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APPLYING DIRECTOR THEORY TO THE MODELLING OF FLUID FLOW IN STRAIGHT AND CURVED PIPES

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SUMMARY

This work involves applying director theory to the modelling of fluid flow in straight and curved pipes. Director theory assumes that the velocity of the fluid can be approximated by a summation of weighting functions multiplied by vectors that are dependent on the coordinate following the centreline of the pipe. The aim is to investigate whether this could be a useful approach for modelling blood flow in the human cardiovascular system. The approach allows for a more realistic geometric model than classical 1D approaches, but is computationally cheaper than full 3D simulations.

Key words: *director theory, fluid, modelling, cardiovascular*

1 INTRODUCTION

This work is about applying director theory (also known as Cosserat theory) to the modelling of fluid flow in pipes. The motivation behind this project is to explore if director theory could help the modelling of blood flow in the human cardiovascular system become a viable clinical tool. While it is possible to reconstruct 3D computational models of individual patients using non-invasive medical imaging techniques, only a section of this can be used for 3D CFD simulations, otherwise the computational effort becomes too great. This can be coupled with 1D modelling for the arterial branching. The aim is to find a good balance between computational cost and accuracy. The director theory approach is computationally cheaper than full 3D modelling but retains more information about the geometry of the pipes than classical 1D models.

2 METHODOLOGY

The idea behind director theory is to assume that the velocity of the fluid flow can be approximated by a series expansion of shape functions, which are generally polynomials of the cross-sectional coordinates, multiplied by vectors (called director velocities) that depend only on the coordinate following the centreline of the pipe and on time, as shown by Eq. (1). Integrated versions of the conservation of momentum and 'director momentum' equations are imposed.

$$\mathbf{v}(\zeta^1,\zeta^2,\zeta^3) = \sum_{N=1}^{K} \lambda_N^i(\zeta^1,\zeta^2)\mathbf{w}_N^i(\zeta^3,t) \tag{1}$$

There are a number of advantages to the director theory approach. The system of equations is closed at each order, meaning that no assumptions need to be made about the form of the nonlinear and viscous terms. It is a hierarchical theory, so the accuracy can be improved, at the cost of the simplicity of the equations. The theory allows for the description of curvature and torsion, as well as non-circular cross-sections. This results in a more accurate solution of the flow field as compared to classical 1D models which are effectively straight rods. Preliminary discussion and comparison of director theory to classical 1D models are outlined in Robertson and Sequeira [1].

We began following the work of Caulk and Naghdi [2], taking a director approach to modelling fluid flow in a straight cylindrical pipe, where the radius of the pipe could vary along the coaxial direction. Satisfying boundary, incompressibility and integrated (over the cross-section) versions of the momentum equations, leads to a system of six partial differential equations dependent on the coaxial coordinate and time. These equations can be simplified for particular geometries or conditions, which in some cases leaves a system which can be solved analytically.

Then we began applying the theory to curved pipes, starting with a toroidally curved pipe, initially following the work of Green et al [3], later diverging from their approach in how the appropriate equations of motions are derived. We set up a local coordinate system that follows the curve of the pipe. The shape functions are assumed to be of polynomial form. To determine their exact form, the velocity expansion, for a chosen order, is substituted into incompressibility and boundary condition equations. The shape functions for $N = 1$ to $N = 6$ are shown in Fig. 2. As can be seen, cross-sectional flow does not enter until the sixth order. Once the shape functions have been determined, the director velocities are determined through the satisfaction of the integrated momentum equations. We employ Powell's hybrid method to find solutions to these equations.

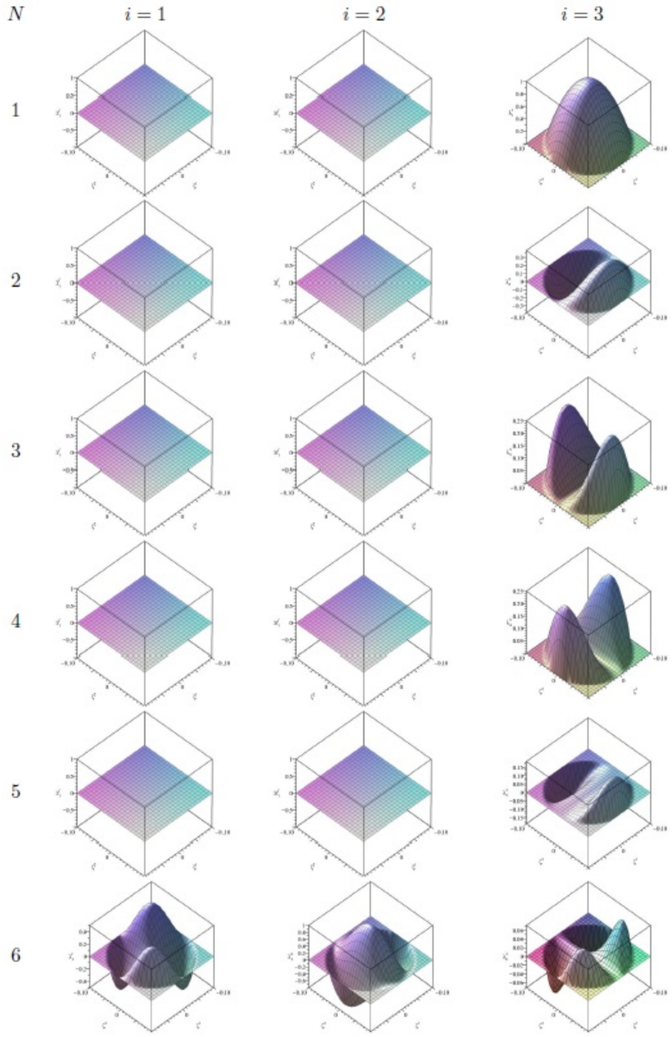

Figure 1: The shape functions for order $N = 1$ to $N = 6$ in each of the ζ^i directions.

3 RESULTS AND CONCLUSIONS

In the most simple case of a straight cylindrical pipe of constant cross-section, following the work of Caulk and Naghdi, we were able to recover the Poiseuille solution through the director theory approach. We were also able to recover a decaying swirling solution in the straight pipe as seen in Fig. 2. This shows the advantage of the director theory approach over classical 1D modelling, where it is not possible to capture flow in the cross-section.

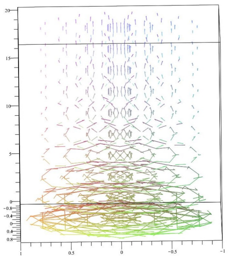

Figure 2: Decaying swirling solution in a straight pipe.

In the toroidally curved pipe, the flow field from our solutions, which can be seen in Fig. 3, matches the flow field from a 3D simulation created using STAR-CCM+. This is with a curvature ratio of 0.01 and a Reynolds number of approximately 2.

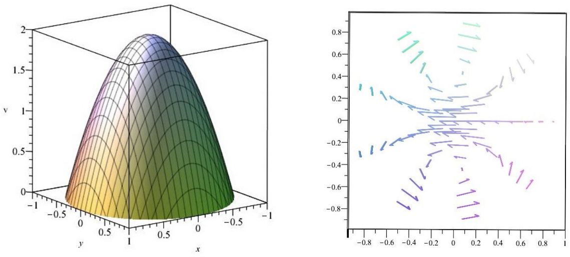

Figure 3: Solution for a toroidally curved pipe of curvature ratio 0.01.

Figures 4 - 6 show the contours of the velocity in the ζ^3, ζ^1 and ζ^2 directions respectively in a cross-section of the toroidal pipe. The left images are from a 3D simulation created using STAR-CCM+. The right images are the solutions obtained through the director theory approach.

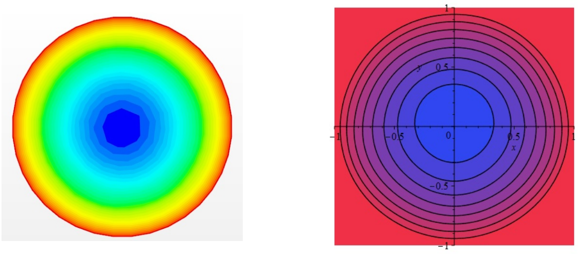

Figure 4: Coaxial velocity contours in the cross-section of the pipe. Left: STAR-CCM+ simulation. Right: director theory approach.

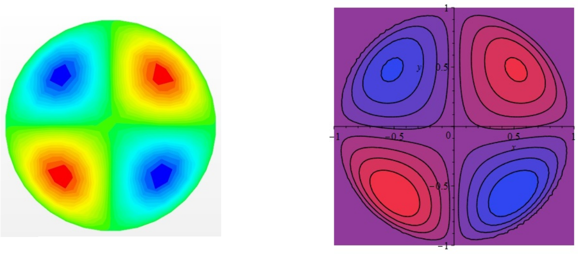

Figure 5: Vertical velocity contours in the cross-section of the pipe. Left: STAR-CCM+ simulation. Right: director theory approach.

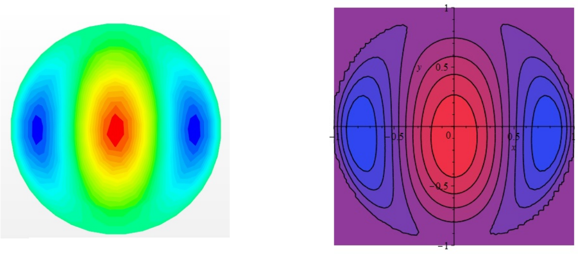

Figure 6: Radial velocity contours in the cross-section of the pipe. Left: STAR-CCM+ simulation. Right: director theory approach.

This provides a promising start to applying this approach to curved pipes. We can now begin to look at increasing the complexity of the curvature and including torsion into this theory, with the hope of building up to realistic geometries for blood vessels.

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