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Pseudoconvex domains: Diederich - Fornæss index and the invariant metrics near the boundary points

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Abstract

This Thesis deals with some problems related to the pseudoconvex domain.

The first chapter presents some results on the theory on plurisubharmonic defining function. Interested in the relation of the Diederich - Fornæss index with the estimate for $\bar{\partial}$- Neumann operator in [Koh99], the author found a more general boundary condition for the pseudoconvex domain Ω to get the Diederich - Fornæss index goto 1 arbitrarily. These condition and the coresponding index give an estimate for $\bar{\partial}$- Neumann operator on $H^s_{(0,q)}(\Omega)$ for s goto infinity. The author also generalize the results by finding the index and its applications on general q-pseudoconvex domains.

The second part of the thesis is studying the invariant metrics, more precise, the Kobayashi metric, near infinite boundary points. Diederich and Fornæss on showed us how fast the Kobayashi metric of a point go to infinity when it comes near the boundary of a pseudoconvex domain that has real analytic boundary. Remove that cruel assumption, the author prove the result in more general class domains. The last part of this chapter gave an estimate the metric at points near the boundary point of infinite type for a special case. With the estimate for the Kobayashi metric, we can prove a proper holomorphic mapping theorem and have a Holder estimate for it.

Chapter 1

Bounded strictly plurisubharmonic exhaustion function.

1.1 Introduction

In the analytic of strictly pseudoconvex domains, there are three elementary properties of these domains play a fundamential role. They are:

1. Strictly pseudoconvex domain is stable with respect to a small C^2- pertubation.
2. A strictly pseudoconvex domain is locally biholomorphically equivalent to strictly convex domains.
3. If $\Omega \subset\subset \mathbb{C}^n$ is strictly pseudoconvex with a smooth C^2- boundary, then there is a neighborhood U of $\bar{\Omega}$ and a stricty plurisubharmonic C^2 function ρ on U such that $d\rho(p) = 0$ for $p \in b\Omega$ and $\Omega = \{p|\rho(p) < 0\}$.
   And in particular, the domains $\Omega_\epsilon = \{p|\rho(p) < \epsilon\}$, with $|\epsilon|$ small enought are strictly pseudoconvex and approximate Ω from the inside ($\epsilon < 0$) and outside ($\epsilon > 0$).

For a real C^2 function φ on a neighborhood p of $\mathbb{C}^n$, we define the Levi form of φ at p as the Hermitian form

$$i\partial\bar{\partial}\varphi(p) = \sum_{ij} \partial_{z_i}\partial_{\bar{z}_j}\varphi dz_i \otimes d\bar{z}_j.$$

If $\Omega \subset \mathbb{C}^n$, $q \in b\Omega$, U a neighborhood of q and $-\infty \leq a < b \leq \infty$ are given, a continous function $\varphi : \Omega \cap U \to (a, b)$ is called a local exhaustion function of Ω at q, if for all c, d satisfy $a < c < d < b$, we have $\overline{\varphi^{-1}([c, d]) \cap b\Omega} = \emptyset$.

We call a Diederich - Fornæss index of a domain $\Omega \subset\subset \mathbb{C}^n$ is a number $\eta \in (0,1]$ for which there exists a smooth defining function ρ of Ω so that $\hat{\rho} = -(-\rho)^\eta$ is strictly plurisubharmonic on Ω.

The Diederich - Fornæss index was study by some author and have some applications. One of the most important application is the quantitative estimate shown by J.J. Kohn on [Koh99] that show us the relation beween Diederich - Fornæss index and the regularity of $\bar{\partial}$-Neumann problems. For this relation, an attractive problem is find a condition for the domain Ω, for which the index go to 1, and let the H^s-regular holds for that domain with s goes to infinity.

Follow that idea, I prove the existence of the index for q-pseudoconvex domains and find a condition to let the index go to 1.

Theorem 1.1.1. *Let Ω be a bounded q-pseudoconvex domain on $\mathbb{C}^n$ with C^2-boundary then for each point z_0 on the boundary, there exist a neighborhood U of z_0 and $0 < \eta_0 < 1$ such that for $0 < \eta < \eta_0$ there exists a defining function φ such that $-(-\varphi)^\eta$ is q-plurisubharmonic on $U \cap \Omega$.*

Theorem 1.1.2. *Let Ω be a bounded domain on $\mathbb{C}^n$ and satisfies property P_q. Then for each point z_0 on the boundary, for each $0 < \eta_0 < 1$, there exist a neighborhood U of z_0 such that there is a defining function φ_η satisfying $-(-\varphi_\eta)^\eta$ is q-plurisubharmonic on $U \cap \Omega$ for $0 < \eta \leq \eta_0$.*

The prove of these theorems can be found on Section 1.4. After that, a result of G. Zampieri and S. Pinton that generalize the H^s-regualarity on $q-$pseudoconvex domains as an application.

1.2 Background

We study here the complex valued functions $f : \mathbb{C}^n \to \mathbb{C}$. These functions can be identified with functions $f : \mathbb{R}^{2n} \to \mathbb{R}^2$. For a choice of a root $i = \sqrt{-1}$ we write the coordinates in $\mathbb{C}^n$ as

$$z = x + iy \quad \text{for } (x, y) \in \mathbb{R}^{2n};$$

here $z = (z_1, \ldots, z_n)$, $x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We obtains an identification $\mathbb{R}^{2n} \to \mathbb{C}^n$ given by

$$(x, y) \mapsto z = x + iy.$$

We can describe the real structure underlying $\mathbb{C}^n$ through the correspondence

$$(x, y) \mapsto (z, \bar{z}) = (x + iy, x - iy)$$

and the inverse become

$$(z, \bar{z}) \mapsto (x, y) = \left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Now, we can consider the derivatives and differentials behavios. From the change and its inverse, and also the chain rule, we define ∂_z and $\partial_{\bar{z}}$ by

$$\begin{cases} \partial_x = \frac{\partial z}{\partial x}\partial_z + \frac{\partial \bar{z}}{\partial x}\partial_{\bar{z}} = \partial_z + \partial_{\bar{z}}, \\ \partial_y = \frac{\partial z}{\partial y}\partial_z + \frac{\partial \bar{z}}{\partial y}\partial_{\bar{z}} = i(\partial_z - \partial_{\bar{z}}), \end{cases}$$

and by inversion

$$\begin{cases} \partial_z = \frac{\partial x}{\partial z}\partial_x + \frac{\partial y}{\partial z}\partial_y = \frac{1}{2}(\partial_x - i\partial_y), \\ \partial_{\bar{z}} = \frac{\partial x}{\partial \bar{z}}\partial_x + \frac{\partial y}{\partial \bar{z}}\partial_y = \frac{1}{2}(\partial_x + i\partial_y), \end{cases}$$

Using the notation $\partial_x = (\partial_{x_j})_j$, $\frac{\partial z}{\partial x} = \left(\frac{\partial z_i}{\partial x_j}\right)_{ij}$ and similiarly for other variables y, z, $\bar{z}$. We have the dual basis of differentials correspondences

$$\begin{cases} \partial x = \frac{dz+d\bar{z}}{2}, \\ \partial y = \frac{dz-d\bar{z}}{2i}, \end{cases}$$

and the inverse

$$\begin{cases} dz = dx + idy, \\ d\bar{z} = dx - idy, \end{cases}$$

Write dx for $(dx_j)_j$ and similarly for the other variables. A function $f : \mathbb{C}^n \to \mathbb{C}$ identified with a function $f : \mathbb{R}^{2n} \to \mathbb{C}$, we have

$$\partial_x f dx + \partial_y f dy = \partial_z f dz + \partial_{\bar{z}} f d\bar{z}$$

and we write

$$df = \partial f + \bar{\partial} f.$$

We call that ∂f is $(1,0)$ form of f and $\bar{\partial} f$ is (0,1) form.

1.2.1 Plurisubharmonic function.

First, we consider the case complex one dimentional variable. We recall that a C^2 function h on a domain $\Omega \subset \mathbb{C}$ is said to be harmonic when $\partial_z \partial_{\bar{z}} h = 0$.

Definition 1.2.1. A real function φ on $\Omega \subset \mathbb{C}$ with values in $[-\infty, +\infty)$ is subharmonic when

i. The function φ is upper semicontinuous, i.e. for any z_0, $\varphi(z_0) \geq \limsup_{z \to z_0} \varphi(z)$;

ii. for any subset $K \subset\subset \Omega$ and for any h continous on K and harmonic on $\operatorname{int}(K)$

$$\varphi|_{bK} \leq h|_{bK} \text{ implies } \varphi|_K \leq h|_K.$$

Let Δ be the standard disc in $\mathbb{C}$ and $\Delta_{z_0,r}$ the disc of center z_0 and radius r. We have the main characterization of subharmonic function.

Theorem 1.2.2. *Let $\varphi : \Omega \to [-\infty, +\infty)$ be an upper semicontinuos function. The followings are equivalent,*

i. φ is subharmonic function on Ω,

ii. For any disc $\Delta_{z_0,r} \subset\subset \Omega$ and for any polynomial $P = P(z)$

$$\varphi|_{b\Delta_{z_0,r}} \leq ReP|_{b\Delta_{z_0,r}} \quad \textit{implies} \quad \varphi|_{\bar{\Delta}_{z_0,r}} \leq ReP|_{\bar{\Delta}_{z_0,r}}$$

iii. (Spherical submean) For any $\bar{\Delta}_{z_0,r} \subset\subset \Omega$, we have

$$\varphi(z_0) \leq \frac{1}{2i\pi r} \int_{b\Delta_{r_0,r}} \varphi ds$$

where ds is the element unit of the arc.

iv. (Solid submean) For any $\bar{\Delta}_{z_0,r} \subset\subset \Omega$, we have

$$\varphi(z_0) \leq -\frac{1}{2i\pi r^2} \iint_{\Delta_{z_0,r}} \varphi d\tau \wedge d\bar{\tau}.$$

v. (*Local solid submean*) *For any* z_0 *there is* $\tau_0 = \mathrm{dist}(z_0, b\Omega)$ *such that for any* $r < r_0$ *iv. holds.*

The proof of above Theorem can be easily found on many books about complex analysis. We also have a differential description of subharmonicity.

Theorem 1.2.3. *Let* $\varphi \in C^2(\Omega)$*, then* φ *is subharmonic if and only if*

$$\partial_z \partial_{\bar{z}} \varphi \geq 0.$$

Now we can define on case $\mathbb{C}^n$ with $n \geq 1$

Definition 1.2.4. A real upper semicontinuos function φ in $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic if its restriction to any disc $A = \varphi(\Delta) \subset \Omega$ is subharmonic.

For a real C^2 function r in a domain of $\mathbb{C}^n$, we define the *Levi form* of r at z_0 as the Hermitian form

$$i\partial\bar{\partial} r(z_0) = \sum_{ij} \partial_{z_i} \partial_{\bar{z}_j} r(z_0) dz_i \otimes d\bar{z}_j.$$

From the definition of plurisubharmonic functions, if φ is of class C^2, the plurisubharmonicity is characterized by $\partial_{z_i} \partial_{\bar{z}_j} \geq 0$ for any i, j. We just only have the subharmonicity along straight discs $\tau \mapsto w_0 + \tau w$, in which $w = (\ldots, 0, 1, 0, \ldots)$. However we can prove the following fact, that implies that it sufficient to check the subharmonicity along each straight discs.

Proposition 1.2.5. *Let* $\varphi \not\equiv -\infty$ *be upper semicontinuos and subharmonic along each cartesian ray. Then there is a sequence* $\{\varphi_\nu\}_\nu$ *of* C^∞ *plurisuharmonic functions on* $\Omega_\nu := \{z \in \Omega : \mathrm{dist}(z, b\Omega) > \frac{1}{\nu}$ *such that* $\varphi_\nu \searrow \varphi$*. In particular,* φ *is plurisubharmonic.*

1.2.2 q-pseudoconvex domain.

Let Ω is a domain on $\mathbb{C}^n$. A defining function of Ω is a real function $r : \mathbb{C}^n \to \mathbb{R}$ satisfies $\rho < 0$ on Ω and $\partial\rho \neq 0$ at each points where $\rho = 0$. We also have that the boundary $b\Omega$ is a hypersurface defined by the equation $\rho = 0$. We denote by $T^{\mathbb{C}} b\Omega$ the complex tangent bundle to $b\Omega$ defined by $T^{\mathbb{C}} b\Omega = Tb\Omega \cap iTb\Omega$. For any $z \in b\Omega$ its fiber $T^{\mathbb{C}}_z b\Omega$ is the space of vectors orthogonal to $\partial r(z)$ under the hermitian product.

On $b\Omega$, let (ρ_{ij}) be the matrix of the Levi form of ρ when restrict to $T^{\mathbb{C}}b\Omega$ under a choice of a C^2 orthonormal basis of $(1,0)$ form $w_1, \dots w_n$ with $w_n = \partial\rho$. Let $\lambda_1(z) \leq \dots \leq \lambda_{n-1}$ is ordered eigenvalues of (ρ_{ij}). Let $q \leq n-2$, we introduce

Definition 1.2.6. We say that $b\Omega$ is q-pseudoconvex (for the orientation from Ω) if there exists a covering of the boundary and, on each patch, a C^2 smooth bundle $\mathcal{V} \in T^{1,0}b\Omega$ of rank $q_0 \leq q$, say $\mathcal{V} = \text{span}\{\partial_{w_1}, \dots, \partial_{w_{q_0}}$ such that

$$\sum_{j=1}^{q+1} \lambda_j(z) - \sum_{j=1}^{q_0} r_{jj}(z) \geq 0.$$

The index q_0 may vary on different patches.

We consider more precisely on the case $q = 1$, and we say $b\Omega$ is pseudoconvex. In this case we can simply define the pseudoconvexity of $b\Omega$ whether the Levi form $i\partial\bar{\partial}\rho(z)$ is positive definite when restrict to $T^{\mathbb{C}}_z b\Omega$ for all $z \in b\Omega$. We define on a neighborhood of $b\Omega$ on $\mathbb{C}^n$

$$\delta_{b\Omega}(p) = \begin{cases} -\text{dist}(p, b\Omega) & \text{for } q \in \Omega \\ \text{dist}(p, b\Omega) & \text{for } q \notin \Omega \end{cases}$$

where *dist* means the Euclidean distance. If ρ is smooth, by shrinking the neighborhood of $b\Omega$ we can get $\delta_{b\Omega}$ is also a smooth function. Then we have the following important result.

Theorem 1.2.7 (Oka's lemma)**.** *Let $b\Omega$ be a hypersurface and $z_0 \in b\Omega$. Let $\delta_{b\Omega}$ be the distance to $b\Omega$ as above. Then, $b\Omega$ is pseudoconvex at z_0 if and only if there is an open neighborhood U of z_0 such $-\log(-\delta_{b\Omega})$ is plurisubharmonic.*

1.3 Boundary strictly plurisubharmonic exhaustion function

1.3.1 The existence of Diederich - Fornæss index.

The existence of Diederich - Fornæss index η for smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ holds by the following theorem.

Theorem 1.3.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$. Then there are a C^2 defining function ρ for Ω on a neighborhood of $\bar{\Omega}$, and $0 < \eta_0 < 1$, such that for any $0 < \eta \leq \eta_0$, the function $\rho = -(-\rho)^\eta$ is a strictly plurisubharmonic bounded exhausion function on Ω.*

Since Ω is pseudoconvex, the Levi form of any defining function is non negative at any boundary point when we apply it to the tangential space. Therefore, in order to construct an plurisubharmonic defining function (the Levi form is non negative for all direction), we need to consider the Levi form in the normal direction only. The following Lemma is a consequence of Oka's lemma.

Lemma 1.3.2. *There is a C^k defining function σ of Ω on a neighborhood U of $b\Omega$ in $\mathbb{C}^n$, such that the function $-\log(-\sigma)$ is a plurisubharmonic exhaustion function of Ω on $U \cap \Omega$. For any such function, after shrinking U if necessary, there exists a constant $C > 0$ such that*

$$i\partial\bar{\partial}\sigma(p;t) \geq -C\, |t|_p \, |\langle \partial\sigma_p, t\rangle| \tag{1.3.1}$$

for all $t \in T_p^{10}\mathbb{C}^n$ and all $p \in U \cap \Omega$.

Proof. According to Oka's lemma, the pseudoconvexity of Ω implies that $-\log(-\delta_{b\Omega})$ is plurisubharmonic on Ω.
For the inequality, we have some computation

$$\begin{aligned}\partial_{z_i}\partial_{\bar{z}_j}(-\log(-\sigma))(p) &= -\partial_{\bar{z}_j}\left(\frac{1}{\sigma}\partial_{z_i}\sigma\right)(p)\\ &= \left(-\frac{1}{\sigma}\partial_{z_i}\bar{\partial}_{\bar{z}_j}\sigma + \frac{1}{\sigma^2}\partial_{z_i}\sigma\partial_{\bar{z}_i}\sigma\right)(p),\end{aligned}$$

then

$$i\partial\bar{\partial}(-\log(-\sigma))(p;t,t) = \frac{1}{\sigma^2(p)}\left[(-\sigma(p))\, i\partial\bar{\partial}\sigma(p;t,t) + |\langle\partial\sigma, u\rangle|^2\right].$$

By definition, the Levi form of $-\log(-\delta_\Omega)$ is non-negative for $p \in U \cap \Omega$, then we get

$$i\partial\bar{\partial}\sigma(p;t,t) \geq 0$$

for $t \in T_p^1 = \left\{t \in T_p^{10} |\, \langle\partial\sigma, t\rangle = 0\right\}$. Let T_p^2 be the orthogonal complement of T_p^1 in T_p^{10}, then for any vector $t \in T_p^{10}$, it can be decomposed into $t = t^\tau + t^\nu \in T_p^1 \oplus T_p^2$. Then we have

$$\begin{aligned}i\partial\bar{\partial}\sigma(p;t,) &= i\partial\bar{\partial}\sigma(p;t^\tau,t^\tau) + 2\mathbf{Re}\, i\partial\bar{\partial}\sigma(p;t^\tau,t^\nu) + i\partial\bar{\partial}\sigma(p;t^\nu,t^\nu)\\ &\geq 2\mathbf{Re}\, i\partial\bar{\partial}\sigma(p;t^\tau,t^\nu) + i\partial\bar{\partial}\sigma(p;t^\nu,t^\nu).\end{aligned} \tag{1.3.2}$$

Since σ is a C^2 function on U then there exists a constant $C_1 > 0$ such that after shrinking U

$$\left| i\partial\bar{\partial}\sigma(p)(u, t^\nu) \right| \leq C_1 \left| u \right|_p \left| t^\nu \right|_p$$

for all $t \in T_p^{10}$, $t^\nu \in T_p^2$ and $p \in U$. Furthermore, there is $C_2 > 0$ such that

$$\left| \langle \partial\sigma_p, t \rangle \right| = \left| \langle \partial\sigma_p, t^\nu \rangle \right| \geq C_2 \left| t^\nu \right|_p$$

with $t = t^\tau + \nu \in T_p^1 \oplus T_p^2$ and $p \in U$. Combine with (1.3.2), we obtain

$$i\partial\bar{\partial}\sigma(p; t, t) \geq -C_1' \left| u \right| \left| t^\nu \right|_p \geq -C \left| t \right| \left| \langle \partial\sigma_p, t \rangle \right|$$

Thus, inequality (1.3.1) was proved. □

Proof of Theorem 1.3.1. Let σ and U be chosen as in Lemma 1.3.2. We fix a strictly plurisubharmonic C^∞ function ψ on $\mathbb{C}^n$ and define the function ρ on U

$$\rho = \sigma e^{-L\psi}$$

Denote $\hat{\rho} = -(-\rho)^\eta$ where $0 < \eta < 1$ and $L > 0$, which will be chosen later. Then we have that ρ is a C^k-defining function of Ω on $U \cap \Omega$.
We have some computation

$$\begin{aligned}
\partial_{z_i}\partial_{\bar{z}_j}\left(-(-\sigma)^\eta e^{-L\eta\psi}\right) &= -\partial_{\bar{z}_j}\left\{e^{-L\eta\psi}\left[-\eta\left(-\sigma\right)^{\eta-1}\partial_{z_i}\sigma - L\eta\left(-\sigma\right)^\eta\partial_{z_i}\psi\right]\right\} \\
&= e^{-L\eta\psi}\big\{\big[-\eta(\eta-1)(-\sigma)^{\eta-2}\partial_{z_i}\sigma\partial_{\bar{z}_j}\sigma + \eta(-\sigma)^{\eta-1}\partial_{z_i}\partial_{\bar{z}_j}\sigma \\
&\qquad - L\eta^2(-\sigma)^{\eta-1}\partial_{\bar{z}_j}\sigma\partial_{z_i}\psi + L\eta(-\sigma)^\eta\partial_{z_i}\partial_{\bar{z}_j}\psi\big] \\
&\qquad\quad - L\eta\bar{\partial}_{z_j}\psi\big[\eta(-\sigma)^{\eta-1}\partial_{z_i}\sigma + L\eta(-\sigma)^\eta\partial_{z_i}\psi\big]\big\}
\end{aligned}$$

and then

$$\begin{aligned}
i\partial\bar{\partial}\hat{\rho}(t,t) &= \eta(-\sigma)^{\eta-2}e^{-L\eta\psi}\big[(L\sigma^2\left(i\partial\bar{\partial}\psi(t,t) - \eta L\left|\langle\partial\psi, t\rangle\right|^2\right) \\
&\qquad + (-\sigma)\left(i\partial\bar{\partial}\sigma(t,t) - 2L\eta\mathbf{Re}\left(\langle\partial\sigma, t\rangle\,\overline{\langle\partial\psi, t\rangle}\right)\right) + (1-\eta)\left|\langle\partial\sigma, t\rangle\right|^2\big]
\end{aligned}$$

for $t \in T^{10}(U \cap \Omega)$. We see that the Levi form $i\partial\bar{\partial}\hat{\rho}$ will be positive if the expression in the bracket [ ], which is denoted by $D(t)$, is strictly positive there.
Apply Cauchy-Schwarz inequality to obtain

$$2\eta L\sigma|\mathbf{Re}\left(\langle\partial\sigma, t\rangle\,\overline{\langle\partial\psi, t\rangle}\right)| \leq 2\frac{L^2\sigma^2}{1-\eta}|\,\langle\partial\psi, t\rangle\,|^2 + \frac{1-\eta}{2}|\,\langle\partial\sigma, t\rangle\,|^2.$$

Combining with the expression of $D(t)$ and by Lemma 1.3.2 we get

$$D(t) \geq L\sigma^2 \left(i\partial\bar{\partial}\psi(t,t) - \eta L |\langle \partial\psi, t\rangle|^2 - 2\frac{L\eta^2}{1-\eta} |\langle \partial\psi, t\rangle|^2 \right) - \sigma C |t| |\langle \partial\sigma, t\rangle| + \frac{1-\eta}{2} |\langle \partial\sigma, t\rangle|^2. \quad (1.3.3)$$

Since ψ is a smooth strictly plurisubharmonic function, one can choose positive numbers C_1 and C_2 such that $i\partial\bar{\partial}\psi(t,t) \geq C_1|t|_p$ and $|\langle \partial\psi, t\rangle|^2 \leq C_2|t|^2$ for $t \in T^{10}(W)$. And then we have a quadratic form for $D(t)$ as

$$D(t) \geq L\sigma^2 \left(C_1 - C_2 \left(\eta L + 2\frac{L\eta^2}{1-\eta} \right) \right) |t|^2 - \sigma C |t| |\langle \partial\sigma, t\rangle| + \frac{1-\eta}{2} |\langle \partial\sigma, t\rangle|^2$$

and then

$$D(t) \geq L\sigma^2 \left(C_1 - C_2 \left(\eta L + 2\frac{L\eta^2}{1-\eta} \right) \right) |t|^2 - \frac{4C^2\sigma^2}{1-\eta} |t|^2 + \frac{1-\eta}{4} |\langle \partial\sigma, t\rangle|^2.$$

From (1.3.3) we can see that if the plurisubharmonic ψ on U satisfies $|\langle \partial\psi, t\rangle|^2$ is small and the Levi form $i\partial\bar{\partial}\psi(t,t)$ is large. Moreover we can let η far from 0 by choosing a sufficiently large constant L and then $\hat{\rho}$ become a strictly plurisubharmonic function.
For an arbitrary plurisubharmonic function ψ, we can find a constant η_0 small, and find $C_2L\eta < C_1$, we can also assume that $C_2\eta L < \frac{C_1}{2}$, and obtain

$$D(t) \geq \left(\frac{C_1}{2} L - \frac{C_1\eta}{2C_2(1-\eta)} - \frac{C^2}{2(1-\eta)} \right) \sigma^2 |t|^2.$$

One can choose η_0 and then L satisfies

$$\frac{C_1}{C_2\eta_0} > \frac{C_1}{2} L > \frac{C_1\eta_0}{2C_2(1-\eta_0)} + \frac{C^2}{2(1-\eta_0)}$$

thus $D(t) > 0$.
The only work is filling in the possible hole $\Omega \setminus U$ in the defining function of ρ. It can be done by replacing the function σ in the above arguments by $-\exp(-\lambda(\log(-\sigma)^{-1}))$ where λ is a convex increasing function on the real axis with $\lambda(t) = t$ for large t and λ constant before a suitable value of t. □

If we only consider in a small neighborhood U of a boundary point of Ω, we also can find a Diederich - Fornæss index for Ω, which is for some $0 < \eta < 1$ depends on U, we can find on a neighorhood U a defining function of Ω satisfies $-(-\rho)^\eta$ is plurisubharmonic. More precisely, we prove the following Theorem.

Theorem 1.3.3. *Let Ω be a pseudoconvex domain on $\mathbb{C}^n$ with C^2-boundary. For any boundary point of Ω, there exist a neighborhood U of p and an index η_0 such that we can find a defining function ρ of Ω on U such that for any $0 < \eta < \eta_0$ we have $-(-r)^\eta$ is plurisubharmonic.*

For the proof of the Theorem, we only need to find an estimate for the Levi form of defining function for Ω as in Lemma 1.3.2.

Lemma 1.3.4. *. Let Ω be a pseudoconvex domain on $\mathbb{C}^n$ with C^2-boundary. For any $p_0 \in b\Omega$ there is a defining function r of Ω on a neighborhood U of p such that*

$$i\partial\bar{\partial}r(p;t,t) \geq -C|t||\langle \partial r, t\rangle|$$

for all $t \in T^{10}_q b\Omega$ and $p \in U \cap \Omega$.

Proof. For $p_0 \in b\Omega$ and a neighborhood U we can choose an orthonormal coordinates system $z_1, \dots, z_n$ such that $p_0 = 0$ and the normal outward unit at p is $\frac{\partial}{\partial y_n}$ where $z_n = x_n + iy_n$, the boundary of Ω near p_0 is obtained by the graph $y_n = g(z', x_n)$ where $z' = (z_1, \dots, z_{n-1})$ and g is of class C^2. Then $r = y^n - g(z', x^n)$ is a defining function for Ω on U.
Let $t = t^\tau \oplus t^\nu \in T^1 \oplus T^2$ as in Lemma 1.3.2. Then we have for any p

$$i\partial\bar{\partial}r(p;t,t) = i\partial\bar{\partial}r(p;t^\tau,t^\tau) + 2\mathbf{Re}(i\partial\bar{\partial}r(p;t^\tau,t^\nu)) + i\partial\bar{\partial}r(p;t^\nu,t^\nu).$$

Moreover, the boundary of Ω is a graph on U, and Ω is pseudoconvex then we have

$$i\partial\bar{\partial}r(p;t^\tau,t^\tau) = i\partial\bar{\partial}r(\pi(p);t^\tau,t^\tau) \geq 0$$

where $\pi(z)$ is the projection of $z \in U$ to $b\Omega$.
Since r is a C^2 function on U, after shrinking U if necessary, there exists a constant $C_1 > 0$ such that

$$|i\partial\bar{\partial}r(p;t,t^\nu)| \leq C_1|t|_p|t^\nu|_p$$

for all $t \in T^{10}_p$, $t^\nu \in T^2_p$ and $q \in U$. And then we get

$$i\partial\bar{\partial}r(p;t,t) \geq -C|t||\langle \partial r_{p_0}, t\rangle|$$

□

Following the proof of Theorem 1.3.1 we obtain the conclusion of Theorem 1.3.3.

For the case $b\Omega$ is just slightly smoother, namely of class C^3, we can get a somewhat more general version and a simpler proof of Theorem 1.3.1.

Theorem 1.3.5. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with C^3 boundary $b\Omega$, such that there exists a C^∞ strictly plurisubharmonic function ψ defined on a neighborhood of $b\Omega$. Let σ be any C^3 defining function for Ω. Then there are a neighborhood U of $b\Omega$ and constants $K > 0$, $0 < \eta_0 < 1$ such that for $0 < \eta \leq \eta_0$, the function $\rho = -(-\sigma e^{-K\psi})^\eta$ is strictly plurisubharmonic on $\Omega \cap U$. If there exists ψ strictly plurisubharmonc on $\bar{\Omega}$, then ρ is strictly plurisubharmonic on Ω.*

Proof. We choose a product neighborhood $U = b\Omega \times (-\epsilon, \epsilon)$ of $b\Omega$ and let $\pi : U \to b\Omega$ be the projection to $b\Omega$. For fixed $u \in \mathbb{C}^n$, it can be decomposed by $u = t_p^\tau + t_p^\nu \in T_p^1 \oplus T_p^2$ where $T_p^1 = \{u \in T_p^{10} | \langle \sigma, u \rangle = 0\}$ and T_p^2 is the orthogonal complement of T_p^1 in T_p^{10}. The function $L_r(p; t^\tau)$ is of class C^1 in $p \in U$ (from the assumtion $b\Omega$, and hence r of class C^3). Therefore, for $p \in U$,

$$i\partial\bar{\partial}\sigma(p; t_p^\tau, t_p^\tau) - i\partial\bar{\partial}\sigma(\pi(p); t_{\pi(p)}^\tau, t_{\pi(p)}^\tau) = \mathcal{O}(|\sigma(p)|) \left|t_p\right|_p^2.$$

Since Ω is pseudoconvex domain, we get

$$i\partial\bar{\partial}\sigma(p; t_p^\tau, t_p^\tau) \geq \mathcal{O}(|\sigma(p)|) \left|t\right|_p^2 \tag{1.3.4}$$

for $p \in \Omega \cap U$ and $t \in T_p^{10}$. Furthermore, (1.3.4) can be estimated by

$$\begin{aligned} i\partial\bar{\partial}\sigma(t,t) &= i\partial\bar{\partial}\sigma(t^\tau) + 2\mathbf{Re}(i\partial\bar{\partial}\sigma(t^\tau, t^\nu)) + i\partial\bar{\partial}\sigma(t^\nu, t^\nu) \\ &= i\partial\bar{\partial}\sigma(t^\tau, t^\tau) + \mathcal{O}(\left|t\right|_p \left|t^\nu\right|_p) \end{aligned}$$

and

$$\left|t^\nu\right|_p = \mathcal{O}(|\langle \partial\sigma_p, t\rangle|)$$

That imply, for some constant $A > 0$,

$$i\partial\bar{\partial}\sigma(p; t, t) \geq -A\left|\sigma(p)\right| \left|t\right|_p^2 - A\left|t\right|_p \left|\langle \partial\sigma_p, t\rangle\right|$$

for $p \in \Omega \cap U$ and $t \in T_p^{10}$.

$$\begin{aligned} i\partial\bar{\partial}\hat{\rho}(t,t) = \eta(-\sigma)^{\eta-2} e^{-K\eta\psi} \big[& K\sigma^2 \left(i\partial\bar{\partial}\psi(t,t) - \eta K \left|\langle \partial\psi, t\rangle\right|^2 \right) \\ & + (-\sigma)\left(i\partial\bar{\partial}\sigma(t,t) - 2K\eta\mathbf{Re}\left(\langle \partial\sigma, t\rangle \overline{\langle \partial\psi, t\rangle} \right) \right) \\ & + (1-\eta)\left|\langle \partial\sigma, t\rangle\right|^2 \big] \end{aligned} \tag{1.3.5}$$

for $t \in T^{10}(\Omega \cap U)$. Let $D(t)$ be the expression in [ ], we going to show that one can choose η and K so that $D(t) > 0$ for $t \neq 0$. Similiar to the proof of Theorem 1.3.1, we can find A_1, A_2 that are positive and independent of K, η and $t \in T_p^{10}(\Omega \cap U)$ such that the lower estimate of $D(t)$ holds

$$D(t) \geq \sigma^2 \left[KA_1/2 - A_2\right] \left|t\right|^2.$$

By the choice $K > 2A_2/A_1$ and $\eta_0 = \eta_0(K)$, we get the desired result.

In the case ψ is strictly plurisubharmonic on $\bar{\Omega}$, $i\partial\bar{\partial}\psi(t,t) \geq A_3 |t|^2$ for $t \in T^{10}(\bar{\Omega})$ and $A_3 > 0$; also $\sigma^2 \geq \epsilon > 0$ on ther compact set $\Omega \setminus U$. From (1.3.5) which now can holds on Ω, $D(t) \geq K\epsilon^2 A_3 |t|^2 - A_4 |t|^2$ for $t \in T^{10}(\Omega)$. Now we can choose $K > \max\{2A_2/A_1, A_6/\epsilon^2 A_5\}$. □

Remark 1.3.6. For fixed $q \in b\Omega$ and fixed η, $0 < \eta < 1$, we can always find a neighborhood U and a C^k defining function ρ of Ω on U such that $-(-\rho)^\eta$ is strictly plurisubharmonic on $\Omega \cap U$. In fact, we can assume that $q = 0$, $U \subset B(0,\epsilon)$, from (1.3.5) and let $\psi = |z|^2$, we have

$$i\partial\bar{\partial}\hat{\rho}(t,t) = \eta(-\sigma)^{\eta-2}e^{-K\eta\psi}\big[K\sigma^2\left(i\partial\bar{\partial}\psi(t,t) - \eta K\,|\langle\bar{z},t\rangle|^2\right) + (-\sigma)\left(i\partial\bar{\partial}\sigma(t,t) - 2K\eta\mathbf{Re}\left(\langle\partial\sigma,t\rangle\,\overline{\langle\bar{z},t\rangle}\right)\right) + (1-\eta)\,|\langle\partial\sigma,t\rangle|^2\big].$$

Let $D(t)$ be the term [ ], since Ω is bounded, we have

$$D(t) \geq K\sigma^2(1-\eta\epsilon K)|t|^2 - \sigma(i\partial\bar{\partial}\sigma(t) - 2K\eta\epsilon|t|\,|\langle\partial,t\rangle|) + (1-\eta)\,|\langle\sigma,t\rangle|^2$$

We can choose ϵ small such that $\eta\epsilon K < 1/2$ and from Lemma 1.3.2

$$D(u) \geq \frac{1}{2}K\sigma^2|t|^2 + \sigma(C+1)\,|t|\,|\langle\partial\sigma,t\rangle| + (1-\eta)\,|\langle\partial\sigma,t\rangle|^2 \geq 0.$$

for $K > \frac{(C+1)^2}{1-\eta}$. And we obtain that $\hat{\rho}$ is plurisubharmonic on the neighborhood U of q.

1.3.2 Property $\tilde{P}$ and Diederich - Fornæss index.

We already show that in a neighborhood of a boundary point of Ω, the Diederich-Fornæss index exists and it can be close arbitrarily to 1. However, in genetal case, the index for global to all Ω is need to be chosen sufficiently small. Our purpose is finding condition for the domain Ω such that the Diederich - Fornæss index can be chosen arbitrarily close to 1.
First, we recall the classical property $\tilde{P}_q$ introduced by McNeal. We start with the definition.

Definition 1.3.7. Let $\Omega \subset\subset \mathbb{C}^n$ be the smoothly bounded domain. We said that the function $f \in C^2(\Omega) \cap PSH(\Omega)$ has a *self-bounded complex gradient* if there exists a constant C such that

$$\left|\sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(z) t_k\right|^2 \leq C \sum_{k,l=1}^{n} \frac{\partial^2 f}{\partial z_k \bar{\partial} z_l}(z) t_k \bar{t}_l \tag{1.3.6}$$

for all $t \in \mathbb{C}^n$ and $z \in \Omega$. When (1.3.6) holds, we write $|\partial f|^2_{i\partial\bar{\partial} f} \leq C$.

Under the scaling $f \to \alpha f$, $t > 0$, a factor of α^2 appears on the left-hand side of (1.3.6) while the right-hand side has a factor of α. The size of C should thus should be viewed as extrinsic.
Consider $g = -e^{-\frac{1}{C} f}$, we have

$$i\partial\bar{\partial} g(z; t, t) = \frac{1}{C} e^{-\frac{1}{C} f} \left[i\partial\bar{\partial} f(z; t, t) - \frac{1}{C} \left|\langle \partial f, t\rangle (z)\right|^2 \right],$$

then we can get that (1.3.6) is equivalent to the statement that

$$-e^{\frac{1}{C} f} \in PSH(\Omega).$$

The reformulation shows that the self-boundedness notion should be interpreted for non smooth functions. If $\phi \in C^2(\Omega)$ is bounded and plurisubharmonic, then $f = e^{\phi}$ satisfies (1.3.6) with $C = \sup e^{\phi}$. And we note that (1.3.6) does not force that f to be bounded.

Definition 1.3.8. We say that the domain Ω has property $\tilde{P}_q$ if, for every $M > 0$, there exists $\phi = \phi_M \in C^2(\bar{\Omega})$ such that

i. $|\partial \phi|_{i\partial\bar{\partial}\phi} \leq 1$,

ii. for any forms u of degree $k \geq q$.

$$\sum_{|K|=k-1}' \sum_{ij=1}^{n} \phi_{ij} u_{iK} \bar{u}_{jK} - \sum_{|J|=k}' \sum_{j=1}^{q} \phi_{jj} |u_J|^2 \geq M|u|^2, \tag{1.3.7}$$

It follows that $\tilde{P}_1 \Rightarrow \tilde{P}_2 \Rightarrow \cdots \Rightarrow \tilde{P}_n$. And we can see that (ii) in Definition 1.3.8 implies that for $z \in b\Omega$

$$i\partial\bar{\partial}\phi(z)(u, u) = \sum_{I,J}' \sum_{k,l}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{I,kJ} \bar{u}_{I,lJ} \geq M\,|u|^2$$

if $u \in \Lambda^{p,q}(\Omega)$.
Recall Catlin's property P_1, for a domain Ω: for every $M > 0$ there exists $\phi \in C^2(\Omega)$ such that

i. $|\phi| \leq 1$ on Ω,

ii. $\sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l}(p) u_k \bar{u}_l \geq M \|u\|^2$ for $p \in b\Omega$ and $u \in \mathbb{C}^n$

It's clear that property P_1 implies property $\tilde{P}_1$. In fact, let ϕ satisfies property P then let $\hat{\phi} = e^{\phi}$ than we have

$$\left|\left\langle \partial\hat{\phi}, u \right\rangle\right|^2 \leq C\partial\bar{\partial}\hat{\phi}(u,u)$$

where $C = \sup e^{\phi}$ and $t \in \mathbb{C}^n$.
From the assumption that Ω satisfies property $\tilde{P}$, we have the following Theorem

Theorem 1.3.9. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded domain satisfies property $\tilde{P}$. Then for any $\eta \in (0,1)$ there is a smooth defining function r such that $-(-r)^{\eta}$ is strictly plurisubharmonic on Ω.*

Proof. For a point p in the boundary, since $b\Omega$ is smooth we can choose orthonormal coordinates $z_1, \ldots, z_n$ such that $p = 0$ and the unit outward normal at p is $\frac{\partial}{\partial y_n}$ where $y_n = \mathbf{Im} z_n$. We can represent $b\Omega$ locally as the graph of function g, $\mathbf{Im} z_n = g(z', \mathbf{Re} z_n)$ where $z' = (z_1, \ldots, z_{n-1})$ then we have $\sigma = \mathbf{Im} z_n - g(z', \mathbf{Re} z_n)$ is a smooth defining function for Ω. Let $\psi = \psi_M$ satisfy property $\tilde{P}$ on Ω

i. $|\partial\psi|_{i\partial\bar{\partial}\psi} \leq \epsilon_M$

ii. $\sum_{k,l=1}^{n} \frac{\partial^2 \psi}{\partial z_k \partial \bar{z}_l}(p) u_k \bar{u}_l \geq M \|u\|^2$ for $p \in b\Omega$ and $u \in \mathbb{C}^n$

which will be scaled later. We consider the smooth defining function

$$r = \sigma e^{-\psi}.$$

Following the proof of Theorem 1.3.1 we have

$$\begin{aligned} i\partial\bar{\partial}\hat{\rho}(u) = \eta(-\sigma)^{\eta-2}e^{-\eta\psi}\big[&(-\sigma)i\partial\bar{\partial}\sigma(u) + (1-\eta)\left|\langle\partial\sigma, u\rangle\right|^2 \\ &+ \eta\sigma 2\mathbf{Re}\left(\langle\partial\sigma, u\rangle\,\overline{\langle\partial\psi, u\rangle}\right) + \sigma^2 i\partial\bar{\partial}\psi(u) - \eta\sigma^2\left|\langle\partial\psi, u\rangle\right|^2\big]. \end{aligned}$$

For η small enough, the assumtion Ω satisfies property $\tilde{P}$ is not necessary since Theorem 1.3.3. When η is far from 0, the term $|\langle\partial\psi, u\rangle|^2$ have more effect on making $i\partial\bar{\partial}\hat{\rho}(u,u)$ negative. Then we use i. of property $\tilde{P}$ to limit this effect.
We write $D(t)$ the expression in [ ] and we are going to find a good control for $D(u) > 0$ for a neighborhood of $b\Omega$ and arbitrary $u \in \mathbb{C}^n$. We decompose

$$i\partial\bar{\partial}\sigma(u,u) = i\partial\bar{\partial}\sigma(u^{\tau}, u^{\tau}) + 2\mathbf{Re}(i\partial\bar{\partial}\sigma(u^{\tau}, u^{\nu})) + i\partial\bar{\partial}\sigma(u^{\nu}, u^{\nu})$$

where $u^\tau \in T^1 = \{u \in T^{10} | \langle \partial\sigma, u\rangle = 0\}$ and $u^\nu \in T^2$ the orthogonal complement of T^1 in T^{10}. Near p, the boudary is the graph g then similar to Lemma 1.3.4 we have

$$|i\partial\bar{\partial}\sigma(u,u)| \leq C|u||\,\langle \partial\sigma, u\rangle\,|,$$

furthermore

$$\langle \partial\sigma, u\rangle \leq C|u|$$

for some $C > 0$. Since ψ satifies property$\tilde{P}$, after shrinking the neighborhood U of p, we have

$$D(u) \geq \frac{1}{2}\,|\sigma|^2\, i\partial\bar{\partial}\psi(u,u) - 2|\sigma|\,\Big[C|u||\,\langle \partial\sigma, u\rangle\,| + |\,\langle \partial\sigma, u\rangle\,|\overline{\langle \partial\psi, u\rangle}|\Big] + (1-\eta)\,|\langle \partial\sigma, u\rangle|^2\,.$$

By Cauchy-Schwarz inequality

$$2\sigma|\,\langle \partial\sigma, u\rangle\,||\,\langle \partial\psi, u\rangle\,| \leq \sigma^2\frac{1}{\lambda}|\,\langle \partial\psi, u\rangle\,|^2 + \lambda|\,\langle \partial\sigma, u\rangle\,|^2.$$

let $\lambda = \frac{1-\eta}{2}$, we can shrink U to obtain

$$D(u) \geq \frac{1}{4}\,|\sigma|^2\, i\partial\bar{\partial}\psi(u,u) - 2C|\sigma||u||\,\langle \partial\sigma, u\rangle\,| + \frac{1-\eta}{2}\,|\langle \partial\sigma, u\rangle|^2\,.$$

From property $\tilde{P}$, choose ψ_M satisfies $M > 4\frac{C^2}{1-\eta}$ we obtain $D(u) \geq 0$ and thus $i\partial\bar{\partial}\hat{\rho}(u)$ be. □

1.3.3 Pseudoconvex domains with plurisubharmonic defining function.

We are going to find another condition of pseudoconvex domain Ω such that the Diederich - Fornæss index η can be chosen arbitrarily close to 1. Assume that for such a pseudoconvex domain Ω there is a smooth defining function which is plurisubharmonic on the boundary $b\Omega$ we can let η go arbitrary close to 1. For that purpose, we need to have a better estimate for the Levi form of a defining function of Ω.

We consider the case Ω has a plurisubharmonic defining function, [FH07, FH08] that means, on the boundary, the Levi form of the defining function is non-negative for all direction. Hence, we predict that the Levi form of the defining function is less negative when we move to the inside of the domain. Precisely, we will estimate the Levi form in normal direction to inside of the domain as in Theorem 1.3.10. In that Theorem, we show that the Levi form of a defining function at points near the boundary can be estimate by a small term of the distance and a controlable term in normal direction.

Theorem 1.3.10. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Suppose Ω has a smooth defining function which is plurisubharmonic on the boundary $b\Omega$. Then for any $\epsilon > 0$, there exist a neighborhood U of $b\Omega$ and a smooth defining functions r such that*

$$i\partial\bar{\partial}r(p; u, u) \geq -\epsilon\left[|r(p)||u|^2 + \frac{1}{|r(p)|}|\langle\partial r(p), u\rangle|^2\right] \tag{1.3.8}$$

holds for all $q \in \Omega \cap U$, $u \in \mathbb{C}^n$.

In the case $n = 2$, Theorem 1.3.10 can be proved directly since the tangential and normal direction has only one-dimention. And then they can be shown in explicit formulas. Furthermore, we can get a stronger estimate for the Levi form of defining function r. The Theorem in this case is stated as follow

Theorem 1.3.11. *Let Ω be a smoothly bounded domain in $\mathbb{C}^2$. Suppose Ω has a smooth defining function which is plurisubharmonic on the boundary $b\Omega$. Then for any $\epsilon > 0$, there exist a neighborhood U of $b\Omega$ and a smooth defining functions $r = r_{\epsilon,K}$ such that*

$$i\partial\bar{\partial}r(p; u, u) \geq -\epsilon|r(p)||u|^2 + K|\langle\partial r(p), u\rangle|^2 \tag{1.3.9}$$

holds for all $p \in \Omega \cap U$, $u \in \mathbb{C}^n$.

For the proof of above Theorems, we first estimate the Levi form of a defining function at a point near the boundary. From our assumption, the domain Ω has a plurisubharmonic defining function, its Levi form is non-negative at any point in the boundary. Then we can simply use Taylor's formula to estimate the Levi form at neighbor points.

Since $b\Omega$ is smooth, there exists a neighborhood U of $b\Omega$ such that the projection π from $\bar{\Omega} \cap U$ to the boundary is smooth. For $p \in \Omega \cap U$ there is $p_0 \in b\Omega$ such that p_0 lies on the real line normal to $b\Omega$ passing through p , and $|p - p_0|$ is equal to the Euclidean distance $d_{b\Omega}(p)$ to $b\Omega$. Then we have

$$p = p_0 - d_{b\Omega}(p)N(p_0).$$

where

$$N(z) = \frac{1}{|\partial\rho(z)|}\sum_{j=1}^{n}\frac{\partial\rho}{\partial\bar{z}_j}(z)\frac{\partial}{\partial z_j}.$$

and ρ be any defining function of Ω. N is the unit outward normal vector at z. If f is a smooth function on U, then from Taylor's Theorem

$$f(p) = f(p_0) - 2d_{b\Omega}(p)(\mathbf{Re}N)(f)(p_0) + \mathcal{O}(d^2_{b\Omega}(p)) \quad \text{for } p \in \bar{\Omega} \cap U.$$

If $p \in \Omega \cap U$ with $\pi(p) = p_0$ and apply Taylor formula to the Levi form $i\partial\bar{\partial}\rho$, we obtain

$$i\partial\bar{\partial}\rho(p; w, w) = i\partial\bar{\partial}\rho(p_0; w, w) - 2d_{b\Omega}(p)(\mathbf{Re}N)(i\partial\bar{\partial}\rho(p_0; w, w)) + \mathcal{O}(d^2_{b\Omega}(p))|w|^2. \quad (1.3.10)$$

for any vector $w \in \mathbb{C}^n$. Since $i\partial\bar{\partial}\rho(w, w)$ is a real value function, we have

$$(\mathbf{Re}N)(i\partial\bar{\partial}\rho(w, w)) = \mathbf{Re}(N(i\partial\bar{\partial}\rho(w, w))).$$

And from the assumption Ω has a smooth plurisubharmonic defining function, $i\partial\bar{\partial}\rho(w, w)$ is non-negative on the boundary. Denote by Ω_W the set of all point $p \in \Omega \cap U$ for which $\pi(p) = p_0$ is weakly pseudoconvex boundary point. Let $w \in \mathbb{C}^n$ be a vector in a weakly pseudoconvex direction at p_0, i.e., $\langle \partial\rho(p_0), w\rangle = 0$ and $i\partial\bar{\partial}\rho(p_0; w, w) = 0$. Therefore, for $p \in \Omega_W$ and w is a weakly pseudoconvex tangent direction, then $i\partial\bar{\partial}\rho(w, w) \geq 0$ and equals to 0 at p_0. Thus, any tangential derivative of $i\partial\bar{\partial}\rho(w, w)$ vanishing at p_0 since $i\partial\bar{\partial}\rho(w, w)|_{b\Omega}$ attains minimum at these points. Since $N - \bar{N}$ is tangential to $b\Omega$, we obtains

$$(N - \bar{N})(i\partial\bar{\partial}\rho(p_0; w, w)) = 0,$$

and then $N(i\partial\bar{\partial}\rho)(w, w)$ is real at p_0 and

$$\mathbf{Re}(N(i\partial\bar{\partial}\rho(p_0; w, w))) = N(i\partial\bar{\partial}\rho(p_0; w, w)) = (N(i\partial\bar{\partial}\rho))(p_0; w, w)$$

where the last equation holds since w is a fixed vector. Hence, (1.3.10) becomes

$$i\partial\bar{\partial}\rho(p; w, w) = -d_{b\Omega}(p)(N(i\partial\bar{\partial}\rho))(p_0; w, w) + \mathcal{O}(d^2_{b\Omega})(p). \quad (1.3.11)$$

Here we see that the problem for attaining conclusion of Theorem 1.3.10 and 1.3.11 is that when $(Ni\partial\bar{\partial}\rho)(p_0; w, w)$ is strictly positive, that mean when moving inward along the real normal line to $b\Omega$ at p_0, the Levi form $i\partial\bar{\partial}\rho(w, w)$ is strictly decreasing. Hence, $i\partial\bar{\partial}\rho(w, w)$ is negative here and then (1.3.8) and (1.3.9) cannot hold for $i\partial\bar{\partial}\rho(w, w)$ when ϵ is sufficiently small.
To solve the problem and get the desired estimate, we must find another defining function r of Ω such that $(Ni\partial\bar{\partial}r)(p_0; w, w)$ is less than $(Ni\partial\bar{\partial}\rho)(p_0; w, w)$. The construction of r is straightforward when $n = 2$. In higher dimensions, the difficulty is that the the Levi form can vanish in more than one complex tangential directions at a point p_0 in the boundary.

Now we will prove Theorem 1.3.11, which is more simplier and straingthforward. In the proof we will use the following Lemma

Lemma 1.3.12. *Let Ω be a pseudoconvex domain in $\mathbb{C}^2$. Suppose X is a smooth vector field, which is complex tangential to $b\Omega$. Furthermore, suppose $b\Omega$ is weakly pseudoconvex at some boundary point p_0. Then $Y = \sum_{j=1}^2 \bar{X}_j \frac{\partial X_k}{\partial z_j} \frac{\partial}{\partial z_k}$ is a weak complex tangential to $b\Omega$ at p_0.*

Proof. Since X is tangential to $b\Omega$, let ρ be a smooth defining function for Ω, then $X(\rho) = 0$ holds on $b\Omega$. Moveover we have $\bar{X}(X(\rho)) = 0$ on $b\Omega$. Then we have

$$0 = \bar{X}(X(\rho))(p_0) = \sum_{j=1}^{2} \bar{X}_j \frac{\partial}{\partial \bar{z}_j} \left(\sum_{k}^{2} X_k \frac{\partial \rho}{\partial z_k} \right)(p_0)$$
$$= \sum_{j,k=1}^{2} \bar{X}_j \frac{\partial X_k}{\partial \bar{z}_j} \frac{\partial \rho}{\partial z_k}(p_0) + \sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_j} X_k \bar{X}_j(p) = Y(\rho)(p_0)$$

where the last equation holds since p_0 is a weak pseudoconvex boundary point. Then we have Y is a complex tangential direction at p_0 and $i\partial\bar{\partial}\rho(p_0; Y, Y) = 0$. □

Proof of Theorem 1.3.11. Since Ω is smooth, we can find a neighborhood U of $b\Omega$ such that the smooth vector fields

$$T = \frac{1}{|\partial\rho|}\left[\frac{\partial\rho}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial\rho}{\partial z_1}\frac{\partial}{\partial z_2}\right] \quad \text{and} \quad N = \frac{1}{|\partial\rho|}\left[\frac{\partial\rho}{\partial \bar{z}_1}\frac{\partial}{\partial z_1} + \frac{\partial\rho}{\partial \bar{z}_2}\frac{\partial}{\partial z_2}\right]$$

are defined on $\bar{\Omega} \cap U$. We know that $\{T, N\}$ spans $\mathbb{C}^2$. And we have

$$T\rho = 0 = \langle T, N\rangle \quad \text{and} \quad |T| = 1 = |N| \text{ on } \bar{\Omega} \cap U.$$

Let ϵ be fixed, for $p \in \Omega_W \cap U$ and $u \in \mathbb{C}^2$, we can find constants $a_{p,u}$ and $b_{p,u}$ such that u is decomposed as

$$u = a_{p,u} T(q) + b_{p,u} N(q).$$

We drop the subscript p, u. Then the Levi form of ρ can be decomposed as

$$i\partial\bar{\partial}\rho(p; u, u) = |a|^2 i\partial\bar{\partial}\rho(p; T, T) + 2\mathbf{Re}\left(a\bar{b}(i\partial\bar{\partial}\rho)(p; T, N)\right) + |b|^2 (i\partial\bar{\partial}\rho)(p; N, N).$$

Note that we are considering $p \in \Omega_W \cap U$ in the cases $\mathbb{C}^2$, then T must be the weakly pseudoconvex tangential. From (1.3.11) we have

$$\begin{aligned} i\partial\bar{\partial}\rho(p; u, u) =& |a|^2 \left(-2d_{b\Omega}(p)(N(i\partial\bar{\partial}\rho)(p_0; T, T)) + \mathcal{O}(d_{b\Omega}^2(p))\right) \\ &+ 2\mathbf{Re}\left(a\bar{b} i\partial\bar{\partial}\rho(p; T, N)\right) + |b|^2 i\partial\bar{\partial}\rho(p; N, N). \end{aligned}$$

The Cauchy-Schwarz inequality gives

$$2|a\bar{b}\mathbf{Re}(i\partial\bar{\partial}\rho(p;T,N))| \leq |a|^2|\rho|^2 + \frac{|b|^2}{|\rho|^2} i\partial\bar{\partial}\rho(p;T,N)$$

and then

$$\begin{aligned} i\partial\bar{\partial}\rho(p;u,u) \geq |a|^2 \left(-2d_{b\Omega}(p)(N(i\partial\bar{\partial}\rho)(p_0;T,T)) - |\rho|^2 + \mathcal{O}(d^2_{b\Omega}(p))\right) \\ + |b|^2 \left(-\frac{1}{|\rho|^2} i\partial\bar{\partial}\rho(p_0;T,N) + i\partial\bar{\partial}\rho(p;N,N)\right). \end{aligned} \tag{1.3.12}$$

Since Ω is smooth, after shrinking U we can assume that

$$-|\rho(p)|^2 + \mathcal{O}(d^2_{b\Omega}(p)) \geq \frac{\epsilon}{4}\rho(p)$$

for $p \in \Omega_W \cap U$. Since ρ is plurisubharmonic on $\bar{\Omega} \cap b\Omega$, we have

$$|i\partial\bar{\partial}\rho(T,N)|^2 \leq |i\partial\bar{\partial}\rho(T,T)||i\partial\bar{\partial}\rho(N,N)|$$

holds on $\bar{\Omega} \cap b\Omega$. For $p \in \Omega_W \cap U$, $\pi(p) = p_0$ is a weakly pseudoconvex boundary point, we get that $i\partial\bar{\partial}\rho(p_0;T,N) = 0$. Therefore, there exists a constant $C_1 > 0$ depend on ρ such that

$$|i\partial\bar{\partial}\rho(p;T,N)|^2 \leq C_1|\rho(p)|^2.$$

Thus we can have a lower estimate for $i\partial\bar{\partial}\rho(p;u,u)$

$$\begin{aligned} i\partial\bar{\partial}\rho(p;u,u) \geq & |a|^2 \left(-2d_{b\Omega}(p)(Ni\partial\bar{\partial}\rho(p_0;T,T)) + \frac{\epsilon}{4}\rho(p)\right) \\ & + |b|^2 \left(C_1 - i\partial\bar{\partial}\rho(p;N,N)\right), \end{aligned}$$

and we can find some C_2 depending on ρ such that

$$i\partial\bar{\partial}\rho(p;u,u) \geq |a|^2 \left(-2d_{b\Omega}(p)(Ni\partial\bar{\partial}\rho(p_0;T,T)) + \frac{\epsilon}{4}\rho(p)\right) - C_2|b|^2 \tag{1.3.13}$$

holds for $p \in \Omega_W \cap U$.

We know that the problem with obtaining the conclusion is that when $(Ni\partial\bar{\partial}\rho(T,T))$ is strictly positive at p_0, (1.3.9) cannot hold when ϵ is suficient small. Therefore we need to construct a smooth defining function r of Ω such that $(Ni\partial\bar{\partial}r(p_0;T,T))$ is less than $(Ni\partial\bar{\partial}\rho(p_0;T,T))$.
Let $C > 0$ be a large constant, which will be chosen later. We consider a smooth defining function

$$r_c = r = \rho e^{-C\sigma}$$

where $\sigma = |i\partial\bar{\partial}\rho(N,T)|^2$. According to the new defining function, we define the vector fields

$$T^r = \frac{1}{|\partial r|}\left[\frac{\partial r}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2}\right] \quad \text{and} \quad N^r = \frac{1}{|\partial r|}\left[\frac{\partial r}{\partial \bar{z}_1}\frac{\partial}{\partial z_1} + \frac{\partial r}{\partial \bar{z}_2}\frac{\partial}{\partial z_2}\right].$$

We note that r and ρ are defining functions of a same domain, then we have $T^r = T$ and $N^r = N$ on the boundary $b\Omega$.
As before, for $p \in \Omega_W \cap U$, for each vector $u \in \mathbb{C}^2$ can be decomposed as $u = a_{p,u}T^r + b_{p,u}N^r$, and we drop the subscripts p, u. Since $i\partial\bar{\partial}\rho(p_0; N, T) = 0$, it follows that not only σ but also any derivative of σ vanishes at p_0, then by straightforward computation we have

$$\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} = e^{-C\sigma}\left[-C\frac{\partial \sigma}{\partial \bar{z}_k}\left(\frac{\partial \rho}{\partial z_j} - C\rho\frac{\partial \sigma}{\partial z_j}\right) + \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} - C\frac{\partial \rho}{\partial \bar{z}_k}\frac{\partial \sigma}{\partial z_j} - C\rho\frac{\partial^2 \sigma}{\partial z_j \partial \bar{z}_k}\right]$$

and then

$$\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p_0) = \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p_0).$$

Moreover, r is plurisubharmonic at p_0 and $i\partial\bar{\partial}r(p_0; T^r, N^r) = 0$, we can apply (1.3.13) for r with constant C_2 depends on r

$$i\partial\bar{\partial}r(p; u, u) \geq |a|^2\left(-2d_{b\Omega}(p)(N^r i\partial\bar{\partial}r(p_0; T^r, T^r)) + \frac{\epsilon}{4}r(p)\right) - C_2|b|^2 \qquad (1.3.14)$$

for all $p \in \Omega_W \cap U$ after shrinking U. We will show the relation between $(N^r i\partial\bar{\partial}r(p_0; T^r, T^r))$ and $(N i\partial\bar{\partial}\rho(p_0; T, T))$. We claim that

$$N^r i\partial\bar{\partial}r(p_0; T^r, T^r) \leq \left[N i\partial\bar{\partial}\rho(p_0; T, T) - C|\partial\rho|.(N i\partial\bar{\partial}\rho(p_0; T, T))^2\right].$$

We have $N^r = N$ on $b\Omega$, it implies

$$N^r i\partial\bar{\partial}r(p_0; T^r, T^r) = N i\partial\bar{\partial}r(p_0; T^r, T^r) = \sum_{l=1}^{2} N_l \frac{1}{\partial z_l}\left(\sum_{j,k=1}^{2}\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}T_j^r \bar{T}_k^r\right)$$

on the boundary. Since r is plurisubharmonic at p and T^r is the weak complex tangential at p_0, we have

$$\sum_{j,k=1}^{2}\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}\left(\sum_{l=1}^{2} N_l \frac{\partial T_j^r}{\partial z_l}\right)\bar{T}_k^r = 0 = \sum_{j,k=1}^{2}\frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}L_j^r\left(\sum_{l=1}^{2} N_l \frac{\partial \bar{T}_k^r}{\partial z_l}\right).$$

Furthermore, $T^r(p_0) = T(p_0)$ on $b\Omega$, we have

$$N^r i\partial\bar\partial r(p_0; T^r, T^r) = \sum_{j,k,l=1}^{2} \frac{\partial^3 r}{\partial z_j \partial \bar z_k \partial z_l}(p_0) T_j \bar T_k N_l.$$

By straightforward computation we get

$$\begin{aligned}
&\frac{\partial^3 r}{\partial z_j \partial \bar z_k \partial z_l} \\
&= e^{-C\sigma}\Bigg[\frac{\partial^3 \rho}{\partial z_j \partial \bar z_k \partial z_l} - C\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar z_k}\frac{\partial \sigma}{\partial z_l} + \frac{\partial \rho}{\partial \bar z_k}\frac{\partial^2 \sigma}{\partial z_j \partial z_l} + \frac{\partial \rho}{\partial z_j}\frac{\partial^2 \sigma}{\partial \bar z_k \partial z_l} + \rho\frac{\partial^3 \sigma}{\partial z_j \partial \bar z_k \partial z_l}\right) \\
&\quad - C\frac{\partial^2 \sigma}{\partial z_j \partial \bar z_k}\left(\frac{\partial \rho}{\partial z_l} - C\rho\frac{\partial \sigma}{\partial z_l}\right) - C\frac{\partial \sigma}{\partial \bar z_k}\frac{\partial}{\partial z_j}\left(\frac{\partial \rho}{\partial z_l} - C\rho\frac{\partial \sigma}{\partial z_l}\right) \\
&\quad - C\frac{\partial \sigma}{\partial z_j}\left(\frac{\partial^2 \rho}{\partial \bar z_k \partial z_l} - C\left(\frac{\partial \rho}{\partial \bar z_k}\frac{\partial \sigma}{\partial z_l} + C\rho\frac{\partial^2 \sigma}{\partial \bar z_k \partial z_l}\right) - C\frac{\partial \sigma}{\partial \bar z_k}\left(\frac{\partial \rho}{\partial z_l} - C\rho\frac{\partial \sigma}{\partial z_l}\right)\right)\Bigg]
\end{aligned}$$

Since $p_0 \in b\Omega$ we have ρ, σ and all the first derivative of σ vanish at p_0. And T is the weak complex tangential to $b\Omega$ at p_0, we have $\left\langle \frac{\partial \rho}{\partial z_j}, T \right\rangle = 0$. Thus we get

$$\begin{aligned}
\left(\sum_{j,k,l=1}^{2} \frac{\partial^3 r}{\partial z_j \partial \bar z_k \partial z_l}(p_0) T_j \bar T_k N_l\right) &= \left(\sum_{j,k,l=1}^{2} \frac{\partial^3 \rho}{\partial z_j \partial \bar z_k \partial z_l}(p_0) T_j \bar T_k N_l\right) \\
&\quad - C\left(\sum_{l=1}^{2} \frac{\partial}{\partial z_l}(p_0) N_l\right)\left(\sum_{j,k=1}^{2} \frac{\partial^2 \sigma}{\partial z_j \partial \bar z_k}(p_0) T_j \bar T_k\right)
\end{aligned}$$

or

$$(N^r i\partial\bar\partial r(p_0; T^r, T^r)) = (N i\partial\bar\partial\rho(p_0; T, T) - C\langle\partial\rho\rangle\, i\partial\bar\partial\sigma(p_0; T, T)). \tag{1.3.15}$$

Since $\sigma = |i\partial\bar\partial\rho(p_0; N, T)|^2$ and $i\partial\bar\partial\rho(p_0; N, T) = 0$, we have

$$\begin{aligned}
i\partial\bar\partial\sigma(p_0; T, T) &= i\partial\bar\partial\sigma(p_0; T, T) \\
&= i\left(\bar\partial i\partial\bar\partial\rho(p_0; N, T)\partial\overline{i\partial\bar\partial\rho(p_0; N, T)} + \partial i\partial\bar\partial\rho(p_0; N, T)\bar\partial\overline{i\partial\bar\partial\rho(p_0; N, T)}\right)(p_0; T, T) \\
&= |\left\langle \partial i\partial\bar\partial\rho(p_0; N, T), T\right\rangle|^2 + |\left\langle \bar\partial i\partial\bar\partial\rho(p_0; N, T), \bar T\right\rangle|^2 \\
&\geq |\left\langle \partial i\partial\bar\partial\rho(p_0; N, T), T\right\rangle|^2
\end{aligned}$$

Compute further, we ger

$$\left\langle \partial i\partial\bar\partial\rho(p_0;N,T),T\right\rangle = \sum_{j=1}^{2} T_j \frac{\partial}{\partial z_j}\left(\sum_{k,l=1}^{2}\frac{\partial^2\rho}{\partial\bar z^k\partial z_l}\bar T_k N_l\right)$$
$$= \sum_{j,k,l=1}^{2}\frac{\partial^3\rho}{\partial z_j\partial\bar z_k\partial z_l}T_j\bar T_k N_l + \sum_{k,l=1}^{2}\frac{\partial^2\rho}{\partial\bar z_k\partial z_l}\left(T_j\frac{\partial}{\partial z_j}(\bar T_k N_l)\right)$$

Since T is a weak complex tangential direction at p_0, it follows that

$$\left\langle \partial i\partial\bar\partial\rho(p_0;N,T),T\right\rangle = Ni\partial\bar\partial\rho(p_0;T,T) + \sum_{k,l=1}^{2}\frac{\partial^2\rho}{\partial z_l\partial\bar z_k}N_l\left(\sum_{j=1}^{2}T_j\frac{\partial\bar T_k}{\partial z_j}(p_0)\right).$$

From Lemma 1.3.12 we know that $\sum_{j=1}^{2} T_j\dfrac{\partial\bar T_k}{\partial z_j}\dfrac{\partial}{\partial z_j}$ is a complex tangential to $b\Omega$ at p_0. And since ρ is plurisubharmonic and p_0 is weak pseudoconvex boundary point the last term on the right-hand side vanishes. Since $\langle\partial\rho,N\rangle(p_0) = |\partial\rho(p_0)|$, from (1.3.15) we obtain

$$N^r i\partial\bar\partial r(p_0;T^r,T^r) \leq \left[Ni\partial\bar\partial\rho(p_0;T,T) - C|\partial\rho|(Ni\partial\bar\partial\rho(p_0;T,T))^2\right].$$

Hence, the estimate (1.3.13) for r becomes

$$i\partial\bar\partial r(p;u,u) \geq |a|^2\left[2d_{b\Omega}(p)\left(CC_3(Ni\partial\bar\partial\rho(p_0;T,T))^2 - Ni\partial\bar\partial\rho(p_0;T,T)\right) + \frac{\epsilon}{4}r(p)\right] - C_2|b|^2 \quad (1.3.16)$$

for $p\in\Omega_W\cap U$, where $C_3>0$ satisfies $|\partial\rho|\geq C_3$ on $b\Omega$.
We need to show that there exist a $C>0$ and a neighborhood U_C of $b\Omega$ such that

$$2d_{b\Omega}(p)\left(CC_3(Ni\partial\bar\partial\rho(p_0;T,T))^2 - Ni\partial\bar\partial\rho(p_0;T,T)\right) \geq \frac{\epsilon}{4}r(p). \quad (1.3.17)$$

For easier notation, we write $A_{p_0} = Ni\partial\bar\partial\rho(p_0;T,T)$, then (1.3.17) becomes

$$2d_{b\Omega}(p)\left(CC_3A_{p_0}^2 - A_{p_0}\right)(p_0) \geq \frac{\epsilon}{4}r(p).$$

If $CC_3A_{p_0}^2 - A_{p_0} > 0$ then (1.3.17) holds trivially for all $C>0$. We consider the case $CC_3A_{p_0}^2 - A_{p_0} < 0$. Since Ω is smooth, there exists a constant C_4 such that $d_{b\Omega}(p)\leq C_4|\rho(p)|$ for $p\in\Omega\cap U$. Note that $\rho = re^{C\sigma}$, and we only need to find such C and the neighborhood U_C such that

$$2C_4|r(p)|e^{C\sigma}\left(CC_3A_{p_0}^2 - A_{p_0}\right) \geq -\frac{\epsilon}{4}r(p).$$

Since σ is smooth, we can find a neighborhood U_C of $b\Omega$ such that for $z \in \Omega \cap U_C$ we have $e^{C\sigma(z)} \leq e^{2C\sigma(\pi(z))}$. Moreover, $q \in \Omega_W \cap U_C$ and then $\pi(q)$ is weakly pseudoconvex boundary point, then we have $e^{C\sigma(z)} \leq 2$ and then to obtain (1.3.17) we only need to find C such that

$$CC_3 A_{p_0}^2 - A_{p_0} \geq -\frac{\epsilon}{16C4}$$

holds on $\Omega_W \cap U_C$. Note that neither C_3, C_4 nor A_{p_0} depends on the choice of C. Thus, by choosing

$$C = \max\left\{0, \max_{p_0 \in b\Omega_W} \frac{-\frac{\epsilon}{16C_4} + A_{p_0}}{C_3 A_{p_0}^2}\right\}$$

we prove (1.3.17) on $\Omega_W \cap U_C$, which implies that

$$i\partial\bar{\partial}r(p; u, u) \geq \frac{\epsilon}{2} r(p)|u|^2 - C_2 | \langle \partial r(p), u \rangle |^2 \tag{1.3.18}$$

holds on $\Omega_W \cap U$. We will find an estiamate similar to (1.3.18) holds for a neighborhood of $\Omega_W \cap U$. For above computation we get that $N^r i\partial\bar{\partial}r(T^r, T^r) \leq \frac{\epsilon}{16}$ holds on the set of weakly pseudoconvex boundary points of Ω. Since r is smooth, there is a neighborhood W of weakly pseudoconvex boundary points such that $\mathbf{Re}(N^r i\partial\bar{\partial}^r(T^r, T^r) \leq \frac{\epsilon}{8}$ on $W \cap b\Omega$. And we can assume that $W \subset U_C$ such that if $p \in W \cap \Omega$ then $\pi(p) \in W \cap b\Omega$. Using Taylor's formula after shrinking W, we have

$$\begin{aligned} i\partial\bar{\partial}r(p; T^r, T^r) &= i\partial\bar{\partial}r(\pi(p); T^r, T^r) + 2d_{b\Omega}(p)\mathbf{Re}(N^r i\partial\bar{\partial}r(\pi(p); T^r, T^r)) + \mathcal{O}(d_{b\Omega}^2(p)) \\ &\geq i\partial\bar{\partial}r(\pi(p); T^r, T^r) + \frac{\epsilon}{4} r(p) + \mathcal{O}(r^2(p)) \\ &\geq i\partial\bar{\partial}r(\pi(p); T^r, T^r) + \frac{\epsilon}{2} r(p) \end{aligned}$$

holds for $p \in W \cap \Omega$. And we also have

$$\begin{aligned} i\partial\bar{\partial}r(u, u) \geq\ & |a|^2 \left[i\partial\bar{\partial}r(\pi(p); T^r, T^r) + \frac{\epsilon}{2} r(p)\right] + |b|^2 i\partial\bar{\partial}r(p; N^r, N^r) \\ & - 2|a||b| \left[i\partial\bar{\partial}r(\pi(p); T^r, N^r) + \mathcal{O}(r(p))\right] \\ \geq\ & |a|^2 \left[i\partial\bar{\partial}r(\pi(p); T^r, T^r) + \epsilon r(p)\right] - C_5 | \langle \partial r(p), u \rangle |^2 \\ & - 2|a||b| i\partial\bar{\partial}r(\pi(p); T^r, N^r). \end{aligned}$$

for a large positive number C_5. In the last step we used Cauchy - Schwartz inequality and since r is smooth. By definition $i\partial\bar{\partial}r(T^r, T^r)$ is positive, we only need to estimate $i\partial\bar{\partial}r(T^r, N^r)$. We know that r is not necessary plurisubharmonic on $b\Omega$ at strictly pseudoconvex boundary points. However since ρ is plurisubharmonic on $b\Omega$ and $r = \rho e^{-C\sigma}$ and since any derivative of σ is $\mathcal{O}(i\partial\bar{\partial}\rho(N^r, T^r)$, we have on the boundary of Ω for a constant $C_6 > 0$

$$i\partial\bar{\partial}r(T^r, N^r) = C_6 i\partial\bar{\partial}r(T^r, T^r)\left[i\partial\bar{\partial}r(N^r, N^r) + C_6\right]$$

By Cauchy-Schwarz inequality we can find a constant $C_7 > 0$ such that

$$i\partial\bar{\partial}r(u, u) \geq \epsilon r(q)|u|^2 - C_7|\left\langle \partial r(p), u\right\rangle|^2$$

for $p \in W \cap \Omega$. Let $\tilde{r} = r + Kr^2$ for some $K > 2C_7$. Then we have

$$i\partial\bar{\partial}\tilde{r}(p; u, u) = (1 + 2Kr)i\partial\bar{\partial}r(p; u, u) + 2K|\left\langle \partial r(p), u\right\rangle|^2$$

Let $U_K = \left\{z \in W | 1 + 2Kr(z) \geq \frac{1}{2} \text{ and } K|r|^2 < \frac{\epsilon}{2}\right\}$, then we get, for $p \in U_K \cap \Omega$

$$\begin{aligned} i\partial\bar{\partial}\tilde{r}(u, u) &\geq \frac{1}{2}\left(\epsilon r(p)|u|^2 - C_7|\left\langle \partial r, u\right\rangle|^2\right) + 2K|\left\langle \partial r(p), u\right\rangle|^2 \\ &\geq \epsilon\tilde{r}(p)|u|^2 + K|\left\langle \partial\tilde{r}(p), u\right\rangle|^2. \end{aligned}$$

We denote $S = b\Omega \setminus (W \cap b\Omega)$, the closed subset of the the set of strictly pseudoconvex boundary points. For K is chosen sufficiently large, there exists a neighborhood U_S of S such that $\tilde{r}$ is strictly plurisubharmonic on $\Omega \cap U_S$. In particular there exists a neighborhood V of $b\Omega$ such that

$$i\partial\bar{\partial}\tilde{r}(p; u, u) \geq \epsilon\tilde{r}(p)|u|^2 + K|\left\langle \partial\tilde{r}(p), u\right\rangle|^2$$

for all $p \in \Omega \cap V$ and $u \in \mathbb{C}^2$. □

Our purpose is find a condition of the pseudoconvex domain Ω such that the Diederich-Fornæss compornent can be chosen arbitrary closed to 1. As a consequence of Theorem 1.3.10 and 1.3.11 we have

Theorem 1.3.13. *Suppose the hypothesis of Theorem 1.3.10 holds. Then for any $\eta \in (0, 1)$ there exists a smooth defining function $\hat{r}$ such that $-(-\hat{r})^\eta$ is strictly plurisubharmonic on Ω.*

Proof. Let $\eta \in (0, 1)$ fixed. For our hypothesis, let r be the defining function of Ω as in Theorem 1.3.10. Set $\hat{r} = re^{-L|z|^2}$. We will prove that there exists a neighborhood U of $b\Omega$ such that $g = -(-\hat{r})^\eta$ is strictly plurisubharmonic on $\Omega \cap U$ for a large constant L. Denote $\psi(z) = |z|^2$ then we have

$$\begin{aligned} i\partial\bar{\partial}\hat{r}(u, u) = \eta(-r)^{\eta-2}e^{-L\eta\psi}\big[&Lr^2\left(i\partial\bar{\partial}\psi(u, u) - \eta L\,|\langle\partial\psi, u\rangle|^2\right) \\ &+ (-r)\left(i\partial\bar{\partial}r(u, u) - 2L\eta\mathbf{Re}\left(\langle\partial r, u\rangle\,\overline{\langle\partial\psi, u\rangle}\right)\right) + (1-\eta)\,|\langle\partial r, u\rangle|^2\big]. \end{aligned}$$

and we denote $D(u)$ be the term between [ ]. By Cauchy-Schwarz inequality we have

$$2(-r)L\eta \mathrm{Re}\,\langle \partial r, u\rangle\,\overline{\langle \partial\psi, u\rangle} \geq -r^2\frac{2L^2\eta^2}{1-\eta}|\,\langle \partial\psi, u\rangle\,|^2 - \frac{1-\eta}{2}|\,\langle \partial r, u\rangle\,|^2$$

and it follows

$$D(u) \geq Lr^2\left(i\partial\bar{\partial}\psi(u,u) - L\frac{\eta+\eta^2}{1-\eta}|\,\langle \partial\psi, u\rangle\,|^2\right) + (-r)i\partial\bar{\partial}r(u,u) + \frac{1-\eta}{2}|\,\langle \partial r, u\rangle\,|^2.$$

Since $\psi = |z|^2$, we compute further the first term of $D(u)$ and note that Ω is bounded, we have

$$i\partial\bar{\partial}_{\psi}(u,u) - L\frac{\eta+\eta^2}{1-\eta}|\,\langle \partial\psi, u\rangle\,|^2 = |u|^2 - L\frac{\eta+\eta^2}{1-\eta}A|u|^2$$

where $A = \max_{z\in\bar{\Omega}}|z|^2$. Now if we choose $L = \frac{1-\eta}{2(\eta+\eta^2)A}$ then we have

$$D(u) \geq \frac{L}{2}r^2|u|^2 - ri\partial\bar{\partial}r(u,u) + \frac{1-\eta}{2}|\,\langle \partial r, u\rangle\,|^2$$

holds on Ω. We see that when η go to 1 then L is choose near to 0.
In Theorem 1.3.10, set $\epsilon = \min\left\{\frac{L}{4}, \frac{1-\eta}{4}\right\}$ then there exist a neighborhood U of $b\Omega$ and a smooth defining function r such that

$$i\partial\bar{\partial}r(p;u,u) \geq -\epsilon\left(|r(p)||u|^2 + \frac{1}{|r(p)|}|\,\langle \partial r(p), u\rangle\,|^2\right)$$

holds for all $p \in \Omega\cap U$. Then we obtain

$$i\partial\bar{\partial}\hat{r}(p;u,u) \geq \eta(-r)^{\eta-2}e^{-\eta\psi}\frac{L}{4}r^2|u|^2$$

for $p \in \Omega\cap U$, $u \in \mathbb{C}$. And we can extend $\hat{r}$ to $\Omega\setminus U$ such that $-(-\hat{r})^{\eta}$ is strictly plurisubharmonic on Ω. This prove Theorem 1.3.13 . □

1.4 Diederich-Fornæss index in q-pseudoconvex domains.

Let Ω be a domain in $\mathbb{C}^n$ with boundary $b\Omega = M$. Let z_0 be a point in M, U be a neighborhood of z_0. We assume that $b\Omega$ is of class C^2. Let ρ be a defining function and denote $\rho_{ij}(z)$ the matrix of the Levi form $i\partial\bar{\partial}\rho(z)$ in the basis $\{\omega_1,\ldots,\omega_n\}$ with $\omega_n = \partial\rho$. We assume that, for a suitable choice of $\{\omega_1,\ldots,\omega_n\}$, the eigenvalues of the Levi form is ordered as $\lambda_1(z) \leq \lambda_2(z) \leq \ldots \leq \lambda_{n-1}(z)$. We recall the definition of $q-$pseudoconvex domain

Definition 1.4.1. We say that M is $q-$pseudoconvex if there exists a covering of the boundary and, on each patch, a C^2 smooth bundle $\mathcal{V} \subset T^{1,0}M$ of rank $q_0 \leq q$, say $\mathcal{V} = \text{Span}\{\partial_{\omega_1}, \dots, \partial_{w_{q_0}}$, such that

$$\sum_{j=1}^{q+1} \lambda_j(z) - \sum_{j=1}^{q_0} \rho_{jj}(z) \leq 0. \tag{1.4.1}$$

Lemma 1.4.2. *Assume that* (1.4.1) *is satisfied. Then for a suitable ρ, we have*

$$\varphi = -\log(-\rho)(z) + \lambda'|z|^2$$

(λ' positive) is an exhaustion function of Ω at z_0 such that for suitable λ' and for any $k \geq q+1$, the following holds

$$\sum_{|K|=k-1}' \sum_{ij=1}^{n} \varphi_{ij}(z) u_{iK} \bar{u}_{jK} - \sum_{|J|=k}' \sum_{j=1}^{q} \varphi_{jj}(z)|u_J|^2 \geq \lambda |u|^2, \tag{1.4.2}$$

for $z \in \Omega \cap U$ and for any forms u of degree $k \geq q$.

Proof. In condition (1.4.2), the Levi form is evaluated at point of Ω, whereas in assumtion (1.4.1) it is evaluated at $b\Omega$. We represent $b\Omega$ as a graph $x_n = h(z', y_n)$ and we can get the defining function $\rho = x_n - h(z', y_n)$. We denote $z \mapsto z^*$ the projection on $b\Omega$ along the x_n axis. Then we can have

$$\partial\rho^{\perp}(z) = \partial\rho^{\perp}(z^*)$$

and

$$\partial\bar{\partial}\rho(z) = \partial\bar{\partial}\rho(z^*).$$

We shall forget z in the following and always suppose it ranges through Ω. We shall use the notation $\omega^{\tau} = (\omega_1, \dots, \omega_{n-1})$, $\omega_n = \partial\rho$. Let $\lambda_1 \leq \lambda_2 \leq \dots$ and $\mu_1 \leq \mu_2 \leq \dots$ be eigenvalues of $\partial\bar{\partial}\varphi$ and $\partial\bar{\partial}\rho\big|_{\partial\rho^{\perp}}$, respectively. We have some computations

$$\begin{aligned} \partial\bar{\partial}\varphi &= \partial\bar{\partial}\left(-\log(-\rho) + \lambda'|z|^2\right), \\ &= \partial\left(\frac{1}{-\rho}\bar{\partial}\rho\right) + \lambda'\delta_{ij}, \\ &= \frac{1}{(-\rho)^2}\partial\rho \otimes \bar{\partial}\rho + \frac{1}{(-\rho)}\partial\bar{\partial}\rho + \lambda'\delta_{ij}. \end{aligned}$$

then we have $\partial\bar\partial\varphi = |\rho|^{-1}\partial\bar\partial\rho + |\rho|^{-2}\omega_n\otimes\bar\omega_n + \lambda'\omega\otimes\bar\omega$. Thus $|\rho|^{-1}\mu_i + \lambda'$ are the eigenvalues of $\partial\bar\partial\varphi\big|_{\partial\rho^\perp}$. It is clear that

$$\sum_{|K|=k-1}' \sum_{ij=1,\dots,n}' \varphi_{ij}(z) u_{iK}\bar u_{jK} \geq \left(\sum_{i=1}^k \lambda_i\right)|u|^2; \tag{1.4.3}$$

and

$$\sum_{|J|=k}' \sum_{i\leq q}' \varphi_{ii}(z)|u_J|^2 = \left((-\rho)^{-1}\left(\sum_{i=1}^q \mu_i\right) + \lambda' q\right)|u|^2. \tag{1.4.4}$$

We claim that for suitable $c > 0$,

$$\sum_{i=1}^k \lambda_i - \frac{1}{-\rho}\sum_{i=1}^q \mu_i - \lambda' q \geq ((k-q)\lambda' - kc) =: \lambda. \tag{1.4.5}$$

(where λ positive for suitable λ'. In fact, we decompose ω into ω^τ and ω_n then we have

$$\begin{aligned}\partial\bar\partial\varphi =& \frac{1}{(-\rho)}\partial\bar\partial\rho + \frac{1}{(-\rho)^2}\omega_n\otimes\bar\omega_n + \lambda'\omega\otimes\bar\omega\\ =& \frac{1}{(-\rho)}\partial^\tau\bar\partial^\tau\rho + \frac{1}{(-\rho)}\partial_{\omega_n}\bar\partial_{\omega_n}\rho + 2\mathrm{Re}\frac{1}{(-\rho)}\partial^\tau\bar\partial_{w_n}\rho + \frac{1}{(-\rho)^2}\omega_n\otimes\bar\omega_n + \lambda'\omega\otimes\bar\omega,\\ =& \frac{1}{(-\rho)}\partial^\tau\bar\partial^\tau\rho + \left[\left(\frac{1}{(-\rho)^2} + \frac{C_1}{(-\rho)^2}\right)\omega_n\otimes\bar\omega_n + 2\mathrm{Re}\frac{1}{(-\rho)}\partial^\tau\partial_{\omega_n}\rho + c\omega^\tau\otimes\bar\omega^\tau\right],\\ & - c\omega^\tau\otimes\bar\omega^\tau + \lambda'\omega\otimes\bar\omega. \end{aligned} \tag{1.4.6}$$

After shrinking U, for suitable c we can make the term between brackets [.] in (1.4.6) to be positive. Thus we get

$$\partial\bar\partial\varphi \geq \frac{1}{(-\rho)}\partial^\tau\bar\partial^\tau\rho - c\omega^\tau\otimes\bar\omega^\tau + \lambda'\omega\otimes\bar\omega. \tag{1.4.7}$$

Let N_k describe a family of complex k-dimentional planes in $\mathbb{C}^n$. We have

$$\begin{aligned}\sum_{i=1}^k \lambda_i =& \inf_{N_k} \mathrm{trace}\left(\partial\bar\partial\varphi\big|_{N_k}\right)\\ &\geq \inf_{N_k}\mathrm{trace}\left(\left(\frac{1}{(-\rho)}\partial^\tau\bar\partial^\tau\rho - c\omega^\tau\otimes\bar\omega^\tau + \lambda'\omega\otimes\bar\omega\right)\Big|_{N_k}\right),\\ &\geq \sum_{i=1}^k \frac{1}{(-\rho)}\mu_i + (k\lambda' - kc).\end{aligned}$$

Thus, we have

$$\sum_{i=1}^{k}\lambda_i - \frac{1}{-\rho}\sum_{i=1}^{q}\mu_i - \lambda' q \geq ((k-q)\lambda' - kc) = \lambda.$$

Combine (1.4.3), (1.4.4) and (1.4.5) we prove the lemma. □

In Lemma 1.4.2, we have proved that for q-pseudoconvex domain Ω we can find an exhausion function that is q-plurisubharmonic. For Ω is a pseudoconvex domain ($q = 0$), there are some results in which there are Diederich - Fornæss index η such that for suitable defining function φ, $-(-\varphi)^\eta$ is a plurisubharmonic defining function of Ω. The question is that is there a Diedrich - Fornæss index for q-pseudoconvex domain Ω, $q > 0$. In the following Theorem we show that there is such a η index that satisfies in a neighborhood of each point on the boundary.

Theorem 1.4.3. *Let Ω be a bounded q-pseudoconvex domain on $\mathbb{C}^n$ with C^2-boundary then for each point z_0 on the boundary, there exist a neighborhood U of z_0 and $0 < \eta_0 < 1$ such that for $0 < \eta < \eta_0$ there exists a defining function φ such that $-(-\varphi)^\eta$ is q-plurisubharmonic on $U \cap \Omega$.*

Proof. On the neighborhood U of z_0, assume that the Levi form we choose an orthogonal basis of $(1,0)$-forms $\omega_1, \dots, \omega_n$ on U and a defining function ρ as in Lemma 1.4.2. We also assume

$$(\omega_1, \dots, \omega_n)^T = (V_{ij})\,(dz_1, \dots, dz_n)^T.$$

where (V_{ij}) is an unitary matrix. We have that $(\omega_1, \dots, \omega_q)$ span the negative eigenspace of $\partial\bar{\partial}\rho\big|_{\partial\rho^\perp}$. Let S_{ij} be a $n \times n$ diagonal matrix with $S_{ii} = L_1$ with $i = 1, \dots, q$ and $S_{ii} = -L_2$ with $i = q+1, \dots n-1$ and $S_{nn} = 0$ We define on U

$$\varphi = \rho e^{\psi}.$$

where

$$\psi(z) = (\bar{z}_1, \dots, \bar{z}_n)\,\overline{(V_{ij})}^T\,(S_{ij})\,(V_{ij})\,(z_1, \dots, z_n)\,.$$

where $L_1, L_2 > 0$ will be choosen later. Denote $\hat{\varphi} = -(-\varphi)^\eta$ with $0 < \eta < 1$. Then we have that $\hat{\varphi}$ is a C^∞ defining function of Ω on $U \cap \bar{\Omega}$. After some computations, we obtain

$$\begin{aligned}\partial\bar{\partial}\hat{\varphi} =& \eta(-\rho)^{\eta-2}e^{-\eta\psi}\times \\ & \left[(1-\eta)\partial\rho\otimes\bar{\partial}\rho + (-\rho)\left(\partial\bar{\partial}\rho + 2\eta\mathrm{Re}\left(\partial\rho\otimes\bar{\partial}\psi\right)\right) - \rho^2\left(\partial\bar{\partial}\psi + \eta\partial\psi\otimes\bar{\partial}\psi\right)\right].\end{aligned} \tag{1.4.8}$$

Denote $A = \eta(-\rho)^{\eta-2}e^{-\eta\psi}$ a positive term. Let $\partial_\omega\bar{\partial}_\omega\hat{\varphi}$ be the matrix of $(\partial\bar{\partial}\rho)$ in the basis of $\{\omega_i\}$. Notice that when restrict on $\partial\rho^\perp$ we have

$$\partial\bar{\partial}\hat{\varphi}\big|_{\partial\rho^\perp} = A\left[(-\rho)\partial\bar{\partial}\rho - \rho^2(\partial\bar{\partial}\psi + \eta\partial\psi\otimes\bar{\partial}\psi)\right]\big|_{\partial\rho^\perp}. \tag{1.4.9}$$

Let $\partial^\tau_\omega\bar{\partial}^\tau_\omega$ be the restriction of $\partial_\omega\bar{\partial}_\omega\rho$ to the plane orthogonal to ω_n. We have

$$\begin{aligned}
\partial_\omega\bar{\partial}_\omega\hat{\varphi} =& A\Bigg[(1-\eta)\omega_n\otimes\bar{\omega}_n + (-\rho)\left(\partial_\omega\bar{\partial}_\omega\rho + 2\eta\mathrm{Re}\left(\omega_n\otimes\bar{\partial}_\omega\psi\right)\right) - \rho^2\left(\partial_\omega\bar{\partial}_\omega\psi + \eta\partial_\omega\psi\otimes\bar{\partial}_\omega\psi\right)\Bigg],\\
=& A\Bigg[(-\rho)\partial^\tau_\omega\bar{\partial}^\tau_\omega\rho - \rho^2\left(\partial_\omega\bar{\partial}_\omega\psi + \eta\partial_\omega\psi\otimes\bar{\partial}_\omega\psi\right)\\
&+ (-\rho)\left(\partial_{\omega_n}\bar{\partial}_{\omega_n}\rho + 2\mathrm{Re}\ \partial^\tau_\omega\bar{\partial}_{\omega_n}\rho + 2\eta\mathrm{Re}\ (\omega_n\otimes\bar{\partial}_\omega\psi\right) + (1-\eta)\omega_n\otimes\bar{\omega}_n\Bigg]\\
=& A\Bigg[(-\rho)\partial^\tau_\omega\bar{\partial}^\tau_\omega\rho - \rho^2\left(\partial_\omega\bar{\partial}_\omega\psi + \eta\partial_\omega\psi\otimes\bar{\partial}_\omega\psi\right)\\
&+ (-\rho)\left(\partial_{\omega_n}\bar{\partial}_{\omega_n}\rho + \frac{(1-\eta)}{(-\rho)}\omega_n\otimes\bar{\omega}_n + 2\mathrm{Re}\ \partial^\tau_\omega\bar{\partial}_{\omega_n}\rho + 2\eta\mathrm{Re}\ \omega_n\otimes\bar{\partial}_\omega\psi\right)\Bigg].
\end{aligned} \tag{1.4.10}$$

Shrink the neighborhood U if necessary, we can have the term $\left(\partial_n\bar{\partial}_n\rho + \frac{1-\eta}{(-\rho)}\right)$ is large. We also can choose that ηL_1 and ηL_2 small, then we can insert a small term $C(-\rho)^2$ for a suitable constant C to get the second line of (1.4.10) positive. Hence we obtain

$$\partial_\omega\bar{\partial}_\omega\hat{\varphi} \geq A\left[(-\rho)\partial^\tau_\omega\bar{\partial}^\tau_\omega\rho - \rho^2\left(\partial_\omega\bar{\partial}_\omega\psi + \eta\partial_\omega\psi\otimes\bar{\partial}_\omega\psi\right) - C\rho^2\omega^\tau\otimes\bar{\omega}^\tau\right]. \tag{1.4.11}$$

Let $\lambda_1 \leq \lambda_2 \leq \ldots$ and $\mu_1 \leq \mu_2 \leq \ldots$ be the eigenvalues of $\partial\bar{\partial}\varphi$ and $\partial\bar{\partial}\rho\big|_{\partial\rho^\perp}$, respectively, in the basis $\{\omega_i\}$. We note that in the basis $\{\omega_i\}$ we have $\left(\partial_\omega\bar{\partial}_\omega\psi\right) = S$. The Kronecker product $\partial_\omega\psi\otimes\bar{\partial}_\omega\psi$ forms a Hermitian, the eigenvalues of it are real and we can compute that it has $(n-1)$ zero-eigenvalues and one is

$$\lambda_L = L_1\sum_{i=1}^{q}\left|\sum_{j=1}^{n}V_{ij}z_j\right|^2 + L_2\sum_{j=q+1}^{n-1}\left|\sum_{j=1}^{n}V_{ij}z_j\right|^2,$$

and we also have that ω_n is the eigenvector corresponding to a zero-eigenvalue.
By (1.4.11) and above remark, we have

$$\sum_{|K|=k-1}^{\prime}\sum_{ij=1}^{n}\partial_{\omega_i}\partial_{\bar{\omega}_j}\hat{\varphi}w_{iK}\bar{w}_{jK} \geq$$
$$A\left[(-\rho)\sum_{j=1}^{k}\mu_j + \rho^2(k-q)L_2 - \rho^2\left(qL_1 + \eta\lambda_L + C\right)\right]|w|^2. \tag{1.4.12}$$

On the other hand, we have

$$\sum_{|J|=k}' \sum_{j=1}^{q} \partial_{\omega_j}\bar{\partial}_{\omega_j}\hat{\varphi}|w_J|^2 \leq A\left[(-\rho)\left(\sum_{j=1}^{q}\partial_{\omega_j}\bar{\partial}_{\omega_j}\rho\right) - qL_1\rho^2\right]|w|^2. \tag{1.4.13}$$

From (1.4.12) and (1.4.13) and a choice of L_1 and η_0 small and L_2 large enough, we obtain

$$\sum_{|K|=k-1}' \sum_{ij=1}^{n} \partial_{\omega_i}\partial_{\bar{\omega}_j}\hat{\varphi} w_{iK}\bar{w}_{jK} \geq \sum_{|J|=k}' \sum_{j=1}^{q} \partial_{\omega_j}\bar{\partial}_{\omega_j}\hat{\varphi}|w_J|^2.$$

Hence, we proved the Theorem that $-(-\varphi)^\eta$ is q-plurisubharmonic for all $0 < \eta \leq \eta_0$. □

In Theorem 1.4.3 we proved that for each q-pseudoconvex domain Ω we can find the Diedrich - Fornæss index η_0 such that there exists a defining function $\varphi = \varphi_\eta$ satisfies $-(-\varphi)^\eta$ is q-pseudosubharmonic for any $\eta \leq \eta_0$. Similar to the case Ω is pseudoconvex (means q=0), we going to find a condition for the boundary $b\Omega$ such that η can be closed to 1 arbitrarily. We remind the definition of self-bounded complex gradient and property P_q the domain Ω.

Definition 1.4.4. Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded domain. We said that a function $\psi \in C^2$ has a self-bounded complex gradient if there exists a constant C such that

$$\left|\sum_{j=1}^{n} \frac{\partial \psi}{\partial z_j}(z)u_j\right|^2 \leq C\sum_{i,j=1}^{n} \frac{\partial^2\psi}{\partial z_i \partial \bar{z}_j}(z)u_i\bar{u}_j \tag{1.4.14}$$

for all $u \in \mathbb{C}^n$ and $z \in \Omega$. When (1.4.14) holds, we write $|\partial\psi|^2_{\partial\bar{\partial}\psi} \leq C$.

Under the scaling $\psi \to t\psi$ for $t > 0$, a factor of t^2 appears on the left hand side of (1.4.14) while the right hand side has a factor of t. The size of the constant C can be choosen as we need.

Definition 1.4.5. We say that Ω has the property P_q if for any positive number M, there is a function $\psi = \psi_M \in C^2(\bar{\Omega})$ with

i. $|\psi| \leq 1$ on $\bar{\Omega}$;

and such that, if we denote by $\lambda_1^\psi \leq \lambda_2^\psi \leq \ldots \leq \lambda_{n-1}^\psi$ the ordered eigenvalues of the Levi form $(\partial\bar{\partial}\psi)$, we have

ii. $\sum_{j=1}^{q} \lambda_j^{\psi} - \sum_{j=1}^{q_0} \dfrac{\partial^2 \psi}{\partial z_j \partial \bar{z}_j} \geq M$ on $b\Omega$;

where the constant $c > 0$ does not depend on M.

Remark 1.4.6. Let Ω satisfies property P with function $\varphi = \varphi_M$. Let $\psi = e^{\varphi}$ then we have by computation

$$\frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} = \mathrm{e}^{\varphi} \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} + \frac{\partial \varphi}{\partial z_i} \otimes \frac{\partial \varphi}{\partial \bar{z}_j} \right);$$

and

$$\frac{\partial \psi}{\partial z_i} \otimes \frac{\partial \psi}{\partial \bar{z}_j} = \mathrm{e}^{2\varphi} \frac{\partial \varphi}{\partial z_i} \otimes \frac{\partial \varphi}{\partial \bar{z}_j}.$$

For any k form u with $k \geq q$ we have

$$\begin{aligned}
\sum_{|K|=k-1}^{\prime} \left| \sum_{j=1}^{n} \frac{\partial \psi}{\partial z_j} u_{jK} \right|^2 =& \mathrm{e}^{\varphi} \sum_{|K|=k-1}^{\prime} \left(\sum_{ij=1}^{n} \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} u_{iK} \bar{u}_{jK} - e^{\varphi} \sum_{ij=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} u_{iK} \bar{u}_{jK} \right) \\
=& \mathrm{e}^{\varphi} \sum_{|K|=k-1}^{\prime} \left[\sum_{ij=1}^{n} \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} u_{iK} \bar{u}_{jK} - (1-\epsilon) \sum_{j=1}^{q_0} \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_j} |u_{jK}|^2 \right. \\
& \left. - \mathrm{e}^{\varphi} \left(\sum_{ij=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} u_{iK} \bar{u}_{jK} - (1-\epsilon) \sum_{j=1}^{q_0} \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_j} |u_{jK}|^2 \right) \right].
\end{aligned}$$

Write the derivative of ψ in the last line in term of derivative of φ and obtain

$$\begin{aligned}
\sum_{|K|=k-1}^{\prime} & \left(\epsilon \left| \sum_{j=1}^{q_0} \frac{\partial \psi}{\partial z_j} u_{jK} \right|^2 + \left| \sum_{j=q_0+1}^{n} \frac{\partial \psi}{\partial z_j} u_{jK} \right|^2 \right) \\
&= \mathrm{e}^{\varphi} \sum_{|K|=k-1}^{\prime} \left[\sum_{ij=1}^{n} \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} u_{iK} \bar{u}_{jK} - (1-\epsilon) \sum_{j=1}^{q_0} \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_j} |u_{jK}|^2 \right. \\
& \qquad \left. - \mathrm{e}^{\varphi} \left(\sum_{ij=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} u_{iK} \bar{u}_{jK} - (1-\epsilon) \sum_{j=1}^{q_0} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j} |u_{jK}|^2 \right) \right].
\end{aligned}$$

From condition ii., if we choose ϵ small enough, the term in the second line of above equality can be positive. And we obtain

$$\sum_{|K|=k-1}^{\prime} \left| \sum_{j=1}^{n} \frac{\partial \psi}{\partial z_j} u_{jK} \right|^2 \leq \frac{\mathrm{e}^{\varphi}}{\epsilon} \sum_{|K|=q-1}^{\prime} \left[\sum_{ij=1}^{n} \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} u_{iK} \bar{u}_{jK} - (1-\epsilon) \sum_{j=1}^{q_0} \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_j} |u_{jK}|^2 \right]. \tag{1.4.15}$$

We have an estimate look like the self-bounded for the gradient.

Theorem 1.4.7. *Let Ω be a bounded domain on $\mathbb{C}^n$ and satisfies property P_q. Then for each point z_0 on the boundary, for each $0 < \eta_0 < 1$, there exist a neighborhood U of z_0 such that there is a defining function φ_η satisfying $-(-\varphi_\eta)^\eta$ is q-plurisubharmonic on $U \cap \Omega$ for $0 < \eta \leq \eta_0$.*

Proof. For a point p in the boundary, we assume a neighborhood U and choose a orthogonal basis of $(1,0)$ forms $\{\omega\}$ and a defining function ρ as in Lemma 1.4.2. Let $\psi = \psi_M$ satisfy property P_q on Ω, that means there exist $M > 0$ and

i. $|\psi| \leq 1$ on Ω;

ii. $\sum_{j=1}^{q} \lambda_j^\psi - \sum_{j=1}^{q_0} \dfrac{\partial^2 \psi}{\partial_{\omega_j} \partial_{\bar{\omega}_j}} \geq M$.

We can also replace $\psi = e^\psi$ if necessary that ψ can satisfy the above conditions (with a new constant M) and the estimate for the gradient as in above remark

$$\sum_{|K|=k-1}^{\prime} \left| \sum_{j=1}^{n} \frac{\partial \psi}{\partial \omega_j} u_{jK} \right|^2 \leq \frac{1}{\epsilon_0} \sum_{|K|=k-1}^{\prime} \left[\sum_{ij=1}^{n} \frac{\partial^2 \psi}{\partial \omega_i \partial \bar{\omega}_j} u_{iK} \bar{u}_{jK} - (1-\epsilon_0) \sum_{j=1}^{q_0} \frac{\partial^2 \psi}{\partial \omega_j \partial \bar{\omega}_j} |u_{jK}|^2 \right],$$

for some ϵ_0 small enough.
Let $\varphi = \rho e^{-\psi}$ and denote $\hat{\varphi} = -(-\varphi)^\eta$ for $0 < \eta < 1$.
Similar to Theorem 1.4.3 we can compute

$$\begin{aligned} \partial\bar{\partial}\hat{\varphi} = A\big[(1-\eta)\partial\rho \otimes \bar{\partial}\rho + (-\rho)\left(\partial\bar{\partial}\rho + 2\eta \mathrm{Re}\left(\partial\rho \otimes \bar{\partial}\psi\right)\right) \\ + \rho^2 \left(\partial\bar{\partial}\psi - \eta \partial\psi \otimes \bar{\partial}\psi\right)\big], \end{aligned} \tag{1.4.16}$$

with $A = \eta(-\rho)^{\eta-2} \mathrm{e}^{-\eta\psi}$ is a positive term.
Following the proof in Theorem 1.4.3, by restrict $\partial\bar{\partial}\hat{\varphi}$ to the plane orthogonal to ω_n, we get

$$\begin{aligned} \partial_\omega \bar{\partial}_\omega \hat{\varphi} = & A\Bigg[(-\rho)\partial_\omega^\tau \bar{\partial}_\omega^\tau \rho + \rho^2 \left(\partial_\omega \bar{\partial}_\omega \psi - \eta \partial_\omega \psi \otimes \bar{\partial}_\omega \psi\right) \\ & + (-\rho)\left(\partial_{\omega_n} \bar{\partial}_{\omega_n} \rho + \frac{(1-\eta)}{(-\rho)} \omega_n \otimes \bar{\omega}_n + 2\mathrm{Re}\ \partial_\omega^\tau \bar{\partial}_{\omega_n} \rho + 2\eta \mathrm{Re}\ \omega_n \otimes \bar{\partial}_\omega \psi\right)\Bigg]. \end{aligned} \tag{1.4.17}$$

In the second line of (1.4.17), we see that the Cauchy-Schwart inequation gives

$$2\mathrm{Re}\ \omega_n \otimes \bar{\partial}_\omega \psi \geq -\frac{1-\eta_0}{2(-\rho)} \omega_n \otimes \bar{\omega}_n - \frac{2(-\rho)}{1-\eta_0} \partial_\omega \psi \otimes \bar{\partial}_\omega \psi. \tag{1.4.18}$$

Shrink the neighborhood U if necessary makes $\frac{1}{(-\rho)}$ become a large term. By combining (1.4.17) and (1.4.18) and also find a suitable constant $C = C_{\eta_0,U}$ that makes the remain term be positive. Then (1.4.17) becomes

$$\partial_\omega \bar{\partial}_\omega \hat{\varphi} \geq A\left[(-\rho)\partial^\tau_\omega \bar{\partial}^\tau_\omega \rho + \rho^2\left(\partial_\omega \bar{\partial}_\omega \psi - 2\eta \partial_\omega \psi \otimes \bar{\partial}_\omega \psi\right) - C\rho^2 \omega^\tau \otimes \bar{\omega}^\tau\right]. \tag{1.4.19}$$

For η_0 small, the assumption property P_q for Ω is not necessary since we can find a weight ψ as in Theorem 1.4.3. With the asumption Ω has property P_q, we will find a weight $\psi = \psi_\eta$ such that for arbitrary $0 < \eta_0 < 1$ we can have $\hat{\varphi}$ satisfies the Theorem with $0 < \eta \leq \eta_0$.
By replace ψ with $t\psi$ for $t > 0$, and let t be larger, we can make $\epsilon_\eta = \frac{2\eta}{t\epsilon_0}$ as small as we need, and also M increases by a multiplication of t.
Let $\lambda^\varphi_1 \leq \lambda^\varphi_2 \leq \dots$ and $\lambda^\psi_1 \leq \lambda^\psi_2 \leq \dots$ be the eigenvalues of $\partial\bar{\partial}\varphi$ and $\partial\bar{\partial}\psi$ in the basis of $\{\omega\}$, respectively. Then we obtain

$$\begin{aligned}&\sum_{|K|=k-1}^{\prime} \sum_{i,j=1}^{k} \frac{\partial^2 \hat{\varphi}}{\partial \omega_i \partial \bar{\omega}_j} u_{iK} \bar{u}_{jK} \\ &\geq A\left[(-\rho)\sum_{j=1}^{k} \lambda^\rho_j + \rho^2\left((1-\epsilon_\eta)\sum_{j=1}^{k} \lambda^\psi_j + \epsilon_\eta(1-\epsilon_0)\sum_{j=1}^{q_0} \frac{\partial^2 \psi}{\partial \omega_j \partial \bar{\omega}_j} - C\right)\right]|u_J|^2.\end{aligned} \tag{1.4.20}$$

Furthermore, we have

$$\sum_{|K|=k-1}^{\prime} \sum_{j=1}^{q_0} \frac{\partial^2 \hat{\varphi}}{\partial \omega_j \partial \bar{\omega}_j} u_{jK} \bar{u}_{jK} \leq A\left[(-\rho)\sum_{j=1}^{q_0} \frac{\partial^2 \rho}{\partial \omega_j \partial \bar{\omega}_j} + \rho^2 \sum_{j=1}^{q_0} \frac{\partial^2 \psi}{\partial \omega_j \partial \bar{\omega}_j}\right]|u_J|^2. \tag{1.4.21}$$

From here, we can choose t large enough to obtain

$$(1-\epsilon_\eta)\sum_{|K|=k-1}^{\prime} \sum_{ij=1}^{n} \frac{\partial^2 \psi}{\partial \omega_i \partial \bar{\omega}_j} u_{iK} \bar{u}_{jK} - (1-\epsilon_\eta(1-\epsilon_0))\sum_{j=1}^{q_0} \frac{\partial^2 \psi}{\partial \omega_j \partial \bar{\omega}_j}|u_J|^2 \geq C|u_J|^2. \tag{1.4.22}$$

Combine (1.4.20)-(1.4.22) we have

$$\sum_{|K|=k-1}^{\prime} \sum_{i,j=1}^{k} \frac{\partial^2 \hat{\varphi}}{\partial \omega_i \partial \bar{\omega}_j}(z) u_{iK} \bar{u}_{jK} \geq \sum_{|K|=k-1}^{\prime} \sum_{j=1}^{q_0} \frac{\partial^2 \hat{\varphi}}{\partial \omega_j \partial \bar{\omega}_j}(z) u_{jK} \bar{u}_{jK}, \tag{1.4.23}$$

for any form u of degree $k \geq q$ and that proves the Theorem. □

1.5 Applications

There are some applications for the Diederich - Fornæss index.

1.5.1 An embedding Lemma

Theorem 1.5.1. *Let X be a Stein manifold and $\Omega \subset\subset X$ be a pseudoconvex domain with C^2 boundary. Let X be embedded as a closed submanifold into some $\mathbb{C}^n$. Let $\pi : U \to X$ be a holomorphic retraction from a Stein neighborhood U of X onto X. Then there is a bounded pseudoconvex domain $\hat{\Omega} \subset\subset U$ with C^2 boundary such that $\hat{\Omega} \cap X = \Omega$ and $\pi(b\hat{\Omega}) = \bar{\Omega}$. The domain $\hat{\Omega}$ can be chosen to be strictly pseudoconvex outside X.*

Proof. Let $f_1, \dots, f_s$ be holomorphic functions on $\mathbb{C}^n$, which generate the ideal sheaf of X on $\mathbb{C}^n$. Then for any vector $t = \sum_{k=1}^n t^k \frac{\partial}{\partial z_k}$ that is not tangent to X, we have

$$\sum_{i=1}^{s} \left| \sum_{k=1}^{n} \frac{\partial f_i(p)}{\partial z_k} t_k \right|^2 > 0 \tag{1.5.1}$$

for all $p \in X$. We see that, for each $p \in X$, $\pi^{-1} \circ \pi(p)$ is a submanifold tranversal to X at p. Let t_p be tangent to $\pi^{-1} \circ \pi(p)$ at p, t_p is not tangent to X, then by shrinking U, we can assume that (1.5.1) holds for all $p \in X$ and such these t.
Let ρ be a defining function of Ω on a neighborhood U of $\bar{\Omega}$ correspond to the Diederich - Fornæss index $\eta = \frac{1}{l}$ for $l \in \mathbb{N}$. Let $\hat{\rho} = -(-\rho)^\eta$ on Ω and

$$\varphi = \hat{\rho} \circ \pi + L \sum_{j=1}^{s} |f_j|^2$$

on $\pi^{-1}(\Omega)$ with a constant $L > 0$. Since $\hat{\rho}$ is exhaust, we can choose L large enough and obtain

$$\hat{\Omega} = \{p \in \pi^{-1}(\Omega) | \varphi(p) < 0\} \subset\subset U$$

and

$$\partial\varphi = \pi^*(d\hat{\rho}) + L \sum_{i=1}^{s} (\bar{f}_i \partial f_i + f_i \partial \bar{f}_i) \neq 0 \quad \text{on } b\hat{\Omega} \setminus X.$$

On the boundary of $\hat{\Omega}$ we have

$$-\hat{\rho} \circ \pi = L \sum_{j=1}^{s} |f_j|^2$$

therefore it can be described by

$$\psi = \rho \circ \pi + L^l \left(\sum_{j=1}^{s} |f_j|^2 \right)^l.$$

We have that ψ is a C^2 function and $d\psi = \pi^*(d\rho) \neq 0$ on $b\Omega$. Hence, $b\hat{\Omega}$ is smooth at $b\Omega$. The Levi form of φ at $q \in \pi^{-1}(\Omega)$ is

$$i\partial\bar{\partial}\varphi(p;t) = i\partial\bar{\partial}\hat{\rho}(\pi(p);\pi_*t) + L\sum_{j=1}^{s}\left|\sum_{k=1}^{n}\frac{\partial f_j(p)}{\partial z_k}t_k\right|^2.$$

The strict plurisubharmonicity of $\hat{\rho}$ implies the strictly plurisubharmonic on $\pi^{-1}(\Omega)$. Thus the proof is complete. □

1.5.2 Global regularity of the $\bar{\partial}$- Neumann problem.

For any pseudoconvex domain Ω, let η be the Diederich - Fornæss index relate to the defining function r_η. We get that r_η has the form $r_\eta = g_\eta r$ for some g_η. For general pseudoconvex domain Ω, the index η may near to 0. On the other hand, it's shown that for a given Sobolev index s that goes to 0, one can find a domain Ω in which B_k fails H^s-regularity. So, the relation beween s and η is an interesting problem. The problem was stated by Kohn in [Koh99] and the result has been improved by Pinton and Zampieri in [PZ11].

For this result, we assume that the domain Ω is q-convex, which means that for the ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1}$ of the Levi form $i\partial\bar{\partial}r$ when restrict to the tangent space $\partial r^\perp$, we have

$$\sum_{j=1}^{q}\lambda_j \geq 0.$$

By a same process as for q-pseudoconvex domain, we can prove that there also have Diederich - Fornæss index for q-convex domain.

First we introduce some notations. For an operator F, we define Q_F by

$$Q_F(u,u) = \|F\bar{\partial}u\|^2 + \|F\bar{\partial}^*u\|^2.$$

We also write $Q_s(u,u) = \|\bar{\partial}u\|_s^2 + \|\bar{\partial}^*u\|_s^2$. For any function g, let $r_g = gr$ and we denote

$$N_g = \frac{1}{|\partial r_g|^2}\sum_j\frac{\partial r_g}{\partial\bar{z}_j}\frac{\partial}{\partial z_j}, \qquad L_{g,j} = \frac{\partial}{\partial z_j} - \frac{\partial r_g}{\partial z_j}N_g \quad \text{and} \quad T_g = -i(N_g - \bar{N}_g).$$

Then we can have that $L_{g,j}$ are complex tangential and T_g is the complementary real tangential vector field. Now we use the notation

$$\bar{\theta}_j := -\frac{1}{|\partial r_g|^2} \sum \frac{\partial^2 r_g}{\partial z_i \partial \bar{z}_j} \frac{\partial r_g}{\partial \bar{z}_i}$$

and introduce the Euclidean derivatives

$$\begin{cases} \bar{\Theta}_g u = \sum'_{|K|=k-1} \sum_{ij} \left(\bar{\theta}_j u_{iK} - \bar{\theta}_j u_{jK}\right) + \text{error} \\ \bar{\Theta}^*_g u = \sum'_{|K|=k-1} \sum_j \theta_j u_{jK} + \text{error.} \end{cases}$$

Then we can get a realation

$$[\bar{\partial}, T_g] = \bar{\Theta}_g T_g.$$

For the commutator with $\bar{\partial}^*$, we need to modify T_g by $\tilde{T}_g$

$$(\tilde{T}_g u)_{jK} = T_g u_{jK} + \frac{1}{|\partial r|^2} \frac{\partial r_g}{\partial \bar{z}_j} \sum_i \left[T_g, \frac{\partial r_g}{\partial z_i}\right] u_{iK}$$

Thus $u \in \operatorname{dom}(\bar{\partial}^*)$ implies $\tilde{T}_g u \in \operatorname{dom}(\bar{\partial}^*)$ and we also have

$$[\bar{\partial}^*, \tilde{T}_g] = \bar{\Theta}^*_g \tilde{T}_g.$$

Definition 1.5.2. Let s be a positive integer and let $1 \leq q \leq n-1$. We say that T^s_g well commutes with $\bar{\partial}^*$ in degree $\geq q$ when

$$\|[\bar{\partial}^*, \tilde{T}_g]u\|^2 \leq \epsilon_{s,g} Q(u,u) + c_g \|u\|^2_{-1}$$

for any u of degree $\geq q$ and for $\epsilon_{s,g} \leq c_1^2 e^{-2c_2 s \text{diam}(\Omega)^2 \inf\left(\frac{1}{|g|^s}\right)^{-1}}$ where c_1 is a small constant and c_2 is controlled by the C^2-norm of r_g.

Theorem 1.5.3. *Let Ω be q-convex and assume that for some g, T^s_g well commutes with $\bar{\partial}^*$ in degree $\geq q$. Assume also that this property of good commutation holds, with a uniform constant $\epsilon_{s,g}$ for a strongly q-pseudoconvex exhaustion of Ω. Then for any form $f \in H^s$ we have that $B_k f \in H^s$ and*

$$||B_k f||_s \leq c||f||_s \quad \textit{for any } k \geq q-1.$$

We also use some inequalities come from [Koh99] by Kohn.

Theorem 1.5.4. *If Ω is q-convex and has a Diederich-Fornæss index $\eta = \eta_s$ which controls the commators of $(\bar\partial, \bar\partial^*)$ with Ω^s in degree $k \geq q$, then B_k is s-regular for $k \geq q$.*

Proof. For any form u we can decompose into $u = u^\tau + u^\nu$. We have

$$\|u^\nu\|_1^2 \leq \sum \|\partial_{z_j} u^\nu\|_0^2 \lesssim Q(u, u)$$

and

$$Q(u^\tau, u^\tau) \leq Q(u, u) + Q(u^\nu, u^\nu) \lesssim Q(u, u) + \|u^\nu\|_1^2 \lesssim Q(u, u).$$

We going to show that the existence of Diederich-Fornæss index in q-convex domain implies the good commutation with $\bar\partial^*$ in defree $\geq q$.

Step 1. First, the q-convex domain Ω has the Diederich-Fornæss index η corresponds to the defining function r_η. We can assume that η is bounded away from 0 (in some cases, it approches 1 and we expect that $(1-\eta)^{\frac{1}{2}} \leq \epsilon_{s,g}$).

Denote $i\partial\bar\partial r_\eta(u, \partial r_\eta) := \sum_{K=k-1}' \left(\sum_{ij=1,\dots,n} \frac{r_\eta}{\partial z_i \partial \bar z_j} u_{iK} \frac{r_\eta}{\partial \bar z_j} \right) d\bar z_K$. We consider a k-form $w_\eta := \sum_{|K|=k-1}' \bar\partial r_\eta \wedge d\bar z_K$. Note that $|\partial r_\eta| \sim |\omega_\eta|$. For shorter notation, let $\hat r_\eta = -(-r_\eta)^\eta$, then we get

$$\begin{aligned}
i\partial\bar\partial r_\eta(u, \partial r_\eta) &= \frac{1}{\eta}(-r_\eta)^{1-\eta} i\partial\bar\partial \hat r_\eta(u, \partial r_\eta) - (-r_\eta)^{-1}(1-\eta)\partial r_\eta \otimes \bar\partial r_\eta(u, \partial r_\eta) \\
&\lesssim (-r_\eta)^{1-\eta}(i\partial\bar\partial \hat r_\eta)(u, \partial r_\eta) + |(-r_\eta)^{-1}(1-\eta)\partial r_\eta \otimes \bar\partial r_\eta(u, \partial r_\eta)| \\
&\lesssim (-r_\eta)^{1-\eta}\left(i\partial\bar\partial \hat r_\eta\right)(u, \partial r_\eta) + (-r_\eta)^{1-\eta} \sum_{j=1}^{q_0} \frac{\partial^2 \hat r_\eta}{\partial z_j \partial \bar z_j}(u, \partial r_\eta) \\
&\qquad + |(-r_\eta)^{-1}(1-\eta)\partial r_\eta \otimes \bar\partial r_\eta(u, \partial r_\eta)| \\
&\lesssim (-r_\eta)^{1-\eta}\left(i\partial\bar\partial \hat r_\eta\right)(u, w_\eta) + (-r_\eta)^{1-\eta}|\partial r_\eta.u| \\
&\qquad + (-r_\eta)^{1-\eta} \sum_{j=1}^{q_0} \frac{\partial^2 \hat r_\eta}{\partial z_j \partial \bar z_j}(u, \partial r_\eta) + |(r_\eta)^{-1}(1-\eta)\partial r_\eta \otimes \bar\partial r_\eta(u, \partial r_\eta)| \\
&\lesssim (-r_\eta)^{1-\eta}\left[\left(i\partial\bar\partial \hat r_\eta\right)(u, u)\right]^{\frac{1}{2}} \left[\left(i\partial\bar\partial \hat r_\eta\right)(w_\eta, w_\eta)\right]^{\frac{1}{2}} \\
&\qquad + (-r_\eta)^{1-\eta}|\partial r_\eta.u| + (-r_\eta)^{1-\eta} \sum_{j=1}^{q_0} \frac{\partial^2 \hat r_\eta}{\partial z_j \partial \bar z_j}(u, \partial r_\eta) \\
&\qquad + |(r_\eta)^{-1}(1-\eta)\partial r_\eta \otimes \bar\partial r_\eta(u, \partial r_\eta)|.
\end{aligned} \tag{1.5.2}$$

In (1.5.2), we apply the same way as in (2.26) of [PZ11] with $\left(i\partial\bar{\partial}\hat{r}_\eta\right)$ is a positive form. Continue (1.5.2), shrink the neighborhood if necesary, we can get

$$\begin{aligned}
&i\partial\bar{\partial}r_\eta(u,\partial r_\eta)\\
&\quad\lesssim (-r_\eta)^{1-\eta}\left[\left(i\partial\bar{\partial}\hat{r}_\eta\right)(u,u)\right]^{\frac{1}{2}}\left((-r_\eta)^{-2+\eta}(1-\eta)|\partial r_\eta|^4+\mathcal{O}((-r_\eta)^{-1+\eta})\right)^{\frac{1}{2}}\\
&\qquad\quad + (1-\eta)|\partial r_\eta|^2(-r_\eta)^{-1}|\partial r_\eta.u|\\
&\quad\lesssim (-r_\eta)^{1-\eta}\left[\left(i\partial\bar{\partial}\hat{r}_\eta\right)(u,u)\right]^{\frac{1}{2}}\left((1-\eta)^{\frac{1}{2}}(-r_\eta)^{-\frac{\eta}{2}}|\partial r_\eta|^2+\mathcal{O}(-r_\eta)^{\frac{1}{2}-\frac{\eta}{2}}\right)\\
&\qquad\quad + (1-\eta)|\partial r_\eta|^2(-r_\eta)^{-1}|\partial r_\eta.u|.
\end{aligned} \tag{1.5.3}$$

Shrink the neighborhood, we can get

$$\begin{aligned}
\left|i\partial\bar{\partial}r_\eta(u,\partial r_\eta)\right| \lesssim (1-\eta)^{\frac{1}{2}}(-r_\eta)^{-\frac{\eta}{2}}|\partial r_\eta|^2\left[\left(i\partial\bar{\partial}\hat{r}_\eta\right)(u,u)\right]^{\frac{1}{2}}&\\
+ (1-\eta)|\partial r_\eta|^3(-r_\eta)^{-1}|u^\nu|.&
\end{aligned} \tag{1.5.4}$$

Apply (1.5.4) for $u=u^\tau$ yields

$$\left|i\partial\bar{\partial}r_\eta(u^\tau,\partial r_\eta)\right| \lesssim (1-\eta)^{\frac{1}{2}}(-r_\eta)^{-\frac{\eta}{2}}\left[\left(i\partial\bar{\partial}\hat{r}_\eta\right)(u^\tau,u^\tau)\right]^{\frac{1}{2}}$$

Step 2.

$$\begin{aligned}
\|(-r_\eta)^{\frac{\eta}{2}}[\bar{\partial}^*,\tilde{T}_g]u\|^2 &\simeq \int_\Omega (-r_\eta)^\eta\left|i\partial\bar{\partial}r_\eta(u,\partial r_\eta)\right|^2 dV\\
&\lesssim (1-\eta)\sup|\partial r_\eta|^4\int_\Omega(-r)^{\frac{\eta}{2}}\left(i\partial\bar{\partial}\hat{r}_\eta\right)(u,u)dV\\
&\qquad + (1-\eta)^2\sup|\partial r_\eta|^4\|(-r)^{-1}(-r_\eta)^{\frac{\eta}{2}}u^\nu\|^2\\
&\lesssim (1-\eta)\sup|\partial r_\eta|^4\left[Q_{(-r_\eta)^{\frac{\eta}{2}}}(u,u)+\|(-r_\eta)^{-1+\frac{\eta}{2}}|\partial r_\eta|u^\nu\|^2\right]\\
&\qquad + (1-\eta)^2\sup|\partial r_\eta|^4\|(-r)^{-1}(-r_\eta)^{\frac{\eta}{2}}u^\nu\|^2\\
&\lesssim \epsilon_{s,g}\left[Q_{(-r_\eta)^{\frac{\eta}{2}}}(u,u)+\sup|g|^\eta\left(Q_{T^{-\frac{\eta}{2}}}(u,u)+\|T^{-\frac{\eta}{2}}u\|^2\right)\right].
\end{aligned}$$

Here $r_\eta = gr$. This implies the good commutation of T_g with $\bar{\partial}^*$.

Step 3. First we have

$$\|u\|_s^2 \lesssim Q_{s-1}(u,u)+\|T_g^s u\|^2+\|u\|_s\|u\|_{s-1}.$$

For n-form that is 0 at $b\Omega$, B_{n-1} is regular. By induction, we asume that B_k is s-regular and we shall prove that it is true for B_{k-1}. Let $f\in C^\infty(\bar{D})$ be a test function. We have

$$\begin{aligned}
\|T_g^sB_{k-1}f\|^2 &= (T_g^sB_{k-1}f,T_g^sf)-(T_g^sB_{k-1}f,T_g^s\bar{\partial}^*N_k\bar{\partial}f)\\
&= (T_g^sB_{k-1}f,T_g^sf)-(T_g^{s*}T_g^s\bar{\partial}B_{k-1}f,N_k\bar{\partial}f)-([\bar{\partial},T_g^{s*}T_g^s]B_{k-1}f,N_k\bar{\partial}f).
\end{aligned} \tag{1.5.5}$$

and then apply to $T^{s-\frac{\eta}{2}}$, we get

$$\begin{aligned}\|T^{s-\frac{\eta}{2}}B_{k-1}f\| &\le (\sup|g|)^{s-\frac{\eta}{2}}\|T_g^{s-\frac{\eta}{2}}B_{k-1}f\| + \text{good term}\\ &\lesssim (\sup|g|)^{s-\frac{\eta}{2}}\left(\epsilon\|T_g^{s-\frac{\eta}{2}}B_{k-1}f\| + \frac{1}{\epsilon}\left(\|T_g^{s-\frac{\eta}{2}}f\| + \|[\bar\partial^*, T_g^{s-\frac{\eta}{2}}]N_k\bar\partial f\|\right)\right) + \text{good term.}\end{aligned}$$

We first consider the last term in the right. By spliting it into (h) and (0) component, where $f = f^{(h)} + f^{(0)}$, $f^{(h)}$ is the harmonic extension of the boundary trace $f|_{b\Omega}$, and note that $f^{(0)}|_{b\Omega} = 0$. We have

$$\|[\bar\partial^*, T_g^{s-\frac{\eta}{2}}](N_k\bar\partial f)\|^{(0)} \le c_g\|T^{s-\frac{\eta}{2}-2}\Delta(N_k\bar\partial f)\| \le c_g(\|T^{s-\frac{\eta}{2}-1}f\| + \|T^{s-\frac{\eta}{2}-2}Sf\|)$$

where the right hand side is good terms.

For the (h)-component, denote $\mathcal{E}^{(0)} := \|(-r_\eta)^{\frac{\eta}{2}}\bar\Theta_g^*\tilde T_g^s(\bar\partial M_{k-1}f)^{(0)}$.

We will use the following result for the estimate of $\|[\bar\partial^*, T_g^{s-\frac{\eta}{2}}](N_k\bar\partial f)\|^{(h)}\|$

Proposition 1.5.5. *We have*

$$\|[\tilde T^{s-\frac{\eta}{2}}, \bar\partial^*]v^{(h)} \lesssim \|(-r)^{\frac{\eta}{2}}[\tilde T^s, \bar\partial^*]v^{(h)}\| + Op^{s-\frac{\eta}{2}-1}\|, \quad v \in C^\infty(\bar D \cap U), \tag{1.5.6}$$

where $Op^{s-\frac{\eta}{2}-1}$ denotes an operator of order $s - \frac{\eta}{2} - 1$ in $\mathbb{R}^{2n}$.

$$\begin{aligned}\sup|g|^{2s-\eta}\|[\bar\partial^*, \tilde T_g^{s-\frac{\eta}{2}}](\bar\partial N_{k-1}f)^{(h)}\|^2 &\lesssim \sup|g|^{2s-\eta}\|(-r_\eta)^{\frac{\eta}{2}})[\tilde T_g^s, \bar\partial^*](\bar\partial N_{k-1}f)^{(h)}\|^2\\ &\lesssim |g|^{2s-\eta}s^2\|(-r_\eta)^{\frac{\eta}{2}}\bar\Theta_g^*\tilde T_g^s(\bar\partial N_{k-1}f)^{(h)}\|^2 + \text{error}\\ &\lesssim |g|^{2s-\eta}\frac{1}{|g|^{2s}}s^2\|(-r_\eta)^{\frac{\eta}{2}}\bar\Theta_g^*\tilde T_g^s(\bar\partial N_{k-1}f)^{(h)}\|^2 + \mathcal{E}^{(0)} + \text{error}\\ &\lesssim \sup|g|^{2s-\eta}\sup|g|^\eta\sup\frac{1}{|g|^{2s}}s^2\mathcal{E}_{s,g}\big[Q_{(-r)^{\frac{\eta}{2}}\tilde T^s}(\bar\partial N_{k-1}f, \bar\partial N_{k-1}f)\\ &\qquad + Q_{\tilde T^{s-\frac{\eta}{2}}}(\bar\partial N_{k-1}f, \bar\partial N_{k-1}f) + \|\tilde T^{s-\frac{\eta}{2}}\bar\partial N_{k-1}\|^2\\ &\qquad + \|(-r_\eta)^{\frac{\eta}{2}}[\bar\partial, \tilde T^s]\bar\partial N_{k-1}f\|^2 + \|(-r_\eta)^{\frac{\eta}{2}}[\bar\partial^*, \tilde T^s]\bar\partial N_{k-1}f\|^2\\ &\qquad + \|\tilde T^{-\frac{\eta}{2}}[\bar\partial, \tilde T^s]\bar\partial N_{k-1}f\|^2 + \|\tilde T^{-\frac{\eta}{2}}[\bar\partial^*, \tilde T^s]\bar N_{k-1}f\|^2\big] + \mathcal{E}^{(0)} + \text{error}\\ &\lesssim \epsilon\left(\|(-r)^{\frac{\eta}{2}}T^s\bar\partial^*\bar\partial N_{k-1}f\|^2 + \|T^{s-\frac{\eta}{2}}f\|^2 + \|T^{s-\frac{\eta}{2}}\bar\partial^*\bar\partial N_{k-1}f\|^2\right) + \mathcal{E}^{(0)} + \text{error.}\end{aligned} \tag{1.5.7}$$

where the term $\|T^{s-\frac{\eta}{2}}\bar\partial^*\bar\partial N_{k-1}f\|^2$ can be absorbed. The detail of above calculation can be found on [PZ11].

For the first term of (1.5.7), we start from $f^{(h)}$. We have

$$\|T^{s-\frac{\eta}{2}}B_{k-1}f^{(h)}\|^2 \lesssim \epsilon\|(-r)^{\frac{\eta}{2}}T^s\bar\partial^*\bar\partial N_{k-1}f^{(h)}\|^2 + \|T^{s-\frac{\eta}{2}}f^{(h)}\|^2 + \mathcal{E}^{(0)} + \text{error} \\ + c_g\left(\|T^{s-\frac{\eta}{2}-1}f^{(h)}\| + \|T^{s-\frac{\eta}{2}-2Sf^{(h)}}\|^2\right). \quad (1.5.8)$$

and the last term of (1.5.8) is a good term.

We have from [Koh99], the first term on the right could be described as

$$\|(-r)^{\frac{\eta}{2}}T^s\bar\partial^*\bar\partial N_{k-1}f^{(h)}\| \lesssim \|T^{s-\frac{\eta}{2}}\bar\partial^*\bar\partial N_{k-1}f^{(h)}\| + \|-rT^{s-\frac{\eta}{2}-1}\Delta\bar\partial^*\bar\partial N_{k-1}f^{(h)}\| \\ = 1^{st}\text{ term} + 2^{nd}\text{ term}$$

For the 1^{st} term

$$1^{st}\text{ term} \lesssim \|T^{s-\frac{\eta}{2}}f^{(h)}\|^2 + \|T^{s-\frac{\eta}{2}}B_{k-1}f^{(h)}\|^2.$$

The first term in the right is good. The second term canbe absorbed with a small constant. For the 2^{nd} term

$$\begin{aligned} 2^{nd}\text{ term} &= \|-rT^{s-\frac{\eta}{2}-1}(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)\bar\partial^*\bar\partial N_k f^{(h)}\| + \text{error} \\ &= \|-rT^{s-\frac{\eta}{2}-1}\bar\partial^*\bar\partial f^{(h)}\| + \text{error} \\ &= \|-rT^{s-\frac{\eta}{2}-1}(T^2 + \partial rT + \Delta)f^{(h)}\| \\ &\le \|-rT^{s-\frac{\eta}{2}+1}f^{(h)}\| + \|-rT^{s-\frac{\eta}{2}}\partial rf^{(h)}\| \\ &\lesssim \|T^{s-\frac{\eta}{2}}f^{(h)}\| \end{aligned}$$

which is good.

For the last term that is related to $f^{(0)}$, we must consider is the regularity of $B_{k-1}f^{(0)}$. From elliptic regularity

$$\|T^{s-\frac{\eta}{2}}N_{k-1}f^{(0)}\| \lesssim \|T^{s-\frac{\eta}{2}-2}f^{(0)}\|.$$

Applying Boas-Straube formula gives

$$\begin{aligned} \|(-r_\eta)^{\frac{\eta}{2}}[\bar\partial^*, \tilde{T}^s_g](\bar\partial N_{k-1}f)^{(0)}\|^2 &\le c_g\left(\|T^{s-\frac{\eta}{2}}(\bar\partial N_{k-1}f)(0)\|^2 + \|rT^{s-\frac{\eta}{2}-1}\Delta(\bar\partial N_{k-1}f)^{(0)}\|^2\right) \\ &\le c_g\left(\|T^{s-\frac{\eta}{2}-2}\Delta(\bar\partial N_{k-1}f)^{(0)}\|^2 + \|rT^{s-\frac{\eta}{2}-1}\Delta(\bar\partial N_{k-1}f)^{(0)}\|^2\right) \\ &\le c_g\left(\|T^{s-\frac{\eta}{2}-2}\Delta(\bar\partial N_{k-1}f)\|^2 + \|rT^{s-\frac{\eta}{2}-1}\Delta(\bar\partial N_{k-1}f)\|^2\right) \\ &\le c_g\left(\|T^{s-\frac{\eta}{2}-1}f\|^2 + \|T^{s-\frac{\eta}{2}-2}Sf\|^2 + \|rT^{s-\frac{\eta}{2}-1Sf}\|^2 + \|rT^{s-\frac{\eta}{2}}f\|^2\right). \end{aligned} \quad (1.5.9)$$

The elliptic regularity as in (1.5.9) also controls $\|T^{-\frac{\eta}{2}}[\bar{\partial}^*, T^s](\bar{\partial}N_k f)^{(0)}\|$ and $\|(-r)^{\frac{\eta}{2}}[\bar{\partial}^*, T^s](\bar{\partial}N_k f)^{\nu}\|$. Therefore we get

$$\|B_k f\| \leq c\|f\|_s.$$

$\square$

Chapter 2

Boundary behavior of the Kobayashi metric

2.1 Introduction.

This chapter deals with the question of the boundary behavior of the Kobayashi metric on weakly pseudoconvex domains. When Ω is either strongly pseudoconvex in $\mathbb{C}^n$, or pseudoconvex of finite type in $\mathbb{C}^2$, or convex of finite type in $\mathbb{C}^n$, the size of Kobayashi metric in the "small constant-large constant" sense has been proven by I. Graham [Gra75], D. Catlin [Cat89], L. Lee [Lee08], respectively. In these classes of domains, there exists a quantitative $M(z, X)$ satisfying the asymtotic formula

$$\lim_{z \to b\Omega} M(z, X) = \delta_\Omega^{-1/m}(z)|X^\tau| + \delta_\Omega^{-1}(z)|X^\nu|$$

and positive constant c and C such that

$$cM(z, X) \leq K(z, X) \leq CM(z, X).$$

Here, X^τ and X^ν are the tangential and normal components of X and $\delta_\Omega(z)$ is the distance from z to the boundary.

For generally pseudoconvex domain in $\mathbb{C}^n$, K. Diederich and J. E. Fornaess proved that there is a $\epsilon > 0$ such that $K(z, X) \geq \delta(z)^{-\epsilon}|X|$ (see [DF79]) on real analytic case of finite type by using Kohn's algorithm [Koh79]. In [Cho92], S. Cho improved the result in [DF79] for domains without real analytic condition by using the method of Catlin in [Cat87, Cat89].

However, not much is known in the case when the domain is not finite type except from the recent results by S. Lee [Lee01] for exponentially-flat infinite type.

For $\delta > 0$, denote $S_\delta = \{z \in \Omega : -\delta < r < 0\}$

Definition 2.1.1. Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and U be a local path of its boundary. We say that Ω has Property (f-$\mathcal{P}$) on U if there exist a family of $C^2(U \cap S_\delta)$ functions $\{\phi_\delta\}$ such that

$$\begin{cases} |\phi_\delta| \leq 1 \\ |D\phi_\delta| \lesssim \delta^{-1} \\ i\partial\bar{\partial}\phi_\delta(L, \bar{L}) \gtrsim f(\delta^{-1})^2 |L|^2 \end{cases} \qquad \text{on } U \cap S_\delta$$

for any $L \in T^{1,0}(U \cap b\Omega)$.

Our main result is the following theorem.

Theorem 2.1.2. *Let Ω, U, $\delta(z)$, and $X = (X^\tau, X^\nu)$ be defined as above. Assume that Ω has Property (f-P) on U. Then we have*

$$K(z, X) \gtrsim f(\delta_\Omega^{-1}(z))|X^\tau| + \delta_\Omega^{-1}(z)|X^\nu|$$

for any $z \in U$ and $X \in T_z^{1,0}$

Remark 2.1.3. The asymptotic rate $f(\delta_{b\Omega}^{-1}(z))$ in tangential directions is optimal.

An important tool for this lower bound follows the bumping function, which migh also be useful for other purposes. It says, roughly speaking, for any the boundary point w on Ω which satisfies Property (f-P) that one can find a pseudoconvex hypersurface touching $\bar{\Omega}$ exactly at w from the outside and the distants from $z \in \Omega$ to new hypersurface is exactly controlled by the rate depending on f of $|z - w|$.

Theorem 2.1.4. *Let Ω be pseudoconvex and U be a local path of the boundary. Assume that Ω has Property (f-P) on U. Then for any open set $V \subset U$, there exist a real C^2 function ρ on $V \times (V \cap b\Omega)$ with following properties:*

1. *$\rho(w, w) = 0$.*
2. *$\rho(z, w) \lesssim -F(|z - w|)$ for any $(z, w) \in (V \cap \Omega) \times (V \cap b\Omega)$ where $F(\delta) = f^*(\delta^{-1})^{-1}$.*
3. *$\rho(z, \pi(z)) \gtrsim -\delta(z, b\Omega)$ for any $z \in V \cap \Omega$ where $\pi(z)$ is the projection of z to the boundary.*

4. *For each fixed $w \in V \cap b\Omega$, denote $S_w = \{z \in V : \rho(z, w) = 0\}$. One has:*
   (a) *$d_z\rho(z, w) \neq 0$ everywhere on S_w.*
   (b) *S_w is pseudoconvex. In fact, one can choose ρ such that S_w is strictly pseudoconvex outside of w.*
   (c) *S_w touching $\bar{\Omega}$ exactly at w from outside.*

The proof of Theorem 2.1.4 combines the technical of weighted function in [McN02, KZ10, Kha10] and construction the bumping function in [DF79, Cat89, Cho92]. The details of the proof are given in Section 2.

Using the theory of existence exhaustion function, we obtain the plurisubharmonic peak functions with good estimates. More precisely we obtain the following theorem

Theorem 2.1.5. *Assume that there exists a family of bumping functions on local path V of the boundary as in the conclusion of Theorem 2.1.4. Fix any $0 < \eta < 1$, Then for any $w \in V \cap b\Omega$ there is a plurisuhharmonic functions $\psi_w(z)$ on $V \setminus \{w\}$ verifying*

1. $|\psi_w(z) - \psi_w(z')| \lesssim |z - z'|^\eta$
2. $\psi_w(z) \lesssim -F^\eta(|z - w|)$
3. $\psi_\pi(z)(w) \gtrsim -\delta(z, b\Omega)^\eta$

for all z and z' in $V \cap \bar{\Omega}$.

The lower bound of Kobayashi metric follows the size of general estimates of plurisubharmonic peak function.

Theorem 2.1.6. *Let Ω be a pseudoconvex domain and V be a local path of the boundary. Assume that for any $w \in V \cap b\Omega$, there is a plurisubharmonic function $\psi_w(z)$ such that $\psi_w(z) \lesssim -F_1(d(z, w))$ and $\psi_{\pi(z)}(z) \gtrsim -F_2(\delta(z))$ for all $z \in V \cap \bar{\Omega}$. Then the Kobayashi metric has the lower bound*

$$K_\Omega(z, X) \gtrsim (F_1^* F_2(\delta(z)))^{-1}|X|$$

for all $z \in V$, $X \in T_z^{1,0}$.

The proof of Theorem 2.1.6 can be found in Section 3.5. A domain $\Omega \subset \mathbb{C}^n$ is a connected, open set. Let $D \subset \mathbb{C}$ denote the unit disc and $D_r = \{z \in \mathbb{C} : |z| < r\}$. We also let $U_1(U_2)$ denote the collection of holomorphic mappings from U_2 to U_1. The Kobayashi metric on Ω is defined for $z \in \Omega$ and $X \in \mathbb{C}^n$, to be

$$\begin{aligned} F_K^\Omega(z, X) &= \inf\{\alpha : \alpha > 0 \text{ and } \exists f \in \Omega(D) \text{ with } f(0) = z, f'(0) = \alpha^{-1}X\} \\ &= \inf\{r^{-1} : \exists f \in \Omega(D_r) \text{ with } f(0) = z, f'(0) = X\} \end{aligned}$$

2.2 Pseudoconvex domain with real-analytic boundary.

In this section, we recall the result of Diederich - Fornæss of estimating the Kobayashi metric. In there result, they depend deeply on the condition that the domain Ω has a real-analytic defining function. For that they use J. J. Kohn technique on subelliptic multiplier to construct a bumping function. And they get the following Theorem to estimate the Kobayashi metric of a point near the boundary.

Theorem 2.2.1. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain with real-analytic boundary. Let r be any smooth defining function of Ω. Then there exist a constants $c > 0$ and $\epsilon > 0$ such that*

$$F_\Omega(p, X) \geq c\frac{|X|}{|r(p)|^\epsilon}$$

for all $p \in \Omega$ and $X \in \mathbb{C}^n$.

2.2.1 The bumping theorem.

In [DF79], Diederich and Fornæss introduced a kind of bumping function. It says that one can find for any point w on the boundary of a pseudoconvex domain Ω with real-analytic boundary a pseudoconvex hypersurface near w touching $\bar{\Omega}$ exactly at w from the outside.

Theorem 2.2.2. *Let $U \subset \mathbb{C}^n$ be an open neighborhood of 0 and r a real-analytic function on U such that $dr \neq 0$ everywhere on U, $r(0) = 0$ and the hypersurface $S = \{z \in U : r(z) = 0\}$ is pseudoconvex from the side $r < 0$. Suppose, furthermore that S does not contain any positive dimensional germs of complex analytic subvarieties. Then there exists an open neighborhood $V \subset U$ of 0 and a real-analytic function ρ on $V \times (V \cap S)$ with the following properties*

(*i*) $\rho(w, w) = 0$.

(*ii*) *One has $d_z\rho(z, w) \neq 0$ everywhere on V for each fixed $w \in V \cap S$.*

(*iii*) *The hypersurface $S_w := \{z \in V : \rho(z, w) = 0\}$ is pseudoconvex from the side $\rho(., w) < 0$ for each fixed $w \in V \cap S$. Moreover we can choose ρ and V such that S_w is strictly pseudoconvex outside of w.*

(*iv*) *One has $r > 0$ on $S_w \setminus \{w\}$.*

The proof of Theorem (2.2.2) bases on ideal theory of subelliptic multipliers developed by J. J. Kohn on [Koh79]. To construct the bumping function ρ we have to use the following results

Lemma 2.2.3. *There exists an open neighborhood $U \subset\subset U$ of 0 and a collection of real-analytic functions $f_1^1, \dots, f_{s_1}^1; \dots; f_1^l, \dots, f_{s_l}^l$ on U_1 with the following properties: Put*

$$g_0 := \textit{coeff.}(\partial r \wedge \bar{\partial} r \wedge (\partial\bar{\partial} r)^{n-1})$$

and

$$g_j := \textit{coeff.}(\partial r \wedge \bar{\partial} r \wedge \partial f_1^j \wedge \bar{\partial} f_1^j \wedge \dots \wedge \partial f_{s_j}^J \wedge \bar{\partial} f_{s_j}^j \wedge (\partial\bar{\partial} r)^{n-s_j-1}),$$

$j = 1, \dots, l$; then there is an integer $N \geq 1$ with

$$|f_k^j|^{2N} \leq r^2 + g_0^2 + \dots + g_{j-1}^2 \tag{2.2.1}$$

for all $j = 1, \dots, l$, $k = 1, \dots, s_j$. Furthermore, $g_l \neq 0$ everywhere on $\bar{U}_1$.

The pseudoconvexity of S implies $g_j \geq 0$ on $U_1 \cap S$ for $j = 0, \dots l$. When S is strictly pseudoconvex, then we also have $g_0(0) > 0$ and then f_k^j will not be needed.
We also have some additions to simplify our later calculations.

1. *The functions $f_k^j, j = 1, \dots, l; k = 1 \dots, s_l$ can be chosen to be real-valued.*

2. *Assume that we can choose a coordinates in $\mathbb{C}^n$ and U such that the definiton function r of S can be written in the form*

   $$r(z) = \text{Re} z_1 + G(\text{Im} z_1, z')$$

   with G is a real-analytic function on $I \times U' \subset I \times \mathbb{C}^{n-1}$, an open neighborhood of 0, and $G(0) = 0$, $dG(0) = 0$. Then all functions $f_k^j, g_j, g_0, j = 1, \dots, l; k = 1 \dots, s_l$ can be chosen to depend only on the variables $(\text{Im} z_1, z') \in I_1 \times U_1' = U_1'$ with $I_1' \subset\subset I$ and $U_1' \subset\subset U'$, open neighborhoods of 0.

For the proof of the bumping theorem 2.2.2, we may assume that

$$g_0(0) = \dots g_{l-1}(0) = 0.$$

Then by induction over j we can construct a real-analytic function

$$r_j(z, w) : U_{j+2} \times (U_{j+2} \cap S) \to \mathbb{R}; j = 1, \dots, l$$

on a neighborhood $U_{j+2} \subset U_1$ of 0 satisfies the following properties

a) $d_z r_j \neq 0$ everywhere on U_{j+2}.

b) The hypersurface

$$S_{j,w} := \{z \in U_j + 2 : r_j(z,w) = 0\}$$

is pseudoconvex from the side $r_j(z,w) < 0$.

c) $r_j(.,w) = 0$ on $\{w\} \cup \{r^2 + g_0^2 + \ldots + g_j^2 = 0\}$.

d) $r_j(.,w) < 0$ on $(U_{j+2} \cap S) \setminus (\{w\} \cup \{r^2 + g_0^2 + \ldots + g_j^2 = 0\})$.

e) $S_{j,w}$ is strictly pseudoconvex except at $\{w\} \cup \{r^2 + g_0^2 + \ldots + g_j^2 = 0\}$.

f) $S_{j,w}$ has at least 4^{th} order contact with S on $\{w\} \cup \{r^2 + g_0^2 + \ldots + g_j^2 = 0\}$.

g) All functions $r_j(.,w)$ are of the form

$$r_j(z,w) = \mathrm{Re} z_1 + G(\mathrm{Im} z_1, z', w).$$

We denote by

$$d_w(z) = d(z,w) := |\mathrm{Im} z_1 - \mathrm{Im\ w}|^2 \sum_{k=2}^{n} |z_k - w_k|^2.$$

Step 1. For $j = 0$, we define on an open neighborhood $U_2 \subset\subset U_1$ of 0

$$r_0(z,w) : U_2 \times (U_2 \cap S) \to R$$
$$r_0(z,w) := r(z) - \epsilon_0 (g_0(z) d_w(z))^{N_0}$$

with $\epsilon_0 > 0$ and an integer N_0 will be chosen later. We already have $\partial_z r \neq 0$, hence if we choose ϵ sufficent small, a) is satisfied. Properties c), d), f), g) is trivial. Now we need to prove the pseudoconvexity of $S_{0,w}$. Let fixed $w \in U_2 \cap S$ and $z = (x + iy, z') \in S_{0,w}$. For $t = (t_1, \ldots, t_n) \in t^{1,0}(S_{0.w} \cap U_2)$, then we have

$$t_1 = -\sum_{k=2}^{n} \frac{\partial r_0 / \partial z_k}{\partial r_0 / \partial z_1} t_k$$

Therefore, after some computation we can get the Levi form of $r_0(.,w)$ at z applied to t

$$\begin{aligned}
& i\partial_z\bar\partial_z r_0(z,w)(t,t)\\
&= \frac{\partial^2 r(z)}{\partial z_1\bar\partial z_1}\left|-\sum_{k=2}^n \frac{\partial r_0/\partial z_k}{\partial r_0/\partial z_1}t_k\right|^2 + \sum_{j,k=2}^n \frac{\partial^2 r}{\partial z_j\bar\partial z_k}t_j\bar t_k\\
&\quad - 2\mathrm{Re}\left(\sum_{j=2}^n \frac{\partial^2 r}{\partial z_1\bar\partial z_j}\bar t_j \sum_{k=2}^n \frac{\partial r_0/\partial z_k}{\partial r_0/\partial z_k}t_k\right) - \epsilon_0\sum_{j,k=1}^n \frac{\partial^2}{\partial z_j\bar\partial z_k}(g_0(z)d_w(z))^{N_0}\\
&= \frac{\partial^2 r(z)}{\partial z_1\bar\partial z_1}\left|-\sum_{k=2}^n \frac{\partial r/\partial z_k}{\partial r/\partial z_1}t_k\right|^2 + \sum_{j,k=2}^n \frac{\partial^2 r}{\partial z_j\bar\partial z_k}t_j\bar t_k\\
&\quad - 2\mathrm{Re}\left(\sum_{j=2}^n \frac{\partial^2 r}{\partial z_1\bar\partial z_j}\bar t_j \sum_{k=2}^n \frac{\partial r/\partial z_k}{\partial r/\partial z_k}t_k\right) + \epsilon_0\mathcal{O}(g_0(z)d_w(y,z'))^{N_0-2}|t|^2\\
&= i\partial_z\bar\partial_z r(\hat z,w)(\hat t,\hat t) + \epsilon_0\mathcal{O}(g_0(z)d_w(y,z'))^{N_0-2}|t|^2.
\end{aligned}$$

with $\hat z = (G(y,z')+iy,z')$ the projection of z on S, and $\hat t$ satisfies $\hat t_k = t_k$ for $k=2,\dots,n$; and $\hat t_1 = -\sum_{k=2}^n \frac{\partial r/\partial z_k}{\partial r/\partial z_1}(\hat z)t_k$. From the definition of g_0, we can find a constand $c>0$ such that

$$i\partial_z\bar\partial_z r_0(z,w)(t,t) \ge cg_0^2(y,z')|T^2| + \epsilon_0\mathcal{O}(g_0(z)d_w(y,z'))^{N_0-2}|t|^2 \ge \frac{c}{2}g_0^2(y,z')|t^2| \quad (2.2.2)$$

if we choose ϵ_0 small enough and $N_0\ge 4$. This implies that b) is satisfied. For property g), we see that the Levi form $i\partial\bar\partial r_0(.,w)=0$ only if $g_0(z)=0$. Then we also have $r_0(z,w)=r(z)=0$.

Step 2. Suppose there is r_{j-1} satisfied above properties for some $j=1,\dots,j$. We define on a open neighborhood $U_{j=2}\subset U_{j+1}$

$$r_j(z,w) := r_{j-1}(z,w) - \epsilon_j(g_j^2(z)d_w(z))^{N_j} + \epsilon_j(g_j^2(z)d_w(z))^{N_j-n_j}\sum_{k=1}^{s_j}(f_k^j(z))^2 \quad (2.2.3)$$

with the integers $N_j \gg n_j \gg 1$ and $\epsilon_j>0$ will be chosen later. As in the case $j=0$, for ϵ_j small enough, property a) is satisfied; properties f) and g) is also trivial. Property c) holds since the induction hypothesis.

For the property d), by induction hypothesis we have on $U_{j+2}\cap S$

$$r_{j+1}(z,w) < 0 \text{ when } z\notin \{w\}\cup\{g_0^2+\ldots+g_{j-1}^2=0\}$$

For such (z,w), we distinguish between two cases

1. For $\sum_{k=1}^{s_j}(f_k^j(z))^2 < (g_j^2 d_w(z))^{n_j}$, then $r_j(z,w) < 0$ because the contruction of r_j.

2. For $\sum_{k=1}^{s_j}(f_k^j(z))^2 \geq (g_j^2 d_w(z))^{n_j}$.
   Shrinking U_2 if necessary, there is a constant $K > 0$ such that $g_j^2(z) \leq K g_j(z)$, and therefore

$$\begin{aligned} g_j^2(z)d_w(z) \leq K g_j(z) d_w(z) &\leq K(g_j^2(z)d_w(z))^{\frac{1}{2}} d_w(z)^{\frac{1}{2}} \\ &\leq K \left(\sum_{k=1}^{s_j} (f_k^j(z)) \right)^{1/2n_j} d_w(z)^{\frac{1}{2}}. \end{aligned}$$

   After shrinking U_{j+2} if necessary, by using Lojasiewicz inequality, we get from (2.2.1) that

$$\epsilon_j \big(\sum_{k=1}^{s_j} (f_k^j(z))^2 d_w(z) \big)^M < |r_{j-1}(z,w)|$$

   for some M is large enough and ϵ_j small. Then put everything into (2.2.3) to get

$$r_j(z,w) \leq r_{j-1}(z,w) + \epsilon_j \left[K \sum_{k=1}^{s_j} (f_k^j(z))^2 d_w(z) \right]^{(N_j - n_j)/2n_j}.$$

   if we choose $N_j \gg n_j \gg 1$ such that $\frac{N_j - n_j}{2n_j} \geq M$. Then we can get d) after choosing a sufficiently small ϵ_j.

Now, for fixed $w \in S \cap U_{j+2}$ and $z \in S_{j,w}$ and let $0 \neq T \in T_z^{10} S_{j,w}$ be an arbitrary vector. Then we compute the Levi forms as in the case $j = 0$

$$\begin{aligned} i\partial_z \bar{\partial}_z r_j(z,w)(t,t) = \sum_{k,l=2}^{n} \frac{\partial^2 r_j}{\partial z_k \bar{\partial} z_l} t_k \bar{t}_l + \frac{\partial^2 r_j}{\partial z_1 \bar{\partial} z_1} \left| \sum_{k=2}^{n} \frac{\partial z_j / \partial z_k}{\partial r_j / \partial z_1} t_k \right|^2 \\ - 2\mathrm{Re} \sum_{l=2}^{n} \frac{\partial^2 r_j}{\partial z_1 \bar{\partial} z_l} \bar{t}_l \sum_{k=2}^{n} \frac{\partial r_j / \partial z_k}{\partial z_j / \partial z_1} t_k \end{aligned}$$

As before, the derivatives of $r_j(.,w)$ depend only on $y = \mathrm{Im} z_1, z'$ and if we replace t by $\hat{t} := (\hat{t}_1, \hat{t}') \in T^{10} S_{j-1,w}$, $\hat{t}_k = t_k$ for $k = 2 \ldots n$ and then after a straightforward calculation, we have

$$i\partial_z\bar{\partial}_z r_j(z,w)(t,t) = i\partial_z\bar{\partial}_z r_{j-1}(\hat{z},w)(\hat{t},\hat{t}) + 2\epsilon_j(g_j^2(z)d_w(y,z))^{N_j-n_j}\sum_{k=1}^{s_j}\left|\sum_{k=1}^{n}\frac{\partial f_k^j}{\partial z_k}(z)t'_k\right|^2$$
$$+\epsilon_j\mathcal{O}(g_j^2(z)d_w(y,z))^{N_j-n_j-2}\sum_{k=1}^{s_j}|f_k^j(z)||t|^2$$
$$+\epsilon_j\mathcal{O}(g_j^2(z)d_w(y,z))^{N_j-2}|t|^2. \quad (2.2.4)$$

The first term is the Levi form of r_{j-1} at $\hat{z}\in S_{j-1,w}$ with $\hat{z}$ is the image when we project z to $S_{j-1,w}$ in x direction, and it applies to $\hat{t} := (\hat{t}_1, \hat{t}' \in T^{10}S_{j-1,w}$ with $\hat{t}' = t'$. From induction hypothesis, if $z \notin \{w\}\cup\{r^2+g_0^2+\ldots+g_{j-1}^2 = 0\}$ then this term is strictly positive.

We denote

$$K := \{(\hat{z},w,t') \in U_{j+2}\times(S\cap U_{j+2}\times\mathbb{C}^{n-1}|\hat{z}\in S_{j-1,w}, |t'|=1, \hat{\}}$$
$$F(\hat{z},w,t') := i\partial_z\bar{\partial}_z r_{j-1}(y,z')(\hat{t},\hat{t})$$

and

$$H(\hat{z},w,t') := \sum_{k=1}^{s_j}\left|\sum_{l=1}^{n}\frac{\partial f_k^j(y,z')}{\partial z_l}\hat{t}_l\right|^2$$

with $\hat{t}\in T_{\hat{z}}^{10}S_{j-1,w}$ and $\hat{t}_k = t'_{k-1}$ for $k=2,\ldots,n$; $(z,w,t')\in K$.
We will prove the following claim:

Claim. $(F+G)(\hat{z},w,t') > 0$ for all $(\hat{z},w,t')\in K$ with

$$\hat{z}\notin\{w\}\cup\{r^2+g_0^2+\ldots+g_j^2=0\}.$$

From induction hypothesis, $F(\hat{z},w,t')$ is strictly positive for $\hat{z}\notin\{w\}\cup\{r^2+g_0^2+\ldots+g_{j-1}^2 = 0\}$. So without loss generality, we can assume that $\hat{z}\in X := \{\zeta\in U_{j+2}|(r^2+g_0^2+\ldots+g_{j-1}^2)(\zeta)=0\}$ but $g_j(\hat{z})\neq 0$. We can find a neighborhood V of $\hat{z}$ such that $g_j(\zeta)\neq 0$ for all $\zeta\in V$.

We define

$$M := \{\zeta\in V|r(\zeta) = f_1^j(\zeta) = \ldots = f_{s_j}^j(\zeta) = 0\}\subset S.$$

By the construction of f_j^k and g_j, we have

$$|f_k^j|^{2N}\leq r^2+g_0^2+\ldots+g_{j-1}^2$$

therefore $X \cap V \subset M$. Furthermore, for $\zeta \in M \subset V$

$$0 \neq g_j(\zeta) = \text{coeff}(\partial r \wedge \bar{\partial} r \wedge \partial f_1^j \wedge \bar{\partial} f_1^j \wedge \ldots \wedge \bar{\partial} f_{s_j}^j \wedge (\partial\bar{\partial} r)^{n-s_j-1})(\zeta))$$

then $\partial f_1^j(\zeta), \ldots, \bar{\partial} f_{s_j}^j$ are linearly independent over $\mathbb{C}$. And hence M is a real-analytic manifold. Since by definition, $T^{10}M \subset T^{10}S$ and S is pseudoconvex, we have the Levi form of r is positive definite on $T^{10}M$. Once again, by induction hypothesis, $S_{j,w}$ has at least 4^{th} order of contact with S on $\{w\} \cup \{r^2 + g_0^2 + \ldots + g_{j-1}^2\}$, then we can get the Levi form of $r_{j-1,w}$ is also positive definite on $T^{10}M$. Moreover, if $t \in T^{10}_{\hat{z}} S_{j-1,w}$ such that $i\partial_z \bar{\partial}_z r_{j-1,w}(\hat{z}, w) = 0$, since $g_j \neq 0$, $t \notin T^{10}M$ and then $H(\hat{z}, w, t') \neq 0$. The claim is proven. Now, we can prove property e) . Let $\Delta_{\hat{z},w}$ be the Euclidean distance between $\hat{z}$ and the set

$$\{w\} \cup \{\zeta \in U_{j+2} | (r^2 + g_0^2 + \ldots + g_j^2)(\zeta) = 0\}$$

with $w \in S \cap U_{j+2}$ and $\hat{z} \in S_{j-1,w}$. From the claim and Lojasiewicz inequality, there is a constant $c_1 > 0$ and an integer $\alpha \leq 1$ such that

$$(F + H)(\hat{z}, w, t') \geq c\Delta^{\alpha}_{\hat{z},w}$$

for all $(\hat{z}, w, t') \in K$. Moreover, on $Y := \{(\hat{z}, w, t') \in K | F(\hat{z}, w, t') = 0)\}$, we have the estimate

$$H(\hat{z}, w, t') \geq c_1 \Delta^{\alpha}_{\hat{z},w}.$$

We can modify V as a neighborhood of Y on K such that

$$V := \{(\hat{z}, w, t') \in K | \delta((\hat{z}, w, t'), Y) < c_1' \Delta^a lpha_{\hat{z},w}\}$$

and since H is smooth, we can choose $0 < c_1' \ll c_1$ small enough to obtain

$$H(\hat{z}, w, t') \geq \frac{c_1}{2} \Delta^{\alpha}_{\hat{z},w}.$$

And then, apply this estimate to the Levi form of $r_{j,w}$ in (2.2.4), we get

$$i\partial_z \bar{\partial}_z r_j(z, w)(t, t) \geq \epsilon_j (g_j^2(\hat{z}) d_w(\hat{z}))^{N_j - n_j} c_1 \Delta^{\alpha}_{\hat{z},w} |t|^2 + \epsilon_j \mathcal{O}(g_j^2(\hat{z}) d_w(\hat{z}))^{N_j - 2} |t|^2$$
$$\epsilon_j \mathcal{O}(g_j^2(\hat{z}) d_w(\hat{z}))^{N_j - n_j - 2} \sum_{k=1}^{s_j} |f_k^j(\hat{z})| |t|^2$$

If we choose a pair $N_j \gg n_j \gg 1$ and shrinking U_{j+2} if necessary, we get

$$i\partial_z \bar{\partial}_z r_j(z, w)(t, t) \geq \epsilon_j \frac{c_1}{2} (g_j^2(\hat{z}) d_w(\hat{z}))^{N_j - n_j} \Delta^{\alpha}_{\hat{z},w} |t|^2$$
$$+ \epsilon_j \mathcal{O}(g_j^2(\hat{z}) d_w(\hat{z}))^{N_j - n_j - 2} \sum_{k=1}^{s_j} |f_k^j(\hat{z})| |t|^2.$$

Now we can shrink the neighborhood V again such that for each element $(\hat{z}, w, t') \in V$, $(\hat{z}, w, t') \in K$ and $\text{dist}((\hat{z}, w, t'), Y) \leq c'\Delta^{\alpha'}_{\hat{z},w}$ for some $\alpha' \gg \alpha$. Then we obtain on V

$$i\partial_z\bar{\partial}_z r_j(z, w)(t, t) \geq \frac{1}{4}\epsilon_j c_1 (g_j^2(\hat{z}) d_w(\hat{z}))^{N_j - n_j} \Delta^{\alpha'}_{\hat{z},w}.$$

and then the Levi form $i\partial_z\bar{\partial}_z r_j$ is strictly positive for

$$z \in V \setminus \{w\} \cup \{\zeta \in U_{j+2} | (r^2 + g_0^2 + \ldots + g_j^2)(\zeta) = 0$$

and $w \in S \cap U_{j+2}$ apply to $t = (t_1, t')$ satisfies $t \in T^{10}_{\hat{z}} S_{j,w}$, $|t'| = 1$. To finish the proof of property e), we need only to consider the case that point is in $K \setminus V$. From the definition of Y and apply Lojasiewickz inquality to F on K, then we can find $c_2 > 0$ and an integer $\beta \gg 1$ such that

$$F(\hat{z}, w, t') \geq c_2 (\text{dist}((\hat{z}, w, t'), Y)^{\beta}.$$

Therefore, on $K \setminus V$ we have

$$F(\hat{z}, w, t') \geq c_3 \Delta^{\beta'}_{\hat{z},w}$$

with $c_3 > 0$ and $\beta' = \alpha'\beta$. Again, we apply these estimates to (2.2.4) and obtain

$$i\partial_z\bar{\partial}_z r_j(z, w)(t, t) \geq -c_3\epsilon_j \Delta^{N_j - n_j - 2}(\hat{z}, w) + c_2 \Delta^{\beta'}(\hat{z}, w)$$

for all $(\hat{z}, w, t') \in K \setminus V$, $t = (t_1, t') \in T^{10} S_{j,w}$. The right hand side is strictly positive if we choose the pair N_j, n_j such that $N_j - n_j$ is large enough and ϵ_j is sufficiently small. Then e) is proven. Combine e) and f) we can get b). Then we have constructed r_j satisfies these properties for $j = 1, \ldots, l$.
Then also prove Theorem 2.2.2 if choose $\rho = r_l$.

2.2.2 Boundary behavior of Kobayashi metric.

To estimate the Kobayashi metric for points near the boundary, we separate in several steps.

Step 1. We have already constructed for each boundary point w in a neighborhood of 0 a bumping fucntion $\rho(z, w)$. For that, the bumping function $\rho(., w)$ is pseudoconvex from the side $\rho(., w) < 0$. Furthermore, since $\rho(z, w)$ is a real analytic function on the set $(\Omega \cap V') \times (b\Omega \cap V')$ and also ρ vanishes on this set exactly for $z = w$, Lojasiewicz inequality gives us that

$$d^N(z, w) \leq |\rho(z, w)| \tag{2.2.5}$$

with $d(z,w) = |z-w|^2$ for some integer N. Now, we need to construct an extension of ϕ_w on all Ω that preserve the plurisubharmonicity and also (2.2.5). First we notice that for all $z \in bU \cap \bar{\Omega}$ and $w \in b\Omega \cap V$ it holds that

$$|\rho(z,w)| \geq c > 0.$$

Let

$$L := \sup\{d^N(z,w)|z \in \bar{\Omega}, w \in V \cap b\Omega.\}$$

By multiplying $\rho(z,w)$ with a large fixed number, we can get a new function, still denoted as $\rho(z,w)$ satisfies

$$|\rho(z,w)| \geq L+1.$$

Choose a convex funtion $\chi(x)$ on $\mathbb{R}^+$ which satisfies

$$\chi(x) = \begin{cases} -x & \text{for } 0 \leq x \leq L; \\ -L' & \text{for } x \geq L+1 \text{ with } -L' < -L. \end{cases}$$

And then, for any fixed $w \in V \cap b\Omega$, we denote $\varphi_w(z)$ as an extension of $\rho(z,w)$ to all Ω that is defined as

$$\varphi_w(z) := \begin{cases} \chi(|\rho(z,w)|) & \text{for } z \in V' \cap \Omega; \\ -L' & \text{for } z \in \Omega \setminus V' \end{cases}$$

for $w \in V' \cap b\Omega$. Then it easy to see that ψ_w satisfies the following properties

1. $\varphi_w(w) = 0$.
2. $d_z(\varphi_w(z)) \neq 0$ for all $z \in V \cap b\Omega$.
3. For fix $w \in V \cap \Omega$, $\varphi_w(z)$ is pseudoconvex from the side $\varphi_w < 0$.
4. For $z \in \Omega$ and $w \in V \cap b\Omega$
$$(z) \leq d^N(z,w) \tag{2.2.6}$$
5. There exists $K' > 0$ such that
$$K' d^{\frac{1}{2}}(z, \pi(z)) \leq \varphi_{\pi(z)}(z) \tag{2.2.7}$$
for $z \in V'$.

Step 2. Now, apply plurisubharmonic exhaustion function theory, we have the following Lemma

Lemma 2.2.4. *There exists an open neighborhood $V \subset U$ of 0 such that for any $w \in V \cap b\Omega$, for any $0 < \eta < 1$, we can find a strictly plurisubharmonic function ψ_w on $V' = \{z \in V | \varphi_w(z) < 0\}$ and a constant $K \geq 1$ satisfies*

$$-K|\varphi_w(z)|^\eta < \psi(z) < -1/K|\varphi(z)|^\eta$$

for $z \in V'$.

From the Lemma and above estimates for $\varphi_w(z)$, we obtain

$$-Kd^{\eta/2}(z, \pi(z)) \leq \psi_{\pi(z)}(z) \tag{2.2.8}$$

Step 3. To get the desired estimate of F_Ω on $V \cap \Omega$. Lets fix a point $z \in V' \cap \Omega$ and let $w = \pi(z)$. We now assume that $f = (f_1, \ldots, f_n) : \bar{D} \to \Omega$ is a holomorphic map of the closed unit disc into Ω with $f(0) = z$.
The mean value inequality of the subharmonic function $\psi_w \circ f(t)$ on $\bar{D}$ gives us

$$\psi_w \circ f(0) = \psi_w(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi_w \circ f(e^{i\theta}) d\theta.$$

therefore

$$\frac{1}{2\pi} \int_0^{2\pi} -\psi_w \circ f(e^{i\theta}) d\theta \leq Kd^{\eta/2}(z, w). \tag{2.2.9}$$

From (2.2.8) we have

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} -\psi_w \circ f(e^{i\theta} d\theta &= \frac{1}{2\pi} \int_0^{2\pi} (-\psi_w \circ f(e^{i\theta}) - d^{N\eta}(f(e^{i\theta}), w)) d\theta \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} d^{N\eta}(f(e^{i\theta}), w) d\theta \\
&\geq \frac{1}{2\pi} \int_0^{2\pi} d^{N\eta}(f(e^{i\theta}), w) d\theta \\
&\geq \frac{1}{2\pi} \int_0^{2\pi} |f_k(e^{i\theta}) - w_k)|^{2N\eta} d\theta.
\end{aligned}$$

for all $k = 1, \ldots, n$. Combine with (2.2.9) gives for $k = 1, \ldots, n$

$$\frac{1}{2\pi} \int_0^{2\pi} |f_k(e^{i\theta}) - w_k|^{2N\eta} d\theta \leq Kd^{\eta/2}(z, w).$$

Consequence, we have

$$|f_k'(0)| \leq Cd^{1/4N}(z, \pi(z)), \quad k = 1, \ldots, n$$

with $C > 0$ is a constant does not depend on f and z. And then there exists $C' > 0$ such that

$$|f_k'(0)| \leq C'|r(z)|^{1/2N}$$

for a fixed defining function r of Ω and for all $z \in \Omega \cap V$. The estimate is true for all holomorphic mapping $f : \bar{D} \to \Omega$ satisfies $f(0) = z$. By the definition of $F_\Omega(z, X)$, it can be shown that for all $X \in \mathbb{C}^n$

$$F_\Omega(z, X) \geq c\frac{|X|}{|r(z)|}^{1/2N}$$

for all $z \in V \cap \Omega$ with a constant $c > 0$ independent of z and X. Since Ω can be covered by finite many of such neighborhood V'. Theorem 2.2.1 is proved.

2.3 The pseudoconvex domain in more general cases

Remove the asumption that Ω has a real analytic defining function, follow the idea to construct the bumping function in [Cho92], we are going to find the Kobayashi metric for pseudoconvex domains that satisfy property $(f - P)$.
First, we re-introduce the definition of property $(f - P)$ and some remarks.

2.3.1 Property $(f\text{-}P)$

For $\delta > 0$, denote $S_\delta = \{z \in \Omega : -\delta < r < 0\}$

Definition 2.3.1. Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and U be a local path of its boundary. We say that Ω has Property $(f\text{-}\mathcal{P})$ on U if there exist a family of $C^2(U \cap S_\delta)$ functions $\{\phi_\delta\}$ such that

$$\begin{cases} |\phi_\delta| \leq 1 \\ |D\phi_\delta| \lesssim \delta^{-1} & \text{on } U \cap S_\delta \\ i\partial\bar{\partial}\phi_\delta(L, \bar{L}) \gtrsim f(\delta^{-1})^2|L|^2 \end{cases}$$

for any $L \in T^{1,0}(U \cap b\Omega)$.

We show the equivalent of Property $(f\text{-}P)$ between pseudoconvex and pseudoconcave side of hypersurface.

Since the hypersurface defined by each bumping function lies outside domain except one point and Property $(f\text{-}P)$ happens on strips inside domain, so we first show property

(f-P) still holds outside domain.

We use notation $S_\delta^+ := S_\delta$ and $S_\delta^- := \{z \in \mathbb{C}^n | 0 < r(z) < \delta\}$. We define property ($f$-$P$)$^+$ and (f-P)$^-$ in obvious sense.

Lemma 2.3.2. *Property* (f-P)$^+ \iff$ *Property* (f-P)$^-$.

Proof. Assume that Property (f-P)$^+$ holds on U ,that is, for any $\delta > 0$ there exist a $C^2(U \cap S_\delta^+)$ function ϕ_δ^+ such that

$$\begin{cases} |\phi_\delta^+| \le 1 \\ |D\phi_\delta^+| \lesssim \delta^{-1} \\ i\partial\bar\partial\phi_\delta^+(L, \bar L) \gtrsim f^2(\delta^{-1})|L|^2 \end{cases} \quad \text{on } U \cap S_\delta. \tag{2.3.1}$$

for any $L \in T^{1,0}(U \cap b\Omega)$. Without loss of generality, we can assume that the original point z_o in $U \cap b\Omega$. We choose the special coordinate $z = (x, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$ at z_o. Set $\phi_\delta^-(x, r) = \phi_\delta^+(x, -r)$. Then $i\partial\bar\partial\phi_\delta^-(L, \bar L) = i\partial\bar\partial\phi_\delta^+(L, \bar L)$ on $-\delta < -r < 0$ for any $L \in T^{1,0}(U \cap b\Omega)$. That means Property ($f$-$P$)$^-$ holds. That is completed the proof of this lemma. □

2.3.2 The bumping function

In this section, we give the proof of Theorem 2.1.4. The proof is divided several steps. We already have the equivalent of Property (f-P) between pseudoconvex and pseudoconcave side of hypersurface. We extend the weights in Property (f-P) to be self-gadient bounded in step 1. In step 2, we construct plurisubharmonic function with good estimates of size. The properties of bumping function is checked on step 3.

Step 1. In this step, we will show that there exist a family of functions such that negative and self-gradient bounded. Define $\Phi_\delta = e^{\phi_\delta^- - 1} - 2$ on $U \cap S_\delta^-$. Then $\Phi^\delta \in C^2(U \cap S_\delta^-)$ and satisfies

$$\begin{cases} -2 \le \Phi_\delta \le -1 \\ |D\Phi_\delta| \lesssim \delta^{-1} \\ i\partial\bar\partial\Phi_\delta(L, L) \gtrsim f^2(\delta^{-1})|L|^2 + |L\Phi_\delta|^2 \end{cases} \quad \text{on} \quad U \cap S_\delta^- \tag{2.3.2}$$

for any $L \in T^{1,0}(U \cap b\Omega)$. We also can extend Φ_δ to be negatively bounded on whole U such that the second line of (2.3.2) holds. We still call the new function is Φ_δ.

Step 2. Let $\chi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function defined by

$$\chi(t) = \begin{cases} 1 & \text{for } t \in \left[\frac{1}{4}, 2\right], \\ 0 & \text{elsewhere} \end{cases}$$

such that $\dfrac{|\dot{\chi}|^2}{\chi}$ is bounded on support of χ.

For fixed $w \in U \cap b\Omega$, for $z \in U$, we define

$$P(z, w) = \sum_{k=N}^{\infty} F(2^{-k})\chi(2^{k+1}|z - w|)\Phi_{F(2^{-k})}(z) \tag{2.3.3}$$

Here, we recall again that $F(\delta) = \left(f^*(\delta^{-1})\right)^{-1}$. Denote

$$A_k(w) := \{z \in U : \frac{1}{2} \leq 2^{k+1}|z - w| \leq 2\}$$

be an annulus center w with radii 2^{-k-1} and 2^{-k}. Since $U = \overset{\infty}{\underset{k=1}{\cup}} A_k(w)$, so for any $z \in U$, there is an integer j such that $z \in A_j(w)$. We have

$$\begin{aligned} P(z, w) &= \sum_{k=j-1}^{j+1} F(2^{-k})\chi(2^k|z - w|)\Phi_{F(2^{-k})}(z) \\ &\lesssim - F(2^{-j}) \\ &\leq - F(|z - w|) \end{aligned} \tag{2.3.4}$$

where the first inequality follows by the fact that Φ_δ is negative bounded and $\chi(2^j|z-w|) > 0$ on $A_j(z)$; the last one follows by $|z - w| \leq 2^{-j}$. Moreover, we also get

$$P(z, w) \gtrsim -F(2^{-j+1}) \geq -F(4|z - w|). \tag{2.3.5}$$

Furthemore, we know that for $z \in A_j(w) \cap S_{F(2^{-j})}$. There exists $c > 0$ such that $\chi(2^j|z - w|) \geq c > 0$ for $z \in A_j(w)$.

Then for $L \in T^{1,0}(U \cap b\Omega)$ we have

$$\begin{aligned}
& i\partial_z\bar{\partial}_z(\chi(2^j|z-w|)\Phi_{F(2^{-j})}(z))(L,\bar{L}) \\
&\qquad = i\partial\bar{\partial}\chi(2^j|z-w|)(L,\bar{L})\Phi_{F(2^{-j})}(z) + 2\mathrm{Re}(L(\chi(2^j|z-w|))\bar{L}(\Phi_{F(2^{-j})}(z))) \\
&\qquad\quad + \chi(2^j|z-w|)\partial_z\bar{\partial}_z\Phi_{F(2^{-j})}(z)(L,\bar{L}); \\
&\qquad \geq \partial_z\bar{\partial}_z\chi(2^j|z-w|)(L,\bar{L})\Phi_{F(2^{-j})}(z) - 2\frac{1}{\chi(2^j|z-w|)}|L(\chi(2^j|z-w|))|^2 \\
&\qquad\quad - \frac{1}{2}\chi(2^j|z-w|)|L(\Phi_{F(2^{-j})}(z)|^2 + \chi(2^j|z-w|)\partial_z\bar{\partial}_z\Phi_{F(2^{-k})}(z)(L,\bar{L}); \\
&\qquad \gtrsim \left(-\ddot{\chi}2^{2j} - 2^{2j}\frac{|\dot{\chi}|^2}{\chi}\right)|L|^2 + \frac{1}{2}c\partial_z\bar{\partial}_z\Phi_{F(2^{-j})}(z)(L,\bar{L})
\end{aligned}$$

Similarly, for $k = j-1$ and $k = j+1$ we obtain

$$i\partial_z\bar{\partial}_z(\chi(2^k|z-w|)\Phi_{F(2^{-k})}(z))(L,\bar{L}) \gtrsim (-\ddot{\chi}2^{2j} - \frac{|\dot{\chi}|^2}{\chi}2^{2j})|L|^2. \tag{2.3.6}$$

In these two terms, we do not have the Hessian of Φ since χ is bounded from below by 0.

Therefore we get

$$\begin{aligned}
\partial_z\bar{\partial}_z P(z,w)(X,X) &= \sum_{k=j-1}^{j+1} F(2^{-k-1})\partial_z\bar{\partial}_z(\chi(2^k|z-w|)\Phi_{F(2^{-k})}(z))(X,\bar{X}) \\
&\gtrsim \sum_{k=j-1}^{j+1} F(2^{-k-1})\left(2^{2k}|X|^2) + \chi_k^w\partial_z\bar{\partial}_z\Phi_{F(2^{-k})}(z)(X,\bar{X})\right) \\
&\gtrsim \left(-\epsilon F(2^{-j})2^{2(j-1)} + F(2^{-j-1})2^{2j} - \epsilon F(2^{-j-2})2^{2(j+1)}\right)|X|^2 \\
&\qquad + \chi(2^j|z-w|)\partial_z\bar{\partial}_z\Phi_{F(2^{-j})}(z)(X,\bar{X})
\end{aligned} \tag{2.3.7}$$

for $z \in A_j(w) \cap S_{F(2^{-j})}$ and $X \in T^{1,0}(b\Omega \cap U)$.

Step 3. Again for $z \in U$ and $w \in U \cap b\Omega$, we define

$$\rho(z,w) = r(z) + \epsilon P(z,w).$$

Let $S_w = \{z \in U | \rho(z,w) = 0\}$ be hypersurface defined by $\rho(z,w) = 0$ when w is fixed. We will prove that ρ satisfies the following properties:

(i) $\rho(w,w) = 0$.

(ii) $\rho(z,w) \lesssim -F(|z-w|)$ for $z \in U \cap \Omega$ and $w \in U \cap b\Omega$.

(iii) $\rho(z,\pi(z)) \gtrsim -\delta_{b\Omega}(z)$ for $z \in U \cap \Omega$, where $\pi(z)$ is the projection of z to the boundary.

(iv) $|d_z\rho(z,w)| \approx 1$ on S_w.

(v) S_w is pseudoconvex.

We see that (i) is obvious; (ii) comes from the estimate (2.3.4). For (iii), we have

$$\rho(z,w) \gtrsim -\delta_{b\Omega}(z) - F(4\delta_{b\Omega}(z)) \gtrsim -\delta_{b\Omega}(z).$$

Here, the first inequality follows by (2.3.5), the second follows by the fact that $F(4\delta) \leq (4\delta)^2 << \delta$.

For any $z \in (S_w \setminus \{w\}) \cap A_j(w)$, from (2.3.5), we have

$$0 < r(z) = -\epsilon P(z,w) \lesssim \epsilon F(2^{-j}) \leq F(2^{-j}).$$

That implies $z \in S_{F(2^{-j})}$. Therefore, we obtain $S_w \subset \bigcup_{k=N}^{\infty} \big(A_k(w) \cap S_{F(2^{-k})}\big)$.
To prove (iv), notice that since r is a defining function of Ω then $|D_z r| \approx 1$ on U. We only need to consider $D_z P(z,w)$ for all $z \in S_w$. By above argument, for any $z \in S_w$ there exists $j \in \mathcal{N}$ such that $z \in A_j(w) \cap S_{F(2^{-j})}$. We have

$$\begin{aligned}|D_z P(z,w)| &= \left|\sum_{k=j-1}^{j+1} F(2^{-k}) D_z\big(\chi(2^k|z-w|)\big)\Phi_{F(2^{-k})}(z) + F(2^{-k})\chi(2^k|z-w|) D_z\big(\Phi_{F(2^{-k})}(z)\big)\right| \\ &\lesssim \sum_{k=j-1}^{j+1} F(2^{-k})2^k + F(2^{-k})(F(2^{-k}))^{-1} \lesssim C \qquad (2.3.8)\end{aligned}$$

Here, we use the fact that $|D\Phi_{F(2^{-k})}| \lesssim (F(2^{-1}))^{-1}$ and $F(2^{-k}) \lesssim 2^{-2k}$. For ϵ be sufficiently small, we have $|D_z\rho(z,w)| \approx 1$ and (iv) is satisfied.

Now we have that $S_w(z)$ is a hypersurface, and it easy to see that $r > 0$ on $S_w(z) \cap (U \setminus \{w\})$. We need to prove that $S_w(z)$ are pseudoconvex, more preciously, $S_w(z)$ is strictly pseudconvex on $S_w(z) \cap (U \setminus \{w\})$.

Additional assumption: We assume that the function F satisfies $\frac{F(2t)}{F(t)}$ is bounded.

For this case, from (2.3.7) we can let ϵ small enough such that $\frac{1}{2}F(2^{-j-1})2^j \geq \epsilon F(2^{-j})2^{j-1}$. Furthermore we have that $\frac{F(|z|)}{|z|^2}$ is a increasing function. Then we obtain a lower bound for the Hessian of $P(.,w)$

$$i\partial_z\bar{\partial}_z P(z,w)(X,X) \gtrsim F(2^{-j-1})2^{2j} + \chi(2^j|z-w|)\partial_z\bar{\partial}_z\Phi_{F(2^{-j})}(z)(X,\bar{X}) \qquad (2.3.9)$$

for $z \in A_j(w) \cap S_{F(2^{-j})}$ and $X \in T^{1,0}(b\Omega \cap U)$.
Now we will prove that the hypersurface $S_w = \{z \in U | \rho(z,w) = 0\}$ are pseudoconvex at each point $z \in S_w$.
Suppose $T_w(\rho(z,w)) = 0$ and $|T_w| = 1$. Then T_w can be written as

$$T_w = T + \alpha N.$$

where $T \in T^{1,0}(b\Omega \cap U)$ and $Nr > 0$ on U. We have that

$$\begin{aligned} T_w(\rho(z,w) &= (T + \alpha N)(r(z) + \epsilon P(z,w)) \\ &= \epsilon T P(z,w) + \alpha(Nr(z) + \epsilon N P(z,w)) = 0. \end{aligned} \tag{2.3.10}$$

From property (3), we have that $NP(z,w) \approx 1$, hence we can get $|\alpha| \lesssim \epsilon |TP(z,w)| \ll \frac{1}{2}$.
For $z \in A_k(w) \cap S_{F(2^{-k})}$, (2.3.9) and the hypothesis of $\Phi_{F(2^{-k})}$ imply

$$\partial_z \bar{\partial}_z P(z,w)(T,\bar{T}) \gtrsim F(2^{-k}) \partial\bar{\partial}\Phi_{F(2^{-k})}(z)(T,\bar{T}) \gtrsim F(2^{-k}) |T\Phi_{F(2^{-k})}(z)|^2 \tag{2.3.11}$$

and

$$\partial_z \bar{\partial}_z P(z,w)(T,\bar{T}) \gtrsim F(2^{-k}) \partial_z\bar{\partial}_z\Phi_{F(2^{-k})}(z)(T,\bar{T}) \gtrsim 2^k F(2^{-k}). \tag{2.3.12}$$

From the definition of χ and (2.3.12) we have

$$|T\chi(2^k|z-w|)| \lesssim 2^k \lesssim F(2^{-k})^{-\frac{1}{2}} (\partial_z \bar{\partial}_z P(z,w)(T,\bar{T}))^{\frac{1}{2}}. \tag{2.3.13}$$

And from (2.3.11) we also have

$$|T\Phi_{F(2^{-k})}(z)| \lesssim F(2^{-k})^{-\frac{1}{2}} (\partial_z \bar{\partial}_z P(z,w)(T,\bar{T}))^{\frac{1}{2}}. \tag{2.3.14}$$

Combine (2.3.10), (2.3.13) and (2.3.14) we obtain that

$$|\alpha| \lesssim \epsilon |TP(z,w)| \lesssim \epsilon F(2^{-k})^{-\frac{1}{2}} (\partial_z \bar{\partial}_z P(z,w)(T,\bar{T}))^{\frac{1}{2}}. \tag{2.3.15}$$

Now we have all material to prove the pseudoconvexity of the hypersurface S_w.

$$\begin{aligned} \partial_z\bar{\partial}_z\rho(z,w)(T_w,\bar{T}_w) &= \partial_z\bar{\partial}_z r(z)(T_w,\bar{T}_w) + \epsilon\partial_z\bar{\partial}_z P(z,w)(T_w,\bar{T}_w) \\ &= \partial_z\bar{\partial}_z r(z)(T,\bar{T}) + \epsilon\partial_z\bar{\partial}_z P(z,w)(T,\bar{T}) + \mathcal{O}(\alpha) \\ &\geq \epsilon\partial_z\bar{\partial}_z P(z,w)(T,\bar{T}) - \epsilon C F(2^{-k})^{-\frac{1}{2}} (\partial_z\bar{\partial}_z P(z,w)(T,\bar{T})^{\frac{1}{2}} \\ &\geq \epsilon(\partial_z\bar{\partial}_z P(z,w)(T,\bar{T}))^{\frac{1}{2}} \left((\partial_z\bar{\partial}_z P(z,w)(T,\bar{T}))^{\frac{1}{2}} - CF(2^{-k})^{-\frac{1}{2}} \right) \\ &\gtrsim \epsilon(\partial_z\bar{\partial}_z P(z,w)(T,\bar{T}))^{\frac{1}{2}} \left(2^k - CF(2^{-k})^{-\frac{1}{2}} \right) \\ &\gtrsim \epsilon|T|^2 \gtrsim \epsilon|T_w|^2. \end{aligned}$$

In the second to the last inequality, it is true if we have k is large enough, it means we choose N to be sufficiently large. Then (v) is satisfied.

Remark 2.3.3. The additional assumption $\frac{F(2z)}{F(z)}$ is bounded characterized by finite type domain.
For the case $\frac{F(2z)}{F(z)}$ is unbounded, we will consider the problem for some special infinite type domains later.

2.3.3 Boundary behavior of the Kobayashi metric

After constructing the bumping function for any points w in the boundary of Ω, the hypersurfaces S_w are pseudoconvex and touch the boundary of Ω at w only. Now, we can apply plurisubharmonic defining function theory to extend to the neighborhood of S_w.

Theorem 2.3.4. *Let r be a real-valued-C^2-function on a neighborhood $U \subset \mathbb{C}^n$ of 0 with the following properties:*

1. *$r(0) = 0$.*
2. *$dr \neq 0$ everywhere on U.*
3. *The hypersurface $S = \{z \in U | r(z) = 0\}$ is pseudoconvex from the side $r < 0$.*

Then, for every $\eta > 0$, $0 < \eta < 1$, there exists an open neighborhood $V \subset U$ of 0, a strictly plurisubharmonic function ρ on $V' = \{z \in V : r(z) < 0\}$ and a constant $K \geq 1$ such that $-K|r|^\eta < \rho < -1/K|r|^\eta$ on V'. Furthermore, the data K and V can be chosen independently of small C^2-perturbation of r on U satisfying condition (1), (2), (3) from above.

Now we can prove the following theorem to get an estimate for the Kobayashi metric of a point near the boundary $b\Omega$.

Theorem 2.3.5. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and U be a neighborhood of given point z_o in the boundary. For each $w \in U \cap b\Omega$, assume that there is a plurisubharmonic function $\psi_w(z)$ such that*

i) $\psi_w(z) \leq -F_1(\alpha_1|z_j - w_j|)$ for $z \in U \cap \Omega$

ii) $\psi_{\pi(z)}(z) \geq -c_2F_2(\alpha_2\delta_\Omega(z))$ for $z \in U \cap \Omega$. Here $\pi(z)$ is the orthogonal projection of z to $U \cap b\Omega$.

Then

$$K_\Omega(z, X) \geq \alpha_1(F_1^*(c_2F_2(\alpha_2\delta_\Omega(z)))^{-1}|X_j|$$

for all $z \in V$, $X \in T_z^{1,0}\mathbb{C}^n$ where $V \subset U$.

Proof. We now fix a point $z \in V \cap \Omega$, put $w = \pi(z)$ and assume that $g = (g_1, \ldots, g_n) : \overline{\Delta} \to \Omega$ is a holomorphic map of the closed unit disc into Ω with $g(0) = z$.

By applying the mean value inequality to the subharmonic function $\psi_w(g(t))$ on $\overline{\Delta}$ we get

$$\psi_w(z) = \psi_w(g(0)) \leq \int_0^1 \psi_w \circ g(e^{i2\pi\theta}) d\theta$$

The hypothesis (ii) gives

$$c_2 F_2(\alpha_2 \delta(z))) \geq \int_0^1 -\psi_w \circ g(e^{i2\pi\theta}) d\theta \tag{2.3.16}$$

We now use the hypothesis (i) of ψ_w,

$$\begin{aligned} \int_0^1 -\psi_w \circ f(e^{i2\pi\theta}) d\theta &= \int_0^1 \big(-\psi_w \circ g(e^{i2\pi\theta}) - F_1(\alpha_1 |g_j(e^{i2\pi\theta}) - w_j|) \big) d\theta \\ &\quad + \int_0^1 F_1\big(\alpha_1 |g_j(e^{i2\pi\theta}) - w_j|\big) d\theta \\ &\geq \int_0^1 F_1\big(\alpha_1 |g_j(e^{i2\pi\theta}) - w_j|\big) d\theta. \end{aligned} \tag{2.3.17}$$

Using the Jensen inequality for the increasing, convex function F_1, we get

$$F_1(\alpha_1 |g_j'(0)|) \leq F_1\Big(\alpha_1 \int_0^1 |g_j(e^{i2\pi\theta}) - w_j| d\theta\Big) \leq \int_0^1 F_1\big(\alpha_1 |g_j(e^{i2\pi\theta}) - w_j|\big) d\theta.$$

Combining above inequality with (2.3.16) and (2.3.17), we obtain

$$F_1(\alpha_1 |g_j'(0)|) \leq c_2 F_2(\alpha_2 \delta_\Omega(z)).$$

An immediate consequence of this is

$$|g_j'(0)| \leq \frac{1}{\alpha_1} F_1^*(c_2 F_2(\alpha_2 \delta_\Omega(z)).$$

By the definition of $K(z, X)$ shows immediately that one must have for all $X \in T^{1,0}\mathbb{C}^n$

$$K(z, X) \geq \alpha_1 (F_1^*(c_2 F_2(\alpha_2 \delta_\Omega(z)))^{-1} |X_j|.$$

$\square$

2.4 Application to proper holomorphic maps

We introduce a general Hardy-Littlewood Lemma for $f-$Hölder estimates

Theorem 2.4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\delta_{b\Omega}(x)$ denote the distance function from x to the boundary of Ω. Let $G:\mathbb{R}^+\to\mathbb{R}^+$ be an increasing function such that $\frac{G(t)}{t}$ is decreasing and $\int_0^d \frac{G(t)}{t}\leq 0$ for $d>0$ small enought. If $u\in C^1(\Omega)$ such that*

$$|\Delta u(x)|\lesssim \frac{G(\delta_{b\Omega(x)})}{\delta_{b\Omega}(x)} \quad \text{for every } x\in\Omega.$$

Then $|u(x)-u(y)|\lesssim f(|x-y|^{-1})^{-1}$, for $x,y\in\Omega, x\neq y$ where $f(d^{-1})=\left(\int_0^d \frac{G(t)}{t}dt\right)^{-1}$.

If $G(t)=t^\alpha$, Theorem 2.4.1 is the Hardy-Littlewood for domains of finite type. In [?], T.V.Khanh provide a general proof for some kind the infinite type domains.

Theorem 2.4.2. *Let Ω and Ω' be pseudoconvex domains. Let $\eta, 0<\eta\leq 1$ such that there is a C^2 defining function r of Ω such that $-(-r)^\eta$ is strictly plurisubharmonic on Ω. Assume that Ω' has property (f-$\mathcal{P}$). Then any proper holomorphic map $\Phi:\Omega\to\Omega'$ can be extended a general Hölder continuous map $\hat\Phi:\bar\Omega\to\bar\Omega'$ with a rate $\tilde f$, that is,*

$$|\hat\Phi(z)-\hat\Phi(w)|\lesssim \tilde f(|z-w|^{-1})^{-1}$$

for any $z,w\in\Omega$. Here, $\tilde f$ is defined by

$$\tilde f(d^{-1})=\left(\int_0^d \frac{1}{t}\left(f(\frac{1}{t^\eta})\right)^{-1}dt\right)^{-1} \tag{2.4.1}$$

Proof. Using Theorem 2.1.2 for Ω', the Schwarz-Pick lemma for the Kobayashi metric, and the upper bound of Kobayashi metric, we obtain the following estimate

$$f\left(\delta^{-1}_{\Omega'}(\Phi(z))\right)|\Phi'(z)X|\lesssim K_{\Omega'}(\Phi(z),\Phi'(z)X)\leq K_\Omega(z,X)\lesssim \delta^{-1}_\Omega(z)|X| \tag{2.4.2}$$

for any $z\in\Omega$ and $X\in T^{1,0}\mathbb{C}^n$. Moreover, by the fact that $-(-r)^\eta$ is strictly plurisubharmonic on Ω, one has $\delta_{\Omega'}(\Phi(z))\lesssim\delta^\eta_\Omega(z)$ for any $z\in\Omega$ (Lemma 8 in [DF79]). Therefore,

$$|\Phi'(z)X|\lesssim \delta^{-1}_\Omega(z)f^{-1}(\delta^{-\eta}_\Omega(z))|X|$$

for any $z\in\Omega$ and $X\in T^{1,0}\mathbb{C}^n$. Using Hardy-Littlewood Lemma for general Hölder estimates (see Theorem 5.1 in [?]) then gives that Φ can be extended to a general Hölder continuous map $\hat\Phi:\bar\Omega\to\bar\Omega'$ with the rate $\tilde f$ defined in (2.4.1)

□

2.5 The bumping function for the domain $\Omega = \{z \in \mathbb{C}^2 | r(z) = |z_2|^2 - F(|z_1|) < 1\}$

Let Ω be bounded pseudoconvex domain in $\mathbb{C}^2$ with boundary $b\Omega$ and $0 \in b\Omega$. Assume that in a neighborhood U of 0, Ω has the form

$$\Omega \cap U = \{z \in U | r(z) = -F(|z_1|) + |z_2|^2 < 1\} \tag{2.5.1}$$

where F is strictly increasing, convex function and $F(t)/t^2$ is incresing.

The case that we consider the most is when $F(t) = \exp(-1/t^{2\alpha})$. We can see that Ω is pseudoconvex and $\{(z_1, z_2) | z_1 = 0\}$ is the points of infinity type.

For simplizing our later computation, we only consider the case that $\alpha = 1$. And we are going to find a family of functions Φ_δ on the strip $S_\delta = \{z \in \Omega | -\delta \le r(z) \le 0\}$ such that $(f-P)$- property holds

$$\sum_{i,j=1}^{2} \frac{\partial^2 \Phi_\delta}{\partial z_i \partial \bar{z}_j} \gtrsim f(\delta^{-1})^2 |u|^2$$

on $S_\delta \cap U$ where U is a neighborhood of the origin and for any $(0,1)$ form u.

For any $\delta > 0$, we define

$$\Phi_\delta(z) := \exp\left(\frac{r(z)}{\delta} + 1\right) - \exp\left(-\frac{|z_1|^2}{2F^*(\delta)}\right).$$

It is easy to see that Φ_δ are absolutely bounded on $S_\delta \cap U$. The Levi form of Φ_δ

$$\begin{aligned}
i\partial\bar{\partial}\Phi_\delta(z; u, u) = & \frac{1}{\delta}\left(\sum_{i,j=1} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j} u_i \bar{u}_j + \frac{1}{\delta}\left|\sum_i \frac{\partial r}{\partial z_i} u_i\right|^2\right) \exp\left(\frac{r}{\delta} + 1\right) \\
& + \frac{1}{F^*(\delta)}\left(1 - \frac{|z_1|^2}{2F^*(\delta)}\right) \exp\left(-\frac{|z_1|^2}{2F^*(\delta)}\right) |u_1|^2 \\
\gtrsim & \left(\frac{1}{\delta}\frac{F(|z_1|^2)}{|z_1|^2} + \frac{1}{2F^*(\delta)}\left(1 - \frac{|z_1|^2}{2F^*(\delta)}\right) \exp\left(-\frac{|z_1|^2}{2F^*(\delta)}\right)\right) |u_1|^2 \\
& + \frac{1}{F^*(\delta)} |u_2|^2.
\end{aligned}$$

Let

$$A = \frac{1}{\delta}\frac{F(|z_1|^2)}{|z_1|^2}, \quad \text{and} \quad B = \frac{1}{2F^*(\delta)}\left(1 - \frac{|z_1|^2}{F^*(\delta)}\right) \exp\left(-\frac{|z_1|^2}{F^*(\delta)}\right).$$

We have two cases.

1^{st} *case.* When $|z_1|^2 \le F^*(\delta)$. We have $B \gtrsim \frac{1}{F^*(\delta)}$. Thus, $A + B \ge \frac{1}{F^*(|z_1|^2}$.
2^{nd} *case.* When $|z_1|^2 \le F^*(\delta)$. Since $\frac{F(t)}{t^2}$ is an increasing function. Term A can be estimated by

$$A = \frac{1}{\delta}\frac{F(|z_1|^2)}{|z_1|^2} \ge \frac{1}{\delta}\frac{F(F^*(\delta))}{F^*(\delta)} = \frac{1}{\delta}\frac{\delta}{F^*(\delta)} = \frac{1}{F^*(\delta)}.$$

Term B in this case can be negative. However, consider the function $g(t) = (1-t)e^{-t}$, it attains minimum $\min_{t>1/2} g(t) = -e^{-2}$ when $t = \frac{|z_1|^2}{F^*(\delta)} = 2$. Thus, $B \ge -\frac{e^{-2}}{F^*(\delta)}$ and it implies $A + B \gtrsim \frac{1}{F^*(\delta)}$.

We note that in some cases, for instant: $F(t) = \exp(-1/t^{2\alpha})$ then $F(2^{-j+1})/F(2^{-j})$ is unbounded when j goto infinity. Thus, start from (2.3.7), we cannot find such an ϵ to let $F(2^{-j}) \gtrsim F(2^{-j+1})$ for all $j \in \mathbb{N}$.

2.5.1 The bumping function

By similar arguments as above, let us fix $w = (w_1, w_2) \in b\Omega$ and $w_1 \ne 0$. Denote by $A_k(w)$ the subset of U satisfies

$$A_k(w) = \left\{ z \in U | \frac{1}{2} \le 2^k |z_1 - w_1| \le 1) \right\}$$

and let

$$\psi_k(z, w) = \chi(2^k|z_1 - w_1|) + \frac{1}{\text{diam}U}|z_2 - w_2|.$$

We define

$$P(z, w) = \sum_{k=N}^{\infty} F(2^{-k-1})\psi_k(z, w)\Phi_{F(2^{-k})}(z)$$

and

$$\rho(z, w) = r(z) + \epsilon P(z, w).$$

and we need to prove that S_w is a pseudoconvex hypersurface. Compute the Hessian of $P(z, w)$, similiar as (2.3.7) we can get

$$\begin{aligned} i\partial_z\bar{\partial}_z P \gtrsim \left(-F(2^{-k})2^{2(k-1)} + cF(2^{-k-1})2^{2k}\right)|u_1|^2 - F(2^{-k-1})|u_2|^2 \\ + \chi(2^k|z_1 - w_1|)\partial_z\bar{\partial}_z\Phi_{F(2^{-k-1})}(z)(u, \bar{u}) \quad (2.5.2) \end{aligned}$$

for $z \in A_k(w) \cap S_{F(2^{-k})}$. For any $z \in (S_w \setminus \{w\}) \cap A_j(w)$, we have

$$0 < r(z) = -\epsilon P(z, w) \lesssim \epsilon F(2^{-j}) \leq F(2^{-j}).$$

That implies $z \in S_{F(2^{-j})}$. Therefore, we also obtain $S_w \subset \bigcup_{k=N}^{\infty} \big(A_k(w) \cap S_{F(2^{-k})}\big)$. Hence, compute the Hessian of $\rho(z, w)$, we obtain

$$\begin{aligned} i\partial_z\bar{\partial}_z\rho(z,w)(u,\bar{u}) &= \big(i\partial\bar{\partial}r(z) + \epsilon i\partial_z\bar{\partial}_z P(z,w)\big)(u,\bar{u}) \\ &= \left(\frac{\partial^2 F(|z_1|)}{\partial z_1 \partial \bar{z}_1} + \epsilon\left(-F(2^{-k})2^{2(k-1)} + cF(2^{-k-1})2^{2k}\right)\right)|u_1|^2 \qquad (2.5.3) \\ &\quad + \left(1 + \epsilon\left(-F(2^{-k-1})\right)\right)|u_2|^2. \qquad (2.5.4) \end{aligned}$$

We see that it is suffiticient to consider the Hessian of $\rho(., w)$ on z_1 variable since the term in the last line (2.5.4) is positive when we choose a sufficiently small ϵ.

For any $z \in U$, there exists k such that z belongs to the annulus $A_k(w)$. Therefore, we can write the Hessian of ρ in z_1 variable as

$$\frac{\partial^2 \rho(z,w)}{\partial z_1 \partial \bar{z}_1} \gtrsim \frac{F(|z_1|)}{|z_1|^2} - \epsilon F(2^{-k})2^{2(k-1)} + F(2^{-k-1})2^{2k}$$

for some $k \in \mathbb{N}$.

For $w \neq 0$, we can find on $\mathbb{C}$ such a z_1^* satisfies $|z_1^*| = |z_1^* - w_1| = \frac{1}{2}|w_1|$; z_1^* is the center of the real line connecting 0 and w_1. Let $k_w \in \mathbb{N}$ satisfy $2^{-k_w} < |w_1 - z_1^*| \leq 2^{-k_w - 1}$. Denote by $B_w \subset \mathbb{C}$ the ball center at w_1 with radian 2^{-k_w}. Note that for any $z_1 \in B_w$, we have $|z_1| > 2^{-k_w}$ by triangle inequality.

We know that $F(t)/F(2t)$ is an increasing function, moreover, and $F(t)/F(2t)|_{t=0} = 0$; that implies if we choose a small ϵ such that $\epsilon := F\left(\frac{|w_1|}{4}\right)/F\left(\frac{|w_1|}{2}\right) = F(\frac{|w_1|}{\sqrt{12}})$ then we can get for any $k < k_w$, $F(2^{-k}) > \epsilon F(2^{k-1})$. Now, consider $z \in S_w$, there are two cases.

For the first case, if $z_1 \notin B_w$ then $|z_1 - w_1| > 2^{-k_w}$. It implies that z belongs to an annulus $A_k(w)$ for which $k < k_w$ and then $-\epsilon F(2^{-k+1})2^{2k} + F(2^{-k})2^{2(k+1)}$ is positive.
On the other hand, if $z \in S_w$ satisfies $z_1 \in B_w$ then $z_1 \in A_k(w)$ for such $k > k_w$. Furthermore, following above notation we can have $|z_1| > 2^{-k_w}$ and it gives $\frac{F(|z_1|)}{|z_1|^2} > \frac{F(|2^{-k}|)}{2^{2k}}$ since $F(t)/t^2$ is an increasing function.

In conclude, we have the Hessian of $\rho(z,w)$ is positive for $0 \neq w \in b\Omega$. It means S_w is pseudoconvex.

For $w_1 = 0$, we define

$$\rho(z,w) = r(z) - \epsilon\left(F^2(|z_1|) + |z_2 - w_2|^2\right)$$

for any $\epsilon > 0$ then we can have the hypersurface $S_0 = \{z \in U | \rho(z,0) = 0\}$ is also pseudoconvex and it touches $\bar{\Omega}$ from outside of Ω at $z = 0$.

By the construction of $\rho(z,w)$ and following the proof on Section 2.3, we can prove that ρ satisfies the following properties:

(i) $\rho(w,w) = 0$.

(ii) $\rho(z,w) \lesssim -F(|w_1|)F(|z_1 - w_1|)$ for $z \in U \cap \Omega$ and $w \in U \cap (b\Omega \setminus \{(0,w_2)\})$.
$\rho(z,w) \lesssim -F(|z_1 - w_1|)^2$ for $z \in U \cap \Omega$ and $w \in U \cap (b\Omega \cap \{(0,w_2)\})$.

(iii) $\rho(z,\pi(z)) \gtrsim -\delta_{b\Omega}(z)$ for $z \in U \cap \Omega$, where $\pi(z)$ is the projection of z to the boundary.

(iv) $|d_z\rho(z,w)| \approx 1$ on S_w.

(v) S_w is pseudoconvex.

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