoming GW observatories. The GW memory could be detected within the next decade after LIGO (as well as Virgo and KAGRA) detects hundreds of binary-black-hole mergers [64]. This is possible because the effect enters at leading (Newtonian) order in the waveform, and it has a distinctive dependence on time and on angular coordinates (it is nonoscillatory, and enters into the $m = 0$ and $l = 2, 4$ modes of the gravitational waveform for nonspinning, quasicircular compact binaries at leading order). The spin memory effect also has distinctive time and angular dependencies (it enters into the $l = 3$, $m = 0$ mode of the time-integrated gravitational waveform for nonspinning, quasicircular compact binaries at leading order); however, the related terms in the GWs are of 2.5PN order in the waveform. This means that it will likely be too weak to be detected by the current generation of ground-based detectors, but it could conceivably be observed by the next generation of ground-based GW detectors, like the Einstein Telescope [19].

The GW modes related to the CM memory effect seem much more difficult to detect. For the nonlinear part, the modes appear as a 3PN order term in the waveform. Specifically, for the $l = 3$ modes of nonspinning, quasicircular compact binaries, they are a 2.5PN-order correction to GW modes that vanish when the components of the binary have the same mass (and thus are themselves a correction to the leading quadrupole waveform). While the small amplitude of the effect will make detecting it challenging, there are two other properties of the PN waveform that seem to prohibit being able to identify the terms in the GWs that produce the nonlinear null part of the CM memory effect. First, although we showed that $U^{(\mathrm{IN})}_{lm}$ can be reexpressed as $U'^{(\mathrm{IN})}_{lm}$ [or equivalently $U^{(\mathrm{CM})}_{lm}$] for harmonics with $l = 3$, 5 and m odd in a stationary-to-stationary transition, outside of this context, $U^{(\mathrm{IN})}_{lm}$ and $U^{(\mathrm{CM})}_{lm}$ can be different. Second, at 3PN order, there are additional terms that arise from nonlinear interactions in the near zone of the compact binary that produce effects in the gravitational waveform that have (at least at this PN order) the same time dependence as those responsible for the CM memory (but they would have a different dependence on angular coordinates). The full gravitational waveform is a sum of these different contributions, and it is not clear how observationally to separate out the part related to the nonlinear null CM memory effect from these other similar effects from a given compact-binary source.

Next, we consider the ordinary part of the CM memory effect. For nonspinning, quasicircular compact binaries, it is a 4PN correction to the same GW multipole moments in which the GW memory appears. While it is of a high PN order, it has a different angular dependence (the ratio of the $l = 2$ and $l = 4$ modes differs from that of the GW memory). Perhaps more importantly, it also has a different time dependence than the GW memory does. It grows with time like the instantaneous flux of energy does, unlike the GW memory, which grows with time as the total radiated energy does.

We can roughly estimate whether these modes are detectable by computing the signal-to-noise ratio of the part of the GWs that produce the ordinary CM memory effect. For our source, we choose a binary, like the first GW150914 detection by LIGO [5], and for our detector, we use the Einstein Telescope (specifically the analytical fit for the ET-B noise curve given in [65]). An event like GW150914 will likely be one of the loudest events to be observed by the Einstein Telescope, because its signal-to-noise ratio could be in the thousands [19]. Following a procedure similar to that described in [19] to compute the signal-to-noise ratio, we find that the GW modes that produce the ordinary part of the CM memory effect have a signal-to-noise ratio that is several orders of magnitude less than unity. Thus, it is difficult to imagine that it will be detected by ground-based GW detectors from individual events. Attempting to stack multiple events to build evidence for the CM memory also seems difficult, because the amplitude of the effect in the GWs is significantly smaller than the background noise in the detector. The prospects for other detectors like the space-based LISA mission [66] or pulsar timing arrays (e.g., [67]) we expect will be similar.

V. CONCLUSIONS

In this paper, we investigated the flux of (super) angular momentum in asymptotically flat spacetimes. We showed that within the context of stationary-to-stationary transitions the change in the (super) angular momentum between two cuts is not invariant under supertranslations. The difference produced by a supertranslation is related to the change in supermomentum, the GW memory, and the supertranslation itself. Next, we focused on the flux of the center-of-mass part of the angular momentum. We argued that the change in the (super) CM angular momentum (although not invariant under supertranslations) contains additional information about an isolated system that is not contained in the change in the 4-momentum, intrinsic (super) angular momentum, supermomentum, GW memory, or spin memory. We then derived a new multipolar expression for the flux of CM angular momentum in terms of a set of radiative multipole moments of the GW strain.

The next part of the paper was devoted to defining the CM memory effect. The effect is related to the time integral of the electric-parity part of the GW strain, with the part that grows linearly with retarded time (from the ordinary GW memory) removed. The quantity we defined is invariant under infinitesimal supertranslations. We then derived an expression for the multipole moments of this CM memory effect in terms of the radiative multipoles of the GW strain and the multipole moments of the change in the super-CM angular momentum.

The final part of the paper was devoted to analyzing nonspinning, quasicircular compact binaries, which we treated in the post-Newtonian approximation. We showed that binaries with components with unequal masses will typically have a nonzero flux of CM angular momentum. The effect was quite weak (of a high PN order), because with the freedom to shift the reference time about which the flux is computed, it was possible to set the change in the CM angular momentum to be zero through 2.5PN order (which corresponds to a 6PN-order effect in the near-zone equations of motion).

Lastly, we computed the CM memory effect for these binaries, and we found that the ordinary part of the CM memory was a larger (lower PN-order) effect than the nonlinear null part of the memory. The opposite is true for the GW memory and the spin memory effects. The nonlinear, null part of the CM memory arises from a 3PN term in the GWs, which we could identify with a certain part of the 3PN gravitational waveform from nonspinning, quasicircular compact binaries. The ordinary part of the CM memory comes from a 4PN term in the GWs, which has not yet been computed in PN theory. The null part of the CM memory effect turned out to be degenerate with other nonlinear terms in the PN waveform, which made it seem difficult to identify and measure the effect with current or future GW detectors. The ordinary part of the CM memory is measurable in principle, but it was sufficiently weak that it seemed unlikely that any upcoming GW interferometers or a pulsar timing array would be able to observe the effect. Thus, we suspect that the results in this paper will be more pertinent for helping to understand the theoretical properties of the extended BMS charges than for highlighting observable GW effects related to the changes in these charges from compact binaries.

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