There is no invariant, four-dimensional stuff

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Many people say that Einstein's special theory of relativity (STR) favors a four-dimensional ontology. Some philosophers have expressed this view via the following precise thesis:

> Four-dimensional stuff is invariant, whereas three-dimensional stuff is not.

(Here I am using "stuff" as a vague term including both physical objects and facts.) For example, Yuri Balashov claims that "an object viewed as a 4d being is relativistically invariant in a sense in which its 3d parts are not" (1999, p 659).[1] Similarly, Thomas Sattig says that "there is a permanent shape standing behind the different three-dimensional shapes of the object, namely, an invariant four-dimensional shape, rendering the various three-dimensional shapes different perspectival representations of the single invariant shape" (2015, p 220). Finally, Thomas Hofweber and Marc Lange argue against Kit Fine's fragmentalist interpretation of STR on the basis that "the spacetime interval, as a frame-invariant fact, is the reality, whereas the facts related by the coordinate transformations are frame-dependent facts and hence are appearances of that reality" (2017, p 876).

In this note, I show that these claims are false. First I show the precise sense in which there are no invariant four-dimensional objects. Then I argue that there are no frame-invariant facts.

A four-dimensional object is represented by a region of Minkowski spacetime. Typically one thinks of a "spacetime worm" that represents the region swept out over time by a spatially extended object. The following result shows that no non-trivial subset of Minkowski spacetime is Lorentz invariant, *a fortiori* spacetime worms are not Lorentz invariant.

[1] Balashov's claim was contested by Davidson (2013), who argues that 4d objects themselves fail to be relativistically invariant. However, Balashov (2014) and Calosi (2015) argue that Davidson's conclusion and the reasoning behind it are in error. I show here that Davidson's conclusion is correct.

Proposition. *Let O be a region in Minkowski spacetime M that is invariant under all Lorentz transformations. Then O is either M or the empty set.*

This result follows from the fact that the Lorentz group acts transitively on Minkowski spacetime. In particular:

Lemma. *For any two points $p, q \in M$, there are Lorentz transformations L_1 and L_2 such that $L_1 p = L_2 q$.*

Sketch of proof. Suppose first that p and q are spacelike related. In this case, we let L_2 be the identity. Let $\overline{pq}$ be the spacelike line segment connecting p and q, and let m be its midpoint. Now let ℓ be a (timelike) line passing through m and orthogonal to $\overline{pq}$. If one follows ℓ backwards in time, eventually there is a point $s \in \ell$ that is timelike related to both p and q; and there is a Lorentz boost L_1 based at s such that $L_1 p = q$.

Now for the general case, for any two points $p, q \in M$, there is a point $r \in M$ that is spacelike related to both p and q. By the previous argument, there are Lorentz transformations L_1 and L_2 such that $L_1 p = r$ and $L_2 q = r$ □

Thus, there is nothing in Minkowski spacetime that could be called a physical object — whether three or four dimensional — and that is relativistically invariant.

Let's turn our attention to the facts. Hofweber and Lange claim that, "the spacetime interval . . . [is] a frame-invariant fact." This is an interesting way to use the English language — i.e. to apply the predicate "is a fact" to a noun, and what's more, to a noun that does not occur in everyday usage. What is this "spacetime interval" of which Hofweber and Lange speak? To be honest, I am not sure, but I will try my hand at a couple of disambiguations.

First, I will try to interpret "spacetime interval" so that "the spacetime interval is relativistically invariant" comes out true. When I look around, the closest thing I can find is the Minkowski distance function itself. In other words, we could interpret "spacetime interval" to be a function from pairs of spacetime points to real numbers. However, even on the most charitable interpretation, a function does not represent a fact. (Recalling what we learned in elementary logic: to turn a function f into a fact, one needs either some names or some variables and quantifiers.) So in the case that "spacetime interval" is a function, it is not true that "the spacetime interval is a fact"; a fortiori it is not true that "the spacetime interval is a frame-invariant fact."

Second, I will try to interpret "spacetime interval" so that "the spacetime interval is a fact" comes out true. When I look around, the most plausible facts are of the following form:

(D) The Minkowski distance between points p and q is λ.

But is there any sense in which D is relativistically invariant? I am open to being convinced that it is, but so far, I have been unable to understand what it could mean to say that D is relativistically invariant.

Let's first cash out "D is relativistically invariant" in a straightforward mathematical way: "D is invariant under Lorentz transformations." In that case, D is *not* relativistically invariant: a Lorentz transformation L maps the points p and q to points Lp and Lq, and hence D transforms to a completely different fact. (The two facts are not even about the same things.)

Let's now interpret "D is relativistically invariant" to mean that "the truth-value of D is independent of the context." Certainly the surface syntactic form of D makes it seem that it is not context sensitive. After all, D looks like a proposition (with no free parameters).

But that's not quite right. The symbols "p" and "q" are constants, and for D to come out true, an agent S would need to successfully use p and q to refer to spacetime points. What's more, which points S refers to will depend on S's frame of reference — for if S doesn't live in space and time, then S can't refer to anything. Therefore, D contains hidden parameters that are frame-dependent, and it is not relativistically invariant.

To be clear, the special theory of relativity does not have constant symbols — in particular, it doesn't have names for spacetime points. And that is not an accident: if the theory had such names, then they would break the symmetries of spacetime. Hence, constant symbols like p and q are not part of the theory proper, and the theory doesn't fix their reference. Instead, a user of the theory adds constant symbols, and fixes their reference. Sentences, such as D, that are built with these constant symbols are not observer-independent; they depend on the reference of p and q. The overall result is that STR does *not* include assertions that are both contentful (about specific, local matters of fact), and that are invariant under changes of reference frame.

The issues here can become obscured by the technicalities of relativistic physics, so let's turn to a much simpler example. Consider the Euclidean theory T of three-dimensional space. The key fact about this theory — or about its models — is that any point can be moved by a symmetry, but the Euclidean distance between any two points is invariant. Does it follow that, in this theory, the Euclidean distance between points is an invariant fact?

Once again, T doesn't actually have names for points (and if we added them, then T would lose its symmetries). So the theory T does not itself assert any facts of the form:

(E) The distance between p and q is λ.

A particular inhabitant of Euclidean space could supply names p and q, and could use them to refer to points. But then the assertion's truth-value is relative to her reference fixing act.

In conclusion, lose talk about invariance has led some people to think that special relativity favors a four-dimensional ontology. I have shown that this lose talk falls apart when subjected to critical scrutiny.

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