Distributionally Robust Mechanism Design

Çağıl Koçyiğit

Risk Analytics and Optimization Chair, École Polytechnique Fédérale de Lausanne, Lausanne 1015, Switzerland,

cagil.kocyigit@epfl.ch

Garud Iyengar

Department of Industrial Engineering and Operations Research, Columbia University, New York 10027, USA,

garud@ieor.columbia.edu

Daniel Kuhn

Risk Analytics and Optimization Chair, École Polytechnique Fédérale de Lausanne, Lausanne 1015, Switzerland,

daniel.kuhn@epfl.ch

Wolfram Wiesemann

Imperial College Business School, Imperial College London, London SW7 2AZ, United Kingdom,

ww@imperial.ac.uk

We study a mechanism design problem where an indivisible good is auctioned to multiple bidders, for each of whom it has a private value that is unknown to the seller and the other bidders. The agents perceive the ensemble of all bidder values as a random vector governed by an ambiguous probability distribution, which belongs to a commonly known ambiguity set. The seller aims to design a revenue maximizing mechanism that is not only immunized against the ambiguity of the bidder values but also against the uncertainty about the bidders' attitude towards ambiguity. We argue that the seller achieves this goal by maximizing the worst-case expected revenue across all value distributions in the ambiguity set and by positing that the bidders have Knightian preferences. For ambiguity sets containing all distributions supported on a hypercube, we show that the Vickrey auction is the unique mechanism that is optimal, efficient and Pareto robustly optimal. If the bidders' values are additionally known to be independent, then the revenue of the (unknown) optimal mechanism does not exceed that of a second price auction with only one additional bidder. For ambiguity sets under which the bidders' values are dependent and characterized through moment bounds, on the other hand, we provide a new class of randomized mechanisms, the highest-bidder-lotteries, whose revenues cannot be matched by any second price auction with a constant number of additional bidders. Moreover, we show that the optimal highest-bidder-lottery is a 2-approximation of the (unknown) optimal mechanism, whereas the best second price auction fails to provide any constant-factor approximation guarantee.

Key words: auction, mechanism design, distributionally robust optimization, ambiguity aversion,

  Knightian preferences

1. Introduction

When traders from the Ottoman Empire first brought tulip bulbs to Holland in the seventeenth century, the combination of a limited supply and a rapidly increasing popularity led to highly non-stationary and volatile prices. Faced with the challenge of selling scarce items with a largely unknown demand, the flower exchange invented the Dutch auction, in which an artificially high asking price is gradually decreased until the first participant is willing to accept the trade. Nowadays, auctions are routinely used in economic transactions that are characterized by demand uncertainty, ranging from the sale of financial instruments (*e.g.*, U.S. Treasury bills), antiques, collectibles and commodities (*e.g.*, radio spectra, electricity and carbon emissions) to livestock and holidays.

Despite their long history, the scientific study of auctions only started in the sixties of the last century when the then emerging discipline of mechanism design began to model auctions as incomplete information games between rational but self-interested agents. In the most basic such game, a seller wishes to auction a single product to multiple bidders. Each bidder is fully aware of the value that he attaches to the good, whereas the other bidders and the seller only know the probability distribution from which this value has been drawn. This information structure is referred to as the *private value* setting. The seller aims to design a mechanism that allocates the good and charges the bidders based on a single-shot or iterative bidding process so as to maximize her expected revenues (*optimal* mechanism design), sometimes under the additional constraint that the resulting allocation should maximize the overall welfare (*efficient* mechanism design). The bidders, in turn, seek to submit bids that maximize their expected utility arising from the difference of the value obtained from receiving the good (if they do so) and the charges incurred.

In the private value setting outlined above, the bidders' values for the good are typically modeled as independent random variables. Under this assumption, Vickrey (1961) argues that the second price auction without reserve price, which allocates the good to the highest bidder and charges him the value of the second highest bid, generates maximum revenues among all efficient mechanisms. Myerson (1981) proves that in the same setting, the second price auction maximizes the seller's revenues if it is augmented with a suitable reserve price. In this case, however, efficiency is typically lost since the good resides with the seller whenever the highest bid falls short of the reserve price. Cremer and McLean (1988) show that if the bidders' values are described by *correlated* random variables, then second price auctions no longer maximize the seller's revenues, and the seller can extract all surplus by combining an auction with a menu of side bets with the bidders. For a review of the mechanism design literature, we refer to Klemperer (1999) and Krishna (2009).

Traditionally, the mechanism design literature models the bidders' values as a random vector that is governed by a probability distribution which is known precisely by all participants. Although this assumption greatly facilitates the analysis, the existence and common knowledge of such a

distribution may be difficult to justify in settings where the demand is poorly understood, which arguably form the auctions' *raison d'être*. The literature on robust mechanism design addresses this concern by assuming that the bidders' willingness to pay is only known to be governed by some probability distribution from within an *ambiguity set*. In this setting, the agents take decisions that maximize their expected utility under the most adverse value distribution in the ambiguity set.

The early literature on robust mechanism design has studied the impact of ambiguity on traditional auction schemes. Salo and Weber (1995) show that the experimentally observed deviations from the theoretically optimal bidding strategy in a first price auction can be explained by the presence of ambiguity as well as ambiguity averse decision-making on behalf of the agents. In a similar study, Chen et al. (2007) show that the presence of ambiguity leads to lower bids in first price auctions. Lo (1998) and Ozdenoren (2002) derive the optimal bidding strategy for an ambiguity averse bidder in a first price auction, and they show that in contrast to the traditional theory, first price and second price auctions yield different revenues in the presence of ambiguity. Chiesa et al. (2015) study a variant of the private value setting where the bidders are unsure about both their own value and the other bidders' values for the auctioned good, and they show that the Vickrey mechanism maximizes the worst-case social welfare in this setting.

More recently, the robust mechanism design literature has focused on characterizing revenue maximizing auctions for different variants of the mechanism design problem under ambiguity. Bose et al. (2006) show that full insurance mechanisms, which either make the seller or the bidders indifferent between the possible bids of the (other) bidders, maximize the seller's worst-case revenues in several variants of the optimal auction design problem under ambiguity. Bodoh-Creed (2012) generalizes a well-known payoff equivalence result to ambiguous auctions, and he uses it to provide further intuition about the optimality of full insurance mechanisms. Bose and Daripa (2009) show that for certain classes of ϵ-contamination ambiguity sets, the seller can extract almost all surplus by a variant of the Dutch auction.

Bose et al. (2006), Bose and Daripa (2009) and Bodoh-Creed (2012) all model the bidders' values as independent random variables, and they assume that the agents exhibit maxmin preferences, that is, the agents judge actions in view of their expected utility under the worst probability distribution in the ambiguity set. In contrast, Lopomo et al. (2014) consider agents that exhibit Knightian preferences, that is, an action A is preferred over an action B only if A yields a weakly higher expected utility than B under every probability distribution in the ambiguity set. They derive necessary and sufficient conditions for full surplus extraction under ambiguity in a mechanism design problem where a principal interacts with a single agent. In a similar spirit, Koçyiğit et al. (2018) study a single-item auction where the bidders' values are private and ambiguous, and the agents exhibit Knightian preferences. The authors show that second price auctions are no longer

optimal in this setting, and they develop a numerical solution scheme for determining the optimal mechanism under the premise that the ambiguity set consists of two distributions.

Bandi and Bertsimas (2014) point out that the mechanism design problem is amenable to a formulation as a robust optimization problem (Ben-Tal et al. 2009, Bertsimas et al. 2011). To this end, they model the bidders' values as a deterministic vector that is chosen adversely from an uncertainty set. They show that in this setting, the second price auction with item and bidder dependent reserve prices is optimal for multi-item auctions with budget constrained buyers. They also show that the optimal reserve prices can be calculated through an optimization problem.

In this paper, we study the single-item auction design problem under ambiguity, where we follow the approach of Lopomo et al. (2014) and assume that the agents exhibit Knightian preferences. We show that this assumption not only protects the seller against the ambiguity of the bidders' values, but it also immunizes her against the bidders' attitude towards this ambiguity. We then argue that the resulting mechanism design problem under ambiguity is amenable to a formulation as a *distributionally* robust optimization problem (Delage and Ye 2010, Wiesemann et al. 2014). We use this insight to study three popular classes of ambiguity sets: *(i)* support-only ambiguity sets containing all distributions supported on a hypercube, *(ii)* independence ambiguity sets comprising symmetric and regular distributions supported on a hypercube under which the bidder values are independent, and *(iii)* Markov ambiguity sets containing all distributions that are supported on a hypercube and satisfy a first-order moment constraint.

The contributions of this paper to the three classes of ambiguity sets are summarized below.

1. For support-only ambiguity sets, we show that the Vickrey auction is not only optimal but also Pareto robustly optimal over *all* (efficient and inefficient) mechanisms in the sense that there exists no other feasible mechanism that generates higher revenues to the seller under some realization of the bidders' values while generating at least the same revenues under every other realization. Moreover, we show that the Vickrey auction generates the highest revenues among all efficient mechanisms under every possible realization of the bidders' values.

2. For independence ambiguity sets, we prove that the Vickrey auction generates the highest worst-case expected revenue among all *efficient* (but not necessarily inefficient) mechanisms. We also show that the added value of the (to date unknown) optimal mechanism over the Vickrey auction is offset by attracting just one additional bidder.

3. For Markov ambiguity sets, we specify the best second price auction with reserve price. We show that while this auction asymptotically maximizes the worst-case expected seller revenues as the number of bidders grows, it may extract an arbitrarily small fraction of the optimal mechanism's revenues for a finite number of bidders. We also propose a new class of auctions—the *highest-bidder-lotteries*—in which the seller offers the highest bidder a lottery that determines the allocation

of the good as well as the payment. We analytically determine the best highest-bidder-lottery, and we show that it can generate significantly higher revenues than the best second price auction with reserve price. Specifically, we show that the proposed mechanism is a 2-approximation of the (unknown) optimal one. We also prove that the revenues of the optimal highest-bidder-lottery cannot be matched by any second price auction with a constant number of additional bidders.

Apart from the robust mechanism design literature, relaxations of the assumptions underlying the information structure of auction problems have been studied in the related fields of prior-free and prior-independent mechanism design. While the informational assumptions of prior-free mechanisms are akin to those of our robust mechanisms in Section 3, their goal is to minimize the worst-case regret (in terms of the actual revenues) over all scenarios relative to a judiciously chosen benchmark, rather than to maximize the worst-case revenues. Notable examples are discussed by Goldberg et al. (2006), who provide a 4-approximation to the revenues generated by the optimal posted price in a digital goods environment, as well as Hartline and Roughgarden (2008), who derive an O(1)-approximation to the maximum revenue achieved by any Bayesian optimal mechanism in a general symmetric auction framework. For further details, we refer to Nisan et al. (2007).

In a similar spirit, prior-independent mechanisms aim to minimize the worst-case regret (now in terms of the expected revenues) over all value distributions contained in an ambiguity set and relative to the respective Bayesian optimal mechanism. Since the bidders' values (and/or the received signals) are assumed to be independent, this literature stream relates to the material in Section 4. Closest to our work, Dhangwatnotai et al. (2015) propose the Single Sample mechanism, which is a second price auction with a random reserve price, and show that this mechanism provides constant-factor approximations in different auction settings. This mechanism has subsequently been generalized by Roughgarden and Talgam-Cohen (2013) to an interdependent values setting, where the bidders' values are determined through private signals.

Finally, we note that there is a related but distinct branch of the mechanism design literature that is also referred to as robust mechanism design, see, e.g., Bergemann and Morris (2005), Chung and Ely (2007) and Bergemann et al. (2016). Contrary to our setting, which assumes that the value distribution is ambiguous but the ambiguity set is common knowledge, this literature stream exclusively works with non-ambiguous value distributions, but relaxes the common knowledge assumption in the sense that the agents may be unsure about the (higher-order) beliefs of the other agents. As a consequence, the findings in the two literature streams are complementary due to the different informational assumptions made. For a review of this literature stream, we refer to Bergemann and Morris (2013).

The remainder of the paper is structured as follows. Section 2 defines the auction design problem of interest, it establishes preliminary results required in the remainder of the paper, and it shows

that the assumption of Knightian preferences insures the seller against both the ambiguity in the bidders' values as well as their attitude towards this ambiguity. Sections 3, 4 and 5 study the auction design problem for support-only, independence and Markov ambiguity sets, respectively, and Section 6 concludes. Lengthy and technical proofs are deferred to the appendix.

Notation. For any $\boldsymbol{v} \in \mathbb{R}^I$ we denote by v_i its ith component and by $\boldsymbol{v}_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_I)$ its subvector excluding v_i. The vector of ones is denoted by $\mathbf{e}$. Random variables are designated by tilde signs (*e.g.*, $\tilde{v}$) and their realizations by the same symbols without tildes (*e.g.*, v). For any Borel set $\mathcal{A} \in \mathcal{B}(\mathbb{R}^I)$ we use $\mathcal{P}_0(\mathcal{A})$ to represent the set of all probability distributions on $\mathcal{A}$. The family of all bounded Borel-measurable functions from $\mathcal{A} \in \mathcal{B}(\mathbb{R}^n)$ to $\mathcal{C} \in \mathcal{B}(\mathbb{R}^m)$ is denoted by $\mathcal{L}_\infty(\mathcal{A}, \mathcal{C})$. For $\mathcal{A} \in \mathcal{B}(\mathbb{R}^I)$, $A_i \in \mathcal{B}(\mathbb{R})$, $f, g \in \mathcal{L}_\infty(\mathcal{A}, \mathbb{R})$ and $\mathcal{P} \subseteq \mathcal{P}_0(\mathcal{A})$, statements of the form $\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[f(\tilde{\boldsymbol{v}}) | \tilde{v}_i = v_i] \geq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[g(\tilde{\boldsymbol{v}}) | \tilde{v}_i = v_i] \; \forall v_i \in \mathcal{A}_i$, which are not well-defined because conditional expectations under $\mathbb{P}$ are only defined up to sets of $\mathbb{P}$-measure zero, should be interpreted as $\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[f(\tilde{\boldsymbol{v}}) h(\tilde{v}_i)] \geq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[g(\tilde{\boldsymbol{v}}) h(\tilde{v}_i)] \; \forall h \in \mathcal{L}_\infty(\mathcal{A}_i, \mathbb{R}_+)$. The latter statement is well-defined but cumbersome.

2. Problem Formulation and Preliminaries

We consider the following mechanism design problem. A seller aims to sell an indivisible good which is of zero value to her. There are $I \geq 2$ potential buyers (or bidders) indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$. The buyers' values for the good are modeled as a random vector $\tilde{\boldsymbol{v}}$ that follows a probability distribution $\mathbb{P}^0$ in some ambiguity set $\mathcal{P} \subseteq \mathcal{P}_0(\mathbb{R}_+^I)$. We denote the realizations of $\tilde{\boldsymbol{v}}$ by $\boldsymbol{v}$ and refer to them as scenarios. The probability distribution $\mathbb{P}^0$ is unknown to the agents, but the ambiguity set $\mathcal{P}$ is common knowledge. We assume that the smallest closed set that has probability 1 under every distribution $\mathbb{P} \in \mathcal{P}$ is of the form $\mathcal{V}^I = \mathcal{V} \times \cdots \times \mathcal{V}$ with marginal projections $\mathcal{V} \subseteq \mathbb{R}_+$; this is a standard assumption in the mechanism design literature (McAfee and McMillan 1987).

The seller aims to determine a mechanism for selling the good. A mechanism $(\mathcal{B}_1, \ldots, \mathcal{B}_I, \boldsymbol{q}, \boldsymbol{m})$ consists of a set $\mathcal{B}_i$ of messages (or bids) available to each buyer i, an allocation rule $\boldsymbol{q} : \mathcal{B}_1 \times \cdots \times \mathcal{B}_I \to \mathbb{R}_+^I$ and a payment rule $\boldsymbol{m} : \mathcal{B}_1 \times \cdots \times \mathcal{B}_I \mapsto \mathbb{R}^I$. Depending on his value v_i, each buyer i reports a message $b_i \in \mathcal{B}_i$ to the seller. Once all messages are collected, the seller allocates the good to buyer i with probability $q_i(\boldsymbol{b})$ and charges this buyer an amount $m_i(\boldsymbol{b})$, where $\boldsymbol{b} = (b_1, \ldots, b_I)$.

EXAMPLE 1 (FIRST PRICE SEALED BID AUCTION). The first price sealed bid auction is a widely used mechanism, where bidders simultaneously report their bids $b_i \in \mathcal{B}_i = \mathbb{R}_+$, $i \in \mathcal{I}$. The highest bidder wins the good with probability 1 and pays an amount equal to his bid, whereas all other bidders win the good with probability 0 and do not make a payment. If there is a tie (the highest bidder is not unique), then the winner is determined at random (or by some other tie-breaking rule).

We assume that all agents are risk-neutral with respect to the uncertainty of the allocation.

DEFINITION 1 (EX-POST UTILITY). The *ex-post utility* of bidder i with value v_i and reporting message b_i is defined as

$$u_i(b_i; v_i, \boldsymbol{b}_{-i}) = q_i(b_i, \boldsymbol{b}_{-i})v_i - m_i(b_i, \boldsymbol{b}_{-i}),$$

where $\boldsymbol{b}_{-i}$ denotes the vector of messages reported by the other bidders.

The ex-post utility of a bidder quantifies his expected payoff after all messages are revealed. Note that the ex-post utility depends critically on the allocation and payment rules of the mechanism at hand. We will suppress this dependence notationally, however, in order to avoid clutter.

We assume that the buyers have incomplete preferences as in Knightian decision theory, see, *e.g.*, Knight (1921) and Bewley (2002). In this setting, a buyer prefers an action to another one if it results in a higher expected utility to him under every distribution $\mathbb{P} \in \mathcal{P}$.

Given a mechanism, the buyers play a game of incomplete information and select their bids strategically to induce the most desirable outcome in view of their individual preferences. Recall that buyer i selects a message depending on his value v_i. Thus, his strategy must be modeled as a function $\beta_i : \mathcal{V} \to \mathcal{B}_i$ that maps each of his possible values to a message. An I-tuple of strategies $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_I)$ constitutes an equilibrium for a given mechanism if no agent i has an incentive to unilaterally change his strategy β_i.

DEFINITION 2 (KNIGHTIAN NASH EQUILIBRIUM). An I-tuple of strategies $\beta_i : \mathcal{V} \to \mathcal{B}_i$, $i \in \mathcal{I}$, constitutes a *Knightian Nash equilibrium* for a mechanism $(\mathcal{B}_1, \ldots, \mathcal{B}_I, \boldsymbol{q}, \boldsymbol{m})$ if

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u_i(\beta_i(v_i); v_i, \boldsymbol{\beta}_{-i}(\tilde{\boldsymbol{v}}_{-i})) - u_i(b_i; v_i, \boldsymbol{\beta}_{-i}(\tilde{\boldsymbol{v}}_{-i})) \,|\, \tilde{v}_i = v_i \right] \geq 0 \quad \forall i \in \mathcal{I}, \forall v_i \in \mathcal{V}, \forall b_i \in \mathcal{B}_i.$$

In the absence of ambiguity, that is, for $\mathcal{P} = \{\mathbb{P}^0\}$, a Knightian Nash equilibrium collapses to a Bayesian Nash equilibrium as introduced by Harsanyi (1967). If $\mathcal{P} = \mathcal{P}_0(\mathcal{V}^I)$, on the other hand, then the Knightian Nash equilibrium reduces to an ex-post Nash equilibrium (Fudenberg and Tirole 1991, Section 1.2). Note also that every ex-post Nash equilibrium is automatically a Knightian Nash equilibrium, but the converse implication is generally wrong.

The mechanism design problem is the decision problem of the seller. We assume that the seller is ambiguity averse in the sense that she aims to maximize the worst-case expected revenue in view of all distributions $\mathbb{P} \in \mathcal{P}$. However, she may not know how ambiguity is perceived by the bidders and may wish to hedge against uncertainty in the buyers' preferences. We will argue later that this is achieved by adopting the view that the buyers have Knightian preferences, which in a sense represent the worst-case buyer preferences from the seller's perspective. Hence, the seller is interested in selecting allocation and payment rules that maximize her worst-case expected

revenue, anticipating that the buyers' strategies will be in a Knightian Nash equilibrium. Note that a mechanism is of interest only if it has a Knightian Nash equilibrium because, otherwise, its outcome is unpredictable.

We assume that bidder i with value $v_i \in \mathcal{V}$ will walk away from a mechanism if his expected utility under a Knightian Nash equilibrium is negative for some $\mathbb{P} \in \mathcal{P}$. Nevertheless, the seller only needs to consider mechanisms that attract all buyers. Indeed, imagine that bidder i with value v_i prefers to walk away under the mechanism $(\boldsymbol{q}, \boldsymbol{m})$. The same outcome is achieved by setting $q_i(\beta_i(v_i), \boldsymbol{\beta}_{-i}(\boldsymbol{v}_{-i})) = 0$ and $m_i(\beta_i(v_i), \boldsymbol{\beta}_{-i}(\boldsymbol{v}_{-i})) = 0$ for all $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$, which results in an ex-post utility of zero to him so that participating remains weakly dominant.

The set of all mechanisms is extremely large. An important subset is the family of *direct* mechanisms in which the set of messages available to buyer i is equal to the set of his values, that is, $\mathcal{B}_i = \mathcal{V}$ for all $i \in \mathcal{I}$. Yet a smaller subset is the family of *truthful* direct mechanisms, in which it is optimal for each buyer to report his true value. In fact, due to the celebrated revelation principle by Myerson (1981), we can restrict attention to truthful direct mechanisms without loss of generality.

THEOREM 1 (**The Revelation Principle**). *Given any mechanism $(\mathcal{B}_1, \ldots, \mathcal{B}_I, \boldsymbol{q}, \boldsymbol{m})$ with a corresponding Knightian Nash equilibrium $\beta_i : \mathcal{V} \to \mathcal{B}_i$, $i \in \mathcal{I}$, there exists a truthful direct mechanism resulting in the same ex-post utilities for the bidders and the same ex-post revenue for the seller for every $\boldsymbol{v} \in \mathcal{V}^I$.*

The proof is a straightforward adaptation of the proof of Proposition 5.1 in Krishna (2009). The intuition is as follows. Consider any mechanism $(\mathcal{B}_1, \ldots, \mathcal{B}_I, \boldsymbol{q}, \boldsymbol{m})$ as well as an equilibrium $\boldsymbol{\beta}$ for this mechanism. Then, the seller can construct an equivalent truthful direct mechanism $(\mathcal{V}, \ldots, \mathcal{V}, \boldsymbol{q}', \boldsymbol{m}')$ by asking the bidders to report their true values, allocating the good according to the rule $\boldsymbol{q}'(\boldsymbol{v}) = \boldsymbol{q}(\boldsymbol{\beta}(\boldsymbol{v}))$ and charging payments $\boldsymbol{m}'(\boldsymbol{v}) = \boldsymbol{m}(\boldsymbol{\beta}(\boldsymbol{v}))$ as if the bidders had implemented their equilibrium strategies for the original mechanism. In this case, the bidders have no incentive to misreport their true values because truthful bidding is the equilibrium strategy for the new mechanism by construction. Also, the ex-post revenue of the seller and the ex-post utilities of the bidders do not change.

From now on, we focus exclusively on truthful direct mechanisms and use the shorthand $(\boldsymbol{q}, \boldsymbol{m})$ to denote $(\mathcal{V}, \ldots, \mathcal{V}, \boldsymbol{q}, \boldsymbol{m})$ because the set of messages available to each buyer is always equal to the interval of his possible values. A direct mechanism is truthful under Knightian preferences if and only if it is distributionally robust incentive compatible.

DEFINITION 3 (DISTRIBUTIONALLY ROBUST INCENTIVE COMPATIBILITY).
A mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is called *distributionally robust incentive compatible* if

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) - u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right] \geq 0 \quad \forall i \in \mathcal{I}, \forall v_i, w_i \in \mathcal{V}. \tag{IC-D}$$

Distributionally robust incentive compatibility ensures that reporting the true value $v_i \in \mathcal{V}$ is a dominant strategy for bidder i under Knightian preferences.

Recall that the seller only needs to consider mechanisms that attract all bidders. If the bidders are willing to participate in a given mechanism, the corresponding truthful direct mechanism will be distributionally robust individually rational.

DEFINITION 4 (DISTRIBUTIONALLY ROBUST INDIVIDUAL RATIONALITY).
A mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is called *distributionally robust individually rational* if

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i \right] \geq 0 \quad \forall i \in \mathcal{I}, \, \forall v_i \in \mathcal{V}. \tag{IR-D}$$

Distributionally robust individual rationality ensures that the expected utility of bidder i conditional on his own value v_i is non-negative under truthful bidding for any possible value $v_i \in \mathcal{V}$ and any possible probability distribution $\mathbb{P} \in \mathcal{P}$.

We can now formalize the seller's problem of finding the best truthful direct mechanism as

$$\sup_{\boldsymbol{q} \in \mathcal{Q}, \boldsymbol{m} \in \mathcal{M}} \quad \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[\sum_{i \in \mathcal{I}} m_i(\tilde{\boldsymbol{v}}) \right] \tag{$\mathcal{MDP}$}$$

$$\text{s.t.} \quad \text{(IC-D), (IR-D),}$$

where

$$\mathcal{Q} = \{ \boldsymbol{q} \in \mathcal{L}_\infty(\mathcal{V}^I, \mathbb{R}_+^I) \mid \sum_{i \in \mathcal{I}} q_i(\boldsymbol{v}) \leq 1 \ \ \forall \boldsymbol{v} \in \mathcal{V}^I \}$$

is the set of all possible allocation rules of direct mechanisms. The definition of $\mathcal{Q}$ captures the idea that the seller can sell the good at most once. Similarly, $\mathcal{M} = \mathcal{L}_\infty(\mathcal{V}^I, \mathbb{R}^I)$ denotes the set of all possible payment rules of direct mechanisms. By the revelation principle, solving ($\mathcal{MDP}$) is equivalent to finding the best mechanism among all direct *and* indirect mechanisms.

Sometimes we will further restrict problem ($\mathcal{MDP}$) to optimize only over efficient mechanisms.

DEFINITION 5 (EFFICIENCY). A mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is called *efficient* if $\boldsymbol{q} \in \mathcal{Q}^{\text{eff}}$, where

$$\mathcal{Q}^{\text{eff}} = \left\{ \boldsymbol{q} \in \mathcal{Q} \mid q_i(\boldsymbol{v}) > 0 \implies v_i = \max_{j \in \mathcal{I}} v_j \quad \forall i \in \mathcal{I}, \quad \sum_{i \in \mathcal{I}} q_i(\boldsymbol{v}) = 1 \quad \forall \boldsymbol{v} \in \mathcal{V}^I \right\}.$$

An efficient mechanism allocates the good with probability 1 to a bidder who values it most. Hence, it maximizes the ex-post total social welfare across all agents (seller and bidders), which coincides with the highest bidder value because the payments of the bidders and the revenue of the seller cancel each other. Recall that the good has zero value to the seller. Allocative efficiency plays a crucial role in the sale of public goods such as railway lines, plots of public land or specific bands of the electromagnetic spectrum. Efficient allocations do not normally emerge from mechanisms with inefficient allocation rules, even if we allow for the existence of an aftermarket with zero transaction costs (Krishna 2009, Section 1.4).

We will now demonstrate that distributionally robust incentive compatible mechanisms protect the seller against uncertainty in the bidders' attitude towards ambiguity. Indeed, depending on the bidders' preferences, one can envisage other types of incentive compatibility.

DEFINITION 6. A mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is called

(i) *ex-post incentive compatible* if for all $i \in \mathcal{I}$, $\boldsymbol{v} \in \mathcal{V}^I$, $w_i \in \mathcal{V}$,

$$u_i(v_i; v_i, \boldsymbol{v}_{-i}) \geq u_i(w_i; v_i, \boldsymbol{v}_{-i}),$$

(ii) *maxmin incentive compatible* if for all $i \in \mathcal{I}$, $v_i, w_i \in \mathcal{V}$,

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \, | \, \tilde{v}_i = v_i \right] \geq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \, | \, \tilde{v}_i = v_i \right],$$

(iii) *Hurwicz incentive compatible* with respect to $\alpha \in (0,1)$ if for all $i \in \mathcal{I}$, $v_i, w_i \in \mathcal{V}$,

$$\alpha \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \, | \, \tilde{v}_i = v_i \right] + (1 - \alpha) \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \, | \, \tilde{v}_i = v_i \right]$$

$$\geq \alpha \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \, | \, \tilde{v}_i = v_i \right] + (1 - \alpha) \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \, | \, \tilde{v}_i = v_i \right],$$

(iv) *Bayesian incentive compatible* with respect to a Borel distribution $\mathbb{Q}$ on $\mathcal{P}$ (where $\mathcal{P}$ is equipped with its weak topology) if for all $i \in \mathcal{I}$, $v_i, w_i \in \mathcal{V}$,

$$\mathbb{E}_{\mathbb{Q}} \left[\mathbb{E}_{\tilde{\mathbb{P}}} \left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \, | \, \tilde{v}_i = v_i \right] \right] \geq \mathbb{E}_{\mathbb{Q}} \left[\mathbb{E}_{\tilde{\mathbb{P}}} \left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \, | \, \tilde{v}_i = v_i \right] \right],$$

where $\tilde{\mathbb{P}} \sim \mathbb{Q}$ is a random value distribution.

One can define individual rationality with respect to other preferences analogously.

PROPOSITION 1. *Ex-post incentive compatibility implies distributionally robust incentive compatibility, whereas distributionally robust incentive compatibility implies maxmin incentive compatibility, Hurwicz incentive compatibility and Bayesian incentive compatibility.*

Using similar arguments as in Proposition 1, one can verify that ex-post individual rationality implies distributionally robust individual rationality, and that distributionally robust individual rationality implies maxmin, Hurwicz and Bayesian individual rationality. Thus, agents have no incentive to walk away from a distributionally robust individually rational mechanism or to misreport their values in a distributionally robust incentive compatible mechanism even if they have maxmin, Bayesian or Hurwicz preferences. Hence, adopting a distributionally robust perspective allows the seller to hedge against uncertainty about the bidders' attitude towards ambiguity. Use of ex-post individual rationality and incentive compatibility would provide even stronger protection against uncertainty in the bidders' preferences, but it would also lead to more conservative mechanisms that do not benefit from any distributional information that might be available.

So far, the literature on mechanism design has used almost exclusively the maxmin criterion to model the ambiguity aversion of the bidders. However, while being less conservative, the resulting mechanism design problems may fail to protect against the uncertainty about the bidders' attitude towards ambiguity.

EXAMPLE 2. Consider an auction with two bidders whose values are governed by a probability distribution from the ambiguity set $\mathcal{P} = \{\mathbb{P}_1, \mathbb{P}_2\}$ over $\mathcal{V}^2 = \{0, 4\} \times \{0, 4\}$. The probabilities of the four scenarios under $\mathbb{P}_1$ and $\mathbb{P}_2$ are given in the following table.

$\boldsymbol{v}$	$(0,0)$	$(4,0)$	$(0,4)$	$(4,4)$
$\mathbb{P}_1$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$
$\mathbb{P}_2$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

We consider an all-pay mechanism where the highest bidder wins, and every bidder pays half of his bid (irrespective of whether the bid was successful or not). Ties are broken lexicographically, *i.e.*, the first bidder wins if there is a tie. One can verify that this mechanism is maxmin incentive compatible over $\mathcal{P}$. Below we list the expected utilities of bidder 1 with true value 4 with respect to $\mathbb{P}_1$ and $\mathbb{P}_2$ when he reports the values 4 and 0, respectively.

$$\mathbb{E}_{\mathbb{P}_1}\left[u_1(4; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right] = \mathbb{E}_{\mathbb{P}_2}\left[u_1(4; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right] = 2$$
$$\mathbb{E}_{\mathbb{P}_1}\left[u_1(0; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right] = \frac{10}{3}, \quad \mathbb{E}_{\mathbb{P}_2}\left[u_1(0; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right] = 2$$

Hence, we have

$$\mathbb{E}_{\mathbb{P}_1}\left[u_1(4; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right] = 2 < \frac{10}{3} = \mathbb{E}_{\mathbb{P}_1}\left[u_1(0; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right],$$

that is, the all-pay mechanism is not distributionally robust incentive compatible.

For $\alpha = 1/2$, Hurwicz incentive compatibility is violated because

$$\alpha \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u_1(4; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right] + (1-\alpha) \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u_1(4; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right] = 2$$
$$< \frac{8}{3} = \alpha \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u_1(0; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right] + (1-\alpha) \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u_1(0; \tilde{v}_1, \tilde{v}_2) \mid \tilde{v}_1 = 4\right].$$

One similarly verifies that Bayesian incentive compatibility is violated if $\mathbb{Q}(\mathbb{P}_1) = \mathbb{Q}(\mathbb{P}_2) = 1/2$.

In the following sections, we will investigate the *optimal mechanisms*, which maximize the worst-case expected revenues without any restrictions on the allocation rule, and the *best efficient mechanisms*, which maximize only over efficient allocation rules, for different classes of ambiguity sets $\mathcal{P}$. Before that, we review and extend some important results from the literature that will be used throughout the paper.

2.1. The Revenue Equivalence

We first review a cornerstone result from the mechanism design literature stating that the payment rule of an ex-post incentive compatible mechanism is uniquely determined by the allocation rule up to an additive constant for each bidder. In addition, we derive a related result for distributionally robust incentive compatible mechanisms. When applicable, these results will help us to simplify problem ($\mathcal{MDP}$) by substituting out the payment rule. For ease of exposition, we will henceforth assume that $\mathcal{V} = [\underline{v}, \overline{v}]$.

We first define monotonicity of allocation rules.

DEFINITION 7 (MONOTONE ALLOCATION RULE). An allocation rule is called

(i) *ex-post monotone* if it belongs to the set

$$\mathcal{Q}^{\text{m-p}} = \{\boldsymbol{q} \in \mathcal{Q} \mid q_i(v_i, \boldsymbol{v}_{-i}) - q_i(w_i, \boldsymbol{v}_{-i}) \geq 0 \quad \forall i \in \mathcal{I}, \forall v_i, w_i \in \mathcal{V} : v_i \geq w_i, \forall \boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}\},$$

(ii) *distributionally robust monotone* if it belongs to the set

$$\mathcal{Q}^{\text{m-d}} = \{\boldsymbol{q} \in \mathcal{Q} \mid \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[q_i(v_i, \tilde{\boldsymbol{v}}_{-i}) - q_i(w_i, \tilde{\boldsymbol{v}}_{-i})] \geq 0 \quad \forall i \in \mathcal{I}, \forall v_i, w_i \in \mathcal{V} : v_i \geq w_i\}.$$

Ex-post monotonicity implies that bidder i's probability to win the good is non-decreasing in his value v_i if $\boldsymbol{v}_{-i}$ is kept constant. On the other hand, distributionally robust monotonicity ensures that the expected allocation to bidder i is non-decreasing in v_i under all distributions $\mathbb{P} \in \mathcal{P}$. Note that $\mathcal{Q}^{\text{m-p}} \subseteq \mathcal{Q}^{\text{m-d}}$ by construction.

Some results below will rely on the assumption that the bidders' values are independent.

DEFINITION 8 (INDEPENDENCE). We say that the bidders' values are *independent* if the random variables $\tilde{v}_i$, $i \in \mathcal{I}$, are mutually independent under every $\mathbb{P} \in \mathcal{P}$.

From now on, we use the shorthand $u_i(v_i, \boldsymbol{v}_{-i})$ to denote the ex-post utility $u_i(v_i; v_i, \boldsymbol{v}_{-i})$ under truthful bidding. The next proposition shows that the actual (expected) payment of each bidder under an ex-post (distributionally robust) incentive compatible mechanism is completely determined by the allocation rule and the ex-post (expected) utility of the bidder under his lowest value.

PROPOSITION 2. *We have the following equivalent characterizations of incentive compatibility.*

(i) A mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is ex-post incentive compatible if and only if $\boldsymbol{q} \in \mathcal{Q}^{\text{m-p}}$ and

$$m_i(v_i, \boldsymbol{v}_{-i}) = q_i(v_i, \boldsymbol{v}_{-i})v_i - u_i(\underline{v}, \boldsymbol{v}_{-i}) - \int_{\underline{v}}^{v_i} q_i(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I. \tag{1}$$

(ii) If the bidders' values are independent, then a mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is distributionally robust incentive compatible if and only if $\boldsymbol{q} \in \mathcal{Q}^{\text{m-d}}$ and

$$\mathbb{E}_{\mathbb{P}}[m_i(v_i, \tilde{\boldsymbol{v}}_{-i})] = \mathbb{E}_{\mathbb{P}}\left[q_i(v_i, \tilde{\boldsymbol{v}}_{-i})v_i - u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) - \int_{\underline{v}}^{v_i} q_i(x, \tilde{\boldsymbol{v}}_{-i}) \, \mathrm{d}x\right] \quad \forall i \in \mathcal{I}, \forall v_i \in \mathcal{V}, \forall \mathbb{P} \in \mathcal{P}. \tag{2}$$

Proof. Assertion (*ii*) follows directly from Krishna (2009, Section 5.1.2), and thus its proof is omitted. Assertion (*i*), on the other hand, is an immediate consequence of assertion (*ii*). This can be seen by defining $\mathcal{P}$ as the set of all distributions supported on $\mathcal{V}^I$ under which the bidders' values are independent. Since $\mathcal{P}$ contains all Dirac distributions supported on $\mathcal{V}^I$, equation (2) implies equation (1). □

Proposition 2 is the main ingredient for the following generalized revenue equivalence theorem, which is an extension of the revenue equivalence theorem by Myerson (1981) and Riley and Samuelson (1981).

THEOREM 2 (**The Revenue Equivalence**). *If the bidders' values are independent, then all distributionally robust individually rational and incentive compatible mechanisms with the same allocation rule $\boldsymbol{q}$, for which the ex-post utility of each bidder under his lowest value is 0, result in the same worst-case expected revenue for the seller.*

The revenue equivalence theorem naturally extends to all *indirect* mechanisms by virtue of the revelation principle (see Theorem 1). Note that the assertion (*ii*) of Proposition 2 ceases to hold if the bidders' values are dependent, even if the ambiguity set is a singleton, which implies that the revenue equivalence breaks down (Milgrom and Weber 1982).

2.2.   Second Price Auctions with Reserve Prices

The most widely used incentive compatible mechanisms are the second price auctions with reserve prices. From now on, we let $\mathcal{W}^i$, $i \in \mathcal{I}$, be any partition of $\mathcal{V}^I$ such that $\mathcal{W}^i$ contains scenarios $\boldsymbol{v}$ for which i is among the highest bidders. In scenarios where there are multiple highest bidders, an arbitrary tie-breaking rule may be applied (*e.g.*, the lexicographic tie-breaker assigns $\boldsymbol{v}$ to $\mathcal{W}^i$ if and only if $i = \min \arg\max_{j \in \mathcal{I}} v_j$).

DEFINITION 9 (SECOND PRICE AUCTION WITH RESERVE PRICE). A mechanism $(\boldsymbol{q}^{\mathrm{sp}}, \boldsymbol{m}^{\mathrm{sp}})$ is called a *second price auction with a reserve price r* if $\forall i \in \mathcal{I}$, $\forall \boldsymbol{v} \in \mathcal{V}^I$,

$$q_i^{\mathrm{sp}}(v_i, \boldsymbol{v}_{-i}) = \begin{cases} 1 & \text{if } \boldsymbol{v} \in \mathcal{W}^i \text{ and } v_i \geq r, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$m_i^{\mathrm{sp}}(v_i, \boldsymbol{v}_{-i}) = \begin{cases} \max\left\{ \max_{j \neq i} v_j,\, r \right\} & \text{if } q_i^{\mathrm{sp}}(v_i, \boldsymbol{v}_{-i}) = 1, \\ 0 & \text{otherwise}. \end{cases}$$

The allocation rule $\boldsymbol{q}^{\mathrm{sp}}$ depends on the tie-breaker that was used in the definition of the sets $\mathcal{W}^i$, $i \in \mathcal{I}$. Our subsequent results will not depend on the particular choice of the tie-breaker. Intuitively, in a second price auction with a reserve price, the good is allocated to the highest

bidder provided that his value exceeds the reserve price r, and the winner pays an amount equal to the maximum of the second highest bid and r. Second price auctions with reserve prices are known to be incentive compatible and individually rational at the ex-post stage (Krishna 2009, Section 2.2). Since ex-post individual rationality and incentive compatibility imply distributionally robust individual rationality and incentive compatibility (see Proposition 1 and the subsequent discussion), second price auctions with reserve prices are feasible in ($\mathcal{MDP}$).

A second price auction with a reserve price is not necessarily efficient. Indeed, the seller may keep the object for herself if the highest bid falls short of the reserve price. The Vickrey mechanism is an instance of an *efficient* second price auction.

DEFINITION 10 (VICKREY MECHANISM). The *Vickrey mechanism* is the second price auction $(\boldsymbol{q}^{\mathrm{v}}, \boldsymbol{m}^{\mathrm{v}})$ with reserve price $r = 0$.

Under the Vickrey mechanism, the highest bidder always receives the good (using any tie-breaking rule). As the Vickrey mechanism is a special instance of a second price auction with reserve price, it is ex-post individually rational and incentive compatible.

COROLLARY 1. *The Vickrey mechanism is ex-post individually rational and incentive compatible.*

Proof.   The claim follows from Section 2.2 in Krishna (2009).   □

3.   Robust Mechanism Design

Assume that the seller and the bidders only know the support $\mathcal{V}^I$ of the values that are possible but have no information about their probabilities. In this case, the ambiguity set reduces to $\mathcal{P} = \mathcal{P}_0(\mathcal{V}^I)$.

PROPOSITION 3. *If $\mathcal{P} = \mathcal{P}_0(\mathcal{V}^I)$, then the optimal mechanism design problem ($\mathcal{MDP}$) reduces to the robust optimization problem*

$$
\begin{aligned}
\sup_{\boldsymbol{q} \in \mathcal{Q}, \boldsymbol{m} \in \mathcal{M}} \quad & \inf_{\boldsymbol{v} \in \mathcal{V}^I} \sum_{i \in \mathcal{I}} m_i(v_i, \boldsymbol{v}_{-i}) \\
\text{s.t.} \quad & u_i(v_i; v_i, \boldsymbol{v}_{-i}) - u_i(w_i; v_i, \boldsymbol{v}_{-i}) \geq 0 \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I, \forall w_i \in \mathcal{V} \\
& u_i(v_i; v_i, \boldsymbol{v}_{-i}) \geq 0 \quad\quad\quad\quad\quad\quad\quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I.
\end{aligned}
\tag{$\mathcal{RMDP}$}
$$

The proof of Proposition 3 is elementary and therefore omitted. Note that under a support-only ambiguity set, distributionally robust individual rationality and incentive compatibility reduce to ex-post individual rationality and incentive compatibility, respectively.

Bandi and Bertsimas (2014, Section 3) show that the Vickrey mechanism solves ($\mathcal{RMDP}$). Thus, ($\mathcal{RMDP}$) admits an optimal mechanism that is efficient even though efficiency was not imposed. This is unusual because requiring efficiency generically reduces revenues (see, *e.g.*, Krishna 2009, Section 2.5). We formalize this result in the following theorem.

THEOREM 3 **(Bandi and Bertsimas (2014))**. *The Vickrey mechanism is optimal in* $(\mathcal{RMDP})$.

Bandi and Bertsimas (2014) show that Theorem 3 generalizes to any non-rectangular bounded support sets. However, the rectangularity assumption is essential for Propositions 4 and 5 below, which establish that the Vickrey mechanism is not only efficient but also happens to display two other useful properties that were not imposed in $(\mathcal{RMDP})$. First, we demonstrate that the Vickrey mechanism is Pareto robustly optimal with respect to the theory of Pareto optimality in robust optimization due to Iancu and Trichakis (2013).

DEFINITION 11 (PARETO ROBUST OPTIMALITY). An ex-post individually rational and incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is called *Pareto robustly optimal* if there exists no ex-post individually rational and incentive compatible mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ such that

$$\sum_{i \in \mathcal{I}} m_i'(\boldsymbol{v}) \geq \sum_{i \in \mathcal{I}} m_i(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{V}^I$$

and the above inequality is strict for at least one $\boldsymbol{v} \in \mathcal{V}^I$.

PROPOSITION 4. *The Vickrey mechanism is Pareto robustly optimal.*

The idea of the proof can be summarized as follows. If the Vickrey mechanism was not Pareto robustly optimal, then there would exist a mechanism $(\boldsymbol{q}, \boldsymbol{m})$ that generates higher revenues to the seller under some scenario $\boldsymbol{v}'$ while generating at least the same revenues under every other scenario. Note that the revenues generated by the Vickrey mechanism equal the second highest value under every scenario. At the same time, $(\boldsymbol{q}, \boldsymbol{m})$ cannot charge any bidder more than his value due to individual rationality. Thus, in order to generate higher revenues than the Vickrey auction in scenario $\boldsymbol{v}'$, the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ must allocate the item with strictly positive probability to the highest bidder i^1 and charge him more than the probability of him winning times the second highest value.

Given that scenario $\boldsymbol{v}'$ exists, we can construct another scenario $\boldsymbol{w}$ in which we reduce the value of i^1 (while ensuring that his bid remains the highest one) but keep all other values unchanged. In scenario $\boldsymbol{w}$, the probability of i^1 winning cannot have increased due to the assumed incentive compatibility of $(\boldsymbol{q}, \boldsymbol{m})$, see Proposition 2(i). Due to individual rationality, on the other hand, i^1's payment has to decrease in scenario $\boldsymbol{w}$. Thus, to match the revenues generated by the Vickrey auction, the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ has to assign the good to the second highest bidder i^2 with a strictly positive probability.

Finally, we construct a third scenario $\boldsymbol{w}'$ in which we increase the value of i^2 to the value of i^1. In scenario $\boldsymbol{w}'$, the probability of i^2 winning cannot have decreased due to Proposition 2(i). To match the revenues of the Vickrey auction, the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ has to charge bidder i^2 his probability
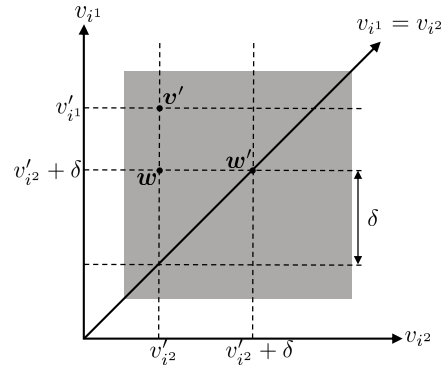
Figure 1   **Visualization of v', w and w'. The gray rectangle represents $\mathcal{V}^2$ whereas v'_{i^1} and v'_{i^2} are the values of the highest bidder i^1 and the second highest bidder i^2 in scenario v', respectively.**

of winning the good times his value. Thus, the utility of bidder i^2 is 0 in scenario w'. Since i^2 would have also won the good with a positive probability when reporting w_{i^2} (while being charged less due to individual rationality), however, he receives a strictly positive utility when he misreports his value. Since this contradicts incentive compatibility, we conclude that no such mechanism (q, m) can exist. Figure 1 visualizes the scenarios v', w and w'.

Pareto optimality is important in classical robust optimization because there are typically multiple optimal solutions. Too see this, consider the mechanism that allocates the good to the first bidder with probability 1 and that charges this bidder $\underline{v}$. One can show that this mechanism is ex-post individually rational and incentive compatible. Moreover, it generates the same worst-case revenue $\underline{v}$ for the seller as the Vickrey mechanism and is thus optimal in $(\mathcal{RMDP})$. However, the Vickrey mechanism generates weakly higher revenues in *every* fixed scenario $v \in \mathcal{V}^I$, and strictly higher revenues in every scenario in which the second highest bid exceeds $\underline{v}$.

The Vickrey mechanism also displays a powerful Pareto dominance property among all *efficient* ex-post individually rational and incentive compatible mechanisms.

PROPOSITION 5. *Among all efficient ex-post individually rational and incentive compatible mechanisms, the Vickrey mechanism generates the highest revenues in every fixed scenario $v \in \mathcal{V}^I$.*

Proposition 5 provides a stronger result than Pareto robust optimality. It shows that among all efficient solutions to $(\mathcal{RMDP})$ the Vickrey mechanism offers the highest revenues in *every* fixed scenario. One can show that the Pareto dominance property of Proposition 5 ceases to hold if we compare the Vickrey mechanism against every (not necessarily efficient) ex-post individually rational and incentive compatible mechanism.

Theorem 3, Proposition 4 and Proposition 5 imply the following corollary.

COROLLARY 2. *The Vickrey mechanism is the unique mechanism that is optimal, efficient and Pareto robustly optimal.*

4. Mechanism Design under Independent Values

Throughout this section, we assume that the bidders' values are independent in the sense of Definition 8. Some results of this section will further require that each $\mathbb{P} \in \mathcal{P}$ is symmetric and regular.

DEFINITION 12 (SYMMETRY). A distribution $\mathbb{P} \in \mathcal{P}_0(\mathcal{V}^I)$ is called *symmetric* if the random variables $\tilde{v}_i$, $i \in \mathcal{I}$, share the same marginal distribution under $\mathbb{P}$.

DEFINITION 13 (REGULARITY). A distribution $\mathbb{P} \in \mathcal{P}_0(\mathcal{V}^I)$ is called *regular* if the marginal density $\rho_i^{\mathbb{P}}(v_i)$ of $\tilde{v}_i$ under $\mathbb{P}$ exists and is strictly positive for all $v_i \in \mathcal{V}$, while the *virtual valuation*

$$\psi_i^{\mathbb{P}}(v_i) = v_i - \frac{1 - \int_{\underline{v}}^{v_i} \rho_i^{\mathbb{P}}(x)\,\mathrm{d}x}{\rho_i^{\mathbb{P}}(v_i)}$$

is non-decreasing in v_i for all $i \in \mathcal{I}$.

Independence, symmetry and regularity are standard assumptions of the benchmark model for auctions as defined by McAfee and McMillan (1987). The virtual valuations can be interpreted as marginal revenues (see Krishna 2009, Section 5.2.3). They were first introduced by Myerson (1981) in order to solve the optimal mechanism design problem in the absence of ambiguity. Note that a sufficient condition for regularity is that the hazard function $\frac{\rho_i^{\mathbb{P}}(v_i)}{1 - \int_{\underline{v}}^{v_i} \rho_i^{\mathbb{P}}(x)\,\mathrm{d}x}$ is non-decreasing in v_i.

PROPOSITION 6. *If the bidders' values are independent, then the optimal mechanism design problem* ($\mathcal{MDP}$) *reduces to*

$$
\begin{aligned}
\sup_{\boldsymbol{q} \in \mathcal{Q}^{\mathrm{m\text{-}d}}, \boldsymbol{m} \in \mathcal{M}} \quad & \inf_{\mathbb{P} \in \mathcal{P}} \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbb{P}}\Big[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) - \int_{\underline{v}}^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x \Big] \\
\text{s.t.} \quad & \mathbb{E}_{\mathbb{P}}\left[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \right] \\
& = \mathbb{E}_{\mathbb{P}}\Big[u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) + \int_{\underline{v}}^{v_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x \Big] \quad \forall i \in \mathcal{I}, \forall v_i \in \mathcal{V}, \forall \mathbb{P} \in \mathcal{P} \\
& \mathbb{E}_{\mathbb{P}}\left[u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) \right] \geq 0 \quad \forall i \in \mathcal{I}, \forall \mathbb{P} \in \mathcal{P}.
\end{aligned}
\tag{$\mathcal{IMDP}$}
$$

Proof. By Proposition 2(*ii*), distributionally robust incentive compatibility is equivalent to the first constraint of ($\mathcal{IMDP}$) and the requirement that $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{m\text{-}d}}$. The first constraint in ($\mathcal{IMDP}$) then implies that distributionally robust individual rationality simplifies to

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{P}}\left[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \right] = \mathbb{E}_{\mathbb{P}}\Big[u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) + \int_{\underline{v}}^{v_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x \Big] \geq 0 \quad \forall i \in \mathcal{I}, \forall v_i \in \mathcal{V}, \forall \mathbb{P} \in \mathcal{P} \\
& \iff \mathbb{E}_{\mathbb{P}}\left[u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) \right] \geq 0 \qquad\qquad\qquad\qquad\qquad\quad \forall i \in \mathcal{I}, \forall \mathbb{P} \in \mathcal{P},
\end{aligned}
$$

where the equivalence holds because the integral in the first line is always non-negative.

To see that the objective function of ($\mathcal{MDP}$) reduces to the objective function of ($\mathcal{IMDP}$), we proceed as in the proof of Theorem 2. Details are omitted for brevity. $\square$

We now show that the Vickrey mechanism is the best *efficient* mechanism in ($\mathcal{IMDP}$).

THEOREM 4. *The Vickrey mechanism generates the highest worst-case expected revenue in* $(\mathcal{IMDP})$ *among all efficient mechanisms.*

Proof. By Corollary 1, the Vickrey mechanism is ex-post individually rational and incentive compatible. Hence, by Proposition 1, it is also distributionally robust individually rational and incentive compatible. We conclude that the Vickrey mechanism is feasible in $(\mathcal{MDP})$ and thus, by Proposition 6, in $(\mathcal{IMDP})$.

We now show that the ex-post utility of each bidder i with value $\underline{v}$ vanishes under the Vickrey mechanism. If bidder i with value $\underline{v}$ does not win the good, then he does not have to make a payment, and his ex-post utility is zero. Otherwise, if he wins the good, then we have

$$u_i(\underline{v}, \boldsymbol{v}_{-i}) = q_i(\underline{v}, \boldsymbol{v}_{-i})\underline{v} - m_i(\underline{v}, \boldsymbol{v}_{-i}) = \underline{v} - \max_{j \neq i} v_j = \underline{v} - \underline{v} = 0,$$

where the third equality holds because $\underline{v} \leq \max_{j \neq i} v_j \leq v_i = \underline{v}$. Thus, the ex-post utility of each bidder i with value $\underline{v}$ is always zero.

Next, we show that the Vickrey mechanism generates a weakly higher worst-case expected revenue than any other efficient mechanism $(\boldsymbol{q}', \boldsymbol{m}') \in \mathcal{Q}^{\mathrm{eff}} \times \mathcal{M}$ that is feasible in $(\mathcal{IMDP})$. Indeed, as the ex-post utility of each bidder i with value $\underline{v}$ vanishes under the Vickrey mechanism, the objective value of the Vickrey mechanism in $(\mathcal{IMDP})$ satisfies

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i \in \mathcal{I}} q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i}) \, \mathrm{d}x\Big] = \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i \in \mathcal{I}} q_i'(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i'(x, \tilde{\boldsymbol{v}}_{-i}) \, \mathrm{d}x\Big]$$

$$\geq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i \in \mathcal{I}} q_i'(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i'(x, \tilde{\boldsymbol{v}}_{-i}) \, \mathrm{d}x\Big] - \mathbb{E}_{\mathbb{P}}\Big[\sum_{i \in \mathcal{I}} q_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i})\underline{v} - m_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i})\Big].$$

Here, the first equality follows from efficiency, which implies that $\sum_{i \in \mathcal{I}} q_i(\boldsymbol{v}) = \sum_{i \in \mathcal{I}} q_i'(\boldsymbol{v}) = 1$ and that $\sum_{i \in \mathcal{I}} q_i(\boldsymbol{v})v_i = \sum_{i \in \mathcal{I}} q_i'(\boldsymbol{v})v_i = \max_{i \in \mathcal{I}} v_i$, while the inequality is due to the second constraint of $(\mathcal{IMDP})$. The claim then follows because the last line of the above expression represents the objective function value of $(\boldsymbol{q}', \boldsymbol{m}')$ in $(\mathcal{IMDP})$. $\square$

In order to examine the properties of the optimal (not necessarily efficient) mechanism, we reformulate problem $(\mathcal{IMDP})$ in terms of the virtual valuations introduced in Definition 13. The following proposition extends Lemma 3 by Myerson (1981) to ambiguous value distributions. Its proof is relegated to the appendix.

PROPOSITION 7. *If the bidders' values are independent and each $\mathbb{P} \in \mathcal{P}$ is regular, then problem* $(\mathcal{IMDP})$ *is equivalent to*

$$\sup_{\boldsymbol{q} \in \mathcal{Q}^{\mathrm{m\text{-}d}}} \quad \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\Big[\sum_{i \in \mathcal{I}} \psi_i^{\mathbb{P}}(\tilde{v}_i) q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\Big]. \tag{3}$$

We now review a celebrated result by Myerson (1981), which asserts that in the absence of ambiguity ($\mathcal{P} = \{\mathbb{P}\}$), a second price auction with a reserve price is optimal if the bidders' values are independent and the distribution $\mathbb{P}$ is symmetric and regular.

THEOREM 5 (**Myerson (1981)**). *If $\mathcal{P} = \{\mathbb{P}\}$, the bidders' values are independent, and the distribution $\mathbb{P}$ is symmetric and regular, then the allocation rule*

$$q_i^\star(\boldsymbol{v}) = \begin{cases} 1 & \text{if } \boldsymbol{v} \in \mathcal{W}^i \text{ and } \psi_1^{\mathbb{P}}(v_i) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in \mathcal{I}$ and $\boldsymbol{v} \in \mathcal{V}^I$, is optimal in (3) *and generates expected revenues of*

$$\mathbb{E}_{\mathbb{P}}\Big[\max\Big\{ \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}}(\tilde{v}_i), 0 \Big\} \Big]. \tag{4}$$

The allocation rule $\boldsymbol{q}^\star$ can be used to construct a payment rule $\boldsymbol{m}^\star$ defined through

$$m_i^\star(v_i, \boldsymbol{v}_{-i}) = q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - \int_{\underline{v}}^{v_i} q_i^\star(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I.$$

One can show that the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is optimal in $(\mathcal{IMDP})$ when $\mathcal{P}$ is a singleton.

Note that if the virtual valuation $\psi_1^{\mathbb{P}}$ is continuous, then the optimal mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is the second price auction with reserve price $r = \inf\{v_1 \in \mathcal{V} : \psi_1^{\mathbb{P}}(v_1) \geq 0\}$. Note also that this mechanism can be inefficient because $\psi_1^{\mathbb{P}}(v_1)$ can be negative.

Koçyiğit et al. (2018) show that second price auctions with reserve prices are generally suboptimal as soon as $\mathcal{P}$ contains two distributions, even if the bidders' values are independent and each $\mathbb{P} \in \mathcal{P}$ is symmetric and regular. Unfortunately, we are unable to solve problem (3) analytically unless $\mathcal{P}$ is a singleton. Even though the optimal mechanism remains elusive, Dhangwatnotai et al. (2015) have identified a mechanism, called the modified Single Sample mechanism, which is guaranteed to generate at least half of the expected revenue of the optimal mechanism under every distribution $\mathbb{P} \in \mathcal{P}$. This mechanism can be viewed as a second price auction with a random reserve price, and the corresponding constant-factor approximation guarantee critically relies on the independence of the bidder values. As we will argue below, a simple second price auction without reserve price also offers compelling optimality guarantees, which suggest that the added value of the unknown optimal mechanism is negligible.

In the absence of ambiguity, Bulow and Klemperer (1996) demonstrate that the second price auction without reserve price for $I + 1$ bidders yields higher expected revenues than the optimal auction for I bidders. In the non-ambiguous case, the optimal mechanism for I bidders is known to be a second price auction with a reserve price (see Theorem 5). The following theorem generalizes the result by Bulow and Klemperer (1996) to mechanism design problems under ambiguity even though the optimal mechanism remains unknown in this setting.

THEOREM 6. *Assume that the bidders' values are independent and that each distribution in the ambiguity set is symmetric and regular. Then, a second price auction without reserve price for $I+1$ bidders yields a weakly higher worst-case expected revenue than an optimal auction for I bidders.*

Theorem 6 shows that the added value of the optimal mechanism over a simple second price auction without reserve price is offset by just attracting one additional bidder. Theorem 6 critically relies on the independence of the bidders' values, which facilitates the reformulation (3). In the next section, we will investigate the mechanism design problem under moment ambiguity sets where the bidders' values may be correlated. In this case, the added value of the optimal mechanism over even the best second price auction can be significant.

5. Mechanism Design under Moment Information

While commonly employed in the mechanism design literature, the assumption of independent bidder values can be restrictive in practice, where bidders may interact with one another or share common information sources. This motivates us to investigate settings where the bidders' values can be dependent. Specifically, we assume that the agents have information about some (generalized) moments of the value distribution. We thus consider moment ambiguity sets of the form

$$\mathcal{P} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}_+^I) : \mathbb{P}(\tilde{\boldsymbol{v}} \in \mathcal{V}^I) = 1, \, \mathbb{E}_{\mathbb{P}}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] \geq \boldsymbol{\mu} \right\}, \tag{5}$$

where $\boldsymbol{h} = (h_1, \ldots, h_J)$ represents a vector of generalized moment functions $h_j : \mathcal{V}^I \to \mathbb{R}$, and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_J)$ denotes a vector of given moment bounds $\mu_j \in \mathbb{R}$. The following non-restrictive technical condition will be assumed to hold throughout this section.

ASSUMPTION 1 (**Slater Condition**). *There exists a Slater point $\mathbb{P}_s \in \mathcal{P}$ with $\mathbb{E}_{\mathbb{P}_s}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] > \boldsymbol{\mu}$.*

The following proposition shows that if $\mathcal{P}$ is of the form (5), the bidders will require ex-post individual rationality and incentive compatibility.

PROPOSITION 8. *If $\mathcal{P}$ is a moment ambiguity set of the form (5) and Assumption 1 holds, then distributionally robust individual rationality and incentive compatibility simplify to ex-post individual rationality and incentive compatibility, respectively.*

Proof. Select an arbitrary bidder $i \in \mathcal{I}$ with value $v_i \in \mathcal{V}$, and note that the inequality $\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \,|\, \tilde{v}_i = v_i] \geq \inf_{\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}} u_i(v_i, \boldsymbol{v}_{-i})$ is trivially satisfied. To establish the converse inequality, we use Assumption 1, whereby there exists $\mathbb{P}_s \in \mathcal{P}$ with $\mathbb{E}_{\mathbb{P}_s}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] > \boldsymbol{\mu}$. By the Richter-Rogosinski theorem, we can assume without loss of generality that $\mathbb{P}_s$ is discrete and representable as

$$\mathbb{P}_s = \sum_{j=1}^{J+1} p_j \delta_{\boldsymbol{v}^{(j)}} \text{ with } \sum_{j=1}^{J+1} p_j = 1, \, p_j \geq 0 \text{ and } \boldsymbol{v}^{(j)} \in \mathcal{V}^I \,\, \forall j = 1, \ldots, J+1,$$

where $\delta_{\boldsymbol{v}^{(j)}}$ denotes the Dirac point mass at $\boldsymbol{v}^{(j)}$, see Theorem 7.23 in Shapiro et al. (2014). Moreover, by a standard perturbation argument, we can assume without loss of generality that $v_i^{(j)} \neq v_i$ for all $j = 1, \ldots, J+1$.

Select now an arbitrary $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$ and set $\boldsymbol{v} = (v_i, \boldsymbol{v}_{-i})$ as usual. As $\mathbb{E}_{\mathbb{P}_{\mathrm{s}}}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] > \boldsymbol{\mu}$, there exists $\lambda \in (0,1)$ small enough such that the distribution $\mathbb{P}_{\boldsymbol{v}_{-i}} = \lambda \delta_{\boldsymbol{v}} + (1-\lambda)\mathbb{P}_{\mathrm{s}}$ satisfies

$$\mathbb{E}_{\mathbb{P}_{\boldsymbol{v}_{-i}}}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] = \lambda \boldsymbol{h}(\boldsymbol{v}) + (1-\lambda)\mathbb{E}_{\mathbb{P}_{\mathrm{s}}}[\boldsymbol{h}(\tilde{\boldsymbol{v}})] \geq \boldsymbol{\mu}.$$

Hence, $\mathbb{P}_{\boldsymbol{v}_{-i}} \in \mathcal{P}$. This implies that

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \leq \mathbb{E}_{\mathbb{P}_{\boldsymbol{v}_{-i}}}\left[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] = u_i(v_i, \boldsymbol{v}_{-i}),$$

where the inequality holds because $\mathbb{P}_{\boldsymbol{v}_{-i}} \in \mathcal{P}$ and the equality holds due to construction of $\mathbb{P}_{\boldsymbol{v}_{-i}}$. Since $\boldsymbol{v}_{-i}$ was chosen arbitrarily, the above inequality holds for all $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$, that is, $\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[u_i(v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i] \leq \inf_{\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}} u_i(v_i, \boldsymbol{v}_{-i})$. Thus, distributionally robust individual rationality simplifies to ex-post individual rationality.

Using similar arguments, one can also prove the assertion about incentive compatibility. Details are omitted for brevity. $\square$

We now use the above proposition to simplify problem $(\mathcal{MDP})$.

PROPOSITION 9. *If $\mathcal{P}$ is a moment ambiguity set of the form* (5) *and Assumption 1 holds, then the optimal mechanism design problem* $(\mathcal{MDP})$ *reduces to*

$$\sup_{\boldsymbol{q} \in \mathcal{Q}^{\mathrm{m\text{-}p}}, \boldsymbol{m} \in \mathcal{M}} \quad \inf_{\mathbb{P} \in \mathcal{P}} \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbb{P}}\left[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - u_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) - \int_{\underline{v}}^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x\right]$$

$$\text{s.t.} \quad u_i(v_i, \boldsymbol{v}_{-i}) = u_i(\underline{v}, \boldsymbol{v}_{-i}) + \int_{\underline{v}}^{v_i} q_i(x, \boldsymbol{v}_{-i})\,\mathrm{d}x \qquad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I \qquad (\mathcal{MMDP})$$

$$u_i(\underline{v}, \boldsymbol{v}_{-i}) \geq 0 \qquad \forall i \in \mathcal{I}, \forall \boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}.$$

Proof. By Proposition 8, distributionally robust individual rationality and incentive compatibility simplify to ex-post individual rationality and incentive compatibility, respectively. By Proposition 2 (i), ex-post incentive compatibility is equivalent to the first constraint of $(\mathcal{MMDP})$ and the requirement that $\boldsymbol{q} \in \mathcal{Q}^{\mathrm{m\text{-}p}}$. The reformulation of the objective function immediately follows from the first constraint in $(\mathcal{MMDP})$. This constraint also implies that ex-post individual rationality simplifies to

$$u_i(v_i, \boldsymbol{v}_{-i}) = u_i(\underline{v}, \boldsymbol{v}_{-i}) + \int_{\underline{v}}^{v_i} q_i(x, \boldsymbol{v}_{-i}) \geq 0 \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I$$

$$\Longleftrightarrow \quad u_i(\underline{v}, \boldsymbol{v}_{-i}) \geq 0 \qquad\qquad\qquad \forall i \in \mathcal{I}, \forall \boldsymbol{v}_{-i} \in \mathcal{V}^{I-1},$$

where the equivalence holds because the integral in the first line is always non-negative. $\square$

THEOREM 7. *The Vickrey mechanism generates the highest worst-case expected revenue in* $(\mathcal{MMDP})$ *among all efficient mechanisms.*

Proof. The proof is immediate from Proposition 5 and Proposition 8. □

5.1. Markov Ambiguity Sets

If the efficiency condition is relaxed, then the Vickrey mechanism is suboptimal for generic moment ambiguity sets. To show this, we will henceforth focus on Markov ambiguity sets with first-order moment information of the form

$$\mathcal{P} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}_+^I) : \mathbb{P}(\tilde{\boldsymbol{v}} \in [0,1]^I) = 1, \, \mathbb{E}_{\mathbb{P}}[\tilde{v}_i] \geq \mu, \, \forall i \in \mathcal{I} \right\}, \tag{6}$$

where $\mu \in [0,1]$. The Markov ambiguity set stipulates that the bidder values range over the unit interval $[0,1]$ and are not smaller than μ in expectation. Markov ambiguity sets are intuitively appealing as they only require the specification of the smallest, the highest and the most likely bidder value for the good. As the seller's revenue is non-decreasing in the bidder values, we could actually require $\mathbb{E}_{\mathbb{P}}[\tilde{v}_i] = \mu$, $i \in \mathcal{I}$, without affecting the objective function of problem $(\mathcal{MMDP})$. We prefer to work with inequality constraints, however, to ensure that $\mathcal{P}$ admits a Slater point. Note also that, although the description of the Markov ambiguity set (6) is permutation symmetric, it contains distributions that are not symmetric.

It is instructive to investigate what would happen if the seller knew the bidder values from the outset. In this case, the seller's optimal strategy would be to give the good to the highest bidder and to charge him an amount equal to his value. In this manner, the seller could both maximize and appropriate the total social welfare. In other words, the seller could extract full surplus.

DEFINITION 14 (WORST-CASE EXPECTED FULL SURPLUS). The *worst-case expected full surplus* corresponding to an ambiguity set $\mathcal{P}$ is $\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[\max_{i \in \mathcal{I}} \tilde{v}_i]$.

The worst-case expected full surplus clearly provides an upper bound on the worst-case expected revenue the seller can obtain by implementing any ex-post individually rational and incentive compatible mechanism. This is because, by ex-post individual rationality, the seller cannot charge the winner more than his value.

PROPOSITION 10. *If $\mathcal{P}$ is a Markov ambiguity set of the form* (6), *then the worst-case expected full surplus is equal to μ.*

Proof. Let $\delta_{\mu\mathbf{e}}$ be the Dirac point mass at $\mu\mathbf{e}$. Since $\delta_{\mu\mathbf{e}} \in \mathcal{P}$, we have

$$\mu = \mathbb{E}_{\delta_{\mu\mathbf{e}}}\left[\max_{i \in \mathcal{I}} \tilde{v}_i\right] \geq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[\max_{i \in \mathcal{I}} \tilde{v}_i\right] \geq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[\tilde{v}_1] = \mu,$$

and thus the claim follows. □

If $\mu = 1$, then the Markov ambiguity set is a singleton that contains only the Dirac distribution at $\boldsymbol{v} = \mathbf{e}$. In this case, the seller can implement a second price auction without reserve price to extract the worst-case full surplus μ. On the other hand, if $\mu = 0$, then the Dirac distribution at $\boldsymbol{v} = \mathbf{0}$ is contained in the Markov ambiguity set. In this case, the worst-case expected revenue is 0 independent of the mechanism implemented. To exclude these trivial special cases, we will henceforth assume that $\mu \in (0, 1)$.

We now offer two equivalent reformulations of the worst-case expected revenues in $(\mathcal{MMDP})$ when $\mathcal{P}$ is a Markov ambiguity set.

PROPOSITION 11. *If $\mathcal{P}$ is a Markov ambiguity set of the form* (6) *and $\mu \in (0,1)$, then the objective function value of a fixed allocation rule $\boldsymbol{q} \in \mathcal{Q}^{\text{m-p}}$ in $(\mathcal{MMDP})$ coincides with the (equal) optimal values of the primal and dual semi-infinite linear programs*

$$\inf_{\mathbb{P} \in \mathcal{P}} \sum_{i \in \mathcal{I}} \int_{[0,1]^I} \left[q_i(v_i, \boldsymbol{v}_{-i}) v_i - \int_0^{v_i} q_i(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \right] \mathrm{d}\mathbb{P}(\boldsymbol{v}) \tag{7}$$

and

$$\sup_{\boldsymbol{\sigma} \in \mathbb{R}_+^I, \lambda \in \mathbb{R}} \quad \lambda + \sum_{i \in \mathcal{I}} \sigma_i \mu$$
$$\text{s.t.} \quad \sum_{i \in \mathcal{I}} \left[q_i(v_i, \boldsymbol{v}_{-i}) v_i - \int_0^{v_i} q_i(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \right] \geq \lambda + \sum_{i \in \mathcal{I}} \sigma_i v_i \quad \forall \boldsymbol{v} \in [0,1]^I. \tag{8}$$

Proof. Note that $u_i(\underline{v}, \boldsymbol{v}_{-i}) = 0$ for all $i \in \mathcal{I}$, $\boldsymbol{v}_{-i} \in [0,1]^{I-1}$ because $\underline{v} = 0$. Hence, the objective value of $\boldsymbol{q} \in \mathcal{Q}^{\text{m-p}}$ in $(\mathcal{MMDP})$ is equal to (7).

By the definition of the Markov ambiguity set in (6), problem (7) can be represented as the generalized moment problem

$$\inf_{\mathbb{P} \in \mathcal{P}_0(\mathbb{R}_+^I)} \quad \sum_{i \in \mathcal{I}} \int_{[0,1]^I} \left[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i}) \tilde{v}_i - \int_0^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i}) \, \mathrm{d}x \right] \mathrm{d}\mathbb{P}(\boldsymbol{v})$$
$$\text{s.t.} \quad \int_{[0,1]^I} \mathrm{d}\mathbb{P}(\boldsymbol{v}) = 1$$
$$\int_{[0,1]^I} v_i \, \mathrm{d}\mathbb{P}(\boldsymbol{v}) \geq \mu \quad \forall i \in \mathcal{I}.$$

The Lagrangian dual of this moment problem is given by the semi-infinite linear program (8). Strong duality holds due to Proposition 3.4 in Shapiro (2001), which is applicable because $\mu \in (0,1)$. Hence, problems (7) and (8) share the same optimal value. $\square$

Note that, by Proposition 11, two equivalent reformulations of problem $(\mathcal{MMDP})$ are obtained by maximizing (7) or (8) over $\boldsymbol{q} \in \mathcal{Q}^{\text{m-p}}$. Solving either of these problems yields an optimal allocation rule. The corresponding optimal payment rule can then be recovered from the first constraint in $(\mathcal{MMDP})$.

5.2. The Optimal Second Price Auction with Reserve Price

Consider problem ($\mathcal{MMDP}$) with a Markov ambiguity set of the form (6) with $\mu \in (0,1)$, and assume for now that the seller aims to find the best second price auction $(\boldsymbol{q}^{\mathrm{sp}}, \boldsymbol{m}^{\mathrm{sp}})$ with reserve price $r \in [0,1]$. Recall that all second price auctions with reserve prices are indeed ex-post individually rational and incentive compatible and thus feasible in ($\mathcal{MMDP}$), see Section 2.2 in Krishna (2009). As $\underline{v} = 0$ for all $i \in \mathcal{I}$, it follows from Proposition 2(i) that

$$m_i^{\mathrm{sp}}(v_i, \boldsymbol{v}_{-i}) = q_i^{\mathrm{sp}}(v_i, \boldsymbol{v}_{-i}) v_i - \int_0^{v_i} q_i^{\mathrm{sp}}(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \quad \forall i \in \mathcal{I}, \, \forall \boldsymbol{v} \in [0,1]^I. \tag{9}$$

The correctness of (9) can also be checked directly. Imagine that bidder i wins the good in scenario $\boldsymbol{v}$. Thus, the first term on the right-hand side of (9) reduces to v_i. As $q_i^{\mathrm{sp}}(x, \boldsymbol{v}_{-i}) = 1$ only if $x \geq \max\{\max_{j \neq i} v_j, r\}$ and $q_i(x, \boldsymbol{v}_{-i}) = 0$ whenever $x < \max\{\max_{j \neq i} v_j, r\}$, the integral in (9) evaluates to the difference between v_i and $\max\{\max_{j \neq i} v_j, r\}$. As expected, the payment of bidder i is therefore equal to the maximum of the second highest value and the reserve price.

We now calculate the worst-case expected revenue generated by a fixed second price auction $(\boldsymbol{q}^{\mathrm{sp}}, \boldsymbol{m}^{\mathrm{sp}})$ with reserve price $r \in [0,1]$, which coincides with the (equal) optimal values of the problems (7) and (8) for $\boldsymbol{q} = \boldsymbol{q}^{\mathrm{sp}}$ (see Proposition 11).

Assume first that $r > \mu$. In this case, the worst-case expected revenue is 0, which is attained by the Dirac distribution at $\boldsymbol{v} = \mu \mathbf{e}$. Therefore, the seller will only consider reserve prices $r \leq \mu$. The subsequent discussion is based on the following partition of the interval $[0, \mu]$ of all reasonable candidate reserve prices.

$$\mathcal{R}_1 = \left\{ r \in \mathbb{R}_+ : \min\left\{\frac{1}{I}, \frac{I\mu - 1}{I - 1}\right\} \geq r \right\}$$

$$\mathcal{R}_2 = \left\{ r \in \mathbb{R}_+ : \min\left\{\mu, \frac{1}{I}\right\} \geq r > \frac{I\mu - 1}{I - 1} \right\}$$

$$\mathcal{R}_3 = \left\{ r \in \mathbb{R}_+ : \mu \geq r > \frac{1}{I} \right\}$$

One can verify that

$$\mu \geq \frac{I\mu - 1}{I - 1} \quad \forall \mu \in (0,1), \, I \in \mathbb{N}, \tag{10}$$

which ensures that $\mathcal{R}_1 \subseteq [0, \mu]$ as desired. Later in this section, we will show that $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ indeed form a partition of the interval $[0, \mu]$.

We now show that the structure of the worst-case distribution in problem (7) depends on whether the reserve price belongs to $\mathcal{R}_1$, $\mathcal{R}_2$ or $\mathcal{R}_3$. To this end, consider the semi-infinite constraint in the dual problem (8). By equation (9), the left-hand side of this constraint quantifies the total revenue in scenario $\boldsymbol{v}$. On the other hand, the right-hand side represents a linear function of $\boldsymbol{v}$. The objective function of (8) tries to push this linear function upwards. At optimality, the linear function touches the total revenue function at a finite number of points in $\mathcal{V}^I$. By complementary

(a) $r \in \mathcal{R}_1$ (b) $r \in \mathcal{R}_2$ (c) $r \in \mathcal{R}_3$
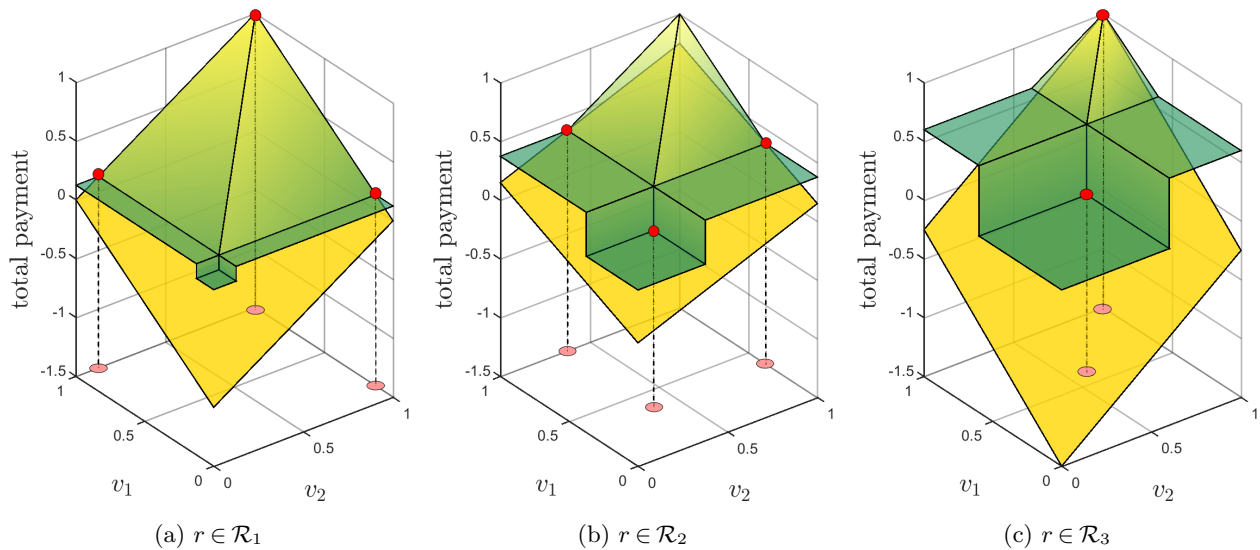
Figure 2   **Complementary slackness between the distribution $\mathbb{P}$ in (7) and the semi-infinite constraint in (8) for two bidders and $\mu = \frac{5}{8}$. The green (dark shaded) area and the yellow (light shaded) area represent the left-hand side and right-hand side values of the semi-infinite constraint in (8), respectively. The atoms of the distribution $\mathbb{P}$ in (7) are visualized by the red dots.**

slackness, the support of the worst-case distribution that solves (7), if it exists, is confined to these discrete points. The following propositions provide explicit formulas for these extremal distributions and the corresponding worst-case expected revenues.

PROPOSITION 12. *If $\mathcal{P}$ is a Markov ambiguity set of the form (6) and $\mu \in (0,1)$, then the worst-case expected revenue of a second price auction with reserve price $r \in \mathcal{R}_1$ amounts to*

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \Big[\sum_{i \in \mathcal{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}}) \Big] = \frac{I\mu - 1}{I - 1}$$

and is attained by the extremal distribution

$$\mathbb{Q}^{(1)} = \left(1 - \frac{I(\mu - 1)}{(I-1)(r-1)} \right) \delta_{\mathbf{e}} + \sum_{i \in \mathcal{I}} \frac{\mu - 1}{(I-1)(r-1)} \, \delta_{\mathbf{e}_i + r \mathbf{e}_{-i}}.$$

The atoms of the distribution $\mathbb{Q}^{(1)}$ are visualized by the red dots in Figure 2a. Note that the worst-case expected revenue is independent of the reserve price r as long as $r \in \mathcal{R}_1$. This independence emerges because two opposite effects offset each other: When r increases, the probability of the scenario $\boldsymbol{v} = \mathbf{e}$, in which the seller earns the highest payments, decreases so that the expected value of $\tilde{v}_i$ is preserved at μ. At the same time, the payments in all other scenarios increase due to the change in r.

PROPOSITION 13. *If $\mathcal{P}$ is a Markov ambiguity set of the form (6) and $\mu \in (0,1)$, then the worst-case expected revenue of a second price auction with reserve price $r \in \mathcal{R}_2$ is equal to*

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \Big[\sum_{i \in \mathcal{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}}) \Big] = Ir \frac{\mu - r}{1 - r},$$

which is attained asymptotically by the sequence of distributions

$$\mathbb{Q}_\epsilon^{(2)} = \left(1 - \frac{I(\mu - (r - \epsilon))}{1 - (r - \epsilon)}\right)\delta_{(r-\epsilon)\mathbf{e}} + \sum_{i\in\mathcal{I}} \frac{\mu - (r - \epsilon)}{1 - (r - \epsilon)}\,\delta_{\mathbf{e}_i + (r-\epsilon)\mathbf{e}_{-i}}$$

for $\epsilon \downarrow 0$.

The atoms of the distribution $\mathbb{Q}_\epsilon^{(2)}$ (for ϵ close to 0) are visualized by the red dots in Figure 2b. Note that the probabilities assigned to the scenarios $\mathbf{e}_i + (r - \epsilon)\mathbf{e}_{-i}$, $i \in \mathcal{I}$, are independent of the number of bidders. These scenarios each contribute an ex-post revenue of r. This explains why the worst-case expected revenue increases linearly in the number of bidders.

PROPOSITION 14. *If $\mathcal{P}$ is a Markov ambiguity set of the form (6) and $\mu \in (0,1)$, then the worst-case expected revenue of a second price auction with reserve price $r \in \mathcal{R}_3$ amounts to*

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}\Big[\sum_{i\in\mathcal{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\Big] = \frac{\mu - r}{1 - r},$$

which is attained asymptotically by the sequence of distributions

$$\mathbb{Q}_\epsilon^{(3)} = \left(1 - \frac{1 - \mu}{1 - (r - \epsilon)}\right)\delta_{\mathbf{e}} + \frac{1 - \mu}{1 - (r - \epsilon)}\,\delta_{(r-\epsilon)\mathbf{e}}$$

for $\epsilon \downarrow 0$.

The atoms of the distribution $\mathbb{Q}_\epsilon^{(3)}$ (for ϵ close to 0) are indicated by the red dots in Figure 2c. In this case, the worst-case expected revenue does not depend on the number of bidders because $\mathbb{Q}_\epsilon^{(3)}$ is itself independent of the number of bidders.

We are now ready to determine the optimal reserve price as a function of μ and the number of bidders I. Recall from (10) that μ is always larger than or equal to $\frac{I\mu-1}{I-1}$. However, $\frac{1}{I}$ can be larger than μ, between $\frac{I\mu-1}{I-1}$ and μ or smaller than $\frac{I\mu-1}{I-1}$. As $\frac{1}{I}$ is greater (smaller) than or equal to $\frac{I\mu-1}{I-1}$ if and only if $\frac{2I-1}{I^2}$ is greater (smaller) than or equal to μ, the interval $(0,1)$ of possible mean values μ can be partitioned into the following disjoint subsets.

$$\mathcal{M}_1 = \left\{\mu\in\mathbb{R}_+ : 0 < \mu \le \frac{1}{I}\right\}$$
$$\mathcal{M}_2 = \left\{\mu\in\mathbb{R}_+ : \frac{1}{I} < \mu \le \frac{2I-1}{I^2}\right\}$$
$$\mathcal{M}_3 = \left\{\mu\in\mathbb{R}_+ : \frac{2I-1}{I^2} < \mu < 1\right\}$$

The intervals $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, and their relations to $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are visualized in Figure 3. If $\mu \in \mathcal{M}_1$, then $\frac{I\mu-1}{I-1}$ is non-positive. Hence, $\mathcal{R}_1$ is empty unless $\mu = \frac{1}{I}$, in which case we have $\mathcal{R}_1 = \{0\}$. Moreover, $\mathcal{R}_2$ is nonempty and $\mathcal{R}_3$ is empty. If $\mu \in \mathcal{M}_2$, then $\frac{I\mu-1}{I-1} \le \frac{1}{I}$, which implies that $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are all nonempty. Finally, if $\mu \in \mathcal{M}_3$, then $\frac{I\mu-1}{I-1} > \frac{1}{I}$, in which case $\mathcal{R}_2$ is empty while $\mathcal{R}_1$ and $\mathcal{R}_3$ are nonempty. In particular, Figure 3 illustrates that $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ form a partition of $[0,\mu]$ for any value of μ.
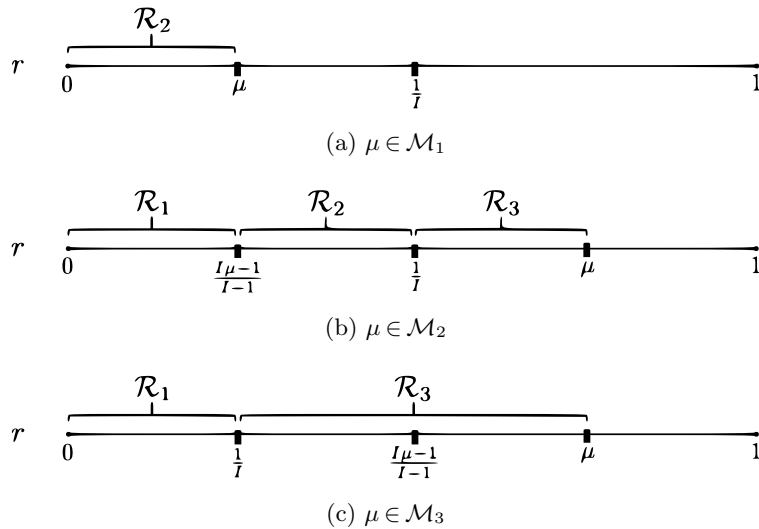
Figure 3 Relation between $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ and $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$.

THEOREM 8. *Assume that $\mathcal{P}$ is a Markov ambiguity set of the form* (6) *with $\mu \in (0,1)$, and let $r^\star$ and $z^\star$ denote the optimal reserve price and the corresponding worst-case expected revenue, respectively.*

(i) If $\mu \in \mathcal{M}_1 \cup \mathcal{M}_2$, then $r^\star = 1 - \sqrt{1-\mu}$ and $z^\star = I(1-\sqrt{1-\mu})^2$.

(ii) If $\mu \in \mathcal{M}_3$, then any reserve price $r^\star \in \mathcal{R}_1$ is optimal and $z^\star = \frac{I\mu-1}{I-1}$.

Recall from Theorem 5 that a second price auction with reserve price is optimal in ($\mathcal{MDP}$) if $\mathcal{P} = \{\mathbb{P}\}$ is a singleton, the bidders' values are independent and $\mathbb{P}$ is symmetric and regular. Moreover, in this case, the optimal reserve price is independent of the number of bidders. In contrast, Theorem 8 asserts that, for a Markov ambiguity set, the optimal reserve price depends on the number of bidders through the sets $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$. Specifically, for $\mu \in \mathcal{M}_1 \cup \mathcal{M}_2$, the optimal reserve price depends on μ but not on the number of bidders I. However, as I increases, $\mathcal{M}_3$ will eventually cover μ, which results in a decrease of the optimal reserve price. In this case, an interval of the reserve prices becomes optimal. We note that while each reserve price in this interval maximizes the worst-case revenues, the choice $r^\star = 0$ is preferable in practice as it additionally ensures efficiency.

We close this section by proving that the second price auction without reserve price is asymptotically optimal in ($\mathcal{MMDP}$) as the number of bidders tends to infinity.

PROPOSITION 15. *If $\mathcal{P}$ is a Markov ambiguity set of the form* (6) *with $\mu \in (0,1)$, then for every $\epsilon > 0$ there exists $I_\epsilon \in \mathbb{N}$ such that the second price auction without reserve price is ϵ-suboptimal in* ($\mathcal{MMDP}$) *for all $I \geq I_\epsilon$.*

Proof. Fix any $\epsilon > 0$ and select $I_\epsilon \in \mathbb{N}$ such that $\frac{I_\epsilon \mu - 1}{I_\epsilon - 1} \geq \max\{\mu - \epsilon, 0\}$. Thus, $0 \in \mathcal{R}_1$ whenever $I \geq I_\epsilon$. This implies via Proposition 12 that the objective value of the second price auction without

reserve price in $(\mathcal{MMDP})$ is at least $\mu - \epsilon$ for all $I \geq I_\epsilon$. The claim then follows because the optimal value of $(\mathcal{MMDP})$ is at most μ by Proposition 10. $\square$

Proposition 15 does not imply that second price auctions are optimal when the number of bidders is finite. Indeed, this is not the case as we will see in the next section.

5.3. The Optimal Highest-Bidder-Lottery

Consider again problem $(\mathcal{MMDP})$ with a Markov ambiguity set of the form (6) and $\mu \in (0,1)$. Assume that the seller aims to optimize over all mechanisms in which only the highest bidder has a chance to win the good. Note that these mechanisms are not necessarily efficient because the seller can keep the good or assign the good to the highest bidder with some probability smaller than 1.

By Proposition 11, the mechanism design problem $(\mathcal{MMDP})$ can thus be reformulated as

$$\sup_{\boldsymbol{q} \in \mathcal{Q}^{\text{m-p}}, \boldsymbol{\sigma} \in \mathbb{R}_+^I, \lambda \in \mathbb{R}} \quad \lambda + \sum_{i \in \mathcal{I}} \sigma_i \mu \tag{11a}$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{I}} \left[q_i(v_i, \boldsymbol{v}_{-i}) v_i - \int_0^{v_i} q_i(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \right] \geq \lambda + \sum_{i \in \mathcal{I}} \sigma_i v_i \quad \forall \boldsymbol{v} \in [0,1]^I \tag{11b}$$

$$q_i(\boldsymbol{v}) = 0 \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in [0,1]^I : \boldsymbol{v} \notin \mathcal{W}^i, \tag{11c}$$

where the last constraint ensures that only the highest bidder (with respect to some prescribed tie-breaker) has a chance to win the good. Thus, we refer to the mechanisms feasible in (11) as *highest-bidder-lotteries*.

THEOREM 9. *Assume that $\mathcal{P}$ is a Markov ambiguity set of the form (6) with $\mu \in (0,1)$, and set $\sigma^\star = -(W_{-1}(-\mu I e^{-I}) + 1)^{-1}$, where W_{-1} denotes the lower branch of the Lambert-W function (Corless et al. 1996). Moreover, set $r = e^{(I-1-\frac{1}{\sigma^\star})}$, $\lambda^\star = -\sigma^\star r$ and*

$$q_i^\star(\boldsymbol{v}) = \begin{cases} \sigma^\star \log\left(\dfrac{v_i}{\max\limits_{j \neq i} v_j}\right) + I\sigma^\star - \dfrac{\sigma^\star r}{\max\limits_{j \neq i} v_j} & \text{if } \boldsymbol{v} \in \mathcal{W}^i \text{ and } v_i \geq \max\limits_{j \neq i} v_j \geq r, \tag{12a} \\[2ex] \sigma^\star \log(v_i) + 1 & \text{if } v_i \geq r > \max\limits_{j \neq i} v_j, \tag{12b} \\[1ex] (I-1)\sigma^\star & \text{if } \boldsymbol{v} \in \mathcal{W}^i \text{ and } r > v_i \geq \max\limits_{j \neq i} v_j, \tag{12c} \\[1ex] 0 & \text{if } \boldsymbol{v} \notin \mathcal{W}^i. \tag{12d} \end{cases}$$

Then, $(\boldsymbol{q}^\star, \sigma^\star \mathbf{e}, \lambda^\star)$ is optimal in (11) with corresponding objective value $r = e^{(I-1-\frac{1}{\sigma^\star})}$.

By Proposition 2(i), we can construct a payment rule $\boldsymbol{m}^\star$ from the allocation rule $\boldsymbol{q}^\star$ by

$$m_i^\star(\boldsymbol{v}) = q_i^\star(\boldsymbol{v}) v_i - \int_0^{v_i} q_i^\star(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in [0,1]^I. \tag{13}$$

Theorem 9 implies that the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is an optimal highest-bidder-lottery in $(\mathcal{MMDP})$. The optimal allocation rule $\boldsymbol{q}^\star$ is randomized and can be interpreted as follows. The highest bidder
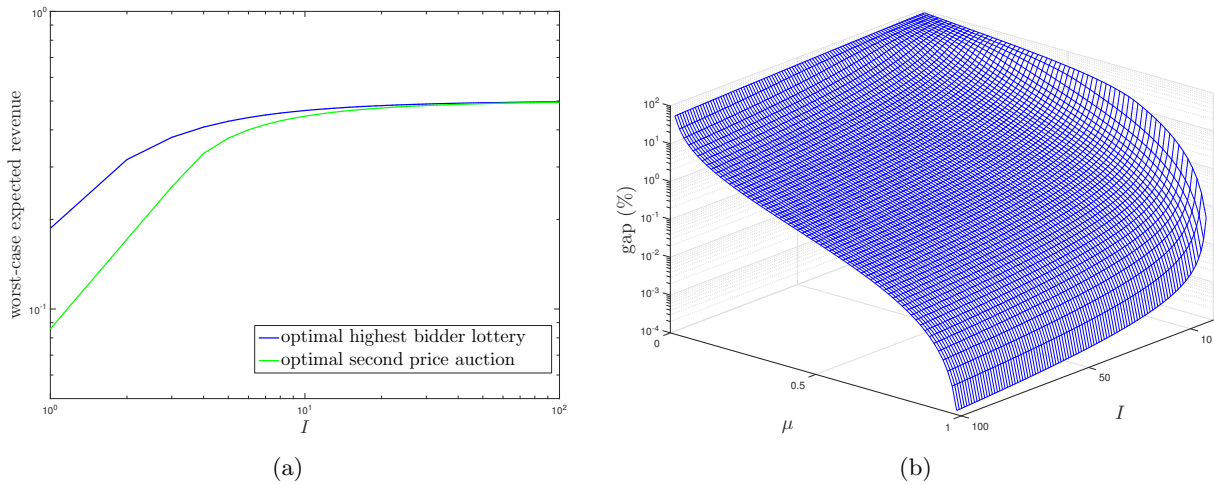
Figure 4   **Optimal highest-bidder-lottery versus optimal second price auction.**

i earns the right to participate in a lottery, which allows him to win the object with probability $q_i^\star(\boldsymbol{v})$. The probability $q_i^\star(\boldsymbol{v})$ is increasing in v_i if $v_i \geq r$ and constant otherwise. Moreover, $q_i^\star(\boldsymbol{v})$ is constant in $\boldsymbol{v}_{-i}$ if $\max_{j \neq i} v_j \leq r$. Finally, $q_i^\star(\boldsymbol{v})$ is decreasing in the second highest bid as long as both exceed r. It is perhaps surprising that the optimal allocation rule is randomized. As shown by Delage et al. (2016), however, agents with maxmin preferences can derive substantial benefits from randomization when facing a discrete choice (such as choosing a buyer out of I bidders).

PROPOSITION 16.   *As the number of bidders tends to infinity, the optimal highest-bidder-lottery* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ *converges uniformly to the second price auction without reserve price.*

Figure 4 compares the optimal highest-bidder-lottery against the best second price auction. Figure 4a shows the worst-case expected revenues generated by the optimal highest-bidder-lottery and the optimal second price auction with reserve price as a function of the number of bidders for $\mu = 0.5$. The gap between them relative to the worst-case expected revenue of the optimal highest-bidder-lottery is visualized in Figure 4b. We observe that the optimal highest-bidder-lottery generates substantially higher revenues when μ or I are small.

Even though the optimal highest-bidder-lottery was derived under the restriction that the good can be allocated only to the highest bidder, Proposition 15 implies that it is asymptotically optimal in $(\mathcal{MMDP})$. Next, we show that the optimal highest-bidder-lottery also offers a constant-factor approximation guarantee, which holds for any number of bidders.

THEOREM 10.   *The optimal highest-bidder-lottery extracts at least* 50% *of the worst-case expected revenue of the unknown optimal mechanism.*

Theorem 10 shows that the disadvantage of the optimal highest-bidder-lottery relative to the unknown optimal mechanism is bounded. On the other hand, the seller can be arbitrarily worse off by using the best second price auction instead of the optimal highest-bidder-lottery.

THEOREM 11. *For every $\epsilon > 0$ and every fixed $I \in \mathbb{N}$, there exists $\mu \in (0,1)$ such that the optimal second price auction extracts less than $\epsilon \cdot 100\%$ of the worst-case expected revenue of the optimal highest-bidder-lottery.*

Since the worst-case expected revenue of the optimal highest-bidder-lottery provides a lower bound on the worst-case expected revenue of the optimal mechanism, the following corollary holds.

COROLLARY 3. *For every $\epsilon > 0$ and every fixed $I \in \mathbb{N}$, there exists $\mu \in (0,1)$ such that the optimal second price auction extracts less than $\epsilon \cdot 100\%$ of the worst-case expected revenue of the unknown optimal mechanism.*

In Section 4 we have seen that if the bidders' values are independent and each distribution in the ambiguity set is symmetric and regular, then the added value of the optimal mechanism over a simple second price auction without reserve price is offset by just attracting one additional bidder. We now demonstrate that this result ceases to hold if the bidders' values can be dependent, as is the case under some distributions in a Markov ambiguity set. To this end, we denote by $\Delta(I) \in \mathbb{N}$ the least number of additional bidders needed by the best second price auction (which generates higher revenues than the second price auction without reserve price) to outperform the optimal highest-bidder-lottery with I bidders (which may be suboptimal in $(\mathcal{MMDP})$). Figure 5 shows that $\Delta(I)$ can be much larger than 1. In fact, we can even prove that there does not exist any finite upper bound on $\Delta(I)$ that holds uniformly across all $I \in \mathbb{N}$.

PROPOSITION 17. *If $\mu \in (0,1)$, then the set $\{\Delta(I) : I \in \mathbb{N}\}$ does not have a finite upper bound.*

Proposition 17 suggests that a Bulow and Klemperer (1996) type result does not hold in the setting of this section even if it is relaxed to any finite additional number of bidders.

Unlike $\boldsymbol{q}^\star$, the optimal payment rule $\boldsymbol{m}^\star$ is deterministic, implying that the highest bidder has to make a payment even if he is unlucky in the lottery. That is, he is charged a fee for the right to participate in the lottery. In the following, we construct a new mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ equivalent to $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, where both the allocation rule and the payment rule are randomized, and which charges the highest bidder only if he actually receives the good. To this end, we assume that the seller has access to a randomization device which generates a uniformly distributed sample $\tilde{u}$ from the interval $[0,1]$ that is independent of $\tilde{\boldsymbol{v}}$. Then, we define the new mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ through

$$q'_i(\boldsymbol{v}, u) = \begin{cases} 1 & \text{if } \boldsymbol{v} \in \mathcal{W}^i \text{ and } u \leq q_i^\star(\boldsymbol{v}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$m'_i(\boldsymbol{v}, u) = \begin{cases} v_i - \displaystyle\int_0^{v_i} \frac{q_i^\star(x, \boldsymbol{v}_{-i})}{q_i^\star(v_i, \boldsymbol{v}_{-i})} \, \mathrm{d}x & \text{if } \boldsymbol{v} \in \mathcal{W}^i \text{ and } u \leq q_i^\star(\boldsymbol{v}) \\ 0 & \text{otherwise} \end{cases}$$
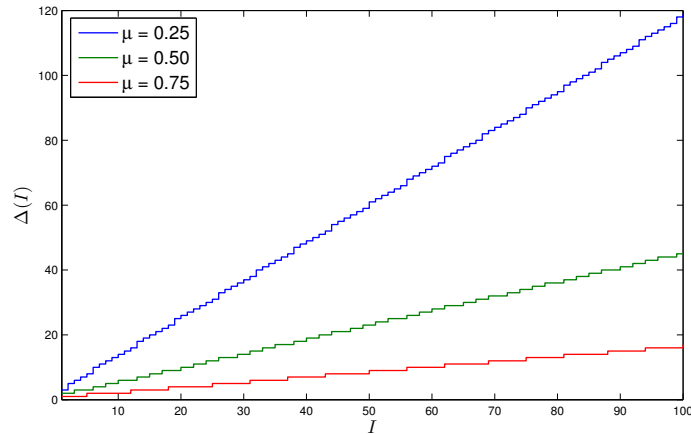
Figure 5 **The minimum number of additional bidders necessary to outperform the optimal highest-bidder-lottery with the best second price auction.**

for every $i \in \mathcal{I}$, $\boldsymbol{v} \in [0,1]^I$, and $u \in [0,1]$. Note that both the allocation rule $\boldsymbol{q}'$ and the payment rule $\boldsymbol{m}'$ depend on the outcome u of the randomization device and are thus randomized. By construction, however, the winner is not required to pay unless he receives the good in the lottery.

It is easy to verify that $(\boldsymbol{q}', \boldsymbol{m}')$ is equivalent to $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$. Indeed, we have

$$\mathbb{E}[q_i'(\boldsymbol{v}, \tilde{u})] = q_i^\star(\boldsymbol{v}), \quad \mathbb{E}[m_i'(\boldsymbol{v}, \tilde{u})] = q_i^\star(\boldsymbol{v})v_i - \int_0^{v_i} q_i^\star(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x = m_i^\star(\boldsymbol{v}) \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in [0,1]^I,$$

which implies that the expected revenues of the seller and the expected utilities of the bidders are identical under $(\boldsymbol{q}', \boldsymbol{m}')$ and $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, irrespective of the value distribution $\mathbb{P} \in \mathcal{P}$.

In this section, we introduced the optimal highest-bidder-lottery, a randomized mechanism that offers to the seller significantly higher revenues than the best second price auction. Based on numerical experiments, we conjecture that the best second price auction from Section 5.2 is also the best deterministic mechanism for problem ($\mathcal{MMDP}$). We emphasize that Theorem 11, Corollary 3 and Proposition 17 remain valid even if the bidders display maxmin preferences instead of Knightian preferences. Indeed, relaxing the Knightian incentive compatibility to maxmin incentive compatibility constraints increases the set of admissible mechanisms in ($\mathcal{MMDP}$); see Proposition 1. As all second price auctions are ex-post incentive compatible, however, this relaxation does not alter the best second price auction. Thus, the highest-bidder-lottery $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ continues to outperform the best second price auction if the bidders have maxmin preferences.

6. Conclusion

The standard assumption in the classical mechanism design literature whereby the seller has full knowledge of the bidders' value distribution is not tenable in practice. This prompts us to question whether the popular second price auctions remain optimal in the presence of distributional ambiguity. If not, we aim to identify the optimal mechanism or, if that is not possible, a near-optimal

mechanism that offers strong performance guarantees. If only the range of all possible bidder values is known, we prove that the Vickrey auction is the unique optimal, efficient and Pareto robustly optimal mechanism. Thus, sellers who have no information about the bidders' values or have little trust in their information, as is typically the case for one-off auctions, might settle for a simple second price auction without reserve price. If the seller lacks information about the distribution but knows that the bidders' values are independent, the globally optimal auction is unknown, but second price auctions remain near-optimal: there exists a second price auction with a random reserve price that offers a 2-approximation for the mechanism design problem, and even the naïve second price auction without reserve price generates higher revenues than the unknown optimal mechanism if the former attracts only one additional bidder. However, independence can rarely be ascertained in reality as the bidders might interact with each other or might share common information sources. This concern motivates us to study Markov ambiguity sets, which allow the bidders' values to be correlated. Markov ambiguity sets capture the knowledge of more informed sellers who not only know the range of possible bidder values but are also aware of the expected or 'most likely' values. In this setting, which makes honest assumptions about the information that is realistically available to the seller, the performance of second price auctions drops significantly. Indeed, we prove that even the best second price auction, which we can characterize analytically, fails to offer any constant-factor approximation guarantee. While the globally optimal mechanism remains unknown, we can explicitly construct a randomized near-optimal mechanism, the optimal highest-bidder-lottery, which offers a 2-approximation for the mechanism design problem. Moreover, we demonstrate that the number of additional bidders needed by the best second price auction to match the revenues of the optimal highest-bidder-lottery (and, a fortiori, of the unknown globally optimal mechanism) is unbounded. Under realistic assumptions about the available distributional information, second price auction can therefore be severely suboptimal.

References

Bandi C, Bertsimas D (2014) Optimal design for multi-item auctions: A robust optimization approach. *Mathematics of Operations Research* 39(4):1012–1038.

Ben-Tal A, Ghaoui LE, Nemirovski A (2009) *Robust Optimization* (Princeton University Press).

Bergemann D, Brooks BA, Morris S (2016) Informationally robust optimal auction design. Technical report.

Bergemann D, Morris S (2005) Robust mechanism design. *Econometrica* 73(6):1771–1813.

Bergemann D, Morris S (2013) An introduction to robust mechanism design. *Foundations and Trends® in Microeconomics* 8(3):169–230.

Bertsimas D, Brown DB, Caramanis C (2011) Theory and applications of robust optimization. *SIAM Review* 53(3):464–501.

Bewley TF (2002) Knightian decision theory. Part I. *Decisions in Economics and Finance* 25(2):79–110.

Bodoh-Creed AL (2012) Ambiguous beliefs and mechanism design. *Games and Economic Behavior* 75(2):518–537.

Bose S, Daripa A (2009) A dynamic mechanism and surplus extraction under ambiguity. *Journal of Economic Theory* 144(5):2084–2114.

Bose S, Ozdenoren E, Pape A (2006) Optimal auctions with ambiguity. *Theoretical Economics* 1(4):411–438.

Bulow J, Klemperer P (1996) Auctions versus negotiations. *The American Economic Review* 86(1):180–194.

Chen Y, Katuščák P, Ozdenoren E (2007) Sealed bid auctions with ambiguity: Theory and experiments. *Journal of Economic Theory* 136(1):513–535.

Chiesa A, Micali S, Zhu ZA (2015) Knightian analysis of the Vickrey mechanism. *Econometrica* 83(5):1727–1754.

Chung KS, Ely JC (2007) Foundations of dominant-strategy mechanisms. *The Review of Economic Studies* 74(2):447–476.

Corless RM, Gonnet GH, Hare DE, Jeffrey DJ, Knuth DE (1996) On the Lambert W function. *Advances in Computational Mathematics* 5(1):329–359.

Cremer J, McLean RP (1988) Full extraction of the surplus in Bayesian and dominant strategy auctions. *Econometrica 56(6):* 1247–1257.

Delage E, Kuhn D, Wiesemann W (2016) "Dice"-sion making under uncertainty: When can a random decision reduce risk? Technical report.

Delage E, Ye Y (2010) Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Operations Research* 58(3):596–612.

Dhangwatnotai P, Roughgarden T, Yan Q (2015) Revenue maximization with a single sample. *Games and Economic Behavior* 91:318–333.

Fudenberg D, Tirole J (1991) *Game Theory* (MIT Press).

Goldberg AV, Hartline JD, Karlin AR, Saks M, Wright A (2006) Competitive auctions. *Games and Economic Behavior* 55(2):242–269.

Harsanyi JC (1967) Games with incomplete information played by "Bayesian" players, I–III: Part I. The basic model. *Management Science* 14(3):159–182.

Hartline JD, Roughgarden T (2008) Optimal mechanism design and money burning. *Proceedings of the Fortieth Annual ACM Symposium on Theory of Computing*, 75–84 (ACM).

Iancu DA, Trichakis N (2013) Pareto efficiency in robust optimization. *Management Science* 60(1):130–147.

Klemperer P (1999) Auction theory: A guide to the literature. *Journal of Economic Surveys* 13(3):227–286.

Knight FH (1921) *Risk, Uncertainty and Profit* (Hart, Schaffner and Marx).

Koçyiğit Ç, Bayrak HI, Pınar MÇ (2018) Robust auction design under multiple priors by linear and integer programming. *Annals of Operations Research* 260(1-2):233–253.

Krishna V (2009) *Auction Theory* (Academic Press), 2nd edition.

Lo KC (1998) Sealed bid auctions with uncertainty averse bidders. *Economic Theory* 12(1):1–20.

Lopomo G, Rigotti L, Shannon C (2014) Uncertainty in mechanism design. Technical report.

McAfee RP, McMillan J (1987) Auctions with entry. *Economics Letters* 23(4):343–347.

Milgrom PR, Weber RJ (1982) A theory of auctions and competitive bidding. *Econometrica* 50(5):1089–1122.

Myerson RB (1981) Optimal auction design. *Mathematics of Operations Research* 6(1):58–73.

Nisan N, Roughgarden T, Tardos E, Vazirani VV (2007) *Algorithmic Game Theory*, volume 1 (Cambridge University Press Cambridge).

Ozdenoren E (2002) Auctions and bargaining with a set of priors. Technical report.

Riley JG, Samuelson WF (1981) Optimal auctions. *The American Economic Review* 71(3):381–392.

Roughgarden T, Talgam-Cohen I (2013) Optimal and near-optimal mechanism design with interdependent values. *Proceedings of the Fourteenth ACM Conference on Electronic Commerce*, 767–784 (ACM).

Salo A, Weber M (1995) Ambiguity aversion in first-price sealed-bid auctions. *Journal of Risk and Uncertainty* 11(2):123–137.

Shapiro A (2001) On duality theory of conic linear problems. In M. A. Goberna and M. A. Lopez, editors, *Semi-Infinite Programming*, 135–165 (Kluwer Academic Publishers).

Shapiro A, Dentcheva D, Ruszczyński A (2014) *Lectures on Stochastic Programming: Modeling and Theory* (SIAM).

Vickrey W (1961) Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance* 16(1):8–37.

Wiesemann W, Kuhn D, Sim M (2014) Distributionally robust convex optimization. *Operations Research* 62(6):1358–1376.

Appendix. Proofs

Proof of Proposition 1. It is easy to verify that ex-post incentive compatibility implies distributionally robust incentive compatibility.

For any fixed $i \in \mathcal{I}$ and $v_i, w_i \in \mathcal{V}$, distributionally robust incentive compatibility requires that

$$\mathbb{E}_{\mathbb{P}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \mathbb{E}_{\mathbb{P}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \quad \forall \mathbb{P} \in \mathcal{P}. \tag{14}$$

Since (14) holds for all $\mathbb{P} \in \mathcal{P}$, we have

$$\mathbb{E}_{\mathbb{P}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \quad \forall \mathbb{P} \in \mathcal{P}.$$

Now, taking the infimum over $\mathbb{P} \in \mathcal{P}$ on the left-hand side yields

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right]. \tag{15}$$

This establishes maxmin incentive compatibility.

Note that (14) implies

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \mathbb{E}_{\mathbb{P}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \quad \forall \mathbb{P} \in \mathcal{P}.$$

Since this condition holds for all $\mathbb{P} \in \mathcal{P}$, we can take the supremum over $\mathbb{P} \in \mathcal{P}$ on the right-hand side to obtain

$$\sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[u_i(v_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right] \geq \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[u_i(w_i; v_i, \tilde{\boldsymbol{v}}_{-i}) \mid \tilde{v}_i = v_i\right]. \tag{16}$$

Summing α times (15) and $1 - \alpha$ times (16) for any $\alpha \in (0, 1)$ yields Hurwicz incentive compatibility.

By taking expectations on both sides of (14) with respect to the distribution $\mathbb{Q}$ on $\mathcal{P}$, finally, one concludes that Bayesian incentive compatibility holds as well. □

Proof of Theorem 2. By Proposition 2(*ii*) and by the assumption that $u_i(\underline{v}, \boldsymbol{v}_{-i}) = 0$ for all $i \in \mathcal{I}$ and $\boldsymbol{v}_{-i} \in \mathcal{V}$, we have

$$\mathbb{E}_{\mathbb{P}}\left[m_i(v_i, \tilde{\boldsymbol{v}}_{-i})\right] = \mathbb{E}_{\mathbb{P}}\left[q_i(v_i, \tilde{\boldsymbol{v}}_{-i}) v_i - \int_{\underline{v}}^{v_i} q_i(x, \tilde{\boldsymbol{v}}_{-i}) \, \mathrm{d}x\right] \quad \forall i \in \mathcal{I}, \forall v_i \in \mathcal{V}, \forall \mathbb{P} \in \mathcal{P}. \tag{17}$$

Hence, the expected revenue of the seller with respect to some $\mathbb{P} \in \mathcal{P}$ is equal to

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{i \in \mathcal{I}} m_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\right] = \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}[m_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i}) | \tilde{v}_i]\right] = \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbb{P}}\left[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i}) \tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i}) \, \mathrm{d}x\right],$$

where the second equality follows from (17). This indicates that the seller's expected revenue under any $\mathbb{P}$ is determined solely by the allocation rule $\boldsymbol{q}$. Hence, the seller earns the same worst-case expected revenue from all mechanisms with identical allocation rules. □

Proof of Proposition 4. We will prove the claim by contradiction. Assume that the Vickrey mechanism is not Pareto robustly optimal. Thus, there exists a mechanism $(\boldsymbol{q}, \boldsymbol{m})$ feasible in $(\mathcal{RMDP})$ with $\sum_{i \in \mathcal{I}} m_i(\boldsymbol{v}) \geq \sum_{i \in \mathcal{I}} m_i^{\mathrm{v}}(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathcal{V}^I$, and there exists $\boldsymbol{v}' \in \mathcal{V}^I$ with $\sum_{i \in \mathcal{I}} m_i(\boldsymbol{v}') > \sum_{i \in \mathcal{I}} m_i^{\mathrm{v}}(\boldsymbol{v}')$. Let $i^1 \in \mathcal{I}$ be the winning bidder in the Vickrey mechanism under scenario $\boldsymbol{v}'$, which implies that $i^1 \in \arg\max_{j \in \mathcal{I}} v_j'$, and let $i^2 \in \arg\max_{j \neq i^1} v_j'$ be any second highest bidder.

If $v'_{i^1} = v'_{i^2}$, then we have

$$\sum_{j \in \mathcal{I}} m_j(\boldsymbol{v}') > \sum_{j \in \mathcal{I}} m_j^{\mathrm{v}}(\boldsymbol{v}') = v'_{i^2} = v'_{i^1} \geq \sum_{j \in \mathcal{I}} q_j(\boldsymbol{v}') v'_j,$$

where the last inequality holds because $\sum_{j \in \mathcal{I}} q_j(\boldsymbol{v}') \leq 1$. This contradicts ex-post individual rationality, which is imposed in $(\mathcal{RMDP})$ and which implies that $\sum_{j \in \mathcal{I}} m_j(\boldsymbol{v}') \leq \sum_{j \in \mathcal{I}} q_j(\boldsymbol{v}') v'_j$.

Assume from now on that $v'_{i^1} > v'_{i^2}$. Our analysis is divided into three steps. In the first step, we construct a scenario $\boldsymbol{w} \in \mathcal{V}^I$ such that $w_{i^1} = v'_{i^2} + \delta$ for a judiciously chosen $\delta > 0$ and $w_j = v'_j$ for all $j \neq i^1$. In the second step, we show that the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ has to assign the good to a second highest bidder i^2 with strictly positive probability under scenario $\boldsymbol{w}$ in order to satisfy individual rationality and incentive compatibility at the ex-post stage. In the third step, we construct a scenario $\boldsymbol{w}' \in \mathcal{V}^I$ such that $w'_{i^1} = w'_{i^2} = v'_{i^2} + \delta$ and $w'_j = v'_j$ for all $j \notin \{i^1, i^2\}$. Leveraging the results from Steps 1 and 2, we then show that if $(\boldsymbol{q}, \boldsymbol{m})$ satisfies ex-post individual rationality in scenario $\boldsymbol{w}'$, then it has to violate ex-post incentive compatibility because bidder i^2 has an incentive to misreport his value as w_{i^2} instead of w'_{i^2}. We will conclude that the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ cannot simultaneously satisfy individual rationality and incentive compatibility at the ex-post stage, which contradicts our assumption that $(\boldsymbol{q}, \boldsymbol{m})$ is feasible in $(\mathcal{RMDP})$. Figure 1 visualizes the scenarios $\boldsymbol{v}', \boldsymbol{w}$ and $\boldsymbol{w}'$ constructed below.

Step 1: Let $\epsilon = \frac{m_{i^1}(\boldsymbol{v}')}{q_{i^1}(\boldsymbol{v}')} - v'_{i^2}$ and $\gamma = \epsilon / 2 v'_{i^1}$. Note that ϵ is well-defined since $q_{i^1}(\boldsymbol{v}') > 0$ must hold for $(\boldsymbol{q}, \boldsymbol{m})$ to generate higher revenues than the Vickrey mechanism in scenario $\boldsymbol{v}'$. Note also that γ is well-defined due to assumption $v'_{i^1} > v'_{i^2}$ and because the bidders' values are non-negative. Define $\boldsymbol{w}$ such that $w_{i^1} = v'_{i^2} + \delta$ and $w_j = v'_j$ for all $j \neq i^1$, where

$$\delta = \min \left\{ \frac{\epsilon}{2}, \frac{v'_{i^2} - v'_{i^3}}{q_{i^1}(\boldsymbol{v}')(1 - \gamma)} - (v'_{i^2} - v'_{i^3}) \right\} \tag{18}$$

and $i^3 \in \arg\max_{j \in \mathcal{I}}\{v_j : v_j < v'_{i^2}\}$. If $\arg\max_{j \in \mathcal{I}}\{v_j : v_j < v'_{i^2}\} = \emptyset$, in which $\boldsymbol{v}'$ accounts only for two different bids, we may set $v'_{i^3} = 0$. In this step, we show that $\delta \in (0, v'_{i^1} - v'_{i^2})$, which ensures that $\boldsymbol{w} \in \mathcal{V}^I$. Specifically, we show that both terms inside the minimum in (18) are positive and that the first term is smaller than $v'_{i^1} - v'_{i^2}$.

We start by showing that $\frac{\epsilon}{2} \in (0, v'_{i^1} - v'_{i^2})$. By assumption, we have $\sum_{j \in \mathcal{I}} m_j(\boldsymbol{v}') > \sum_{j \in \mathcal{I}} m_j^{\mathrm{v}}(\boldsymbol{v}') = v'_{i^2}$. Moreover, by individual rationality, for all $j \neq i^1$, we have

$$m_j(\boldsymbol{v}') \leq q_j(\boldsymbol{v}') v'_j \leq q_j(\boldsymbol{v}') v'_{i^2},$$

where the second inequality holds because $v'_j \leq v'_{i^2}$. As $\sum_{j \in \mathcal{J}} q_j(\boldsymbol{v}') \leq 1$, our initial assumption that $\sum_{j \in \mathcal{I}} m_j(\boldsymbol{v}') > v'_{i^2}$ can hold only if $m_{i^1}(\boldsymbol{v}') > q_{i^1}(\boldsymbol{v}') v'_{i^2}$. Thus $\epsilon = \frac{m_{i^1}(\boldsymbol{v}')}{q_{i^1}(\boldsymbol{v}')} - v'_{i^2} > 0$. We also have that $\frac{\epsilon}{2} < v'_{i^1} - v'_{i^2}$ because $\frac{\epsilon}{2} < \epsilon = \frac{m_{i^1}(\boldsymbol{v}')}{q_{i^1}(\boldsymbol{v}')} - v'_{i^2} \leq v'_{i^1} - v'_{i^2}$ as individual rationality requires that $m_{i^1}(\boldsymbol{v}') \leq q_{i^1}(\boldsymbol{v}') v'_{i^1}$. Finally, the second term inside the minimum in (18) is greater than 0 since

$$\frac{v'_{i^2} - v'_{i^3}}{q_{i^1}(\boldsymbol{v}')(1 - \gamma)} > \frac{v'_{i^2} - v'_{i^3}}{q_{i^1}(\boldsymbol{v}')} \geq v'_{i^2} - v'_{i^3},$$

where the first inequality holds because $0 < \gamma < 1$ as $0 < \frac{\epsilon}{2} < v'_{i^1} - v'_{i^2} < v'_{i^1}$, and the second inequality follows from $0 < q_{i^1}(\boldsymbol{v}') \leq 1$. As both terms inside the minimum in (18) are positive and the first term is smaller than $v'_{i^1} - v'_{i^2}$, we conclude that $\delta \in (0, v'_{i^1} - v'_{i^2})$.

Step 2: Assume without loss of generality that amongst all second highest bidders, i^2 has the highest probability to win under scenario $\boldsymbol{w}$. In this step, we show that $q_{i^2}(\boldsymbol{w}) > 0$. Assume to the contrary that $q_{i^2}(\boldsymbol{w}) = 0$, which implies that the probability to win for any second highest bidder equals 0. By ex-post individual rationality and the assumption that the Vickrey mechanism is not Pareto robustly optimal, we have

$$q_{i^1}(\boldsymbol{w})w_{i^1} + \sum_{j \neq i^1} q_j(\boldsymbol{w})w_j = q_{i^1}(\boldsymbol{w})(v'_{i^2} + \delta) + \sum_{j \neq i^1} q_j(\boldsymbol{w})w_j \geq \sum_{j \in \mathcal{I}} m_j(\boldsymbol{w}) \geq \sum_{j \in \mathcal{I}} m_j^{\mathrm{v}}(\boldsymbol{w}) = v'_{i^2}. \tag{19}$$

Since $q_{i^2}(\boldsymbol{w}) = 0$, equation (19) can only hold if $q_{i^1}(\boldsymbol{w})(v'_{i^2} + \delta) + (1 - q_{i^1}(\boldsymbol{w}))v'_{i^3} \geq v'_{i^2}$. Reordering term, this implies that

$$\delta \geq \frac{v'_{i^2} - v'_{i^3}}{q_{i^1}(\boldsymbol{w})} - (v'_{i^2} - v'_{i^3}). \tag{20}$$

We next prove that (20) contradicts the definition of δ if $(\boldsymbol{q}, \boldsymbol{m})$ is ex-post individually rational and incentive compatible. Note that we have $m_{i^1}(\boldsymbol{w}) \leq q_{i^1}(\boldsymbol{w})(v'_{i^2} + \delta) < q_{i^1}(\boldsymbol{v}')(v'_{i^2} + \epsilon) = m_{i^1}(\boldsymbol{v}')$, where the first inequality follows from individual rationality, and the second inequality holds because $\epsilon > \delta$ by definition and $q_{i^1}(\boldsymbol{v}') \geq q_{i^1}(\boldsymbol{w})$ by Proposition 2(i). Combining the inequality $m_{i^1}(\boldsymbol{w}) < m_{i^1}(\boldsymbol{v}')$ with the incentive compatibility of $(\boldsymbol{q}, \boldsymbol{m})$, which requires that $q_{i^1}(\boldsymbol{v}')v'_{i^1} - m_{i^1}(\boldsymbol{v}') \geq q_{i^1}(\boldsymbol{w})v'_{i^1} - m_{i^1}(\boldsymbol{w})$, implies that $1 \geq q_{i^1}(\boldsymbol{v}') > q_{i^1}(\boldsymbol{w})$ and

$$q_{i^1}(\boldsymbol{v}') - q_{i^1}(\boldsymbol{w}) \geq \frac{m_{i^1}(\boldsymbol{v}') - m_{i^1}(\boldsymbol{w})}{v'_{i^1}} > \frac{q_{i^1}(\boldsymbol{v}')\epsilon}{2v'_{i^1}} = q_{i^1}(\boldsymbol{v}')\gamma, \tag{21}$$

where the second inequality holds because $m_{i^1}(\boldsymbol{v}') = q_{i^1}(\boldsymbol{v}')(v'_{i^2} + \epsilon)$ by definition of ϵ and $m_{i^1}(\boldsymbol{w}) \leq q_{i^1}(\boldsymbol{w})(v'_{i^2} + \delta) < q_{i^1}(\boldsymbol{v}')(v'_{i^2} + \frac{\epsilon}{2})$ due to $q_{i^1}(\boldsymbol{v}') > q_{i^1}(\boldsymbol{w})$ and $\delta \leq \frac{\epsilon}{2}$. The inequalities (20) and (21) imply that

$$\delta \geq \frac{v'_{i^2} - v'_{i^3}}{q_{i^1}(\boldsymbol{w})} - (v'_{i^2} - v'_{i^3}) > \frac{v'_{i^2} - v'_{i^3}}{q_{i^1}(\boldsymbol{v}')(1 - \gamma)} - (v'_{i^2} - v'_{i^3}),$$

which contradicts the definition of δ. Thus $q_{i^2}(\boldsymbol{w}) > 0$, which concludes Step 2.

Step 3: Define $\boldsymbol{w}'$ such that $w'_{i^1} = w'_{i^2} = v'_{i^2} + \delta$ and $w'_j = v'_j$ for all $j \notin \{i^1, i^2\}$. Note that $\boldsymbol{w}' \in \mathcal{V}^I$ because $\mathcal{V} = [\underline{v}, \bar{v}]$ and $v'_{i^2} + \delta \leq v'_{i^1} \leq \bar{v}$ by construction. Note that we have

$$q_{i^2}(\boldsymbol{w})(v'_{i^2} + \delta) - m_{i^2}(\boldsymbol{w}) > q_{i^2}(\boldsymbol{w})v'_{i^2} - m_{i^2}(\boldsymbol{w}) \geq 0, \tag{22}$$

where the first inequality holds because $\delta > 0$ due to Step 1 and $q_{i^2}(\boldsymbol{w}) > 0$ due to Step 2, and the second inequality holds due to ex-post individual rationality at scenario $\boldsymbol{w}$. Next, we prove that $q_{i^2}(\boldsymbol{w}')(v'_{i^2} + \delta) - m_{i^2}(\boldsymbol{w}') = 0$ if $(\boldsymbol{q}, \boldsymbol{m})$ is ex-post individually rational at scenario $\boldsymbol{w}'$. By Proposition 2(i), we have that $q_{i^2}(\boldsymbol{w}') \geq q_{i^2}(\boldsymbol{w}) > 0$. Moreover, by ex-post individual rationality, we have that $\sum_{j \in \mathcal{I}} m_j(\boldsymbol{w}') \leq \sum_{j \in \mathcal{I}} q_j(\boldsymbol{w}')w'_j \leq \sum_{j \in \mathcal{I}} q_j(\boldsymbol{w}')(v'_{i^2} + \delta)$. Since $\sum_{j \in \mathcal{I}} q_j(\boldsymbol{w}') \leq 1$, the assumption $\sum_{j \in \mathcal{I}} m_j(\boldsymbol{w}') \geq \sum_{j \in \mathcal{I}} m_j^{\mathrm{v}}(\boldsymbol{w}') = v'_{i^2} + \delta$ thus can hold only if $m_{i^2}(\boldsymbol{w}') = q_{i^2}(\boldsymbol{w}')(v'_{i^2} + \delta)$, which contradicts the incentive compatibility of $(\boldsymbol{q}, \boldsymbol{m})$ as we have

$$q_{i^2}(\boldsymbol{w}')(v'_{i^2} + \delta) - m_{i^2}(\boldsymbol{w}') = 0 < q_{i^2}(\boldsymbol{w})(v'_{i^2} + \delta) - m_{i^2}(\boldsymbol{w}),$$

where the inequality follows from (22). The claim thus follows. □

Proof of Proposition 5. Select any efficient ex-post individually rational and incentive compatible mechanism $(\boldsymbol{q}, \boldsymbol{m})$. Suppose that $\sum_{j \in \mathcal{I}} m_j(\boldsymbol{v}) > \sum_{j \in \mathcal{I}} m_j^{\mathrm{v}}(\boldsymbol{v})$ for some fixed $\boldsymbol{v} \in \mathcal{V}^I$, and note that $\boldsymbol{v} \in \mathcal{W}^i$ for some $i \in \mathcal{I}$. Then, we have

$$\sum_{j \in \mathcal{I}} m_j(\boldsymbol{v}) > \sum_{j \in \mathcal{I}} m_j^{\mathrm{v}}(\boldsymbol{v}) = m_i^{\mathrm{v}}(\boldsymbol{v}) = \max_{j \neq i} v_j.$$

We will show that if the above strict inequality holds, then $(\boldsymbol{q}, \boldsymbol{m})$ cannot simultaneously satisfy ex-post individual rationality, ex-post incentive compatibility and efficiency.

If the second highest bid equals the highest bid v_i, then we have

$$\sum_{j \in \mathcal{I}} m_j(\boldsymbol{v}) > \max_{j \neq i} v_j = v_i = \sum_{j \in \mathcal{I}} q_j(\boldsymbol{v}) v_i \geq \sum_{j \in \mathcal{I}} q_j(\boldsymbol{v}) v_j,$$

where the second inequality holds because $\sum_{j \in \mathcal{I}} q_j(\boldsymbol{v}) = 1$. This contradicts ex-post individual rationality, which implies that $\sum_{j \in \mathcal{I}} m_j(\boldsymbol{v}) \leq \sum_{j \in \mathcal{I}} q_j(\boldsymbol{v}) v_j$.

If there is no tie such that $v_i > \max_{j \neq i} v_j$, then $m_i(\boldsymbol{v}) = \sum_{j \in \mathcal{I}} m_j(\boldsymbol{v}) > \sum_{j \in \mathcal{I}} m_j^{\mathrm{v}}(\boldsymbol{v}) = m_i^{\mathrm{v}}(\boldsymbol{v})$, where the first equality holds because the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ is efficient and ex-post individually rational. Select $\epsilon > 0$ small enough such that $\epsilon < v_i - \max_{j \neq i} v_j$ and $\epsilon < m_i(\boldsymbol{v}) - m_i^{\mathrm{v}}(\boldsymbol{v})$. Moreover, set $v_i' = \max_{j \neq i} v_j + \epsilon$ and note that $v_i' < v_i$ and that $(v_i', \boldsymbol{v}_{-i}) \in \mathcal{V}^I$ because $\mathcal{V} = [\underline{v}, \bar{v}]$. Then, the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ violates ex-post incentive compatibility because

$$q_i(\boldsymbol{v}) v_i - m_i(\boldsymbol{v}) < v_i - v_i' \leq q_i(v_i', \boldsymbol{v}_{-i}) v_i - m_i(v_i', \boldsymbol{v}_{-i}),$$

where the first inequality holds because $m_i(\boldsymbol{v}) - m_i^{\mathrm{v}}(\boldsymbol{v}) > \epsilon$, which implies that $m_i(\boldsymbol{v}) > v_i'$. The second inequality holds because $v_i' > \max_{j \neq i} v_j$, which implies that $q_i(v_i', \boldsymbol{v}_{-i}) = 1$ due to efficiency, and because $m_i(v_i', \boldsymbol{v}_{-i}) \leq q_i(v_i', \boldsymbol{v}_{-i}) v_i' = v_i'$ due to ex-post individual rationality. Thus $(\boldsymbol{q}, \boldsymbol{m})$ violates ex-post incentive compatibility if it is ex-post individually rational and efficient. $\square$

We need the following auxiliary result to prove Proposition 7.

LEMMA 1. *For each mechanism $(\boldsymbol{q}, \boldsymbol{m})$ feasible in $(\mathcal{IMDP})$, there exists a mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ with*

$$\mathbb{E}_{\mathbb{P}}[q_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) \underline{v} - m_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i})] = 0 \quad \forall i \in \mathcal{I}, \forall \mathbb{P} \in \mathcal{P} \tag{23}$$

that is also feasible in $(\mathcal{IMDP})$ and results in a weakly higher worst-case expected revenue to the seller.

Proof. Construct $(\boldsymbol{q}', \boldsymbol{m}')$ by setting $\boldsymbol{q}' = \boldsymbol{q}$ and

$$m_i'(v_i, \boldsymbol{v}_{-i}) = q_i'(v_i, \boldsymbol{v}_{-i}) v_i - \int_{\underline{v}}^{v_i} q_i'(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I. \tag{24}$$

Note that the ex-post utility under mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ satisfies $q_i'(\underline{v}, \boldsymbol{v}_{-i}) \underline{v} - m_i'(\underline{v}, \boldsymbol{v}_{-i}) = 0$ for all $i \in \mathcal{I}$, $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$, which implies that $\mathbb{E}_{\mathbb{P}}[q_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i}) \underline{v} - m_i'(\underline{v}, \tilde{\boldsymbol{v}}_{-i})] = 0$ for all $i \in \mathcal{I}$, $\mathbb{P} \in \mathcal{P}$. Thus, $(\boldsymbol{q}', \boldsymbol{m}')$ satisfies the second constraint in $(\mathcal{IMDP})$. Moreover, we have

$$q_i'(v_i, \boldsymbol{v}_{-i}) v_i - m_i'(v_i, \boldsymbol{v}_{-i}) = \int_{\underline{v}}^{v_i} q_i'(x, \boldsymbol{v}_{-i}) \, \mathrm{d}x \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I,$$

which implies that $(\boldsymbol{q}', \boldsymbol{m}')$ satisfies also the first constraint in $(\mathcal{IMDP})$.

We next show that $(\boldsymbol{q}', \boldsymbol{m}')$ results in a weakly higher worst-case expected revenue to the seller than $(\boldsymbol{q}, \boldsymbol{m})$. As the ex-post utility satisfies $q_i'(\underline{v}, \boldsymbol{v}_{-i})\underline{v} - m_i'(\underline{v}, \boldsymbol{v}_{-i}) = 0$ for all $i \in \mathcal{I}$, $\boldsymbol{v}_{-i} \in \mathcal{V}^{I-1}$, the objective function of $(\boldsymbol{q}', \boldsymbol{m}')$ in $(\mathcal{IMDP})$ reduces to

$$\inf_{\mathbb{P} \in \mathcal{P}} \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbb{P}}\left[q_i'(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i'(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x \right]$$

$$\geq \inf_{\mathbb{P} \in \mathcal{P}} \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbb{P}}\left[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x \right] - \mathbb{E}_{\mathbb{P}}[q_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i})\underline{v} - m_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i})],$$

where the inequality holds because $\boldsymbol{q}' = \boldsymbol{q}$ and $\mathbb{E}_{\mathbb{P}}[q_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i})\underline{v} - m_i(\underline{v}, \tilde{\boldsymbol{v}}_{-i})] \geq 0$ for all $i \in \mathcal{I}$ and $\mathbb{P} \in \mathcal{P}$ due to the second constraint in $(\mathcal{IMDP})$. The statement now follows from the fact that right-hand side of this equation coincides with the objective function value of $(\boldsymbol{q}, \boldsymbol{m})$ in $(\mathcal{IMDP})$. $\square$

Proof of Proposition 7. Denote by $\rho^{\mathbb{P}}$ the density function of $\tilde{\boldsymbol{v}}$ and by $\rho_i^{\mathbb{P}}$ the marginal density function of $\tilde{v}_i$, $i \in \mathcal{I}$, under $\mathbb{P} \in \mathcal{P}$. By Lemma 1, without loss of generality, we can restrict the feasible set of $(\mathcal{IMDP})$ to mechanisms that satisfy (23). Fix now an arbitrary mechanism $(\boldsymbol{q}, \boldsymbol{m})$ feasible in this restriction of $(\mathcal{IMDP})$.

Proposition 2(*ii*) implies that the expected payment of bidder i under $\mathbb{P}$ can be expressed as

$$\mathbb{E}_{\mathbb{P}}\left[q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i})\tilde{v}_i - \int_{\underline{v}}^{\tilde{v}_i} q_i(x, \tilde{\boldsymbol{v}}_{-i})\,\mathrm{d}x \right]$$

$$= \int_{\mathcal{V}^{I-1}} \int_{\mathcal{V}} q_i(v_i, \boldsymbol{v}_{-i})v_i\, \rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i} - \int_{\mathcal{V}^{I-1}} \int_{\mathcal{V}} \int_{\underline{v}}^{v_i} q_i(x, \boldsymbol{v}_{-i})\,\mathrm{d}x\, \rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i}.$$

Using Fubini's theorem, we can re-write the second term as

$$\int_{\mathcal{V}^{I-1}} \int_{\mathcal{V}} \int_{\underline{v}}^{v_i} q_i(x, \boldsymbol{v}_{-i})\,\mathrm{d}x\, \rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i} = \int_{\mathcal{V}^{I-1}} \int_{\underline{v}}^{\bar{v}} q_i(x, \boldsymbol{v}_{-i}) \left(\int_{v_i}^{\bar{v}} \rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}x \right) \mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i}.$$

Thus, the expected payment of bidder i under $\mathbb{P}$ simplifies to

$$\int_{\mathcal{V}^{I-1}} \int_{\mathcal{V}} \left(v_i - \frac{\int_{v_i}^{\bar{v}} \rho^{\mathbb{P}}(x, \boldsymbol{v}_{-i})\,\mathrm{d}x}{\rho^{\mathbb{P}}(\boldsymbol{v})} \right) q_i(v_i, \boldsymbol{v}_{-i})\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i}$$

$$= \int_{\mathcal{V}^{I-1}} \int_{\mathcal{V}} \left(v_i - \frac{1 - \int_{\underline{v}}^{v_i} \rho_i^{\mathbb{P}}(x)\,\mathrm{d}x}{\rho_i^{\mathbb{P}}(v_i)} \right) q_i(v_i, \boldsymbol{v}_{-i})\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i},$$

where the equality holds because the bidders' values are independent.

Recalling the definition of the virtual valuation

$$\psi_i^{\mathbb{P}}(v_i) = v_i - \frac{1 - \int_{\underline{v}}^{v_i} \rho_i^{\mathbb{P}}(x)\,\mathrm{d}x}{\rho_i^{\mathbb{P}}(v_i)},$$

we can now rewrite the objective function of $(\boldsymbol{q}, \boldsymbol{m})$ in $(\mathcal{IMDP})$ as

$$\inf_{\mathbb{P} \in \mathcal{P}} \sum_{i \in \mathcal{I}} \left[\int_{\mathcal{V}^{I-1}} \int_{\mathcal{V}} \psi_i^{\mathbb{P}}(v_i)q_i(v_i, \boldsymbol{v}_{-i})\rho^{\mathbb{P}}(\boldsymbol{v})\,\mathrm{d}v_i\,\mathrm{d}\boldsymbol{v}_{-i} \right] = \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[\sum_{i \in \mathcal{I}} \psi_i^{\mathbb{P}}(\tilde{v}_i)q_i(\tilde{v}_i, \tilde{\boldsymbol{v}}_{-i}) \right].$$

Thus, the claim follows. $\square$

Proof of Theorem 6. Throughout this proof, we write $\mathcal{Q}_I^{\text{m-d}}$ instead of $\mathcal{Q}^{\text{m-d}}$ and $\mathcal{P}_I$ instead of $\mathcal{P}$ in order to highlight the dependence on the number of bidders. Moreover, we denote by $f_I(\boldsymbol{q}, \mathbb{P})$ the objective function value of an allocation rule $\boldsymbol{q} \in \mathcal{Q}_I^{\text{m-d}}$ and distribution $\mathbb{P} \in \mathcal{P}_I$ in problem (3).

Select an arbitrary $\epsilon > 0$. Assume first that the seller attracts $I+1$ bidders, and denote by $\boldsymbol{q}^{\mathrm{sp}} \in \mathcal{Q}_{I+1}^{\mathrm{m\text{-}d}}$ the allocation rule of the second price auction without reserve price for $I+1$ bidders. Then, there exists an ϵ-worst-case distribution $\mathbb{P}_\epsilon \in \mathcal{P}_{I+1}$ such that

$$f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}_\epsilon) < \inf_{\mathbb{P} \in \mathcal{P}_{I+1}} f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}) + \epsilon. \tag{25}$$

Denote by $\rho_1^{\mathbb{P}_\epsilon}$ the common marginal density function of the values $\tilde{v}_i$ under the distribution $\mathbb{P}_\epsilon$, $i \in \mathcal{I}$. Note that the virtual valuation $\psi_1^{\mathbb{P}_\epsilon}(v_i)$ is non-decreasing in v_i because $\mathbb{P}_\epsilon \in \mathcal{P}_{I+1}$ is regular. As second price auctions allocate the good to the highest bidder, Proposition 7 implies that

$$f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}_\epsilon) = \mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max\Big\{ \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1}) \Big\} \Big].$$

Next, we derive a lower bound on $f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}_\epsilon)$ by conditioning the above expectation separately on the events $\max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \geq 0$ and $\max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0$. First, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max\Big\{ \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1}) \Big\} \Big| \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \geq 0 \Big] \\
&\geq \mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \Big| \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \geq 0 \Big] \\
&= \mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max\Big\{ \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), 0 \Big\} \Big| \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \geq 0 \Big].
\end{aligned} \tag{26}$$

Similarly, we find

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max\Big\{ \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1}) \Big\} \Big| \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0 \Big] \\
&\geq \max\Big\{ \mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \Big| \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0 \Big], \mathbb{E}_{\mathbb{P}_\epsilon}\big[\psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1}) \big] \Big\} \\
&= \max\Big\{ \mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) \Big| \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0 \Big], 0 \Big\} = 0 \\
&= \mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max\Big\{ \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), 0 \Big\} \Big| \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i) < 0 \Big],
\end{aligned} \tag{27}$$

where the inequality follows from Jensen's inequality and the independence of the bidders' values. In the third line, we use the fact that $\mathbb{E}_{\mathbb{P}_\epsilon}[\psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1})] = 0$, which can be verified through a direct calculation using integration by parts. By combining (26) and (27), we then obtain

$$f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}_\epsilon) = \mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max\Big\{ \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_{I+1}) \Big\} \Big] \geq \mathbb{E}_{\mathbb{P}_\epsilon}\Big[\max\Big\{ \max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), 0 \Big\} \Big]. \tag{28}$$

Consider now the mechanism design problem with I bidders. There exists an ϵ-suboptimal allocation rule $\boldsymbol{q}_\epsilon \in \mathcal{Q}_I^{\mathrm{m\text{-}d}}$ with

$$\inf_{\mathbb{P} \in \mathcal{P}_I} f_I(\boldsymbol{q}_\epsilon, \mathbb{P}) > \sup_{\boldsymbol{q} \in \mathcal{Q}_I^{\mathrm{m\text{-}d}}} \inf_{\mathbb{P} \in \mathcal{P}_I} f_I(\boldsymbol{q}, \mathbb{P}) - \epsilon. \tag{29}$$

Denote by $\mathbb{P}_\epsilon^-$ the marginal distribution of $(\tilde{v}_1, \ldots, \tilde{v}_I)$ under $\mathbb{P}_\epsilon$, and observe that $\mathbb{P}_\epsilon^- \in \mathcal{P}_I$ because the bidders' values are independent and $\mathbb{P}_\epsilon$ is symmetric. Let $\boldsymbol{q}^{\mathbb{P}_\epsilon^-} \in \mathcal{Q}_I^{\mathrm{m\text{-}d}}$ be the allocation rule that maximizes the expected revenues in problem (3) under the distribution $\mathbb{P}_\epsilon^-$. Note that this allocation rule exists due to Theorem 5. Thus, we have

$$\sup_{\boldsymbol{q} \in \mathcal{Q}_I^{\mathrm{m\text{-}d}}} \inf_{\mathbb{P} \in \mathcal{P}_I} f_I(\boldsymbol{q}, \mathbb{P}) - \epsilon < \inf_{\mathbb{P} \in \mathcal{P}_I} f_I(\boldsymbol{q}_\epsilon, \mathbb{P}) \leq f_I(\boldsymbol{q}_\epsilon, \mathbb{P}_\epsilon^-) \leq \sup_{\boldsymbol{q} \in \mathcal{Q}_I^{\mathrm{m\text{-}d}}} f_I(\boldsymbol{q}, \mathbb{P}_\epsilon^-) = f_I(\boldsymbol{q}^{\mathbb{P}_\epsilon^-}, \mathbb{P}_\epsilon^-).$$

Here, the first inequality holds by the construction of $\boldsymbol{q}_\epsilon$, and the equality follows from the optimality of $\boldsymbol{q}^{\mathbb{P}_\epsilon^-}$ for the given distribution $\mathbb{P}_\epsilon^-$. Hence,

$$f_I(\boldsymbol{q}^{\mathbb{P}_\epsilon^-}, \mathbb{P}_\epsilon^-) = \mathbb{E}_{\mathbb{P}_\epsilon^-}\left[\max\left\{\max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon^-}(\tilde{v}_i), 0\right\}\right] = \mathbb{E}_{\mathbb{P}_\epsilon}\left[\max\left\{\max_{i \in \mathcal{I}} \psi_1^{\mathbb{P}_\epsilon}(\tilde{v}_i), 0\right\}\right]$$

$$\leq f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}_\epsilon) < \inf_{\mathbb{P} \in \mathcal{P}_{I+1}} f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}) + \epsilon,$$

where the first equality holds due to Theorem 5, the second equality follows from the definition of $\mathbb{P}_\epsilon^-$, and the inequalities follow from (28) and (25), respectively.

Since ϵ was chosen arbitrarily, the above implies that

$$\sup_{\boldsymbol{q} \in \mathcal{Q}_I^{\mathrm{m\text{-}d}}} \inf_{\mathbb{P} \in \mathcal{P}_I} f_I(\boldsymbol{q}, \mathbb{P}) \leq \inf_{\mathbb{P} \in \mathcal{P}_{I+1}} f_{I+1}(\boldsymbol{q}^{\mathrm{sp}}, \mathbb{P}),$$

and thus the claim follows. □

We need the following auxiliary results to prove Proposition 12.

LEMMA 2. *If $r \in \mathcal{R}_1$ and $\mathcal{P}$ is a Markov ambiguity set of the form (6) with $\mu \in (0, 1)$, then $\mathbb{Q}^{(1)} \in \mathcal{P}$ and*

$$\mathbb{E}_{\mathbb{Q}^{(1)}}\left[\sum_{i \in \mathcal{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\right] = \frac{I\mu - 1}{I - 1}.$$

Proof. Using the inequalities $0 \leq r \leq \frac{I\mu - 1}{I - 1} < 1$, which hold because $r \in \mathcal{R}_1$, one can show that the atoms of $\mathbb{Q}^{(1)}$ have non-negative probabilities that add up to 1. Moreover, we have

$$\mathbb{E}_{\mathbb{Q}^{(1)}}[\tilde{v}_i] = 1\left(1 - \frac{I(\mu - 1)}{(I - 1)(r - 1)}\right) + 1\frac{(\mu - 1)}{(I - 1)(r - 1)} + r(I - 1)\frac{(\mu - 1)}{(I - 1)(r - 1)}$$

$$= 1 - \frac{(I - 1)(\mu - 1)}{(I - 1)(r - 1)} + \frac{r(I - 1)(\mu - 1)}{(I - 1)(r - 1)} = 1 + \frac{(r - 1)(\mu - 1)}{(r - 1)} = \mu \quad \forall i \in \mathcal{I}.$$

This confirms that $\mathbb{Q}^{(1)} \in \mathcal{P}$. A direct calculation further yields

$$\mathbb{E}_{\mathbb{Q}^{(1)}}\left[\sum_{i \in \mathcal{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\right] = 1 - \frac{I(\mu - 1)}{(I - 1)(r - 1)} + I\frac{(\mu - 1)}{(I - 1)(r - 1)}r = 1 + \frac{I(r - 1)(\mu - 1)}{(I - 1)(r - 1)}$$

$$= \frac{(I - 1) + I(\mu - 1)}{(I - 1)} = \frac{I\mu - 1}{I - 1},$$

and thus the claim follows. □

LEMMA 3. *If $r \in \mathcal{R}_1$ and $\mathcal{P}$ is a Markov ambiguity set of the form (6) with $\mu \in (0, 1)$, then $\boldsymbol{\sigma}^{(1)} = \frac{1}{I-1}\mathbf{e}$ and $\lambda^{(1)} = 1 - \frac{I}{I-1}$ are feasible in (8) with objective value $\frac{I\mu - 1}{I - 1}$.*

Proof. Select an arbitrary $\boldsymbol{v} \in \mathcal{V}^I$ and assume without loss of generality that $\boldsymbol{v} \in \mathcal{W}^i$. Due to (9) and the convention that in a second price auction only the winner makes a payment, the left-hand side of the semi-infinite constraint in (8) reduces to $m_i^{\mathrm{sp}}(\boldsymbol{v})$. Moreover, by construction of $\boldsymbol{\sigma}^{(1)}$ and $\lambda^{(1)}$, the right-hand side of the semi-infinite constraint reduces to

$$\lambda^{(1)} + \sum_{j \in \mathcal{I}} \sigma_j^{(1)} v_j = 1 - \frac{I}{I - 1} + \sum_{j \in \mathcal{I}}\left(\frac{1}{I - 1}\right)v_j.$$

If $v_i < r$, we then have

$$m_i^{\mathrm{sp}}(\boldsymbol{v}) = 0 \geq \frac{Ir - 1}{I - 1} = 1 - \frac{I}{I - 1} + I\left(\frac{1}{I - 1}\right)r \geq 1 - \frac{I}{I - 1} + \sum_{j \in \mathcal{I}}\left(\frac{1}{I - 1}\right)v_j,$$

where the first inequality holds because $r \leq \frac{1}{I}$. If $v_i \geq r$, on the other hand, we have

$$m_i^{\mathrm{sp}}(\boldsymbol{v}) = \max\{\max_{j \neq i} v_j, r\} \geq 1 - \frac{I}{I-1} + \left(\frac{1}{I-1}\right)v_i + (I-1)\left(\frac{1}{I-1}\right)\max\{\max_{j \neq i} v_j, r\}$$
$$\geq 1 - \frac{I}{I-1} + \sum_{j \in \mathcal{I}}\left(\frac{1}{I-1}\right)v_j,$$

where the first inequality exploits that $v_i \leq 1$. Finally, the objective value of $(\boldsymbol{\sigma}^{(1)}, \lambda^{(1)})$ in (8) amounts to

$$\lambda^{(1)} + \sum_{i \in \mathcal{I}} \sigma_i^{(1)} \mu = 1 - \frac{I}{I-1} + I\left(\frac{1}{I-1}\right)\mu = \frac{I\mu - 1}{I-1},$$

and thus the claim follows. $\square$

Proof of Proposition 12. The distribution $\mathbb{Q}^{(1)}$ is feasible in (7) due to Lemma 2, and $(\boldsymbol{\sigma}^{(1)}, \lambda^{(1)})$ is feasible in (8) due to Lemma 3. Since the objective value of $\mathbb{Q}^{(1)}$ in (7) is equal to the objective value of $(\boldsymbol{\sigma}^{(1)}, \lambda^{(1)})$ in the dual problem (8) (see Lemmas 2 and 3), $\mathbb{Q}^{(1)}$ is optimal in (7) by weak duality, implying that the worst-case expected revenue amounts to $\frac{I\mu - 1}{I-1}$. $\square$

The proof of Proposition 13 requires the following auxiliary results.

LEMMA 4. *If $r \in \mathcal{R}_2$ and $\mathcal{P}$ is a Markov ambiguity set of the form (6) with $\mu \in (0,1)$, then $\mathbb{Q}_\epsilon^{(2)} \in \mathcal{P}$ for every sufficiently small $\epsilon > 0$, and we have*

$$\lim_{\epsilon \downarrow 0} \mathbb{E}_{\mathbb{Q}_\epsilon^{(2)}}\left[\sum_{i \in \mathcal{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\right] = Ir\frac{\mu - r}{1 - r}.$$

Proof. One can show that the atoms of $\mathbb{Q}_\epsilon^{(2)}$ have non-negative probabilities that add up to 1 for every $\epsilon \leq r - \frac{I\mu - 1}{I-1}$. Note that this upper bound on ϵ is strictly positive because $r \in \mathcal{R}_2$. Moreover, we have

$$\mathbb{E}_{\mathbb{Q}_\epsilon^{(2)}}[\tilde{v}_i] = (r-\epsilon)\left(1 - \frac{I(\mu - (r-\epsilon))}{1 - (r-\epsilon)}\right) + 1\frac{\mu - (r-\epsilon)}{1 - (r-\epsilon)} + (r-\epsilon)(I-1)\frac{\mu - (r-\epsilon)}{1 - (r-\epsilon)}$$
$$= (r-\epsilon) - (r-\epsilon)\frac{(\mu - (r-\epsilon))}{1 - (r-\epsilon)} + \frac{\mu - (r-\epsilon)}{1 - (r-\epsilon)} = (r-\epsilon) + (1 - (r-\epsilon))\frac{(\mu - (r-\epsilon))}{1 - (r-\epsilon)} = \mu \quad \forall i \in \mathcal{I}.$$

This confirms that $\mathbb{Q}_\epsilon^{(2)} \in \mathcal{P}$ for every sufficiently small $\epsilon > 0$.

Note that the ith bidder receives the good only in scenario $\boldsymbol{v} = \mathrm{e}_i + (r-\epsilon)\mathrm{e}_{-i}$, in which case he has to pay the reserve price r. Note also that all other bids are below r in this scenario. Thus, we find

$$\lim_{\epsilon \downarrow 0} \mathbb{E}_{\mathbb{Q}_\epsilon^{(2)}}\left[\sum_{i \in \mathcal{I}} m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\right] = \lim_{\epsilon \downarrow 0} Ir\frac{\mu - (r-\epsilon)}{1 - (r-\epsilon)} = Ir\frac{\mu - r}{1 - r}.$$

This observation completes the proof. $\square$

LEMMA 5. *If $r \in \mathcal{R}_2$ and $\mathcal{P}$ is a Markov ambiguity set of the form (6) with $\mu \in (0,1)$, then $\boldsymbol{\sigma}^{(2)} = \frac{r}{1-r}\mathrm{e}$ and $\lambda^{(2)} = \frac{Ir^2}{r-1}$ are feasible in problem (8) with objective value $Ir\frac{\mu - r}{1 - r}$.*

Proof. Select an arbitrary $\boldsymbol{v} \in \mathcal{V}^I$ and assume without loss of generality that $\boldsymbol{v} \in \mathcal{W}^i$. Recall from the proof of Lemma 3 that the left-hand side of the semi-infinite constraint in (8) reduces to $m_i^{\mathrm{sp}}(\boldsymbol{v})$. Using the definitions of $\boldsymbol{\sigma}^{(2)}$ and $\lambda^{(2)}$, we can further rewrite the right-hand side of the semi-infinite constraint as

$$\lambda^{(2)} + \sum_{j \in \mathcal{I}} \sigma_j^{(2)} v_j = \frac{Ir^2}{r-1} + \sum_{j \in \mathcal{I}}\left(\frac{r}{1-r}\right)v_j.$$

If $v_i < r$, then we have

$$m_i^{\mathrm{sp}}(\boldsymbol{v}) = 0 = \frac{Ir^2}{r-1} + I\left(\frac{r}{1-r}\right)r \geq \frac{Ir^2}{r-1} + \sum_{j\in\mathcal{I}}\left(\frac{r}{1-r}\right)v_j.$$

If $v_i \geq r$, on the other hand, note that

$$r \leq \max\{\max_{j\neq i} v_j, r\} = m_i^{\mathrm{sp}}(\boldsymbol{v})$$

$$\iff (1 - Ir)r \leq (1 - Ir)m_i^{\mathrm{sp}}(\boldsymbol{v})$$

$$\iff \frac{r}{1-r}[(1 - Ir) + (I - 1)m_i^{\mathrm{sp}}(\boldsymbol{v})] \leq m_i^{\mathrm{sp}}(\boldsymbol{v}),$$

where the first equivalence holds because $r \leq \frac{1}{I}$. Hence, we obtain

$$m_i^{\mathrm{sp}}(\boldsymbol{v}) \geq \frac{r}{1-r}[(1 - Ir) + (I - 1)m_i^{\mathrm{sp}}(\boldsymbol{v})] \geq \frac{Ir^2}{r-1} + \frac{r}{1-r}v_i + (I-1)\left(\frac{r}{1-r}\right)\max\{\max_{j\neq i} v_j, r\}$$

$$\geq \frac{Ir^2}{r-1} + \sum_{j\in\mathcal{I}}\left(\frac{r}{1-r}\right)v_j,$$

where the second inequality holds because $v_i \leq 1$ and $m_i(\boldsymbol{v}) = \max\{\max_{j\neq i} v_j, r\}$.

Finally, the objective value of $(\boldsymbol{\sigma}^{(2)}, \lambda^{(2)})$ in problem (8) amounts to

$$\lambda^{(2)} + \sum_{i\in\mathcal{I}}\sigma_i^{(2)}\mu = \frac{Ir^2}{r-1} + I\mu\frac{r}{1-r} = Ir\frac{\mu-r}{1-r},$$

and thus the claim follows. $\quad\square$

Proof of Proposition 13. For every $\epsilon > 0$ small enough, the discrete distribution $\mathbb{Q}_\epsilon^{(2)}$ is feasible in (7) by Lemma 4, and $(\boldsymbol{\sigma}^{(2)}, \lambda^{(2)})$ is feasible in (8) by Lemma 5. Since the limiting objective value of the distributions $\mathbb{Q}_\epsilon^{(2)}$ in (7) for $\epsilon \downarrow 0$ coincides with the objective value of $(\boldsymbol{\sigma}^{(2)}, \lambda^{(2)})$ in the dual problem (8) (see Lemmas 4 and 5), we conclude via weak duality that the distributions $\mathbb{Q}_\epsilon^{(2)}$, $\epsilon \downarrow 0$, are asymptotically optimal in (7), implying that the worst-case expected revenue amounts to $Ir\frac{\mu-r}{1-r}$. $\quad\square$

To prove Proposition 14, we will need the following results.

LEMMA 6. *If $r \in \mathcal{R}_3$ and $\mathcal{P}$ is a Markov ambiguity set of the form* (6) *with $\mu \in (0,1)$, then $\mathbb{Q}_\epsilon^{(3)} \in \mathcal{P}$ for every $\epsilon > 0$, and we have*

$$\lim_{\epsilon\downarrow 0}\mathbb{E}_{\mathbb{Q}_\epsilon^{(3)}}\left[\sum_{i\in\mathcal{I}}m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\right] = \frac{\mu-r}{1-r}.$$

Proof. One can show that the atoms of $\mathbb{Q}_\epsilon^{(3)}$ have non-negative probabilities that add up to 1 because $r \in \mathcal{R}_3$ implies that $r \leq \mu < 1$. Moreover, we have

$$\mathbb{E}_{\mathbb{Q}_\epsilon^{(3)}}[\tilde{v}_i] = 1\left(1 - \frac{1-\mu}{1-(r-\epsilon)}\right) + (r-\epsilon)\left(\frac{1-\mu}{1-(r-\epsilon)}\right) = 1 - \frac{(1-(r-\epsilon))(1-\mu)}{1-(r-\epsilon)} = \mu \quad \forall i \in \mathcal{I}.$$

This confirms that $\mathbb{Q}_\epsilon^{(3)} \in \mathcal{P}$ for every $\epsilon > 0$.

Note that the good is allocated only if $\boldsymbol{v} = \mathbf{e}$, in which case the winner pays an amount equal to 1, that is, the second highest bid. Therefore, we find

$$\lim_{\epsilon\downarrow 0}\mathbb{E}_{\mathbb{Q}_\epsilon^{(3)}}\left[\sum_{i\in\mathcal{I}}m_i^{\mathrm{sp}}(\tilde{\boldsymbol{v}})\right] = \lim_{\epsilon\downarrow 0} 1 - \frac{1-\mu}{1-(r-\epsilon)} = \lim_{\epsilon\downarrow 0}\frac{\mu-(r-\epsilon)}{1-(r-\epsilon)} = \frac{\mu-r}{1-r}.$$

This observation completes the proof. $\quad\square$

LEMMA 7. *If $r \in \mathcal{R}_3$ and $\mathcal{P}$ is a Markov ambiguity set of the form (6) with $\mu \in (0,1)$, then $\boldsymbol{\sigma}^{(3)} = \frac{1}{I(1-r)}\mathbf{e}$ and $\lambda^{(3)} = \frac{r}{r-1}$ are feasible in problem (8) with objective value $\frac{\mu-r}{1-r}$.*

Proof. Select an arbitrary $\boldsymbol{v} \in \mathcal{V}^I$ and assume without loss of generality that $\boldsymbol{v} \in \mathcal{W}^i$. Recall from the proof of Lemma 3 that the left-hand side of the semi-infinite constraint in (8) reduces to $m_i^{\mathrm{sp}}(\boldsymbol{v})$. Using the definitions of $\boldsymbol{\sigma}^{(3)}$ and $\lambda^{(3)}$, we can rewrite the right-hand side of the semi-infinite constraint as

$$\lambda^{(3)} + \sum_{j \in \mathcal{I}} \sigma_j^{(3)} v_j \;=\; \frac{r}{r-1} + \sum_{j \in \mathcal{I}} \frac{1}{I(1-r)} v_j.$$

If $v_i < r$, then we find

$$m_i^{\mathrm{sp}}(\boldsymbol{v}) = 0 = \frac{r}{r-1} + I\frac{1}{I(1-r)}r \;\geq\; \frac{r}{r-1} + \sum_{j \in \mathcal{I}} \frac{1}{I(1-r)} v_j.$$

If $v_i \geq r$, on the other hand, we have

$$1 \geq \max\{\max_{j \neq i} v_j, r\} \;=\; m_i^{\mathrm{sp}}(\boldsymbol{v})$$

$$\iff 1 - Ir \leq (1-Ir)m_i^{\mathrm{sp}}(\boldsymbol{v})$$

$$\iff \frac{1}{I(1-r)}[1 - Ir + (I-1)m_i^{\mathrm{sp}}(\boldsymbol{v})] \leq m_i^{\mathrm{sp}}(\boldsymbol{v}),$$

where the first equivalence holds because $r > \frac{1}{I}$, which implies that $(1 - Ir) < 0$. Hence, we have

$$m_i^{\mathrm{sp}}(\boldsymbol{v}) \geq \frac{1}{I(1-r)}[1 - Ir + (I-1)m_i^{\mathrm{sp}}(\boldsymbol{v})]$$

$$\geq \frac{r}{r-1} + \frac{1}{I(1-r)} v_i + (I-1)\frac{1}{I(1-r)}\max\{\max_{j \neq i} v_j, r\} \geq \frac{r}{r-1} + \sum_{j \in \mathcal{I}} \frac{1}{I(1-r)} v_j.$$

Finally, the objective value of $(\boldsymbol{\sigma}^{(3)}, \lambda^{(3)})$ in problem (8) amounts to

$$\lambda^{(3)} + \sum_{i \in \mathcal{I}} \sigma_i^{(3)} \mu \;=\; \frac{r}{r-1} + \frac{I\mu}{I(1-r)} \;=\; \frac{\mu-r}{1-r},$$

and thus the claim follows. $\square$

Proof of Proposition 14. For every $\epsilon > 0$, the discrete distribution $\mathbb{Q}_\epsilon^{(3)}$ is feasible in (7) by Lemma 6, and $(\boldsymbol{\sigma}^{(3)}, \lambda^{(3)})$ is feasible in (8) by Lemma 7. Since the limiting objective value of the distributions $\mathbb{Q}_\epsilon^{(3)}$, $\epsilon \downarrow 0$, in (7) coincides with the objective value of $(\boldsymbol{\sigma}^{(3)}, \lambda^{(3)})$ in the dual problem (8) (see Lemmas 6 and 7), we conclude via weak duality that the distributions $\mathbb{Q}_\epsilon^{(3)}$, $\epsilon \downarrow 0$, are asymptotically optimal in (7), implying that the worst-case expected revenue amounts to $\frac{\mu-r}{1-r}$. $\square$

Proof of Theorem 8. As for (i), assume first that $\mu \in \mathcal{M}_1$. In this case $\mathcal{R}_3$ is empty. Moreover, the interval $\mathcal{R}_1$ is nonempty only if $\mu = \frac{1}{I}$, which implies that $\frac{I\mu-1}{I-1} = 0$ and leads to a worst-case expected revenue of 0. By the definition of second price auctions, their worst-case expected revenue is at least 0 since $\boldsymbol{m}^{\mathrm{sp}}$ is non-negative. Hence, the optimal reserve price must reside within $\mathcal{R}_2$.

We know from Proposition 13 that, for $r \in \mathcal{R}_2$, the worst-case expected revenue amounts to $Ir\frac{\mu-r}{1-r}$. Elementary calculus shows that

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(Ir\frac{\mu-r}{1-r}\Big) = I - \frac{I(1-\mu)}{(1-r)^2} \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big(Ir\frac{\mu-r}{1-r}\Big) = \frac{2(r-1)I(1-\mu)}{(1-r)^4}.$$

Thus, the worst-case expected revenue is strictly concave and maximized by $r^\star = 1 - \sqrt{1-\mu}$. Note that $r^\star$ is indeed an element of $\mathcal{R}_2$ and results in a worst-case expected revenue of $I(1 - \sqrt{1-\mu})^2$.

Assume next that $\mu \in \mathcal{M}_2$. In this case, the sets $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are all nonempty. If $r \in \mathcal{R}_1$, by Proposition 12, the seller's worst-case expected revenue amounts to $\frac{I\mu-1}{I-1}$ irrespective of r. If $r \in \mathcal{R}_3$, on the other hand, by Proposition 14 the worst-case expected revenue is given by $\frac{\mu-r}{1-r}$, which is a decreasing function of r because $\mu < 1$. Hence, the highest possible worst-case expected revenue corresponding to any reserve price $r \in \mathcal{R}_3$ is given by $\frac{I\mu-1}{I-1}$, which is attained asymptotically as r tends to $\frac{1}{I}$, the left boundary of $\mathcal{R}_3$. If $r \in \mathcal{R}_2$, finally, by Proposition 13 the worst-case expected revenue amounts to $Ir\frac{\mu-r}{1-r}$. At both boundary points $r = \frac{I\mu-1}{I-1}$ and $r = \frac{1}{I}$ of $\mathcal{R}_2$, this function evaluates to $\frac{I\mu-1}{I-1}$. Inside interval $\mathcal{R}_2$ this function is concave and attains its maximum at $r^\star = 1 - \sqrt{1-\mu}$, resulting in a worst-case expected revenue of $I(1 - \sqrt{1-\mu})^2$. Hence, the seller obtains a worst-case expected revenue of $I(1 - \sqrt{1-\mu})^2$ by imposing the optimal reserve price $r^\star = 1 - \sqrt{1-\mu}$ whenever $\mu \in \mathcal{M}_1 \cup \mathcal{M}_2$.

As for (ii), recall that $\mathcal{R}_2$ is empty if $\mu \in \mathcal{M}_3$. We already know that for $r \in \mathcal{R}_3$, the worst-case expected revenue amounts to $\frac{\mu-r}{1-r}$ which is decreasing in r and attains its maximum $\frac{I\mu-1}{I-1}$ as r tends to $\frac{1}{I}$. For $r \in \mathcal{R}_1$ the worst-case expected revenue $\frac{I\mu-1}{I-1}$ does not depend on the reserve price. Hence, if $\mu \in \mathcal{M}_3$, the seller earns a worst-case expected revenue of $\frac{I\mu-1}{I-1}$ by imposing any reserve price $r \in \mathcal{R}_1$. $\qquad\square$

Before proving Theorem 9, we first show that $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ is feasible in problem (11). To this end, we need the following auxiliary result.

LEMMA 8. *For any fixed $\mu \in (0,1)$, $\sigma^\star = -(W_{-1}(-\mu Ie^{-I})+1)^{-1}$ is the unique solution of the equation*

$$\left(\frac{1+\sigma}{I\sigma}\right)e^{(I-1-\frac{1}{\sigma})} = \mu, \tag{30}$$

in the interval $(0, \frac{1}{I-1})$, where W_{-1} denotes the lower branch of the Lambert-W function.

Proof. Set $f(\sigma) = \frac{1+\sigma}{\sigma}e^{(I-1-\frac{1}{\sigma})}$, and note that $\lim_{\sigma \downarrow 0} f(\sigma) = 0$, which follows from L'Hôpital's rule, and that $f(\frac{1}{I-1}) = 1$. Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}f(\sigma) = \frac{1}{I\sigma^3}e^{(I-1-\frac{1}{\sigma})} > 0 \quad \forall \sigma \in \left(0, \frac{1}{I-1}\right).$$

Thus, for any $\mu \in (0,1)$ the equation $f(\sigma) = \mu$ has a unique solution in the interval $(0, \frac{1}{I-1})$.

Equation (30) is equivalent to

$$-\left(\frac{1+\sigma}{\sigma}\right)e^{-\frac{1+\sigma}{\sigma}} = -\mu Ie^{-I} \iff \frac{1+\sigma}{\sigma} = W(-\mu Ie^{-I}) \iff \sigma = -\frac{1}{W(-\mu Ie^{-I})+1},$$

where the first equivalence follows from the definition of the Lambert-W function (Corless et al. 1996). As we are interested in finding a solution of (30) in the interval $(0, \frac{1}{I-1})$ and as the lower branch of the Lambert-W function is at most -1, we thus have that $\sigma^\star = -(W_{-1}(-\mu Ie^{-I})+1)^{-1}$. $\qquad\square$

LEMMA 9. *The solution $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ is feasible in problem (11).*

Proof. Note that $r \in [0,1]$ because $\sigma^\star \in [0, \frac{1}{I-1}]$. Note also that constraint (11c) trivially holds by the construction of $\boldsymbol{q}^\star$. Similarly, it is easy to see that $q_i^\star(\boldsymbol{v})$ is non-decreasing in v_i for every $i \in \mathcal{I}$. Thus, we

only have to show that the proposed solution satisfies constraint (11b) and that the elements of $q^\star(v)$ are non-negative and sum up to at most 1.

Select an arbitrary $v \in \mathcal{V}^I$ and assume without loss of generality that $v \in \mathcal{W}^i$ so that bidder i is the winner. We denote the second highest bid by $v_{j^\star} = \max_{j \neq i} v_j$, where $j^\star$ represents an arbitrary second highest bidder, i.e., $j^\star \in \arg\max_{j \neq i} v_j$. Using the definitions of $\lambda^\star$ and the highest-bidder-lottery allocation rule $q^\star$, we can rewrite (11b) in scenario v as

$$q_i^\star(v_i, v_{-i})v_i - \int_{v_{j^\star}}^{v_i} q_i^\star(x, v_{-i}) \, \mathrm{d}x \geq \sigma^\star \Big(\sum_{j \in \mathcal{I}} v_j \Big) - \sigma^\star r. \tag{31}$$

In the remainder of the proof, we show that $(q^\star, \sigma^\star e, \lambda^\star)$ satisfies (31), $q^\star(v) \geq 0$ and $e^\top q^\star(v) \leq 1$ when scenario v satisfies the conditions in (12a), (12b) and (12c), respectively.

Case 1 ($v_{j^\star} \geq r$): In this case, $q_i^\star(v)$ is given by (12a). Using integration by parts, we can rewrite (31) as

$$q_i^\star(v_i, v_{-i})v_i - x q_i^\star(x, v_{-i}) \Big|_{v_{j^\star}}^{v_i} + \int_{v_{j^\star}}^{v_i} x \partial_x q_i^\star(x, v_{-i}) \, \mathrm{d}x = v_{j^\star} \Big[\sigma^\star \log(1) + I\sigma^\star - \frac{\sigma^\star r}{v_{j^\star}} \Big] + \sigma^\star(v_i - v_{j^\star})$$

$$= \sigma^\star(v_i + (I-1)v_{j^\star}) - \sigma^\star r \geq \sigma^\star \Big(\sum_{j \in \mathcal{I}} v_j \Big) - \sigma^\star r.$$

The first equality holds because the allocation $q_i^\star(x, v_{-i})$ is of the form (12a) for all $x \in [v_{j^\star}, v_i]$. The last inequality holds as $v_{j^\star} \geq v_j$ for all $j \neq i$.

Next, we prove that $q_j^\star(v) \geq 0$ for all $j \in \mathcal{I}$. By construction, we have $q_j^\star(v) = 0$ for all $j \neq i$. To prove that $q_i^\star(v) \geq 0$, we observe that

$$q_i^\star(v) = \sigma^\star \log \Big(\frac{v_i}{v_{j^\star}} \Big) + I\sigma^\star - \frac{\sigma^\star r}{v_{j^\star}} = \sigma^\star \log \Big(\frac{v_i}{v_{j^\star}} \Big) + \sigma^\star \Big(\frac{Iv_{j^\star} - r}{v_{j^\star}} \Big) \geq 0,$$

where the inequality holds because $\sigma^\star \geq 0$, $v_i \geq v_{j^\star}$, and $v_{j^\star} \geq r$.

To prove that the sum of the allocation probabilities is at most 1, we note that

$$\sum_{j \in \mathcal{I}} q_j^\star(v) = q_i^\star(v) = \sigma^\star[\log(v_i) - \log(v_{j^\star})] + I\sigma^\star - \frac{\sigma^\star r}{v_{j^\star}}$$

$$= \sigma^\star \log(v_i) - \sigma^\star \Big(\frac{v_{j^\star} \log(v_{j^\star}) + r}{v_{j^\star}} \Big) + I\sigma^\star$$

$$\leq I\sigma^\star - \sigma^\star \Big(\frac{v_{j^\star} \log(v_{j^\star}) + r}{v_{j^\star}} \Big) \leq I\sigma^\star - \sigma^\star \Big(I - \frac{1}{\sigma^\star} \Big) = 1,$$

where the first equality follows from the definition of the highest-bidder-lottery allocation rule $q^\star$, the first inequality holds because $v_i \leq 1$, and the second inequality holds because the expression in the third line is non-increasing in $v_{j^\star} \in [r, v_i]$, while $r = e^{(I-1-\frac{1}{\sigma^\star})}$.

Case 2 ($v_i \geq r > v_{j^\star}$): In this case, $q_i^\star(v)$ is of the form (12b), whereby (31) reduces to

$$q_i^\star(v_i, v_{-i})v_i - \int_{v_{j^\star}}^{r} q_i^\star(x, v_{-i}) \, \mathrm{d}x - \int_{r}^{v_i} q_i^\star(x, v_{-i}) \, \mathrm{d}x \geq \sigma^\star \Big(\sum_{j \in \mathcal{I}} v_j \Big) - \sigma^\star r. \tag{32}$$

Note that $q_i^\star(x, v_{-i})$ is of the form (12c) for all $x \in [v_{j^\star}, r)$ and of the form (12b) for all $x \in [r, v_i]$. Using integration by parts and recalling that $r = e^{(I-1-\frac{1}{\sigma^\star})}$, we thus obtain

$$\int_{v_{j^\star}}^{r} q_i^\star(x, v_{-i}) \, \mathrm{d}x = x q_i^\star(x, v_{-i}) \Big|_{v_{j^\star}}^{r} - \int_{v_{j^\star}}^{r} x \partial_x q_i^\star(x, v_{-i}) \, \mathrm{d}x = (I-1)\sigma^\star(r - v_{j^\star})$$

and

$$q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - \int_r^{v_i} q_i^\star(x, \boldsymbol{v}_{-i})\, \mathrm{d}x = q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - xq_i^\star(x, \boldsymbol{v}_{-i})\Big|_r^{v_i} + \int_r^{v_i} x\partial_x q_i^\star(x, \boldsymbol{v}_{-i})\, \mathrm{d}x$$

$$= r\Big[\sigma^\star\Big(I - 1 - \frac{1}{\sigma^\star}\Big) + 1\Big] + \sigma^\star(v_i - r) = \sigma^\star v_i + (I-2)\sigma^\star r.$$

Hence, the left-hand side of (32) is equal to

$$\sigma^\star v_i + (I-2)\sigma^\star r - (I-1)\sigma^\star(r - v_{j^\star}) = \sigma^\star v_i + (I-1)\sigma^\star v_{j^\star} - \sigma^\star r.$$

The inequality (32) then follows because $v_{j^\star} \geq v_j$ for all $j \neq i$.

To show that the allocation probabilities are non-negative and that their sum is at most 1, we first note that $q_j^\star(\boldsymbol{v}) = 0$ for all $j \neq i$. Moreover, we have

$$q_i^\star(\boldsymbol{v}) = \sigma^\star \log(v_i) + 1 \geq \sigma^\star(I-1) \geq 0,$$

where the first inequality holds because $v_i \geq r$ and $r = e^{(I-1-\frac{1}{\sigma^\star})}$, while the last inequality follows from Lemma 8. Finally, since $v_i \leq 1$, we obtain

$$\sum_{j \in \mathcal{I}} q_j^\star(\boldsymbol{v}) = q_i^\star(\boldsymbol{v}) = \sigma^\star \log(v_i) + 1 \leq 1.$$

Case 3 ($r > v_i \geq v_{j^\star}$): In this case, $q_i^\star(\boldsymbol{v})$ is given by (12c). Moreover, $q_i^\star(x, \boldsymbol{v}_{-i})$ is of the form (12c) for all $x \in [v_{j^\star}, v_i]$. Using integration by parts, we can thus rewrite the left-hand side of (31) as

$$q_i^\star(v_i, \boldsymbol{v}_{-i})v_i - xq_i^\star(x, \boldsymbol{v}_{-i})\Big|_{v_{j^\star}}^{v_i} + \int_{v_{j^\star}}^{v_i} x\partial_x q_i^\star(x, \boldsymbol{v}_{-i})\, \mathrm{d}x = q_i^\star(v_{j^\star}, \boldsymbol{v}_{-i})v_{j^\star} = (I-1)\sigma^\star v_{j^\star}.$$

We conclude that the inequality (31) is equivalent to

$$(I-1)\sigma^\star v_{j^\star} \geq \sigma^\star\Big(\sum_{j \in \mathcal{I}} v_j\Big) - \sigma^\star r \iff 0 \geq \sigma^\star\Big(v_i - r - (I-1)v_{j^\star} + \sum_{j \neq i} v_j\Big),$$

which is manifestly satisfied because $\sigma^\star \geq 0$, $v_i < r$ and $v_{j^\star} \geq v_j$ for all $j \neq i$.

As $\sigma^\star \in [0, \frac{1}{I-1}]$ by Lemma 8, it is easy to see that the allocation probabilities are non-negative and their sum is at most 1. $\square$

LEMMA 10. *The objective value of* $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ *in problem* (11) *amounts to* r.

Proof. By using (30) and the definition of r, we find

$$\mu = \Big(\frac{1+\sigma^\star}{I\sigma^\star}\Big)e^{(I-1-\frac{1}{\sigma^\star})} = \Big(\frac{1+\sigma^\star}{I\sigma^\star}\Big)r.$$

Recalling that $\lambda^\star = -\sigma^\star r$, the objective value of $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ in (11) can then be expressed as

$$\lambda^\star + \sum_{j \in \mathcal{I}} \sigma^\star \mu = I\sigma^\star\Big(\frac{1+\sigma^\star}{I\sigma^\star}\Big)r - \sigma^\star r = r,$$

and thus the claim follows. $\square$

To prove Theorem 9, we first ignore the monotonicity condition on the allocation rule and show that $(\boldsymbol{q}^\star, \sigma^\star \boldsymbol{e}, \lambda^\star)$ is an optimal solution to the relaxed problem (11) where $\mathcal{Q}^{\text{m-p}}$ is replaced with $\mathcal{Q}$. As $\boldsymbol{q}^\star$ happens to be ex-post monotone, we may then conclude that this solution is also optimal in (11).

LEMMA 11. *The Lagrangian dual of problem* (11) *with* $\mathcal{Q}$ *in lieu of* $\mathcal{Q}^{\mathrm{m\text{-}p}}$ *is equal to*

$$
\inf_{\alpha \in \mathcal{L}_\infty(\mathcal{V}, \mathbb{R}_+)} \quad \int_{[0,1]^I} \max\Big\{0, \sum_{i \in \mathcal{I}} \mathbb{1}_{\mathcal{W}^i}(\boldsymbol{v})\Big(\alpha(\boldsymbol{v})v_i - \int_{v_i}^1 \alpha(x, \boldsymbol{v}_{-i})\, \mathrm{d}x\Big)\Big\}\, \mathrm{d}\boldsymbol{v}
$$

$$
s.t. \quad \int_{[0,1]^I} \alpha(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} = 1 \tag{33}
$$

$$
\int_{[0,1]^I} \alpha(\boldsymbol{v})v_i\, \mathrm{d}\boldsymbol{v} = \mu \qquad\qquad \forall i \in \mathcal{I}.
$$

Proof. The Lagrangian dual of problem (11) with $\mathcal{Q}$ in lieu of $\mathcal{Q}^{\mathrm{m\text{-}p}}$ is given by

$$
\inf_{\alpha, \beta \in \mathcal{L}_\infty(\mathcal{V}, \mathbb{R}_+)} \quad \int_{[0,1]^I} \beta(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v}
$$

$$
\text{s.t.} \quad \alpha(\boldsymbol{v})v_i - \int_{v_i}^1 \alpha(x, \boldsymbol{v}_{-i})\, \mathrm{d}x \le \beta(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{W}^i,\ \forall i \in \mathcal{I}
$$

$$
\int_{[0,1]^I} \alpha(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v} = 1
$$

$$
\int_{[0,1]^I} \alpha(\boldsymbol{v})v_i\, \mathrm{d}\boldsymbol{v} = \mu \qquad\qquad \forall i \in \mathcal{I}.
$$

It is clear that β plays the role of an epigraphical variable. Indeed, for any $\boldsymbol{v} \in \mathcal{W}^i$, $\beta(\boldsymbol{v})$ will be equal to the maximum of 0 and $\alpha(\boldsymbol{v})v_i - \int_{v_i}^1 \alpha(x, \boldsymbol{v}_{-i})\, \mathrm{d}x$ at optimality. We can thus eliminate β and rewrite the above dual problem as (33). $\quad\square$

Note that α can be viewed as the density function of some probability distribution on $[0,1]^I$ with mean μ.

THEOREM 12. *The optimal objective value of problem* (33) *is asymptotically attained by the sequence of density functions*

$$
\alpha_\epsilon(\boldsymbol{v}) = \begin{cases} \rho_\epsilon \dfrac{r}{I\, v_i^2\, \epsilon^{(I-1)}} + \delta_\epsilon \dfrac{r}{I\, \epsilon^I} & \textit{if } \exists i \in \mathcal{I} \textit{ with } \boldsymbol{v} \in \mathcal{W}^i,\ 1 \ge v_i \ge 1 - \epsilon \ \textit{ and } \ r \ge v_j \ge r - \epsilon \ \forall j \neq i, \\[3mm] \rho_\epsilon \dfrac{r}{I\, v_i^2\, \epsilon^{(I-1)}} & \textit{if } \exists i \in \mathcal{I} \textit{ with } \boldsymbol{v} \in \mathcal{W}^i, 1 - \epsilon > v_i \ge r \ \textit{ and } \ r \ge v_j \ge r - \epsilon \ \forall j \neq i, \\[3mm] 0 & \textit{otherwise,} \end{cases} \tag{34}
$$

where

$$
\delta_\epsilon = \frac{1 - r\big(\frac{1}{\sigma^\star} - (I-2)\big) - \frac{(I-1)}{2}(1 - r^{-1})\epsilon}{1 - \frac{\epsilon}{2} - r\big(\frac{1}{\sigma^\star} - \frac{\epsilon}{2} - (I-2)\big)} \quad and \quad \rho_\epsilon = \frac{1 - \delta_\epsilon r}{1 - r}
$$

for $\epsilon \downarrow 0$.

One can verify that $\delta_\epsilon > 1$ and $0 < \rho_\epsilon < 1$ for small enough $\epsilon > 0$. Figure 6 visualizes α_ϵ.

We prove Theorem 12 together with Theorem 9. The proof relies on the following auxiliary results.

LEMMA 12. *The function* α_ϵ *defined in* (34) *is feasible in* (33) *for every* $\epsilon > 0$ *small enough.*

Proof. Note first that δ_ϵ and ρ_ϵ are positive for $\epsilon > 0$ small enough because $r \in [0,1]$. Thus, we have $\alpha_\epsilon(\boldsymbol{v}) \ge 0$ for all $\boldsymbol{v} \in [0,1]^I$. It remains to be shown that α_ϵ satisfies the normalization and mean constraints in (33).
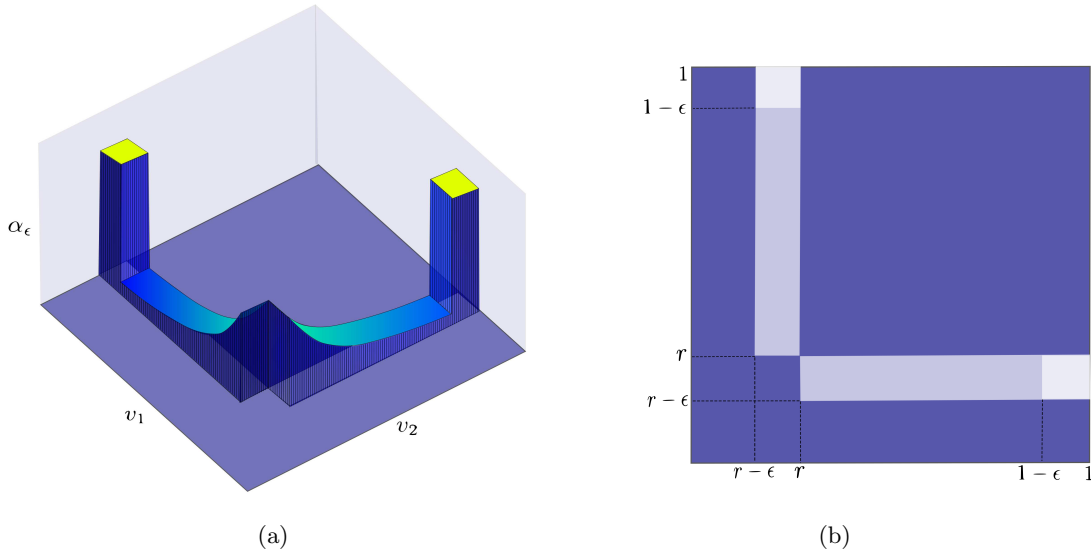
Figure 6   Visualization of α_ϵ.

As for the normalization constraint, we have

$$\int_{[0,1]^I} \alpha_\epsilon(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} = \sum_{i \in \mathcal{I}} \int_{\mathcal{W}^i} \alpha_\epsilon(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} = \sum_{i \in \mathcal{I}} \int_r^1 \int_{[r-\epsilon,r]^{I-1}} \alpha_\epsilon(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v}_{-i} \, \mathrm{d}v_i = \sum_{i \in \mathcal{I}} \int_r^1 \epsilon^{(I-1)} \alpha_\epsilon(\boldsymbol{v}) \, \mathrm{d}v_i$$

$$= \sum_{i \in \mathcal{I}} \left[\int_r^1 \rho_\epsilon \frac{r}{I \, v_i^2} \, \mathrm{d}v_i + \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{I \, \epsilon} \, \mathrm{d}v_i \right] = \int_r^1 \rho_\epsilon \frac{r}{v_1^2} \, \mathrm{d}v_1 + \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{\epsilon} \, \mathrm{d}v_1. \tag{35}$$

The second equality holds because, for $\boldsymbol{v} \in \mathcal{W}^i$, $\alpha_\epsilon(\boldsymbol{v})$ is non-zero only if $v_i \in [r,1]$ and $\boldsymbol{v}_{-i} \in [r-\epsilon, r]^{(I-1)}$, while the third equality holds because $\alpha(\boldsymbol{v})$ is constant in $\boldsymbol{v}_{-i}$ as long as $\boldsymbol{v} \in \mathcal{W}^i$. The last equality exploits the permutation symmetry of α_ϵ. By explicitly calculating the integrals, (35) simplifies to

$$I \left(\rho_\epsilon \frac{(1-r)}{I} + \delta_\epsilon \frac{r}{I} \right) = \frac{1 - \delta_\epsilon r}{1-r} (1-r) + \delta_\epsilon r = 1,$$

where the first equality follows from the definition of ρ_ϵ.

Next, we verify that α_ϵ satisfies the mean constraint. For an arbitrary $i \in \mathcal{I}$, we have

$$\int_{[0,1]^I} \alpha_\epsilon(\boldsymbol{v}) v_i \, \mathrm{d}\boldsymbol{v} = \sum_{j \in \mathcal{I}} \int_{\mathcal{W}^j} \alpha_\epsilon(\boldsymbol{v}) v_i \, \mathrm{d}\boldsymbol{v} = \int_r^1 \int_{[r-\epsilon,r]^{I-1}} \alpha_\epsilon(\boldsymbol{v}) v_i \, \mathrm{d}\boldsymbol{v}_{-i} \, \mathrm{d}v_i + \sum_{j \neq i} \int_r^1 \int_{[r-\epsilon,r]^{I-1}} \alpha_\epsilon(\boldsymbol{v}) v_i \, \mathrm{d}\boldsymbol{v}_{-j} \, \mathrm{d}v_j$$

$$= \int_r^1 \rho_\epsilon \frac{r}{I \, v_i} \, \mathrm{d}v_i + \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{I \, \epsilon} v_i \, \mathrm{d}v_i + \sum_{j \neq i} \left[\int_r^1 \int_{(r-\epsilon)}^r \rho_\epsilon \frac{r}{I \, v_j^2 \, \epsilon} v_i \, \mathrm{d}v_i \, \mathrm{d}v_j + \int_{1-\epsilon}^1 \int_{(r-\epsilon)}^r \delta_\epsilon \frac{r}{I \, \epsilon^2} v_i \, \mathrm{d}v_i \, \mathrm{d}v_j \right]. \tag{36}$$

The second equality holds because, for $\boldsymbol{v} \in \mathcal{W}^j$, $\alpha_\epsilon(\boldsymbol{v})$ is non-zero only if $v_j \in [r,1]$ and $\boldsymbol{v}_{-j} \in [r-\epsilon, r]^{(I-1)}$, while the last equality holds because $\alpha_\epsilon(\boldsymbol{v})$ is constant in $\boldsymbol{v}_{-j}$ as long as $\boldsymbol{v} \in \mathcal{W}^j$.

An explicit calculation yields

$$\int_r^1 \rho_\epsilon \frac{r}{I \, v_i} \, \mathrm{d}v_i + \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{I \, \epsilon} v_i \, \mathrm{d}v_i = \rho_\epsilon \frac{r}{I} \left(\frac{1}{\sigma^\star} - I + 1 \right) + \delta_\epsilon \frac{r}{I} \left(1 - \frac{\epsilon}{2} \right), \tag{37a}$$

where we use the relation $\log(r) = I - 1 - \frac{1}{\sigma^\star}$, which follows from the definition of r. Similarly, for an arbitrary $j \neq i$, we find

$$\int_r^1 \int_{(r-\epsilon)}^r \rho_\epsilon \frac{r}{I \, v_j^2 \, \epsilon} v_i \, \mathrm{d}v_i \, \mathrm{d}v_j = \int_r^1 \rho_\epsilon \frac{r}{I \, v_j^2} \left(r - \frac{\epsilon}{2} \right) \mathrm{d}v_j = \rho_\epsilon \frac{(1-r)}{I} \left(r - \frac{\epsilon}{2} \right) \tag{37b}$$

and

$$\int_{1-\epsilon}^{1}\int_{(r-\epsilon)}^{r}\delta_\epsilon\frac{r}{I\,\epsilon^2}v_i\,\mathrm{d}v_i\,\mathrm{d}v_j = \int_{1-\epsilon}^{1}\delta_\epsilon\frac{r}{I\,\epsilon}\Big(r-\frac{\epsilon}{2}\Big)\,\mathrm{d}v_j = \delta_\epsilon\frac{r}{I}\Big(r-\frac{\epsilon}{2}\Big). \tag{37c}$$

Substituting (37) into (36) and using the permutation symmetry of α_ϵ, we obtain

$$\rho_\epsilon\frac{r}{I}\Big(\frac{1}{\sigma^\star}-I+1\Big)+\delta_\epsilon\frac{r}{2I}(2-\epsilon)+(I-1)\Big[\rho_\epsilon\frac{(1-r)}{I}\Big(r-\frac{\epsilon}{2}\Big)+\delta_\epsilon\frac{r}{I}\Big(r-\frac{\epsilon}{2}\Big)\Big]$$

$$=\frac{1}{I}\rho_\epsilon\Big(r\Big(\frac{1}{\sigma^\star}-(I-1)r+(I-1)\frac{\epsilon}{2}\Big)-(I-1)\frac{\epsilon}{2}\Big)+\frac{1}{I}\delta_\epsilon r\Big(1-\frac{I\epsilon}{2}+(I-1)r\Big)$$

$$=\frac{1}{I}\Big(\frac{1}{1-r}\Big)\Big(r\Big(\frac{1}{\sigma^\star}-(I-1)r+(I-1)\frac{\epsilon}{2}\Big)-(I-1)\frac{\epsilon}{2}\Big)+\frac{1}{I}\delta_\epsilon r\Big(\frac{1}{1-r}\Big)\Big(1-\frac{\epsilon}{2}-r\Big(2-I-\frac{\epsilon}{2}+\frac{1}{\sigma^\star}\Big)\Big)$$

$$=\frac{1}{I}\Big(\frac{1}{1-r}\Big)\Big(r\Big(\frac{1}{\sigma^\star}-(I-1)r+(I-1)\frac{\epsilon}{2}\Big)-(I-1)\frac{\epsilon}{2}\Big)$$

$$+\frac{1}{I}\Big(1-r\Big(\frac{1}{\sigma^\star}-(I-2)\Big)-\frac{(I-1)}{2}(1-r^{-1})\epsilon\Big)r\Big(\frac{1}{1-r}\Big)$$

$$=\frac{1}{I}\Big(\frac{1}{1-r}\Big)\Big[r\Big(\frac{1}{\sigma^\star}+1-r\Big(\frac{1}{\sigma^\star}+1\Big)\Big)\Big]$$

$$=\frac{1}{I}r\Big(\frac{1}{\sigma^\star}+1\Big)=\mu.$$

Here, the first equality follows from grouping terms that involve ρ_ϵ and terms that involve δ_ϵ. The second equality follows from replacing ρ_ϵ with its definition and rearranging terms. Similarly, the fourth equality follows from replacing δ_ϵ with its definition. The remaining reformulations are based on elementary algebra. $\square$

LEMMA 13. *As ϵ tends to zero, the objective value of α_ϵ in (33) converges to r.*

Proof. Throughout the proof we assume that $\epsilon<\frac{1}{2}$. Substituting α_ϵ into the objective function of (33) yields

$$\int_{[0,1]^I}\max\Big\{0,\sum_{i\in\mathcal{I}}\mathbb{1}_{\mathcal{W}^i}(\boldsymbol{v})\Big(\alpha_\epsilon(\boldsymbol{v})v_i-\int_{v_i}^{1}\alpha_\epsilon(x,\boldsymbol{v}_{-i})\,\mathrm{d}x\Big)\Big\}\,\mathrm{d}\boldsymbol{v}$$

$$=\sum_{i\in\mathcal{I}}\int_{\mathcal{W}^i}\max\Big\{0,\alpha_\epsilon(\boldsymbol{v})v_i-\int_{v_i}^{1}\alpha_\epsilon(x,\boldsymbol{v}_{-i})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}$$

$$=I\int_{r}^{1}\int_{[r-\epsilon,r]^{I-1}}\max\Big\{0,\alpha_\epsilon(\boldsymbol{v})v_1-\int_{v_1}^{1}\alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1. \tag{38}$$

Here, the first equality is obtained by partitioning the integration domain into the subsets $\mathcal{W}^i$, $i\in\mathcal{I}$. The second equality follows from symmetry and because, for $\boldsymbol{v}\in\mathcal{W}^i$, $\alpha_\epsilon(\boldsymbol{v})$ is non-zero only if $v_i\in[r,1]$ and $\boldsymbol{v}_{-i}\in[r-\epsilon,r]^{(I-1)}$. We can decompose the integral in (38) into the two terms

$$\int_{r}^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}}\max\Big\{0,\alpha_\epsilon(\boldsymbol{v})v_1-\int_{v_1}^{1}\alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1$$

$$+\int_{1-\epsilon}^{1}\int_{[r-\epsilon,r]^{I-1}}\max\Big\{0,\alpha_\epsilon(\boldsymbol{v})v_1-\int_{v_1}^{1}\alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1,$$

which we investigate separately below. The first integral reduces to

$$\int_{r}^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}}\max\Big\{0,\alpha_\epsilon(\boldsymbol{v})v_1-\int_{v_1}^{1}\alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1$$

$$=\int_{r}^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}}\max\Big\{0,\rho_\epsilon\frac{r}{I\,v_1\,\epsilon^{(I-1)}}-\int_{v_1}^{1}\rho_\epsilon\frac{r}{I\,x^2\,\epsilon^{(I-1)}}\,\mathrm{d}x-\int_{1-\epsilon}^{1}\delta_\epsilon\frac{r}{I\,\epsilon^I}\,\mathrm{d}x\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1$$

$$=\int_{r}^{1-\epsilon}\int_{[r-\epsilon,r]^{I-1}}\max\Big\{0,\rho_\epsilon\frac{r}{I\,\epsilon^{(I-1)}}-\delta_\epsilon\frac{r}{I\,\epsilon^{(I-1)}}\Big\}\,\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1=0,$$

where the last equality holds because $\rho_\epsilon \leq 1 \leq \delta_\epsilon$. Similarly, the second integral can be rewritten as

$$\int_{1-\epsilon}^1 \int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x, \boldsymbol{v}_{-1})\,\mathrm{d}x\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1$$

$$= \int_{1-\epsilon}^1 \int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \rho_\epsilon \frac{r}{I\,v_1\,\epsilon^{(I-1)}} + \delta_\epsilon \frac{r}{I\,\epsilon^I}v_1 - \int_{v_1}^1 \left(\rho_\epsilon \frac{r}{I\,x^2\,\epsilon^{(I-1)}} + \delta_\epsilon \frac{r}{I\,\epsilon^I}\right)\mathrm{d}x\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1. \quad (39)$$

The second argument of the max function in (39) is equal to

$$\rho_\epsilon \frac{r}{I\,v_1\,\epsilon^{(I-1)}} + \delta_\epsilon \frac{r}{I\,\epsilon^I}v_1 + \rho_\epsilon \frac{r}{I\,x\,\epsilon^{(I-1)}}\Big|_{v_1}^1 - \delta_\epsilon \frac{r}{I\,\epsilon^I}x\Big|_{v_1}^1$$

$$= 2\,\delta_\epsilon \frac{r}{I\,\epsilon^I}v_1 + \rho_\epsilon \frac{r}{I\,\epsilon^{(I-1)}} - \delta_\epsilon \frac{r}{I\,\epsilon^I} = \frac{r}{I\,\epsilon^I}\left(\delta_\epsilon(2v_1-1) + \rho_\epsilon\epsilon\right).$$

Note that the last expression is non-negative for all $v_1 \in [1-\epsilon, 1]$ because ρ_ϵ and δ_ϵ are non-negative and because $\epsilon < \frac{1}{2}$. In summary, (39) thus reduces to

$$\int_{1-\epsilon}^1 \int_{[r-\epsilon,r]^{I-1}} \frac{r}{I\,\epsilon^I}\left(\delta_\epsilon(2v_1-1) + \rho_\epsilon\epsilon\right)\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 = \int_{1-\epsilon}^1 \frac{r}{I\,\epsilon}\left(\delta_\epsilon(2v_1-1) + \rho_\epsilon\epsilon\right)\mathrm{d}v_1$$

$$= \frac{r}{I\,\epsilon}\delta_\epsilon(v_1^2 - v_1)\Big|_{1-\epsilon}^1 + \frac{r}{I}\rho_\epsilon v_1\Big|_{1-\epsilon}^1 = \frac{r}{I}\delta_\epsilon(1-\epsilon) + \frac{r}{I}\rho_\epsilon\epsilon.$$

Therefore, the asymptotic objective value of α_ϵ for small ϵ is given by

$$\lim_{\epsilon\downarrow 0} r\,\delta_\epsilon(1-\epsilon) + r\rho_\epsilon\epsilon = r$$

because both δ_ϵ and ρ_ϵ converge to 1 as ϵ tends to 0. $\quad\square$

We are now ready to prove Theorems 9 and 12.

Proof of Theorems 9 and 12. By Lemmas 9 and 10, $(\boldsymbol{q}^\star, \sigma^\star\boldsymbol{e}, \lambda^\star)$ is feasible in (11) with the objective value r. By Lemmas 12 and 13, on the other hand, α_ϵ is feasible in problem (33) and asymptotically attains the objective value r for $\epsilon \downarrow 0$. As (33) is the dual of a relaxation of (11) (obtained by replacing $\mathcal{Q}^{\text{m-p}}$ with $\mathcal{Q}$), it is a restriction of the dual of (11). Thus, α_ϵ is a feasible solution in the dual of (11) that certifies via weak duality that $(\boldsymbol{q}^\star, \sigma^\star\boldsymbol{e}, \lambda^\star)$ is optimal in (11). The corresponding worst-case expected revenue amounts to $r = e^{(I-1-\frac{1}{\sigma^\star})}$. $\quad\square$

Proof of Proposition 16. For the purpose of this proof we let $\sigma^\star(I) = -\frac{1}{W_{-1}(-\mu I e^{-I})+1}$ denote the value of $\sigma^\star$ from Theorem 9 for a fixed number of bidders I. We first show that $I\sigma^\star(I)$ converges to 1 as I tends to infinity. Since $-\mu I e^{-I}$ drops to 0 as I grows, we obtain

$$W_{-1}(-\mu I e^{-I}) = \log(\mu I e^{-I}) - \log(-\log(\mu I e^{-I})) + o(1) = -I + \log(\mu I) - \log(-\log(\mu I) + I) + o(1), \quad (40)$$

where the first equality follows from a well-known asymptotic expansion of the Lambert-W function (see Corless et al. 1996). Thus, we have

$$\lim_{I\to\infty} I\sigma^\star(I) = \lim_{I\to\infty} \frac{I}{I - \log(\mu I) + \log(-\log(\mu I) + I) + o(1)} = 1,$$

which implies that $\sigma^\star(I)$ converges to 0 as I tends to infinity. By (12a)-(12d) and (13), it is immediate that the optimal highest-bidder-lottery $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ converges uniformly to the second price auction without reserve price. $\quad\square$

To prove Theorem 10, we first derive an upper bound on the supremum of problem ($\mathcal{MMDP}$). Note that ($\mathcal{MMDP}$) is equivalent to problem (11) without the constraint (11c) that restricts attention to highest-bidder-lotteries. An upper bound on ($\mathcal{MMDP}$) is thus obtained by solving problem (11) without the constraint (11c) and by ignoring the monotonicity condition on $\boldsymbol{q}$, which is tantamount to relaxing $\mathcal{Q}^{\text{m-p}}$ to $\mathcal{Q}$. By weak duality, the dual of this relaxation also provides an upper bound on ($\mathcal{MMDP}$).

LEMMA 14. *The Lagrangian dual of problem* (11) *without the constraint* (11c) *and with* $\mathcal{Q}$ *in lieu of* $\mathcal{Q}^{\text{m-p}}$ *is equivalent to*

$$
\inf_{\alpha \in \mathcal{L}_\infty(\mathcal{V}, \mathbb{R}_+)} \quad \int_{[0,1]^I} \max\left\{0, \max_{i \in \mathcal{I}}\left(\alpha(\boldsymbol{v})v_i - \int_{v_i}^1 \alpha(x, \boldsymbol{v}_{-i})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}
$$
$$
s.t. \quad \int_{[0,1]^I} \alpha(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} = 1 \tag{41}
$$
$$
\int_{[0,1]^I} \alpha(\boldsymbol{v})v_i\,\mathrm{d}\boldsymbol{v} = \mu \qquad\qquad \forall i \in \mathcal{I}.
$$

Proof. The proof widely parallels that of Lemma 11. Details are omitted for brevity. □

The feasible set of (41) is equivalent to that of (33). Thus, by Lemma 12, α_ϵ defined in (34) is feasible in (41). We can now use the objective value of α_ϵ in (41) as an upper bound on the supremum of ($\mathcal{MMDP}$).

LEMMA 15. *The objective value of problem* ($\mathcal{MMDP}$) *is bounded above by* $r(2-r)$, *where* r *denotes the worst-case expected revenue of the optimal highest-bidder-lottery as defined in Theorem 9.*

Proof. By Lemma 12 and Lemma 14, α_ϵ defined in (34) is feasible in (41) and its objective value in (41) provides an upper bound on the supremum of problem ($\mathcal{MMDP}$). Throughout the proof we assume that $\epsilon \leq \frac{1-\sqrt{r}}{2}$. Note that $r \leq \mu$ by Proposition 10 and because the worst-case expected full surplus provides an upper bound on the optimal objective value of ($\mathcal{MMDP}$). Recall also that $\mu < 1$ by assumption. This implies that $\frac{1-\sqrt{r}}{2} > 0$. Substituting α_ϵ into the objective function of (41) yields

$$
\int_{[0,1]^I} \max\left\{0, \max_{j \in \mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{\boldsymbol{v}_j}^1 \alpha_\epsilon(x, \boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}
$$
$$
= \sum_{i \in \mathcal{I}} \int_{\mathcal{W}^i} \max\left\{0, \max_{j \in \mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x, \boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}
$$
$$
= I \int_r^1 \int_{[r-\epsilon, r]^{I-1}} \max\left\{0, \max_{j \in \mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x, \boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1. \tag{42}
$$

Here, the first equality is obtained by partitioning the integration domain into the subsets $\mathcal{W}^i$, $i \in \mathcal{I}$. The second equality follows from symmetry and because, for $\boldsymbol{v} \in \mathcal{W}^i$, $\alpha_\epsilon(\boldsymbol{v})$ is non-zero only if $v_i \in [r, 1]$ and $\boldsymbol{v}_{-i} \in [r-\epsilon, r]^{(I-1)}$. We can decompose the integral in (42) into the two terms

$$
\int_r^{1-\epsilon} \int_{[r-\epsilon, r]^{I-1}} \max\left\{0, \max_{j \in \mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x, \boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1
$$
$$
+ \int_{1-\epsilon}^1 \int_{[r-\epsilon, r]^{I-1}} \max\left\{0, \max_{j \in \mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x, \boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1, \tag{43}
$$

which we will investigate separately. We first consider the first integral. To this end, select an arbitrary $\boldsymbol{v} \in \mathcal{W}^1$ such that $v_1 \in [r, 1-\epsilon)$ and $v_j \in [r-\epsilon, r]$ for all $j \neq 1$. We have

$$
\begin{aligned}
\alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x, \boldsymbol{v}_{-1})\,\mathrm{d}x &= \rho_\epsilon \frac{r}{I v_1 \epsilon^{(I-1)}} - \int_{v_1}^1 \rho_\epsilon \frac{r}{I x^2 \epsilon^{(I-1)}}\,\mathrm{d}x - \int_{1-\epsilon}^1 \delta_\epsilon \frac{r}{I \epsilon^I}\,\mathrm{d}x \\
&= \rho_\epsilon \frac{r}{I \epsilon^{(I-1)}} - \delta_\epsilon \frac{r}{I \epsilon^{(I-1)}} \leq 0,
\end{aligned}
$$

where the last equality holds because $\rho_\epsilon \le \delta_\epsilon$. For any $j \ne 1$, we have

$$\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x, \boldsymbol{v}_{-j})\,\mathrm{d}x \;\le\; \alpha_\epsilon(\boldsymbol{v})v_j \;\le\; \alpha_\epsilon(\boldsymbol{v})r \;=\; \rho_\epsilon \frac{r^2}{Iv_1^2\epsilon^{(I-1)}}.$$

In summary, for the first integral in (43), we have

$$\int_r^{1-\epsilon} \int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \max_{j\in\mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1$$

$$\le \int_r^{1-\epsilon} \int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \rho_\epsilon \frac{r^2}{Iv_1^2\epsilon^{(I-1)}}\right)\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1$$

$$= \int_r^{1-\epsilon} \rho_\epsilon \frac{r^2}{Iv_1^2}\,\mathrm{d}v_1 \;=\; -\rho_\epsilon \frac{r^2}{Iv_1}\bigg|_r^{1-\epsilon} \;=\; \rho_\epsilon\frac{r(1-\epsilon-r)}{I(1-\epsilon)}.$$

We now investigate the second integral in (43). To this end, select an arbitrary $\boldsymbol{v} \in \mathcal{W}^1$ such that $v_1 \in [1-\epsilon,1]$ and $v_j \in [r-\epsilon,r]$ for all $j \ne 1$. We have

$$\alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x \;= \rho_\epsilon \frac{r}{I\,v_1\,\epsilon^{(I-1)}} + \delta_\epsilon \frac{r}{I\,\epsilon^I}v_1 - \int_{v_1}^1\left(\rho_\epsilon\frac{r}{I\,x^2\,\epsilon^{(I-1)}} + \delta_\epsilon\frac{r}{I\,\epsilon^I}\right)\mathrm{d}x$$

$$= \rho_\epsilon\frac{r}{I\,v_1\,\epsilon^{(I-1)}} + \delta_\epsilon\frac{r}{I\,\epsilon^I}v_1 + \rho_\epsilon\frac{r}{I\,x\,\epsilon^{(I-1)}}\bigg|_{v_1}^1 - \delta_\epsilon\frac{r}{I\,\epsilon^I}x\bigg|_{v_1}^1$$

$$= 2\,\delta_\epsilon\frac{r}{I\,\epsilon^I}v_1 + \rho_\epsilon\frac{r}{I\,\epsilon^{(I-1)}} - \delta_\epsilon\frac{r}{I\,\epsilon^I} \;=\; \frac{r}{I\,\epsilon^I}\left(\delta_\epsilon(2v_1-1) + \rho_\epsilon\epsilon\right).$$

Similarly, for any $j \ne 1$, we have

$$\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x \;\le\; \alpha_\epsilon(\boldsymbol{v})v_j \;\le\; \alpha_\epsilon(\boldsymbol{v})r \;=\; \frac{r^2}{I\epsilon^{(I-1)}}\left(\rho_\epsilon\frac{1}{v_1^2} + \delta_\epsilon\frac{1}{\epsilon}\right).$$

It is easy to show that $\frac{r}{I\epsilon^I}\left(\delta_\epsilon(2v_1-1)+\rho_\epsilon\epsilon\right) \ge \frac{r^2}{I\epsilon^{(I-1)}}\left(\rho_\epsilon\frac{1}{v_1^2}+\delta_\epsilon\frac{1}{\epsilon}\right)$ as the coefficients of δ_ϵ and ρ_ϵ on the left-hand side are greater than or equal to the respective coefficients on the right-hand side, which follows from $\epsilon \le \frac{1-\sqrt{r}}{2}$. Thus, the second integral in (43) simplifies to

$$\int_{1-\epsilon}^1 \int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \max_{j\in\mathcal{I}}\left(\alpha_\epsilon(\boldsymbol{v})v_j - \int_{v_j}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-j})\,\mathrm{d}x\right)\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1$$

$$= \int_{1-\epsilon}^1 \int_{[r-\epsilon,r]^{I-1}} \max\left\{0, \alpha_\epsilon(\boldsymbol{v})v_1 - \int_{v_1}^1 \alpha_\epsilon(x,\boldsymbol{v}_{-1})\,\mathrm{d}x\right\}\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1$$

$$= \int_{1-\epsilon}^1 \int_{[r-\epsilon,r]^{I-1}} \frac{r}{I\epsilon^I}\left(\delta_\epsilon(2v_1-1)+\rho_\epsilon\epsilon\right)\mathrm{d}\boldsymbol{v}_{-1}\,\mathrm{d}v_1 \;=\; \int_{1-\epsilon}^1 \frac{r}{I\epsilon}\left(\delta_\epsilon(2v_1-1)+\rho_\epsilon\epsilon\right)\mathrm{d}v_1$$

$$= \frac{r}{I\epsilon}\delta_\epsilon(v_1^2-v_1)\bigg|_{1-\epsilon}^1 + \frac{r}{I}\rho_\epsilon v_1\bigg|_{1-\epsilon}^1 \;=\; \frac{r}{I}\delta_\epsilon(1-\epsilon) + \frac{r}{I}\rho_\epsilon\epsilon.$$

As both δ_ϵ and ρ_ϵ converge to 1 when ϵ tends to 0, the asymptotic objective value of α_ϵ for small ϵ is

$$\lim_{\epsilon\downarrow 0}\rho_\epsilon\frac{r(1-\epsilon-r)}{(1-\epsilon)} + r\,\delta_\epsilon(1-\epsilon) + r\rho_\epsilon\epsilon \;=\; r(2-r).$$

Moreover, as (41) is the dual of a relaxation of ($\mathcal{MMDP}$), we conclude via weak duality that the optimal value of ($\mathcal{MMDP}$) is bounded above by $r(2-r)$. $\square$

We are now ready to prove Theorem 10.

Proof of Theorem 10. By Lemma 15, the supremum of problem ($\mathcal{MMDP}$) is bounded above by $r(2-r)$, where r denotes the worst-case expected revenue of the optimal highest-bidder-lottery. Thus, the ratio of worst-case expected revenues of the optimal highest-bidder-lottery and the unknown optimal mechanism is at least $\frac{r}{r(2-r)} = \frac{1}{2-r} \ge \frac{1}{2}$. $\square$

Proof of Theorem 11. Fix a number of bidders $I \in \mathbb{N}$. By Theorem 8, the worst-case expected revenue of the best second price auction amounts to $I(1 - \sqrt{1 - \mu})^2$ for any $\mu \in \mathcal{M}_1 = (0, \frac{1}{I}]$. By Theorem 9, we also know that the worst-case expected revenue of the optimal highest-bidder-lottery equals $e^{(I - 1 - \frac{1}{\sigma^\star})}$, where $\sigma^\star = -(W_{-1}(-\mu I e^{-I}) + 1)^{-1}$.

We will show that, as μ approaches zero, the ratio of worst-case expected revenues of the best second price auction and the optimal highest-bidder-lottery becomes arbitrarily small. Indeed, we have

$$\lim_{\mu \downarrow 0} \frac{I(1 - \sqrt{1 - \mu})^2}{e^{(I - 1 - \frac{1}{\sigma^\star})}} = \lim_{\mu \downarrow 0} \frac{I(1 - \sqrt{1 - \mu})^2}{e^{(\log(\mu I) - \log(-\log(\mu I) + I) + o(1))}} = \lim_{\mu \downarrow 0} \frac{(1 - \sqrt{1 - \mu})^2 (I - \log(\mu I))}{\mu e^{o(1)}},$$

where the first equality follows from the definition of $\sigma^\star$ and the asymptotic expansion (40) of the Lambert-W function. As $\lim_{\mu \downarrow 0} e^{o(1)} = 1$, the last expression equals

$$\lim_{\mu \downarrow 0} \frac{(1 - \sqrt{1 - \mu})^2}{\frac{\mu}{(I - \log(\mu I))}} = \lim_{\mu \downarrow 0} \frac{\frac{1 - \sqrt{1 - \mu}}{\sqrt{1 - \mu}}}{\frac{I + 1 - \log(\mu I)}{(I - \log(\mu I))^2}} = \lim_{\mu \downarrow 0} \frac{1}{2(1 - \mu)^{(3/2)}} \frac{\mu (I - \log(\mu I))^3}{(I + 2 - \log(\mu I))} = 0,$$

where the first and the second equalities both follow from L'Hôpital's rule. The last equality holds because $\lim_{\mu \downarrow 0} \mu (I - \log(\mu I))^3 = 0$, which can also be derived using L'Hôpital's rule. Thus the claim follows. $\quad\square$

Proof of Proposition 17. By Theorem 9, the worst-case expected revenue of the optimal highest-bidder-lottery amounts to

$$e^{(I - 1 - \frac{1}{\sigma^\star(I)})} = e^{(I + W_{-1}(-\mu I e^{-I}))}.$$

Moreover, as $\mu \in (0, 1)$, there exists I_μ such that $\mu \in \mathcal{M}_3$ for all $I \geq I_\mu$. By Theorem 8, the worst-case expected revenue generated by the best second price auction thus amounts to $\frac{I\mu - 1}{I - 1}$ for all $I \geq I_\mu$. This implies that

$$\Delta(I) = \min \left\{ \Delta \in \mathbb{N} : \frac{(I + \Delta)\mu - 1}{(I + \Delta) - 1} \geq e^{(I + W_{-1}(-\mu I e^{-I}))} \right\}$$

$$= \left\lceil \frac{1 - e^{(I + W_{-1}(-\mu I e^{-I}))} - I(\mu - e^{(I + W_{-1}(-\mu I e^{-I}))})}{\mu - e^{(I + W_{-1}(-\mu I e^{-I}))}} \right\rceil$$

for all $I \geq I_\mu$. Assume now that there exists an upper bound $\overline{\Delta} \in \mathbb{N}$ on $\Delta(I)$ for all $I \in \mathbb{N}$. Any such $\overline{\Delta}$ must satisfy

$$\overline{\Delta} \geq \lim_{I \to \infty} \frac{1 - e^{(I + W_{-1}(-\mu I e^{-I}))} - I(\mu - e^{(I + W_{-1}(-\mu I e^{-I}))})}{\mu - e^{(I + W_{-1}(-\mu I e^{-I}))}}. \tag{44}$$

As the best second price auction is an instance of a highest-bidder-lottery, we have

$$\frac{I\mu - 1}{I - 1} \leq e^{(I + W_{-1}(-\mu I e^{-I}))} \leq \mu \tag{45}$$

for all $I \geq I_\mu$, which implies that $e^{(I + W_{-1}(-\mu I e^{-I}))}$ converges from below to μ as I grows. Next, we show that $\lim_{I \to \infty} I(\mu - e^{(I + W_{-1}(-\mu I e^{-I}))}) < 1 - \mu$, which implies that the limit in (44) evaluates to infinity and that there cannot exist any uniform upper bound $\overline{\Delta}$ on $\Delta(I)$. Specifically, we have

$$\lim_{I \to \infty} I(\mu - e^{(I + W_{-1}(-\mu I e^{-I}))}) = \lim_{\epsilon \downarrow 0} \frac{\mu - e^{(\frac{1}{\epsilon} + W_{-1}(-\frac{\mu}{\epsilon} e^{-\frac{1}{\epsilon}}))}}{\epsilon}$$

$$= \lim_{\epsilon \downarrow 0} \left(\frac{1}{\epsilon^2} + \frac{(\epsilon - 1) W_{-1}(-\frac{\mu}{\epsilon} e^{-\frac{1}{\epsilon}})}{\epsilon^2 (W_{-1}(-\frac{\mu}{\epsilon} e^{-\frac{1}{\epsilon}}) + 1)} \right) e^{(\epsilon^{-1} + W_{-1}(-\frac{\mu}{\epsilon} e^{-\frac{1}{\epsilon}}))}$$

$$\leq \lim_{\epsilon \downarrow 0} \left(\frac{1}{\epsilon^2} + \frac{(\epsilon - 1) W_{-1}(-\frac{\mu}{\epsilon} e^{-\frac{1}{\epsilon}})}{\epsilon^2 (W_{-1}(-\frac{\mu}{\epsilon} e^{-\frac{1}{\epsilon}}) + 1)} \right) \mu,$$

where the second equality holds by L'Hôpital's rule and the analytical formula for the derivative of the Lambert-W function (see Corless et al. 1996), while the inequality follows from (45). As $-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}}$ drops to 0 when ϵ tends to 0, we have

$$
\begin{aligned}
W_{-1}(-\tfrac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}}) &= \log(\tfrac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}}) - \log(-\log(\tfrac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})) + o(1) \\
&= -\frac{1}{\epsilon} + \log(\tfrac{\mu}{\epsilon}) - \log(-\log(\tfrac{\mu}{\epsilon}) + \tfrac{1}{\epsilon}) + o(1),
\end{aligned}
\tag{46}
$$

where the first equality follows from a well-known asymptotic expansion of the Lambert-W function (see Corless et al. 1996). We thus have

$$
\begin{aligned}
\lim_{\epsilon\downarrow 0} &\left(\frac{1}{\epsilon^2} + \frac{(\epsilon-1)W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})}{\epsilon^2(W_{-1}(-\frac{\mu}{\epsilon}e^{-\frac{1}{\epsilon}})+1)} \right)\mu = \lim_{\epsilon\downarrow 0}(\log(-\log(\tfrac{\mu}{\epsilon})+\tfrac{1}{\epsilon})) - \log(\tfrac{\mu}{\epsilon}))\mu \\
&= \lim_{\epsilon\downarrow 0}\log\left(\frac{(\tfrac{1}{\epsilon}-\log(\tfrac{\mu}{\epsilon}))\epsilon}{\mu} \right)\mu = \lim_{I\to\infty}\log\left(\frac{1}{\mu} - \epsilon\frac{\log(\tfrac{\mu}{\epsilon})}{\mu} \right)\mu = \log\left(\frac{1}{\mu} \right)\mu < 1-\mu,
\end{aligned}
$$

where the first equality follows from (46) and elementary rearrangements, while the last inequality holds because $\mu \in (0,1)$. This implies that $\lim_{I\to\infty} I(\mu - e^{(I+W_{-1}(-\mu Ie^{-I}))}) < 1-\mu$, and thus the claim follows. $\quad\square$