On the Complexity of Pointer Arithmetic in Separation Logic

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Abstract. We investigate the complexity consequences of adding pointer arithmetic to separation logic. Specifically, we study an extension of the points-to fragment of symbolic-heap separation logic with sets of simple "difference constraints" of the form $x \leq y + k$, where x and y are pointer variables and k is an integer offset. This extension can be considered a practically minimal language for separation logic with pointer arithmetic.

Most significantly, we find that, even for this minimal language, polynomial-time decidability is already impossible: satisfiability becomes NP-complete, while quantifier-free entailment becomes coNP-complete and quantified entailment becomes Π_2^P-complete (where Π_2^P is the second class in the polynomial-time hierarchy).

However, the language does satisfy the small model property, meaning that any satisfiable formula has a model, and any invalid entailment has a countermodel, of polynomial size, whereas this property fails when richer forms of arithmetical constraints are permitted.

Keywords: Separation logic, pointer arithmetic, complexity.

1 Introduction

Separation logic (SL) [23] is a well-known and popular Hoare-style framework for verifying the memory safety of heap-manipulating programs. Its power stems from the use of *separating conjunction* in its assertion language, where $A * B$ denotes a portion of memory that can be split into two disjoint fragments satisfying A and B respectively. Using separating conjunction, the *frame rule* becomes sound [27], capturing the fact that any valid Hoare triple can be extended with the same separate memory in its pre- and postconditions and remain valid, which empowers the framework to scale to large programs (see e.g. [26]). Indeed, separation logic now forms the basis for verification tools used in industrial practice, notably Facebook's Infer [8] and Microsoft's SLAyer [3].

Most separation logic analyses and tools restrict the form of assertions to a simple propositional structure known as *symbolic heaps* [2]. Symbolic heaps are (possibly existentially quantified) pairs of so-called "pure" and "spatial" assertions, where pure assertions mention only equalities and disequalities between variables and spatial formulas are $*$-conjoined lists of pointer formulas $x \mapsto y$ and data structure formulas typically describing (segments of) *linked lists* ($\mathsf{ls}\, x\, y$) or sometimes binary trees. This fragment of the logic enjoys decidability in polynomial time [11] and is therefore highly suitable for use in large-scale analysers. However, in recent years, various authors have investigated the computational complexity of (and/or developed prototype analysers for) many other fragments employing various different assertion constructs, including user-defined inductive predicates [18, 5, 7, 1, 10], pointers with *fractional permissions* [22, 13], arrays [6, 19], separating *implication* ($-\!\!*$) [9, 4], reachability predicates [14] and arithmetic [20, 21].

It is with this last feature, arithmetic, and more specifically *pointer arithmetic*, with which we are concerned in this paper. Although most programming languages do not allow the explicit use of pointer arithmetic (with the exception of C, where it is nevertheless discouraged), it nevertheless occurs *implicitly* in many programming situations, of which the most common are array indexing and structure / union member selection. For example, a C expression like `ptr[i]` implicitly generates an address expression of the form `ptr+(sizeof(*ptr)*i)`. Thus a program analysis performing bounds checking for C arrays or strings, say, must account for such implicit pointer arithmetic. We therefore set out by asking the following question: *How much pointer arithmetic can one include in separation logic and remain within polynomial time?*

Unfortunately, and perhaps surprisingly, the answer turns out to be: essentially none at all.

We study the complexity of symbolic-heap separation logic with points-to formulas, but no other data structure predicates, when pure formulas are extended by a minimal form of pointer arithmetic. Specifically, we permit only conjunctions of "difference constraints" $x \leq y + k$, where x and y are pointer variables and k is an integer. We certainly do *not* claim that this fragment is appropriate for practical program verification; clearly, lacking constructs for lists or other data structures, and using only a very weak form of arithmetic, it will be insufficiently expressive for most purposes (although it might *possibly* be practical e.g. for some concurrent programs that deal only with shared memory buffers of a small fixed size). The point is that any practical fragment of separation logic employing pointer arithmetic will almost inevitably include our minimal language and thus inherit its computational lower bounds.

We establish precise complexity bounds for the satisfiability and entailment problems, in both quantified and quantifier-free forms, for our SL with minimal pointer arithmetic. Perhaps our most striking result is that the satisfiability problem is already NP-complete; the entailment problem becomes coNP-complete for quantifier-free entailments, and Π_2^P-complete for existentially quantified entailments (where Π_2^P is the second class in the *polynomial-time hierarchy* [25]). However, the language does at least enjoy the *small model property*, meaning that any satisfiable symbolic heap A has a model of size polynomial in A, and any invalid entailment $A \models B$ has a countermodel of size polynomial in A and B — a property that fails when richer forms of arithmetical constraints are permitted in the language. In all cases, the lower bounds follow by reduction from the 3-colourability problem or its 2-round variant [15]. The upper bounds are by straightforward encodings into Presburger arithmetic, but the Π_2^P upper bound for quantified entailments is *not* trivial, as it requires us to show that all quantified variables in the resulting Presburger formula can be polynomially bounded; this follows from the small model property.

The remainder of this paper is structured as follows. In Section 2 we define symbolic-heap separation logic with minimal pointer arithmetic. Sections 3 and 4 study the satisfiability and quantifier-free entailment problems, respectively, for this language, and Sections 5 and 6 establish the lower and upper complexity bounds, respectively, for the general entailment problem. Section 7 concludes.

2 Separation logic with minimal pointer arithmetic

Here, we introduce a minimal language for *separation logic with pointer arithmetic* ($\mathsf{SL}_{\mathsf{MPA}}$ for short), a simple variant of the well-known "symbolic heap" fragment over pointers [2].

Our choice of language is influenced primarily by the need to 'balance' the arithmetical part of the language against the spatial part. To show lower complexity bounds, we have to challenge the fact that Σ_1^0 Presburger arithmetic is already NP-hard by itself; thus, to reveal the true memory-related nature of the problem, we restrict the language to a minimal form of pointer arithmetic, which is simple enough that it can be processed in polynomial time. This leads us to consider only conjunctions of "difference constraints", of the form $x = y + k$ and $x \leq y + k$ where x and y are variables and k is an integer (even disequality $x \neq y$ is not permitted). We write bold vector notation to denote sequences of variables, e.g. $\mathbf{x}$ for $x_1, \ldots, x_n$.

Definition 2.1 (Syntax). *A* symbolic heap *is given by*

$$\exists \mathbf{z}.\ \Pi : F$$

where $\mathbf{z}$ *is a tuple of variables from an infinite set* Var*, and* Π *and* F *are respectively* pure *and* spatial *formulas, defined along with* terms t *by:*

$$\begin{aligned} t &::= x \mid x + k \\ \Pi &::= x = t \mid x \leq t \mid \Pi \wedge \Pi \\ F &::= \mathsf{emp} \mid t \mapsto t \mid t \mapsto \mathsf{nil} \mid F * F \end{aligned}$$

where x *ranges over* Var *and* k *over integers* $\mathbb{Z}$*. If* Π *is empty in a symbolic heap* $\exists \mathbf{z}.\ \Pi : F$*, we omit the colon. We sometimes abbreviate* $*$*-conjunctions of spatial formulas using "big star" notation:*

$$\mathop{\Huge *}_{i=1}^{n} F_i \ =_{def}\ F_1 * \ldots * F_n\ ,$$

which is interpreted as emp *if* $n < 1$*.*

In our $\mathsf{SL}_{\mathsf{MPA}}$, the pure part of a symbolic heap is a conjunction of *difference constraints* of the form $x = y + k$ or $x \leq y + k$, where x and y are variables, and k is a fixed offset in $\mathbb{Z}$ (we disallow equalities of the form $x = \mathsf{nil}$ for technical convenience). Thus $x < y + k$ can be encoded as $x \leq y + (k - 1)$, $x \leq y - k$ as $x \leq y + (-k)$ and $x + k \leq y$ as $x \leq y - k$; however, note that unlike the conventional symbolic heap fragment in [2], we *cannot* express disequality $x \neq y$. The satisfiability of such formulas can be decided in polynomial time; see [12]. The crucial observation for polynomial-time decidability is:

Proposition 2.2. *A 'circular' system of difference constraints $x_1 \leq x_2 + k_{12}$, $\ldots$, $x_{m-1} \leq x_m + k_{m-1,m}$, $x_m \leq x_1 + k_{m,m+1}$ implies that $x_1 - x_1 \leq \sum_{i=1}^{m} k_{i,i+1}$, which is a contradiction iff the latter sum is negative.*

Semantics. As usual, we interpret symbolic heaps in a stack-and-heap model of the standard type, as given, e.g., in Reynolds' seminal paper on separation logic [23] (which similarly permits unrestricted pointer arithmetic). For convenience we consider the addressable locations to be the set $\mathbb{N}$ of natural numbers, and values to be either natural numbers or a non-addressable null value *nil*. Thus a *stack* is a function $s \colon \mathsf{Var} \to \mathbb{N} \cup \{nil\}$. We extend stacks to terms by $s(\mathsf{nil}) = nil$ and, insisting that any pointer-offset sum should always be non-negative: $s(x + k) = s(x) + k$ if $s(x) + k \geq 0$, and undefined otherwise. If s is a stack, $z \in \mathsf{Var}$ and v is a value, we write $s[z \mapsto v]$ for the stack defined as s except that $s[z \mapsto v](z) = v$. We extend stacks pointwise over term tuples.

A *heap* is a finite partial function $h \colon \mathbb{N} \rightharpoonup_{\text{fin}} \mathbb{N} \cup \{nil\}$ mapping finitely many locations to values; we write $\mathrm{dom}\,(h)$ for the domain of h, and e for the empty heap that is undefined on all locations. We write $\circ$ for *composition* of domain-disjoint heaps: if h_1 and h_2 are heaps, then $h_1 \circ h_2$ is the union of h_1 and h_2 when $\mathrm{dom}\,(h_1)$ and $\mathrm{dom}\,(h_2)$ are disjoint, and undefined otherwise.

Definition 2.3. *The* satisfaction relation $s, h \models A$*, where s is a stack, h a heap and A a symbolic heap, is defined by structural induction on A.*

$$\begin{array}{lcl}
s, h \models x = t & \Leftrightarrow & s(x) = s(t) \\
s, h \models x \leq t & \Leftrightarrow & s(x) \leq s(t) \\
s, h \models \mathit{\Pi}_1 \wedge \mathit{\Pi}_2 & \Leftrightarrow & s, h \models \mathit{\Pi}_1 \textit{ and } s, h \models \mathit{\Pi}_2 \\
s, h \models \mathsf{emp} & \Leftrightarrow & h = e \\
s, h \models t_1 \mapsto t_2 & \Leftrightarrow & \mathrm{dom}\,(h) = \{s(t_1)\} \textit{ and } h(s(t_1)) = s(t_2) \\
s, h \models F_1 * F_2 & \Leftrightarrow & \exists h_1, h_2.\ h = h_1 \circ h_2 \textit{ and } s, h_1 \models F_1 \textit{ and } s, h_2 \models F_2 \\
s, h \models \exists \mathbf{z}.\ \mathit{\Pi} : F & \Leftrightarrow & \exists \mathbf{m} \in \mathbb{N}^{|\mathbf{z}|}.\ s[\mathbf{z} \mapsto \mathbf{m}], h \models \mathit{\Pi} \textit{ and } s[\mathbf{z} \mapsto \mathbf{m}], h \models F
\end{array}$$

We remark that the satisfaction of pure formulas $\mathit{\Pi}$ does not depend on the heap, which justifies writing $s \models \mathit{\Pi}$ rather than $s, h \models \mathit{\Pi}$.

Remark 2.4. Although our language allows unbounded integer offsets k to be added to pointer variables, we would have exactly the same expressivity even if offsets were restricted to 1 and -1. Namely, a difference constraint $x \leq y + k$ for $k > 0$ can be encoded by introducing k auxiliary variables and k equalities:

$$z_1 = y + 1 \wedge z_2 = z_1 + 1 \wedge \ldots \wedge z_k = z_{k-1} + 1 \wedge x \leq z_k \ .$$

3 Satisfiability and the small model property

In this section we investigate the *satisfiability* problem for our $\mathsf{SL}_{\mathsf{MPA}}$, defined formally as follows:

Satisfiability problem for $\mathsf{SL}_{\mathsf{MPA}}$. *Given a symbolic heap A, decide whether there is a stack s and heap h with $s, h \models A$.*

(Without loss of generality, we may consider A to be quantifier-free in the above problem, because A and $\exists \mathbf{z}.A$ are equisatisfiable.)

We establish three main results about this problem: (a) an NP upper bound; (b) an NP lower bound; and (c) the small model property, meaning that any satisfiable formula has a model of polynomial size.

In fact, the NP upper bound is fairly trivial; there is a simple encoding of the satisfiability problem into Σ^0_1 *Presburger arithmetic* (as is also done for a more complicated *array separation logic* in [6]). Nevertheless, we include the details here, since they will be useful in setting up later results.

Definition 3.1. Presburger arithmetic *(*PbA*) is defined as the first-order theory of the natural numbers $\mathbb{N}$ over the signature $\langle 0, s, +, = \rangle$, where s is the successor function, and $0, +, =$ have their usual interpretations. The relations $\neq$, $\leq$ and $<$ can be straightforwardly encoded (possibly introducing an existential quantifier).*

Note that a stack is just a first-order valuation, and a pure formula in $\mathsf{SL}_{\mathsf{MPA}}$ is also a formula of PbA, with exactly the same interpretation. Thus we overload $\models$ to include the standard first-order satisfaction relation of PbA.

Definition 3.2. *Let A be a quantifier-free symbolic heap, of the general form*

$$\Pi : \mathop{\ast}\nolimits_{i=1}^{m} t_i \mapsto u_i \ .$$

We define a corresponding PbA formula γ_A by enriching the pure part Π with the constraints that the allocated addresses t_i must be distinct:

$$\gamma_A =_{def} \Pi \wedge \bigwedge\nolimits_{1 \leq i < j \leq m} \ t_i \neq t_j \ .$$

The above γ_A can be easily rewritten as a Boolean combination of elementary formulas of the form $x \leq y + k$ where the 'offset' k is an integer.

Lemma 3.3. *For any symbolic heap A in $\mathsf{SL}_{\mathsf{MPA}}$, we have*

$$(\exists h.\ s, h \models A) \ \Leftrightarrow \ s \models \gamma_A \ .$$

Proof. We assume A of the general form given by Definition 3.2.

($\Rightarrow$) By assumption, we have $s \models \Pi$ and $\mathrm{dom}\,(h) = \{s(t_1), \ldots, s(t_m)\}$, which implies that all the t_i are distinct. Hence $s \models \gamma_A$ as required.

($\Leftarrow$) By assumption, we have $s \models \Pi$ and all of $s(t_1), \ldots, s(t_m)$ are distinct. Hence, defining a heap h by $\mathrm{dom}\,(h) = \{s(t_1), \ldots, s(t_m)\}$ and $h(s(t_i)) = u_i$ for each i, we have $s, h \models A$ as required. $\square$

Proposition 3.4. *Satisfiability for* $\mathsf{SL}_{\mathsf{MPA}}$ *is in* NP.

Proof. Follows from Lemma 3.3 and the fact that satisfiability for quantifier-free Presburger arithmetic belongs to NP [24]. □

Next, we tackle the lower bound. Satisfiability is shown NP-hard by reduction from the 3-*colourability problem [15].*

3-colourability problem. *Given an undirected graph with $n \geq 4$ vertices, decide whether there is a "perfect" 3-colouring of the vertices, such that no two adjacent vertices share the same colour.*

Definition 3.5. *Let $G = (V, E)$ be a graph with n vertices $v_1, \ldots, v_n$. We encode a perfect 3-colouring of G with the following symbolic heap A_G.*

First, we introduce n variables $c_1, \ldots, c_n$ to represent the colour (1, 2, or 3) assigned to each vertex. The fact that no two adjacent vertices v_i and v_j share the same colour will be encoded by allocating two cells with base address $e_{ij} \in \mathbb{N}$ and offsets c_i and c_j respectively in A_G. To ensure that all *such pairs of cells are disjoint, the base addresses e_{ij} are defined by:*

$$e_{ij} = i \cdot n^2 + j \cdot n \quad (1 \leq i < j \leq n) \tag{1}$$

We then define A_G to be the following quantifier-free symbolic heap:

$$\textstyle\bigwedge_{i=1}^{n} (a + 1 \leq c_i \wedge c_i \leq a + 3) \colon \; \mathop{\Huge *}_{(v_i, v_j) \in E} (c_i + e_{ij} \mapsto \mathsf{nil} * c_j + e_{ij} \mapsto \mathsf{nil})$$

where a is a "dummy" variable (ensuring that A_G adheres to the strict formatting of pure assertions in $\mathsf{SL}_{\mathsf{MPA}}$*).*

The relevant fact concerning our definition of the base addresses e_{ij} in Definition 3.5 is the following.

Proposition 3.6. *For distinct pairs of numbers (i, j) and (i', j'), with $1 \leq i, i', j, j' \leq n$, we have $|e_{i',j'} - e_{ij}| \geq n$.*

Although for the present purposes we *could* have used a simpler definition of the e_{ij}, such that they are all spaced 4 cells apart, the definition by equation (1) is convenient as it will be re-used later on; see Definition 5.1.

Lemma 3.7. *Let G be an instance of the 3-colouring problem. Then A_G from Definition 3.5 is satisfiable iff there is a perfect 3-colouring of G.*

Proof. Let $G = (V, E)$ have vertices $v_1, \ldots, v_n$, where $n \geq 4$.

($\Leftarrow$) Suppose G has a perfect 3-colouring given by assigning a colour b_i to each vertex v_i, with each $b_i \in \{1, 2, 3\}$. We define a stack s by $s(a) = 0$ and $s(c_i) = b_i$ for each $1 \leq i \leq n$. Note that since $b_i \in \{1, 2, 3\}$ we have $s(a + 1) \leq s(c_i) \leq s(a + 3)$ for each i, and so s satisfies the pure part of A_G. Now define heap h by

$$\mathrm{dom}\,(h) =_{\mathrm{def}} \textstyle\bigcup_{(v_i, v_j) \in E} (\{s(c_i) + e_{ij}\} \cup \{s(c_j) + e_{ij}\})$$

and $h(\ell) = nil$ for all $\ell \in \mathrm{dom}\,(h)$. Clearly, by construction, $s, h \models A_G$ provided that none of the singleton sets involved in the definition of $\mathrm{dom}\,(h)$ are overlapping.

Since we have a perfect 3-colouring of G, for any edge $(v_i, v_j) \in E$ we have $s(c_i) \neq s(c_j)$, so the subsets $\{s(c_i) + e_{ij}\}$ and $\{s(c_j) + e_{ij}\}$ of $\mathrm{dom}\,(h)$ do not overlap. Furthermore, by Proposition 3.6, for any two distinct edges (v_i, v_j) and $(v_{i'}, v_{j'})$ in E, the base addresses e_{ij} and $e_{i'j'}$ are at least 4 cells apart (because $n \geq 4$). Since $1 \leq s(c_i) \leq 3$ for any i, we cannot have $s(c_i) + e_{ij} = s(c_{i'}) + e_{i'j'}$ either. Thus all involved singleton sets are non-overlapping as required.

($\Rightarrow$) Supposing that $s, h \models A_G$, we define a 3-colouring of G by $b_i = s(c_i) - s(a)$ for each $1 \leq i \leq n$. Since $s \models a + 1 \leq c_i \wedge c_i \leq a + 3$ by assumption, we have $b_i \in \{1, 2, 3\}$ for each i, so this is indeed a 3-colouring. To see that it is a *perfect* 3-colouring, let $(v_i, v_j) \in E$. By construction, we have that $s, h' \models c_i + e_{ij} \mapsto \mathsf{nil} * c_j + e_{ij} \mapsto \mathsf{nil}$ for some subheap h' of h. Using the definition of $*$, this means that $s(c_i) + e_{ij} \neq s(c_j) + e_{ij}$, i.e. $s(c_i) \neq s(c_j)$, and so $b_i \neq b_j$ as required. □

In fact, given a graph G with m edges, one can see that the proof above still works by taking the numbers e_{ij} to be $\{0, 4, 8, \ldots, 4(m-1)\}$. Thus Defn. 3.5 encodes the 3-colouring problem for G inside a heap region of size roughly $4m$, i.e., only a linear size expansion.

Theorem 3.8. *Satisfiability for* $\mathsf{SL}_{\mathsf{MPA}}$ *is* NP*-hard.*

Proof. From Lemma 3.7 and the fact that 3-colourability is NP-hard [15]. □

Corollary 3.9. *Satisfiability in* $\mathsf{SL}_{\mathsf{MPA}}$ *is* NP*-complete.*

Proof. From Proposition 3.4 and Theorem 3.8. □

Finally, we tackle the *small model property* for $\mathsf{SL}_{\mathsf{MPA}}$; that is, any satisfiable formula A has a model (s, h) of size polynomial w.r.t. A (see e.g. [1]). Note that, by "size", we do not mean here the number of allocated cells in h (since clearly any model of A only allocates as many cells as there are $\mapsto$-assertions in A) but the sizes of the addresses and/or values involved in their definition. Indeed, this property breaks if we increase the expressivity of our system only slightly.

Remark 3.10. The small model property fails if we allow our symbolic heaps to contain constraints of the form $x \leq y \pm z$ where x, y and z are *all* variables. In that case, we could define, e.g.,

$$A_n =_{\mathrm{def}} \bigwedge_{i=0}^{n-1} x_{i+1} > x_i + x_i \colon \mathop{\Huge *}_{i=1}^{n} x_i \mapsto \mathsf{nil}$$

(Note that the constraint $x_{i+1} > x_i + x_i$ can be expressed in our syntax, e.g., as $x_i \leq x_{i+1} - y_i \wedge y_i = x_i + 1$.) Then, for any model (s, h) of A_n, and for any $i < n$, we have that $s(x_{i+1}) > 2s(x_i)$, which implies $s(x_{i+1}) > 2^{i+1}$. Thus, (the distances between) at least half the addresses in h must be of exponential size.

In order to prove the small model property for our $\mathsf{SL}_{\mathsf{MPA}}$, we need a more workable specification of γ_A:

Definition 3.11. *Given a symbolic heap A , we rewrite the Presburger formula γ_A by replacing every formula $x = y + k$ by $x \le y + k \wedge y \le x - k$, and every formula $t_i \ne t_j$ by $t_i \le t_j - 1 \vee t_j \le t_i - 1$. Then γ_A can be viewed as*

$$\gamma_A \;\equiv\; f_A(Z_1, Z_2, \dots, Z_m) \tag{2}$$

where $f_A(z_1, z_2, .., z_m)$ is a Boolean function, and within (2) the Boolean variable z_i is substituted with a difference constraint Z_i of the form $x_i \le y_i + k_i$ (where k_i is an integer).

Proposition 3.12. *Any model s of γ_A for a symbolic heap A can be conceived of as a non-negative integer solution to the system $\gamma_{A,\bar{\zeta}}$ given by*

$$Z_1 \equiv \zeta_1, \dots, Z_m \equiv \zeta_m \tag{3}$$

where $(\zeta_1, \dots, \zeta_m)$ is a tuple of Boolean values ($\top$ or $\bot$) with $f_A(\zeta_1, .., \zeta_m) = \top$, where $f_A(Z_1, \dots, Z_m)$ is γ_A as a Boolean function over difference constraints, as in Defn. 3.11.

Proof. Rewriting γ_A as $f_A(Z_1, \dots, Z_m)$ as in Defn. 3.11, we can evaluate each difference constraint Z_i as $\top$ or $\bot$ under s, which gives an appropriate value for each ζ_i such that s is a solution to (3). Clearly, $f_A(\zeta_1, \dots, \zeta_m) = \top$.

Conversely, given a non-negative solution to (3), we can view this solution as a stack s and observe that, since $f_A(\zeta_1, \dots, \zeta_m) = \top$, we have $s \models \gamma_A$. □

Definition 3.13. *Given a model (s, h) for symbolic heap A, we further encode the equation system $\gamma_{A,\bar{\zeta}}$ (3) in Proposition 3.12 as a* constraint graph *$G_{A,\bar{\zeta}}$, constructed as follows.*

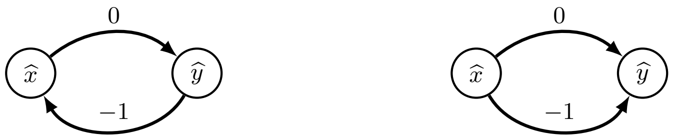
- *For each variable x in $\gamma_{A,\bar{\zeta}}$, we will associate a vertex $\widehat{x}$;*
- *An equation of the form $(x \le y + k) \equiv \top$ in (3) is encoded as an edge from $\widehat{y}$ to $\widehat{x}$ labelled by k: $\widehat{y} \xrightarrow{k} \widehat{x}$.*
- *An equation of the form $(x \le y + k) \equiv \bot$ in (3), meaning that $y \le x - k - 1$, is encoded as an edge from $\widehat{x}$ to $\widehat{y}$ labelled by $(-k-1)$: $\widehat{x} \xrightarrow{-k-1} \widehat{y}$.*
- *Finally, to provide the connectivity we need for models, we always add, if necessary, a "maximum node" $\widehat{x_0}$, with the constraint $x_i \le x_0$, i.e. edges $\widehat{x_0} \xrightarrow{0} \widehat{x}_i$, for all x_i.*

Example 3.14. Let A be the symbolic heap $y \le x\colon x \mapsto \mathsf{nil} * y \mapsto \mathsf{nil}$. We have:

$$\gamma_A \;=\; (y \le x) \wedge ((x \le y - 1) \vee (y \le x - 1)) \ .$$

Following Defn. 3.11, we can view γ_A as $f_A(Z_0, Z_1, Z_2)$, where $f_A(z_0, z_1, z_2)$ is the Boolean function $z_0 \wedge (z_1 \vee z_2)$, and $Z_0 = (y \le x)$, $Z_1 = (x \le y - 1)$ and $Z_2 = (y \le x - 1)$ are difference constraints.

Since Z_1 and Z_2 are mutually exclusive, there are essentially two Boolean vectors $\bar{\zeta} = \zeta_0, \zeta_1, \zeta_2$ such that $f_A(\bar{\zeta}) = \top$:



(a) $\gamma_1 = (y \le x) \wedge (x \le y - 1)$ (b) $\gamma_2 = (y \le x) \wedge (y \le x - 1)$

Fig. 1. The constraint graphs for γ_1 and γ_2 from Example 3.14.

(a) $\bar{\zeta} = \top, \top, \bot$, giving us difference constraints $\gamma_1 =_{\mathrm{def}} (y \le x) \wedge (x \le y - 1)$.
(b) $\bar{\zeta} = \top, \bot, \top$, giving us difference constraints $\gamma_2 =_{\mathrm{def}} (y \le x) \wedge (y \le x - 1)$.

Figure 1 shows the respective constraint graphs for γ_1 and γ_2. Notice that, because of $y \le x$, the node $\widehat{x}$ is a "maximum node" in both cases, and so we do not need to add one.

In the case of (a), we have no solution. Namely, there is a negative cycle of the form $\widehat{x} \xrightarrow{0} \widehat{y} \xrightarrow{-1} \widehat{x}$, which encodes the contradictory $x \le x - 1$.

In the case of (b), the minimal weighted path from $\widehat{x}$ to $\widehat{y}$ has weight -1, which guarantees that $y = x - 1$ is a model for γ_A and thereby for A.

Theorem 3.15 (Small model property). *Let A be a satisfiable symbolic heap in minimal pointer arithmetic. Then we can find a model (s, h) for A in which all values are bounded by $M = \sum_i (|k_i| + 1)$, where k_i ranges over all occurrences of integers in A.*

Proof. According to Proposition 3.12, there is a Boolean vector $\bar{\zeta} = \zeta_1, \zeta_2, .., \zeta_m$ such that the corresponding system, $\gamma_{A,\bar{\zeta}}$, has a solution. Hence, the associated constraint graph $G_{A,\bar{\zeta}}$ has no negative cycles (see Proposition 2.2).

We define our small model with the following mapping s over all variables x_i in A, such that $s \models \gamma_A$. First we define $s(x_0) = M$ for the "maximum node" $\widehat{x_0}$. Then, $s(x_i)$ is defined as $M + d_i$, where d_i is the *minimal weighted path* from $\widehat{x_0}$ to $\widehat{x_i}$; this is well-defined since $G_{A,\bar{\zeta}}$ has no negative-weight cycles. Note that d_i can never be positive, as there is always, trivially, a path from $\widehat{x_0}$ to $\widehat{x_i}$ of weight 0 by construction. Thus s is indeed "small". To see that it is a model of $\gamma_{A,\bar{\zeta}}$, consider e.g. the difference constraint $x \le y + k$; thus there is an edge from $\widehat{y}$ to $\widehat{x}$ with weight k in the graph, and so d_x cannot be greater than $d_y + k$, meaning $s(x) \le s(y) + k$. Hence s satisfies $\gamma_{A,\bar{\zeta}}$ and, by Proposition 3.12, $s \models \gamma_A$. Thus by Lemma 3.3 there is an h such that $s, h \models A$; note that h only uses values given by $s(x_i)$ and thus is also "small'. □

Remark 3.16. In addition, the corresponding polytime sub-procedures are the shortest path procedures with negative weights allowed (e.g., the Bellman-Ford algorithm), which provides polynomials of low degrees.

4 Quantifier-free Entailment

We now turn to the *entailment* problem for our $\mathsf{SL}_{\mathsf{MPA}}$, given as follows:

Entailment in $\mathsf{SL}_{\mathsf{MPA}}$. *Given symbolic heaps A and B, decide whether $s, h \models A$ implies $s, h \models B$ for all stacks s and heaps h (we say $A \models B$ is* valid*).*

Without loss of generality, A may be assumed quantifier-free, and any quantified variables in B assumed disjoint from the free variables in A and B.

In this section, we focus on the case of (entirely) quantifier-free entailments, for which we establish both an upper and a lower bound of coNP.

Definition 4.1. *Let $A \models B$ be an $\mathsf{SL}_{\mathsf{MPA}}$ entailment, where A and B are symbolic heaps of the form*

$$A \;=\; \mathit{\Pi}_A \colon \mathop{\ast}\nolimits_{i=1}^{\ell}\; t_i \mapsto t'_i \qquad \textit{and} \qquad B \;=\; \exists \mathbf{y}.\, \mathit{\Pi}_B \colon \mathop{\ast}\nolimits_{j=1}^{\ell'}\; u_j \mapsto u'_j$$

We define a corresponding PbA formula $\varepsilon_{A,B}$ by:

$$\gamma_A \to \exists \mathbf{y}\, \big(\gamma_B \wedge \textstyle\bigwedge_i \bigvee_j (t_i = u_j \wedge t'_i = u'_j) \wedge \bigwedge_j \bigvee_i (u_j = t_i \wedge u'_j = t'_i)\big) \qquad (4)$$

where $\gamma_{\_}$ is given by Defn. 3.2.

Lemma 4.2. *For any $\mathsf{SL}_{\mathsf{MPA}}$ entailment $A \models B$ and stack s, we have*

$$(\exists h.\; s, h \models A \textit{ implies } s, h \models B) \;\Leftrightarrow\; s \models \varepsilon_{A,B}\;.$$

Proof. We assume A and B of the general form given by Definition 4.1, and assume w.l.o.g. that $\mathbf{y}$ is disjoint from all free variables in A and B. We write $\mathsf{qf}(B)$ for the quantifier-free part of B.

($\Rightarrow$) Assume that $s \models \gamma_A$, the antecedent of (4). By Lemma 3.3 we have h with $s, h \models A$. By assumption, $s, h \models B$; i.e., for some values $\mathbf{v}$ with $|\mathbf{v}| = |\mathbf{y}|$, and defining $s' = s[\mathbf{y} \mapsto \mathbf{v}]$, we have $s', h \models \mathsf{qf}(B)$. Thus $s' \models \gamma_B$ by Lemma 3.3, and $\mathrm{dom}\,(h) = \{s'(u_1), \ldots, s'(u_{\ell'})\}$ (all of which are disjoint), with $h(s'(u_j)) = s'(u'_j)$ for each $1 \leq j \leq \ell'$. Since no variable in $\mathbf{y}$ occurs in A and $s, h \models A$, we also have $s', h \models A$, and so $\mathrm{dom}\,(h) = \{s'(t_1), \ldots, s'(t_\ell)\}$ (all disjoint), with $h(s'(t_i)) = s'(t'_i)$ for each $1 \leq i \leq \ell$. Thus $\ell' = \ell$ and each pair $(s'(t_i), s'(t'_i))$ is equal to some pair $(s'(u_j), s'(u'_j))$. Thus s' satisfies the quantifier-free consequent of (4), meaning that s satisfies the entire consequent, as required.

($\Leftarrow$) Suppose that $s, h \models A$ for some heap h. We have $s \models \gamma_A$ by Lemma 3.3, so, for some $s' = s[\mathbf{y} \mapsto \mathbf{v}]$, we have that s' satisfies the quantifier-free consequent of (4). That is, $s' \models \gamma_B$, so that $s', h' \models \mathsf{qf}(B)$ for some h' by Lemma 3.3. Moreover, for each pair $(s'(t_i), s'(t'_i))$ with $1 \leq i \leq \ell$, there is an equal pair $(s'(u_j), s'(u'_j))$ with $1 \leq j \leq \ell'$, and vice versa. Now, since no variable in $\mathbf{y}$ occurs in A and $s, h \models A$, we also have $s', h \models A$, and so $\mathrm{dom}\,(h) = \{s'(t_1), \ldots, s'(t_\ell)\}$ (all disjoint), with $h(s'(t_i)) = s'(t'_i)$ for each $1 \leq i \leq \ell$. Simultaneously, since $s', h' \models \mathsf{qf}(B)$, we have $\mathrm{dom}\,(h')\,\{s'(u_1), \ldots, s'(u_{\ell'})\}$ (all disjoint), with $h'(s'(u_j)) = s'(u'_j)$ for each $1 \leq j \leq \ell'$. Thus $\ell' = \ell$ and, because of the isomorphism between the pairs $(s'(t_i), s'(t'_i))$ and $(s'(u_j), s'(u'_j))$, we deduce that in fact $h' = h$. Thus $s', h \models \mathsf{qf}(B)$ and so $s, h \models B$, as required. □

As an immediate consequence of Lemma 4.2, the general entailment problem for $\mathsf{SL}_{\mathsf{MPA}}$ is in Π^0_2 Presburger arithmetic, which corresponds to Π^{EXP}_1 in the *exponential-time hierarchy* [17]. However, as it turns out, this bound is exponentially overstated; as we show in Theorem 6.7, the problem also belongs to the much smaller class Π^P_2, the second class in the polynomial time hierarchy [25]. The crucial difference between Presburger Π^0_2 and polynomial Π^P_2 is that, in the latter, all variables must be *polynomially bounded.*

However, the construction above does yield an optimal upper bound for the quantifier-free version of the problem.

Theorem 4.3. *The quantifier-free entailment problem for* $\mathsf{SL}_{\mathsf{MPA}}$ *is in* coNP.

Proof. According to Lemma 4.2, deciding whether $A \models B$ is valid is equivalent to deciding whether the PbA formula $\forall \mathbf{x}.\ \varepsilon_{A,B}$ is valid (where $\mathbf{x}$ is the set of all free variables in A and B). Although the latter is in general a Π^0_2 formula, it becomes a Π^0_1 formula when B is quantifier-free; the validity of such formulas can be decided in coNP time. □

We now turn to the small model property. We note that this property is sensitive to the exact form of our arithmetical constraints, and, similar to Remark 3.10, it fails when we allow the addition of two pointer variables.

Theorem 4.4 (Small model property). *Suppose that the quantifier-free entailment $A \models B$ is not valid. Then we can find a counter-model (s,h) such that $(s,h) \models A$ but $(s,h) \not\models B$, in which all values are bounded by $M = \sum_i(|k_i|+1)$, where k_i ranges over all occurrences of numbers in A and B.*

Proof. (Sketch) The proof follows the structure of the small model property for satisfiability (Theorem 3.15), noting first that we can rewrite the PbA formula $\forall \mathbf{x}.\ \varepsilon_{A,B}$ as a Π^0_2 Boolean combination of difference constraints $x \le y+k$, similar to Defn. 3.11. □

As for the coNP *lower* bound, we use a construction similar to Definition 3.5, based on the complement of 3-colourability.

Definition 4.5. *Given a graph G with n vertices, and reusing notation from Definition 3.5, we introduce a satisfiable symbolic heap A'_G by:*

$$\bigwedge_{i=1}^{n}(a+1 \le c_i \wedge c_i \le d):\ \mathop{\ast}_{(v_i,v_j)\in E} c_i + e_{ij} \mapsto \mathsf{nil} * c_j + e_{ij} \mapsto \mathsf{nil}$$

and a satisfiable symbolic heap B'_G by $d \ge a+4 : A'_G$.

Lemma 4.6. *Let G be an instance of the* 3*-colouring problem, and let A'_G and B'_G be given by Defn. 4.5 above. Then $A'_G \models B'_G$ is not valid iff there is a perfect* 3*-colouring of G.*

Proof. Let $G = (V, E)$ have n vertices $v_1, \ldots, v_n$, where $n \geq 4$.

($\Leftarrow$) Suppose G has a perfect 3-colouring given by assigning colours $b_i \in \{1, 2, 3\}$ to vertices v_i. By the argument in the ($\Leftarrow$) case of the proof of Lemma 3.7, if we define $s(a) = 0$, $s(c_i) = b_i$ and (new here) $s(d) = 3$ then there is a heap h such that $s, h \models A'_G$. However, we do not have $s, h \models B'_G$ because $s \not\models d \geq a + 4$. Thus $A'_G \models B'_G$ is not valid, as required.

($\Rightarrow$) Conversely, suppose $s, h \models A'_G$ but $s, h \not\models B'_G$ for some (s, h). By construction of B'_G, this implies that $s \not\models a \leq d - 4$, which implies $s(d) \leq s(a) + 3$. We can then use this fact together with the fact that $s, h \models A'_G$ to obtain a 3-colouring of G exactly as in the ($\Rightarrow$) case of the proof of Lemma 3.7. □

Theorem 4.7. *The quantifier-free entailment problem for* $\mathsf{SL}_{\mathsf{MPA}}$ *is* coNP*-hard, even when both symbolic heaps are satisfiable.*

Proof. Lemma 4.6 gives a reduction from the complement of the 3-colourability problem, which is coNP-hard, using only satisfiable symbolic heaps. □

Corollary 4.8. *The quantifier-free entailment problem for* $\mathsf{SL}_{\mathsf{MPA}}$ *is* coNP*-complete (even when both symbolic heaps are satisfiable).*

Proof. Theorems 4.3 and 4.7 give the upper and lower bounds respectively. □

5 Quantified entailment: Π_2^P lower bound

In this section, and the following one, we investigate the general form of the entailment problem $A \models B$ for our $\mathsf{SL}_{\mathsf{MPA}}$, where B may contain existential quantifiers. Here, we establish a lower bound for this problem of Π_2^P in the *polynomial-time hierarchy* (see [25]); in the next section we shall establish an identical upper bound.

To prove Π_2^P-hardness, we build a reduction from the so-called *2-round* version of the 3-colourability problem, defined as follows.

2-round 3-colourability problem. *Let $G = (V, E)$ be an undirected graph with $n \geq 4$ vertices and k leaves (vertices of degree 1). The problem is to decide whether every 3-colouring of the leaves can be extended to a perfect 3-colouring of the entire graph, such that no two adjacent vertices share the same colour.*

Definition 5.1. *Let $G = (V, E)$ be an instance graph with n vertices and k leaves. In addition to the variables c_i and a and the numbers e_{ij} which we reuse from Definition 3.5, to each edge (v_i, v_j) we also associate a new variable $\widetilde{c_{ij}}$, representing the colour "complementary" to c_i and c_j.*

To encode the fact that no two adjacent vertices v_i and v_j share the same colour, we shall use c_i, c_j, and $\widetilde{c_{ij}}$ as the addresses, relative to the base-offset e_{ij}, for three consecutive cells within a memory chunk of length 3, which forces c_i, c_j, and $\widetilde{c_{ij}}$ to form a permutation *of* $(1, 2, 3)$.

Formally, we define A''_G to be the following quantifier-free symbolic heap:

$$\bigwedge_{i=1}^{k}(a+1 \le c_i \wedge c_i \le a+3)\colon \ \mathop{\ast}\nolimits_{(v_i,v_j)\in E}^{\ell\in\{1,2,3\}} \ a+(e_{ij}+\ell)\mapsto \mathsf{nil}$$

and B''_G to be the following quantified symbolic heap:

$$\begin{array}{l}\exists \mathbf{z}.\ \bigwedge_{i=1}^{n}(a+1 \le c_i \le a+3) \wedge \bigwedge_{(v_i,v_j)\in E}(a+1 \le \widetilde{c_{ij}} \le a+3) : \\ \quad \mathop{\ast}\nolimits_{(v_i,v_j)\in E} \ c_i+e_{ij}\mapsto \mathsf{nil} \ \ast \ c_j+e_{ij}\mapsto \mathsf{nil} \ \ast \ \widetilde{c_{ij}}+e_{ij}\mapsto \mathsf{nil}\end{array} \tag{5}$$

where the existentially quantified variables $\mathbf{z}$ are all variables occurring in B''_G that are not mentioned explicitly in A''_G; namely, the variables c_i for $k+1 \le i \le n$, and the "complementary colour" variables $\widetilde{c_{ij}}$. Note that both A''_G and B''_G are satisfiable.

Lemma 5.2. *Let G be an instance of the 2-round 3-colouring problem, and let A''_G and B''_G be given by Defn. 5.1 above. Then $A''_G \models B''_G$ is valid iff there is a perfect 3-colouring of G given any 3-colouring of its leaves.*

Proof. Let $G=(V,E)$ have vertices $v_1,\ldots,v_n$ of which the first k are leaves. We assume $n \ge 4$.

($\Leftarrow$) Let (s,h) be a stack-heap pair satisfying $s,h \models A''_G$; we have to show that $s,h \models B''_G$. The spatial part of A''_G yields

$$\mathrm{dom}\,(h) = \bigcup\nolimits_{(v_i,v_j)\in E}^{\ell=1,2,3} \ \{\, s(a)+e_{ij}+\ell \,\} \tag{6}$$

where these locations are all disjoint (and h maps each of them to *nil*); furthermore, $s(a)+1 \le s(c_i) \le s(a)+3$ for each $1 \le i \le k$. Take the 3-colouring of the leaves obtained by assigning colours $b_i = s(c_i)-s(a)$ to each of the leaves $v_1,\ldots,v_k$. According to the winning strategy, we can assign colours b_i to the remaining vertices $v_{k+1},\ldots,v_n$, obtaining a 3-colouring of the whole G such that no two adjacent vertices share the same colour. In addition, we mark each edge (v_i,v_j) by $\widetilde{b}_{ij}$, the colour complementary to b_i and b_j.

We extend the stack s to interpret the existentially quantified variables in B''_G as follows:

$$\begin{array}{rl} s(c_i) = s(a)+b_i & \text{for each } k+1 \le i \le n \\ s(\widetilde{c_{ij}}) = s(a)+6-b_i-b_j & \text{for each } (v_i,v_j)\in E \end{array}$$

The fact that no adjacent vertices v_i and v_j share the same colour means that

$$(s(c_i),\, s(c_j),\, s(\widetilde{c_{ij}})) \ \text{ is a } \textit{permutation} \text{ of } \ (s(a)+1,\, s(a)+2,\, s(a)+3),$$

and, as a result, (s,h) is also a model for B''_G; in particular,

$$s,h \models \mathop{\ast}_{(v_i,v_j)\in E} \ s(c_i)+e_{ij}\mapsto \mathsf{nil} \ \ast \ s(c_j)+e_{ij}\mapsto \mathsf{nil} \ \ast \ s(\widetilde{c_{ij}})+e_{ij}\mapsto \mathsf{nil}\ . \tag{7}$$

($\Rightarrow$) As for the opposite direction, let $A''_G \models B''_G$. Since A''_G is satisfiable, there is a model (s,h) for A''_G so that, in particular, h satisfies (6).

We will construct the required winning strategy in the following way. Assume a 3-colouring of the leaves is given by assigning colours b_i to the leaves $v_1, \ldots, v_k$. We modify our original s to a stack s' by defining $s'(c_i) = s(a) + b_i$ for each $1 \leq i \leq k$. This does not change the heap h, but provides

$$s(a) + 1 \leq s'(c_i) \leq s(a) + 3 \quad \text{for each } 1 \leq i \leq k.$$

It is clear that the modified (s', h) is still a model for A''_G, and, hence, a model for B''_G. Then for some stack s_B, an extension of s' to the existentially quantified variables in B, we get $s_B, h \models B''_G$.

For each $1 \leq i \leq k$, we have $s_B(c_i) = s'(c_i) = s_B(a) + b_i$, which means that these $s_B(c_i)$ represent correctly the original 3-colouring of the leaves. By assigning the colours $b_i = s_B(c_i) - s_B(a)$ to each of the remaining vertices $v_{k+1}, \ldots, v_n$, we obtain a 3-colouring of the whole G.

The spatial part of B''_G, cf. (7), provides that $s_B(c_i) \neq s_B(c_j)$, which implies that no adjacent vertices v_i and v_j can share the same colours b_i and b_j. This means that we have a perfect 3-colouring of G, as required. □

Theorem 5.3. *The general entailment problem for* $\mathsf{SL}_{\mathsf{MPA}}$ *is* Π_2^P*-hard, even when both symbolic heaps are satisfiable.*

Proof. Definition 5.1 and Lemma 5.2 give a reduction from the 2-round 3-colourability problem, which is Π_2^P-hard [15]. □

6 Quantified entailment: $\boldsymbol{\Pi_2^P}$ upper bound

Following the Π_2^P lower bound for quantified entailments in $\mathsf{SL}_{\mathsf{MPA}}$ given in the previous section, we show here that the upper bound is also Π_2^P, as well as establishing the small model property. Indeed, we shall see that the former result follows from the latter one.

Theorem 6.1 (Small model property). *Suppose that $A \models B$, encoded as $\epsilon_{A,B}$ in Definition 4.1, is not valid. Let $x_1, \ldots, x_n$ be the free variables in A and B, and let $y_1, \ldots, y_m$ be the existentially quantified variables in B.*

Then we can find a counter-model (s, h) such that $s, h \models A$ but $s, h \not\models B$, in which all values of $s(x_i)$ are bounded by $(n+1) \cdot M$ and all values of $s(y_j)$ by $(n+m+2) \cdot M$, where $M = \sum_i (|k_i| + 1)$, with k_i ranging over all occurrences of 'offset' integers in A and B.

Proof sketch. Let (s, h) be a counter-model for $A \models B$. For convenience (but without loss of generality) we assume that s orders the variables as follows: $s(x_1) = 0$, and $s(x_1) < s(x_2) < \cdots < s(x_n)$, and $s(x_n) \leq s(y_m)$, and, for all y_j, $s(x_1) \leq s(y_j) \leq s(y_m)$. In particular, note that x_1 is a "zero" variable and y_m a "maximum" variable under the valuation s.

Note that, being a model for A, (s, h) is fully determined by the system:

$$\gamma_{A,s} = \textstyle\bigwedge_{i=1}^{n-1} (x_{i+1} = x_i + d_{i,i+1}) \tag{8}$$

where for all $1 \le i < j \le n$, the d_{ij} is defined as: $d_{ij} = s(x_j) - s(x_i)$.

Following Proposition 3.12, the fact that $s, h \not\models B$ means that for a certain Boolean function $f_{A,B}$, whatever Boolean vector $\bar{\zeta} = \zeta_1, .., \zeta_\ell$ such that $f_{A,B}(\zeta_1, .., \zeta_\ell) = \top$ we take, the following system, $G_{A,B,s,\bar{\zeta}}$, has no integer solution for fixed $s(x_1)$, .., $s(x_n)$ given by $\gamma_{A,s}$ from (8):

$$G_{A,B,s,\bar{\zeta}} \;=\; \gamma_{A,s} \wedge Z_1 \equiv \zeta_1 \wedge \cdots \wedge Z_\ell \equiv \zeta_\ell \tag{9}$$

This constraint system can be seen as a graph, in exactly the same way as is done in Definition 3.13.

Example 6.2 (A running example). Let A and B be the following symbolic heaps:

$$A: \qquad x_1 < x_2 < x_3 < x_4 \colon x_1 \mapsto \mathsf{nil} \ * \ x_4 \mapsto \mathsf{nil}$$

$$B: \qquad \exists y_1 \exists y_4.\ x_2 \le y_1 - 3 \wedge x_3 \le y_4 + 7 \colon y_1 \mapsto \mathsf{nil} \ * \ y_4 \mapsto \mathsf{nil}$$

As a 'large' counter-model for $A \models B$, we take (s, h), where s is defined by

$$\begin{cases} s(x_2) = s(x_1) + 3D, \\ s(x_3) = s(x_2) + 2, \\ s(x_4) = s(x_3) + D, \end{cases}$$

where D is a very large number (say 2^{1024}). To show that (s, h) is not a model for B, the spatial parts provide two cases to be considered: $y_1 = x_1 \wedge y_4 = x_4$ and $y_1 = x_4 \wedge y_4 = x_1$.

(a) In case of $y_1 = x_1 \wedge y_4 = x_4$, the corresponding system $G_{A,B,s,\bar{\zeta}}$ in (9) has no solution, e.g., because of the negative cycle:

$$\widehat{x_1} \xrightarrow{0} \widehat{y_1} \xrightarrow{-3} \widehat{x_2} \xrightarrow{-3D} \widehat{x_1} \tag{10}$$

(b) In case of $y_1 = x_4 \wedge y_4 = x_1$, the corresponding system $G_{A,B,s,\bar{\zeta}}$ in (9) has no solution, e.g., because of the negative cycle:

$$\widehat{x_4} \xrightarrow{0} \widehat{y_1} \xrightarrow{-3} \widehat{x_2} \xrightarrow{-3D} \widehat{x_1} \xrightarrow{0} \widehat{y_4} \xrightarrow{7} \widehat{x_3} \xrightarrow{D} \widehat{x_4} \tag{11}$$

□

The intuitive idea of constructing a small counter-model is as follows.

Definition 6.3. *Given a 'large' counter-model (s, h) and a small M, we construct a small counter-model (s', h') by simply* replacing *all large gaps $d_{i,i+1}$ in (8) with M, as follows:*

$$s'(x_{i+1}) := \begin{cases} s'(x_i) + d_{i,i+1}, & \text{if } d_{i,i+1} \le M \\ s'(x_i) + M, & \text{otherwise} \end{cases}$$

(The heap h' is then obtained simply by updating h to use values given by s' rather than s, in the evident way.)

Lemma 6.4. *We can check that (s', h') is still a model for A.*

A real challenge is to prove that our (s', h') is not a model for B.

Example 6.5 (continuing Example 6.2). To show that $s', h' \not\models B$, we have two cases to be considered: $y_1 = x_1 \wedge y_4 = x_4$, and $y_1 = x_4 \wedge y_4 = x_1$.

(a) In case of $y_1 = x_1 \wedge y_4 = x_4$, the updated $G_{A,B,s',\bar{\zeta}}$ has no solution. E.g., by replacing the large $3D$ in the negative cycle (10) with our modest M, we get a negative cycle in terms of (s', h'):

$$\widehat{x_1} \xrightarrow{0} \widehat{y_1} \xrightarrow{-3} \widehat{x_2} \xrightarrow{-M} \widehat{x_1}$$

(b) In case of $y_1 = x_4 \wedge y_4 = x_1$, however, the same strategy *fails*. Namely, by replacing the large D and $3D$ in the negative cycle (11) with M, we get a cycle in terms of (s', h'):

$$\widehat{x_4} \xrightarrow{0} \widehat{y_1} \xrightarrow{-3} \widehat{x_2} \xrightarrow{-M} \widehat{x_1} \xrightarrow{0} \widehat{y_4} \xrightarrow{7} \widehat{x_3} \xrightarrow{M} \widehat{x_4}$$

but now with *positive* weight. □

The challenge to our construction can be resolved by the following lemma.

Lemma 6.6. *Having got a negative cycle $\mathcal{C}$ for (9), we can extract a smaller negative cycle which is good for (s', h') as well.*

Proof. (Sketch) We introduce the following *reductions* on negative cycles $\mathcal{C}$. We write $\widehat{x_j} \overset{\sigma}{\Longrightarrow}_Y \widehat{x_i}$ to denote a subpath of $\mathcal{C}$ from $\widehat{x_j}$ to $\widehat{x_i}$ with total weight σ and whose intermediate nodes are all of the form $\widehat{y_k}$. Then, assuming $i < j$, we distinguish two cases:

Case: $\mathcal{C}$ contains $\widehat{x_j} \overset{\sigma}{\Longrightarrow}_Y \widehat{x_i}$. We note that $d_{ij} > 0$, because $s(x_j) > s(x_i)$ by assumption. We distinguish two subcases:

Subcase (a1): $-d_{ij} \leq \sigma$. In this subcase, we replace the above path with the single labelled edge $\widehat{x_j} \xrightarrow{-d_{ij}} \widehat{x_i}$, which ensures that the updated $\mathcal{C}$ still has negative weight, but now also contains *fewer nodes of the form $\widehat{y_k}$*.

E.g., within Example 6.5, replacing $\widehat{x_4} \xrightarrow{0} \widehat{y_1} \xrightarrow{-3} \widehat{x_2}$, the cycle (11) can be transformed into the negative cycle:

$$\widehat{x_4} \xrightarrow{-D-2} \widehat{x_2} \xrightarrow{-3D} \widehat{x_1} \xrightarrow{0} \widehat{y_4} \xrightarrow{7} \widehat{x_3} \xrightarrow{D} \widehat{x_4} \qquad (12)$$

Subcase (a2): $-d_{ij} > \sigma$. We identify the negative cycle:

$$\widehat{x_j} \overset{\sigma}{\Longrightarrow}_Y \widehat{x_i} \xrightarrow{d_{ij}} \widehat{x_j}$$

Since $d_{ij} < -\sigma \leq M$, we have $d'_{ij} = d_{ij}$, and hence this smaller negative cycle is good for (s', h') as well. This completes the case.

Case: $\mathcal{C}$ contains $\widehat{x_i} \overset{\sigma}{\Longrightarrow}_Y \widehat{x_j}$. In that case, $d_{ij} < 0$, again because $s(x_j) > s(x_i)$, and we again distinguish two subcases:

Subcase (b1): $d_{ij} \le \sigma$. In this subcase, we replace this path with the edge $\widehat{x_i} \xrightarrow{d_{ij}} \widehat{x_j}$, which ensures that the updated $\mathcal{C}$ remains negative, *but has fewer nodes of the form $\widehat{y_k}$.*

Subcase (b2): $d_{ij} > \sigma$. We identify the negative cycle:

$$\widehat{x_i} \overset{\sigma}{\Longrightarrow}_Y \widehat{x_j} \xrightarrow{-d_{ij}} \widehat{x_i}$$

If $d_{k,k+1} \le M$ for all k such that $i \le k < j$, then $d'_{ij} = d_{ij}$, and hence this smaller negative cycle is good for (s', h'), as well. Otherwise, for some k, $d_{k,k+1} > M$, and thereby by construction $d'_{k,k+1} = M$, and, hence, $d'_{ij} \ge M$. Then the following cycle defined in terms of (s', h'),

$$\widehat{x_i} \overset{\sigma}{\Longrightarrow}_Y \widehat{x_j} \xrightarrow{-d'_{ij}} \widehat{x_i}$$

is of negative weight, since $\sigma - d'_{ij} \le \sigma - M < 0$.

E.g., within Example 6.5 with: $\widehat{x_1} \xrightarrow{0} \widehat{y_4} \xrightarrow{7} \widehat{x_3}$, in (12), we obtain the following cycle in terms of (s', h'):

$$\widehat{x_1} \xrightarrow{0} \widehat{y_4} \xrightarrow{7} \widehat{x_3} \xrightarrow{-2-M} \widehat{x_1}$$

which is guaranteed to be of negative weight.

Finally, we show that any chain of reductions must terminate in one of the subcases (a2) and (b2). To see this, suppose otherwise. Then, having eliminated all nodes of the form $\widehat{y_k}$ in $\mathcal{C}$ via reductions (a1) and (b1), we would obtain a negative cycle $\mathcal{C}$ (by Lemma 6.6) consisting only of nodes of the form $\widehat{x_i}$, e.g.:

$$\widehat{x_i} \xrightarrow{d_{ij}} \widehat{x_j} \xrightarrow{-d_{ij}} \widehat{x_i}$$

However, such a cycle necessarily has weight 0, and is therefore non-negative; contradiction. This concludes the proof of the lemma, and thereby of Theorem 6.1. □

Theorem 6.7. *The entailment problem in $\mathsf{SL}_{\mathsf{MPA}}$ is in Π_2^P.*

Moreover, given A and B, for a certain Boolean combination of difference constraints $R(\mathbf{x}, \mathbf{y})$ defined by A and B as in Defn. 4.1, $A \models B$ is equivalent to

$$\forall \mathbf{x}.\ (\gamma_A(\mathbf{x}) \to \exists \mathbf{y}.\ R(\mathbf{x}, \mathbf{y}))$$

where all x_i in $\mathbf{x}$ and all y_j in $\mathbf{y}$ are bounded in accordance with Theorem 6.1.

Proof. This follows from the small model property provided by Theorem 6.1. □

Remark 6.8. The proof of Theorem 6.1 provides quite efficient procedures for the entailment problem in Theorem 6.7, in which the corresponding polytime sub-procedures are the usual shortest paths procedures with negative weights allowed, providing polynomials of low degrees. Alternatively, Theorem 5.3 and Definition 4.1 give an encoding of entailment as a Π^0_2 sentence of PbA and a polynomial bound for all variables, which could be passed directly to an arithmetic constraint solver.

In fact we prove that the entailment problem is Π^P_2-complete, and enjoys the small model property, even if we allow *any* Boolean combination of difference constraints $x \leq y + k$ in the pure part of our symbolic heaps.

7 Conclusions and future work

In this paper, we study the points-to fragment of symbolic-heap separation logic extended with pointer arithmetic, in a minimal form allowing only conjunctions of difference constraints $x \leq y + k$ for $k \in \mathbb{Z}$.

Perhaps surprisingly, we find that polynomial time algorithms are out of reach even in this minimal case: satisfiability is already NP-complete, quantifier-free entailment is coNP-complete, and quantified entailment is Π^P_2-complete. However, a small consolation is that the *small model property* holds for all three problems.

We note that our upper bound complexity results for satisfiability and quantifier-free entailment can be seen as following already from our earlier results for *array separation logic* [6], where we allow array predicates $\mathrm{array}(x, y)$ as well as pointers and arithmetic constraints. Of course, pointer arithmetic is often an essential feature in reasoning about array-manipulating programs. The main value of our findings, we believe, is in our *lower* bound complexity results, which show that NP-hardness or worse is an inevitable consequence of admitting pointer arithmetic of almost any kind. Moreover, the exact upper bound of Π^P_2 for entailment in $\mathsf{SL}_{\mathsf{MPA}}$ is new, and not straightforward to obtain.

We remark that our lower-bound results do however rely on the presence of *pointer* arithmetic, as opposed to arithmetic *per se*. Where pointers and data values are strictly distinguished and arithmetic is permitted only over data, as is done e.g. in [16], then polynomial-time algorithms may still be achievable in that case. Another possibility might be to impose further restrictions on the version of pointer arithmetic used here by adopting a different memory model, e.g. one that only allows pointers to be compared within specified memory regions (similar to the way pointers are intended to be used in C). To stand any chance of yielding a complexity improvement, such regions would need to be bounded "in advance", since, as we point out in Section 3, one can encode a 3-colourability graph with m edges as a satisfiability problem in $\mathsf{SL}_{\mathsf{MPA}}$ within a heap region of only linear size in m. In any case, however, we are not aware of any such region-aware models in the literature on separation logic.

It is worth mentioning the existence of software security measures that combat attacks like "stack smashing" by deliberately reordering the heap memory.

For programs employing such obfuscatory defensive measures, one typically cannot say anything definitive about the relative ordering of pointers in memory, in which case pointer arithmetic may be of limited utility as a reasoning tool.

Finally, we believe that our complexity results might well extend to the full first-order version of $\mathsf{SL}_{\mathsf{MPA}}$. For the entailment lower bound, the natural approach would be to develop a reduction from the k-round 3-colourability problem to Π_k^0 entailments, building on the reduction from 2-round 3-colourability to Π_2^0 entailments[3] with one alternation in Section 5. For the upper bound, the translation into an equivalent PbA formula in Definition 4.1 extends to quantifiers in the obvious way; but, moreover, we believe that our small-model technique in Section 6 might be also extended to alternating quantifiers, thus obtaining polynomial bounds for all variables. If so, then this would result in Π_k^P-completeness for Π_k^0 entailments in $\mathsf{SL}_{\mathsf{MPA}}$, i.e., the standard polynomial-time hierarchy; but, of course, that remains to be seen.

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[3] Here we view the complexity of $A \models \exists \mathbf{z}.B$ as Π_2^0, noting that the entailment is, implicitly, universally quantified at the outermost level.

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