Independence and counting problems in Combinatorics and Number theory

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Abstract

The method of hypergraph containers has become a very important tool for dealing with problems which can be phrased in the language of independent sets in hypergraphs. This method has applications to numerous problems in combinatorics and other areas. In this thesis we consider examples of such problems; in particular problems concerning sets avoiding solutions to a given system of linear equations $\mathcal{L}$ (known as $\mathcal{L}$-free sets) or graphs avoiding copies of a given graph H (H-free graphs).

First we attack a number of questions relating to $\mathcal{L}$-free sets. For example, we give various bounds on the number of maximal $\mathcal{L}$-free subsets of $[n]$ for three-variable homogeneous linear equations $\mathcal{L}$.

We then use containers to prove results corresponding to problems concerning tuples of disjoint independent sets in hypergraphs. In particular we generalise the random Ramsey theorem of Rödl and Ruciński by providing a resilience analogue. We obtain similar results for $\mathcal{L}$-free sets.

Finally we consider the Maker-Breaker game where Maker's aim is to obtain a solution to a given system of linear equations $\mathcal{L}$ amongst a random set of integers. We determine the threshold probability for this game for a large class of systems $\mathcal{L}$.

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Chapter 1

Introduction

1.1 Independent sets in hypergraphs

Let $\mathcal{X}$ be an algebraic object, such as a group, a field, or a subset of one of these such as the set of positive integers $[n] := \{1, \ldots, n\}$. Many famous questions in additive combinatorics concern the combinatorial properties of a subset $X \subseteq \mathcal{X}$ which satisfy (or fail to satisfy) some given algebraic structure. Two key examples of this are finding the size of an extremal set, and counting the number of such sets.

Consider the example of *sum-free* sets, that is, a set which contains no solutions to $x + y = z$. It is easy to see that the largest sum-free set $X \subseteq [n]$ has size $\lceil n/2 \rceil$. (One example of an extremal set here is the set of odd numbers, which is sum-free since two odd numbers can never sum to another odd number.) Indeed, a classical question is to ask about the size of an extremal set avoiding solutions in $[n]$ to a given system of linear equations; the most famous of these concern sets not containing arithmetic progressions. Note arithmetic progressions can be encoded as solutions to a system of linear equations.

Similarly to above, two famous questions in extremal graph theory concern finding the size of an extremal graph, or counting the number of graphs, which have (or do not have) a given property $\mathcal{P}$. For example, Mantel's theorem [80] says that all triangle-free graphs on n vertices have at most $\lfloor n^2/4 \rfloor$ edges, where the extremal example is a complete bipartite graph with two vertex classes of equal size. Generally, the size of an extremal H-free graph is given asymptotically by the Erdős-Stone-Simonovits theorem [37], while a result of Erdős, Frankl and Rödl [34] determines asymptotically the number of H-free graphs on n vertices.

Many combinatorial problems, such as those described so far, can be phrased in terms of independent sets in hypergraphs. Consider the examples already considered; let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H}) := [n]$ and a hyperedge for each solution to $x + y = z$ in $[n]$ (note that there are edges of size two and three). Then the independent sets in $\mathcal{H}$ (subsets of $V(\mathcal{H})$ which do not contain any edge of $\mathcal{H}$) correspond to the sum-free subsets of $[n]$. Alternatively, let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H}) := E(K_n)$ (the set of all edges of the complete graph K_n) and a hyperedge for each triple of edges in K_n giving rise to a triangle. Here, the independent sets in $\mathcal{H}$ correspond to triangle-free subgraphs of K_n.

The two key questions mentioned in each topic can now be formulated as follows:

(i) What is the size $\alpha(\mathcal{H})$ of the largest independent set in $\mathcal{H}$?

(ii) What is the total number $i(\mathcal{H})$ of independent sets in $\mathcal{H}$?

Observe that since any subset of an independent set is still independent, there are at least $2^{\alpha(\mathcal{H})}$ independent sets in $\mathcal{H}$. A trivial upper bound can be found by counting all of the subsets of $V(\mathcal{H})$ of size at most $\alpha(\mathcal{H})$. For many hypergraphs $\mathcal{H}$, it turns out that the lower bound is closer to the truth, and so most efforts have gone into trying to improve the upper bound. For example, Alon [2] proved a conjecture of Granville (see [2]) from 1988 that when $\mathcal{H}$ is an n-vertex, d-regular graph, then $i(\mathcal{H}) \leqslant 2^{(1+O(d^{-0.1}))n/2}$ (note that here $\alpha(\mathcal{H}) \leqslant n/2$).

1.1.1 Containers

The method of containers, which was first used by Kleitman and Winston [68, 69] and Sapozhenko [103, 104, 105], has become a very good tool for establishing better upper bounds for $i(\mathcal{H})$. In general a 'container theorem' roughly states the following: There exists a family $\mathcal{C}(\mathcal{H})$ of subsets of $V(\mathcal{H})$ known as 'containers' such that

(i) Each container is 'almost' an independent set;

(ii) Every independent set in $\mathcal{H}$ is a subset of a container $C \in \mathcal{C}(\mathcal{H})$;

(iii) There are 'not too many' containers, i.e. $|\mathcal{C}(\mathcal{H})|$ is small;

(iv) No container is 'large'.

So one may choose a container $C \in \mathcal{C}(\mathcal{H})$ and bound the number of independent sets which it contains (in general this number should not be too big since no container is large). Then since every independent set lies in some container, we obtain an upper bound for the total number of independent sets by multiplying by the number of containers. Often this approach gives good upper bounds on $i(\mathcal{H})$, in particular as in many applications of the method there are a sufficiently small number of containers. The container method is often useful in the probabilistic setting for bounding the size of the largest independent set.

The original method used by Kleitman and Winston involved the use of an algorithm, in which a container method for graphs was implicitly stated within. They used it to obtain upper bounds on the number of lattices, and graphs without cycles of length four. In 2004 Green [48] developed a method of containers using Fourier analysis, for counting the number of subsets of $[n]$ avoiding solutions to a given linear equation. Recently Balogh, Morris and Samotij [8], and independently Saxton and Thomason [106] developed very general container theorems for independent sets in hypergraphs; both of which have seen numerous applications to a wide range of problems. These include problems in Ramsey theory, combinatorial number theory, positional games and list colourings of graphs. See the very recent survey article by Balogh, Morris and Samotij [9] for a detailed history and a number of illustrative examples of the method of hypergraph containers. Throughout this thesis we will apply the container method to a range of problems.

1.1.2 Removal lemmas

An important tool which often comes side by side with a container theorem is a so-called 'removal lemma'. This is often used in the step of the container method mentioned above, where one must bound the number of independent sets in a given container. The graph removal lemma states that for any graphs G, H on n, h vertices respectively, and any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, H) > 0$ such that if G contains at most δn^h copies of H, then it may be made H-free by removing at most εn^2 edges. By considering the hypergraph $\mathcal{H}$ with vertex set $E(G)$ and edge set corresponding to copies of H in G, we observe that an independent set in $\mathcal{H}$ is an H-free subgraph of G, and G itself is 'almost' an independent set.

The triangle removal lemma (i.e. H is a triangle) was first used by Ruzsa and Szemerédi [102] in 1976. The general graph removal lemma was first explicitly stated by Alon, Duke, Lefmann, Rödl and Yuster [3] and by Füredi [43] in 1994. As with the container method, there has been much work on extending the graph removal lemma to a result for hypergraphs, most notably by Gowers [44, 45] and independently by Nagle, Rödl, Schacht and Skokan [84, 97]. Green also proved a removal lemma for abelian groups using Fourier analysis [48]. This combined with Green's container lemma gives an immediate result for the number of subsets of $[n]$ avoiding solutions to a given linear equation. This removal lemma was also generalised to systems of linear equations by Král', Serra and Vena [74].

1.2 Our work

In Chapter 2, we state the aforementioned container and removal lemmas of Green, the general container result of Balogh, Morris and Samotij, and the general removal lemma of Král', Serra and Vena. All of these results will explicitly be required for proofs of results within Chapters 3–6. We now describe the content of these four chapters.

1.2.1 Solution-free sets of integers

Given a linear equation $\mathcal{L}$ of the form $a_1x_1 + \cdots + a_kx_k = b$, we call a set $X \subseteq [n]$ *weakly $\mathcal{L}$-free* if it does not contain any 'non-trivial' solutions to $\mathcal{L}$ in $[n]$ (see the notation section below for a full definition). In Chapters 3 and 4 we prove a number of results concerning weakly $\mathcal{L}$-free subsets of $[n]$ where $\mathcal{L}$ is a homogeneous linear equation in *three variables*. In particular, our work is motivated by the following general questions.

(i) What is the size of the largest weakly $\mathcal{L}$-free subset of $[n]$?

(ii) How many weakly $\mathcal{L}$-free subsets of $[n]$ are there?

(iii) How many maximal weakly $\mathcal{L}$-free subsets of $[n]$ are there?

A weakly $\mathcal{L}$-free set $X \subseteq [n]$ is *maximal* if it is not possible to add any other element of $[n]$ to X so that X remains weakly $\mathcal{L}$-free. Notice that (iii) (as we showed with (i) and (ii) earlier) can also be phrased in terms of independent sets in hypergraphs; if $\mathcal{H}$ has vertex set $[n]$ and the edges of $\mathcal{H}$ correspond to solutions to $\mathcal{L}$, then the corresponding question is to count the number of maximal independent sets in $\mathcal{H}$.

We make progress on all three of these questions. For each question we use tools from graph theory; for (i) and (ii) our methods are somewhat elementary. For (iii) our method is more involved and utilises container and removal lemmas of Green [48]. In Chapter 3 we focus mainly on equations of the form $px + qy = z$ for some fixed positive integers

p, q with $p \geqslant 2$. In Chapter 4 we focus on equations of the form $px + qy = rz$ for some fixed positive integers $p \geqslant q \geqslant r$, and also obtain some generalisations of our results to equations with more than three variables. These two chapters are based respectively on the content of two papers with Andrew Treglown [56, 57].

1.2.2 Ramsey properties of graphs and the integers

Rather than forbidding a solution to some system of linear equations, one may instead choose to assign colours to numbers in $[n]$, and then see if there is any guarantee of there being a monochromatic solution. A famous result of Schur [110] states that for any $r \in \mathbb{N}$, if n is sufficiently large, then however $[n]$ is r-coloured, we can find a monochromatic triple $\{x, y, z\}$ with $x, y, z \in [n]$ such that $x + y = z$. Van der Waerden obtained a similar result [120] for arithmetic progressions. This was followed by the generalisation by Rado [88] which describes all systems of linear equations $\mathcal{L}$ for which, for any $r \in \mathbb{N}$, if n is sufficiently large, then however $[n]$ is r-coloured, we can find a monochromatic solution to $\mathcal{L}$ in $[n]$. In the graph setting, Ramsey's theorem [89] states that for any $r \in \mathbb{N}$ and graph H, if n is sufficiently large, then however we r-colour the edges of K_n, there is a monochromatic copy of H.

Recently there has been a trend of obtaining extremal results in sparse random sets or graphs. For example, one may ask about Rado-type properties of $[n]_p$ (each element of $[n]$ is included with probability p independently of all other elements). Given a fixed system of linear equations $\mathcal{L}$ which satisfies Rado's partition theorem above, and integer $r \geqslant 2$, how small must we make p so that with high probability (with probability one as n tends to infinity; we will use the abbreviation w.h.p.) $[n]_p$ no longer has the property that however r-coloured, there is always a monochromatic solution to $\mathcal{L}$? Results of Friedgut, Rödl and Schacht [42] and Rödl and Ruciński [93] together determine a *threshold* p_0 which depends only on $\mathcal{L}$. Similarly, a famous result of Rödl and Ruciński [90, 91, 92] determines the threshold p_0 for a given graph H, for when the Erdős–Rényi random graph $G_{n,p}$ has the property that however its edges are r-coloured, there is always a monochromatic copy of H. (Recall that $G_{n,p}$ has vertex set $[n]$ in which each possible edge is present with probability p, independently of all other edges.)

Consider the hypergraphs with edge sets corresponding to solutions to a given system of linear equations or copies of a given graph H described earlier. Here a tuple of disjoint independent sets describes a tuple of disjoint solution-free sets/disjoint H-free graphs. In Chapter 5, we use the container method to prove results that correspond to problems concerning tuples of disjoint independent sets in hypergraphs. That is, we are able to tackle problems relating to the Rado property of sets of integers, and Ramsey properties of graphs and hypergraphs. In particular, we strengthen the random Rado theorem of Friedgut, Rödl and Schacht [42] described above by proving a so-called resilience version of the result. We also generalise the random Ramsey theorem of Rödl and Ruciński described above by providing a resilience analogue. This result also implies the *random version of Turán's theorem* due to Conlon and Gowers [28] and Schacht [108]. We also obtain hypergraph and asymmetric generalisations and counting results. This chapter is based on joint work with Katherine Staden and Andrew Treglown [55].

1.2.3 The Maker-Breaker Rado game on a random set of integers

Recently there has been a number of results which link extremal results in sparse random graphs or sets to results in Maker-Breaker games (see, for example [75, 86]). This is due in part to it being possible to phrase both problems in the language of independent sets in hypergraphs.

Given a system of linear equations $Ax = b$, the Maker-Breaker (A, b)-game on a set of integers X is the game where Maker and Breaker take turns claiming previously unclaimed integers from X, and Maker's aim is to obtain a solution to $Ax = b$, whereas Breaker's aim is to prevent this. We can view the game as players taking turns claiming vertices from the hypergraph with edges corresponding to solutions to $Ax = b$. Maker's aim here is to obtain an edge of the hypergraph; if Maker fails to claim an edge, her set is an independent set and therefore a solution-free set.

When $X := [n]_p$, we determine the threshold probability p_0 for when the game is Maker or Breaker's win, for a large class of systems of linear equations. This class includes but is not limited to all single linear equations. The Maker's win statement also extends to a much wider class of systems of linear equations, which include those which satisfy Rado's partition theorem. Its proof involves the use of the container method. The proof of the Breaker's win statement draws on the method used by Rödl and Ruciński [93] to prove the *0-statement* of the random Rado theorem. This chapter is based on the author's recently submitted paper [54].

1.3 Notation and preliminaries

1.3.1 Solutions to systems of linear equations

Throughout the thesis, unless otherwise stated, we will assume that A is a fixed integer-valued matrix of dimension $\ell \times k$ and b a fixed integer-valued vector of dimension ℓ. We will let $\mathcal{L}(A, b)$ denote the associated system of linear equations $Ax = b$, noting that for brevity we will simply write $\mathcal{L}$ if it is clear from the context which matrix A and vector b it refers to.

It is important to observe that in the problem of avoiding solutions to a given system of linear equations, there are a few different notions of what constitutes a solution. For example in the case of sum-free sets, should we care or not about whether we include solutions with repetition, e.g. $(3, 3, 6)$, or not? Historically most problems have focused on studying the case where repetition is not allowed, however in some cases (especially

Chapters 3 and 4) we wish to allow so called non-trivial solutions, which we will define shortly.

Given a set of integers X, first let $S(\mathcal{L}^s, X)$ be the set of all vectors $x = (x_1, \dots, x_k) \in X^k$ such that $Ax = b$ (i.e. the vector x is a solution to $\mathcal{L}$ in X). If $S(\mathcal{L}^s, X)$ is empty then we call X *strongly $\mathcal{L}$-free.* If for a solution x, additionally the x_i are pairwise distinct, we call x a *k-distinct solution* to $\mathcal{L}$ in X. Let $S(\mathcal{L}^d, X)$ refer to the set of all k-distinct solutions to $\mathcal{L}$ in X. If $S(\mathcal{L}^d, X)$ is empty then we call X *distinct $\mathcal{L}$-free.*

Generally, we call a system of linear equations $\mathcal{L}$ *homogeneous* if $b = 0$. In the case where we have a $1 \times k$ matrix, we simply have a linear equation of the form

$$a_1x_1 + \cdots + a_kx_k = b \tag{1.3.1}$$

where $a_1, \dots, a_k, b \in \mathbb{Z}$. If

$$\sum_{i\in[k]} a_i = b = 0$$

then we say that $\mathcal{L}$ is *translation-invariant.* Let $\mathcal{L}$ be a translation-invariant linear equation. Then notice that $(x, \dots, x)$ is a 'trivial' solution of (1.3.1) for any x. More generally, a solution $(x_1, \dots, x_k)$ to $\mathcal{L}$ is said to be *trivial* if there exists a partition $P_1, \dots, P_\ell$ of $[k]$ so that:

(i) $x_i = x_j$ for every i, j in the same partition class P_r;

(ii) For each $r \in [\ell]$, $\sum_{i\in P_r} a_i = 0$.

Let $S(\mathcal{L}^w, X)$ refer to the set of all non-trivial solutions to $\mathcal{L}$ in X. If $S(\mathcal{L}^w, X)$ is empty then call X *weakly $\mathcal{L}$-free.* This definition of non-trivial was introduced by Ruzsa in [100]. Note that for non-translation-invariant $\mathcal{L}$ there are no trivial solutions, and so the definition of weakly $\mathcal{L}$-free coincides with the definition of strongly $\mathcal{L}$-free. A definition of non-trivial for $\ell \times k$ matrices is given in [99], though we do not state it here since this case is not studied in this thesis. Note that we have

$$S(\mathcal{L}^d, X) \subseteq S(\mathcal{L}^w, X) \subseteq S(\mathcal{L}^s, X).$$

For shorthand notation, for $t \in \{d, w, s\}$ we will use $\mathcal{L}^t$*-free* for distinct/weakly/strongly $\mathcal{L}$-free sets respectively. We will also use $\mu(n, \mathcal{L}^t)$ to denote the size of the largest $\mathcal{L}^t$-free subset of $[n]$; observe that

$$\mu(n, \mathcal{L}^s) \leqslant \mu(n, \mathcal{L}^w) \leqslant \mu(n, \mathcal{L}^d).$$

If the equation $\mathcal{L}$ (and type of solutions we are interested in) is clear from the context, then we simply say X is *solution-free*. In Chapters 3 and 4 we consider weakly $\mathcal{L}$-free sets, whereas in Chapters 5 and 6 we mainly consider distinct $\mathcal{L}$-free sets.

We also require definitions for r-colourings of sets of integers. Fix $r \in \mathbb{N}$ and $t \in \{d, w, s\}$. For each $i \in [r]$ let A_i be a matrix of dimension $\ell_i \times k_i$ and b_i a vector of dimension ℓ_i. We say a set X is $(\mathcal{L}_1^t, \dots, \mathcal{L}_r^t)$-free if there exists an r-colouring of X so that for each $i \in [r]$, the subset $Y \subseteq X$ with colour i is $\mathcal{L}_i^t$-free. We write $\mu(n, \mathcal{L}_1^t, \dots, \mathcal{L}_r^t)$ to denote the size of the largest $(\mathcal{L}_1^t, \dots, \mathcal{L}_r^t)$-free subset of $[n]$, and write $\mu(n, \mathcal{L}^t, r) := \mu(n, \mathcal{L}_1^t, \dots, \mathcal{L}_r^t)$ if $\mathcal{L} = \mathcal{L}_i$ for all $i \in [r]$.

We also include some defintions which are important in both Chapters 5 and 6. We call a system of linear equations $\mathcal{L}$ (and the matrix A in the case where $b = 0$) *irredundant* if there exists a k-distinct solution to $Ax = b$ in $\mathbb{N}$, and *redundant* otherwise.

Call $\mathcal{L}$ (and again A if $b = 0$) *partition regular* if for any finite colouring of $\mathbb{N}$, there is always a monochromatic solution (of any kind) to $Ax = b$.

Let $(*)$ be the following matrix property:

$(*)$ Under Gaussian elimination A does not have any row which consists of precisely two non-zero rational entries.

Index the columns of an $\ell \times k$ matrix A by $[k]$. For a partition $W \dot\cup \overline{W} = [k]$ of the columns of A, we denote by $A_{\overline{W}}$ the matrix obtained from A by restricting to the columns indexed by $\overline{W}$. Let $\mathrm{rank}(A_{\overline{W}})$ be the rank of $A_{\overline{W}}$, where $\mathrm{rank}(A_{\overline{W}}) = 0$ for $\overline{W} = \emptyset$. We set

$$m(A) := \max_{\substack{W \dot\cup \overline{W} = [k] \\ |W| \geqslant 2}} \frac{|W| - 1}{|W| - 1 + \mathrm{rank}(A_{\overline{W}}) - \mathrm{rank}(A)}. \tag{1.3.2}$$

We remark that the denominator of $m(A)$ is strictly positive provided that A is irredundant and satisfies $(*)$. (Note that in Section 5.3.1 we will show that irredundant partition regular matrices are a strict subclass of irredundant matrices which satisfy $(*)$.)

1.3.2 Notation

Let $\mathcal{H}$ be a (hyper)graph. We write $V(\mathcal{H})$, $E(\mathcal{H})$ and $\mathcal{I}(\mathcal{H})$ to represent the vertex set, edge set and set of independent sets of $\mathcal{H}$, and $v(\mathcal{H})$, $e(\mathcal{H})$ and $i(\mathcal{H})$ for the respective numbers of each of these. Consider any subset $X \subseteq V(\mathcal{H})$. Let $\mathcal{H}[X]$ denote the induced subgraph of $\mathcal{H}$ on the vertex set X and $\mathcal{H} \setminus X$ denote the induced subgraph of $\mathcal{H}$ on the vertex set $V(\mathcal{H}) \setminus X$. For an edge set $Y \subseteq E(\mathcal{H})$, we define $\mathcal{H} - Y$ to be hypergraph with vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H}) \setminus Y$.

For a set X and a positive integer x, we define $\binom{X}{x}$ to be the set of all subsets of X of size x, and we define $\binom{X}{\leqslant x}$ to be the set of all subsets of X of size at most x. We use $\mathcal{P}(X)$ to denote the powerset of X, that is, the set of all subsets of X. If B is a family of subsets of X, then we define $\overline{B}$ to be the complement family, that is, precisely the subsets of X which are not in B.

Given a hypergraph $\mathcal{H}$, for each $T \subseteq V(\mathcal{H})$, we define $\deg_{\mathcal{H}}(T) := |\{e \in E(\mathcal{H}) : T \subseteq e\}|$, and let $\Delta_\ell(\mathcal{H}) := \max\{\deg_{\mathcal{H}}(T) : T \subseteq V(\mathcal{H}) \text{ and } |T| = \ell\}$.

We write $x = a \pm b$ to say that the value of x is some real number in the interval $[a - b, a + b]$.

For two constants $\alpha, \beta > 0$ we use the notation $\alpha \ll \beta$ (often within a hierarchy of constants) to mean that α is bounded by some unspecified function of β, so that the calculations we wish to hold concerning α and β do indeed hold.

1.3.3 Probabilistic tools

We will need the Markov inequality and Chernoff bounds of the following form (see e.g. [64, Theorem 2.1, Corollary 2.3]).

Proposition 1.1. *Let X be a non-negative random variable. Then for all $t > 0$ we have $\mathbb{P}[X \geqslant t] \leqslant \frac{\mathbb{E}[X]}{t}$.*

Proposition 1.2. *Suppose X has binomial distribution.*

(i) For every $\lambda \geqslant 0$, we have

$$\mathbb{P}[X > \mathbb{E}[X] + \lambda] \leqslant \exp\left(-\frac{\lambda^2}{2(\mathbb{E}[X] + \lambda/3)}\right).$$

(ii) For every $0 < \varepsilon \leqslant 3/2$, we have

$$\mathbb{P}[|X - \mathbb{E}[X]| \geqslant \varepsilon\mathbb{E}[X]] \leqslant 2\exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right).$$

Chapter 2

Container and removal lemmas

Here we introduce the container and removal lemmas which will be required in the thesis. Recall that the method of hypergraph containers was greatly advanced by the papers of Balogh, Morris and Samotij [8] and Saxton and Thomason [106]. The first result we require here is that of Balogh, Morris and Samotij. First we require some definitions. Let $\mathcal{H}$ be a k-uniform hypergraph. A family of sets $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ is called *increasing* if it is closed under taking supersets; in other words for every $A, B \subseteq V(\mathcal{H})$, if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$. Suppose $\mathcal{F}$ is an increasing family of subsets of $V(\mathcal{H})$ and let $\varepsilon \in (0,1]$. We say that $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-*dense* if $e(\mathcal{H}[A]) \geqslant \varepsilon e(\mathcal{H})$ for every $A \in \mathcal{F}$.

Theorem 2.1 ([8], Theorem 2.2). *For every $k \in \mathbb{N}$ and all positive c and ε, there exists a positive constant C such that the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph and let $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ be an increasing family of sets such that $|A| \geqslant \varepsilon v(\mathcal{H})$ for all $A \in \mathcal{F}$. Suppose that $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-dense and $p \in (0,1)$ is such that, for every $\ell \in [k]$,*

$$\Delta_\ell(\mathcal{H}) \leqslant c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

Then there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leqslant Cp \cdot v(\mathcal{H})}$ and functions $f : \mathcal{S} \to \overline{\mathcal{F}}$ and $g : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for every $I \in \mathcal{I}(\mathcal{H})$, we have that $g(I) \subseteq I$ and $I \setminus g(I) \subseteq f(g(I))$.

Using the above notation, we refer to the set $\mathcal{C} := \{f(g(I)) \cup g(I) : I \in \mathcal{I}(\mathcal{H})\}$ as a set of *containers* and the $g(I) \in \mathcal{S}$ as *fingerprints*. We use this result to derive an analogous result for r-tuples of disjoint independent sets (see Theorems 5.14 and 5.15 in Chapter 5).

As mentioned in the introduction, container results often come side by side with a removal lemma or supersaturation lemma. The removal lemma from which other results we will require are a consequence of, is the following result of Král', Serra and Vena [74].

Lemma 2.2 ([74], Theorem 2). *Let A be an $\ell \times k$ integer matrix of rank ℓ and b an integer vector of dimension ℓ. For every $\delta > 0$ there exist $n_0, \varepsilon > 0$ with the following property. Suppose $n \geqslant n_0$ is an integer and for $X \subseteq [n]$, there are at most $\varepsilon n^{k-\ell}$ solutions in $S(\mathcal{L}^s, X)$. Then we can write $X = B \cup C$ where B is strongly $\mathcal{L}$-free and $|C| \leqslant \delta n$.*

The result is stated with $b = 0$ in [74], though clearly can be extended to general b as stated above. Also within Lemma 2.2, since B is $\mathcal{L}^s$-free, it is also $\mathcal{L}^w$-free and $\mathcal{L}^d$-free. Thus, using little-o notation, we have $|X| \leqslant \mu(n, \mathcal{L}^w) + o(n)$ and $|X| \leqslant \mu(n, \mathcal{L}^d) + o(n)$. Hence we obtain the following r-colour supersaturation lemma.

Lemma 2.3. *Fix $r \in \mathbb{N}$, $t \in \{d, w, s\}$ and for each $i \in [r]$, let A_i be an $\ell_i \times k_i$ integer matrix of rank ℓ_i and b_i be an integer vector of dimension ℓ_i. For every $\delta > 0$ there exist $n_0, \varepsilon > 0$ with the following property. Suppose $n \geqslant n_0$ is an integer and $X \subseteq [n]$ is r-coloured, and $|X| \geqslant \mu(n, \mathcal{L}_1^t, \dots, \mathcal{L}_r^t) + \delta n$. Then there exists an $i \in [r]$ such that there are more than $\varepsilon n^{k_i - \ell_i}$ solutions in $S(\mathcal{L}_i^t, X)$ in colour i.*

We use Lemma 2.3 in Chapters 5 and 6. Note that Lemma 2.2 extends an earlier removal lemma of Green for single linear equations, which we state here since we will use it directly in Chapters 3 and 4.

Lemma 2.4 ([48]). *Fix a k-variable homogeneous linear equation $\mathcal{L}$. Suppose that $A \subseteq [n]$ is a set containing $o(n^{k-1})$ non-trivial solutions to $\mathcal{L}$. Then there exist B and C such that $A = B \cup C$ where B is weakly $\mathcal{L}$-free and $|C| = o(n)$.*

Finally, we also state Green's container lemma for single linear equations (Proposition 9.1 of [48]), since it will also be directly used in Chapters 3 and 4. Lemma 2.5(i)–(iii) is stated explicitly in Proposition 9.1 of [48]. Lemma 2.5(iv) follows as an immediate consequence of Lemma 2.5(i) and Lemma 2.4 above.

Lemma 2.5 ([48]). *Fix a k-variable homogeneous linear equation $\mathcal{L}$. There exists a family $\mathcal{F}$ of subsets of $[n]$ with the following properties:*

(i) Every $F \in \mathcal{F}$ has at most $o(n^{k-1})$ non-trivial solutions to $\mathcal{L}$.

(ii) If $S \subseteq [n]$ is weakly $\mathcal{L}$-free, then S is a subset of some $F \in \mathcal{F}$.

(iii) $|\mathcal{F}| = 2^{o(n)}$.

(iv) Every $F \in \mathcal{F}$ has size at most $\mu(n, \mathcal{L}^w) + o(n)$.

Note that Lemma 2.5 can be recovered from Theorem 2.1 and Lemma 2.2. In fact by using our r-colour container result (Theorem 5.15) we obtain an r-colour version of Lemma 2.5 (see Theorem 5.21).

Chapter 3

Solution-free sets of integers

3.1 Introduction

Recall from the introduction that a set $A \subseteq [n]$ is *weakly $\mathcal{L}$-free* if A does not contain any non-trivial solutions to $\mathcal{L}$ in $[n]$. We will use the notation $\mathcal{L}^w$-free for weakly $\mathcal{L}$-free sets, and throughout this chapter and the next, only consider weakly $\mathcal{L}$-free sets.

The notion of an $\mathcal{L}^w$-free set encapsulates many fundamental topics in combinatorial number theory. Recall that in the case when $\mathcal{L}$ is $x_1 + x_2 = x_3$ we call an $\mathcal{L}^w$-free set (or equivalently an $\mathcal{L}^s$-free set, since $x_1 + x_2 = x_3$ is not translation-invariant) a *sum-free set*. This is a notion that dates back to 1916 when Schur [110] proved that, if n is sufficiently large, any r-colouring of $[n]$ yields a monochromatic triple x, y, z such that $x + y = z$. *Sidon sets* (when $\mathcal{L}$ is $x_1 + x_2 = x_3 + x_4$) have also been extensively studied. For example, a classical result of Erdős and Turán [38] asserts that the largest Sidon set in $[n]$ has size $(1+o(1))\sqrt{n}$. In the case when $\mathcal{L}$ is $x_1+x_2 = 2x_3$ an $\mathcal{L}^w$-free set is simply a *progression-free* set. Roth's theorem [98] states that the largest progression-free subset of $[n]$ has size $o(n)$. In [100, 101], Ruzsa instigated the study of solution-free sets for general linear equations.

In this chapter we prove a number of results concerning $\mathcal{L}^w$-free subsets of $[n]$ where $\mathcal{L}$

is a homogeneous linear equation in *three variables*. In particular, our work is motivated by the following general questions:

(i) What is the size of the largest $\mathcal{L}^w$-free subset of $[n]$?

(ii) How many $\mathcal{L}^w$-free subsets of $[n]$ are there?

(iii) How many maximal $\mathcal{L}^w$-free subsets of $[n]$ are there?

We make progress on all three of these questions. For each question we use tools from graph theory; for (i) and (ii) our methods are somewhat elementary. For (iii) our method is more involved and utilises container and removal lemmas of Green [48].

3.1.1 The size of the largest solution-free set

As highlighted above, a central question in the study of $\mathcal{L}^w$-free sets is to establish the size $\mu(n, \mathcal{L}^w)$ of the largest $\mathcal{L}^w$-free subset of $[n]$. It is not difficult to see that the largest sum-free subset of $[n]$ has size $\lceil n/2 \rceil$, and this bound is attained by the set of odd numbers in $[n]$ and by the interval $[\lfloor n/2 \rfloor + 1, n]$.

When $\mathcal{L}$ is $x_1 + x_2 = 2x_3$, $\mu(n, \mathcal{L}^w) = o(n)$ by Roth's theorem. In fact, very recently Bloom [18] proved that there is a constant C such that every set $A \subseteq [n]$ with $|A| \geqslant Cn(\log \log n)^4 / \log n$ contains a three-term arithmetic progression. On the other hand, Behrend [15] showed that there is a constant $c > 0$ so that $\mu(n, \mathcal{L}^w) \geqslant n \exp(-c\sqrt{\log n})$. See [32, 50] for the best known lower bound on $\mu(n, \mathcal{L}^w)$ in this case.

More generally, it is known that $\mu(n, \mathcal{L}^w) = o(n)$ if $\mathcal{L}$ is homogeneous and translation-invariant, and $\mu(n, \mathcal{L}^w) = \Omega(n)$ otherwise (see [100]). For other (exact) bounds on $\mu(n, \mathcal{L}^w)$ for various linear equations $\mathcal{L}$ see, for example, [100, 101, 11, 31, 53].

In this chapter we mainly focus on $\mathcal{L}^w$-free subsets of $[n]$ for linear equations $\mathcal{L}$ of the form $px + qy = z$ where $p \geqslant 2$ and $q \geqslant 1$ are fixed integers. For such equations the set of $\mathcal{L}^w$-free sets are precisely the same as the set of $\mathcal{L}^s$-free sets since $\mathcal{L}$ is not translation-invariant. Also notice that for such $\mathcal{L}$, the interval $[\lfloor n/(p+q)\rfloor + 1, n]$ is an $\mathcal{L}^w$-free set. Our first result implies that this is the largest such $\mathcal{L}^w$-free subset of $[n]$.

Theorem 3.1. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q$ and $p \geqslant 2$, $p, q \in \mathbb{N}$. Let S be an $\mathcal{L}^w$-free subset of $[n]$, and let $\min(S) = \lfloor \frac{n}{p+q} \rfloor - t$ where t is a non-negative integer.*

(i) If $0 \leqslant t < (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$ then $|S| \leqslant \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q} t \rfloor$.

(ii) If $t \geqslant (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$ then $|S| \leqslant \frac{(q^2+1)n}{q^2+q+1}$ provided that $n \geqslant \frac{3(q^2+q+1)(q^3+p(q^2+q+1))}{q^2+1}$ and $n \geqslant \frac{5(q^2+q+1)(q^5+p(q^4+q^3+q^2+q+1))}{q^4+(p-1)q^3+q^2+1}$.

In both cases of Theorem 3.1 we observe that $|S| \leqslant n - \lfloor \frac{n}{p+q} \rfloor$, hence the following corollary holds.

Corollary 3.2. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q$ and $p \geqslant 2$, $p, q \in \mathbb{N}$. If n is sufficiently large depending on p and q, then $\mu(n, \mathcal{L}^w) = n - \lfloor \frac{n}{p+q} \rfloor$.*

Roughly, Theorem 3.1 implies that every $\mathcal{L}^w$-free subset of $[n]$ is 'interval like' or 'small'. In the case of sum-free subsets (i.e. when $p = q = 1$), a result of Deshouillers, Freiman, Sós and Temkin [30] provides very precise structural information on the sum-free subsets of $[n]$. Loosely speaking, they showed that a sum-free subset of $[n]$ is 'interval like', 'small' or consists entirely of odd numbers.

In the case when $p = q$, Corollary 3.2 was proven by Hegarty [53] (without a lower bound on n).

3.1.2 The number of solution-free sets

Write $f(n, \mathcal{L}^w)$ for the number of $\mathcal{L}^w$-free subsets of $[n]$. In the case when $\mathcal{L}$ is $x + y = z$, define $f(n) := f(n, \mathcal{L}^w)$.

By considering all possible subsets of $[n]$ consisting of odd numbers, one observes that there are at least $2^{n/2}$ sum-free subsets of $[n]$. Cameron and Erdős [23] conjectured that in fact $f(n) = \Theta(2^{n/2})$. This conjecture was proven independently by Green [47] and Sapozhenko [104]. In fact, they showed that there are constants C_1 and C_2 such that $f(n) = (C_i + o(1))2^{n/2}$ for all $n \equiv i \mod 2$.

Results from [71, 107] imply that there are between $2^{(1.16+o(1))\sqrt{n}}$ and $2^{(6.45+o(1))\sqrt{n}}$ Sidon sets in $[n]$. There are also several results concerning the number of so-called (k,ℓ)-sum-free subsets of $[n]$ (see, e.g., [17, 22, 109]).

More generally, given a linear equation $\mathcal{L}$, there are at least $2^{\mu(n,\mathcal{L}^w)}$ $\mathcal{L}^w$-free subsets of $[n]$. In light of the situation for sum-free sets one may ask whether, in general, $f(n,\mathcal{L}^w) = \Theta(2^{\mu(n,\mathcal{L}^w)})$. However, Cameron and Erdős [23] observed that this is false for homogeneous translation-invariant $\mathcal{L}$. In particular, given such an $\mathcal{L}^w$-free set, any translation of it is also $\mathcal{L}^w$-free.

Green [48] though showed that given a homogeneous linear equation $\mathcal{L}$, $f(n,\mathcal{L}^w) = 2^{\mu(n,\mathcal{L}^w)+o(n)}$ (where here the $o(n)$ may depend on $\mathcal{L}$). Our next result implies that one can omit the term $o(n)$ in the exponent for certain types of linear equation $\mathcal{L}$.

Theorem 3.3. *Fix $p, q \in \mathbb{N}$ where (i) $q \geqslant 2$ and $p > q(3q+2)/(2q-2)$ or (ii) $q = 1$ and $p \geqslant 3$. Let $\mathcal{L}$ denote the equation $px + qy = z$. Then*

$$f(n,\mathcal{L}^w) = \Theta(2^{\mu(n,\mathcal{L}^w)}).$$

3.1.3 The number of maximal solution-free sets

Given a linear equation $\mathcal{L}$, we say that $S \subseteq [n]$ is a *maximal $\mathcal{L}^w$-free subset of* $[n]$ if it is $\mathcal{L}^w$-free and it is not properly contained in another $\mathcal{L}^w$-free subset of $[n]$. Write $f_{\max}(n,\mathcal{L}^w)$ for the number of maximal $\mathcal{L}^w$-free subsets of $[n]$. In the case when $\mathcal{L}$ is $x + y = z$, define $f_{\max}(n) := f_{\max}(n,\mathcal{L}^w)$.

A significant proportion of the sum-free subsets of $[n]$ lie in just two maximal sum-free sets, namely the set of odd numbers in $[n]$ and the interval $[\lfloor n/2 \rfloor + 1, n]$. This led

Cameron and Erdős [24] to ask whether $f_{\max}(n) = o(f(n))$ or even $f_{\max}(n) \leqslant f(n)/2^{\varepsilon n}$ for some constant $\varepsilon > 0$. Łuczak and Schoen [79] answered this question in the affirmative, showing that $f_{\max}(n) \leqslant 2^{n/2-2^{-28}n}$ for sufficiently large n. Later, Wolfovitz [121] proved that $f_{\max}(n) \leqslant 2^{3n/8+o(n)}$. Very recently, Balogh, Liu, Sharifzadeh and Treglown [6, 7] proved the following: For each $1 \leqslant i \leqslant 4$, there is a constant C_i such that, given any $n \equiv i \mod 4$, $f_{\max}(n) = (C_i + o(1))2^{n/4}$.

Except for sum-free sets, the problem of determining the number of maximal solution-free subsets of $[n]$ remains wide open. In this chapter we give a number of bounds on $f_{\max}(n, \mathcal{L}^w)$ for homogeneous linear equations $\mathcal{L}$ in three variables. The next result gives a general upper bound for such $\mathcal{L}$. Given a three-variable linear equation $\mathcal{L}$, an *$\mathcal{L}$-triple* is a multiset $\{x, y, z\}$ which forms a solution to $\mathcal{L}$. (In other words, the set of all $\mathcal{L}$-triples in X corresponds to $S(\mathcal{L}^s, X)$.) Let $\mu^*(n, \mathcal{L})$ denote the number of elements $x \in [n]$ that do not lie in *any* $\mathcal{L}$-triple in $[n]$.

Theorem 3.4. *Let $\mathcal{L}$ be a fixed homogenous three-variable linear equation. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 3^{(\mu(n,\mathcal{L}^w)-\mu^*(n,\mathcal{L}))/3+o(n)}.$$

Theorem 3.4 together with the aforementioned result of Green shows that $f_{\max}(n, \mathcal{L}^w)$ is significantly smaller than $f(n, \mathcal{L}^w)$ for all homogeneous three-variable linear equations $\mathcal{L}$ that are not translation-invariant. So in this sense it can be viewed as a generalisation of the result of Łuczak and Schoen. The proof of Theorem 3.4 is a simple application of container and removal lemmas of Green [48]. The same idea was used to prove results in [10, 6, 7]. Although at first sight the bound in Theorem 3.4 may seem crude, perhaps surprisingly there are equations $\mathcal{L}$ where the value of $f_{\max}(n, \mathcal{L}^w)$ is close to this bound (see Proposition 3.19 in Section 3.5).

On the other hand, the following result shows that there are linear equations where the bound in Theorem 3.4 is far from tight.

Theorem 3.5. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q \geqslant 2$ are integers so that $p \leqslant q^2 - q$ and $\gcd(p, q) = q$. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 2^{(\mu(n,\mathcal{L}^w)-\mu^*(n,\mathcal{L}))/2+o(n)}.$$

In the case when $\mathcal{L}$ is the equation $2x + 2y = z$ we provide a matching lower bound. Again though, we suspect there are equations $\mathcal{L}$ where the bound in Theorem 3.5 is far from tight. The proof of Theorem 3.5 applies Theorem 3.1 as well as the container and removal lemmas of Green [48].

We also provide another upper bound on $f_{\max}(n, \mathcal{L}^w)$ for a more general class of linear equations.

Theorem 3.6. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q$, $p \geqslant 2$ and $p, q \in \mathbb{N}$. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 2^{\mu(\lfloor\frac{n-p}{q}\rfloor,\mathcal{L}^w)+o(n)}.$$

Further, if $q \geqslant 2$ and $p > q(3q - 2)/(2q - 2)$ or $q = 1$ and $p \geqslant 3$ then

$$f_{\max}(n, \mathcal{L}^w) = O(2^{\mu(\lfloor\frac{n-p}{q}\rfloor,\mathcal{L}^w)}).$$

In Section 3.5 we provide lower bounds on $f_{\max}(n, \mathcal{L}^w)$ for all equations $\mathcal{L}$ of the form $px + qy = z$ where $p, q \geqslant 2$ are integers; see Proposition 3.21.

Our results suggest that, in contrast to the case of $f(n, \mathcal{L}^w)$, it is unlikely there is a 'simple' general asymptotic formula for $f_{\max}(n, \mathcal{L}^w)$ for all homogeneous linear equations $\mathcal{L}$. It would be extremely interesting to make further progress on this problem.

The chapter is organised as follows. In the next section we collect together a number of useful tools. In Section 3.3 we prove Theorem 3.1. Theorem 3.3 is proven in Section 3.4.

We prove our results on the number of maximal $\mathcal{L}^w$-free sets in Section 3.5. In Chapter 4 we obtain further results on the number of maximal $\mathcal{L}^w$-free sets, and so see Section 4.6 for a note on which of our results produces the best upper bound on $f_{\max}(n, \mathcal{L}^w)$ for a given equation $\mathcal{L}$.

3.2 Containers and independent sets in graphs

3.2.1 Container and removal lemmas

Recall from the introduction that we can phrase the problem in terms of independent sets in hypergraphs. Let $\mathcal{H}$ denote the hypergraph with vertex set $[n]$ and edges corresponding to non-trivial solutions to $\mathcal{L}$. Then an independent set in $\mathcal{H}$ is precisely an $\mathcal{L}^w$-free set. We will use the removal and container results of Green from Chapter 2 (that is, Lemmas 2.4 and 2.5), and the following result (which is an immediate consequence).

Theorem 3.7 ([48]). *Fix a homogeneous linear equation $\mathcal{L}$. Then $f(n, \mathcal{L}^w) = 2^{\mu(n,\mathcal{L}^w)+o(n)}$.*

We will use these results to deduce upper bounds on the number of maximal $\mathcal{L}^w$-free sets (Theorems 3.4, 3.5 and 3.6).

3.2.2 Independent sets in graphs

First observe the following obvious bound on the number of independent sets in a graph.

Fact 3.8. *Let G be a graph and let $A_1, \dots, A_r$ be a partition of $V(G)$. Then $i(G) \leqslant \prod_{i=1}^{r} i(G[A_i])$.*

The following simple lemma will be used in the proof of Theorem 3.3.

Lemma 3.9. *Let G be a graph on n vertices and M be a matching in G which consists of e edges. Suppose that $v \in V(G)$ lies in M. Then the number of independent sets in G which contain v is at most $3^{e-1} \cdot 2^{n-2e}$.*

Proof. First note that the number of independent sets in G which contain v is at most $i(G \setminus X)$ where X consists of v and its neighbour in M. Let $A_1, \ldots, A_e$ be a partition of the vertex set $V(G \setminus X)$, where if $1 \leqslant i \leqslant e-1$ then A_i contains precisely the two vertices from some edge in M. So $|A_e| = n - 2e$. Clearly $i(G[A_i]) = 3$ for $1 \leqslant i \leqslant e-1$ and $i(G[A_e]) \leqslant 2^{n-2e}$. The result then follows by Fact 3.8. □

3.2.3 Link graphs and maximal independent sets

We obtain many of our results by counting the number of maximal independent sets in various auxiliary graphs. Similar techniques were used in [121, 6, 7], and in the graph setting in [10, 5]. To be more precise, let B and S be disjoint subsets of $[n]$ and fix a three-variable linear equation $\mathcal{L}$. The *link graph* $L_S[B]$ *of* S *on* B has vertex set B, and an edge set consisting of the following two types of edges:

(i) Two vertices x and y are adjacent if there exists an element $z \in S$ such that $\{x, y, z\}$ is an $\mathcal{L}$-triple;

(ii) There is a loop at a vertex x if there exists an element $z \in S$ or elements $z, z' \in S$ such that $\{x, x, z\}$ or $\{x, z, z'\}$ is an $\mathcal{L}$-triple.

Notice that since the only possible trivial solutions to a three-variable linear equation $\mathcal{L}$ are of the form $\{x, x, x\}$, all the edges in $L_S[B]$ correspond to non-trivial $\mathcal{L}$-triples.

The following simple lemma was stated in [6, 7] for sum-free sets, but extends to three-variable linear equations.

Lemma 3.10. *Fix a three-variable linear equation $\mathcal{L}$. Suppose that B, S are disjoint $\mathcal{L}^w$-free subsets of $[n]$. If $I \subseteq B$ is such that $S \cup I$ is a maximal $\mathcal{L}^w$-free subset of $[n]$, then I is a maximal independent set in $G := L_S[B]$.*

Let $\mathrm{MIS}(G)$ denote the number of maximal independent sets in G. Suppose we have a container $F \in \mathcal{F}$ as in Lemma 2.5 and suppose $F = A \cup B$ where B is $\mathcal{L}^w$-free.

Observe that any maximal $\mathcal{L}^w$-free subset of $[n]$ in F can be found by first choosing an $\mathcal{L}^w$-free set $S \subseteq A$, and then extending S in B. Note that by Lemma 3.10, the number of possible extensions of S in B (which we shall refer to as $N(S, B)$) is bounded from above by the number of maximal independent sets in the link graph $L_S[B]$ (i.e. we have $N(S, B) \leqslant \mathrm{MIS}(L_S[B])$). Hence Lemma 3.10 is a useful tool for bounding the number of maximal $\mathcal{L}^w$-free subsets of $[n]$.

In particular, we will apply the following result in combination with Lemma 3.10. The first part was proven by Moon and Moser [82] and the second part by Hujter and Tuza [62]. We use the first condition in the proof of Theorems 3.4 and 3.5.

Theorem 3.11. *Suppose that G is a graph on n vertices possibly with loops. Then the following bounds hold.*

(i) $\mathrm{MIS}(G) \leqslant 3^{n/3}$*;*

(ii) $\mathrm{MIS}(G) \leqslant 2^{n/2}$ *if G is additionally triangle-free.*

To prove Theorem 3.5 we will combine Theorem 3.11(ii) and the following result.

Lemma 3.12. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q \geqslant 2$ and $p, q \in \mathbb{N}$. Let $A \subseteq [1, u]$ and let $B \subseteq [u + 1, n]$ for some $u \in [n]$. Consider the link graph $G := L_A[B]$ of A on B. If $q^2 \geqslant p + q$ then G is triangle-free.*

Proof. Suppose that $q^2 \geqslant p + q$ and suppose for a contradiction there is a triangle in G with vertices $b_1 < b_2 < b_3$. By definition of the link graph, there exist $s_1, s_2, s_3 \in A$ such that $\{b_1, b_2, s_1\}, \{b_2, b_3, s_2\}, \{b_1, b_3, s_3\}$ are $\mathcal{L}$-triples.

Since all numbers in A are smaller than all numbers in B we have $1 \leqslant s_1, s_2, s_3 < b_1 < b_2 < b_3$. Also, since $p \geqslant q \geqslant 2$, for each of our $\mathcal{L}$-triples $\{b_i, b_j, s_k\}$ (where $b_i < b_j$) it follows that b_j must play the role of z in $\mathcal{L}$.

Let $r_i \in \{p,q\}$ for each $i \in [6]$, where $r_1 \neq r_2$, $r_3 \neq r_4$ and $r_5 \neq r_6$, and consider the three equations $r_1 b_1 + r_2 s_1 = b_2, r_3 b_2 + r_4 s_2 = b_3$ and $r_5 b_1 + r_6 s_3 = b_3$; observe that each of the possible ordered tuples $(r_1, \ldots, r_6)$ correspond to possible solutions. Combining the second and third equations gives $b_2 = (r_5 b_1 + r_6 s_3 - r_4 s_2)/r_3$. Then combining this with the first equation gives $(r_1 r_3 - r_5) b_1 + r_2 r_3 s_1 + r_4 s_2 = r_6 s_3$. Now since $s_3 < b_1$ and all terms are at least 1, for such an inequality to hold we must have $r_1 r_3 - r_5 < r_6$. Since $r_5 \neq r_6$ this means we have $r_1 r_3 < p+q$. Hence as $r_1, r_3 \in \{p,q\}$, in order for G to have a triangle at least one of $p^2 < p+q$, $q^2 < p+q$ and $pq < p+q$ must be satisfied. Since $p \geqslant q \geqslant 2$, the first and third are not true and so we must have $q^2 < p+q$, a contradiction. $\square$

We also use link graphs as a means to obtain lower bounds on the number of maximal $\mathcal{L}^w$-free sets. We apply the following result in Propositions 3.19 and 3.21.

Lemma 3.13. *Fix a three-variable linear equation $\mathcal{L}$. Suppose that B, S are disjoint $\mathcal{L}^w$-free subsets of $[n]$. Let H be an induced subgraph of the link graph $L_S[B]$. Then $f_{\max}(n, \mathcal{L}^w) \geqslant \mathrm{MIS}(H)$.*

Proof. Suppose I and J are different maximal independent sets in H. First note that $S \cup I$ and $S \cup J$ are $\mathcal{L}^w$-free by definition of the link graph. Both cannot lie in the same maximal $\mathcal{L}^w$-free subset of $[n]$. To see this, observe by definition of I and J, there exists $i \in I \setminus J$. There must exist $s \in S$, $j \in J$ such that $\{i, j, s\}$ forms an $\mathcal{L}$-triple, else $J \cup \{i\}$ would be an independent set in H, which contradicts the maximality of J. Hence any maximal $\mathcal{L}^w$-free subset of $[n]$ containing $S \cup J$ does not contain i. Similarly there exists $j \in J \setminus I$ such that any maximal $\mathcal{L}^w$-free subset of $[n]$ containing $S \cup I$ does not contain j. The result immediately follows. $\square$

3.3 The size of the largest solution-free set

Throughout this section, $\mathcal{L}$ will denote the equation $px + qy = z$ where $p \geqslant q$ and $p \geqslant 2$, $p, q \in \mathbb{N}$. The aim of this section is to determine the size of the largest $\mathcal{L}^w$-free subset of $[n]$. In fact, we will prove a richer structural result on $\mathcal{L}^w$-free sets (Theorem 3.1). For this, we will introduce the following auxiliary graph G_m: Let $m \in [n]$ be fixed. We define the graph G_m to have vertex set $[m, n]$ and edges between c and $pm + qc$ for all $c \in [m, n]$ such that $pm + qc \leqslant n$. We will also make use of these auxiliary graphs in Section 3.4.

Fact 3.14.

(i) The size of the largest $\mathcal{L}^w$-free subset S of $[n]$ with $\min(S) = m$ is at most the size of the largest independent set in G_m which contains m.

(ii) The number of $\mathcal{L}^w$-free subsets S of $[n]$ with $\min(S) = m$ is at most the number of independent sets in G_m which contain m.

Proof. Let S be an $\mathcal{L}^w$-free subset of $[n]$ with $\min(S) = m$. Since $\{m, c, pm + qc\}$ is an $\mathcal{L}$-triple contained in $[n]$ for all $c \in [m, n]$ such that $pm + qc \leqslant n$, S cannot contain both c and $pm + qc$. Hence any $\mathcal{L}^w$-free subset of $[n]$ with minimum element m is also an independent set in G_m which contains m (although the converse does not necessarily hold). This immediately implies (i) and (ii). $\square$

Note that G_m is a union of disjoint paths (and possibly isolated vertices). We refer to the connected components of G_m as the *path components*. Given G_m, we define $y_0 := n$, and for $i \geqslant 1$ define $y_i := \max\{v \in V(G_m) \,|\, pm + qv \leqslant y_{i-1}\}$. Thus we have $y_i = \lfloor \frac{y_{i-1} - pm}{q} \rfloor$. For G_m we also define k to be the largest i such that $y_i \in [m, n]$, and refer to k as the *path parameter* of G_m. We define the *size* of a path component to be the number of *vertices* in it, and we define $N(G_m, i)$ to be the number of path components of size i in G_m.

Fact 3.15. *The graph G_m consists entirely of disjoint path components, where for each $1 \leqslant i \leqslant k-1$ there are $y_{i-1}+y_{i+1}-2y_i$ path components of size i, there are $y_{k-1}-2y_k+m-1$ path components of size k and $y_k - m + 1$ path components of size $k+1$.*

Proof. Every vertex $c \in V(G_m)$ satisfying $y_{j+1} < c \leqslant y_j$ for some $0 \leqslant j \leqslant k-1$ is in a path in G_m which contains precisely j vertices which are larger than it, whereas every vertex $c > y_j$ is not in such a path. All the vertices in $[m, y_k]$ are in paths which contain precisely k vertices which are larger than it, all vertices in $[y_k+1, y_{k-1}]$ are in paths which contain precisely $k-1$ vertices which are larger than it, and so on.

Let A_i be the interval $[y_i+1, y_{i-1}]$ for $1 \leqslant i \leqslant k$ and let A_{k+1} be the interval $[m, y_k]$. There are $|[m, y_k]| = y_k - m + 1$ path components of size $k+1$ in G_m. For $i \leqslant k$ all vertices in A_i are the smallest vertex in a path on i vertices, however they may not be the smallest vertex in their path component. In fact, by definition of the y_i, all paths which start in A_j for some j must include precisely one vertex from each set $A_{j-1}, A_{j-2}, \ldots, A_1$. This means that for $i \leqslant k$, the number of path components of size i in G_m is precisely $|A_i| - |A_{i+1}|$. For $i \leqslant k-1$ this is $y_{i-1} + y_{i+1} - 2y_i$ and for $i = k$ this is $y_{k-1} - 2y_k + m - 1$. □

We use the graphs G_m and the above facts to obtain the bound for the size of the largest $\mathcal{L}^w$-free subset of $[n]$ as stated in Theorem 3.1. For (ii), we will show that a largest independent set in G_m has size at most $(q^2+1)n/(q^2+q+1)$. Before going into full details of the proof of Theorem 3.1, we briefly explain why this is a reasonable target for an upper bound.

Since G_m consists of path components of different sizes, one picks an independent set of maximum size by selecting $\lceil k/2 \rceil$ vertices from each path component of size k. (That is, we select one vertex from a path component of size 1 or 2, two vertices from a path component of size 3 or 4, etc.) Note that the ratio of vertices selected (percentage of vertices chosen from a path of given size) is always $1/2$ if k is even, while it tends towards $1/2$ from above if k is odd and increasing. We show that there is at least one path component of size 3, and thus the 'worst possible' case should be having path components of size at most 3. The relative number of path components of size 3 compared to those of size 2 and 1 leads to the bound $(q^2+1)n/(q^2+q+1)$. For an example, let $\mathcal{L}$ be the equation $2x+2y=z$, suppose $n=21$ and $m=1$; see Figure 3.1 below. Here we can see that a largest independent set is $[1,3]\cup[10,21]$; the ratio of selected vertices here is $15/21=5/7=(q^2+1)/(q^2+q+1)$. We now proceed with the formal proof.

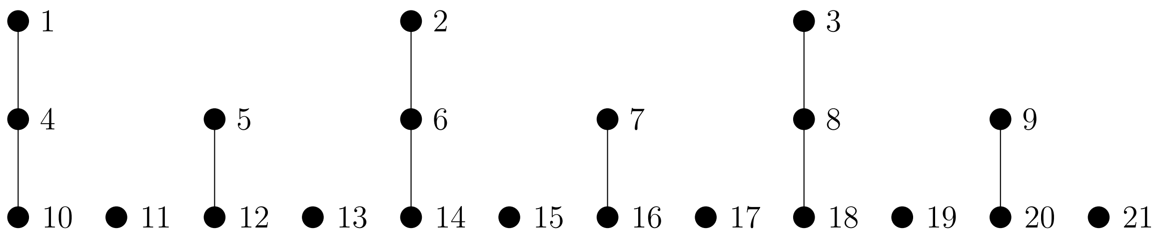


Figure 3.1: G_m where $\mathcal{L}$ is $2x+2y=z$, $n=21$ and $m=1$.

Proof of Theorem 3.1. Let t be a non-negative integer. To prove (i) suppose that $t<(\frac{p+q-1}{p+q+p/q})\lfloor\frac{n}{p+q}\rfloor$. Suppose S is an $\mathcal{L}^w$-free set contained in $[\lfloor\frac{n}{p+q}\rfloor-t,n]$ where $m:=\lfloor\frac{n}{p+q}\rfloor-t\in S$. By Fact 3.14(i) it suffices to prove that the largest independent set in G_m containing m has size at most $\lceil\frac{(p+q-1)n}{p+q}\rceil-\lfloor\frac{p}{q}t\rfloor$. Since $|V(G_m)|=\lceil\frac{(p+q-1)n}{p+q}\rceil+t+1$ it suffices to show that any independent set I in G_m satisfies $|V(G_m)\setminus I|\geqslant\lfloor(p+q)t/q\rfloor+1$.

For $0\leqslant i\leqslant\lfloor(p+q)t/q\rfloor$, there is an edge between $m+i$ and $(p+q)m+qi$. Note that since $i\leqslant\lfloor(p+q)t/q\rfloor$ and $q\leqslant p$ we have that the largest vertex in any of these edges is indeed at most n:

$$(p+q)(\lfloor\tfrac{n}{p+q}\rfloor-t)+qi\leqslant n-(p+q)t+q\lfloor(p+q)t/q\rfloor\leqslant n-(p+q)t+q(p+q)t/q=n.$$

Since I can only contain one vertex from each of these edges, we have proven (i), provided that these edges are disjoint. It suffices to show that $\lfloor\frac{n}{p+q}\rfloor+\lfloor pt/q\rfloor<(p+q)m=(p+q)(\lfloor\frac{n}{p+q}\rfloor-t)$ since the left hand side is the largest element of the set $\{m+i:0\leqslant i\leqslant\lfloor(p+q)t/q\rfloor\}$. But this immediately follows since $t<(\frac{p+q-1}{p+q+p/q})\lfloor\frac{n}{p+q}\rfloor$.

To prove (ii) let $t \geqslant (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$ and suppose S is an $\mathcal{L}^w$-free subset of $[n]$ with $m := \min(S) = \lfloor \frac{n}{p+q} \rfloor - t$. By Fact 3.14(i) $|S|$ is at most the size of the largest independent set in G_m which contains m. We will first show that G_m has path parameter $k \geqslant 2$, and then the case $q = 1$ follows easily. Define $\ell := \lfloor k/2 \rfloor$ and

$$C_k := \left(\frac{\sum_{i=0}^{2\ell+1} (-1)(-q)^i + p \sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1} + p \sum_{i=0}^{2\ell} q^i} \right).$$

We will show that if $q \geqslant 2$ then the largest independent set in G_m has size at most $C_k n + k$. We then further bound this from above by $(q^2+1)n/(q^2+q+1)$ for n sufficiently large.

Note that by Fact 3.15, to prove that $k \geqslant 2$ for G_m it suffices to show that there is a path on 3 vertices in G_m. By definition of k, m lies on a path P on $k+1$ vertices. Write $P = v_0 v_1 \cdots v_k$ where $m = v_0$ and observe that $v_j = (q^j + p \sum_{i=0}^{j-1} q^i)m$ for $0 \leqslant j \leqslant k$. To prove $k \geqslant 2$ it suffices to show that there is indeed a vertex $(q^2 + pq + p)m$ in $V(G_m)$, i.e. $(q^2 + pq + p)m \leqslant n$. Note that since $t \geqslant (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$, we have $m = \lfloor \frac{n}{p+q} \rfloor - t \leqslant (\frac{p+q+p/q-p-q+1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor = (\frac{p+q}{q^2+pq+p})\lfloor \frac{n}{p+q} \rfloor$. Hence $(q^2+pq+p)m \leqslant n$ as desired.

When $q = 1$ observe that $y_i = y_{i-1} - pm$, so for $i \leqslant k-1$ by Fact 3.15 we have $N(G_m, i) = y_{i-1} + y_{i+1} - 2y_i = y_i + pm + y_i - pm - 2y_i = 0$. Hence G_m consists entirely of a union of path components of size either k or $k+1$. Since at most $\lceil i/2 \rceil$ vertices of a path on i vertices can be in an independent set and $k \geqslant 2$, the largest independent set in G_m has size at most $2n/3 = (q^2+1)n/(q^2+q+1)$ in this case, as desired. So now consider the case when $q \geqslant 2$. We calculate the maximum size of an independent set in G_m:

$$\sum_{i=1}^{k+1} \lceil i/2 \rceil \cdot N(G_m, i)$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{k-1} \lceil i/2 \rceil \cdot (y_{i-1} + y_{i+1} - 2y_i) \right) + \lceil k/2 \rceil (y_{k-1} + m - 1 - 2y_k) \\
&\quad + \lceil (k+1)/2 \rceil (y_k - m + 1) \\
&= y_0 + \left(\sum_{i=1}^{k} (\lceil (i-1)/2 \rceil - 2\lceil i/2 \rceil + \lceil (i+1)/2 \rceil) y_i \right) + (m-1)(\lceil k/2 \rceil - \lceil (k+1)/2 \rceil).
\end{aligned} \tag{3.3.1}
$$

Here we used Fact 3.15 in the first equality. For i odd, the coefficient of y_i in (3.3.1) is $(i-1)/2 - 2(i+1)/2 + (i+1)/2 = -1$. For i even, the coefficient of y_i in (3.3.1) is $i/2 - 2i/2 + (i+2)/2 = 1$.

The following bounds are obtained from the definition of y_i and k:

$$
\begin{aligned}
&\text{(a)} \;\; \left(n - q^j + 1 - pm \sum_{i=0}^{j-1} q^i \right) / q^j \leqslant y_j \leqslant \left(n - pm \sum_{i=0}^{j-1} q^i \right) / q^j; \\
&\text{(b)} \;\; n / \left(q^{k+1} + p \sum_{i=0}^{k} q^i \right) < m \leqslant n / \left(q^k + p \sum_{i=0}^{k-1} q^i \right).
\end{aligned}
$$

Let $\ell := \lfloor k/2 \rfloor$ (note $k \geqslant 2$ so $\ell \geqslant 1$). First suppose k is odd, i.e. $k = 2\ell + 1$. Using (3.3.1), the size of the largest independent set in G_m is bounded above by

$$
\begin{aligned}
&y_0 + \left(\sum_{i=1}^{k} (\lceil (i-1)/2 \rceil - 2\lceil i/2 \rceil + \lceil (i+1)/2 \rceil) y_i \right) + (m-1)(\lceil k/2 \rceil - \lceil (k+1)/2 \rceil) \\
&= y_0 - y_1 + y_2 - y_3 + \cdots + y_{2\ell} - y_{2\ell+1} \\
&\overset{\text{(a)}}{\leqslant} n - \left(\frac{n - pm - q + 1}{q} \right) + \left(\frac{n - pm(1+q)}{q^2} \right) - \left(\frac{n - pm(1+q+q^2) - q^3 + 1}{q^3} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \cdots - \left(\frac{n - \left(pm\sum_{i=0}^{2\ell} q^i\right) - q^{2\ell+1} + 1}{q^{2\ell+1}}\right)\\
&= n\left(1 - \frac{1}{q} + \frac{1}{q^2} - \cdots - \frac{1}{q^{2\ell+1}}\right) + m\left(\frac{p}{q} + \frac{p}{q^3} + \cdots + \frac{p}{q^{2\ell+1}}\right) + \frac{q-1}{q} + \frac{q^3-1}{q^3}\\
&\quad + \cdots + \frac{q^{2\ell+1}-1}{q^{2\ell+1}}\\
&\overset{(b)}{\leqslant} \frac{n}{q^{2\ell+1}}\left(\sum_{i=0}^{2\ell+1}(-1)(-q)^i\right) + \left(\frac{n}{q^{2\ell+1} + p\sum_{i=0}^{2\ell} q^i}\right)\left(\frac{p\sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1}}\right) + \frac{k+1}{2}\\
&= \left(\frac{\left[\sum_{i=0}^{2\ell+1}(-1)(-q)^i\right](q^{2\ell+1} + p\sum_{i=0}^{2\ell} q^i) + p\sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1}(q^{2\ell+1} + p\sum_{i=0}^{2\ell} q^i)}\right)n + \frac{k+1}{2}\\
&= \left(\frac{\sum_{i=0}^{2\ell+1}(-q)^{i+2\ell+1} + p\sum_{i=0}^{\ell} q^{2i+2\ell+1}}{q^{2\ell+1}(q^{2\ell+1} + p\sum_{i=0}^{2\ell} q^i)}\right)n + \frac{k+1}{2} = \left(\frac{\sum_{i=0}^{2\ell+1}(-1)(-q)^i + p\sum_{i=0}^{\ell} q^{2i}}{q^{2\ell+1} + p\sum_{i=0}^{2\ell} q^i}\right)n + \frac{k+1}{2}\\
&= C_k n + \frac{k+1}{2} \leqslant C_k n + k.
\end{aligned}
$$

By definition, $m \geqslant y_{k+1} + 1$ and for k even, we have $C_k = C_{k+1}$. So if k is even ($k = 2\ell$) then we have

$$
\begin{aligned}
&y_0 + \left(\sum_{i=1}^{k}(\lceil (i-1)/2\rceil - 2\lceil i/2\rceil + \lceil (i+1)/2\rceil)y_i\right) + (m-1)(\lceil k/2\rceil - \lceil (k+1)/2\rceil)\\
&= y_0 - y_1 + y_2 - y_3 + ... + y_{2\ell} - m + 1 \leqslant y_0 - y_1 + y_2 - y_3 + ... + y_{2\ell} - y_{2\ell+1}\\
&\leqslant C_{k+1}n + \frac{k+2}{2} \leqslant C_k n + k.
\end{aligned}
$$

The penultimate inequality follows by using calculations from the odd case. The last inequality follows since $k \geqslant 2$ and $C_k = C_{k+1}$. Thus we have shown that $|S| \leqslant C_k n + k$ and we know that $k \geqslant 2$. It remains to show that

$$C_k n + k \leqslant \frac{(q^2+1)n}{q^2+q+1} \tag{3.3.2}$$

for $k \geqslant 2$ and n sufficiently large.

We know that $m \leqslant n/(q^k + p\sum_{i=0}^{k-1} q^i)$ and so $n \geqslant q^k + p\sum_{i=0}^{k-1} q^i$, therefore condition (3.3.2) is met if

$$\left(\frac{q^2+1}{q^2+q+1} - C_k\right)\left(q^k + p\sum_{i=0}^{k-1} q^i\right) \geqslant k. \tag{3.3.3}$$

Claim 3.16. *For $k \geqslant 6$, (3.3.3) holds.*

Proof. We use induction on k. Recall that $p \geqslant q \geqslant 2$. For the base case $k = 6$ we directly calculate (3.3.3). First note that

$$\begin{aligned}&\frac{q^2+1}{q^2+q+1} - \frac{q^7-q^6+q^5-q^4+q^3-q^2+q-1+p(q^6+q^4+q^2+1)}{q^7+p(q^6+q^5+q^4+q^3+q^2+q+1)}\\ =&\frac{(q^6+(p-1)q^5+q^4+(p-1)q^3+q^2+1)}{(q^2+q+1)(q^7+p(q^6+q^5+q^4+q^3+q^2+q+1))},\end{aligned}$$

and so we have

$$\left(\frac{q^2+1}{q^2+q+1} - C_6\right)\left(q^6 + p(q^5+q^4+q^3+q^2+q+1)\right)$$

$$=\frac{(q^6+(p-1)q^5+q^4+(p-1)q^3+q^2+1)(q^6+p(q^5+q^4+q^3+q^2+q+1))}{(q^2+q+1)(q^7+p(q^6+q^5+q^4+q^3+q^2+q+1))}.$$

Since $p \geqslant q \geqslant 2$ every power of q in the numerator has a coefficient of at least 1 in both expressions, hence the numerator as a single polynomial in q has positive coefficients. Hence we can make our fraction smaller by dropping lower powers of q. We then make further use of $p \geqslant q \geqslant 2$ to get the desired result:

$$\begin{aligned}
&\frac{(q^6+(p-1)q^5+q^4+(p-1)q^3+q^2+1)(q^6+p(q^5+q^4+q^3+q^2+q+1))}{(q^2+q+1)(q^7+p(q^6+q^5+q^4+q^3+q^2+q+1))}\\
\geqslant&\frac{q^{12}+(2p-1)q^{11}+(p^2+1)q^{10}+(p^2+2p-1)q^9}{(q^2+q+1)(q^7+p(q^6+q^5+q^4+q^3+q^2+q+1))}\\
\geqslant&\frac{q^{12}+(2p-1)q^{11}+(p^2+1)q^{10}+(p^2+2p-1)q^9}{(p+1)q^{10}}\\
=&\frac{q^2+(2p-1)q+(p^2+1)}{p+1}+\frac{p^2+2p-1}{(p+1)q}\\
\geqslant&\frac{p^2+4p+3}{p+1}+\frac{p^2+p}{(p+1)q}=p+3+p/q\geqslant 6=k.
\end{aligned}$$

For the inductive step, assume that (3.3.3) holds for k. It suffices to show that $C_k \geqslant C_{k+1}$ as then the result holds for $k+1$:

$$\begin{aligned}
&\Big(\frac{q^2+1}{q^2+q+1}-C_{k+1}\Big)\Big(q^{k+1}+p\sum_{i=0}^{k}q^i\Big)\geqslant\Big(\frac{q^2+1}{q^2+q+1}-C_k\Big)\Big(q^{k+1}+p\sum_{i=0}^{k}q^i\Big)\\
\geqslant& q\Big(\frac{q^2+1}{q^2+q+1}-C_k\Big)\Big(q^k+p\sum_{i=0}^{k-1}q^i\Big)\geqslant qk\geqslant k+1.
\end{aligned}$$

For k even, we have $C_k = C_{k+1}$ by definition. For k odd, consider the following calculations:

(i) $D_1 := q^{k+2}\Big(\sum_{i=0}^{k}(-1)(-q)^i\Big) - q^k\Big(\sum_{i=0}^{k+2}(-1)(-q)^i\Big) = -q^{k+1} + q^k,$

(ii) $D_2 := pq^{k+2}\Big(\sum_{i=0}^{(k-1)/2} q^{2i}\Big) - pq^k\Big(\sum_{i=0}^{(k+1)/2} q^{2i}\Big) = -pq^k,$

(iii) $D_3 := p\Big(\sum_{i=0}^{k+1} q^i\Big)\Big(\sum_{i=0}^{k}(-1)(-q)^i\Big) - p\Big(\sum_{i=0}^{k-1} q^i\Big)\Big(\sum_{i=0}^{k+2}(-1)(-q)^i\Big) = pq^{k+1} - pq^k,$

(iv) $D_4 := p^2\Big(\sum_{i=0}^{k+1} q^i\Big)\Big(\sum_{i=0}^{(k-1)/2} q^{2i}\Big) - p^2\Big(\sum_{i=0}^{k-1} q^i\Big)\Big(\sum_{i=0}^{(k+1)/2} q^{2i}\Big) = p^2q^k.$

Using these we have

$$\begin{aligned}
C_k - C_{k+1} &= \frac{\Big(\sum_{i=0}^{k}(-1)(-q)^i\Big) + p\Big(\sum_{i=0}^{(k-1)/2} q^{2i}\Big)}{q^k + p\Big(\sum_{i=0}^{k-1} q^i\Big)} - \frac{\Big(\sum_{i=0}^{k+2}(-1)(-q)^i\Big) + p\Big(\sum_{i=0}^{(k+1)/2} q^{2i}\Big)}{q^{k+2} + p\Big(\sum_{i=0}^{k+1} q^i\Big)} \\
&= \frac{D_1 + D_2 + D_3 + D_4}{\Big(q^k + p\Big(\sum_{i=0}^{k-1} q^i\Big)\Big)\Big(q^{k+2} + p\Big(\sum_{i=0}^{k+1} q^i\Big)\Big)} \\
&= \frac{(p-1)q^{k+1} + (p^2 - 2p + 1)q^k}{\Big(q^k + p\Big(\sum_{i=0}^{k-1} q^i\Big)\Big)\Big(q^{k+2} + p\Big(\sum_{i=0}^{k+1} q^i\Big)\Big)} \geqslant 0,
\end{aligned}$$

where the last inequality follows since $p, q \geqslant 2$. $\square$

The claim is not a result which generally holds for $2 \leqslant k \leqslant 5$ so instead we directly calculate how large n should be to satisfy (3.3.2) in these cases. For $k = 3$ and $k = 5$ we obtain $n \geqslant \frac{3(q^3+p(q^2+q+1))(q^2+q+1)}{q^2+1}$ and $n \geqslant \frac{5(q^5+p(q^4+q^3+q^2+q+1))(q^2+q+1)}{q^4+(p-1)q^3+q^2+1}$ respectively. For

$k = 2$ and $k = 4$ we obtain weaker bounds. Hence taking n to be sufficiently large (larger than these two bounds), we have $C_k n + k \leqslant \frac{(q^2+1)n}{q^2+q+1}$ for all $k \geqslant 2$.

$\square$

3.4 The number of solution-free sets

Recall Theorem 3.7 states that $f(n, \mathcal{L}^w) = 2^{\mu(n,\mathcal{L}^w)+o(n)}$ for any fixed homogeneous linear equation $\mathcal{L}$. The aim of this section is to replace the term $o(n)$ here with a constant for many equations $\mathcal{L}$. This will be achieved in Theorem 3.18, which immediately implies Theorem 3.3. Denote by $f(n, \mathcal{L}^w, m)$ the number of $\mathcal{L}^w$-free subsets of $[n]$ with minimum element m. We first give bounds on $f(n, \mathcal{L}^w, m)$ for linear equations $\mathcal{L}$ of the form $px + qy = z$.

Lemma 3.17. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q$ and $p \geqslant 2$, $p, q \in \mathbb{N}$.*

(i) If $m \geqslant \lfloor \frac{n}{p+q} \rfloor + 1$ then $f(n, \mathcal{L}^w, m) = 2^{n-m}$.

(ii) If n is sufficiently large depending on p and q and $m = \lfloor \frac{n}{p+q} \rfloor$ then $f(n, \mathcal{L}^w, m) \leqslant 2^{\mu(n,\mathcal{L}^w)-1}$.

(iii) If n is sufficiently large depending on p and q, $q \geqslant 2$, $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some positive integer t and G_m has path parameter 1, then $f(n, \mathcal{L}^w, m) \leqslant 2^{\mu(n,\mathcal{L}^w)-3/5+t(3q-2p)/(5q)}$.

(iv) If $q \geqslant 2$, $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some positive integer t and G_m has path parameter $k \geqslant 2$, then $f(n, \mathcal{L}^w, m) \leqslant (4/3) \cdot 2^{(5q^2-2q+2)n/(5q^2)}$.

(v) If $q = 1$, G_m has path parameter ℓ, and $m = \lfloor \frac{n}{\ell p+1} \rfloor - t$ for some integer t, then $f(n, \mathcal{L}^w, m) \leqslant 2^{(7\ell p+3p)n/(10\ell p+10)+(t(7-3p)+7)/10}$.

Proof. First note that (i) is trivial since all subsets $S \subseteq [n]$ with $\min(S) \geqslant \lfloor \frac{n}{p+q} \rfloor + 1$ are $\mathcal{L}^w$-free. By Fact 3.14(ii) we know that $f(n, \mathcal{L}^w, m)$ is at most the number of independent sets in G_m which contain m. For (ii), there is one edge between $m = \lfloor \frac{n}{p+q} \rfloor$ and $(p+q)m \leqslant n$ in G_m, hence there are at most $2^{n-\lfloor \frac{n}{p+q} \rfloor -1} = 2^{\mu(n,\mathcal{L}^w)-1}$ independent sets in G_m containing m.

For (iii) suppose $q \geqslant 2$ and $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some $t \in \mathbb{N}$. Notice that G_m contains a matching on $y_1 - m + 1$ edges, namely there is an edge between c and $pm + qc$ for $c \in [m, y_1]$. Observe that $3/4 \leqslant 2^{-2/5}$ and also

$$y_1 - m = \left\lfloor \frac{n - pm}{q} \right\rfloor - m \geqslant \frac{n - (p+q)m - q}{q} \geqslant \frac{t(p+q)}{q} - 1.$$

Hence by Lemma 3.9 the total number of independent sets in G_m which contain m is at most

$$\begin{aligned} 2^{n-m-2(y_1-m)-1}3^{y_1-m} &\leqslant 2^{\mu(n,\mathcal{L}^w)-1+t}(3/4)^{y_1-m} \\ \leqslant & 2^{\mu(n,\mathcal{L}^w)-1+t}(3/4)^{t(p+q)/q-1} \leqslant 2^{\mu(n,\mathcal{L}^w)-3/5+t(3q-2p)/(5q)}, \end{aligned}$$

as desired.

For (iv) suppose $q \geqslant 2$, $m = \lfloor \frac{n}{p+q} \rfloor - t$ for some positive integer t and G_m has path parameter $k \geqslant 2$. First note that

$$\begin{aligned} y_1 - y_2 = & \left\lfloor \frac{n - pm}{q} \right\rfloor - \left\lfloor \frac{\lfloor \frac{n-pm}{q} \rfloor - pm}{q} \right\rfloor \geqslant \frac{n - pm - q}{q} - \frac{n - pm - qpm}{q^2} \\ = & \frac{(q-1)n + pm - q^2}{q^2} \geqslant \frac{(q-1)n}{q^2} - 1. \end{aligned}$$

Define $F(i)$ to be the ith Fibonacci number where $F(1) = F(2) = 1$. There are $F(i+2)$ independent sets (including the empty set) in a path of length i. Recall the following Fibonacci identity: $F(i+2)F(i) - F(i+1)^2 = (-1)^{i+1}$. If i is even and $a > b$

then

$$\left(\frac{F(i)F(i+2)}{F(i+1)^2}\right)^a\left(\frac{F(i+1)F(i+3)}{F(i+2)^2}\right)^b=\left(\frac{F(i+1)^2-1}{F(i+1)^2}\right)^a\left(\frac{F(i+2)^2+1}{F(i+2)^2}\right)^b\leqslant 1.$$

Also observe that by omitting $(F(i+1)F(i+3)/F(i+2)^2)^b$ the inequality still holds. By use of Fact 3.15 and applying the above bounds, we can bound from above the number of independent sets in G_m as required:

$$\begin{aligned}
&2^{y_0+y_2-2y_1}3^{y_1+y_3-2y_2}5^{y_2+y_4-2y_3}\dots F(k+1)^{y_{k-2}+y_k-2y_{k-1}}F(k+2)^{y_{k-1}+m-2y_k-1}F(k+3)^{y_k-m+1}\\
&=2^{y_0+y_2-2y_1}3^{y_1-2y_2}5^{y_2}\left(\frac{3\cdot 8}{5^2}\right)^{y_3}\left(\frac{5\cdot 13}{8^2}\right)^{y_4}\cdots\left(\frac{F(k+1)\cdot F(k+3)}{F(k+2)^2}\right)^{y_k}\left(\frac{F(k+2)}{F(k+3)}\right)^{m-1}\\
&\leqslant 2^{y_0+y_2-2y_1}3^{y_1-2y_2}5^{y_2}\leqslant 2^{y_0+y_2-2y_1+y_2}3^{y_1-y_2}=2^{y_0}(3/4)^{y_1-y_2}\leqslant 2^n(3/4)^{(q-1)n/q^2-1}\\
&\leqslant(4/3)\cdot 2^{n-2(q-1)n/(5q^2)}=(4/3)\cdot 2^{(5q^2-2q+2)n/(5q^2)}.
\end{aligned}$$

For (v), since $y_i=n-ipm$ Fact 3.15 implies that if G_m has path parameter ℓ, then G_m is a union of paths of length ℓ and $\ell+1$. We use the bound $F(i)\leqslant 2^{(7i-11)/10}$ (a simple proof by induction which holds for $i\geqslant 2$). Since $m\leqslant y_\ell=n-\ell pm$ we can write $m=\lfloor\frac{n}{\ell p+1}\rfloor-t$ for some integer $t\geqslant 0$. Now using these bounds, we have

$$\begin{aligned}
&F(\ell+2)^{y_{\ell-1}-2y_\ell+m-1}F(\ell+3)^{y_\ell-m+1}=F(\ell+2)^{(\ell p+p+1)m-n-1}F(\ell+3)^{n-(\ell p+1)m+1}\\
&\leqslant 2^{(3+7\ell)((\ell p+p+1)m-n-1)/10+(10+7\ell)(n-(\ell p+1)m+1)/10}=2^{(7n+(3p-7)m+7)/10}\\
&\leqslant 2^{(7+7n+(3p-7)(n/(\ell p+1)-t))/10}=2^{(7\ell p+3p)n/(10\ell p+10)+(t(7-3p)+7)/10}.
\end{aligned}$$

$\square$

Theorem 3.18. *Let $\mathcal{L}$ denote the equation $px+qy=z$ where $p,q\in\mathbb{N}$ and* *(i)* $q \geqslant 2$ *and* $p > q(3q+2)/(2q-2)$ *or;*

(ii) $q = 1$ *and* $p \geqslant 3$*.*

Then $f(n, \mathcal{L}^w) \leqslant (3/2 + o(1) + C)2^{\mu(n,\mathcal{L}^w)}$ *where for (i)* $C := \frac{2^{-2p/(5q)}}{1-2^{(3q-2p)/(5q)}}$ *and for (ii)* $C := \frac{2^{(14-3p)/10}}{1-2^{(7-3p)/10}}$.

Proof. For both cases by Lemma 3.17(i)–(ii) there are at most $3 \cdot 2^{\mu(n,\mathcal{L}^w)-1}$ $\mathcal{L}^w$-free subsets S of $[n]$ where $\min(S) \geqslant \lfloor \frac{n}{p+q} \rfloor$. For (i), first consider $\mathcal{L}^w$-free subsets arising from Lemma 3.17(iv). Since $k \geqslant 2$,

$$m \leqslant y_2 = \left\lfloor \frac{\lfloor \frac{n-pm}{q} \rfloor - pm}{q} \right\rfloor \leqslant \frac{n - pm - qpm}{q^2}$$

and so $m \leqslant n/(q^2 + pq + p)$. Now as $n \to \infty$,

$$\frac{n/(q^2+pq+p) \cdot (4/3) \cdot 2^{(5q^2-2q+2)n/(5q^2)}}{2^{\mu(n,\mathcal{L}^w)}} = \frac{2^{\log_2(4n/(3(q^2+pq+p)))+(5q^2-2q+2)n/(5q^2)}}{2^{\mu(n,\mathcal{L}^w)}} \to 0,$$

as long as we have $2^{(5q^2-2q+2)n/(5q^2)} \lll 2^{\mu(n,\mathcal{L}^w)}$. This is satisfied if $(5q^2 - 2q + 2)/(5q^2) < (p + q - 1)/(p + q)$ which when rearranged, gives $p > q(3q + 2)/(2q - 2)$.

For $\mathcal{L}^w$-free subsets arising from Lemma 3.17(iii), set $a := 2^{\mu(n,\mathcal{L}^w)-3/5}$, $r := 2^{(3q-2p)/(5q)}$ and let u be the largest t such that G_m with $m = \lfloor \frac{n}{p+q} \rfloor - t$ has path parameter 1. Then since $p > q(3q + 2)/(2q - 2) > 3q/2$ we have $|r| < 1$ and so

$$\sum_{t=1}^{u} 2^{\mu(n,\mathcal{L}^w)-3/5+t(3q-2p)/(5q)} \leqslant \sum_{t=1}^{\infty} ar^t = \sum_{t=0}^{\infty} (ar)r^t = \frac{ar}{1-r} = \frac{2^{\mu(n,\mathcal{L}^w)-2p/(5q)}}{1-2^{(3q-2p)/(5q)}}.$$

Altogether this implies that $f(n, \mathcal{L}^w) \leqslant (3/2 + o(1) + C)2^{\mu(n,\mathcal{L}^w)}$ where $C := \frac{2^{-2p/(5q)}}{1-2^{(3q-2p)/(5q)}}$.

For (ii), since $p \geqslant 3$, if G_m has path parameter $k \geqslant 2$ then we have $f(n, \mathcal{L}, m) \leqslant 2^{17pn/(20p+10)+7/10}$. We also have $m \leqslant y_2 = n - 2pm$ and so $m \leqslant n/(2p + 1)$. Now as

$n \to \infty$,

$$\frac{n/(2p+1) \cdot 2^{17pn/(20p+10)+7/10}}{2^{\mu(n,\mathcal{L}^w)}} = \frac{2^{\log_2(n/(2p+1))+17pn/(20p+10)+7/10}}{2^{\mu(n,\mathcal{L}^w)}} \to 0,$$

since $2^{17pn/(20p+10)+7/10} \ll 2^{\mu(n,\mathcal{L}^w)}$.

Now consider G_m with path parameter $k = 1$. Set $a := 2^{pn/(p+1)+7/10}$, set $r := 2^{(7-3p)/10}$ and let u be the largest t such that G_m with $m := \lfloor \frac{n}{p+q} \rfloor - t$ has path parameter 1. Since $p \geqslant 3$ we have $|r| < 1$ and so

$$\begin{aligned} \sum_{t=1}^{u} 2^{pn/(p+1)+t(7-3p)/10+7/10} &\leqslant \sum_{t=1}^{\infty} ar^t = \sum_{t=0}^{\infty} (ar)r^t = \frac{ar}{1-r} \\ = \frac{2^{pn/(p+1)+(14-3p)/10}}{1-2^{(7-3p)/10}} &\leqslant \frac{2^{\mu(n,\mathcal{L}^w)+(14-3p)/10}}{1-2^{(7-3p)/10}}. \end{aligned}$$

Therefore, Lemma 3.17 implies that $f(n, \mathcal{L}^w) \leqslant (3/2 + o(1) + C)2^{\mu(n,\mathcal{L}^w)}$ where $C := \frac{2^{(14-3p)/10}}{1-2^{(7-3p)/10}}$. □

3.5 The number of maximal solution-free sets

3.5.1 A general upper bound

Let $\mathcal{L}$ be a three-variable linear equation. Let $\mathcal{M}(n, \mathcal{L})$ denote the set of elements $x \in [n]$ such that $x \in [n]$ does not lie in *any* $\mathcal{L}$-triple in $[n]$. Define $\mu^*(n, \mathcal{L}) := |\mathcal{M}(n, \mathcal{L})|$. For example, if $\mathcal{L}$ is translation-invariant then $\{x, x, x\}$ is an $\mathcal{L}$-triple for all $x \in [n]$ so $\mathcal{M}(n, \mathcal{L}) = \emptyset$ and $\mu^*(n, \mathcal{L}) = 0$.

Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant 2$, $p \geqslant q$ and $p, q \in \mathbb{N}$. Write $t := \gcd(p, q)$. Then notice that $\mathcal{M}(n, \mathcal{L}) \supseteq \{s \in [n] : s > \lfloor (n-p)/q \rfloor, t \nmid s\}$. This follows since if $s > \lfloor (n-p)/q \rfloor$ then $ps + q \geqslant qs + p > n$ and so s cannot play the role of x or y in an $\mathcal{L}$-triple in $[n]$. If $t \nmid s$ then as $t|(px + qy)$ for any $x, y \in [n]$ we have that

s cannot play the role of z in an $\mathcal{L}$-triple in $[n]$. Actually, for large enough n we have $\mathcal{M}(n, \mathcal{L}) = \{s \in [n] : s > \lfloor (n-p)/q \rfloor, t \nmid s\}$ for all such $\mathcal{L}$.

We need to show that if $u \in [n]$ satisfies $u \notin \{s \in [n] : s > \lfloor (n-p)/q \rfloor, t \nmid s\}$ then u lies in an $\mathcal{L}$-triple. If $u \leqslant \lfloor (n-p)/q \rfloor$ then $p + qu \leqslant n$ so u lies in the $\mathcal{L}$-triple $\{1, u, p+qu\}$. So suppose $u > \lfloor (n-p)/q \rfloor$ and u is divisible by t. Then since n is sufficiently large, u is sufficiently large and hence can be written as $u = pa + qb$ where a, b are positive integers. Therefore u lies in the $\mathcal{L}$-triple $\{u, a, b\}$.

To prove that such a, b exist, observe that first since u, p, q are all divisible by t, write $u' := u/t$, $r_1 := p/t$ and $r_2 := q/t$. Then $u' = r_1 a + r_2 b$; the largest u' such that there does not exist a, b positive integers such that $r_1 a + r_2 b = s$ is $r_1 r_2$ (see [116]). So we simply require $u > \lfloor (n-p)/q \rfloor \geqslant t(r_1 r_2 + 1)$.

We now prove Theorem 3.4.

Theorem 3.4. *Let $\mathcal{L}$ be a fixed homogenous three-variable linear equation. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 3^{(\mu(n,\mathcal{L}^w) - \mu^*(n,\mathcal{L}))/3 + o(n)}.$$

Proof. Let $\mathcal{F}$ denote the set of containers obtained by applying Lemma 2.5. Since every $\mathcal{L}^w$-free subset of $[n]$ lies in at least one of the $2^{o(n)}$ containers, it suffices to show that every $F \in \mathcal{F}$ houses at most $3^{(\mu(n,\mathcal{L}^w) - \mu^*(n,\mathcal{L}))/3 + o(n)}$ maximal $\mathcal{L}^w$-free subsets.

Let $F \in \mathcal{F}$. By Lemmas 2.5(i) and 2.4, $F = A \cup B$ where $|A| = o(n)$, $|B| \leqslant \mu(n, \mathcal{L}^w)$ and B is $\mathcal{L}^w$-free. Note that we can add all the elements of $\mathcal{M}(n, \mathcal{L})$ to B (and thus F) whilst ensuring that $|B| \leqslant \mu(n, \mathcal{L}^w)$ and B is $\mathcal{L}^w$-free. So we may assume that $\mathcal{M}(n, \mathcal{L}) \subseteq B$.

Each maximal $\mathcal{L}^w$-free subset of $[n]$ in F can be found by picking a subset $S \subseteq A$ which is $\mathcal{L}^w$-free, and extending it in B. The number of ways of doing this is the number of ways of choosing the subset S multiplied by the number of ways of extending a fixed S in B, which we denote by $N(S, B)$. Since $|A| = o(n)$, there are $2^{o(n)}$ choices for S. It therefore suffices to show that for any $S \subseteq A$, we have $N(S,B) \leqslant 3^{(\mu(n,\mathcal{L}^w)-\mu^*(n,\mathcal{L}))/3}$.

Consider the link graph $G := L_S[B]$. Then by definition, $\mathcal{M}(n,\mathcal{L})$ is an independent set in G. Thus, $\mathrm{MIS}(G) = \mathrm{MIS}(G \setminus \mathcal{M}(n,\mathcal{L}))$. Further, Lemma 3.10 and Theorem 3.11(i) imply that

$$N(S,B) \leqslant \mathrm{MIS}(G) = \mathrm{MIS}(G \setminus \mathcal{M}(n,\mathcal{L})) \leqslant 3^{|B \setminus \mathcal{M}(n,\mathcal{L})|/3} \leqslant 3^{(\mu(n,\mathcal{L}^w)-\mu^*(n,\mathcal{L}))/3},$$

as desired. □

As mentioned in the introduction of this chapter, Theorem 3.4 together with Theorem 3.7 shows that $f_{\max}(n,\mathcal{L}^w)$ is significantly smaller than $f(n,\mathcal{L}^w)$ for all homogeneous three-variable linear equations $\mathcal{L}$ that are not translation-invariant. So in this sense it can be viewed as a generalisation of a result of Łuczak and Schoen [79] on sum-free sets.

Let $\mathcal{L}$ denote the equation $px + y = z$ for some $p \in \mathbb{N}$. Notice that in this case we have $\mu^*(n,\mathcal{L}) = 0$ for $n > p$. The next result implies that if p is large then $f_{\max}(n,\mathcal{L}^w)$ is close to the bound in Theorem 3.4. So for such equations $\mathcal{L}$, Theorem 3.4 is close to best possible.

Proposition 3.19. *Given $p \in \mathbb{N}$ where $p \geqslant 2$, let $\mathcal{L}$ denote the equation $px + y = z$ and let n be sufficiently large depending on p. Then*

$$f_{\max}(n,\mathcal{L}^w) \geqslant 3^{\mu(n,\mathcal{L}^w)/3 - 2pn/(3(p+1)(3p^2-1)) - p - 5}.$$

Proof. Given $p, n \in \mathbb{N}$, let $\mathcal{L}$ denote the equation $px + y = z$. Set $s := \lfloor \frac{(p-1)n}{3p^2-1} \rfloor$ and $a := \lfloor \frac{n-s}{p} \rfloor$. Consider the link graph $G := L_{\{s,2s\}}[a+1, a+3ps]$. Observe that:

$$2s \leqslant \frac{(2p-2)n}{3p^2-1} < \frac{n}{p+1} < \frac{(3p-1)n}{3p^2-1} = \frac{n}{p} - \frac{(p-1)n}{3p^3-p} \leqslant \frac{n-s}{p} < a+1;$$
$$a + 3ps = \left\lfloor \frac{n-s}{p} \right\rfloor + 3ps \leqslant \frac{n}{p} + \left(3p - \frac{1}{p}\right)s = \frac{n}{p} + \frac{3p^2-1}{p}\left\lfloor \frac{(p-1)n}{3p^2-1} \right\rfloor$$

$$\leqslant \frac{n+n(p-1)}{p} = n.$$

As a consequence, the sets $\{s, 2s\}$ and $[a+1, a+3ps]$ (a subset of $[\lfloor \frac{n}{p+1} \rfloor + 1, n]$) are disjoint $\mathcal{L}^w$-free sets in $[n]$, and so Lemma 3.13 implies that $f_{\max}(n, \mathcal{L}^w) \geqslant \mathrm{MIS}(G)$. It remains to show that G contains at least $3^{\mu(n,\mathcal{L}^w)/3-2pn/(3(p+1)(3p^2-1))-6}$ maximal independent sets.

Observe that for each $i \in [ps]$ there is an edge in G between $a+i$ and $a+ps+i$ (since $\{s, a+i, a+i+ps\}$ is an $\mathcal{L}$-triple), an edge between $a+i+ps$ and $a+i+2ps$ (since $\{s, a+i+ps, a+i+2ps\}$ is an $\mathcal{L}$-triple) and an edge between $a+i$ and $a+i+2ps$ (since $\{2s, a+i, a+i+2ps\}$ is an $\mathcal{L}$-triple). Also since $a > (n-s)/p-1$, we have $p(a+1)+s > n$ and hence there are no further edges in G.

Hence G is a collection of ps disjoint triangles, where 4 vertices in G have loops ($(p+1)s$, $(p+2)s$, $(2p+1)s$ and $(2p+2)s$). So G has at least 3^{ps-4} maximal independent sets. Now observe:

$$\begin{aligned}
ps - 4 - \frac{\mu(n,\mathcal{L}^w)}{3} &= p\left\lfloor \frac{(p-1)n}{3p^2-1} \right\rfloor - 4 - \frac{n}{3} + \frac{1}{3}\left\lfloor \frac{n}{p+1} \right\rfloor \\
&\geqslant \left(\frac{p^2-p}{3p^2-1} - \frac{1}{3} + \frac{1}{3(p+1)} \right) n - p - 5 \\
&= \left(\frac{-2p}{3(p+1)(3p^2-1)} \right) n - p - 5,
\end{aligned}$$

as required. □

3.5.2 Upper bounds for $px + qy = z$

Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q$, $p \geqslant 2$ and $p, q \in \mathbb{N}$. For such $\mathcal{L}$, the next simple result provides an alternative bound to Theorem 3.4.

Lemma 3.20. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q$, $p \geqslant 2$ and $p, q \in \mathbb{N}$.*

Then $f_{\max}(n, \mathcal{L}^w) \leqslant f(\lfloor (n-p)/q \rfloor, \mathcal{L}^w)$.

Proof. Set $C := [\lfloor \frac{n-p}{q} \rfloor]$ and $B := [\lfloor \frac{n-p}{q} \rfloor + 1, n]$. In particular, B is $\mathcal{L}^w$-free. Notice that every maximal $\mathcal{L}^w$-free subset of $[n]$ can be found by selecting an $\mathcal{L}^w$-free subset $S \subseteq C$ and then extending it in B to a maximal one. Suppose we have such an $\mathcal{L}^w$-free subset S. By Lemma 3.10, the number of such extensions of S is at most $\mathrm{MIS}(L_S[B])$.

For any $\mathcal{L}$-triple $\{x, y, z\}$ in $[n]$ satisfying $px + qy = z$, since $z \leqslant n$, we must have $x \leqslant \frac{n-q}{p}$ and $y \leqslant \frac{n-p}{q}$. Hence $x, y \in C$. This means that there are no $\mathcal{L}$-triples in $[n]$ which contain more than one element from B. Thus the link graph $L_S[B]$ must only contain isolated vertices and loops. So $L_S[B]$ has precisely one maximal independent set. Hence the number of maximal $\mathcal{L}^w$-free subsets of $[n]$ is bounded by the number of choices of S in C which are $\mathcal{L}^w$-free, i.e. $f(\lfloor (n-p)/q \rfloor, \mathcal{L}^w)$. $\square$

Lemma 3.20 together with Theorems 3.3 and 3.7 immediately implies Theorem 3.6.

The next result gives a further upper bound on $f_{\max}(n, \mathcal{L}^w)$ for certain linear equations $\mathcal{L}$. Notice that for such $\mathcal{L}$, Theorem 3.5 yields a better bound than Theorem 3.4.

Theorem 3.5. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q \geqslant 2$ are integers so that $p \leqslant q^2 - q$ and $\gcd(p, q) = q$. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 2^{(\mu(n,\mathcal{L}^w) - \mu^*(n,\mathcal{L}))/2 + o(n)}.$$

Proof. Let $\mathcal{F}$ denote the set of containers obtained by applying Lemma 2.5. Since every $\mathcal{L}^w$-free subset of $[n]$ lies in at least one of the $2^{o(n)}$ containers, it suffices to show that every $F \in \mathcal{F}$ houses at most $2^{(\mu(n,\mathcal{L}^w) - \mu^*(n,\mathcal{L}))/2 + o(n)}$ $\mathcal{L}^w$-free sets.

Let $F \in \mathcal{F}$. By Lemmas 2.5(i) and 2.4, $F = A \cup B$ where $|A| = o(n)$, $|B| \leqslant \mu(n, \mathcal{L}^w)$ and B is $\mathcal{L}^w$-free. Note that we can add all the elements of $\mathcal{M}(n, \mathcal{L})$ to B (and thus F) whilst ensuring that $|B| \leqslant \mu(n, \mathcal{L}^w)$ and B is $\mathcal{L}^w$-free. So we may assume that $\mathcal{M}(n, \mathcal{L}) \subseteq B$. Either we have $\min(B) > \lfloor \frac{n}{p+q} \rfloor$ or we use Theorem 3.1 to say that either $\min(B) = \lfloor \frac{n}{p+q} \rfloor - t$ for some non-negative integer $t < (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$ and $|B| \leqslant \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q} t \rfloor$, or $|B| \leqslant \frac{(q^2+1)n}{q^2+q+1}$.

Case 1: $\min(B) = \lfloor \frac{n}{p+q} \rfloor - t$ for $0 \leqslant t < (\frac{p+q-1}{p+q+p/q})\lfloor \frac{n}{p+q} \rfloor$, or $\min(B) > \lfloor \frac{n}{p+q} \rfloor$ (in which case set $t := 0$). Write $F = X \cup Y$ where $Y \subseteq [\lfloor \frac{n}{p+q} \rfloor + 1, n]$ is $\mathcal{L}^w$-free, and $X \subseteq [1, \lfloor \frac{n}{p+q} \rfloor]$. Note that $|X| = t' + o(n)$ and $|Y| \leqslant \lceil \frac{(p+q-1)n}{p+q} \rceil - \lfloor \frac{p}{q} t \rfloor - t' + o(n)$ where $0 \leqslant t' \leqslant t$. Also $\mathcal{M}(n, \mathcal{L}) \subseteq Y$. Choose $S \subseteq X$ to be $\mathcal{L}^w$-free. Consider the link graph $L_S[Y]$ and observe that by Lemma 3.10, $N(S, Y) \leqslant \mathrm{MIS}(L_S[Y])$. (Recall $N(S, Y)$ denotes the number of extensions of S in Y to a maximal $\mathcal{L}^w$-free set.)

Since $p \leqslant q^2 - q$, by Lemma 3.12 $L_S[Y]$ is triangle-free. By definition, $\mathcal{M}(n, \mathcal{L})$ is an independent set in $L_S[Y]$ and so $\mathrm{MIS}(L_S[Y])$=$\mathrm{MIS}(L_S[Y \setminus \mathcal{M}(n, \mathcal{L})])$. Therefore Theorem 3.11(ii) implies that $\mathrm{MIS}(L_S[Y])\leqslant 2^{(|Y| - |\mathcal{M}(n,\mathcal{L})|)/2}$. Overall, this implies that the number of $\mathcal{L}^w$-free sets contained in F is at most

$$2^{|X|} \times 2^{(|Y| - |\mathcal{M}(n,\mathcal{L})|)/2} \leqslant 2^{t' + o(n) + (\mu(n,\mathcal{L}^w) - \mu^*(n,\mathcal{L}) - \lfloor \frac{p}{q} t \rfloor - t')/2} \leqslant 2^{(\mu(n,\mathcal{L}^w) - \mu^*(n,\mathcal{L}))/2 + o(n)},$$

as desired.

Case 2: $|B| \leqslant \frac{(q^2+1)n}{q^2+q+1}$. In this case $|F| \leqslant \frac{(q^2+1)n}{q^2+q+1} + o(n)$. Choose any $\mathcal{L}^w$-free $S \subseteq A$ (note there are at most $2^{o(n)}$ choices for S). Consider the link graph $L_S[B]$ and observe by Lemma 3.10 that $N(S, B) \leqslant \mathrm{MIS}(L_S[B])$. Similarly as in Case 1 we have that $\mathrm{MIS}(L_S[B])$=$\mathrm{MIS}(L_S[B'])$ where $B' := B \setminus \mathcal{M}(n, \mathcal{L})$. By Theorem 3.11(i),

$$\mathrm{MIS}(L_S[B']) \leqslant 3^{|B'|/3} \leqslant 3^{((q^2+1)n/(3(q^2+q+1)) - \mu^*(n,\mathcal{L})/3)} \leqslant 2^{(\mu(n,\mathcal{L}^w) - \mu^*(n,\mathcal{L}))/2 + o(n)}.$$

The last inequality follows since $\mu(n, \mathcal{L}^w) = n - \lfloor n/(p+q) \rfloor$ and $\mathcal{M}(n, \mathcal{L}) = \{s : s > \lfloor (n-p)/q \rfloor, q \nmid s\}$ since $\gcd(p, q) = q$.

To see this, first note that

$$\mu^*(n,\mathcal{L}) = \frac{(q-1)^2 n}{q^2} - o(n).$$

Hence for the inequality to hold we require that

$$9^{((q^2+1)/(q^2+q+1)-(q^2-2q+1)/(q^2))} < 8^{((p+q-1)/(p+q)-(q^2-2q+1)/(q^2))}.$$

Let $a := \log_9 8$. This rearranges to give

$$p > \frac{(1-a)(q^4-q)+q^3+q^2}{(2a-1)q^3+(a-1)(q^2+q-1)}.$$

Since $p \geqslant q$ it suffices to show that $(3a-2)q^3+(a-2)(q^2+q)+(2-2a) > 0$. This indeed holds since $q \geqslant 2$.

Overall, this implies that the number of $\mathcal{L}^w$-free sets contained in F is at most $2^{(\mu(n,\mathcal{L}^w)-\mu^*(n,\mathcal{L}))/2+o(n)}$, as desired. □

The proof of Theorem 3.5 actually generalises to some other equations $px+qy=z$ where $\gcd(p,q) \neq q$ (but still $p \leqslant q^2-q$). However, in these cases Theorem 3.6 produces a better upper bound on $f_{\max}(n,\mathcal{L}^w)$.

3.5.3 Lower bounds for $px+qy=z$

The following result provides lower bounds on $f_{\max}(n,\mathcal{L}^w)$ for all equations $\mathcal{L}$ of the form $px+qy=z$ where $p \geqslant q \geqslant 2$.

Proposition 3.21. *Let $\mathcal{L}$ denote the equation $px+qy=z$ where $p \geqslant q \geqslant 2$ are integers. Suppose that $n > 2p$. In each case $f_{\max}(n,\mathcal{L}^w) \geqslant 2^{\ell}$ where ℓ is defined as follows:*

(i) $\ell := (n(q-1)-pq+q-2q^2)/q^2$ *if* $p \geqslant q^2$,

(ii) $\ell := (n(p-q) - p^2 + q^2 - 2pq)/(pq)$ *if* $q < p < q^2$,

(iii) $\ell := (n - 6q)/2q$ *if* $p = q$.

Proof. For each case, we shall let $B := [\lfloor \frac{n}{p+q} \rfloor + 1, n]$, and consider the link graph $G := L_{\{1\}}[B]$. Since B and $\{1\}$ are $\mathcal{L}^w$-free, by Lemma 3.13 it suffices to show that there is an induced subgraph of G which contains at least 2^ℓ maximal independent sets. For each case we will find an induced perfect matching on 2ℓ vertices in G. (Note there are 2^ℓ maximal independent sets in such a matching.)

More specifically, for each case we shall find an interval $I := [a, b]$ for some $a, b \in V(G)$ and let $J := \{qi + p | \, i \in I\}$. Note that all edges in G (other than at most one loop) are of the form $\{i, qi + p\}$ and $\{i, pi + q\}$. By our choice of I and J, $G[I \cup J]$ will form a perfect matching on $2|I|$ vertices if the following conditions hold:

(1) $qa + p > b$ (which ensures that $I \cap J = \emptyset$),

(2) $qb + p \leqslant n$ (which ensures that $J \subseteq [n]$),

(3) $pa + q > n$ (which ensures that the only edges in $G[I \cup J]$ are of the form $\{i, qi + p\}$),

(4) $p + q < a$ (which ensures that there is no loop at a vertex in $G[I \cup J]$).

Notice that actually we do not require condition (3) to hold in the case when $p = q$. Indeed, this is because in this case an edge $\{i, pi + q\}$ in G is the same as the edge $\{i, qi + p\}$. Further, there is at most one loop in G (if $p + q \in B$). So even if (4) does not hold we will obtain an induced matching in G on $2|I| - 2$ vertices.

Thus, to obtain an induced matching in G on $2|I| - 2$ vertices it suffices to choose a and b so that (1)–(3) hold except when $p = q$ when we only require that (1) and (2) hold.

By choosing $b := \lfloor (n-p)/q \rfloor$, (2) holds since $qb + p = q\lfloor (n-p)/q \rfloor + p \leqslant q(n-p)/q + p = n$.

If $p \geqslant q^2$ then set $a := \lfloor (n-q)/q^2 \rfloor + 1$. Then $a \in B$ and further $pa + q \geqslant q^2 a + q > q^2((n-q)/q^2) + q = n$ and $qa + p \geqslant qa + q^2 > q((n-q)/q^2) + q^2 = n/q - 1 + q^2 > \lfloor (n-p)/q \rfloor = b$. So (1) and (3) hold.

If $q < p < q^2$ then set $a := \lfloor (n-q)/p \rfloor + 1$. So $a \in B$. Further, $pa+q > p((n-q)/p)+q = n$ and $qa+p > q((n-q)/p)+p = qn/p - q^2/p + p > qn/q^2 - q + p > n/q > \lfloor (n-p)/q \rfloor = b$. So (1) and (3) hold.

If $p = q$ set $a := \lfloor n/(p+q) \rfloor + 1 = \lfloor n/(2q) \rfloor + 1 \in B$. Observe that $qa+q > qn/2q+q > n/2 > \lfloor (n-q)/q \rfloor = b$ since $q \geqslant 2$. So (1) holds.

Now calculating the size of the interval $I = [a, b]$ in each case proves the result:

- If $a = \lfloor (n-q)/q^2 \rfloor + 1$, then $|I| - 1 = \lfloor (n-p)/q \rfloor - (\lfloor (n-q)/q^2 \rfloor + 1) \geqslant (n-p)/q - 1 - (n-q)/q^2 - 1 = (n(q-1) - pq + q - 2q^2)/q^2$.

- If $a = \lfloor (n-q)/p \rfloor + 1$, then $|I| - 1 = \lfloor (n-p)/q \rfloor - (\lfloor (n-q)/p \rfloor + 1) \geqslant (n-p)/q - 1 - (n-q)/p - 1 = (n(p-q) - p^2 + q^2 - 2pq)/(pq)$.

- If $a = \lfloor n/(p+q) \rfloor + 1$ and $p = q$, then $|I| - 1 = \lfloor (n-p)/q \rfloor - (\lfloor n/(p+q) \rfloor + 1) \geqslant (n-p)/q - 1 - n/(p+q) - 1 = (pn - (p+2q)(p+q))/(q(p+q)) = (qn - 6q^2)/(2q^2) = (n-6q)/2q$.

□

Although the lower bounds in Proposition 3.21 do not meet the upper bounds in Theorems 3.5 and 3.6 in general, Theorem 3.5 and Proposition 3.21(iii) do immediately imply the following, where we determine $\log(f_{\max}(n, \mathcal{L}^w))$ asymptotically.

Theorem 3.22. *Let $\mathcal{L}$ denote the equation $2x + 2y = z$. Then $f_{\max}(n, \mathcal{L}^w) = 2^{n/4+o(n)}$.*

In the next chapter, we give a general upper bound on $f_{\max}(n, \mathcal{L}^w)$ for equations $\mathcal{L}$ of the form $px + qy = rz$ where $p \geqslant q \geqslant r$ are fixed positive integers (see Theorem 4.5). In particular, our result shows that in the case when $p = q \geqslant 2$, $r = 1$ the lower bound in Proposition 3.21(iii) is correct up to an $o(n)$ term in the exponent.

3.6 Concluding remarks

The results in the chapter show that the parameter $f_{\max}(n, \mathcal{L}^w)$ can exhibit very different behaviour depending on the linear equation $\mathcal{L}$. Indeed, Theorem 3.4 gives a 'crude' general upper bound on $f_{\max}(n, \mathcal{L}^w)$ for all homogeneous three-variable linear equations $\mathcal{L}$. (It is crude in the sense that, in the proof, we do not use any structural information about the link graphs.) However, this bound is close to the correct value of $f_{\max}(n, \mathcal{L}^w)$ for certain equations $\mathcal{L}$ (Proposition 3.19). On the other hand, for many equations this bound is far from tight (Theorem 3.5). Further, for some equations ($x + y = z$ and $2x + 2y = z$) the value of $f_{\max}(n, \mathcal{L}^w)$ is tied to the property that any triangle-free graph on n vertices contains at most $2^{n/2}$ maximal independent sets. Theorem 3.6 and upper bounds we obtain in the next chapter suggest though that the value of $f_{\max}(n, \mathcal{L}^w)$ for other equations $\mathcal{L}$ may depend on completely different factors. Further progress on understanding the possible behaviour of $f_{\max}(n, \mathcal{L}^w)$ would be extremely interesting.

Chapter 4

More on solution-free sets of integers

4.1 Introduction

In the last chapter we obtained results concerning $\mathcal{L}^w$-free subsets of $[n]$ where $\mathcal{L}$ is a fixed linear equation. We made progress on all three of our general motivating questions:

(i) What is the size of the largest $\mathcal{L}^w$-free subset of $[n]$?

(ii) How many $\mathcal{L}^w$-free subsets of $[n]$ are there?

(iii) How many maximal $\mathcal{L}^w$-free subsets of $[n]$ are there?

Recall that we denote the answers to the above questions by $\mu(n, \mathcal{L}^w)$, $f(n, \mathcal{L}^w)$ and $f_{\max}(n, \mathcal{L}^w)$ respectively. In this chapter we use a new lemma to obtain further results for the above three questions. In particular we obtain results for equations of the form $px + qy = rz$ where $p \geqslant q \geqslant r$ and $p, q, r \in \mathbb{N}$. In Chapter 3 we used various auxiliary graphs as a means to bound the size and number of elements in a solution-free subset of $[n]$. In Section 4.2 we introduce a new auxiliary graph which similarly can be used for this purpose.

We also extend our results to equations in more than three variables. The following simple but key proposition allows us to easily do this.

Proposition 4.1. *Let $\mathcal{L}_1$ denote the equation $p_1x_1+\cdots+p_kx_k = b$ where $p_1, \dots, p_k, b \in \mathbb{Z}$ and let $\mathcal{L}_2$ denote the equation $(p_1+p_2)x_1 + p_3x_2 + \cdots + p_kx_{k-1} = b$. Then $\mu(n, \mathcal{L}_1^w) \leqslant \mu(n, \mathcal{L}_2^w)$ and $f(n, \mathcal{L}_1^w) \leqslant f(n, \mathcal{L}_2^w)$.*

The proposition is just a simple consequence of the observation that any solution to the equation $\mathcal{L}_2$ gives rise to a solution to the equation $\mathcal{L}_1$. So all $\mathcal{L}_1^w$-free subsets of $[n]$ are also $\mathcal{L}_2^w$-free.

4.1.1 The size of the largest solution-free set

The first key question in the study of $\mathcal{L}^w$-free sets listed in the introduction, is to establish the size $\mu(n, \mathcal{L}^w)$ of the largest $\mathcal{L}^w$-free subset of $[n]$. When $\mathcal{L}$ is a homogeneous equation in two variables, the value of $\mu(n, \mathcal{L}^w)$ is known exactly and an extremal $\mathcal{L}^w$-free set can be found by greedy choice. See [53] for further details. For homogeneous linear equations in three variables, the picture is not as clear. First note we may assume without loss of generality that the equation is of the form $px + qy = rz$, where p, q, r are fixed positive integers, and $\gcd(p, q, r) = 1$.

Now consider the following two natural candidates for extremal sets. Let $t := \gcd(p, q)$ and let a be the unique non-negative integer $0 \leqslant a < t$ such that $n - a$ is divisible by t. The interval

$$I_n := \left[\left\lfloor \frac{r(n-a)}{p+q} \right\rfloor + 1, n\right]$$

is $\mathcal{L}^w$-free. To see this observe that since $\gcd(p, q, r) = 1$ and $\gcd(p, q) = t$, any solution (x, y, z) to $\mathcal{L}$ with $x, y, z \in I_n$ must have z divisible by t. Since $px + qy > r(n-a)$, z must lie in $[n-a+1, n]$; however then z is not divisible by t and so I_n is $\mathcal{L}^w$-free. Note that when $r = 1$, $I_n := [\lfloor r(n-a)/(p+q) \rfloor + 1, n] = [\lfloor rn/(p+q) \rfloor + 1, n]$, though this does not hold in general when $r > 1$. Notice also that I_n is only a candidate for an extremal set if r is 'small'. Indeed, if $r > p + q$ and n is sufficiently large then $I_n = \emptyset$. The set

$$T_n := \{x \in [n] : x \not\equiv 0 \bmod t\}$$

is also $\mathcal{L}^w$-free: note that in any solution (x, y, z) to $\mathcal{L}$, z must be divisible by t since $\gcd(r, t) = 1$. But T_n contains no elements divisible by t.

This raises the following question.

Question 4.2. *For which $\mathcal{L}$ do we have $\mu(n, \mathcal{L}^w) = \max\{|I_n|, |T_n|\}$?*

When $\mathcal{L}$ is $x + y = z$ it is easy to see that $\mu(n, \mathcal{L}^w) = \lceil n/2 \rceil$ and the interval $I_n = [\lfloor n/2 \rfloor + 1, n]$ is an extremal set of this size. In the previous chapter we established that when $\mathcal{L}$ is the equation $px+qy = z$ with $p, q \in \mathbb{N}$, $p \geqslant 2$, we have $\mu(n, \mathcal{L}^w) = n - \lfloor n/(p+q) \rfloor$ for sufficiently large n. Again this bound is attained by the interval I_n. Our first result of this chapter determines a further class of equations (of the form $px + qy = rz$) for which I_n or T_n gives an $\mathcal{L}$-free set of maximum size.

Theorem 4.3. *Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$ and p, q, r are fixed positive integers satisfying $\gcd(p, q, r) = 1$. Let $t := \gcd(p, q)$. Write $r_1 := p/t$ and $r_2 := q/t$.*

(i) If q divides p and $p + q \leqslant rq$ then $\mu(n, \mathcal{L}^w) = |T_n| = \lceil (q-1)n/q \rceil$;

(ii) If q divides p and $p+q \geqslant rq$ then $\mu(n, \mathcal{L}^w) = |I_n| = \lceil (p+q-r)(n-a)/(p+q) \rceil + a$ where a is the unique non-negative integer $0 \leqslant a < q$ such that $n - a$ is divisible by q;

(iii) If q does not divide p, $t > 1$ and

$$r > (r_1 r_2 - r_1 - r_2 + 4) r_2 \left(r_1 + 1 + \frac{r_2 - 1}{r_1^2 + (r_1 - 1)(r_2 - 1)} \right)$$

then $\mu(n, \mathcal{L}^w) = |T_n| = \lceil (t-1)n/t \rceil$.

Theorem 4.3(ii) was already proven (for large enough n) in the last chapter (see Corollary 3.2) in the special case when $r = 1$. (Note though that Corollary 3.2 determines $\mu(n, \mathcal{L}^w)$ for many equations $\mathcal{L}$ not covered by Theorem 4.3.) Previously, Hegarty [53] proved Theorem 4.3(i) in the case when $p = q$. In Section 4.3 we also give a generalisation of Theorem 4.3 concerning some linear equations with more variables (see Corollary 4.14).

Notice that in the case when q divides p, Theorem 4.3 gives a dichotomy for the value of $\mu(n, \mathcal{L}^w)$: when $p + q \leqslant rq$ the set T_n is a largest $\mathcal{L}^w$-free subset of $[n]$, whilst when $p+q \geqslant rq$ the interval I_n is a largest $\mathcal{L}^w$-free subset of $[n]$. Theorem 4.3 does not provide us with as much information for the case when q does not divide p; note though it is not true that a similar dichotomy occurs in this case. Take the equation $3x + 2y = 2z$; here we have $|I_n| \approx 3n/5$ and $|T_n| = 0$. However the set $A_n := \{x \in [n] : x \not\equiv 0 \bmod 2$ or $x > 2n/3\}$ has size $|A_n| \approx 2n/3$ and is $\mathcal{L}^w$-free, since any solution (x, y, z) to $\mathcal{L}$ must have that x is even and $x \leqslant 2n/3$. It would be very interesting to fully resolve the case where $p \geqslant q \geqslant r$ and q does not divide p.

For equations $px+qy = rz$ where r is bigger than p, q, there are a range of cases where an extremal set is known and it is neither I_n nor T_n; see [11, 31, 53] for these, and also other results on the size of the largest $\mathcal{L}^w$-free subset of $[n]$ for various $\mathcal{L}$.

4.1.2 The number of solution-free sets

Recall Green [48] proved that $f(n, \mathcal{L}^w) = 2^{\mu(n,\mathcal{L}^w)+o(n)}$ for any fixed homogeneous linear equation $\mathcal{L}$. In the last chapter we were able to replace the term $o(n)$ in the exponent with a constant for certain types of linear equation $\mathcal{L}$. In Section 4.4 we find further linear equations where we can omit the term $o(n)$:

Theorem 4.4. *Let* $\mathcal{L}$ *denote the equation* $px + qy = rz$ *where* $p \geqslant q \geqslant r$, p, q, r *are fixed positive integers satisfying* $\gcd(p, q, r) = 1$. *Let* $t := \gcd(p, q)$. *Write* $r_1 := p/t$ *and*

$r_2 := q/t$. *If*

$$r > (r_1r_2 - r_1 - r_2 + 4)r_2\left(r_1 + 1 + \frac{r_2 - 1}{r_1^2 + (r_1 - 1)(r_2 - 1)}\right)$$

then $f(n, \mathcal{L}^w) = \Theta(2^{\mu(n,\mathcal{L}^w)})$.

By applying Proposition 4.1 we also obtain equations $\mathcal{L}$ with more than three variables for which $f(n, \mathcal{L}^w) = \Theta(2^{\mu(n,\mathcal{L}^w)})$.

4.1.3 The number of maximal solution-free sets

In this chapter we prove the following result.

Theorem 4.5. *Let* $\mathcal{L}$ *denote the equation* $px + qy = rz$ *where* $p \geqslant q \geqslant r$ *and* p, q, r *are fixed positive integers satisfying* $\gcd(p, q, r) = 1$. *Let* $t := \gcd(p, q)$. *Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 2^{\frac{Crn}{q}+o(n)} \text{ where } C := 1 - \frac{t}{p+q}\left(\frac{p^2 + (p-t)(q-t)}{p^2}\right).$$

For a wide class of equations $\mathcal{L}$ this is the current best known upper bound on $f_{\max}(n, \mathcal{L}^w)$; see Section 4.6 for more details. In the case when $p = q \geqslant 2$ and $r = 1$, the upper bound given by Theorem 4.5 is actually exact up to the error term in the exponent.

Theorem 4.6. *Let* $\mathcal{L}$ *denote the equation* $qx + qy = z$ *where* $q \geqslant 2$ *is an integer. Then*

$$f_{\max}(n, \mathcal{L}^w) = 2^{n/2q+o(n)}.$$

In Section 4.5 we will also generalise Theorem 4.5 to consider some linear equations with more variables (see Theorem 4.18).

For the proof of both Theorems 4.3 and 4.5, a simple but crucial tool is a result (Lemma 4.9) which ensures a certain auxiliary graph contains a large collection of disjoint edges. As in the previous chapter, we also make use of container and removal lemmas of Green [48].

In the next section we collect together a number of useful tools and lemmas. We prove our results on the size of the largest solution-free subset of $[n]$, on the number of solution-free subsets of $[n]$, and on the number of maximal solution-free subsets of $[n]$, in Sections 4.3, 4.4 and 4.5 respectively.

4.2 Link hypergraphs and the main lemmas

4.2.1 Link hypergraphs

One can turn the problem of counting the number of maximal $\mathcal{L}^w$-free subsets of $[n]$ into one of counting maximal independent sets in an auxiliary graph. Similar techniques were used in [121, 6, 7], and in the graph setting in [10, 5]. Recall that in the previous chapter we defined the link graph, which we used to deal with equations with three variables. Since we will consider equations with more than three variables in this chapter, we need to generalise this definition.

Consider the following generalisation of a link graph $L_S[B]$ to that of a *link hypergraph*: Let B and S be disjoint subsets of $[n]$ and let $\mathcal{L}$ denote the equation $p_1x_1+\cdots+p_kx_k = 0$ where $p_1, \dots, p_k$ are fixed non-zero integers. The *link hypergraph* $L_S[B]$ of S on B has vertex set B; It has an edge set consisting of hyperedges between $s \leqslant k$ distinct vertices $v_1, \dots, v_s$ of B, whenever there is a solution $(x_1, \dots, x_k)$ to $\mathcal{L}$ in which $\{x_1, \dots, x_k\} \subseteq S \cup \{v_1, \dots, v_s\}$ and $\{v_1, \dots, v_s\} \subseteq \{x_1, \dots, x_k\}$. In this definition one could have edges corresponding to trivial solutions. However in our applications, since we only consider non-translation-invariant equations, there are no trivial solutions.

The link graph lemmas used in the previous chapter (Lemmas 3.10 and 3.13) can easily be extended to the hypergraph case.

Lemma 4.7. *Let $\mathcal{L}$ denote a non-translation-invariant linear equation. Suppose that B, S are disjoint $\mathcal{L}^w$-free subsets of $[n]$. If $I \subseteq B$ is such that $S \cup I$ is a maximal $\mathcal{L}^w$-free subset of $[n]$, then I is a maximal independent set in the link hypergraph $L_S[B]$.* $\square$

As with Lemma 3.10 in the previous chapter, the above result can be used in conjunction with the container lemma as follows. Let $F = A \cup B$ be a container as in Lemma 2.5 where $|A| = o(n)$ and B is $\mathcal{L}^w$-free. Observe that any maximal $\mathcal{L}^w$-free subset of $[n]$ in F can be found by first selecting an $\mathcal{L}^w$-free subset $S \subseteq A$, and then extending S in B. Then the number of extensions of S in B is bounded by $\mathrm{MIS}(L_S[B])$ by Lemma 4.7.

We can also use link graphs to obtain lower bounds.

Lemma 4.8. *Let $\mathcal{L}$ denote a non-translation-invariant linear equation. Suppose that B, S are disjoint $\mathcal{L}^w$-free subsets of $[n]$. Let H be an induced subgraph of the link graph $L_S[B]$. Then $f_{\max}(n, \mathcal{L}^w) \geqslant \mathrm{MIS}(H)$.* $\square$

The proof is analogous to that of Lemma 3.13.

4.2.2 The main lemmas

Here we use a specific link graph as a means to bound the number of elements in a solution-free subset of $[n]$.

Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$ and p, q, r are fixed positive integers satisfying $\gcd(p, q, r) = 1$. Let $t := \gcd(p, q)$ and write $r_1 := p/t$, $r_2 := q/t$. Fix $M \in [n]$ such that M is divisible by t. We define the graph G_M to have vertex set $[\lceil rM/q \rceil - 1]$ and an edge between x and y whenever $px + qy = rM$.

Lemma 4.9. *The graph G_M contains a collection E of vertex-disjoint edges where*

$$|E| = \left\lfloor \frac{rM}{r_2(p+q)} \right\rfloor + (r_1 r_2 - r_1 - r_2 + 1) \left\lfloor \left\lfloor \frac{rM}{r_1(p+q)} - \frac{1}{r_2} \right\rfloor \frac{1}{r_1 r_2} \right\rfloor$$

and at most one edge in E is a loop.

Proof. All edges in G_M are pairs of the form $\{s, (rM - sp)/q\}$ for some $s \in \mathbb{N}$ since $ps + q(rM - sp)/q = rM$. Since $p = r_1 t$ and $q = r_2 t$ where r_1 and r_2 are coprime, for a fixed integer s precisely one element in $\{(rM - (s-j)p)/q : 0 \leqslant j < r_2\}$ is an integer. (Note here we are using that M is divisible by t.) In other words there exists a unique $x \in \mathbb{N}$, $1 \leqslant x \leqslant r_2$ such that $(rM - xp)/q$ is an integer, and all edges in G_M are of the form $\{x + ar_2, (rM - xp)/q - ar_1\}$ for some non-negative integer a. In particular there is an edge incident to $x + ar_2$ provided a satisfies $(rM - xp)/q - ar_1 \in \mathbb{N}$.

Write $y := (rM - xp)/q$. Note that if $x + ar_2 \leqslant rM/(p+q)$ then

$$y - ar_1 = \frac{rM - (x + ar_2)p}{q} \geqslant \frac{rM - prM/(p+q)}{q} = \frac{rM}{p+q}.$$

Hence there are $\lfloor rM/(r_2(p+q)) \rfloor$ distinct edges in G_M of the form $\{x + ar_2, y - ar_1\}$ with $x + ar_2 \leqslant rM/(p+q) \leqslant y - ar_1$. Note that one of these edges may be a loop. (This will be at $rM/(p+q)$ in the case when $rM/(p+q) \in \mathbb{N}$.) Call this collection of edges E_1. Our next aim is to find an additional collection E_2 of edges in G_M that is vertex-disjoint from E_1.

Note that $x + ar_2 \equiv x \bmod r_2$ and $y - ar_1 \equiv y \bmod r_1$. Also $p(rM/p) + q(0) = rM$ and $\lceil rM/p \rceil \leqslant \lceil rM/q \rceil$, hence there are at least

$$\left\lfloor \left(\left\lceil \frac{rM}{p} \right\rceil - 1 - \left\lfloor \frac{rM}{p+q} \right\rfloor \right) / r_2 \right\rfloor \geqslant \left\lfloor \left(\frac{rM}{p} - \frac{rM}{p+q} - 1 \right) / r_2 \right\rfloor = \left\lfloor \frac{rM}{r_1(p+q)} - \frac{1}{r_2} \right\rfloor$$

edges in G_M of the form $\{x + ar_2, y - ar_1\}$ with $x + ar_2 > rM/(p+q)$. Consider a set of $r_1 r_2$ edges $\{\{x + ar_2 + br_2, y - ar_1 - br_1\} : 0 \leqslant b < r_1 r_2\}$ for a fixed a. Since r_1 and r_2 are coprime, precisely r_2 of these edges (1 in r_1 of them) have $x + ar_2 + br_2 \equiv y \bmod r_1$, and precisely r_1 of these edges have $y - ar_1 - br_1 \equiv x \bmod r_2$. (Also, precisely 1 edge satisfies both.) In all other cases since $x + ar_2 + br_2 \not\equiv y \bmod r_1$ and $y - ar_1 - br_1 \not\equiv x \bmod r_2$, the edge $\{x + ar_2 + br_2, y - ar_1 - br_1\}$ is vertex-disjoint from E_1. Hence we obtain a set E_2

of at least $(r_1r_2 - r_1 - r_2 + 1)\lfloor \lfloor rM/(r_1(p+q)) - 1/r_2 \rfloor /(r_1r_2)\rfloor$ additional distinct edges. Thus $E := E_1 \cup E_2$ is our desired set. $\square$

Observe that the graph G_M is a link graph $L_S[B]$, where $S := \{M\}$ and $B := [\lceil rM/q \rceil - 1]$. If we wish to extend a solution-free set S into a solution-free subset of $S \cup B$, then we must pick an independent set in $L_S[B]$. Similarly here if we wish to obtain a solution-free subset of $[n]$ which contains M divisible by t, then we must pick an independent set in G_M. This is the idea behind the following key lemma, which allows us to bound the number of elements in such an $\mathcal{L}^w$-free set.

Lemma 4.10. *Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$ and p, q, r are fixed positive integers satisfying $\gcd(p, q, r) = 1$. Let $t := \gcd(p, q)$ and write $r_1 := p/t$ and $r_2 := q/t$. Let S be an $\mathcal{L}^w$-free subset of $[n]$. If $M \in S$ is divisible by t, then S contains at most*

$$\left\lceil \frac{rM}{q} \right\rceil - 1 - \left\lfloor \frac{rM}{r_2(p+q)} \right\rfloor - (r_1r_2 - r_1 - r_2 + 1)\left\lfloor \left\lfloor \frac{rM}{r_1(p+q)} - \frac{1}{r_2} \right\rfloor \frac{1}{r_1r_2} \right\rfloor$$

elements from $[\lceil rM/q \rceil - 1]$.

Proof. Consider the graph G_M and observe that its edges correspond to $\mathcal{L}$-triples: since $p \geqslant q \geqslant r$ there is an edge between x and y precisely when $\{x, y, M\}$ is an $\mathcal{L}$-triple. Hence if $I \subseteq V(G_M)$ is such that $I \cup \{M\}$ is an $\mathcal{L}^w$-free subset of $[n]$ then I is an independent set in G_M. As a consequence if we find a set of vertex-disjoint edges in G_M of size J, then S contains at most $\lceil rM/q \rceil - 1 - J$ elements from $[\lceil rM/q \rceil - 1]$. The result then follows by applying Lemma 4.9. $\square$

First note that if $\mathcal{L}$ denotes the equation $x + y = z$, then in Lemma 4.10 we are simply saying that if a sum-free set S contains M, then it cannot contain both 1 and $M - 1$, it cannot contain both 2 and $M - 2$, and so on. So in a sense this lemma is a generalisation of the proof that sum-free subsets of $[n]$ cannot contain more than $\lceil n/2 \rceil$ elements.

Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$ and p, q, r are fixed positive integers satisfying $\gcd(p, q, r) = 1$ and let $t := \gcd(p, q)$. Recall that $T_n := \{x \in [n] : x \not\equiv 0 \mod t\}$ is $\mathcal{L}^w$-free. Lemma 4.10 roughly implies that every $\mathcal{L}^w$-free subset of $[n]$ must have 'not too many small elements' or must 'look like' T_n. Clearly this lemma gives rise to an upper bound on the size of the largest $\mathcal{L}^w$-free subset of $[n]$. In Section 4.5 we also show that this lemma can be used to obtain an upper bound on the number of maximal $\mathcal{L}^w$-free subsets of $[n]$.

Recall from the introduction that we can use the following simple proposition to extend our results for linear equations with three variables to linear equations with more than three variables.

Proposition 4.1. *Let $\mathcal{L}_1$ denote the equation $p_1x_1 + \cdots + p_kx_k = b$ where $p_1, \ldots, p_k, b \in \mathbb{Z}$ and let $\mathcal{L}_2$ denote the equation $(p_1 + p_2)x_1 + p_3x_2 + \cdots + p_kx_{k-1} = b$. Then $\mu(n, \mathcal{L}_1^w) \leqslant \mu(n, \mathcal{L}_2^w)$ and $f(n, \mathcal{L}_1^w) \leqslant f(n, \mathcal{L}_2^w)$.*

Proof. If $(p_1 + p_2)x_1 + p_3x_2 + \cdots + p_kx_{k-1} = b$ for some $x_i \in [n]$, $1 \leqslant i \leqslant k-1$, then $p_1x_1 + p_2x_1 + p_3x_2 + \cdots + p_kx_{k-1} = b$. Hence any solution to $\mathcal{L}_2$ in $[n]$ gives rise to a solution to $\mathcal{L}_1$ in $[n]$. So if $A \subseteq [n]$ is $\mathcal{L}_1^w$-free, then A is also $\mathcal{L}_2^w$-free. Hence the size of the largest $\mathcal{L}_2^w$-free set is at least the size of the largest $\mathcal{L}_1^w$-free set, and also there are at least as many $\mathcal{L}_2^w$-free sets as there are $\mathcal{L}_1^w$-free sets. $\square$

We will also make use of the following trivial fact.

Fact 4.11. *Suppose $\mathcal{L}_1$ is a linear equation and $\mathcal{L}_2$ is a positive integer multiple of $\mathcal{L}_1$. Then the set of $\mathcal{L}_1^w$-free subsets of $[n]$ is precisely the set of $\mathcal{L}_2^w$-free subsets of $[n]$. In particular $\mu(n, \mathcal{L}_1^w) = \mu(n, \mathcal{L}_2^w)$, $f(n, \mathcal{L}_1^w) = f(n, \mathcal{L}_2^w)$ and $f_{\max}(n, \mathcal{L}_1^w) = f_{\max}(n, \mathcal{L}_2^w)$.*

The two results above allow us to extend the use of Lemma 4.10 to equations with more than three variables.

Lemma 4.12. *Let $\mathcal{L}$ denote the equation $p_1x_1+\cdots+p_kx_k = 0$ where $p_i \in \mathbb{Z}$. Suppose there is a partition of the p_i into three non-empty parts P_1, P_2 and P_3 where $p' := \sum_{p_j \in P_1} p_j$, $q' := \sum_{p_j \in P_2} p_j$ and $r' := -\sum_{p_j \in P_3} p_j$ satisfy $p' \geqslant q' \geqslant r' \geqslant 1$. Let $t' := \gcd(p', q', r')$ and write $p := p'/t'$, $q := q'/t'$ and $r := r'/t'$. Let $t := \gcd(p, q)$ and write $r_1 := p/t$ and $r_2 := q/t$. Let S be an $\mathcal{L}^w$-free subset of $[n]$. If $M \in S$ is divisible by t, then S contains at most*

$$\left\lceil \frac{rM}{q} \right\rceil - 1 - \left\lfloor \frac{rM}{r_2(p+q)} \right\rfloor - (r_1r_2 - r_1 - r_2 + 1) \left\lfloor \left\lfloor \frac{rM}{r_1(p+q)} - \frac{1}{r_2} \right\rfloor \frac{1}{r_1r_2} \right\rfloor$$

elements from $[\lceil rM/q \rceil - 1]$.

Proof. Let $\mathcal{L}_2$ denote the equation $px + qy = rz$. Now observe by repeatedly applying Proposition 4.1 and Fact 4.11 that any $\mathcal{L}^w$-free set is also an $\mathcal{L}_2^w$-free set. Hence S must be $\mathcal{L}_2^w$-free and so we simply apply Lemma 4.10. □

This bounds the number of 'small elements' in solution-free sets for equations with more than three variables, and in Theorem 4.18 we will use this lemma to obtain a result for the number of maximal solution-free sets.

4.3 The size of the largest solution-free set

The aim of this section is to use our results from the previous section to obtain bounds on $\mu(n, \mathcal{L}^w)$ for linear equations $\mathcal{L}$ of the form $px + qy = rz$ with $p \geqslant q \geqslant r$ positive integers and also linear equations with more than three variables. As previously mentioned we can use Lemma 4.10 to obtain a bound on the size of a solution-free set.

Corollary 4.13. *Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$ and p, q, r are fixed positive integers satisfying $\gcd(p, q, r) = 1$. Let S be an $\mathcal{L}^w$-free subset of $[n]$ and suppose M is the largest element of S divisible by $t := \gcd(p, q)$. Write $r_1 := p/t$ and*

$r_2 := q/t$. *Then*

$$|S| \leqslant M - \left\lfloor \frac{rM}{r_2(p+q)} \right\rfloor - (r_1r_2 - r_1 - r_2 + 1) \left\lfloor \left\lfloor \frac{rM}{r_1(p+q)} - \frac{1}{r_2} \right\rfloor \frac{1}{r_1r_2} \right\rfloor + \left\lceil \frac{(n-M)(t-1)}{t} \right\rceil.$$

Proof. By Lemma 4.10, S contains at most $\lceil rM/q \rceil - 1 - \lfloor rM/(r_2(p+q)) \rfloor - (r_1r_2 - r_1 - r_2 + 1) \lfloor \lfloor rM/(r_1(p+q)) - 1/r_2 \rfloor / (r_1r_2) \rfloor$ elements from $[\lceil rM/q \rceil - 1]$. It also cannot contain any element larger than M and divisible by t. $\square$

Note in the statement of Corollary 4.13 we are implicitly assuming that M exists. If it does not then $|S| \leqslant \lceil n(t-1)/t \rceil$.

We are now ready to prove Theorem 4.3, which determines $\mu(n, \mathcal{L}^w)$ for a wide class of equations of the form $px + qy = rz$ where $p \geqslant q \geqslant r$ and p, q, r are fixed positive integers.

Theorem 4.3. *Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$ and p, q, r are fixed positive integers satisfying $\gcd(p, q, r) = 1$. Let $t := \gcd(p, q)$. Write $r_1 := p/t$ and $r_2 := q/t$.*

(i) If q divides p and $p + q \leqslant rq$ then $\mu(n, \mathcal{L}^w) = \lceil (q-1)n/q \rceil$;

(ii) If q divides p and $p + q \geqslant rq$ then $\mu(n, \mathcal{L}^w) = \lceil (p+q-r)(n-a)/(p+q) \rceil + a$ where a is the unique non-negative integer $0 \leqslant a < q$ such that $n - a$ is divisible by q;

(iii) If q does not divide p, $t > 1$ and

$$r > (r_1r_2 - r_1 - r_2 + 4)r_2 \left(r_1 + 1 + \frac{r_2 - 1}{r_1^2 + (r_1 - 1)(r_2 - 1)} \right)$$

then $\mu(n, \mathcal{L}^w) = \lceil (t-1)n/t \rceil$.

Proof. Let S be an $\mathcal{L}^w$-free subset of $[n]$ and suppose M is the largest element of S divisible by t. If S does not contain an element divisible by t, set $M := 0$. If q divides p

then $t = q$ and $r_2 = 1$ and hence by Corollary 4.13 we have

$$|S| \leqslant \left\lceil \frac{(p+q-r)M}{p+q} \right\rceil + \left\lceil \frac{(n-M)(q-1)}{q} \right\rceil. \tag{4.3.1}$$

(This is true even in the case $M = 0$.)

If $p + q \leqslant rq$ then $|S| \leqslant \lceil (q-1)M/q \rceil + \lceil (n-M)(q-1)/q \rceil = \lceil n(q-1)/q \rceil$ since M is divisible by q. Observe that the set $T_n := \{x \in [n] : x \not\equiv 0 \bmod t\}$ is an $\mathcal{L}^w$-free set obtaining this size, and so this proves (i).

For (ii) we will show that (4.3.1) is an increasing function of M (when restricted to running through M divisible by t) and hence it will be maximised by taking $M = n - a$. Then $|S| \leqslant \lceil (p+q-r)(n-a)/(p+q) \rceil + a$. Observe that the interval $I_n := [\lfloor r(n-a)/(p+q) \rfloor + 1, n]$ is an $\mathcal{L}^w$-free set obtaining this size and so this proves (ii), provided (4.3.1) is an increasing function of M.

Since M must be divisible by $t = q$, write $M' := M/q$ and so (4.3.1) can be written as

$$\left\lceil \frac{((r_1+1)q - r)M'}{r_1+1} \right\rceil + \left\lceil \frac{n(q-1)}{q} \right\rceil - M'(q-1) = M' + \left\lceil \frac{-rM'}{r_1+1} \right\rceil + \left\lceil \frac{n(q-1)}{q} \right\rceil.$$

Now observe that the difference between successive terms M' and $M'+1$ is given by

$$M' + 1 + \left\lceil \frac{-r(M'+1)}{r_1+1} \right\rceil - M' - \left\lceil \frac{-rM'}{r_1+1} \right\rceil = 1 + \left\lceil \frac{-rM'}{r_1+1} - \frac{r}{r_1+1} \right\rceil - \left\lceil \frac{-rM'}{r_1+1} \right\rceil \geqslant 0$$

where the inequality follows since $r_1 + 1 \geqslant r$. Hence (4.3.1) is an increasing function of M as required.

For (iii) if $M = 0$ then $|S| \leqslant \lceil n(t-1)/t \rceil$ as required. So assume $M \geqslant t$. Then by Corollary 4.13 we have

$$
\begin{aligned}
|S| &\leqslant M - \left\lfloor \frac{rM}{r_2(p+q)} \right\rfloor - (r_1r_2 - r_1 - r_2 + 1) \left\lfloor \left\lfloor \frac{rM}{r_1(p+q)} - \frac{1}{r_2} \right\rfloor \frac{1}{r_1r_2} \right\rfloor \\
&\quad + \left\lceil \frac{(n-M)(t-1)}{t} \right\rceil \\
&\leqslant M - \frac{rM}{r_2(p+q)} + 1 - \frac{r_1r_2 - r_1 - r_2 + 1}{r_1r_2} \left(\frac{rM}{r_1(p+q)} - \frac{1}{r_2} - 1 \right) + r_1r_2 - r_1 - r_2 + 1 \\
&\quad - \frac{M(t-1)}{t} + \left\lceil \frac{n(t-1)}{t} \right\rceil \\
&\leqslant \left\lceil \frac{n(t-1)}{t} \right\rceil + r_1r_2 - r_1 - r_2 + 3 - M \left(\frac{r(r_1^2 + (r_1-1)(r_2-1))}{tr_1^2r_2(r_1+r_2)} - \frac{1}{t} \right) \\
&= \left\lceil \frac{n(t-1)}{t} \right\rceil + r_1r_2 - r_1 - r_2 + 3 - M \left(\frac{r}{tr_2} \left(r_1 + 1 + \frac{r_2 - 1}{r_1^2 + (r_1-1)(r_2-1)} \right)^{-1} - \frac{1}{t} \right) \\
&\leqslant \left\lceil \frac{n(t-1)}{t} \right\rceil + r_1r_2 - r_1 - r_2 + 3 - M \left(\frac{r_1r_2 - r_1 - r_2 + 4}{t} - \frac{1}{t} \right) \\
&\leqslant \left\lceil \frac{n(t-1)}{t} \right\rceil,
\end{aligned}
$$

where the penultimate inequality follows by our lower bound on r and the last inequality follows by using $M \geqslant t$. □

For Theorem 4.3(iii) it is easy to check that actually given the conditions on r we must always have $t > 1$ (we just state $t > 1$ in the theorem for clarity). As an example, $p := 3t$, $q := 2t$, $r \geqslant 41$, and $t \geqslant r/2$ gives a set of equations which satisfy the conditions of Theorem 4.3(iii).

Theorem 4.3 together with Proposition 4.1 yield results for $\mu(n, \mathcal{L}^w)$ where $\mathcal{L}$ is an equation with more than three variables.

Corollary 4.14. *Let $\mathcal{L}$ denote the equation $a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_\ell y_\ell = c_1z_1 + \cdots + c_mz_m$ where the $a_i, b_i, c_i \in \mathbb{N}$ and $p' := \sum_i a_i$, $q' := \sum_i b_i$ and $r' := \sum_i c_i$ satisfy $p' \geqslant q' \geqslant r'$. Let $t' := \gcd(p', q', r')$ and write $p := p'/t'$, $q := q'/t'$ and $r := r'/t'$. Let $t := \gcd(p, q)$.*

(i) If $m = 1$, $\ell = 1$, $q' = b_1$ divides a_i for all $1 \leqslant i \leqslant k$ and $p+q \leqslant rq$ then $\mu(n,\mathcal{L}^w) = \lceil (q-1)n/q \rceil$;

(ii) If q divides p and $p+q \geqslant rq$ then $\lceil (p+q-r)n/(p+q) \rceil \leqslant \mu(n,\mathcal{L}^w) \leqslant \lceil (p+q-r)(n-a)/(p+q) \rceil + a$ where a is the unique non-negative integer $0 \leqslant a < q$ such that $n-a$ is divisible by q;

(iii) Write $r_1 := p/t$ and $r_2 := q/t$. If q does not divide p, $m = 1$, tt' divides a_i and b_j for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant \ell$ and

$$r > (r_1 r_2 - r_1 - r_2 + 4) r_2 \left(r_1 + 1 + \frac{r_2 - 1}{r_1^2 + (r_1 - 1)(r_2 - 1)} \right)$$

then $\mu(n,\mathcal{L}^w) = \lceil (t-1)n/t \rceil$.

□

Proof. Let $\mathcal{L}_2$ denote the equation $px + qy = rz$. Then we have $\mu(n,\mathcal{L}^w) \leqslant \mu(n,\mathcal{L}_2^w)$ by repeated use of Proposition 4.1 and Fact 4.11. Then the use of Theorem 4.3(i), (ii) and (iii) respectively for each of the three cases stated gives the required upper bound.

For the lower bounds for each case we show that a suitable $\mathcal{L}_2^w$-free set is also $\mathcal{L}^w$-free. For (i) and (iii) consider the $\mathcal{L}_2^w$-free set $T := \{x \in [n] : x \not\equiv 0 \bmod t\}$. Since tt' divides a_i and b_j for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant \ell$ (noting for (i) that $tt' = qt' = q'$), any solution $\{x_1, \dots, x_k, y_1, \dots, y_\ell, z_1\}$ to $\mathcal{L}$ must have z_1 divisible by t. But T contains no elements divisible by t and hence $\mu(n,\mathcal{L}^w) \geqslant \lceil (t-1)n/t \rceil$.

For (ii) consider the $\mathcal{L}_2^w$-free set $[\lfloor rn/(p+q) \rfloor + 1, n]$. Observe that

$$(a_1 + \cdots + a_k + b_1 + \cdots + b_\ell)(\lfloor rn/(p+q) \rfloor + 1) = (p' + q')(\lfloor r'n/(p'+q') \rfloor + 1)$$
$$> r'n = (c_1 + \cdots + c_m)n$$

and so there are no solutions to $\mathcal{L}$ in the interval $[\lfloor rn/(p+q)\rfloor + 1, n]$, and so $\mu(n, \mathcal{L}^w) \geqslant \lceil (p+q-r)n/(p+q)\rceil$. $\square$

Note that for Corollary 4.14(ii) the lower bound is very close to the upper bound, and in particular matches it in the case where $r = 1$.

4.4 The number of solution-free sets

In this section we will apply Lemma 4.9 and Proposition 4.1 to obtain upper bounds on the number of solution-free sets for various equations. Let $f(n, \mathcal{L}^w, M)$ denote the number of $\mathcal{L}^w$-free subsets of $[n]$ with maximum element M.

Lemma 4.15. *Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$ and p, q, r are fixed positive integers satisfying $\gcd(p, q, r) = 1$. Assume that $M \in [n]$ is divisible by $t := \gcd(p, q)$. Write $r_1 := p/t$ and $r_2 := q/t$, and let $c := 2^{(2-\log_2 3)(r_1r_2 - r_1 - r_2 + 4) - 1}$. Then*

$$f(n, \mathcal{L}^w, M) \leqslant c \cdot 2^{M(1 - rt(2-\log_2 3)(p^2 + (p-t)(q-t))/(p^2q(p+q)))}.$$

Proof. Note that if I is an $\mathcal{L}^w$-free set with maximum element M, then $I \cap [\lceil rM/q\rceil - 1]$ is an independent set in G_M. Since M is divisible by t, by Lemma 4.9 there exists a matching H in G_M of size

$$\begin{aligned} J := &\left\lfloor \frac{rM}{r_2(p+q)} \right\rfloor + (r_1r_2 - r_1 - r_2 + 1)\left\lfloor \left\lfloor \frac{rM}{r_1(p+q)} - \frac{1}{r_2} \right\rfloor \frac{1}{r_1r_2} \right\rfloor - 1 \\ \geqslant &\, rtM\Big(\frac{p^2 + (p-t)(q-t)}{p^2q(p+q)}\Big) - r_1r_2 + r_1 + r_2 - 4. \end{aligned}$$

Let $A_1, \dots, A_{J+1}$ be a partition of $V(G_M)$ where if $1 \leqslant i \leqslant J$ then A_i contains precisely the two vertices from some edge in H. So $|A_{J+1}| = \lceil rM/q\rceil - 1 - 2J$. Recalling that we use $i(G)$ for the number of independent sets in G, we have $i(G_M[A_i]) = 3$ for $1 \leqslant i \leqslant J$ and $i(G_M[A_{J+1}]) \leqslant 2^{\lceil rM/q\rceil - 1 - 2J}$. So Lemma 3.8 implies that $i(G_M) \leqslant 3^J 2^{\lceil rM/q\rceil - 1 - 2J}$.

Observe that every $\mathcal{L}^w$-free subset of $[n]$ with maximum element M can be found by picking an independent set $I \subseteq V(G_M)$ and extending it in $[\lceil rM/q \rceil, M-1]$. There are at most $2^{M-\lceil rM/q \rceil}$ ways to form this extension. Hence we have

$$
\begin{aligned}
f(n, \mathcal{L}^w, M) \leqslant i(G_M) \cdot 2^{M-\lceil rM/q \rceil} &\leqslant 3^J 2^{M-2J-1} = 2^{M-(2-\log_2 3)J-1} \\
&\leqslant 2^{M(1-rt(2-\log_2 3)(p^2+(p-t)(q-t))/(p^2q(p+q)))} 2^{(2-\log_2 3)(r_1r_2-r_1-r_2+4)-1}.
\end{aligned}
$$

$\square$

Theorem 4.16. *Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$ and p, q, r are fixed positive integers satisfying* $\gcd(p, q, r) = 1$. *Let $t := \gcd(p, q)$. Suppose*

$$
r > (r_1r_2 - r_1 - r_2 + 4)r_2\left(r_1 + 1 + \frac{r_2 - 1}{r_1^2 + (r_1 - 1)(r_2 - 1)}\right).
$$

Then $f(n, \mathcal{L}^w) \leqslant C \cdot 2^{\mu(n, \mathcal{L}^w)}$ where

$$
C := \frac{2^{(2-\log_2 3)(r_1r_2-r_1-r_2+4)}}{1 - 2^{1-rt^2(2-\log_2 3)(p^2+(p-t)(q-t))/(p^2q(p+q))}}.
$$

Proof. If q does not divide p, since $r > (r_1r_2 - r_1 - r_2 + 4)r_2(r_1 + 1 + (r_2 - 1)/(r_1^2 + (r_1 - 1)(r_2 - 1)))$ we have $\mu(n, \mathcal{L}^w) = \lceil (t-1)n/t \rceil$ by Theorem 4.3(iii). If q divides p then our condition on r implies $rq \geqslant p + q$ and so we have $\mu(n, \mathcal{L}^w) = \lceil (q-1)n/q \rceil = \lceil (t-1)n/t \rceil$ by Theorem 4.3(i).

Every $\mathcal{L}^w$-free subset A of $[n]$ can be written as $A = B \cup D$ where the maximum element M of B is divisible by t and $\min D > M$ and no element of D is divisible by t. (Note here B or D could be empty. If B is empty then we define $M := 0$.) Thus

$$
f(n, \mathcal{L}^w) \leqslant \sum_{\substack{M \equiv 0 \bmod t \\ 0 \leqslant M \leqslant n}} f(n, \mathcal{L}^w, M) \cdot 2^{\lceil (n-M)(t-1)/t \rceil} = \sum_{\substack{M \equiv 0 \bmod t \\ 0 \leqslant M \leqslant n}} f(n, \mathcal{L}^w, M) \cdot 2^{\mu(n, \mathcal{L}^w) - M(t-1)/t}
$$

where we define $f(n, \mathcal{L}^w, 0) := 1$.

Let $b := 1 - rt(2-\log_2 3)(p^2+(p-t)(q-t))/(p^2q(p+q))$, $d := 2^{bt-t+1}$ and $a := c \cdot 2^{\mu(n,\mathcal{L}^w)}$ where c is as stated in Lemma 4.15. Note that $|d| < 1$ if $r > r_2(r_1 + 1 + (r_2 - 1)/(r_1^2 + (r_1 - 1)(r_2 - 1)))/(2 - \log_2 3)$. But this is true since $(r_1r_2 - r_1 - r_2 + 4) > 1/(2 - \log_2 3)$. By using Lemma 4.15 we obtain

$$\begin{aligned}
f(n, \mathcal{L}^w) &\leqslant \sum_{\substack{M \equiv 0 \bmod t \\ 0 \leqslant M \leqslant n}} c \cdot 2^{M(1-rt(2-\log_2 3)(p^2+(p-t)(q-t))/(p^2q(p+q)))} 2^{\mu(n,\mathcal{L}^w)-M(t-1)/t} \\
&= \sum_{\substack{M \equiv 0 \bmod t \\ 0 \leqslant M \leqslant n}} a \cdot 2^{M(b-(t-1)/t)} = \sum_{i=0}^{\lfloor n/q \rfloor} a \cdot 2^{it(b-(t-1)/t)} \\
&= \sum_{i=0}^{\lfloor n/q \rfloor} ad^i \leqslant \sum_{i=0}^{\infty} ad^i = \frac{a}{1-d} \\
&= C \cdot 2^{\mu(n,\mathcal{L}^w)} \text{ where } C := \frac{2^{(2-\log_2 3)(r_1r_2-r_1-r_2+3)}}{1 - 2^{1-rt^2(2-\log_2 3)(p^2+(p-t)(q-t))/(p^2q(p+q))}}.
\end{aligned}$$

□

Note Theorem 4.16 implies Theorem 4.4. Recall Theorem 3.3 from the last chapter. Both of these involve equations with three variables. To prove the following corollary, we use Theorems 4.4 and 3.3 respectively, together with repeated use of Proposition 4.1.

Corollary 4.17. *Let $\mathcal{L}$ denote one of the following equations:*

(i) $a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_\ell y_\ell = r'z$ *where* $a_i, b_i, r' \in \mathbb{N}$ *and* $p' = \sum_i a_i$, $q' = \sum_i b_i$ *satisfy* $p' \geqslant q' \geqslant r'$*. Let* $t' := \gcd(p', q', r')$ *and write* $p := p'/t'$, $q := q'/t'$ *and* $r := r'/t'$*. Write* $r_1 := p/t$ *and* $r_2 := q/t$*. Suppose additionally* tt' *divides* a_i *and* b_j *for* $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant \ell$ *and*

$$r > (r_1r_2 - r_1 - r_2 + 4)r_2\left(r_1 + 1 + \frac{r_2 - 1}{r_1^2 + (r_1 - 1)(r_2 - 1)}\right).$$

(ii) $a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_\ell y_\ell = z$ *where the* $a_i, b_i \in \mathbb{N}$ *and* $p = \sum_i a_i$, $q = \sum_i b_i$ *satisfy* $p \geqslant q$ *and additionally either* $q = 1$ *and* $p \geqslant 3$ *or* $q \geqslant 2$ *and* $p > q(3q-2)/(2q-2)$.

Then $f(n, \mathcal{L}^w) = \Theta(2^{\mu(n,\mathcal{L}^w)})$.

Proof. Let $\mathcal{L}_2$ denote the equation $px+qy = rz$. By repeated use of Proposition 4.1 and Fact 4.11 we have $f(n, \mathcal{L}^w) \leqslant f(n, \mathcal{L}_2^w)$. By Theorems 4.4 and 3.3 respectively for cases (i) and (ii), we have $f(n, \mathcal{L}_2^w) \leqslant C \cdot 2^{\mu(n,\mathcal{L}_2^w)}$ for some constant C. But by Corollary 4.14 we have $\mu(n, \mathcal{L}^w) = \mu(n, \mathcal{L}_2^w)$ and so $f(n, \mathcal{L}^w) \leqslant C \cdot 2^{\mu(n,\mathcal{L}^w)}$. □

4.5 The number of maximal solution-free sets

We start this section with the proof of Theorem 4.5.

Theorem 4.5. *Let* $\mathcal{L}$ *denote the equation* $px + qy = rz$ *where* $p \geqslant q \geqslant r$ *and* p, q, r *are fixed positive integers satisfying* $\gcd(p, q, r) = 1$. *Let* $t := \gcd(p, q)$. *Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 2^{\frac{Crn}{q}+o(n)} \text{ where } C := 1 - \frac{t}{p+q}\Big(\frac{p^2 + (p-t)(q-t)}{p^2}\Big).$$

Proof. First note that C lies between $1/2$ and $1 - t/(p + q)$. To see this, note that if q divides p, then $C = 1 - q/(p + q) \geqslant 1/2$ since $p \geqslant q$. Otherwise, $p > q > t$, and so $(p-t)(q-t) < p^2$. Hence $t(p^2+(p-t)(q-t))/(p^2(p+q)) < 2t/(p+q) \leqslant 2(q/2)/(p+q) < 1/2$ and so $C > 1/2$. We observe that $C \leqslant 1 - t/(p + q)$ since $p \geqslant q \geqslant t$.

Let $\mathcal{F}$ denote the set of containers obtained by applying Lemma 2.5. Since every maximal $\mathcal{L}^w$-free subset of $[n]$ lies in at least one of the $2^{o(n)}$ containers, it suffices to show that every $F \in \mathcal{F}$ houses at most $2^{Crn/q+o(n)}$ maximal $\mathcal{L}^w$-free sets.

Let $F \in \mathcal{F}$. By Lemmas 2.5(i) and 2.4, $F = A \cup B$ where $|A| = o(n)$, $|B| \leqslant \mu(n, \mathcal{L}^w)$ and B is $\mathcal{L}^w$-free. Define $M := \max\{x \in B : x \equiv 0 \bmod t\}$ and $u :=$

$\max\{\lfloor rM/q \rfloor, \lfloor rn/2q \rfloor\}$. Every maximal $\mathcal{L}^w$-free set which lies in such a container can be constructed by:

(i) Picking $S_1 \subseteq A$ to be $\mathcal{L}^w$-free;

(ii) Adding a set $S_2 \subseteq [u] \cap B$ so that $S_1 \cup S_2$ is $\mathcal{L}^w$-free;

(iii) Choosing a set $S_3 \subseteq [u+1, n] \cap B$ so that $S_1 \cup S_2 \cup S_3$ is a maximal $\mathcal{L}^w$-free subset of $[n]$.

There are $2^{o(n)}$ ways to pick S_1. If $M \leqslant n/2$ then $u = \lfloor rn/2q \rfloor$ and so there are at most $2^{rn/2q} \leqslant 2^{Crn/q}$ ways to pick S_2 so that $S_1 \cup S_2$ is $\mathcal{L}^w$-free. Write $r_1 := p/t$ and $r_2 := q/t$. If $M \geqslant n/2$ then since M is divisible by t, we apply Lemma 4.10 to show that

$$\begin{aligned}
|[u] \cap B| &= |[\lfloor rM/q \rfloor] \cap B| \\
&\leqslant \left\lfloor \frac{rM}{q} \right\rfloor - \left\lfloor \frac{rM}{r_2(p+q)} \right\rfloor - (r_1 r_2 - r_1 - r_2 + 1) \left\lfloor \left\lfloor \frac{rM}{r_1(p+q)} - \frac{1}{r_2} \right\rfloor \frac{1}{r_1 r_2} \right\rfloor \\
&= \frac{CrM}{q} + o(n).
\end{aligned}$$

Hence there are at most $2^{CrM/q+o(n)} \leqslant 2^{Crn/q+o(n)}$ ways to pick S_2 so that $S_1 \cup S_2$ is $\mathcal{L}^w$-free.

Let $B' := [u+1, n] \cap B$. For step (iii) we calculate the number of extensions of $S_1 \cup S_2$ into B'. Observe by Lemma 4.7, this is bounded above by $\mathrm{MIS}(L_{S_1 \cup S_2}[B'])$. We will show that this link graph has only one maximal independent set. Then combining steps (i)-(iii) we have that F contains at most $2^{o(n)} \times 2^{Crn/q+o(n)} = 2^{Crn/q+o(n)}$ maximal $\mathcal{L}^w$-free sets as desired.

If the link graph only contains loops and isolated vertices, then it has only one maximal independent set. For it to have an edge between distinct vertices, we either must have $x, z \in B'$, $y \in S_1 \cup S_2$ such that $px + qy = rz$ or $py + qx = rz$, or we must have $x, y \in B'$, $z \in S_1 \cup S_2$ such that $px + qy = rz$.

The first of these events does not occur since otherwise $rz \geqslant q(x+y) > qx \geqslant q(\lfloor rM/q \rfloor + 1) > rM$ and so $z > M$. Note that since z is part of the solution $px+qy = rz$ and $\gcd(p,q,r) = 1$, it must be divisible by t. However this contradicts $z > M$ as we have $z \in B$, but M was defined to be the largest element in B divisible by t.

If $M > n/2$ then the second event does not occur since $rz = px + qy \geqslant q(x+y) \geqslant 2q(\lfloor rM/q \rfloor + 1) > 2rM > rn$ and so $z > n$. If $M \leqslant n/2$ then the second event does not occur since $rz = px + qy \geqslant q(x+y) \geqslant 2q(\lfloor rn/2q \rfloor + 1) > rn$ and so again $z > n$. $\square$

Note that when $r = 1$, Theorem 4.5 gives us new results for equations of the form $px + qy = z$. Recall that in the last chapter we found results for such equations. In Section 4.6 we give a summary describing which result gives the best upper bound for various values of p and q.

When $\mathcal{L}$ denotes the equation $qx + qy = z$ for some positive integer $q \geqslant 2$, Proposition 3.21(iii) gives a lower bound of $f_{\max}(n, \mathcal{L}^w) \geqslant 2^{(n-6q)/2q}$. Combining this with Theorem 4.5 allows us to determine $\log(f_{\max}(n, \mathcal{L}^w))$ asymptotically.

Theorem 4.6. *Let $\mathcal{L}$ denote the equation $qx + qy = z$ where $q \geqslant 2$ is an integer. Then*

$$f_{\max}(n, \mathcal{L}^w) = 2^{n/2q + o(n)}.$$

By adapting the proof of Theorem 4.5 we obtain the following result for $f_{\max}(n, \mathcal{L}^w)$ for some equations with more than three variables.

Theorem 4.18. *Let $\mathcal{L}$ denote the equation $p_1x_1 + \cdots + p_kx_k = rz$ where $p_1, \ldots, p_k, r \in \mathbb{N}$ satisfy $\gcd(p_1, \ldots, p_k, r) = 1$ and $p_1 \geqslant \cdots \geqslant p_k \geqslant r$. Suppose that $p := \sum_{i=1}^{k-1} p_i$ and $q := p_k$ satisfy $t := \gcd(p,q) = \gcd(p_1, \ldots, p_k)$. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 2^{\frac{Crn}{q} + o(n)} \text{ where } C := 1 - \frac{t}{p+q}\Big(\frac{p^2 + (p-t)(q-t)}{p^2}\Big).$$

Proof. We follow the proof used in Theorem 4.5 precisely (except for using Lemma 4.12 instead of Lemma 4.10 in step (ii)) up until counting the number of ways of extending $S_1 \cup S_2$ to a maximal $\mathcal{L}^w$-free set in $B' := [u+1, n] \cap B$. Observe by Lemma 4.7, this is bounded above by $\mathrm{MIS}(L_{S_1 \cup S_2}[B'])$ since B' and $S_1 \cup S_2$ are $\mathcal{L}^w$-free. To see that B' is $\mathcal{L}^w$-free, suppose $(x_1, \dots, x_k, z)$ is a solution within B' and note that $rz = p_1x_1 + \cdots + p_kx_k > p_kx_k \geqslant q(rM/q) = rM$ and so $z > M$. (Here we needed that each p_i is positive.) Since $\gcd(p_1, \dots, p_k, r) = 1$ and $\gcd(p, q) = \gcd(p_1, \dots, p_k)$ we have $\gcd(t, r) = 1$ and so in any solution to $\mathcal{L}$, z must be divisible by t. However this contradicts $z > M$ as we have $z \in B$, but M was defined to be the largest element in B divisible by t.

We will show that this link hypergraph $L_{S_1 \cup S_2}[B']$ has only one maximal independent set (and hence the number of maximal $\mathcal{L}^w$-free sets contained in F is at most $2^{Crn/q+o(n)}$ as required). If the link hypergraph only contains loops and isolated vertices, then it has only one maximal independent set.

For it to have a hyperedge between at least two vertices, there must exist a solution $(x_1, \dots, x_k, z)$ where either there is a hyperedge with distinct vertices $x_i, z \in B'$ for some $1 \leqslant i \leqslant k$ and $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k\} \subseteq B' \cup S_1 \cup S_2$, or there is a hyperedge with distinct vertices $x_i, x_j \in B'$ for some $1 \leqslant i < j \leqslant k$ and

$$\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_k, z\} \subseteq B' \cup S_1 \cup S_2.$$

Suppose the first event occurs with $(x_1, \dots, x_k, z)$. Then $rz = p_1x_1 + \cdots + p_kx_k > p_ix_i \geqslant p_kx_i = qx_i \geqslant q(rM/q) = rM$ and so $z > M$. But since z is part of a solution, it must be divisible by t. This contradicts $z \in B$, since M was defined to be the largest element in B divisible by t.

If $M > n/2$ then the second event does not occur since $rz = p_1x_1 + \cdots + p_kx_k > p_k(x_i + x_j) \geqslant 2q(\lfloor rM/q \rfloor + 1) > 2rM > rn$ and so $z > n$. If $M \leqslant n/2$ then the second event does not occur since $rz = p_1x_1 + \cdots + p_kx_k > p_k(x_i + x_j) \geqslant 2q(\lfloor rn/(2q)\rfloor + 1) > rn$ and so again $z > n$. $\square$

We end the section with a lower bound.

Proposition 4.19. *Let $\mathcal{L}$ denote the equation $qx + qy = rz$ where $q > r$ and q, r are fixed positive integers satisfying* $\gcd(q, r) = 1$*. Then*

$$f_{\max}(n, \mathcal{L}^w) \geqslant 2^{\lceil \lfloor rn/2q - rq/2 \rfloor (q-1)/q \rceil - 1}.$$

Proof. Let B be the $\mathcal{L}^w$-free set $\{z \in [n] : z \not\equiv 0 \bmod q\}$. Let $M := \max\{z \in [n] : rz/q^2 \in [n]\}$; so $M > n - q^2$. Let $S := \{M\}$ and consider the link graph $L_S[B]$. Note that if $i \in B$ where $i < rM/q$ then $rM/q - i \in B$. This follows since $rM/q^2 \in \mathbb{N}$ and so $rM/q - i \not\equiv 0 \bmod q$. Hence there is an edge in $L_S[B]$ between every such i and $rM/q - i$ since $q(i + rM/q - i) = rM$. By running through all $i \in B$ we obtain a total of $\lceil \lfloor rM/2q \rfloor (q-1)/q \rceil$ disjoint edges in $L_S[B]$ of which at most one is a loop (at $rM/2q$ if it is an integer not congruent to 0 modulo q). Hence we obtain an *induced* matching E in $L_S[B]$ of size $\lceil \lfloor rM/2q \rfloor (q-1)/q \rceil - 1 \geqslant \lceil \lfloor rn/2q - rq/2 \rfloor (q-1)/q \rceil - 1$. It is easy to see that the matching E contains $2^{|E|}$ maximal independent sets. Since E is an induced subgraph of $L_S[B]$, by applying Lemma 4.8 we obtain the result. $\square$

Question 4.20. *Let $\mathcal{L}$ denote the equation $qx + qy = rz$ where $q > r \geqslant 2$ and q, r are fixed positive integers satisfying* $\gcd(q, r) = 1$*. Does $f_{\max}(n, \mathcal{L}^w) = 2^{rn(q-1)/(2q^2) + o(n)}$?*

We remark that it turns out that for equations $\mathcal{L}$ as in Question 4.20, we have that $2^{rn(q-1)/(2q^2)+o(n)} = 2^{(\mu(n,\mathcal{L}^w) - \mu^*(n,\mathcal{L}))/2 + o(n)}$.

4.6 Best upper bounds on $f_{\max}(n, \mathcal{L}^w)$

In this section we give a summary of the best known upper bound on $f_{\max}(n, \mathcal{L}^w)$ for equations of the form $px + qy = rz$ where $p \geqslant q \geqslant r$. First we recall some of the results from the previous chapter.

Theorem 3.4. *Let $\mathcal{L}$ be a fixed homogeneous three-variable linear equation. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 3^{(\mu(n,\mathcal{L}^w)-\mu^*(n,\mathcal{L}))/3+o(n)}.$$

Theorem 3.5. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q \geqslant 2$ are integers so that $p \leqslant q^2 - q$ and $\gcd(p, q) = q$. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 2^{(\mu(n,\mathcal{L}^w)-\mu^*(n,\mathcal{L}))/2+o(n)}.$$

Theorem 3.6. *Let $\mathcal{L}$ denote the equation $px + qy = z$ where $p \geqslant q$, $p \geqslant 2$ and $p, q \in \mathbb{N}$. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 2^{\mu(\lfloor \frac{n-p}{q} \rfloor,\mathcal{L}^w)+o(n)}.$$

We can generalise Theorem 3.6 for $r > 1$ by following a very similar proof to that of Lemma 3.20 to obtain the following.

Theorem 4.21. *Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$ and $p, q, r \in \mathbb{N}$. Then*

$$f_{\max}(n, \mathcal{L}^w) \leqslant 2^{\mu(\lfloor \frac{rn}{q} \rfloor,\mathcal{L}^w)+o(n)}.$$

The upper bounds given by Theorems 3.5, 3.6 and 4.21 are all superseded by the bound given by Theorem 4.5. We prove this here and also compare Theorem 4.5 with Theorem 3.4.

Proposition 4.22. *Let $\mathcal{L}$ denote the equation $px + qy = rz$ where $p \geqslant q \geqslant r$, $p \geqslant 2$ and p, q, r are fixed positive integers satisfying $\gcd(p, q, r) = 1$. Let $t := \gcd(p, q)$ and let $a := \log_2 3$. The best upper bound on $f_{\max}(n, \mathcal{L}^w)$ given by Theorems 3.4, 3.5, 3.6, 4.5 and 4.21 is:*

(i) $f_{\max}(n,\mathcal{L}^w) \leqslant 3^{(\mu(n,\mathcal{L}^w)-\mu^*(n,\mathcal{L}))/3+o(n)}$ *if*

(a) $r=1$, $\gcd(p,q)=q$, $p \geqslant \max\{q^2,(q^2-q)a/(q(3-2a)+a)\}$, *and* $q \leqslant 9$*;*

(b) $r \geqslant 2$, $\mu(n,\mathcal{L}^w)=\lceil (t-1)n/t\rceil$, *and additionally (1)* $p \neq q$ *or (2)* $2 \leqslant q \leqslant 18$*;*

(c) $r \geqslant 2$, q *divides* p, $p+q \geqslant rq$ *and additionally (1)* $p \neq q$ *or (2)* $2 \leqslant q \leqslant 18$*;*

(ii) $f_{\max}(n,\mathcal{L}^w) \leqslant 2^{Crn/q+o(n)}$ *where* $C := 1-t(p^2+(p-t)(q-t))/(p^2(p+q))$ *if*

(a) $r=1$, $\gcd(p,q)\neq q$ *or* $q>9$ *or* $p<q^2$ *or* $p<(q^2-q)a/(q(3-2a)+a)$*;*

(b) $r \geqslant 2$, $\mu(n,\mathcal{L}^w)=\lceil (t-1)n/t\rceil$, *and* $p=q\geqslant 19$.

Proof. First suppose that $r=1$ (and so $\mu(n,\mathcal{L}^w)=\lceil (p+q-1)n/(p+q)\rceil$). Note that $C \leqslant 1-t/(p+q)=(p+q-t)/(p+q)\leqslant (p+q-1)/(p+q)$ and so the exponent given by Theorem 4.5 is at most the exponent given by Theorem 3.6. For Theorem 3.5 we require $\gcd(p,q)=q$ and $p\leqslant q^2-q$. In this case $C=p/(p+q)$ and $(p+q-1)/(2(p+q))-(q-1)^2/(2q^2)=(2pq+q^2-p-q)/(2q^2(p+q))\geqslant p/(q(p+q))=C/q$ and so the exponent given by Theorem 4.5 is at most the exponent given by Theorem 3.5.

For $r=1$ it remains to check when the bound given by Theorem 3.4 is better than the bound given by Theorem 4.5. Since Theorem 3.5 gives a better bound than Theorem 3.4 when $\gcd(p,q)=q$ and $p\leqslant q^2-q$, it suffices to consider the case when $\gcd(p,q)=q$ and $p\geqslant q^2$, and when $\gcd(p,q)\neq q$. For the latter, when $t=\gcd(p,q)\neq q$, it certainly suffices to show that $2\mu(\lfloor (n-p)/q\rfloor,\mathcal{L}^w)\leqslant \mu(n,\mathcal{L}^w)-\mu^*(n,\mathcal{L})+o(n)$, since Theorem 4.5 gives a better bound than Theorem 3.6. In this case we have $\mu^*(n,\mathcal{L})=(q-1)(t-1)n/(qt)+o(n)$, and hence it suffices to show that $t\leqslant (pq+q^2-p-q)/(p+2q-2)$. First note that $t\leqslant q/2$ and so $q\neq 1$. Now observe that $t(p+2q-2)\leqslant q(p+2q-2)/2=pq/2+q^2-q\leqslant pq+q^2-p-q$ and so our inequality on t holds as required. Now suppose $\gcd(p,q)=q$ and $p\geqslant q^2$. To prove (i)(a) it suffices to show that

$$3^{\frac{(p+q-1)n}{3(p+q)}-\frac{(q-1)^2n}{3q^2}} \leqslant 2^{\frac{pn}{q(p+q)}},$$

or rearranging

$$p(q(3-2a)+a) \geqslant (q^2-q)a.$$

If $q \geqslant 10$ then $(q(3-2a)+a)$ is negative, but then we would require p negative, a contradiction. Hence we must have $q \leqslant 9$ and then the inequality holds if $p > (q^2 - q)a/(q(3-2a)+a)$.

Now suppose that $r \geqslant 2$ and $\mu(n, \mathcal{L}^w) = \lceil (t-1)n/t \rceil$. Then $\mu(n, \mathcal{L}^w) - \mu^*(n, \mathcal{L}) = r(t-1)n/(qt) + o(n)$ and $3^{x/3} < 2^x$ and so Theorem 3.4 gives a better bound than Theorem 3.6. We wish to know when

$$3^{\frac{r}{q}\frac{t-1}{3t}} < 2^{\frac{r}{q}(1-\frac{t}{p+q}(\frac{p^2+(p-t)(q-t)}{p^2}))}.$$

Write $r_1 := p/t$ and $r_2 := q/t$. The above rearranges to give

$$t((a-3)r_1^2(r_1+r_2) + 3r_1^2 + 3(r_1-1)(r_2-1)) < ar_1^2(r_1+r_2).$$

The right hand side is positive and the left hand side is negative unless $r_1 = r_2 = 1$. In this case $p = q = t$ and so we now require $3^{(q-1)/(3q)} < 2^{1/2}$, which holds when $q \leqslant 18$.

Finally suppose that $r \geqslant 2$, q divides p and $p + q \geqslant rq$ (so $\mu(n, \mathcal{L}^w) = \lceil (p+q-r)n/(p+q) \rceil$). Since q divides p, we have $t = q$ and $p = r_1 q$, and so Theorem 4.5 gives a bound of $2^{rpn/(q(p+q))+o(n)}$. This is better than Theorem 4.21 which gives a bound of $2^{r(p+q-r)n/(q(p+q))+o(n)}$ since $q \geqslant r$. Therefore we wish to know when

$$3^{\frac{p+q-r}{3(p+q)} - \frac{(q-r)(q-1)}{3q^2}} < 2^{\frac{rp}{q(p+q)}}.$$

Rearranging, we require $r_1(a(q+qr-r)/3 - qr) \leqslant a(r-q)/3$. Now note $a(q+qr-r)/3 - qr$ is negative when $r \geqslant 2$, so this rearranges to give $r_1 \geqslant (q-r)/(r-q-rq+3rq/a)$. If $p > q$ (so $r_1 \geqslant 2$), it suffices to have $2 \geqslant (q-r)/(r-q-rq+3rq/a)$ or rearranging,

$q(r(2-6/a)+3) \leqslant 3r$. This holds since $r(2-6/a)+3$ is negative for $r \geqslant 2$. Otherwise $p = q$, and so since $p+q \geqslant rq$, we have that $r = 2$. So we require $1 \geqslant (q-2)/((6/a-3)q+2)$ which holds when $q \leqslant 18$. (In this final case, $\mu(n, \mathcal{L}^w) = \lceil (t-1)n/t \rceil = \lceil (p+q-r)n/(p+q) \rceil = \lceil (q-1)n/q \rceil$.)

$\square$

4.7 Concluding Remarks

The crucial trick used in the proof of Theorems 4.5 and 4.18 was to choose our sets S carefully so that the link hypergraphs $L_S[B]$ each contain precisely one maximal independent set. In other applications of this method (see [6, 7] and Chapter 3) the approach was to instead obtain other structural properties of the link graphs (such as being triangle-free) to ensure there are not too many maximal independent sets in $L_S[B]$. It would be interesting to see if the approach of this chapter can be applied to obtain other results in the area.

Although we have found an initial bound on $f_{\max}(n, \mathcal{L}^w)$ for some equations with more than three variables, we still do not know in general if there are significantly fewer maximal $\mathcal{L}^w$-free subsets of $[n]$ than there are $\mathcal{L}^w$-free subsets of $[n]$. Progress on giving general upper bounds on the number of maximal independent sets in (non-uniform) hypergraphs should (through the method of link hypergraphs) yield results in this direction.

Chapter 5

Ramsey properties of graphs and the integers

5.1 Introduction

In this chapter we use the container method to prove results that correspond to problems concerning tuples of *disjoint* independent sets in hypergraphs. An overarching aim is to demonstrate that with the container method at hand, one can give relatively short and elementary proofs of fundamental results concerning Ramsey properties of graphs and the integers. Moreover, our results give us a precise understanding about how *resiliently* typical graphs and sets of integers of a given density possess a given Ramsey property. In particular, one of our main results is a resilience random Ramsey theorem (Theorem 5.7). This result provides a unified framework for studying both the Ramsey and Turán problems in the setting of random (hyper)graphs. In particular, Theorem 5.7 implies the (so-called 1-statements of the) random Ramsey theorem due to Rödl and Ruciński [90, 91, 92] and the random version of Turán's theorem [28, 108]. Moreover, Theorem 5.7 also resolves a general subcase of the asymmetric random Ramsey conjecture of Kohayakawa and Kreuter [70]. Since Theorem 5.7 unifies and generalises several fundamental results concerning Ramsey and Turán properties of random (hyper)graphs, we survey these topics in Sections 5.1.1.2–5.1.1.4 before we state this result in Section 5.1.1.5.

We also prove a sister result to Theorem 5.7, a resilience strengthening of the random Rado theorem (Theorem 5.11). Again the container method allows us to give a rather short proof of this result. We further provide results on the enumeration of Ramsey graphs (Theorem 5.12) and sets of integers without a given Ramsey property (Theorem 5.13).

As mentioned the results we prove all correspond to problems concerning tuples of disjoint independent sets in hypergraphs. In particular, from the container theorem of Balogh, Morris and Samotij one can easily obtain an analogous result for tuples of independent sets in hypergraphs (see Theorem 5.14). It turns out that many Ramsey-type questions (and other problems) can be naturally phrased in this setting. For example, by Schur's theorem we know that, if n is large, then whenever one r-colours the elements of $[n] := \{1, \dots, n\}$ there is a monochromatic solution to $x + y = z$. This raises the question of how large can a subset $S \subseteq [n]$ be whilst failing to have this property? (This problem was first posed back in 1977 by Abbott and Wang [1].) Let H be the hypergraph with vertex set $[n]$ in which edges precisely correspond to solutions to $x + y = z$. (Note H will have edges of size 2 and 3.) Then sets $S \subseteq [n]$ without this property are precisely the union of r disjoint independent sets in H.

In Section 5.2 we state the container theorem for tuples of independent sets in hypergraphs. In Sections 5.3 and 5.4 we give our applications of this container result to enumeration and resilience questions arising in Ramsey theory for graphs and the integers.

5.1.1 Resilience in hypergraphs and the integers

5.1.1.1 Resilience in graphs

The notion of *graph resilience* has received significant attention in recent years. Roughly speaking, resilience concerns the question of how 'strongly' a graph G satisfies a certain monotone graph property $\mathcal{P}$. *Global resilience* concerns how many edges one can delete and still ensure the resulting graph has property $\mathcal{P}$ whilst *local resilience* considers how many edges one can delete at each vertex whilst ensuring the resulting graph has property $\mathcal{P}$. More precisely, we define the *global resilience of G with respect to $\mathcal{P}$*, $\mathrm{res}(G, \mathcal{P})$, to be the minimum number t such that by deleting t edges from G, one can obtain a set not having $\mathcal{P}$. Many classical results in extremal combinatorics can be rephrased in terms of resilience. For example, Turán's theorem determines the global resilience of K_n with respect to the property of containing K_r (where $r < n$) as a subgraph.

The systematic study of graph resilience was initiated in a paper of Sudakov and Vu [115], though such questions had been studied before this. In particular, a key question in the area is to establish the resilience of various properties of the Erdős–Rényi random graph $G_{n,p}$. (Recall that $G_{n,p}$ has vertex set $[n]$ in which each possible edge is present with probability p, independent of all other choices.) The local resilience of $G_{n,p}$ has been investigated, for example, with respect to Hamiltonicity e.g. [115, 77], almost spanning trees [4] and embedding subgraphs of small bandwidth [20]. See [115] and the surveys [27, 114] for further background on the subject. In this chapter we study the global resilience of $G_{n,p}$ with respect to Ramsey properties (in fact, as we explain later, we will consider its hypergraph analogue $G^{(k)}_{n,p}$ for $k \geqslant 2$). First we will focus on the graph case.

5.1.1.2 Ramsey properties of random graphs

An event occurs in $G_{n,p}$ *with high probability* (w.h.p.) if its probability tends to 1 as $n \to \infty$. For many properties $\mathcal{P}$ of $G_{n,p}$, the probability that $G_{n,p}$ has the property exhibits a phase transition, changing from 0 to 1 over a small interval. That is, there is a *threshold* for $\mathcal{P}$: a function $p_0 = p_0(n)$ such that $G_{n,p}$ has $\mathcal{P}$ w.h.p. when $p \gg p_0$ (the 1-*statement*), while $G_{n,p}$ does not have $\mathcal{P}$ w.h.p. when $p \ll p_0$ (the 0-*statement*). Indeed, Bollobás and Thomason [19] proved that every *monotone* property $\mathcal{P}$ has a threshold.

Given a graph H, set $d_2(H) := 0$ if $e(H) = 0$; $d_2(H) := 1/2$ when H is precisely an edge and define $d_2(H) := (e(H) - 1)/(v(H) - 2)$ otherwise. Then define $m_2(H) := \max_{H' \subseteq H} d_2(H')$ to be the 2-*density* of H. This graph parameter turns out to be very important when determining the threshold for certain properties in $G_{n,p}$ concerning the containment of a small subgraph H, which we explain further below.

Given $\varepsilon > 0$ and a graph H, we say that a graph G is (H, ε)-*Turán* if every subgraph of G with at least $(1 - \frac{1}{\chi(H)-1} + \varepsilon)e(G)$ edges contains a copy of H. Note that the Erdős–Stone theorem implies that K_n is (H, ε)-Turán for any fixed H provided n is sufficiently large. To motivate the definition, consider any graph G. Then by considering a random partition of $V(G)$ into $\chi(H) - 1$ parts (and then removing any edge contained within a part) we see that there is a subgraph G' of G that is $(\chi(H) - 1)$-partite where $e(G') \geqslant (1 - \frac{1}{\chi(H)-1})e(G)$. In particular, $H \not\subseteq G'$. Intuitively speaking, this implies that (up to the ε term), (H, ε)-Turán graphs are those graphs that most *strongly* contain H.

Rephrasing to the language of resilience, we see that if, for any $\varepsilon > 0$, G is (H, ε)-Turán, then $\mathrm{res}(G, G \supseteq H) = (\frac{1}{\chi(H)-1} \pm \varepsilon)e(G)$, and vice versa. (Recall that we write $x = a \pm b$ to say that the value of x is some real number in the interval $[a - b, a + b]$.) The global resilience of $G_{n,p}$ with respect to the Turán problem has been extensively studied. Indeed, a recent trend in combinatorics and probability concerns so-called *sparse random* analogues of extremal theorems (see [27]), and determining when $G_{n,p}$ is (H, ε)-Turán is an example of such a result.

If $p \leqslant cn^{-1/m_2(H)}$ for some small constant c, then it is not hard to show that w.h.p. $G_{n,p}$ is not (H, ε)-Turán. In [58, 59, 72] it was conjectured that w.h.p. $G_{n,p}$ is (H, ε)-Turán provided that $p \geqslant Cn^{-1/m_2(H)}$, where C is a (large) constant. After a number of partial results, this conjecture was confirmed by Schacht [108] and (in the case when H is strictly 2-balanced, i.e. $m_2(H') < m_2(H)$ for all $H' \subset H$) by Conlon and Gowers [28].

Theorem 5.1 ([108, 28]). *For any graph H with $\Delta(H) \geqslant 2$ and any $\varepsilon > 0$, there are positive constants c, C such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \text{ is } (H,\varepsilon)\text{-Turán}] = \begin{cases} 0 & \text{if } p < cn^{-1/m_2(H)}; \\ 1 & \text{if } p > Cn^{-1/m_2(H)}. \end{cases}$$

Given an integer r, an *r-colouring* of a graph G is a function $\sigma : E(G) \to [r]$. (So this is not necessarily a proper colouring.) We say that G is (H, r)*-Ramsey* if every r-colouring of G yields a monochromatic copy of H in G. Observe that being $(H, 1)$-Ramsey is the same as containing H as a subgraph. So the 1-statement of Theorem 5.1 says that, given $\varepsilon > 0$, there exists a positive constant C such that, if $p > Cn^{-1/m_2(H)}$, then

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\operatorname{res}(G_{n,p}, (H,1)\text{-Ramsey})}{e(G_{n,p})} = \frac{1}{\chi(H)-1} \pm \varepsilon\right] = 1. \tag{5.1.1}$$

The following result of Rödl and Ruciński [90, 91, 92] yields a random version of Ramsey's theorem.

Theorem 5.2 ([90, 91, 92]). *Let $r \geqslant 2$ be a positive integer and let H be a graph that is not a forest consisting of stars and paths of length 3. There are positive constants c, C such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \text{ is } (H,r)\text{-Ramsey}] = \begin{cases} 0 & \text{if } p < cn^{-1/m_2(H)}; \\ 1 & \text{if } p > Cn^{-1/m_2(H)}. \end{cases}$$

Thus $n^{-1/m_2(H)}$ is again the threshold for the (H, r)-Ramsey property. Let us provide some intuition as to why. The expected number of copies of H in $G_{n,p}$ is $\Theta(n^{v(H)}p^{e(H)})$, while the expected number of edges in $G_{n,p}$ is $\Theta(pn^2)$. When $p = \Theta(n^{-1/d_2(H)})$, these quantities agree up to a constant. Suppose that H is *2-balanced*, i.e. $d_2(H) = m_2(H)$. For small $c > 0$, when $p < cn^{-1/m_2(H)}$, most copies of H in $G_{n,p}$ contain an edge which appears in no other copy. Thus we can hope to colour these special edges blue and colour the remaining edges red to eliminate all monochromatic copies of H. For large $C > 0$, most edges lie in many copies of H, so the copies of H are highly overlapping and we cannot avoid monochromatic copies. In general, when H is not necessarily 2-balanced, the threshold is $n^{-1/d_2(H')}$ for the 'densest' subgraph H' of H since, roughly speaking, the appearance of H is governed by the appearance of its densest part.

We remark that Nenadov and Steger [85] recently gave a short proof of Theorem 5.2 using the container method.

5.1.1.3 Asymmetric Ramsey properties in random graphs

It is natural to ask for an *asymmetric* analogue of Theorem 5.2. Now, for graphs $H_1, \ldots, H_r$, a graph G is $(H_1, \ldots, H_r)$-*Ramsey* if for any r-colouring of G there is a copy of H_i in colour i for some $i \in [r]$. (This definition coincides with that of (H, r)-Ramsey when $H_1 = \ldots = H_r = H$.) Kohayakawa and Kreuter [70] conjectured an analogue of Theorem 5.2 in the asymmetric case. To state it, we need to introduce the *asymmetric density* of H_1, H_2 where $m_2(H_1) \geqslant m_2(H_2)$ via

$$m_2(H_1, H_2) := \max\left\{\frac{e(H_1')}{v(H_1') - 2 + 1/m_2(H_2)} : H_1' \subseteq H_1 \text{ and } e(H_1') \geqslant 1\right\}. \tag{5.1.2}$$

Conjecture 5.3 ([70]). *For any graphs $H_1, \ldots, H_r$ with $m_2(H_1) \geqslant \ldots \geqslant m_2(H_r) > 1$, there are positive constants $c, C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[G_{n,p} \textit{ is } (H_1, \ldots, H_r)\textit{-Ramsey}\right] = \begin{cases} 0 & \textit{if } p < cn^{-1/m_2(H_1,H_2)}; \\ 1 & \textit{if } p > Cn^{-1/m_2(H_1,H_2)}. \end{cases}$$

So the conjectured threshold only depends on the 'joint density' of the densest two graphs H_1, H_2. The intuition for this threshold is discussed in detail e.g. in Section 1.1 in [51]. One can show that $m_2(H_1) \geqslant m_2(H_1, H_2) \geqslant m_2(H_2)$ with equality if and only if $m_2(H_1) = m_2(H_2)$. Thus Conjecture 5.3 would generalise Theorem 5.2. Kohayakawa and Kreuter [70] have confirmed Conjecture 5.3 when the H_i are cycles. In [81] it was observed that the approach used by Kohayakawa and Kreuter [70] implies the 1-statement of Conjecture 5.3 holds when H_1 is strictly 2-balanced *provided* the so-called KŁR conjecture holds. This latter conjecture was proven by Balogh, Morris and Samotij [8] thereby proving the 1-statement of Conjecture 5.3 holds in this case. The full 1-statement of Conjecture 5.3 has very recently been proven by Mousset, Nenadov and Samotij [83].

5.1.1.4 Ramsey properties of random hypergraphs

Consider now the k-uniform analogue $G^{(k)}_{n,p}$ of $G_{n,p}$ which has vertex set $[n]$ and in which every k-element subset of $[n]$ appears as an edge with probability p, independent of all other choices. Here, we wish to obtain analogues of Theorems 5.1, 5.2 and Conjecture 5.3 by determining the threshold for being (H, ε)-Turán, (H, r)-Ramsey, and more generally being $(H_1, \dots, H_r)$-Ramsey. The definitions of (H, r)-Ramsey and $(H_1, \dots, H_r)$-Ramsey extend from graphs in the obvious way. Given a k-uniform hypergraph H, let $\mathrm{ex}(n; H)$ be the maximum size of an n-vertex H-free hypergraph. A simple averaging argument shows that the limit

$$\pi(H) := \lim_{n\to\infty} \frac{\mathrm{ex}(n; H)}{\binom{n}{k}}$$

exists. Now we say that a k-uniform hypergraph G is (H, ε)-*Turán* if every subhypergraph of G with at least $(\pi(H) + \varepsilon)e(G)$ edges contains a copy of H. (Since $\pi(H) = 1 - \frac{1}{\chi(H)-1}$ when $k = 2$, this generalises the definition we gave earlier.) We also need to generalise the notion of 2-density to k-*density*: Given a k-graph H, define

$$d_k(H) := \begin{cases} 0 & \text{if } e(H) = 0; \\ 1/k & \text{if } v(H) = k \text{ and } e(H) = 1; \\ \frac{e(H)-1}{v(H)-k} & \text{otherwise,} \end{cases}$$

and let

$$m_k(H) := \max_{H' \subseteq H} d_k(H').$$

The techniques of Conlon–Gowers [28] and of Schacht [108] actually extended to a proof of a version of Theorem 5.1 for hypergraphs:

Theorem 5.4 ([28, 108]). *For any k-uniform hypergraph H with maximum vertex degree at least two and any $\varepsilon > 0$, there are positive constants c, C such that*

$$\lim_{n\to\infty} \mathbb{P}[G^{(k)}_{n,p} \textit{ is } (H,\varepsilon)\textit{-Turán}] = \begin{cases} 0 & \textit{if } p < cn^{-1/m_k(H)}; \\ 1 & \textit{if } p > Cn^{-1/m_k(H)}. \end{cases}$$

The 1-statement of Theorem 5.2 was generalised to hypergraphs by Friedgut, Rödl and Schacht [42] and by Conlon and Gowers [28], proving a conjecture of Rödl and Ruciński [94]. (The special cases of the complete 3-uniform hypergraph $K_4^{(3)}$ on four vertices and of k-partite k-uniform hypergraphs were already proved in [94], [95] respectively. Also in [85] Nenadov and Steger remark that their proof of the 1-statement of Theorem 5.2 extends to Theorem 5.5.)

Theorem 5.5 ([28, 42]). *Let $r, k \geqslant 2$ be integers and let H be a k-uniform hypergraph with maximum vertex degree at least two. There is a positive constant C such that*

$$\lim_{n\to\infty} \mathbb{P}[G^{(k)}_{n,p} \textit{ is } (H,r)\textit{-Ramsey}] = 1 \quad \textit{if } p > Cn^{-1/m_k(H)}.$$

In [51], sufficient conditions are given for a corresponding 0-statement. However, the authors further show that, for $k \geqslant 4$, there is a k-uniform hypergraph H such that the threshold for $G^{(k)}_{n,p}$ to be (H,r)-Ramsey is *not* $n^{-1/m_k(H)}$, and *nor* does it correspond to the exceptional case in the graph setting of certain forests, where there is a coarse threshold due to the appearance of small subgraphs.

For the asymmetric Ramsey problem, we need to suitably generalise (5.1.2), in the obvious way: for any k-uniform hypergraphs H_1, H_2 with non-empty edge sets and $m_k(H_1) \geqslant m_k(H_2)$, let

$$m_k(H_1, H_2) := \max\left\{\frac{e(H_1')}{v(H_1') - k + 1/m_k(H_2)} : H_1' \subseteq H_1 \text{ and } e(H_1') \geqslant 1\right\} \tag{5.1.3}$$

be the *asymmetric k-density* of (H_1, H_2). Again,

$$m_k(H_1) \geqslant m_k(H_1, H_2) \geqslant m_k(H_2),$$

so, in particular, $m_k(H_1, H_2) = m_k(H_1)$ if and only if H_1 and H_2 have the same k-density.

Recently, Gugelmann, Nenadov, Person, Steger, Škorić and Thomas [51] generalised the 1-statement of Conjecture 5.3 to k-uniform hypergraphs, in the case when $H_1' = H_1$ is the unique maximiser in (5.1.3), i.e. H_1 is *strictly k-balanced with respect to $m_k(\cdot, H_2)$*.

Theorem 5.6 ([51]). *For all positive integers r, k with $k \geqslant 2$ and k-uniform hypergraphs $H_1, \ldots, H_r$ with $m_k(H_1) \geqslant \ldots \geqslant m_k(H_r)$ where H_1 is strictly k-balanced with respect to $m_k(\cdot, H_2)$, there exists $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[G^{(k)}_{n,p} \text{ is } (H_1, \ldots, H_r)\text{-Ramsey}\right] = 1 \quad \text{if } p > Cn^{-1/m_k(H_1,H_2)}.$$

They further prove a version of Theorem 5.6 with the weaker bound $p > Cn^{-1/m_k(H_1,H_2)} \log n$ when H_1 is not required to be strictly k-balanced with respect to $m_k(\cdot, H_2)$.

5.1.1.5 New resilience result

Our main result here is Theorem 5.7, which generalises, fully and partially, all of the 1-statements of the results discussed in this section, giving a unified setting for both the random Ramsey theorem and the random Turán theorem. Once we have obtained a container theorem for Ramsey graphs (Theorem 5.34), the proof is short (see Section 5.4.6).

For k-uniform hypergraphs $H_1, \dots, H_r$ and a positive integer n, let $\mathrm{ex}^r(n; H_1, \dots, H_r)$ be the maximum size of an n-vertex k-uniform hypergraph G which is not $(H_1, \dots, H_r)$-Ramsey. Define the *r-coloured Turán density*

$$\pi(H_1, \dots, H_r) := \lim_{n\to\infty} \frac{\mathrm{ex}^r(n; H_1, \dots, H_r)}{\binom{n}{k}}. \tag{5.1.4}$$

Observe that $\mathrm{ex}^1(n; H) = \mathrm{ex}(n; H)$ since a hypergraph is H-free if and only if it is not $(H, 1)$-Ramsey. Note further that $\pi(\cdot, \dots, \cdot)$ generalises $\pi(\cdot)$. So when $k = 2$, we have $\pi(H) = 1 - \frac{1}{\chi(H)-1}$. We will show in Section 5.4.2 that the limit in (5.1.4) does indeed exist, so $\pi(\cdot, \dots, \cdot)$ is well-defined. Further, crucially for k-uniform hypergraphs $H_1, \dots, H_r$, there exists an $\varepsilon = \varepsilon(H_1, \dots, H_r) > 0$ so that $\pi(H_1, \dots, H_r) < 1 - \varepsilon$ (see (5.4.3) in Section 5.4.2).

Theorem 5.7 (Resilience for random Ramsey). *Let $\delta > 0$, let r, k be positive integers with $k \geqslant 2$ and let $H_1, \dots, H_r$ be k-uniform hypergraphs each with maximum vertex degree at least two, and such that $m_k(H_1) \geqslant \ldots \geqslant m_k(H_r)$. There exists $C > 0$ such that if $p > Cn^{-1/m_k(H_1)}$, then*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\mathrm{res}\left(G^{(k)}_{n,p}, (H_1, \dots, H_r)\text{-}Ramsey\right)}{e\left(G^{(k)}_{n,p}\right)} = 1 - \pi(H_1, \dots, H_r) \pm \delta\right] = 1.$$

Thus, when $p > Cn^{-1/m_k(H_1)}$, the random hypergraph $G^{(k)}_{n,p}$ is w.h.p such that *every* subhypergraph G' with at least a $\pi(H_1, \dots, H_r) + \Omega(1)$ fraction of the edges is $(H_1, \dots, H_r)$-

Ramsey. Conversely, there is a subgraph of $G^{(k)}_{n,p}$ whose edge density is slightly smaller than this which *does not* have the Ramsey property.

Note that the threshold of $p > Cn^{-1/m_k(H_1)}$ in Theorem 5.7 is tight up to the multiplicative constant C. Indeed, consider the random hypergraph $G^{(k)}_{n,p}$ with $p \ll n^{-1/m_k(H_1)}$. Let $H_1' \subseteq H_1$ be such that $m_k(H_1) = d_k(H_1')$. Then the expected number of copies of H_1' in $G^{(k)}_{n,p}$ is much smaller than the expected number of edges in $G^{(k)}_{n,p}$, so w.h.p. we can delete every copy of H_1' (and therefore H_1) by removing $o(e(G^{(k)}_{n,p}))$ edges. So the hypergraph G that remains has $(1-o(1))e(G^{(k)}_{n,p})$ edges, and is not $(H_1, \ldots, H_r)$-Ramsey because we can colour every edge of G with colour 1. Then, since G is H_1-free, there is no copy of H_i in colour i in G.

Let us describe the importance of Theorem 5.7 (in the case $k = 2$ and $H_1 = \ldots = H_r = H$) in conjunction with Theorem 5.2. The 0-statement of Theorem 5.2 says that a typical sparse graph, i.e. one with density at most $cn^{2-1/m_2(H)}$, is *not* (H, r)-Ramsey. On the other hand, by Theorem 5.7, a typical dense graph, i.e. one with density at least $Cn^{2-1/m_2(H)}$, has the Ramsey property in a sense which is *as strong as possible* with respect to subgraphs: every sufficiently dense subgraph is (H, r)-Ramsey, and this minimum density is the largest we could hope to require.

The relationship between Theorem 5.7 and the previous results stated in this section can be summarised as follows:

- The 1-statement of Theorem 5.1 is recovered when $k = 2$ and $r = 1$. This follows from (5.1.1) and the relation between $\pi(H)$ and $\chi(H)$.

- In the case $k = 2$ and $H_1 = \ldots = H_r = H$, we obtain a stronger statement in place of the 1-statement of Theorem 5.2 as described above.

- Theorem 5.7 proves the 1-statement of Conjecture 5.3 in the case when $m_2(H_1) = m_2(H_2)$ in the same stronger sense as above.

- The 1-statement of Theorem 5.4 is recovered when $r = 1$.
- Theorem 5.7 implies Theorem 5.5, yielding a resilience version of this result.
- Theorem 5.7 implies a version of Theorem 5.6 when $m_k(H_1) = m_k(H_2)$ but now H_1 is not required to be strictly k-balanced with respect to $m_k(\cdot, H_2)$.

Note that even though Theorem 5.7 implies many of the known results concerning Ramsey properties of random (hyper)graphs, often the resilience random Ramsey problem is different to the random Ramsey problem. In particular, we have determined the threshold for the former problem, whilst we have seen above examples of (hyper)graphs $H_1, \ldots, H_r$ where a lower value of p still ensures that $G^{(k)}_{n,p}$ is w.h.p. $(H_1, \ldots, H_r)$-Ramsey.

5.1.1.6 Resilience in the integers

An important branch of Ramsey theory concerns partition properties of sets of integers. Schur's classical theorem [110] states that if $\mathbb{N}$ is r-coloured there exists a monochromatic solution to $x + y = z$; later van der Waerden [120] showed that the same hypothesis ensures a monochromatic arithmetic progression of arbitrary length. More generally, Rado's theorem [88] characterises all those systems of homogeneous linear equations $\mathcal{L}$ for which every finite colouring of $\mathbb{N}$ yields a monochromatic solution to $\mathcal{L}$.

As in the graph case, there has been interest in proving random analogues of such results from arithmetic Ramsey theory.

Before we describe the background of this area we will introduce some notation and definitions. As mentioned in the introduction of this thesis, we will assume A is an $\ell \times k$ integer matrix where $k \geqslant \ell$ of full rank ℓ and b is an integer vector of dimension ℓ. We will let $\mathcal{L}(A, b)$ denote the associated system of linear equations $Ax = b$, noting that for brevity we will simply write $\mathcal{L}$ if it is clear from the context which matrix A and vector b it refers to. If A is stated but b is not, then we assume $\mathcal{L}$ refers to the system of linear equations $Ax = 0$.

Let S be a set of integers. Recall that if a vector $x = (x_1, \dots, x_k) \in S^k$ satisfies $Ax = b$ (i.e. it is a solution to $\mathcal{L}$) and the x_i are distinct we call x a *k-distinct solution* to $\mathcal{L}$ in S. Throughout this chapter, with the exception of Sections 5.3.6 and 5.3.7, we consider $\mathcal{L}^d$-free sets.

We generalise the definition of $\mathcal{L}^d$-free to r colours in the obvious way: we call a set S of integers *$(\mathcal{L}^d, r)$-free* if there exists an r-colouring of S such that it contains no monochromatic k-distinct solution to $\mathcal{L}$. Otherwise we will call S *$(\mathcal{L}^d, r)$-Rado*. In the case when $r = 1$, we write $\mathcal{L}^d$-free instead of $(\mathcal{L}^d, 1)$-free. Define $\mu(n, \mathcal{L}^d, r)$ to be the size of the largest $(\mathcal{L}^d, r)$-free subset of $[n]$.

Recall from the introduction the definitions of irredundant, partition regular, and $m(A)$. As mentioned above Rado's theorem [88] characterises all partition regular systems of linear equations $\mathcal{L}$. The study of random versions of Rado's theorem has focused on irredundant partition regular matrices. This is natural since for every redundant $\ell \times k$ matrix A there exists an irredundant $\ell' \times k'$ matrix A' for some $\ell' < \ell$ and $k' < k$ with the same family of solutions (viewed as sets). See [93, Section 1] for a full explanation. Also note that most historical results concern distinct $\mathcal{L}$-free sets; many methods (including that of this chapter) require the use of a k-uniform hypergraph, so it is easier to work with k-distinct solutions only. However since there are an insignificant number of non k-distinct solutions compared to k-distinct solutions, a removal lemma such as Lemma 2.2 implies that it makes little difference to restrict to only k-distinct solutions.

Another class of matrices that have received attention in relation to this problem are so-called density regular matrices: An irredundant, partition regular matrix A is *density regular* if any subset $F \subseteq \mathbb{N}$ with positive upper density, i.e.,

$$\limsup_{n \to \infty} \frac{|F \cap [n]|}{n} > 0,$$

contains a k-distinct solution to $\mathcal{L}$.

We now describe some random analogues of results from arithmetic Ramsey theory. Recall that $[n]_p$ denotes a set where each element $a \in [n]$ is included with probability p independently of all other elements. Rödl and Ruciński [93] showed that for irredundant partition regular matrices A, $m(A)$ is an important parameter for determining whether $[n]_p$ is $(\mathcal{L}^d, r)$-Rado or $(\mathcal{L}^d, r)$-free.

Theorem 5.8 ([93]). *For all irredundant partition regular full rank matrices A and all positive integers $r \geqslant 2$, there exists a constant $c > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[[n]_p \text{ is } (\mathcal{L}^d, r)\text{-Rado}\right] = 0 \quad \text{if } p < cn^{-1/m(A)}.$$

We remark it is important that $r \geqslant 2$ in Theorem 5.8. That is, the corresponding statement for $r = 1$ is not true in general. Roughly speaking, Theorem 5.8 implies that almost all subsets of $[n]$ with significantly fewer than $n^{1-1/m(A)}$ elements are $(\mathcal{L}^d, r)$-free for any irredundant partition regular matrix A. The following theorem of Friedgut, Rödl and Schacht [42] complements this result, implying that almost all subsets of $[n]$ with significantly more than $n^{1-1/m(A)}$ elements are $(\mathcal{L}^d, r)$-Rado for any irredundant partition regular matrix A.

Theorem 5.9 ([42]). *For all irredundant partition regular full rank matrices A and all positive integers r, there exists a constant $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[[n]_p \text{ is } (\mathcal{L}^d, r)\text{-Rado}\right] = 1 \quad \text{if } p > Cn^{-1/m(A)}.$$

Earlier, Theorem 5.9 was confirmed by Graham, Rödl and Ruciński [46] in the case where $\mathcal{L}$ is $x + y = z$ and $r = 2$, and then by Rödl and Ruciński [93] in the case when A is density regular.

Together Theorems 5.8 and 5.9 show that the threshold for the property of being $(\mathcal{L}^d, r)$-Rado is $p = n^{-1/m(A)}$. In light of this, it is interesting to ask if above this threshold the property of being $(\mathcal{L}^d, r)$-Rado is resilient to the deletion of a significant number of elements. To be precise, given a set S, we define the *resilience of S with respect to $\mathcal{P}$*, $\mathrm{res}(S, \mathcal{P})$, to be the minimum number t such that by deleting t elements from S, one can obtain a set not having $\mathcal{P}$. For example, when $\mathcal{P}$ is the property of containing an arithmetic progression of length k, then Szemerédi's theorem can be phrased in terms of resilience; it states that for all $k \geqslant 3$ and $\varepsilon > 0$, there exists $n_0 > 0$ such that for all integers $n \geqslant n_0$, we have $\mathrm{res}([n], \mathcal{P}) \geqslant (1 - \varepsilon)n$.

The following result of Schacht [108] provides a resilience strengthening of Theorem 5.9 in the case of density regular matrices.

Theorem 5.10 ([108]). *For all irredundant density regular full rank matrices A, all positive integers r and all $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\mathrm{res}([n]_p, (\mathcal{L}^d, r)\text{-}\mathit{Rado})}{|[n]_p|} \geqslant 1 - \varepsilon\right] = 1 \qquad \text{if } p > Cn^{-1/m(A)}.$$

Note that in [108] the result is stated in the $r = 1$ case only, but the general result follows immediately from this special case.

Our next result gives a resilience strengthening of Theorem 5.9 for all irredundant partition regular matrices.

Theorem 5.11. *For all irredundant partition regular full rank matrices A, all positive integers r and all $\delta > 0$, there exists a constant $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\mathrm{res}([n]_p, (\mathcal{L}^d, r)\text{-}\mathit{Rado})}{|[n]_p|} = 1 - \frac{\mu(n, \mathcal{L}^d, r)}{n} \pm \delta\right] = 1 \qquad \text{if } p > Cn^{-1/m(A)}.$$

It is well known that for all irredundant partition regular full rank matrices A and all positive integers r, there exist $n_0 = n_0(A, r), \eta = \eta(A, r) > 0$, such that for all integers $n \geqslant n_0$, we have $\mu(n, \mathcal{L}^d, r) \leqslant (1 - \eta)n$. (This follows from a supersaturation lemma of Frankl, Graham and Rödl [41, Theorem 1].) Thus, Theorem 5.11 does imply Theorem 5.9. Further, in the case when A is density regular, [41, Theorem 2] immediately implies that $\mu(n, \mathcal{L}^d, r) = o(n)$ for any fixed $r \in \mathbb{N}$. Thus Theorem 5.11 implies Theorem 5.10. Theorem 5.11 in the case when $r = 1$ and $\mathcal{L}$ is $x + y = z$ was proved by Schacht [108]. In fact, the method of Schacht can be used to prove the theorem for $r = 1$ and every irredundant partition regular matrix A.

Intuitively, the reader can interpret Theorem 5.11 as stating that almost all subsets of $[n]$ with significantly more than $n^{1-1/m(A)}$ elements strongly possess the property of being $(\mathcal{L}^d, r)$-Rado for any irredundant partition regular matrix A. The 'strength' here depends on the parameter $\mu(n, \mathcal{L}^d, r)$. In light of this it is natural to seek good bounds on $\mu(n, \mathcal{L}^d, r)$ (particularly in the cases when $\mu(n, \mathcal{L}^d, r) = \Omega(n)$). In general, not too much is known about this parameter. However, as mentioned earlier, in the case when $A = (1, 1, -1)$ (i.e. $\mathcal{L}$ is $x + y = z$), this is (by replacing distinct $\mathcal{L}$-free with strongly $\mathcal{L}$-free) a 40-year-old problem of Abbott and Wang [1]. In Section 5.3.6 we give an upper bound on $\mu(n, \mathcal{L}^s, r)$ in this case for all $r \in \mathbb{N}$.

Instead of proving Theorem 5.11 directly, in Section 5.3 we will prove a version of the result that holds for a more general class of matrices A, does not assume $b = 0$, and also deals with the asymmetric case, namely Theorem 5.16. In [112], Spiegel proved the case $r = 1$ of Theorem 5.16 and used the container method to give an alternative proof of Theorem 5.9.

5.1.2 Enumeration questions for Ramsey problems

A fundamental question in combinatorics is to determine the number of structures with a given property. For example, Erdős, Frankl and Rödl [34] proved that the number of

n-vertex H-free graphs is $2^{\binom{n}{2}(1-\frac{1}{r-1}+o(1))}$ for any graph H of chromatic number r. Here the lower bound follows by considering all the subgraphs of the $(r-1)$-partite Turán graph. Given any $k, r, n \in \mathbb{N}$ with $k \geqslant 2$ and k-uniform hypergraphs $H_1, \dots, H_r$, define $\mathrm{Ram}(n; H_1, \dots, H_r)$ to be the collection of all k-uniform hypergraphs on vertex set $[n]$ that are $(H_1, \dots, H_r)$-Ramsey and $\overline{\mathrm{Ram}}(n; H_1, \dots, H_r)$ to be all those k-uniform hypergraphs on $[n]$ that are not $(H_1, \dots, H_r)$-Ramsey. A natural question is to determine the size of $\mathrm{Ram}(n; H_1, \dots, H_r)$. Surprisingly, we are unaware of any *explicit* results in this direction for $r \geqslant 2$. The next application of the container method fully answers this question up to an error term in the exponent.

Theorem 5.12. *Let $k, r, n \in \mathbb{N}$ with $k \geqslant 2$ and $H_1, \dots, H_r$ be k-uniform hypergraphs. Then*

$$|\overline{\mathrm{Ram}}(n; H_1, \dots, H_r)| = 2^{\mathrm{ex}^r(n, H_1, \dots, H_r)+o(n^k)} = 2^{\pi(H_1, \dots, H_r)\binom{n}{k}+o(n^k)}.$$

Note that in the case when $k = 2$ and $r = 1$, Theorem 5.12 is precisely the above mentioned result of Erdős, Frankl and Rödl [34]. In fact, one can also obtain Theorem 5.12 by using the work from [84], a hypergraph analogue of the result in [34]; see Section 5.4.5 for a proof of this. Similar results were obtained also using containers by Falgas-Ravry, O'Connell and Uzzell in [40], and by Terry in [118] who reproved a result of Ishigami [63].

Our final application of the container method determines, up to an error term in the exponent, the number of $(\mathcal{L}^d, r)$-free subsets of $[n]$.

Theorem 5.13. *Let A be an irredundant partition regular matrix of full rank and let $r \in \mathbb{N}$ be fixed. There are $2^{\mu(n,\mathcal{L}^d,r)+o(n)}$ $(\mathcal{L}^d, r)$-free subsets of $[n]$.*

As an illustration, a result of Hu [61] implies that $\mu(n, \mathcal{L}^d, 2) = 4n/5 + o(n)$ in the case when $\mathcal{L}$ is $x + y = z$. Thus, Theorem 5.13 tells us all but $2^{(4/5+o(1))n}$ subsets of $[n]$ are $(\mathcal{L}^d, 2)$-Rado in this case. Related results (in the 1-colour case) were obtained by Green [48] and Saxton and Thomason [107].

5.2 Container results for disjoint independent sets

First recall from Chapter 2 the general hypergraph container theorem of Balogh, Morris and Samotij [8].

Theorem 2.1 ([8]). *For every $k \in \mathbb{N}$ and all positive c and ε, there exists a positive constant C such that the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph and let $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ be an increasing family of sets such that $|A| \geqslant \varepsilon v(\mathcal{H})$ for all $A \in \mathcal{F}$. Suppose that $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-dense and $p \in (0,1)$ is such that, for every $\ell \in [k]$,*

$$\Delta_\ell(\mathcal{H}) \leqslant c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

Then there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leqslant Cp \cdot v(\mathcal{H})}$ and functions $f : \mathcal{S} \to \overline{\mathcal{F}}$ and $g : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for every $I \in \mathcal{I}(\mathcal{H})$, we have that $g(I) \subseteq I$ and $I \setminus g(I) \subseteq f(g(I))$.

Throughout the chapter, when we consider r-tuples of sets, the r-tuples are *always* ordered. For two r-tuples of sets $(I_1, \dots, I_r)$ and $(J_1, \dots, J_r)$ we write $(I_1, \dots, I_r) \subseteq (J_1, \dots, J_r)$ if $I_x \subseteq J_x$ for each $x \in [r]$. We write $(I_1, \dots, I_r) \cup (J_1, \dots, J_r) := (I_1 \cup J_1, \dots, I_r \cup J_r)$.

We write ij to denote the pair $\{i, j\}$. For a hypergraph $\mathcal{H}$ define

$$\mathcal{I}_r(\mathcal{H}) := \left\{ (I_1, \dots, I_r) \in \mathcal{P}(V(\mathcal{H}))^r : I_x \in \mathcal{I}(\mathcal{H}) \text{ and } I_i \cap I_j = \emptyset \text{ for all } x \in [r], ij \in \binom{[r]}{2} \right\}.$$

Whereas Theorem 2.1 provides a set of containers for the independent sets of a hypergraph, the following theorem is an analogous result for the r-tuples of disjoint independent sets of a hypergraph. It is a straightforward consequence of Theorem 2.1.

Theorem 5.14. *For every $k, r \in \mathbb{N}$ and all positive c and ε, there exists a positive constant C such that the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph and let*

$\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ be an increasing family of sets such that $|A| \geqslant \varepsilon v(\mathcal{H})$ for all $A \in \mathcal{F}$. Suppose that $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-dense and $p \in (0,1)$ is such that, for every $\ell \in [k]$,

$$\Delta_\ell(\mathcal{H}) \leqslant c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

Then there exists a family $\mathcal{S}_r \subseteq \mathcal{P}(V(\mathcal{H}))^r$ and functions $f : \mathcal{S}_r \to (\overline{\mathcal{F}})^r$ and $g : \mathcal{I}_r(\mathcal{H}) \to \mathcal{S}_r$ such that the following conditions hold:

(i) If $(S_1, \ldots, S_r) \in \mathcal{S}_r$ then $\sum |S_i| \leqslant Cp \cdot v(\mathcal{H})$;

(ii) every $S \in \mathcal{S}_r$ satisfies $S \in \mathcal{I}_r(\mathcal{H})$;

(iii) for every $(I_1, \ldots, I_r) \in \mathcal{I}_r(\mathcal{H})$, we have that $S \subseteq (I_1, \ldots, I_r) \subseteq S \cup f(S)$ where $S := g(I_1, \ldots, I_r)$.

Proof. Apply Theorem 2.1 with k, c, ε to obtain a positive constant C_1. Let $C := rC_1$. We will show that C has the required properties. Let $\mathcal{H}$ be a k-uniform hypergraph which together with a set $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ satisfies the hypotheses of Theorem 5.14. Since $\mathcal{H}$, $\mathcal{F}$ also satisfy the hypotheses of Theorem 2.1, there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leqslant C_1 p \cdot v(\mathcal{H})}$ and functions $f' : \mathcal{S} \to \overline{\mathcal{F}}$ and $g' : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for every $I \in \mathcal{I}(\mathcal{H})$ we have $g'(I) \subseteq I$ and $I \setminus g'(I) \subseteq f'(g'(I))$. Define

$$\mathcal{S}' := \{S \in \mathcal{S} : \text{ there exists } I \in \mathcal{I}(\mathcal{H}) \text{ such that } g'(I) = S\},$$

and

$$\mathcal{S}_r := \left\{ (S_1, \ldots, S_r) \in \mathcal{P}(V(\mathcal{H}))^r : S_x \in \mathcal{S}' \text{ and } S_i \cap S_j = \emptyset \text{ for all } x \in [r], ij \in \binom{[r]}{2} \right\}.$$

Let $(S_1, \dots, S_r) \in \mathcal{S}_r$. First note that

$$\sum_{x \in [r]} |S_x| \leqslant C_1 r \cdot pv(\mathcal{H}) = Cp \cdot v(\mathcal{H}),$$

so (i) holds. Also since $S_x \in \mathcal{S}'$ for all $x \in [r]$, we have $S_x \in \mathcal{I}(\mathcal{H})$ and so by definition of $\mathcal{S}_r$ we have $\mathcal{S}_r \subseteq \mathcal{I}_r(\mathcal{H})$ proving (ii).

Consider any $(S_1, \dots, S_r) \in \mathcal{S}_r$ and any $(I_1, \dots, I_r) \in \mathcal{I}_r(\mathcal{H})$. Define $f : \mathcal{S}_r \to (\overline{\mathcal{F}})^r$ by setting $f(S_1, \dots, S_r) := (f'(S_1), \dots, f'(S_r))$ and define $g : \mathcal{I}_r(\mathcal{H}) \to \mathcal{S}_r$ by setting $g(I_1, \dots, I_r) := (g'(I_1), \dots, g'(I_r))$.

Note that since $f'(S_x) \in \overline{\mathcal{F}}$, $g'(I_x) \in \mathcal{S}'$ and $g'(I_i) \cap g'(I_j) = \emptyset$ for all $x \in [r]$ and $ij \in \binom{[r]}{2}$, we do indeed have $(f'(S_1), \dots, f'(S_r)) \in (\overline{\mathcal{F}})^r$ and $(g'(I_1), \dots, g'(I_r)) \in \mathcal{S}_r$.

Now for (iii), since $g'(I_x) \subseteq I_x$ and $I_x \setminus g'(I_x) \subseteq f'(g'(I_x))$ for all $x \in [r]$, we have $g(I_1, \dots, I_r) = (g'(I_1), \dots, g'(I_r)) \subseteq (I_1, \dots, I_r)$. Now since $f(g(I_1, \dots, I_r)) = (f'(g'(I_1)), \dots, f'(g'(I_r)))$ we also have $(I_1, \dots, I_r) \subseteq f(g(I_1, \dots, I_r)) \cup g(I_1, \dots, I_r)$ as required. $\square$

In all of our applications of the container method, we will in fact apply the following asymmetric version of Theorem 5.14. In particular, in the proof of e.g. Theorem 5.7, instead of considering tuples of disjoint independent sets from the *same* hypergraph $\mathcal{H}$, we are actually concerned with disjoint independent sets from *different* hypergraphs but which have the *same vertex set*: For all $i \in [r]$, let $\mathcal{H}_i$ be a k_i-uniform hypergraph, each on the same vertex set V, and define $\mathcal{I}(\mathcal{H}_1, \dots, \mathcal{H}_r)$ to be the set of all r-tuples $(I_1, \dots, I_r) \in \prod_{i \in [r]} \mathcal{I}(\mathcal{H}_i)$ such that $I_i \cap I_j = \emptyset$ for all $1 \leqslant i < j \leqslant r$.

We omit the proof of Theorem 5.15 since it follows from Theorem 2.1 as in the proof of Theorem 5.14.

Theorem 5.15. *For every $r, k_1, \dots, k_r \in \mathbb{N}$ with $k_i \geqslant 2$ for all $i \in [r]$, and all $c, \varepsilon > 0$, there exists a positive constant C such that the following holds. For all $i \in [r]$, let $\mathcal{H}_i$ be* *a k_i-uniform hypergraph, each on the same vertex set V. For all $i \in [r]$, let $\mathcal{F}_i \subseteq \mathcal{P}(V)$ be an increasing family of sets such that $|A| \geqslant \varepsilon|V|$ for all $A \in \mathcal{F}_i$. Suppose that each $\mathcal{H}_i$ is $(\mathcal{F}_i, \varepsilon)$-dense. Further suppose $p \in (0,1)$ is such that, for every $i \in [r]$ and $\ell \in [k_i]$,*

$$\Delta_\ell(\mathcal{H}_i) \leqslant c \cdot p^{\ell-1}\frac{e(\mathcal{H}_i)}{|V|}.$$

Then there exists a family $\mathcal{S}_r \subseteq \mathcal{I}(\mathcal{H}_1, \dots, \mathcal{H}_r)$ and functions $f : \mathcal{S}_r \to \prod_{i\in[r]} \overline{\mathcal{F}_i}$ and $g : \mathcal{I}(\mathcal{H}_1, \dots, \mathcal{H}_r) \to \mathcal{S}_r$ such that the following conditions hold:

(i) If $(S_1, \dots, S_r) \in \mathcal{S}_r$ then $\sum |S_i| \leqslant Cp|V|$;

(ii) for every $(I_1, \dots, I_r) \in \mathcal{I}(\mathcal{H}_1, \dots, \mathcal{H}_r)$, we have that $S \subseteq (I_1, \dots, I_r) \subseteq S \cup f(S)$ where $S := g(I_1, \dots, I_r)$.

5.3 Applications of the container method to r-tuples of solution-free sets

In this section we will prove Theorems 5.11 and 5.13 by using the container theorem for r-tuples of disjoint independent sets, applied with irredundant partition regular matrices A. Suppose that we have a k-uniform hypergraph $\mathcal{H}$ whose vertex set is a subset of $\mathbb{N}$ and where the edges correspond to the k-distinct solutions of $\mathcal{L}$. Then in this setting, an $(\mathcal{L}^d, r)$-free set is precisely an r-tuple of disjoint independent sets in $\mathcal{H}$.

Theorems 5.11 and 5.13 will be deduced from a container theorem, Theorem 5.21, which in turn follows from Theorem 5.15. Theorem 5.21 actually holds for a class of irredundant matrices of which partition regular matrices are a subclass. Let $(*)$ be the following matrix property:

$(*)$ Under Gaussian elimination A does not have any row which consists of precisely two non-zero rational entries.

For an integer matrix A and integer vector b, call the system of linear equations $\mathcal{L}$ (and the matrix A in the case when $b = 0$) *r-regular* if all r-colourings of $\mathbb{N}$ yield a monochromatic solution to $Ax = b$. Observe that a system of linear equations $\mathcal{L}$ is r-regular for all $r \in \mathbb{N}$ if and only if it is partition regular. As outlined in the next subsection, given any $r \geqslant 2$, all irredundant r-regular matrices A satisfy $(*)$. We will in fact prove stronger versions of Theorems 5.11 and 5.13 that consider irredundant systems of linear equations $\mathcal{L}$ for which the matrix A satisfies $(*)$.

These general results also consider 'asymmetric' Rado properties: Suppose that $\mathcal{L}_i$ is a system of linear equations for each $1 \leqslant i \leqslant r$ (and, here and elsewhere, A_i and b_i is the matrix and vector such that $\mathcal{L}_i = \mathcal{L}(A_i, b_i)$). We say a set $X \subseteq \mathbb{N}$ is *$(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-free* if there is an r-colouring of X such that there are no k-distinct solutions to $\mathcal{L}_i$ in X in colour i for every $i \in [r]$. Otherwise we say that X is *$(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-Rado*. We denote the size of the largest $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-free subset of $[n]$ by $\mu(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$.

In general it is not known which systems of linear equations $\mathcal{L}_1, \dots, \mathcal{L}_r$ are such that $\mathbb{N}$ is $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-Rado. However, if each $\mathcal{L}_i$ is an r-regular homogenous linear equation, then $\mathbb{N}$ is $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-Rado (see [76, Theorem 9.19]).

We will prove the following strengthenings of Theorems 5.11 and 5.13.

Theorem 5.16. *For all positive integers r, all full rank integer matrices $A_1, \dots, A_r$ of dimension $\ell_i \times k_i$ which satisfy $(*)$ with $m(A_1) \geqslant \ldots \geqslant m(A_r)$, all integer vectors b_i of dimension ℓ_i where $\mathcal{L}_i$ is irredundant, and all $\delta > 0$, there exists a constant $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\mathrm{res}([n]_p, (\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)\text{-}Rado)}{|[n]_p|} = 1 - \frac{\mu(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)}{n} \pm \delta\right] = 1 \qquad \text{if } p > Cn^{-1/m(A_1)}.$$

Theorem 5.17. *For all positive integers r, all full rank integer matrices $A_1, \dots, A_r$ of dimension $\ell_i \times k_i$ which satisfy $(*)$, all integer vectors b_i of dimension ℓ_i where $\mathcal{L}_i$ is* *irredundant, there are* $2^{\mu(n,\mathcal{L}_1^d,\dots,\mathcal{L}_r^d)+o(n)}$ $(\mathcal{L}_1^d,\dots,\mathcal{L}_r^d)$*-free subsets of* $[n]$.

Given a system of linear equations $\mathcal{L}$, recall that a strongly $\mathcal{L}$-free subset of $[n]$ is a subset that contains no solution to $\mathcal{L}$ of any kind. Although this is not quite the same definition as distinct $\mathcal{L}$-free, we remark that Theorem 5.17 implies a result of Green [48, Theorem 9.3] in the case where $k \geqslant 3$, on the number of strongly $\mathcal{L}$-free subsets of $[n]$ for homogeneous linear equations $\mathcal{L}$.

As mentioned in the introduction of this chapter, Spiegel [112] independently proved the case $r = 1$ of Theorem 5.16. (Note in [112] this result is mentioned in terms of *abundant* matrices A. That is every $\ell \times (k-2)$ submatrix of A has rank ℓ. But this is clearly equivalent to $(*)$ in the case of irredundant full rank matrices.)

5.3.1 Matrices which satisfy $(*)$

First we prove that irredundant partition regular matrices are a strict subclass of irredundant matrices which satisfy $(*)$.

Suppose that an irredundant matrix A does not satisfy $(*)$. Then for all solutions $(x_1,\dots,x_k)$ to $\mathcal{L}$, there exists a pair $ij \in \binom{[k]}{2}$ and non-zero rationals α,β such that $\alpha x_i = \beta x_j$. If $\alpha = \beta$ then no solution to $\mathcal{L}$ is k-distinct and so A is redundant, a contradiction. Otherwise, without loss of generality, assume that $\alpha > \beta$, and devise the following 2-colouring of $\mathbb{N}$: greedily colour the numbers $\{1,2,3,...\}$ so that when colouring x, we always give it a different colour to $\beta x/\alpha$ (if $\beta x/\alpha \in \mathbb{N}$). Such a colouring ensures that no solution to $\mathcal{L}$ is monochromatic, and so A is not partition regular.

Note that the converse is not true. An $\ell \times k$ matrix with columns $a^{(1)},\dots,a^{(k)}$ satisfies the *columns property* if there is a partition of $[k]$, say $[k] = D_1 \cup \cdots \cup D_t$ such that

$$\sum_{i\in D_1} a^{(i)} = 0$$

and for every $r \in [t]$ we have

$$\sum_{i \in D_r} a^{(i)} \in \langle a^{(j)} : j \in D_1 \cup \cdots \cup D_{r-1} \rangle.$$

Rado's theorem [88] states that a matrix is partition regular if and only if it satisfies the columns property. Now, for example $A := \begin{pmatrix} 2 & 2 & -1 \end{pmatrix}$ clearly satisfies $(*)$, and additionally does not have the columns property, so is not partition regular.

The argument above actually implies that if an irredundant matrix A is 2-regular, then it satisfies $(*)$. So in the symmetric case, Theorems 5.16 and 5.17 consider *all* pairs $(\mathcal{L}, r)$ such that $\mathcal{L}$ is an irredundant r-regular system of linear equations and $r \geqslant 2$.

5.3.2 Useful matrix lemmas

Before we can prove our container result (Theorem 5.21), we require some matrix lemmas. Note that all of these lemmas hold for irredundant matrices which satisfy $(*)$. As a consequence, Theorem 5.8 was actually implicitly proven for irredundant matrices which satisfy $(*)$, since in [93] the only necessity of the matrix being partition regular was so that the results stated below could be applied.

Recall the definition of $m(A)$ given by (1.3.2). Parts (i) and (ii) of the following proposition were verified for irredundant partition regular matrices by Rödl and Ruciński (see Proposition 2.2 in [93]). In fact their result easily extends to matrices which satisfy $(*)$. We give the full proof for completeness, and add further facts ((iii)–(v)) which will be useful in the proof of Theorem 5.21.

Proposition 5.18. *Let A be an $\ell \times k$ irredundant matrix of full rank ℓ which satisfies $(*)$. Then for every $W \subseteq [k]$, the following hold.*

(i) If $|W| = 1$, then $\operatorname{rank}(A_{\overline{W}}) = \ell$.

(ii) If $|W| \geqslant 2$, then $\ell - \operatorname{rank}(A_{\overline{W}}) + 2 \leqslant |W|$.

(iii) If $|W| \geqslant 2$*, then*

$$|W| + \mathrm{rank}(A_{\overline{W}}) \geqslant \ell + 1 + \frac{|W|-1}{m(A)}.$$

Furthermore,

(iv) $k \geqslant \ell + 2$*;*

(v) $m(A) > 1$*.*

Proof. For (i), suppose that $\mathrm{rank}(A_{\overline{W}}) = \ell - 1$ for some $W \subseteq [k]$ with $|W| = 1$. Since $A_{\overline{W}}$ is an $\ell \times (k-1)$ matrix of rank $\ell - 1$, under Gaussian elimination it must contain a row of zeroes. Hence A under Gaussian elimination contains a row with at most one rational non-zero entry. If there is one, then there are no positive solutions to $\mathcal{L}$, which contradicts A being irredundant. If there are none, then A does not have rank ℓ, also a contradiction.

For (ii) proceed by induction on $|W|$. Assume first that there is a $W \subseteq [k]$ with $|W| = 2$, such that $\mathrm{rank}(A_{\overline{W}}) < \ell$. Using a similar argument to (i), under Gaussian elimination A contains a row with at most two rational non-zero entries. If there are two rational non-zero entries this contradicts A satisfying $(*)$. Otherwise we again get a contradiction to either A being irredundant or of rank ℓ. Assume now that $|W| \geqslant 3$ and that the statement holds for $|W| - 1$. The rank of a matrix drops by at most one when a column is deleted, hence the required inequality follows by induction.

For (iii), note that for $|W| \geqslant 2$, by definition we have $m(A) \geqslant (|W|-1)/(|W|-1+\mathrm{rank}(A_{\overline{W}}) - \ell)$. This can be rearranged to give the required inequality. For (iv), suppose that $k \leqslant \ell + 1$. Then under Gaussian elimination A must have a row with at most two rational non-zero entries. This leads to the same contradiction as the base case of the induction for (ii). For (v), take $W = [k]$. Then by definition $m(A) \geqslant (k-1)/(k-\ell-1) > 1$, where the second inequality follows since the denominator is positive by (iv). $\square$

We will require Lemma 2.3. Finally we need the following well known result (and a simple corollary of it), which gives a useful upper bound on the number of solutions to a system of linear equations. Note that in this lemma only, we *do not* assume A to be necessarily of full rank (as we will apply the result directly to matrices formed by deleting columns from our original matrix of full rank).

Lemma 5.19. *For an $\ell \times k$ matrix A not necessarily of full rank, an ℓ-dimensional integer vector b and a set $X \subseteq [n]$, the system $Ax = b$ has at most $|X|^{k-\mathrm{rank}(A)}$ solutions in X.*

Proof. Let $m := \mathrm{rank}(A)$. Proceed by induction on k. If $k = 1$, then $m = 1$ or $m = 0$. If $m = 1$, then there is a unique solution to $Ax = b$, so $Ax = b$ has at most $1 = |X|^{k-m}$ solution in X. If $m = 0$, then A is the zero matrix, and so each element in X could be a solution (if $b = 0$), but then $Ax = b$ has at most $|X| = |X|^{k-m}$ solutions in X.

For the inductive step, pick any $c \in [k]$ and fix some $x_c \in X$. Form a new system of linear equations $A'x' = b'$, where A' is formed from $A := (a_{ij})$ by removing the c^{th} column, and $b' := (b'_1, \ldots, b'_\ell)$ is formed from $b := (b_1, \ldots, b_\ell) \in \mathbb{Z}^\ell$ by setting $b'_r := b_r - a_{rc}x_c$. (So $x := (x_1, \ldots, x_k)$ is a solution to $Ax = b$ if and only if $x' := (x_1, \ldots, x_{c-1}, x_{c+1}, \ldots, x_k)$ is a solution to $A'x' = b'$.)

Note that A' must have rank m or $m - 1$. If A' has rank m then by the induction hypothesis $A'x' = b'$ has at most $|X|^{(k-1)-\mathrm{rank}(A')} = |X|^{k-m-1}$ solutions in X. Since there are $|X|$ valid choices for x_c, there are at most $|X|^{k-m}$ solutions to $Ax = b$ in X, as required.

Now suppose A' has rank $m - 1$. Then under Gaussian elimination, A' and A have $\ell - m + 1$, and respectively $\ell - m$ rows consisting entirely of zeroes, and in particular, A has a row with precisely one non-zero entry which is in the c^{th} column. Hence there is at most one value x_c can take in any solution $x = (x_1, \ldots, x_k)$ to $Ax = b$. So for this choice of x_c, $Ax = b$ and $A'x' = b'$ have precisely the same number of solutions in X. Since A' is an $\ell \times (k - 1)$ matrix of rank $m - 1$, the induction hypothesis implies that there are

$|X|^{(k-1)-(m-1)} = |X|^{k-\text{rank}(A)}$ solutions to $A'x' = b'$ and thus $Ax = b$, as desired. $\square$

Corollary 5.20. *Consider an $\ell \times k$ matrix A of rank ℓ, a set $X \subseteq [n]$ and an integer $1 \leqslant t \leqslant k$. Fix distinct $y_1, \dots, y_t \in X$ and consider any $W = \{s_1, \dots, s_t\} \subseteq [k]$. The system $Ax = b$ has at most $|X|^{k-t-\text{rank}(A_{\overline{W}})}$ solutions $(x_1, \dots, x_k)$ in X for which $x_{s_j} = y_j$ for each $j \in [t]$. Moreover, if the system $Ax = b$ is irredundant and A satisfies $(*)$ and $t = 1$, then the system $Ax = b$ has at most $|X|^{k-\ell-1}$ solutions $(x_1, \dots, x_k)$ in X for which $x_{s_1} = y_1$.*

Proof. Write $A =: (a_{ij})$. Consider the system of linear equations $A_{\overline{W}}x' = b'$ where, for each $r \in [\ell]$, the r^{th} term in b' is

$$b'_r := b_r - \sum_{s_j \in W} a_{rs_j} y_j.$$

Now by Lemma 5.19 the system of linear equations $A_{\overline{W}}x' = b'$ has at most $|X|^{k-t-\text{rank}(A_{\overline{W}})}$ solutions in X. The first part of the corollary then follows since all solutions $(x_1, \dots, x_k)$ to $Ax = b$ with $x_{s_j} = y_j$ for each $j \in [t]$, rise from a solution x' to $A_{\overline{W}}x' = b'$. For the second part, if $Ax = b$ is irredundant and A satisfies $(*)$ and $t = 1$, then by Proposition 5.18(i), we have $\text{rank}(A_{\overline{W}}) = \ell$ and so the result follows. $\square$

5.3.3 A container theorem for tuples of $\mathcal{L}^d$-free sets

Recall that an $\mathcal{L}^d$-free set is simply an $(\mathcal{L}^d, 1)$-free set. Let $\mathcal{I}(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$ denote the set of all ordered r-tuples $(X_1, \dots, X_r) \in \mathcal{P}([n])^r$ so that each X_i is $\mathcal{L}_i^d$-free and $X_i \cap X_j = \emptyset$ for all distinct $i, j \in [r]$. Note that any $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-free subset X of $[n]$ has a partition $X_1, \dots, X_r$ so that $(X_1, \dots, X_r) \in \mathcal{I}(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$. We now prove a container theorem for the elements of $\mathcal{I}(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$.

Theorem 5.21. *Let $r \in \mathbb{N}$ and $0 < \delta < 1$. For each $i \in [r]$ let A_i be an integer matrix of dimension $\ell_i \times k_i$ which satisfies $(*)$, let b_i be an integer vector of dimension ℓ_i and suppose $\mathcal{L}_i$ is irredundant. Suppose that $m(A_1) \geqslant \ldots \geqslant m(A_r)$. Then there exists $D > 0$ such that the following holds. For all $n \in \mathbb{N}$, there is a collection $\mathcal{S}_r \subseteq \mathcal{P}([n])^r$ and a function $f : \mathcal{S}_r \to \mathcal{P}([n])^r$ such that:*

(i) For all $(I_1, \ldots, I_r) \in \mathcal{I}(n, \mathcal{L}_1^d, \ldots, \mathcal{L}_r^d)$, there exists $S \in \mathcal{S}_r$ such that $S \subseteq (I_1, \ldots, I_r) \subseteq f(S)$.

(ii) If $(S_1, \ldots, S_r) \in \mathcal{S}_r$ then $\sum_{i\in[r]} |S_i| \leqslant Dn^{\frac{m(A_1)-1}{m(A_1)}}$.

(iii) Every $S \in \mathcal{S}_r$ satisfies $S \in \mathcal{I}(n, \mathcal{L}_1^d, \ldots, \mathcal{L}_r^d)$.

(iv) Given any $S = (S_1, \ldots, S_r) \in \mathcal{S}_r$, write $f(S) =: (f(S_1), \ldots, f(S_r))$. Then

(a) for each $1 \leqslant i \leqslant r$, $f(S_i)$ contains at most $\delta n^{k_i-\ell_i}$ k_i-distinct solutions to $\mathcal{L}_i$; and

(b) $|\cup_{i\in[r]} f(S_i)| \leqslant \mu(n, \mathcal{L}_1^d, \ldots, \mathcal{L}_r^d) + \delta n$.

We emphasise that (iv)(b) does not necessarily guarantee $\sum_{i\in[r]} |f(S_i)| \leqslant \mu(n, \mathcal{L}_1^d, \ldots, \mathcal{L}_r^d) + \delta n$. Rather it ensures at most $\mu(n, \mathcal{L}_1^d, \ldots, \mathcal{L}_r^d) + \delta n$ elements of $[n]$ appear in at least one of the co-ordinates of $f(S)$. This property is crucial for our applications.

Proof. First note that since each of the systems of linear equations $\mathcal{L}_i$ are irredundant, a result of Kusch, Rué, Spiegel and Szabó [75] implies that there exists a constant $d > 0$ such that, for each $i \in [r]$, there are at least $dn^{k_i-\ell_i}$ k_i-distinct solutions to $\mathcal{L}_i$ in $[n]$. (This is a generalisation of a result by Janson and Ruciński [65].)

Note that it suffices to prove the theorem in the case when $0 < \delta < d$. So let $0 < \delta < d$ and $r \in \mathbb{N}$ be given and apply Lemma 2.3 to obtain $n_0, \varepsilon > 0$. Without loss of generality we may assume $\varepsilon \leqslant \delta$. Define $k := \max k_i$ and let

$$\varepsilon' := \frac{\varepsilon}{2} \quad \text{and} \quad c := \frac{k!}{\varepsilon'}.$$

Apply Theorem 5.15 with parameters $r, k_1, \ldots, k_r, c, \varepsilon'$ playing the roles of $r, k_1, \ldots, k_r, c, \varepsilon$ respectively to obtain $D_1 > 0$. Increase n_0 if necessary so that

$$0 < \frac{1}{n_0} \ll \frac{1}{D_1}, \frac{1}{k_1}, \ldots, \frac{1}{k_r}, \frac{1}{r}, \varepsilon, \delta.$$

For $n < n_0$, set $\mathcal{S}_r$ to be $\mathcal{I}(n, \mathcal{L}_1^d, \ldots, \mathcal{L}_r^d)$; set f to be the identity function and choose D_2 to be large. By setting D to be the maximum of D_1 and D_2, it remains to prove the result for integers $n \geqslant n_0$. So now fix $n \geqslant n_0$.

For each $i \in [r]$ let $\mathcal{H}_{n,i}$ be the hypergraph with $V(\mathcal{H}_{n,i}) := [n]$ and an edge set which consists of all k_i-distinct solutions to $\mathcal{L}_i$ in $[n]$. Observe that $\mathcal{H}_{n,i}$ is k_i-uniform and an independent set in $\mathcal{H}_{n,i}$ is an $\mathcal{L}_i^d$-free set.

For each $i \in [r]$ we define $\mathcal{F}_{n,i} := \{F \subseteq V(\mathcal{H}_{n,i}) : e(\mathcal{H}_{n,i}[F]) \geqslant \varepsilon' e(\mathcal{H}_{n,i})\}$. Note that since $\varepsilon' < d$, we have

$$\varepsilon' n^{k_i - \ell_i} \leqslant e(\mathcal{H}_{n,i}). \tag{5.3.1}$$

We claim that $\mathcal{H}_{n,i}$ and $\mathcal{F}_{n,i}$ satisfy the hypotheses of Theorem 5.15 with parameters chosen as above with

$$p = p(n) := n^{-1/m(A_1)}.$$

Clearly $\mathcal{F}_{n,i}$ is increasing and $\mathcal{H}_{n,i}$ is $(\mathcal{F}_{n,i}, \varepsilon')$-dense. By Lemma 5.19, a set $F \subseteq V(\mathcal{H}_{n,i})$ contains at most $|F|^{k_i - \ell_i}$ solutions to $\mathcal{L}_i$ (so $e(\mathcal{H}_{n,i}[F]) \leqslant |F|^{k_i - \ell_i}$). Hence for all

$F \in \mathcal{F}_{n,i}$, we have

$$|F| \geqslant e(\mathcal{H}_{n,i}[F])^{\frac{1}{k_i-\ell_i}} \geqslant (\varepsilon' e(\mathcal{H}_{n,i}))^{\frac{1}{k_i-\ell_i}} \overset{(5.3.1)}{\geqslant} ((\varepsilon')^2 n^{k_i-\ell_i})^{\frac{1}{k_i-\ell_i}} \geqslant \varepsilon' n$$

where the last inequality follows by Proposition 5.18(iv).

For each $j \in [k_i]$, we wish to bound the number of hyperedges containing some $\{y_1,\dots,y_j\} \subseteq V(\mathcal{H}_{n,i})$. Suppose $(x_1,\dots,x_{k_i})$ is a k_i-distinct solution to $\mathcal{L}_i$ so that $\{y_1,\dots,y_j\} \subseteq \{x_1,\dots,x_{k_i}\}$. There are $k_i!/(k_i-j)!$ choices for picking the j roles the y_i play in $(x_1,\dots,x_{k_i})$. Let W be one such choice for the set of indices of the x_a used by $\{y_1,\dots,y_j\}$. In this case, Corollary 5.20 implies there are at most $n^{k_i-j-\mathrm{rank}((A_i)_{\overline{W}})}$ such solutions to $\mathcal{L}_i$, and if $j=1$, there are at most $n^{k_i-\ell_i-1}$ such solutions. So for $j=1$ this yields

$$\deg_{\mathcal{H}_{n,i}}(y_1) \leqslant k_i n^{k_i-\ell_i-1} \overset{(5.3.1)}{\leqslant} \frac{k_i}{\varepsilon'} \frac{e(\mathcal{H}_{n,i})}{v(\mathcal{H}_{n,i})} \leqslant c\frac{e(\mathcal{H}_{n,i})}{v(\mathcal{H}_{n,i})}.$$

For $j \geqslant 2$, by Proposition 5.18(iii) we have $k_i-j-\mathrm{rank}((A_i)_{\overline{W}}) \leqslant k_i-\ell_i-1-(j-1)/m(A_i)$. Also $m(A_1) \geqslant m(A_i)$ for all $i \in [r]$ and hence we have

$$\begin{aligned}\deg_{\mathcal{H}_{n,i}}(\{y_1,\dots,y_j\}) &\leqslant k_i! n^{k_i-\ell_i-1-\frac{j-1}{m(A_i)}} \leqslant k_i! n^{k_i-\ell_i-1-\frac{j-1}{m(A_1)}} \\ &\leqslant \frac{k_i!}{\varepsilon'} p^{j-1} \frac{e(\mathcal{H}_{n,i})}{v(\mathcal{H}_{n,i})} \leqslant c p^{j-1} \frac{e(\mathcal{H}_{n,i})}{v(\mathcal{H}_{n,i})}.\end{aligned}$$

Since $\{y_1,\dots,y_j\}$ was arbitrary, we therefore have $\Delta_j(\mathcal{H}_{n,i}) \leqslant cp^{j-1}e(\mathcal{H}_{n,i})/v(\mathcal{H}_{n,i})$, as required. We have therefore shown that $\mathcal{H}_{n,i}$ and $\mathcal{F}_{n,i}$ satisfy the hypotheses of Theorem 5.15 for all $i \in [r]$.

Then Theorem 5.15 implies that there exists a family $\mathcal{S}_r \subseteq \prod_{i\in[r]} \mathcal{P}(V(\mathcal{H}_{n,i})) = \mathcal{P}([n])^r$ and functions $f' : \mathcal{S}_r \to \prod_{i\in[r]} \overline{\mathcal{F}_{n,i}}$ and $g : \mathcal{I}(\mathcal{H}_{n,1},\dots,\mathcal{H}_{n,r}) \to \mathcal{S}_r$ such that the following conditions hold:

(a) If $(S_1, \dots, S_r) \in \mathcal{S}_r$ then $\sum_{i \in [r]} |S_i| \leqslant D_1 pn$;

(b) every $S \in \mathcal{S}_r$ satisfies $S \in \mathcal{I}(\mathcal{H}_{n,1}, \dots, \mathcal{H}_{n,r})$;

(c) for every $(I_1, \dots, I_r) \in \mathcal{I}(\mathcal{H}_{n,1}, \dots, \mathcal{H}_{n,r})$, we have that $S \subseteq (I_1, \dots, I_r) \subseteq S \cup f'(S)$, where $S := g(I_1, \dots, I_r)$.

Note that $\mathcal{I}(\mathcal{H}_{n,1}, \dots, \mathcal{H}_{n,r}) = \mathcal{I}(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$. For each $S \in \mathcal{S}_r$, define

$$f(S) := S \cup f'(S).$$

So $f : \mathcal{S}_r \to \mathcal{P}([n])^r$. Thus, (a)–(c) immediately imply that (i)–(iii) hold.

Given any $S = (S_1, \dots, S_r) \in \mathcal{S}_r$ write $f(S) =: (f(S_1), \dots, f(S_r))$ and $f'(S) =: (f'(S_1), \dots, f'(S_r))$. (Note the slight abuse of the use of the f and f' notation here.) By definition of $\mathcal{F}_{n,i}$ any $F \in \overline{\mathcal{F}_{n,i}}$ contains at most $\varepsilon' n^{k_i - \ell_i}$ k_i-distinct solutions to $\mathcal{L}_i$. By Corollary 5.20, the number of k_i-distinct solutions to $\mathcal{L}_i$ in $[n]$ that use at least one element from S_i is at most $k_i n^{k_i - \ell_i - 1} |S_i|$. Further,

$$k_i n^{k_i - \ell_i - 1} |S_i| \leqslant k_i D_1 p n^{k_i - \ell_i} \leqslant \varepsilon' n^{k_i - \ell_i}.$$

Here, the first inequality holds by (a), and the second since $p = n^{-1/m(A_1)}$ and $m(A_1) > 0$ by Proposition 5.18(v). Thus, in total $f(S_i) = S_i \cup f'(S_i)$ contains at most $2\varepsilon' n^{k_i - \ell_i} \leqslant \delta n^{k_i - \ell_i}$ k_i-distinct solutions to $\mathcal{L}_i$, so (iv)(a) holds.

In fact, the argument above implies that there is an r-colouring of the set $\cup_{i \in [r]} f(S_i)$ so that there are at most $2\varepsilon' n^{k_i - \ell_i} = \varepsilon n^{k_i - \ell_i}$ k_i-distinct solutions to $\mathcal{L}_i$ in colour i, in $\cup_{i \in [r]} f(S_i)$. Hence, Lemma 2.3 ensures (iv)(b), as desired. $\square$

5.3.4 The number of $(\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)$-free subsets of $[n]$

Our first application of Theorem 5.21 yields an enumeration result (Theorem 5.17) for the number of $(\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)$-free subsets of $[n]$.

Proof of Theorem 5.17. By definition of $\mu(n,\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)$ there are at least $2^{\mu(n,\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)}$ $(\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)$-free subsets of $[n]$. So it suffices to prove the upper bound.

For this, note that we may assume n is sufficiently large. Let $0<\delta<1$ be arbitrary and let $D>0$ be obtained from Theorem 5.21 applied to $A_1,\dots,A_r$ with parameter δ. We obtain a collection $\mathcal{S}_r$ and function f as in Theorem 5.21. Consider any $(\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)$-free subset X of $[n]$. Note that X has a partition $X_1,\dots,X_r$ so that $(X_1,\dots,X_r)\in \mathcal{I}(n,\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)$. So by Theorem 5.21(i) this means there is some $S=(S_1,\dots,S_r)\in\mathcal{S}_r$ so that $X\subseteq \cup_{i\in[r]}f(S_i)$.

Further, given any $S=(S_1,\dots,S_r)\in\mathcal{S}_r$, we have that $|\cup_{i\in[r]}f(S_i)|\leqslant \mu(n,\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)+\delta n$. Thus, each such $\cup_{i\in[r]}f(S_i)$ contains at most $2^{\mu(n,\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)+\delta n}$ $(\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)$-free subsets of $[n]$. Note that, by Theorem 5.21(ii),

$$|\mathcal{S}_r|\leqslant \left(\sum_{s=0}^{Dn^{\frac{m(A_1)-1}{m(A_1)}}}\binom{n}{s}\right)^r<2^{\delta n},$$

where the last inequality holds since n is sufficiently large.

Altogether, this implies that the number of $(\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)$-free subsets of $[n]$ is at most

$$2^{\delta n}\times 2^{\mu(n,\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)+\delta n}=2^{\mu(n,\mathcal{L}^d_1,\dots,\mathcal{L}^d_r)+2\delta n}.$$

Since the choice of $0<\delta<1$ was arbitrary this proves the theorem. $\square$

5.3.5 The resilience of being $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-Rado

Recall that the resilience of S with respect to $\mathcal{P}$, $\mathrm{res}(S, \mathcal{P})$, is the minimum number t such that by deleting t elements from S, one can obtain a set not having $\mathcal{P}$. In this section we will determine $\mathrm{res}([n]_p, (\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)\text{-Rado})$ for irredundant systems of linear equations $\mathcal{L}_1, \dots, \mathcal{L}_r$ for which matrices $A_1, \dots, A_r$ which satisfy $(*)$. We now use Theorem 5.21 to deduce Theorem 5.16.

Proof of Theorem 5.16. Let $0 < \delta < 1$, $r \in \mathbb{N}$ and $\mathcal{L}_1, \dots, \mathcal{L}_r$ be the systems of linear equations as in the statement of the theorem. Given n, if $p > n^{-1/m(A_1)}$ then since $m(A_1) > 1$ by Proposition 5.18(v), Proposition 1.2(ii) implies that, w.h.p.,

$$|[n]_p| = \left(1 \pm \frac{\delta}{4}\right) pn. \tag{5.3.2}$$

We first show that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\mathrm{res}([n]_p, (\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)\text{-Rado})}{|[n]_p|} \leqslant 1 - \frac{\mu(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)}{n} + \delta\right] = 1 \quad \text{if } p > n^{-1/m(A_1)}.$$

For this, we must show that the probability of the event that there exists a set $S \subseteq [n]_p$ such that $|S| \geqslant (\mu(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)/n - \delta)|[n]_p|$ and S is $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-free, tends to one as n tends to infinity. This indeed follows: Let T be an $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-free subset of $[n]$ of maximum size $\mu(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$. Then, by Proposition 1.2(ii), w.h.p. we have $|T \cap [n]_p| = (\mu(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)/n \pm \delta)|[n]_p|$, and $T \cap [n]_p$ is $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-free, as required.

For the remainder of the proof, we will focus on the lower bound, namely that there exists $C > 0$ such that whenever $p > Cn^{-1/m(A_1)}$,

$$\mathbb{P}\left[\mathrm{res}([n]_p, (\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)\text{-Rado}) \geqslant \left(1 - \frac{\mu(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)}{n} - \delta\right) |[n]_p|\right] \to 1 \quad \text{as } n \to \infty. \tag{5.3.3}$$

Suppose n is sufficiently large. Apply Theorem 5.21 with parameters $r, \delta/8, \mathcal{L}_1, \dots, \mathcal{L}_r$ to obtain $D > 0$, a collection $\mathcal{S}_r \subseteq \mathcal{P}([n])^r$ and a function f satisfying (i)–(iv). Now choose C such that $0 < 1/C \ll 1/D, \delta, 1/r$. Let $p \geqslant Cn^{-1/m(A_1)}$.

Since (5.3.2) holds with high probability, to prove (5.3.3) holds it suffices to show that the probability $[n]_p$ contains an $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-free subset of size at least $(\frac{\mu(n,\mathcal{L}_1^d,\dots,\mathcal{L}_r^d)}{n}+\delta/2)np$ tends to zero as n tends to infinity.

Suppose that $[n]_p$ does contain an $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-free subset I of size at least $(\frac{\mu(n,\mathcal{L}_1^d,\dots,\mathcal{L}_r^d)}{n}+\delta/2)np$. Note that I has a partition $I_1, \dots, I_r$ so that $(I_1, \dots, I_r) \in \mathcal{I}(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$. Further, there is some $S = (S_1, \dots, S_r) \in \mathcal{S}_r$ such that $S \subseteq (I_1, \dots, I_r) \subseteq f(S)$. Thus, $[n]_p$ must contain $\cup_{i\in[r]} S_i$ as well as at least $(\frac{\mu(n,\mathcal{L}_1^d,\dots,\mathcal{L}_r^d)}{n} + \delta/4)np$ elements from $\big(\cup_{i\in[r]} f(S_i)\big) \setminus \big(\cup_{i\in[r]} S_i\big)$. (Note here we are using that $| \cup_{i\in[r]} S_i| \leqslant \delta np/4$, which holds by Theorem 5.21(ii) and since $0 < 1/C \ll 1/D, \delta$.) Writing $s := | \cup_{i\in[r]} S_i|$, the probability $[n]_p$ contains $\cup_{i\in[r]} S_i$ is p^s. Note that $| \big(\cup_{i\in[r]} f(S_i)\big) \setminus \big(\cup_{i\in[r]} S_i\big) | \leqslant \mu(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d) + \delta n/8$ by Theorem 5.21(iv)(b). So by Proposition 1.2(i), the probability $[n]_p$ contains at least $(\frac{\mu(n,\mathcal{L}_1^d,\dots,\mathcal{L}_r^d)}{n} + \delta/4)np$ elements from $\big(\cup_{i\in[r]} f(S_i)\big) \setminus \big(\cup_{i\in[r]} S_i\big)$, is at most $\exp(-\delta^2 np/256)$.

Write $N := n^{(m(A_1)-1)/m(A_1)}$ and $\gamma := \delta^2/256$. Given some $0 \leqslant s \leqslant DN$, there are at most $r^s\binom{n}{s}$ elements $(S_1, \dots, S_r) \in \mathcal{S}_r$ such that $| \cup_{i\in[r]} S_i| = s$. Indeed, this follows since there are r^s ways to partition a set of size s into r classes. Thus, the probability $[n]_p$ does contain an $(\mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$-free subset I of size at least $(\frac{\mu(n,\mathcal{L}_1^d,\dots,\mathcal{L}_r^d)}{n} + \delta/2)np$ is at most

$$\sum_{s=0}^{DN} r^s \binom{n}{s} \cdot p^s \cdot e^{-\gamma np} \leqslant (DN+1)(rp)^{DN} \binom{n}{DN} e^{-\gamma np} \leqslant (DN+1) \left(\frac{repn}{DN}\right)^{DN} e^{-\gamma np}$$
$$\leqslant (DN+1) \left(\frac{reC}{D}\right)^{DN} e^{-\gamma CN} \leqslant e^{\gamma CN/2} e^{-\gamma CN} = e^{-\gamma CN/2}$$

which tends to zero as n tends to infinity. This completes the proof.

$\square$

5.3.6 The size of the largest $(\mathcal{L}^s, r)$-free set

Both as a natural question in itself, and in light of Theorems 5.16 and 5.17, it is of interest to obtain good bounds on $\mu(n, \mathcal{L}_1^d, \dots, \mathcal{L}_r^d)$. For the rest of this section consider the symmetric homogeneous case ($A := A_1 = \cdots = A_r$ and $b = 0$) and assume that A is a $1 \times k$ matrix, i.e. we are interested in solutions to a linear equation $a_1x_1 + \cdots + a_kx_k = 0$. Recall $S \subseteq [n]$ is *strongly* $(\mathcal{L}, r)$*-free* if there exists an r-colouring of S which contains no monochromatic solutions to $\mathcal{L}$ of any type (that is, solutions are not required to be k-distinct). Note that for any density regular matrix A, $(x, \dots, x)$ is a solution to $\mathcal{L}$ for all $x \in [n]$ (as observed by Frankl, Graham and Rödl [41, Fact 4]) and so we have $\mu(n, \mathcal{L}^s, r) = 0$. (Note that this result implies that all density regular $1 \times k$ matrices give rise to an equation $\mathcal{L}$ which is translation-invariant.) In fact, if A is any $1 \times k$ irredundant integer matrix, then for all $\varepsilon > 0$ there exists an $n_0 > 0$ such that for all integers $n \geqslant n_0$ we have

$$\mu(n, \mathcal{L}^s, r) \leqslant \mu(n, \mathcal{L}^d, r) \leqslant \mu(n, \mathcal{L}^s, r) + \varepsilon n.$$

This follows from Lemma 2.2 since such $\mathcal{L}$ have $o(n^{k-\ell})$ non-k-distinct solutions in $[n]$ (i.e. a solution $(x_1, \dots, x_k)$ where there is an $i \neq j$ such that $x_i = x_j$).

Consequently it is equally interesting to study $\mu(n, \mathcal{L}^s, r)$ in the case when $\mu(n, \mathcal{L}^d, r) = \Omega(n)$. In the case of sum-free sets (where $\mathcal{L}$ is $x + y = z$), the study of $\mu(n, \mathcal{L}^s, r)$ is a classical problem of Abbott and Wang [1]. (Note that the only difference between $\mu(n, \mathcal{L}^d, r)$ and $\mu(n, \mathcal{L}^s, r)$ in this case is that $\mu(n, \mathcal{L}^d, r)$ allows non-distinct sums $x+x = z$ whereas $\mu(n, \mathcal{L}^s, r)$ does not.) Let $\mu(n, r) := \mu(n, \mathcal{L}^s, r)$ where $\mathcal{L}$ is $x + y = z$. An easy proof shows that $\mu(n, 1) = \lceil n/2 \rceil$.

The following definitions help motivate the study of $\mu(n, r)$ for $r \geqslant 2$. Let $f(r)$ denote the largest positive integer m for which there exists a partition of $[m]$ into r sum-free sets, and let $h(r)$ denote the largest positive integer m for which there exists a partition of $[m]$

into r sets which are sum-free modulo $m+1$.

Abbott and Wang [1] conjectured that $h(r) = f(r)$, and showed that $\mu(n,r) \geqslant n - \lfloor n/(h(r)+1) \rfloor$. They also proved the following upper bound.

Theorem 5.22 ([1]). *We have $\mu(n,r) \leqslant n - \lfloor cn/((f(r)+1)\log(f(r)+1)) \rfloor$ where $c := e^{-\gamma} \approx 0.56$ (γ denotes the Euler-Mascheroni constant).*

We provide an alternate upper bound, which is a modification of Hu's [61] proof that $\mu(n,2) = n - \lfloor \frac{n}{5} \rfloor$. (To see why this is a lower bound, consider the set $\{x \in [n] : x \equiv 1 \text{ or } 4 \mod 5\} \cup \{y \in [n] : y \equiv 2 \text{ or } 3 \mod 5\}$.) First we need the following fact. Given $x \in [n]$ and $T \subseteq [n]$, write $x + T := \{x + y : y \in T\}$. Given $S, T \subseteq [n]$, say that T is a *difference set of S* if there exists $x \in S$ such that $x + T \subseteq S$.

Fact 5.23. *Let $n \in \mathbb{N}$ and $S, T, T' \subseteq [n]$.*

(i) If T is a difference set of a sum-free set S, then $S \cap T = \emptyset$.

(ii) If T' is a difference set of T, and T is a difference set of S, then T' is a difference set of S.

Proof. If there exists $x \in S$ such that $x + T \subseteq S$ and moreover there exists $y \in S \cap T$, then $x + y \in S$, proving (i). For (ii), suppose that there is $x' \in T$ and $x \in S$ such that $x' + T' \subseteq T$ and $x + T \subseteq S$. Then $x + x' + T' \subseteq S$ and $x + x' \in x + T \subseteq S$, proving (ii). $\square$

Theorem 5.24. *We have $\mu(n,r) \leqslant n - \lfloor \frac{n}{\lfloor r!e \rfloor} \rfloor$.*

Note that Theorem 5.24 does indeed recover Hu's bound [61] for the case $r = 2$.

Proof. Fix $n, r \in \mathbb{N}$. Let $\ell(0) := 1$. For all integers $i \geqslant 1$, define

$$\ell(i) := i! \left(1 + \sum_{t \in [i]} \frac{1}{t!} \right) = \lfloor i!e \rfloor.$$

Note that $\ell(i) = i\ell(i-1)+1$ for all $i \geqslant 1$. Choose the unique $q \in \mathbb{N} \cup \{0\}$ and $0 \leqslant k \leqslant \ell(r)-1$ such that $n = \ell(r)q+k$. Consider any partition $S_1\dot{\cup}\cdots\dot{\cup}S_r\dot{\cup}R = [n]$, where each S_i is sum-free. We wish to show that $|R| \geqslant q$, since then $\mu(\ell(r)q+k,r) \leqslant (\ell(r)-1)q+k$ and so $\mu(n,r) \leqslant n - \lfloor n/\ell(r)\rfloor$.

Suppose not. We will obtain integers $\{j_1,\ldots,j_r\} = [r]$ and subsets $D_0, D_1, \ldots, D_r$ of $[n]$ such that the following properties hold for all $0 \leqslant i \leqslant r$.

$P_1(i)$ $|D_i| \geqslant \ell(r-i)q$;

$P_2(i)$ D_i is a difference set of S_{j_t} for all $t \in [i]$;

$P_3(i)$ $D_i \cap S_{j_t} = \emptyset$ for all $t \in [i]$.

Let $D_0 := [n]$. Then $P_1(0)$ holds by definition, and $P_2(0)$ and $P_3(0)$ are vacuous. Suppose, for some $0 \leqslant i < r$, we have obtained distinct $\{j_1,\ldots,j_i\} \subseteq [r]$ and $D_0, D_1, \ldots, D_i$ such that $P_1(t)$–$P_3(t)$ hold for all $t \in [i]$.

Suppose that $|D_i \cap \bigcup_{t\in[r]\setminus\{j_1,\ldots,j_i\}} S_t| \leqslant (\ell(r-i)-1)q$. Then we have that

$$|D_i \cap R| \overset{P_3(i)}{\geqslant} |D_i| - (\ell(r-i)-1)q \overset{P_1(i)}{\geqslant} q,$$

a contradiction. So by averaging, there exists $j_{i+1} \in [r] \setminus \{j_1,\ldots,j_i\}$ such that

$$|D_i \cap S_{j_{i+1}}| \geqslant \left\lceil \frac{(\ell(r-i)-1)q+1}{r-i} \right\rceil = \ell(r-i-1)q+1.$$

Thus we can write $D_i \cap S_{j_{i+1}} \supseteq \{s_{i,0} < \ldots < s_{i,\ell(r-i-1)q}\}$. Let $D_{i+1} := \{s_{i,x} - s_{i,0} : x \in [\ell(r-i-1)q]\}$. We claim that $P_1(i+1)$–$P_3(i+1)$ hold. Property $P_1(i+1)$ is clear by definition. For $P_2(i+1)$, note that D_{i+1} is a difference set of both D_i and $S_{j_{i+1}}$. Then Fact 5.23(ii) and $P_2(i)$ imply that additionally D_{i+1} is a difference set of S_{j_t} for all $t \in [i]$. Fact 5.23(i) implies that $D_{i+1} \cap S_{j_t} = \emptyset$ for all $t \in [i+1]$, proving $P_3(i+1)$.

Thus we obtain D_r satisfying $P_1(r)$–$P_3(r)$. By $P_1(r)$ and $P_3(r)$ we have that $|D_r| \geqslant \ell(0)q = q$ and $D_r \subseteq R$, a contradiction. $\square$

5.3.7 Two-coloured analogue of the Cameron–Erdős conjecture

We conclude the section with a problem which, since the paper [55] corresponding to this chapter was first submitted, has been (essentially) solved by Tran [119]. Recall Hu [61] showed that $\mu(n, 2) = n - \lfloor \frac{n}{5} \rfloor$. So in the case when $\mathcal{L}$ is $x + y = z$, Theorem 5.17 implies that there are $2^{4n/5+o(n)}$ $(\mathcal{L}^d, 2)$-free subsets of $[n]$. By considering $(\mathcal{L}^s, 2)$-free subsets of $[n]$ instead, the error term in the exponent here can be replaced by a constant.

Theorem 5.25 ([119]). *Let $\mathcal{L}$ denote $x + y = z$. There are $\Theta(2^{4n/5})$ $(\mathcal{L}^s, 2)$-free subsets of $[n]$.*

Note that Theorem 5.25 can be viewed as a 2-coloured analogue of the Cameron–Erdős conjecture [23] which was famously resolved by Green [47] and independently Sapozhenko [104].

5.4 Applications of the container method to graph Ramsey theory

In this section we answer some questions in hypergraph Ramsey theory, introduced in Sections 5.1.1 and 5.1.2. How many n-vertex hypergraphs are not Ramsey, and what does a typical such hypergraph look like? How dense must the Erdős-Rényi random hypergraph be to have the Ramsey property with high probability, and above this threshold, how strongly does it possess the Ramsey property?

Our main results here are applications of the asymmetric container theorem (Theorem 5.15). For arbitrary k-uniform hypergraphs $H_1, \ldots, H_r$, we first prove Theorem 5.34, a container theorem for non-$(H_1, \ldots, H_r)$-Ramsey k-uniform hypergraphs. To see how one might prove such a theorem, observe that, if $\mathcal{H}_i$ is the hypergraph of copies of H_i on

n vertices (i.e. vertices correspond to k-subsets of $[n]$, and edges correspond to copies of $E(H_i)$), then every non-$(H_1,\ldots,H_r)$-Ramsey k-uniform hypergraph G corresponds to a set in $\mathcal{I}(\mathcal{H}_1,\ldots,\mathcal{H}_r)$. We then use Theorem 5.34 to:

(1) count the number of k-uniform hypergraphs on n vertices which are not $(H_1,\ldots,H_r)$-Ramsey (Theorem 5.12);

(2) determine the global resilience of $G^{(k)}_{n,p}$ with respect to the property of being $(H_1,\ldots,H_r)$-Ramsey (Theorem 5.7). That is, we show that there is a constant C such that whenever $p \geqslant Cn^{-1/m_k(H_1)}$, we obtain a function $t(n,p)$ such that, with high probability, any subhypergraph $G \subseteq G^{(k)}_{n,p}$ with $e(G) > t+\Omega(pn^k)$ is $(H_1,\ldots,H_r)$-Ramsey. Further, there is some $G' \subseteq G^{(k)}_{n,p}$ with $e(G') > t - o(pn^k)$ which is *not* $(H_1,\ldots,H_r)$-Ramsey.

(3) As a corollary of (2), we see that, whenever $p \geqslant Cn^{-1/m_k(H_1)}$, the random hypergraph $G^{(k)}_{n,p}$ is $(H_1,\ldots,H_r)$-Ramsey with high probability.

Notice that each of the statements (1)–(3) involve a common parameter: the maximum size $\mathrm{ex}^r(n;H_1,\ldots,H_r)$ of an n-vertex k-uniform hypergraph which is not $(H_1,\ldots,H_r)$-Ramsey. For this reason, we generalise the classical supersaturation result of Erdős and Simonovits [36] to show that any n-vertex k-uniform hypergraph G with at least $\mathrm{ex}^r(n;H_1,\ldots,H_r)+\Omega(n^k)$ edges is somehow 'strongly' $(H_1,\ldots,H_r)$-Ramsey. In the graph case, an old result of Burr, Erdős and Lovász [21] allows us to quite accurately determine $\mathrm{ex}^r(n;H_1,\ldots,H_r)$.

5.4.1 Definitions and notation

In this section, $k \geqslant 2$ is an integer and we use *k-graph* as shorthand for k-uniform hypergraph. Recall from Section 5.1.1 that, given $r \in \mathbb{N}$ and a k-graph G, an *r-colouring* is a function $\sigma : E(G) \to [r]$. Given k-graphs $H_1,\ldots,H_r$, we say that σ is $(H_1,\ldots,H_r)$-

free if $\sigma^{-1}(i)$ is H_i-free for all $i \in [r]$. Then G is $(H_1, \ldots, H_r)$-*Ramsey* if it has no $(H_1, \ldots, H_r)$-free r-colouring.

Given an integer $\ell \geqslant k$, denote by $K_\ell^{(k)}$ the complete k-graph on ℓ vertices. A k-graph H is *k-partite* if the vertices of H can be k-coloured so that each edge contains one vertex of each colour. Given a k-graph S, recall the definitions

$$d_k(S) := \begin{cases} 0 & \text{if } e(S) = 0; \\ 1/k & \text{if } v(S) = k \text{ and } e(S) = 1; \\ \frac{e(S)-1}{v(S)-k} & \text{otherwise} \end{cases}$$

and

$$m_k(S) := \max_{S' \subseteq S} d_k(S').$$

5.4.2 The maximum density of a hypergraph which is not Ramsey

Given integers $n \geqslant k$ and a k-graph H, we denote by $\mathrm{ex}(n; H)$ the maximum size of an n-vertex H-free k-graph. Define the *Turán density* $\pi(H)$ of H by

$$\pi(H) := \lim_{n \to \infty} \frac{\mathrm{ex}(n; H)}{\binom{n}{k}} \tag{5.4.1}$$

(which exists by a simple averaging argument, see [66]). The so-called supersaturation phenomenon discovered by Erdős and Simonovits [36] asserts that any sufficiently large hypergraph with density greater than $\pi(H)$ contains not just one copy of H, but in fact a positive fraction of $v(H)$-sized sets span a copy of H.

Theorem 5.26 ([36]). *For all $k \in \mathbb{N}$; $\delta > 0$ and all k-graphs H, there exist $n_0, \varepsilon > 0$ such that for all integers $n \geqslant n_0$, every n-vertex k-graph G with $e(G) \geqslant (\pi(H) + \delta)\binom{n}{k}$*

contains at least $\varepsilon\binom{n}{v(H)}$ *copies of* H.

When $k = 2$, the Erdős–Stone–Simonovits theorem [37] says that for all graphs H, the value of $\pi(H)$ is determined by the chromatic number $\chi(H)$ of H, via

$$\pi(H) = 1 - \frac{1}{\chi(H) - 1}. \tag{5.4.2}$$

For $k \geqslant 3$, the value of $\pi(H)$ is only known for a small family of k-graphs H. It remains an open problem to even determine the Turán density of $K_4^{(3)}$, the smallest non-trivial complete 3-graph (the widely-believed conjectured value is $\frac{5}{9}$). For more background on this, the so-called *hypergraph Turán problem*, the interested reader should consult the excellent survey of Keevash [67].

In this section, we generalise Theorem 5.26 from H-free hypergraphs to non-$(H_1, \dots, H_r)$-Ramsey hypergraphs (note that a hypergraph is H-free if and only if it is *not* (H)-Ramsey). Given $\varepsilon > 0$, we say that an n-vertex k-graph G is ε*-strongly* $(H_1, \dots, H_r)$-*Ramsey* if for all r-colourings σ of G there exists an $i \in [r]$ such that the number of copies of H_i in $\sigma^{-1}(i)$ is more than $\varepsilon\binom{n}{v(H_i)}$.

Using a well-known averaging argument of Katona, Nemetz and Simonovits [66], we can show that $\binom{n}{k}^{-1}\mathrm{ex}^r(n; H_1, \dots, H_r)$ converges as n tends to infinity. Indeed, let G be an n-vertex non-$(H_1, \dots, H_r)$-Ramsey graph with $e(G) = \mathrm{ex}^r(n; H_1, \dots, H_r)$. The average density of an $(n-1)$-vertex induced subgraph of G is precisely

$$\binom{n}{n-1}^{-1} \sum_{U \subseteq V(G) : |U| = n-1} \frac{e(G[U])}{\binom{n-1}{k}} = (n-k)^{-1} \cdot \binom{n}{k}^{-1} \sum_{U \subseteq V(G) : |U| = n-1} e(G[U]) = \binom{n}{k}^{-1} e(G).$$

But the left-hand side is at most $\binom{n-1}{k}^{-1} \cdot \mathrm{ex}^r(n-1; H_1, \dots, H_r)$, otherwise G would contain an $(n-1)$-vertex subgraph which is $(H_1, \dots, H_r)$-Ramsey, violating the choice of

G. We have shown that

$$\frac{\mathrm{ex}^r(n; H_1, \ldots, H_r)}{\binom{n}{k}}$$

is a non-increasing function of n (which is bounded below, by 0), and so this function has a limit. Therefore we may define the *r-coloured Turán density* $\pi(H_1, \ldots, H_r)$ of $(H_1, \ldots, H_r)$ by

$$\pi(H_1, \ldots, H_r) := \lim_{n\to\infty} \frac{\mathrm{ex}^r(n; H_1, \ldots, H_r)}{\binom{n}{k}}.$$

As for $k \geqslant 3$, the problem of determining $\pi(H)$ is still out of reach, we certainly cannot evaluate $\pi(H_1, \ldots, H_r)$ in general. However, any non-$(H_1, \ldots, H_r)$-Ramsey graph is $K_s^{(k)}$-free, where $s := R(H_1, \ldots, H_r)$ is the smallest integer m such that $K_m^{(k)}$ is $(H_1, \ldots, H_r)$-Ramsey. Thus

$$\pi(H_1, \ldots, H_r) \leqslant \pi(K_s^{(k)}), \tag{5.4.3}$$

which is at most $1 - \binom{s-1}{k-1}^{-1}$ (de Caen [29]). An interesting question is for which $H_1, \ldots, H_r$ the inequality in (5.4.3) is tight. We discuss the case $k = 2$ in detail in Section 5.4.3.

We now state the main result of this subsection, which generalises Theorem 5.26 to $r \geqslant 1$.

Theorem 5.27. *For all $\delta > 0$, integers $r \geqslant 1$ and $k \geqslant 2$, and k-graphs $H_1, \ldots, H_r$, there exist $n_0, \varepsilon > 0$ such that for all integers $n \geqslant n_0$, every n-vertex k-graph G with $e(G) \geqslant (\pi(H_1, \ldots, H_r) + \delta)\binom{n}{k}$ is ε-strongly $(H_1, \ldots, H_r)$-Ramsey.*

Proof. Let $\delta > 0$ and let r, k be positive integers with $k \geqslant 2$. By the definition of $\pi(\cdot)$, there exists $m_0 > 0$ such that for all integers $m \geqslant m_0$,

$$\mathrm{ex}^r(m; H_1, \ldots, H_r) < \left(\pi(H_1, \ldots, H_r) + \frac{\delta}{2}\right)\binom{m}{k}.$$

Fix an integer $m \geqslant m_0$. Without loss of generality, we may assume that $m \geqslant v(H_i)$ for all $i \in [r]$. Choose $\varepsilon > 0$ to be such that

$$\varepsilon \leqslant \frac{\delta}{2r}\binom{m}{v(H_i)}^{-1}$$

for all $i \in [r]$. Let n be an integer which is sufficiently large compared to m, and let G be a k-graph on n vertices with $e(G) = (\pi(H_1, \ldots, H_r) + \delta)\binom{n}{k}$. We need to show that, for every r-colouring σ of G, there is $i \in [r]$ such that $\sigma^{-1}(i)$ contains at least $\varepsilon\binom{n}{v(H_i)}$ copies of H_i; so fix an arbitrary σ.

Define $\mathcal{M}$ to be the set of $M \in \binom{V(G)}{m}$ such that $e(G[M]) \geqslant (\pi(H_1, \ldots, H_r) + \frac{\delta}{2})\binom{m}{k}$. Then

$$\sum_{U \subseteq V(G):|U|=m} e(G[U]) \leqslant |\mathcal{M}|\binom{m}{k} + \left(\binom{n}{m} - |\mathcal{M}|\right)\left(\pi(H_1, \ldots, H_r) + \frac{\delta}{2}\right)\binom{m}{k}.$$

But for every $e \in E(G)$, there are exactly $\binom{n-k}{m-k}$ sets $U \subseteq V(G)$ with $|U| = m$ such that $e \in E(G[U])$. Thus also

$$\begin{aligned}\sum_{U \subseteq V(G):|U|=m} e(G[U]) \geqslant & \binom{n-k}{m-k}(\pi(H_1, \ldots, H_r) + \delta)\binom{n}{k} \\ = & (\pi(H_1, \ldots, H_r) + \delta)\binom{n}{m}\binom{m}{k},\end{aligned}$$

and so, rearranging, we have $|\mathcal{M}| \geqslant \delta\binom{n}{m}/2$. By the choice of m, for every $M \in \mathcal{M}$, there exists $i = i(M) \in [r]$ such that $\sigma^{-1}(i)$ contains a copy of H_i with vertices in M. Choose $\mathcal{M}' \subseteq \mathcal{M}$ such that the $i(M')$ are equal for all $M' \in \mathcal{M}'$ and $|\mathcal{M}'| \geqslant |\mathcal{M}|/r$. Without loss of generality let us assume that $i(M') = 1$ for all $M' \in \mathcal{M}'$. So for each $M' \in \mathcal{M}'$, there is a copy of $H_1 \subseteq G[M']$ which is monochromatic with colour 1 under σ. Each such copy has vertex set contained in at most $\binom{n-v(H_1)}{m-v(H_1)}$ sets $M' \in \mathcal{M}'$. Thus the number of such monochromatic copies of H_1 in G is at least

$$\frac{\frac{\delta}{2}\cdot\binom{n}{m}}{r\binom{n-v(H_1)}{m-v(H_1)}} = \frac{\delta}{2r}\cdot\binom{m}{v(H_1)}^{-1}\cdot\binom{n}{v(H_1)} \geqslant \varepsilon\binom{n}{v(H_1)}.$$

So G is ε-strongly $(H_1,\dots,H_r)$-Ramsey, as required. $\square$

5.4.3 The special case of graphs: maximum size and typical structure

The intimate connection between forbidden subgraphs and chromatic number when $k=2$ allows us to make some further remarks here. (This section is separate from the remainder of the chapter and the results stated here will not be required later on.)

5.4.3.1 The maximum number of edges in a graph which is not Ramsey

Given $s,n \in \mathbb{N}$, let $T_s(n)$ denote the s-partite Turán (2-)graph on n vertices; that is, the vertex set of $T_s(n)$ has a partition into s parts $V_1,\dots,V_s$ such that $||V_i|-|V_j|| \leqslant 1$ for all $i,j \in [s]$; and xy is an edge of $T_s(n)$ if and only if there are $ij \in \binom{[s]}{2}$ such that $x \in V_i$ and $y \in V_j$. Write $t_s(n) := e(T_s(n))$.

We need to define two notions of Ramsey number.

Given an integer $r \geqslant 1$ and families $\mathcal{H}_1,\dots,\mathcal{H}_r$ of graphs, the *Ramsey number* $R(\mathcal{H}_1,\dots,\mathcal{H}_r)$ is the least m such that any r-colouring of K_m contains an i-coloured copy of H_j for some $i \in [r]$ and some $H_j \in \mathcal{H}_i$. If $\mathcal{H}_i = \{K_{\ell_i}\}$ for all $i \in [r]$ then we instead write $R(\ell_1,\dots,\ell_r)$, and simply $R^r(\ell)$ in the case when $\ell_1 = \ldots = \ell_r =: \ell$. Given graphs $H_1,\dots,H_r$, the *chromatic Ramsey number* $R_\chi(H_1,\dots,H_r)$ is the least m for which there exists an $(H_1,\dots,H_r)$-Ramsey graph with chromatic number m.

Trivially, for any k-graph H, we have that $R_\chi(H) = \chi(H)$. If $H_1,\dots,H_r$ are graphs, then

$$t_{R_\chi(H_1,\dots,H_r)-1}(n) \leqslant \mathrm{ex}^r(n; H_1,\dots,H_r) \leqslant t_{R_\chi(H_1,\dots,H_r)-1}(n) + o(n^2). \tag{5.4.4}$$

Thus

$$\pi(H_1,\dots,H_r) = 1 - \frac{1}{R_\chi(H_1,\dots,H_r)-1} = \pi\left(K_{R_\chi(H_1,\dots,H_r)}\right). \tag{5.4.5}$$

The first inequality in (5.4.4) follows by definition of $\mathrm{ex}^r(n; H_1,\dots,H_r)$; the second from (5.4.2) applied with a graph H which is $(H_1,\dots,H_r)$-Ramsey and has $\chi(H) = R_\chi(H_1,\dots,H_r)$. Clearly, then, $\pi(H_1,\dots,H_r) = \pi(J_1,\dots,J_r)$ if and only if $R_\chi(J_1,\dots,J_r) = R_\chi(H_1,\dots,H_r)$. So, in the graph case, the inequality (5.4.3) is tight when the Ramsey number and chromatic Ramsey number coincide.

As noted by Bialostocki, Caro and Roditty [16], one can determine $\mathrm{ex}^r(n; H_1,\dots,H_r)$ exactly in the case when $H_1,\dots,H_r$ are cliques of equal size.

Theorem 5.28 ([16]). *For all positive integers $\ell, n \geqslant 3$ and $r \geqslant 1$, we have $\mathrm{ex}^r(n; K_\ell,\dots,K_\ell) = t_{R^r(\ell)-1}(n)$.*

Thus in this case (5.4.3) is tight. The chromatic Ramsey number was introduced by Burr, Erdős and Lovász [21] who showed that, in principle, one can determine R_χ given the usual Ramsey number R. A *graph homomorphism* from a graph H to a graph K is a function $\phi : V(H) \to V(K)$ such that $\phi(x)\phi(y) \in E(K)$ whenever $xy \in E(H)$. Let $\mathrm{Hom}(H)$ denote the set of all graphs K such that there exists a graph homomorphism ϕ for which $K = \phi(H)$. Since there exists a homomorphism from H into K_ℓ if and only if $\chi(H) \leqslant \ell$, we also have that $R(\mathrm{Hom}(H)) = \chi(H)$. Thus $R(\mathrm{Hom}(H)) = R_\chi(H)$. In fact this relationship extends to all $r \geqslant 1$.

Lemma 5.29 ([21, 26, 78]). *For all integers $r \in \mathbb{N}$ and graphs $H_1,\dots,H_r$,*

$$R_\chi(H_1,\dots,H_r) = R(\mathrm{Hom}(H_1),\dots,\mathrm{Hom}(H_r)).$$

Moreover, for all integers $\ell_1, \ldots, \ell_r \geqslant 3$, we have that

$$R_\chi(K_{\ell_1}, \ldots, K_{\ell_r}) = R(\ell_1, \ldots, \ell_r).$$

The second statement is a corollary of the first since $\mathrm{Hom}(K_\ell) = \{K_\ell\}$. Another observation (see [21]) is that for all $\ell \in \mathbb{N}$, the chromatic Ramsey number $R_\chi(C_{2\ell+1}, C_{2\ell+1})$ is equal to 5 if $\ell = 2$, and equal to 6 otherwise.

The first inequality in (5.4.4) is not always tight, for example when H is the disjoint union of two copies of some graph G. Indeed, $\mathrm{Hom}(H) \supseteq \mathrm{Hom}(G)$ and so $R_\chi(H, \ldots, H) = R_\chi(G, \ldots, G)$. Let F be an n-vertex graph with $e(F) = \mathrm{ex}^r(n; G, \ldots, G)$ which is not (G, r)-Ramsey. Obtain a graph T by adding an edge e to F. Then there exists an r-colouring of T in which every monochromatic copy of G contains e (the monochromatic-G-free colouring of F, with e arbitrarily coloured). Hence T is not (H, r)-Ramsey and so

$$\mathrm{ex}^r(n; H, \ldots, H) > \mathrm{ex}^r(n; G, \ldots, G) \geqslant t_{R_\chi(G,\ldots,G)}(n) = t_{R_\chi(H,\ldots,H)}(n).$$

We say that a graph H is *(weakly) colour-critical* if there exists $e \in E(H)$ for which $\chi(H - e) < \chi(H)$. Complete graphs and odd cycles are examples of colour-critical graphs. The following conjecture would generalise Theorem 5.28 to provide a large class of graphs where the first inequality in (5.4.4) is tight.

Conjecture 5.30. *Let r be a positive integer and H a colour-critical graph. Then, whenever n is sufficiently large,*

$$\mathrm{ex}^r(n; H, \ldots, H) = t_{R_\chi(H,\ldots,H)-1}(n).$$

If true, this conjecture would also generalise a well-known result of Simonovits [111] which extends Turán's theorem to colour-critical graphs. It would also determine $\mathrm{ex}^r(n; H, \ldots, H)$

explicitly whenever H is an odd cycle.

5.4.3.2 The typical structure of non-Ramsey graphs

There has been much interest in determining the *typical* structure of an H-free graph. For example, Kolaitis, Prömel and Rothschild [73] proved that almost all K_r-free graphs are $(r-1)$-partite. It turns out that one can easily obtain a result on the typical structure of non-Ramsey graphs from a result of Prömel and Steger [87].

Given two families $\mathcal{A}(n), \mathcal{B}(n)$ of n-vertex graphs such that $\mathcal{B}(n) \subseteq \mathcal{A}(n)$, we say that *almost all* n-vertex graphs $G \in \mathcal{A}(n)$ are in $\mathcal{B}(n)$ if

$$\lim_{n\to\infty} \frac{|\mathcal{A}(n)|}{|\mathcal{B}(n)|} = 1.$$

The next result of Prömel and Steger [87] immediately tells us the typical structure of non-Ramsey graphs in certain cases.

Theorem 5.31 ([87]). *For every graph H, the following holds. Almost all H-free graphs are $(\chi(H)-1)$-partite if and only if H is colour-critical.*

Corollary 5.32. *For all integers r and graphs $H_1, \dots, H_r$, if there exists an $(H_1, \dots, H_r)$-Ramsey graph H such that $\chi(H) = R_\chi(H_1, \dots, H_r)$ and H is colour-critical, then almost every non-$(H_1, \dots, H_r)$-Ramsey graph is $(R_\chi(H_1, \dots, H_r) - 1)$-partite.*

Proof. The result follows since every non-$(H_1, \dots, H_r)$-Ramsey graph G is H-free, and every $(R_\chi(H_1, \dots, H_r) - 1)$-partite graph is non-$(H_1, \dots, H_r)$-Ramsey. □

In particular, if in Corollary 5.32, each H_i is a clique, say $H_i = K_{\ell_i}$, then by Lemma 5.29 we can take $H := K_{R(\ell_1,\dots,\ell_r)}$. So, for example, almost every non-$(K_3, 2)$-Ramsey graph is 5-partite.

5.4.4 A container theorem for Ramsey hypergraphs

Recall that $\overline{\mathrm{Ram}}(n; H_1, \ldots, H_r)$ is the set of n-vertex k-graphs which are not $(H_1, \ldots, H_r)$-Ramsey and $\mathrm{Ram}(H_1, \ldots, H_r)$ is the set of $(H_1, \ldots, H_r)$-Ramsey k-graphs (on any number of vertices). Recall further that an H-free k-graph is precisely a non-$(H, 1)$-Ramsey graph. Write $\mathcal{G}_k(n)$ for the set of all k-graphs on vertex set $[n]$. Let $\mathcal{I}_r(n; H_1, \ldots, H_r)$ denote the set of all ordered r-tuples $(G_1, \ldots, G_r) \in (\mathcal{G}_k(n))^r$ of k-graphs such that each G_i is H_i-free and $E(G_i) \cap E(G_j) = \emptyset$ for all distinct $i, j \in [r]$. Note that for any $G \in \overline{\mathrm{Ram}}(n; H_1, \ldots, H_r)$, there exist pairwise edge-disjoint k-graphs $G_1, \ldots, G_r$ such that $\bigcup_{i\in[r]} G_i = G$ and $(G_1, \ldots, G_r) \in \mathcal{I}_r(n; H_1, \ldots, H_r)$. Given an integer $k \geqslant 2$, a k-graph H and positive integer n, the *hypergraph $\mathcal{H}$ of copies of H in $K_n^{(k)}$* has vertex set $V(\mathcal{H}) := \binom{[n]}{k}$, and $E \subseteq \binom{V(\mathcal{H})}{e(H)}$ is an edge of $\mathcal{H}$ if and only if E is isomorphic to $E(H)$.

In this subsection, we prove a container theorem for elements in $\mathcal{I}_r(n; H_1, \ldots, H_r)$. To do so, we will apply Theorem 5.15 to hypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_r$, where $\mathcal{H}_i$ is the hypergraph of copies of H_i In $\mathcal{H}_i$, an independent set corresponds to an H_i-free k-graph.

We will need the following simple proposition from [8].

Proposition 5.33 ([8], Proposition 7.3). *Let H be a k-graph. Then there exists $c > 0$ such that, for all positive integers n, the following holds. Let $\mathcal{H}$ be the $e(H)$-uniform hypergraph of copies of H in $K_n^{(k)}$. Then, letting $p = n^{-1/m_k(H)}$,*

$$\Delta_\ell(\mathcal{H}) \leqslant c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})},$$

for every $\ell \in [e(H)]$.

We can now prove our container theorem for elements in $\mathcal{I}_r(n; H_1, \ldots, H_r)$.

Theorem 5.34. *Let $r, k \in \mathbb{N}$ with $k \geqslant 2$ and $\delta > 0$. Let $H_1, \ldots, H_r$ be k-graphs such that $m_k(H_1) \geqslant \ldots \geqslant m_k(H_r)$ and $\Delta_1(H_i) \geqslant 2$ for all $i \in [r]$. Then there exists $D > 0$ such* *that the following holds. For all $n \in \mathbb{N}$, there is a collection $\mathcal{S}_r \subseteq (\mathcal{G}_k(n))^r$ and a function $f : \mathcal{S}_r \to (\mathcal{G}_k(n))^r$ such that:*

(i) For all $(I_1, \dots, I_r) \in \mathcal{I}_r(n; H_1, \dots, H_r)$, there exists $S \in \mathcal{S}_r$ such that $S \subseteq (I_1, \dots, I_r) \subseteq f(S)$.

(ii) If $(S_1, \dots, S_r) \in \mathcal{S}_r$ then $\sum_{i \in [r]} e(S_i) \leqslant D n^{k-1/m_k(H_1)}$.

(iii) Every $S \in \mathcal{S}_r$ satisfies $S \in \mathcal{I}_r(n; H_1, \dots, H_r)$.

(iv) Given any $S = (S_1, \dots, S_r) \in \mathcal{S}_r$, write $f(S) =: (f(S_1), \dots, f(S_r))$. Then

(a) $\bigcup_{i \in [r]} f(S_i)$ is not δ-strongly $(H_1, \dots, H_r)$-Ramsey; and

(b) $e\left(\bigcup_{i \in [r]} f(S_i)\right) \leqslant \mathrm{ex}^r(n; H_1, \dots, H_r) + \delta\binom{n}{k}$.

Note that if H is a k-graph with $\Delta_1(H) = 1$, then H is a matching, i.e. a set of vertex-disjoint edges.

Proof. We will identify any hypergraph which has vertex set $[n]$ with its edge set. We may further assume that there are no isolated vertices in H_i for any $i \in [r]$.

Apply Proposition 5.33 with input hypergraphs $H_1, \dots, H_r$ to obtain $c > 0$ such that its conclusion holds with H_i playing the role of H, for all $i \in [r]$. Let $\delta > 0$, $r \in \mathbb{N}$ and $k \geqslant 2$ be given and apply Theorem 5.27 (with $\delta/2$ playing the role of δ) to obtain $n_0, \varepsilon > 0$. Without loss of generality we may assume $\varepsilon \leqslant \delta < 1$. For each $i \in [r]$, let $v_i := v(H_i)$ and $m_i := e(H_i)$ for all $i \in [r]$. Set $v := \max_{i \in [r]} v_i$; $m := \max_{i \in [r]} m_i$;

$$\varepsilon' := \frac{\varepsilon}{2 \cdot v!}; \quad \text{and} \quad \varepsilon'' := \frac{\varepsilon'}{\binom{v}{k} \cdot v!}.$$

Apply Theorem 5.15 with parameters $r, m_1, \dots, m_r, c, \varepsilon''$ playing the roles of $r, k_1, \dots, k_r, c, \varepsilon$ respectively to obtain $D_1 > 0$. Increase n_0 if necessary so that $0 < 1/n_0 \ll 1/D_1, 1/k, 1/r, \varepsilon, \delta$. If $n < n_0$, then set $\mathcal{S}_r$ to be $\mathcal{I}_r(n; H_1, \dots, H_r)$; set f to be the identity function and choose

D_2 to be large. By setting D to be the maximum of D_1 and D_2, it remains to prove the result for integers $n \geqslant n_0$. So now fix $n \geqslant n_0$.

Let $\mathcal{H}_{n,i}$ be the hypergraph of copies of H_i in $K_n^{(k)}$. That is, $V(\mathcal{H}_{n,i}) := \binom{[n]}{k}$ and for each m_i-subset E of $\binom{[n]}{k}$, put $E \in E(\mathcal{H}_{n,i})$ if and only if E is isomorphic to a copy of H_i. By definition, $\mathcal{H}_{n,i}$ is an m_i-uniform hypergraph and an independent set in $\mathcal{H}_{n,i}$ corresponds to an H_i-free k-graph with vertex set $[n]$. Since H_i is a k-graph with no isolated vertices,

$$e(\mathcal{H}_{n,i}) = \frac{v_i!}{|\mathrm{Aut}(H_i)|}\binom{n}{v_i} \tag{5.4.6}$$

where $\mathrm{Aut}(H_i)$ is the automorphism group of H_i. For all $i \in [r]$, let

$$\mathcal{F}_{n,i} := \left\{ A \subseteq \binom{[n]}{k} : e(\mathcal{H}_{n,i}[A]) \geqslant \varepsilon' e(\mathcal{H}_{n,i}) \right\}.$$

We claim that $\mathcal{H}_{n,1}, \dots, \mathcal{H}_{n,r}$ and $\mathcal{F}_{n,1}, \dots, \mathcal{F}_{n,r}$ satisfy the hypotheses of Theorem 5.15 with the parameters chosen as above and with

$$p = p(n) := n^{-1/m_k(H_1)}.$$

Clearly each family $\mathcal{F}_{n,i}$ is increasing, and $\mathcal{H}_{n,i}$ is $(\mathcal{F}_{n,i}, \varepsilon')$-dense. Next, we show that $|A| \geqslant \varepsilon'' \binom{n}{k}$ for all $A \in \mathcal{F}_{n,i}$. In any k-graph on n vertices, there are at most $v_i!\binom{n-k}{v_i-k}$ copies of H_i that contain some fixed set $\{x_1, \dots, x_k\}$ of vertices. Therefore, for every $e \in \binom{[n]}{k}$, the number of $E \in E(\mathcal{H}_{n,i})$ containing e is at most

$$v_i!\binom{n-k}{v_i-k}. \tag{5.4.7}$$

Thus every $A \in \mathcal{F}_{n,i}$ satisfies

$$|A| \geqslant \frac{e(\mathcal{H}_{n,i}[A])}{v_i!\binom{n-k}{v_i-k}} \overset{(5.4.6)}{\geqslant} \frac{\varepsilon' v_i!\binom{n}{v_i}}{v_i!\binom{n-k}{v_i-k}|\mathrm{Aut}(H_i)|} = \frac{\varepsilon'}{\binom{v}{k}|\mathrm{Aut}(H_i)|}\binom{n}{k} \geqslant \varepsilon''\binom{n}{k},$$

where, in the final inequality, we used the fact that $|\mathrm{Aut}(H_i)| \leqslant v_i!$. Note that $\varepsilon'' < \varepsilon'$. So $\mathcal{H}_{n,i}$ is $(\mathcal{F}_{n,i}, \varepsilon'')$-dense and $|A| \geqslant \varepsilon''\binom{n}{k}$ for all $A \in \mathcal{F}_{n,i}$.

Certainly $p \geqslant n^{-1/m_k(H_j)}$ for all $j \in [r]$. By the choice of c, we then have

$$\Delta_\ell(\mathcal{H}_{n,i}) \leqslant c \cdot p^{\ell-1}\frac{e(\mathcal{H}_{n,i})}{\binom{n}{k}}$$

for all $i \in [r]$ and $\ell \in [m_i]$. We have shown that $\mathcal{H}_{n,i}$ and $\mathcal{F}_{n,i}$ satisfy the hypotheses of Theorem 5.15 for all $i \in [r]$.

Then Theorem 5.15 implies that there exists a family $\mathcal{S}_r \subseteq \mathcal{I}(\mathcal{H}_{n,1}, \dots, \mathcal{H}_{n,r})$ and functions $f' : \mathcal{S}_r \to \prod_{i\in[r]} \overline{\mathcal{F}_{n,i}}$ and $g : \mathcal{I}(\mathcal{H}_{n,1}, \dots, \mathcal{H}_{n,r}) \to \mathcal{S}_r$ such that the following conditions hold:

(a) If $(S_1, \dots, S_r) \in \mathcal{S}_r$ then $\sum |S_i| \leqslant D_1 p\binom{n}{k}$;

(b) every $S \in \mathcal{S}_r$ satisfies $S \in \mathcal{I}(\mathcal{H}_{n,1}, \dots, \mathcal{H}_{n,r})$;

(c) for every $(I_1, \dots, I_r) \in \mathcal{I}(\mathcal{H}_{n,1}, \dots, \mathcal{H}_{n,r})$, we have that $S \subseteq (I_1, \dots, I_r) \subseteq S \cup f'(S)$, where $S := g(I_1, \dots, I_r)$.

Note that $(G_1, \dots, G_r) \in \mathcal{I}(\mathcal{H}_{n,1}, \dots, \mathcal{H}_{n,r})$ if and only if $(G_1, \dots, G_r) \in \mathcal{I}_r(n; H_1, \dots, H_r)$ (where we recall the identification of graphs and edge sets). For each $S \in \mathcal{S}_r$, define

$$f(S) := S \cup f'(S).$$

So $f : \mathcal{S}_r \to \mathcal{P}(\binom{[n]}{k})^r$. (Note that under the correspondence of graphs and edge sets we can view $\mathcal{P}(\binom{[n]}{k})^r = (\mathcal{G}_k(n))^r$.) Thus (a)–(c) immediately imply that (i) and (iii) hold, and additionally for any $(S_1, \dots, S_r) \in \mathcal{S}_r$ we have

$$\sum_{i\in[r]} e(S_i) \leqslant D_1 p \binom{n}{k} \leqslant D_1 n^{-1/m_k(H_1)} \cdot \frac{n^k}{k!} < D_1 n^{k-1/m_k(H_1)},$$

yielding (ii).

Given any $S = (S_1, \dots, S_r) \in \mathcal{S}_r$ write $f(S) =: (f(S_1), \dots, f(S_r))$ and $f'(S) =: (f'(S_1), \dots, f'(S_r))$. Let $G := \bigcup_{i\in[r]} f(S_i)$; so G is a k-graph with vertex set $[n]$. To prove (iv)(a), we need to exhibit an r-colouring σ of G with the property that $\sigma^{-1}(i)$ contains less than $\varepsilon\binom{n}{v_i}$ copies of H_i for all $i \in [r]$. Indeed, consider the r-colouring σ of G defined by setting $\sigma(e) = i$ when i is the least integer such that $e \in f(S_i)$. Then the subgraph of G coloured i is $\sigma^{-1}(i) \subseteq f(S_i) = S_i \cup f'(S_i)$. Since S_i is an independent set in $\mathcal{H}_{n,i}$, we have that S_i is H_i-free. Every copy of H_i in $\sigma^{-1}(i)$ either contains at least one edge in S_i, or has every edge contained in $f'(S_i)$. Note that $m_k(H_i) \leqslant m$. By (5.4.7), the number of copies of H_i in G containing at least one edge in S_i is at most

$$e(S_i) \cdot v_i! \binom{n-k}{v_i-k} \leqslant D_1 n^{k-1/m_k(H_1)} \cdot v_i!(n-k)^{v_i-k} \leqslant D_1 v_i! \cdot n^{v_i - \frac{1}{m}} < \frac{\varepsilon}{2}\binom{n}{v_i}.$$

For each $i \in [r]$ we have that $f'(S_i) \in \overline{\mathcal{F}_{n,i}}$, and so $e(\mathcal{H}_{n,i}[f'(S_i)]) < \varepsilon' e(\mathcal{H}_{n,i})$. That is, the number of copies of H_i in $f'(S_i)$ is less than

$$\varepsilon' \cdot \frac{v_i!}{|\mathrm{Aut}(H_i)|}\binom{n}{v_i} \leqslant \frac{\varepsilon}{2}\binom{n}{v_i}.$$

Thus, in total $f(S_i) = S_i \cup f'(S_i)$ contains at most $\varepsilon\binom{n}{v_i}$ copies of H_i, so G is not ε-strongly $(H_1, \dots, H_r)$-Ramsey. Since $\varepsilon \leqslant \delta$, this immediately implies (iv)(a), and (iv)(b) follows from Theorem 5.27, our choice of parameters, and since n is sufficiently large. $\square$

As in Theorem 2.1, we will call the elements $S \in \mathcal{S}_r$ *fingerprints*, and each $\bigcup_{i\in[r]} f(S_i)$ with $(S_1, \dots, S_r) \in \mathcal{S}_r$ is a *container*.

5.4.5 The number of hypergraphs which are not Ramsey

Our first application of Theorem 5.34 is an enumeration result for non-$(H_1, \dots, H_r)$-Ramsey hypergraphs (Theorem 5.12), which asymptotically determines the logarithm of $|\overline{\mathrm{Ram}}(n; H_1, \dots, H_r)|$.

Proof of Theorem 5.12. Let $0 < \delta < 1$ be arbitrary, and let $n \in \mathbb{N}$ be sufficiently large. Clearly, $|\overline{\mathrm{Ram}}(n; H_1, \dots, H_r)| \geqslant 2^{\mathrm{ex}^r(n; H_1, \dots, H_r)}$ since no subhypergraph of an n-vertex non-$(H_1, \dots, H_r)$-Ramsey k-graph with $\mathrm{ex}^r(n; H_1, \dots, H_r)$ edges is $(H_1, \dots, H_r)$-Ramsey.

For the upper bound, suppose first that $\Delta_1(H_i) \geqslant 2$ for all $i \in [r]$. Let $D > 0$ be obtained from Theorem 5.34 applied to $H_1, \dots, H_r$ with parameter δ. We obtain a collection $\mathcal{S}_r$ and a function f as in Theorem 5.34. Consider any $G \in \overline{\mathrm{Ram}}(n; H_1, \dots, H_r)$. Note that there are pairwise edge-disjoint k-graphs $G_1, \dots, G_r$ such that $\bigcup_{i \in [r]} G_i = G$ and $(G_1, \dots, G_r) \in \mathcal{I}_r(n; H_1, \dots, H_r)$. So by Theorem 5.34(i) this means there is some $S = (S_1, \dots, S_r) \in \mathcal{S}_r$ so that $G \subseteq \bigcup_{i \in [r]} f(S_i)$. Further, given any $S = (S_1, \dots, S_r) \in \mathcal{S}_r$, we have

$$e\left(\bigcup_{i \in [r]} f(S_i)\right) \leqslant \mathrm{ex}^r(n; H_1, \dots, H_r) + \delta \binom{n}{k}.$$

Thus, each such $\bigcup_{i \in [r]} f(S_i)$ contains at most $2^{\mathrm{ex}^r(n; H_1, \dots, H_r) + \delta\binom{n}{k}}$ k-graphs in $\overline{\mathrm{Ram}}(n; H_1, \dots, H_r)$. Note that, by Theorem 5.34(ii),

$$|\mathcal{S}_r| \leqslant \left(\sum_{s=0}^{Dn^{k-1/m_k(H_1)}} \binom{\binom{n}{k}}{s} \right)^r < 2^{\delta\binom{n}{k}},$$

where the last inequality holds since n is sufficiently large. Altogether, this implies

$$|\overline{\mathrm{Ram}}(n; H_1, \dots, H_r)| \leqslant 2^{\delta\binom{n}{k}} \times 2^{\mathrm{ex}^r(n; H_1, \dots, H_r) + \delta\binom{n}{k}} = 2^{\mathrm{ex}^r(n; H_1, \dots, H_r) + 2\delta\binom{n}{k}}. \tag{5.4.8}$$

Since the choice of $0 < \delta < 1$ was arbitrary, this proves the theorem in the case when

$\Delta_1(H_i) \geqslant 2$ for all $i \in [r]$.

Suppose now that, say, $\Delta_1(H_1) = 1$. Then H_1 is a matching. Certainly every non-$(H_2, \ldots, H_r)$-Ramsey k-graph is non-$(H_1, \ldots, H_r)$-Ramsey. Let $H \in \overline{\mathrm{Ram}}(n; H_1, \ldots, H_r)$. Then there exists an r-colouring σ of H such that $\sigma^{-1}(i)$ is H_i-free for all $i \in [r]$. Thus H is the union of pairwise edge-disjoint k-graphs $J \in \overline{\mathrm{Ram}}(n; H_2, \ldots, H_r)$ and $J' := \sigma^{-1}(1)$. But J' is H_1-free and hence does not contain a matching of size $\lfloor v(H_1)/2 \rfloor =: h$. A result of Erdős [33] (used here in a weaker form) implies that, for sufficiently large n,

$$e(J') \leqslant (h-1)\binom{n-1}{k-1}.$$

Thus, for large n,

$$\begin{aligned}
|\overline{\mathrm{Ram}}(n; H_1, \ldots, H_r)| &\leqslant \sum_{J \in \overline{\mathrm{Ram}}(n; H_2, \ldots, H_r)} \sum_{e(J')=0}^{(h-1)\binom{n-1}{k-1}} \binom{\binom{n}{k}}{e(J')} \\
&= |\overline{\mathrm{Ram}}(n; H_2, \ldots, H_r)| \sum_{e(J')=0}^{\frac{k(h-1)}{n}\binom{n}{k}} \binom{\binom{n}{k}}{e(J')} \\
&\leqslant |\overline{\mathrm{Ram}}(n; H_2, \ldots, H_r)| \cdot 2^{\delta\binom{n}{k}}.
\end{aligned}$$

Iterating this argument, using (5.4.8) and the fact that $0 < \delta < 1$ was arbitrary, we obtain the required upper bound in the general case. $\square$

In fact Theorem 5.12 can be recovered in a different way, which, to the best of our knowledge, has not been explicitly stated elsewhere. Let $\mathcal{F}$ be a (possibly infinite) family of k-graphs, and let $\mathrm{Forb}(n; \mathcal{F})$ be the set of n-vertex k-graphs which contain no copy of any $F \in \mathcal{F}$ as a subhypergraph. The following result of Nagle, Rödl and Schacht [84] asymptotically determines the logarithm of $|\mathrm{Forb}(n; \mathcal{F})|$. (This generalises the corresponding result of Erdős, Frankl and Rödl [34] for graphs.) Let

$$\mathrm{ex}(n;\mathcal{F}) := \max\{e(H) : H \in \mathrm{Forb}(n;\mathcal{F})\}.$$

(So when $\mathcal{F} = \{F\}$ contains a single k-graph, we have $\mathrm{ex}(n;\{F\}) = \mathrm{ex}(n;F)$.)

Theorem 5.35 (Theorem 2.3, [84]). *Let $k \geqslant 2$ be a positive integer and $\mathcal{F}$ be a (possibly infinite) family of k-graphs. Then, for all $n \in \mathbb{N}$,*

$$|\mathrm{Forb}(n;\mathcal{F})| = 2^{\mathrm{ex}(n;\mathcal{F})+o(n^k)}.$$

Since $G \in \overline{\mathrm{Ram}}(n;H_1,\ldots,H_r)$ if and only if G is an n-vertex k-graph without a copy of any $F \in \mathrm{Ram}(H_1,\ldots,H_r)$ as a subhypergraph, Theorem 5.35 immediately implies Theorem 5.12.

We remark that the proof of Nagle, Rödl and Schacht [84] uses hypergraph regularity. Our proof of Theorem 5.12 has the advantage that it is is regularity-free.

5.4.6 The resilience of being $(H_1,\ldots,H_r)$-Ramsey

Recall that $G^{(k)}_{n,p}$ has vertex set $[n]$, where each edge lies in $\binom{[n]}{k}$ and appears with probability p, independently of all other edges. In this section we apply Theorem 5.34 to prove Theorem 5.7, which determines $\mathrm{res}(G^{(k)}_{n,p},(H_1,\ldots,H_r)\text{-Ramsey})$ for given fixed k-graphs $H_1,\ldots,H_r$. Explicitly, $\mathrm{res}(G^{(k)}_{n,p},(H_1,\ldots,H_r)\text{-Ramsey})$ is the minimum integer t such that one can remove t edges from $G^{(k)}_{n,p}$ to obtain a k-graph H which has an $(H_1,\ldots,H_r)$-free r-colouring.

Observe that Theorem 5.7 together with (5.4.3) immediately implies the following corollary.

Corollary 5.36 (Random Ramsey for hypergraphs). *For all positive integers r,k with $k \geqslant 2$ and k-graphs $H_1,\ldots,H_r$ with $m_k(H_1) \geqslant \ldots \geqslant m_k(H_r)$ and $\Delta_1(H_i) \geqslant 2$ for all*

$i \in [r]$, there exists $C > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left[G_{n,p}^{(k)} \textit{ is } (H_1,\dots,H_r)\textit{-Ramsey}\right] = 1 \quad \textit{if } p > Cn^{-1/m_k(H_1)}.$$

In the case when $m_k(H_1) = m_k(H_2)$, Corollary 5.36 generalises Theorem 5.6 since we do not require H_1 to be strictly k-balanced. Further, Corollary 5.36 resolves (the 1-statement part) of Conjecture 5.3 in the case when $m_2(H_1) = m_2(H_2)$.

Proof of Theorem 5.7. Let $0 < \delta < 1$ be arbitrary, $r, k \in \mathbb{N}$ with $k \geqslant 2$, and let $H_1,\dots,H_r$ be k-graphs as in the statement of the theorem. Given $n \in \mathbb{N}$, if $p > n^{-1/m_k(H_1)}$, then $p > n^{-(k-1)}$ since $\Delta_1(H_1) \geqslant 2$. Proposition 1.2(ii) implies that, w.h.p.,

$$e(G_{n,p}^{(k)}) = \left(1 \pm \frac{\delta}{4}\right) p \binom{n}{k}. \tag{5.4.9}$$

For brevity, write $\pi := \pi(H_1,\dots,H_r)$. We will first prove the upper bound

$$\lim_{n\to\infty} \mathbb{P}\left[\mathrm{res}(G_{n,p}^{(k)}, (H_1,\dots,H_r)\text{-Ramsey}) \leqslant (1-\pi+\delta)e(G_{n,p}^{(k)})\right] = 1 \quad \text{if} \quad p > n^{-1/m_k(H_1)}.$$

For this, we must show that the probability of the event that there exists an n-vertex k-graph $G \subseteq G_{n,p}^{(k)}$ such that $e(G) \geqslant (\pi - \delta)e(G_{n,p}^{(k)})$ and $G \in \overline{\mathrm{Ram}}(n; H_1,\dots,H_r)$, tends to one as n tends to infinity. This indeed follows: Let n be sufficiently large so that $\mathrm{ex}^r(n; H_1,\dots,H_r) \geqslant (\pi - \delta/2)\binom{n}{k}$. Let G^* be an n-vertex non-$(H_1,\dots,H_r)$-Ramsey k-graph with $e(G^*) = \mathrm{ex}^r(n; H_1,\dots,H_r)$. Then, by Proposition 1.2(ii), w.h.p. we have $e(G^* \cap G_{n,p}^{(k)}) = (\pi \pm \delta)e(G_{n,p}^{(k)})$, and $G^* \cap G_{n,p}^{(k)} \in \overline{\mathrm{Ram}}(n; H_1,\dots,H_r)$, as required.

For the remainder of the proof, we will focus on the lower bound, namely that there exists $C > 0$ such that whenever $p > Cn^{-1/m_k(H_1)}$,

$$\mathbb{P}\left[\mathrm{res}(G_{n,p}^{(k)}, (H_1,\dots,H_r)\text{-Ramsey}) \geqslant (1-\pi-\delta)e(G_{n,p}^{(k)})\right] \to 1 \quad \text{as} \quad n \to \infty. \tag{5.4.10}$$

Suppose n is sufficiently large. Apply Theorem 5.34 with parameters $r, k, \delta/16, (H_1, \dots, H_r)$ to obtain $D > 0$ and for each $n \in \mathbb{N}$, a collection $\mathcal{S}_r$ and a function f satisfying (i)–(iv). Now choose C such that $0 < 1/C \ll 1/D, \delta, 1/k, 1/r$. Let $p \geqslant Cn^{-1/m_k(H_1)}$.

Since (5.4.9) holds with high probability, to prove (5.4.10) holds it suffices to show that the probability $G^{(k)}_{n,p}$ contains a non-$(H_1, \dots, H_r)$-Ramsey k-graph with at least $(\pi + \delta/2)p\binom{n}{k}$ edges tends to zero as n tends to infinity.

Suppose that $G^{(k)}_{n,p}$ does contain a non-$(H_1, \dots, H_r)$-Ramsey k-graph I with at least $(\pi + \delta/2)p\binom{n}{k}$ edges. Then there exist pairwise edge-disjoint k-graphs $I_1, \dots, I_r$ such that $\bigcup_{i\in[r]} I_i = I$ and $(I_1, \dots, I_r) \in \mathcal{I}_r(n; H_1, \dots, H_r)$. Further, there is some $S = (S_1, \dots, S_r) \in \mathcal{S}_r$ such that $S \subseteq (I_1, \dots, I_r) \subseteq f(S)$. Thus, $G^{(k)}_{n,p}$ must contain (the edges of) $\bigcup_{i\in[r]} S_i$ as well as at least $(\pi + \delta/4)p\binom{n}{k}$ edges from $(\bigcup_{i\in[r]} f(S_i)) \setminus (\bigcup_{i\in[r]} S_i)$. (Note here we are using that $e(\bigcup_{i\in[r]} S_i) \leqslant \delta p\binom{n}{k}/4$, which holds by Theorem 5.34(ii) and since $0 < 1/C \ll 1/D, 1/k, \delta$.) Writing $s := e(\bigcup_{i\in[r]} S_i)$, the probability $G^{(k)}_{n,p}$ contains $\bigcup_{i\in[r]} S_i$ is p^s. Note that $e((\bigcup_{i\in[r]} f(S_i)) \setminus (\bigcup_{i\in[r]} S_i)) \leqslant (\pi + \delta/8)\binom{n}{k}$ by Theorem 5.34(iv)(b) and since n is sufficiently large. So by Proposition 1.2(i), the probability $G^{(k)}_{n,p}$ contains at least $(\pi + \delta/4)p\binom{n}{k}$ edges from $(\bigcup_{i\in[r]} f(S_i)) \setminus (\bigcup_{i\in[r]} S_i)$ is at most $\exp(-\delta^2 p\binom{n}{k}/256) \leqslant \exp(-\delta^2 pn^k/256k^k)$.

Write $N := n^{k-1/m_k(H_1)}$ and $\gamma := \delta^2/256k^k$. Given some integer $0 \leqslant s \leqslant DN$, there are at most $r^s\binom{\binom{n}{k}}{s}$ elements $(S_1, \dots, S_r) \in \mathcal{S}_r$ such that $e(\bigcup_{i\in[r]} S_i) = s$. Indeed, this follows since there are r^s ways to partition a set of size s into r classes. Thus, the probability that $G^{(k)}_{n,p}$ does contain a non-$(H_1, \dots, H_r)$-Ramsey k-graph I with at least $(\pi + \delta/2)p\binom{n}{k}$ edges is at most

$$\begin{aligned}\sum_{s=0}^{DN} r^s \binom{\binom{n}{k}}{s} \cdot p^s \cdot e^{-\gamma n^k p} &\leqslant (DN+1)(rp)^{DN}\binom{\binom{n}{k}}{DN} e^{-\gamma n^k p} \\ &\leqslant (DN+1)\left(\frac{re^{k+1}pn^k}{DNk^k}\right)^{DN} e^{-\gamma n^k p}\end{aligned}$$

$$\leqslant (DN+1)\left(\frac{re^{k+1}C}{Dk^k}\right)^{DN} e^{-\gamma CN} \leqslant e^{\gamma CN/2}e^{-\gamma CN} = e^{-\gamma CN/2},$$

which tends to zero as n tends to infinity. This completes the proof. $\square$

Chapter 6

The Maker-Breaker Rado game on a random set of integers

6.1 Introduction

Given a finite set X and a family of subsets of X, $\mathcal{F} \subseteq \mathcal{P}(X)$, we define the Maker-Breaker game on $(X, \mathcal{F})$ to be the game where Maker and Breaker take turns to select a previously unchosen element $x \in X$, and at the conclusion of the game, if Maker has claimed all of the elements of some $F \in \mathcal{F}$, then Maker wins. Otherwise Breaker has claimed at least one element x in every set $F \subseteq \mathcal{F}$, and Breaker wins. The set X is known as the *board*, and the family $\mathcal{F}$ as the *winning sets*. If Maker has a strategy so that no matter how Breaker plays, Maker can always win, then we call the game *Maker's win*. If Maker has no such strategy, then since there is no draw scenario, the game is *Breaker's win*.

Maker-Breaker games first stemmed from a seminal paper by Erdős and Selfridge [35], where they proved their famous criterion which gives a general winning strategy for Breaker. Some well-known examples of Maker-Breaker games are where the board X is the edge set of a complete graph K_n, and the winning sets $\mathcal{F}$ are all sets of edges which correspond to a perfect matching; a Hamilton cycle; or a fixed subgraph H. All of these games turn out to be Maker's win if n is sufficiently large, therefore an adjustment to the game is required if we wish to make the problem of determining whose win the game is more interesting. This leads to the following two variations of Maker-Breaker board games, which have each received significant attention.

- **Biased board games.** Maker claims one element of the board per turn, whereas Breaker claims b elements per turn, for some fixed $b \in \mathbb{N}$. We call the game the $(1 : b)$ *game on* $(X, \mathcal{F})$. Maker-Breaker games are 'bias-monotone' (see e.g. [52]). This means that there exists a *threshold bias* b_0 such that the $(1 : b)$ game on $(X, \mathcal{F})$ is Maker's win if and only if $b < b_0$.

- **Random board games.** For a fixed probability $p = p(n)$ and game $(X, \mathcal{F})$, let X_p be obtained by including each element $x \in X$ with probability p independently of all other elements, and let $\mathcal{F}_p := \{F \in \mathcal{F} : x \in X_p \text{ for all } x \in F\}$. We then consider the game on the random board $(X_p, \mathcal{F}_p)$, noting that it is a probability space of games. By the monotonicity of the game $(X, \mathcal{F})$ being Maker's win, the existence of a threshold function follows from [19]. That is, there exists a *threshold probability* $p_0 = p_0(n)$ such that

$$\lim_{n\to\infty} \mathbb{P}[\text{The game on } (X_p, \mathcal{F}_p) \text{ is Maker's win}] = \begin{cases} 1 & \text{if } p/p_0 \to \infty \text{ as } n \to \infty; \\ 0 & \text{if } p/p_0 \to 0 \text{ as } n \to \infty. \end{cases}$$

The interesting problem now is to determine the threshold bias and threshold probability for various Maker-Breaker games. For examples and further history of combinatorial board games, see e.g. [12, 52].

Random board games were first introduced by Stojaković and Szabó [113], who considered games played on a random subset of the edges of a complete graph. Note that this precisely corresponds to the edges of the Erdős–Rényi random graph $G_{n,p}$. Here, we focus on the game where Maker's aim is to obtain a solution to a system of linear equations within a random set of integers. To be precise, in our Maker-Breaker game, the board will be a random set of integers $[n]_p$ (recall this is obtained by including each element of $[n]$ with probability p independently of all other elements). The winning sets are all sets of size k which correspond to a k-distinct solution (i.e. $x = (x_1, \dots, x_k)$ has each x_i distinct) to a system of linear equations $Ax = b$, where A is a fixed integer-valued matrix of dimension $\ell \times k$ and b is a fixed integer-valued vector of dimension ℓ. We call such a game played on a set of integers X the *(A, b)-game on X*, or the *$\mathcal{L}$-game on X* (recalling the use of $\mathcal{L}$ to represent the system of linear equations $Ax = b$ used in previous chapters). The class of all $\mathcal{L}$-games are known as *Rado games* (introduced in [75]), due to the intimate link with Rado's partition theorem, which will be discussed shortly. As in the majority of the last chapter, we only care about k-distinct solutions (and therefore distinct $\mathcal{L}$-free sets).

Maker-Breaker games in this setting were first considered by Beck [13], who studied the *van der Waerden game*. Here, Maker's aim is to obtain a k-term arithmetic progression $a, a+r, \dots, a+(k-1)r$ for some $a, r \in \mathbb{N}$ and fixed $k \in \mathbb{N}$. Note that the set of k-term arithmetic progressions in $[n]$ exactly coincides with the set of k-distinct solutions to $Ax = 0$ in $[n]$ where A is the $(k-2) \times k$ matrix given by

$$\begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{pmatrix}.$$

Beck determined that the smallest $n \in \mathbb{N}$ such that the $(A, 0)$-game on $[n]$ is Maker's win is $n = 2^{k(1+o(1))}$.

Here we consider a generalisation of the van der Waerden game using the following definitions which we recall from the previous chapter. Let A be a fixed integer-valued matrix of dimension $\ell \times k$ and b a fixed integer-valued vector of dimension ℓ. We call $\mathcal{L}$ (and the matrix A in the case where $b = 0$) *irredundant* if there exists a k-distinct solution to $Ax = b$ in $\mathbb{N}$, and *partition regular* if for any finite colouring of $\mathbb{N}$, there is always a monochromatic solution (of any kind) to $Ax = b$.

A cornerstone result in the area of Ramsey theory for integers is Rado's theorem [88], which characterises all partition regular systems of linear equations $\mathcal{L}$. In [60], Hindman and Leader extended Rado's theorem to characterise all systems $\mathcal{L}$ for which given any finite colouring of $\mathbb{N}$, there is always a monochromatic k-distinct solution to $Ax = b$ (in particular, if $b = 0$ then A must be irredundant and partition regular). Hindman and Leader's result implies that given such a system $\mathcal{L}$, if n is sufficiently large then however we 2-colour $[n]$, there exists a monochromatic k-distinct solution to $Ax = b$. So in order for Breaker to win the $\mathcal{L}$-game on $[n]$, he must himself obtain a k-distinct solution. But then by the classic strategy-stealing argument, Maker can claim this solution for herself. Thus this game is an (easy) win for Maker. Therefore it is interesting to consider biased and random versions of the $\mathcal{L}$-game on $[n]$. In a very recent paper of Kusch, Rué, Spiegel and Szabó [75], the biased version is considered. In this chapter, we consider the random version.

In fact (as in [75]), we consider a wider class of systems of linear equations $\mathcal{L}$. Recall the definition of $(*)$, the matrix property we introduced in the previous chapter:

$(*)$ Under Gaussian elimination A does not have any row which consists of precisely two non-zero rational entries.

In [75] the term *abundant* is used, which, recall from the previous chapter, is equivalent to $(*)$ in the case of irredundant full rank matrices. Recall that in Section 5.3.1 it is proven that irredundant partition regular matrices are a strict subclass of irredundant matrices which satisfy $(*)$.

Recall the definition of $m(A)$ (see (1.3.2)). The biased game result of [75] is the following.

Theorem 6.1 ([75], Theorem 1.3 and Proposition 1.4). *Let A be a fixed integer-valued matrix of dimension $\ell \times k$ and b a fixed integer-valued vector of dimension ℓ. Given the system of linear equations $\mathcal{L}$ and the matrix A are both irredundant, then we have the following:*

(i) If A satisfies $(*)$ then the threshold bias for the (A,b)-game on $[n]$ is $\Theta(n^{1/m(A)})$;

(ii) If A does not satisfy $(*)$ then the $(1:2)$ (A,b)-game on $[n]$ is Breaker's win.

In this chapter we mainly focus on the case when A satisfies $(*)$, though the case where A does not satisfy $(*)$ does feature in our first result and also Section 6.4.

Our first result gives the threshold for the random $\mathcal{L}$-game whenever $\mathcal{L}$ is a single linear equation.

Theorem 6.2. *Let A be a fixed non-zero-integer-valued matrix of dimension $1 \times k$ and b a fixed integer (i.e. $Ax = b$ corresponds to a single linear equation $a_1x_1 + \cdots + a_kx_k = b$ with the a_i non-zero integers).*

(i) If the system of linear equations $\mathcal{L}$ is irredundant and A is irredundant and satisfies $(*)$, then the $\mathcal{L}$-game on $[n]_p$ has a threshold probability of $\Theta(n^{-\frac{k-2}{k-1}})$;

(ii) If the system of linear equations $\mathcal{L}$ is irredundant and A is irredundant and does not satisfy $(*)$, then the $\mathcal{L}$-game on $[n]_p$ is Maker's win if $p \gg n^{-1/3}$ and Breaker's win if $p \ll n^{-1/3}$;

(iii) If the system of linear equations $\mathcal{L}$ is irredundant and A is not irredundant, then

(a) *the $\mathcal{L}$-game on $[n]_p$ is Breaker's win w.h.p. for any $p = o(1)$ if the coefficients a_i are all positive or all negative;*

(b) *the $\mathcal{L}$-game on $[n]_p$ is Maker's win if $p \gg n^{-1/3}$ and Breaker's win if $p \ll n^{-1/3}$ otherwise;*

(iv) *If the system of linear equations $\mathcal{L}$ is not irredundant, then the $\mathcal{L}$-game on $[n]$ is (trivially) Breaker's win.*

Note that most 'interesting' equations lies in the class of equations given by (i). In particular it includes several natural equations that have been extensively studied, e.g. $x + y = z$, $x + y = z + t$ and $x + y = 2z$. In the $\mathcal{L}$-games corresponding to these equations, Breaker's aim is to restrict Maker's set to being a *sum-free set*, a *Sidon set* and a *progression-free set* respectively. The remaining classes of equations given by (ii)–(iv) are all in some sense 'trivial'; the proofs of these statements appear in Section 6.4.3.

In fact Theorem 6.2(i) will follow immediately from a much more general theorem. First we need some more definitions. We say that an $\ell \times k$ matrix A of full rank ℓ is *strictly balanced* if, for every $W \subseteq [k]$ for which $2 \leqslant |W| < k$, the inequality

$$\frac{|W| - 1}{|W| - 1 + \operatorname{rank}(A_{\overline{W}}) - \ell} < \frac{k-1}{k-1-\ell}$$

holds. In particular note that if A is strictly balanced then $m(A) = \frac{k-1}{k-1-\ell}$ (though the converse is not true). Given an irredundant matrix A which satisfies $(*)$, we define the *associated matrix* $B(A)$ to be a strictly balanced, irredundant matrix of full rank which satisfies $(*)$, which is found by using elementary row operations on A then deleting some rows and columns, and satisfies $m(B(A)) = m(A)$. The fact that such a matrix exists is not entirely obvious, and is essentially proven in [93]. We provide further details in Section 6.3. Also note that if A itself is strictly balanced then we simply have $B(A) = A$.

Recall $\mu(n, \mathcal{L}^d)$ denotes the size of the largest subset of $[n]$ which does not contain a k-distinct solution to $Ax = b$. The main result of this chapter is the following.

Theorem 6.3. *Let A be a fixed integer-valued matrix of dimension $\ell' \times k'$ and b a fixed integer-valued vector of dimension ℓ'. Given the system of linear equations $\mathcal{L}$ is irredundant and A is irredundant and satisfies $(*)$ we have the following:*

(i) Let $\varepsilon > 0$. There exists a positive constant C such that if $p > Cn^{-1/m(A)}$, then for any $R \subseteq [n]_p$ satisfying $|R| \leqslant (1 - \frac{\mu(n,\mathcal{L}^d)}{n} - \varepsilon)np$, we have

$$\lim_{n\to\infty} \mathbb{P}\left(\textit{Maker wins the } \mathcal{L}\textit{-game on } [n]_p \setminus R\right) = 1.$$

(ii) Suppose the associated matrix $B(A)$ is an $\ell \times k$ matrix of full rank ℓ, where ℓ divides $k - 1$. There exists a positive constant c such that if $p < cn^{-1/m(A)}$ then

$$\lim_{n\to\infty} \mathbb{P}\left(\textit{Breaker wins the } \mathcal{L}\textit{-game on } [n]_p\right) = 1.$$

First note that it follows from a supersaturation result (Theorem 6.5) that for all pairs (A, b) as stated in Theorem 6.3, there exist $n_0 = n_0(A, b)$, $\delta = \delta(A, b) > 0$, such that for all integers $n \geqslant n_0$ we have $\mu(n, \mathcal{L}^d) \leqslant (1-\delta)n$. Thus in particular Theorem 6.3(i) implies that there exists a positive constant C such that if $p > Cn^{-1/m(A)}$, then Maker wins the $\mathcal{L}$-game on $[n]_p$ w.h.p. Also, note that if A is a $1 \times k$ matrix with non-zero entries, then it is strictly balanced, and so $B(A) = A$. Thus A is a matrix which satisfies the hypothesis of Theorem 6.3(ii). Theorem 6.2(i) follows immediately from these two comments.

Another example of a class of pairs (A, b) for which Theorem 6.3 gives the threshold probability up to a constant factor are all irredundant systems of linear equations $\mathcal{L}$ for which A is irredundant, has no columns consisting entirely of zeroes, satisfies $(*)$ and is of dimension $2 \times k'$ for some odd k'. This follows since by construction either $B(A) = A$ or $B(A)$ is a $1 \times k$ matrix for some $k < k'$. Either way, $B(A)$ then satisfies the hypothesis of Theorem 6.3(ii).

For the Maker's win statement, the fact that we can delete a certain fraction of elements from $[n]_p$ and still have Maker's win w.h.p. means we have a resilience theorem. Note that in our result, the property is the game being Maker's win w.h.p., and the resilience is best possible in terms of the bound on the size of the set R: Indeed, since the largest subset of $[n]$ with no k-distinct solutions to $Ax = b$ has size $\mu(n, \mathcal{L}^d)$, w.h.p. $[n]_p$ contains a subset S of size $p(\mu(n, \mathcal{L}^d) - \varepsilon n)$ with no k-distinct solutions to $Ax = b$. Thus we can remove $(1 - \frac{\mu(n,\mathcal{L}^d)}{n} + \varepsilon)np$ elements from $[n]_p$ to obtain S (noting that a game on S is trivially Breaker's win).

It is very interesting to note the parallels between our theorem and the following random Rado theorems and the resilience theorem, which appeared in the previous chapter.

Theorem 5.8 ([93]). *For all irredundant partition regular full rank matrices A and all positive integers $r \geqslant 2$, there exists a constant $c > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[[n]_p \textit{ is } (\mathcal{L}^d, r)\textit{-Rado}\right] = 0 \qquad \textit{if } p < cn^{-1/m(A)}.$$

Theorem 5.9 ([42]). *For all irredundant partition regular full rank matrices A and all positive integers r, there exists a constant $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[[n]_p \textit{ is } (\mathcal{L}^d, r)\textit{-Rado}\right] = 1 \qquad \textit{if } p > Cn^{-1/m(A)}.$$

Theorem 5.11. *For all irredundant partition regular full rank matrices A, all positive integers r and all $\delta > 0$, there exists a constant $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\mathrm{res}([n]_p, (\mathcal{L}^d, r)\textit{-Rado})}{|[n]_p|} = 1 - \frac{\mu(n, \mathcal{L}^d, r)}{n} \pm \delta\right] = 1 \qquad \textit{if } p > Cn^{-1/m(A)}.$$

Theorem 5.9 implies that by again using strategy-stealing, we could obtain a proof for the non-resilient version of Theorem 6.3(i) for irredundant partition regular matrices. However our method as already noted achieves the best resilience possible, and further it extends to all irredundant matrices which satisfy $(*)$ (even those for which there exists a 2-colouring of $\mathbb{N}$ with no monochromatic k-distinct solutions to $Ax = b$). Our proof also gives an explicit strategy.

The proof of Theorem 6.3(i) closely follows the method of Theorem 16 in [86]. Here, Nenadov, Steger and Stojaković consider a similar problem: the *H-game* is where the board is the edges of a complete graph, and the winning sets are sets of edges which correspond to a copy of a fixed subgraph H. This game and its related Ramsey problems resemble the $\mathcal{L}$-game as follows: Set $d_2(H) := 0$ if $e(H) = 0$, $d_2(H) := 1/2$ if $e(H) = 1$, and $d_2(H) := (e(H) - 1)/(v(H) - 2)$ otherwise. Then define the *2-density* of H to be $m_2(H) := \max_{H' \subseteq H} d_2(H')$. For most graphs H, the graph analogues of Theorems 5.8, 5.9 and 5.11 (the random Ramsey theorem and resilient subgraphs theorem, see Theorems 5.2 and 5.1) have a threshold of $\Theta(n^{-1/m_2(H)})$. Bednarska and Łuczak [14] showed that the threshold bias for the H-game is $\Theta(n^{1/m_2(H)})$. Thus both the H-games and $\mathcal{L}$-games (in most cases) have a threshold bias which is the inverse of the threshold for the random (respective) Ramsey/Rado theorem and the resilience theorems. Kusch, Rué, Spiegel and Szabó [75] in fact show that there is an intimate link between resilience and the threshold bias, which explains the parameters of $m(A)$ and $m_2(H)$ appearing for both. They refer to this phenomenon as the *probabilistic Turán intuition* for biased Maker-Breaker games; see Section 6.4 of [75] for more details.

An analogous definition of strictly balanced exists for graphs. In [86], Nenadov, Steger and Stojaković show that the threshold probability for the random H-game is $\Theta(n^{-1/m_2(H)})$ when H is strictly balanced (Theorem 2 in [86]). However there are a class of graphs which have a threshold probability different to that of the random Ramsey/resilient subgraph theorem and the inverse of the threshold bias (Theorem 4 in [86]). Indeed, this is one of the main motivations for studying the random $\mathcal{L}$-game: For the proof of Theorem 6.3(ii), we build upon the method used by Rödl and Ruciński [93] to prove Theorem 5.8. Although our Breaker win statement is 'incomplete', its proof does seem to indicate that the threshold probability for the random $\mathcal{L}$-game (for any system of linear equations $\mathcal{L}$ which is irredundant and A irredundant and satisfying $(*)$) should be the same as the random Rado threshold. That is, we hope that there is no need for the assumption that ℓ divides $k-1$ in Theorem 6.3(ii). Also note that if we could prove our Breaker win statement for all strictly balanced matrices, then the full result would follow (see Proposition 6.8 and the paragraph following it). So interestingly in this sense, the random $\mathcal{L}$-game does not resemble the random H-game.

We prove the two parts of Theorem 6.3 in Sections 6.2 and 6.3 respectively, before finishing by proving Theorem 6.2(ii)–(iv) along with making some further remarks in Section 6.4.

6.2 Proof of Maker's win in Theorem 6.3

First we list a few results which are required for the proof. We will use the following simplification of Theorem 5.21. First recall $\mathcal{I}(n, \mathcal{L}^d)$ denote all sets from $\mathcal{P}([n])$ which contain no k-distinct solutions to $Ax = b$.

Theorem 6.4. *Let $0 < \delta < 1$. Let A be a fixed integer-valued matrix of dimension $\ell \times k$ and b a fixed integer-valued vector of dimension ℓ. Suppose the system of linear equations $\mathcal{L}$ is irredundant and A is irredundant and satisfies $(*)$. Then there exists $D > 0$ such that the following holds. For all $n \in \mathbb{N}$, there is a collection $\mathcal{S} \subseteq \mathcal{P}([n])$ and a function $f : \mathcal{S} \to \mathcal{P}([n])$ such that:*

(i) For all $I \in \mathcal{I}(n, \mathcal{L}^d)$, there exists $S \in \mathcal{S}$ such that $S \subseteq I \subseteq f(S)$.

Additionally, every $S \in \mathcal{S}$ satisfies *(ii)* $|S| \leqslant Dn^{\frac{m(A)-1}{m(A)}}$*;*

(iii) $S \in \mathcal{I}(n, \mathcal{L}^d)$*;*

(iv) $f(S)$ *contains at most* $\delta n^{k-\ell}$ *k-distinct solutions to* $Ax = b$*; and*

(v) $|f(S)| \leqslant \mu(n, \mathcal{L}^d) + \delta n$*.*

An upper bound on the size of the largest subset of $[n]$ containing no k-distinct solutions to $Ax = b$ is also required. The following is a consequence of Lemma 4.1 in [75] and Lemma 2.2.

Theorem 6.5. *Let* A *be a fixed integer-valued matrix of dimension* $\ell \times k$ *and* b *a fixed integer-valued vector of dimension* ℓ*. Given the system of linear equations* $\mathcal{L}$ *is irredundant and* A *is irredundant and satisfies* $(*)$*, then there exist* $n_0 \in \mathbb{N}$ *and* $\delta > 0$ *such that for all integers* $n \geqslant n_0$ *we have* $\mu(n, \mathcal{L}^d) \leqslant (1-\delta)n$*.*

Finally, we require the Erdős-Selfridge Criterion, commonly used to prove a Breaker strategy result, which we mentioned in the introduction. (Note that we do mean Breaker here; in our proof, we create an auxiliary game where the original Maker needs to play the role of Breaker!)

Theorem 6.6 ([35]). *Let* X *be a set and let* $\mathcal{F}$ *be a family of subsets of* X*. Then if Breaker has the first move in the game, and*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < 1,$$

then Breaker has a winning strategy for the Maker-Breaker game $(X, \mathcal{F})$*.*

Proof of Theorem 6.3(i). Apply Theorem 6.5 with parameters A, b to obtain $\varepsilon' > 0$ such that $\mu(n, \mathcal{L}^d) \leqslant (1-\varepsilon')n$ for sufficiently large n. Let $\varepsilon > 0$ noting that without loss of generality we can assume $\varepsilon \ll \varepsilon'$. Suppose n is sufficiently large. Apply Theorem 6.4 with parameters $\varepsilon/4, A, b$ to obtain $D > 0$, a collection $\mathcal{S} \subseteq \mathcal{P}([n])$ and a function f satisfying Theorem 6.4(i)–(v). Fix $\delta \ll \varepsilon$ and choose C such that $0 \leqslant 1/C \ll 1/D, \delta, \varepsilon$. Let $p > Cn^{-1/m(A)}$. Note that $m(A) > 1$ (see Proposition 5.18(v)) and thus pn tends to infinity as n tends to infinity. Let R be as in the statement of the theorem and set $X := [n]_p \setminus R$.

Maker's aim is to claim a k-distinct solution to $Ax = b$ within X, and Breaker's aim is to prevent this. If Maker loses, then her set $M \subseteq X$ does not contain a k-distinct solution to $Ax = b$. Hence $M \in \mathcal{I}(n, \mathcal{L}^d)$ and so there exists $S \in \mathcal{S}$ such that $S \subseteq M \subseteq f(S)$ and $S \subseteq X$. Given $S \in \mathcal{S}$ note that if Maker claims one element from $X \setminus f(S)$ then $M \not\subseteq f(S)$; hence consider the auxiliary game $(X, \mathcal{F})$ where

$$\mathcal{F} := \{X \setminus f(S) : S \in \mathcal{S} \text{ and } S \subseteq X\}.$$

Maker can ensure that she wins the $\mathcal{L}$-game on X by picking at least one element from each set in $\mathcal{F}$, that is, she wins the auxiliary game as Breaker. We now make the following claim about the auxiliary game.

Claim 6.7. *(i) For all $S \in \mathcal{S}$ such that $S \subseteq X$, we have $|X \setminus f(S)| \geqslant \varepsilon np/2$ w.h.p.*

(ii) We have $|\mathcal{F}| \leqslant 2^{\varepsilon np/4}$ w.h.p.

Assuming the claim holds, it now easily follows that

$$\sum_{F \in \mathcal{F}} 2^{-|F|} \leqslant 2^{\varepsilon np/4} \cdot 2^{-\varepsilon np/2} < 1,$$

that is, the hypothesis of Theorem 6.6 holds for the game $(X, \mathcal{F})$. Thus Maker wins the game as Breaker in the auxiliary game, and thus wins the original game (the $\mathcal{L}$-game on X). Since this happens w.h.p., it remains to prove the claim.

Proof of Claim 6.7. First we shall count $|\mathcal{F}|$. We wish to count the number of $S \in \mathcal{S}$ such that $S \subseteq X$. Recall that every $S \in \mathcal{S}$ satisfies $|S| \leqslant Dn^{1-1/m(A)} \leqslant Dpn/C$ and there are at most $\binom{n}{s}$ sets $S \in \mathcal{S}$ of size s. Thus we have

$$\mathbb{E}[|\mathcal{F}|] \leqslant \sum_{S\in\mathcal{S}} \mathbb{P}[S \subseteq [n]_p] \leqslant \sum_{S\in\mathcal{S}} p^{|S|} \leqslant \sum_{s=0}^{Dpn/C} \binom{n}{s} p^s \leqslant (Dpn/C+1)\binom{n}{Dpn/C} p^{Dpn/C} \tag{6.2.1}$$

$$\leqslant (Dpn/C+1)\left(\frac{Ce}{D}\right)^{Dpn/C} \leqslant e^{\delta np} \leqslant 2^{\varepsilon np/8},$$

where the last two inequalities follows by our choice of C and since $\delta \ll \varepsilon$ respectively. Thus by Proposition 1.1 we have

$$\mathbb{P}[|\mathcal{F}| \geqslant 2^{\varepsilon np/4}] \leqslant 2^{-\varepsilon np/8},$$

which tends to zero as n tends to infinity, proving (ii).

Now for (i), observe that if we show that the probability that there exists $S \in \mathcal{S}$ such that $S \subseteq X$ and $|X \setminus f(S)| \leqslant \varepsilon np/2$ tends to zero as n tends to infinity, we will be done. First observe by Theorem 6.4 that for all $S \in \mathcal{S}$ we have $|f(S)| \leqslant \mu(n, \mathcal{L}^d) + \varepsilon n/4$ and so $|[n] \setminus f(S)| \geqslant n - \mu(n, \mathcal{L}^d) - \varepsilon n/4$. (In particular note $|[n] \setminus f(S)| = \Omega(n)$.) Let $\gamma := \varepsilon/(4 - 4\mu(n, \mathcal{L}^d)/n - \varepsilon)$ and $Y := [n]_p \setminus f(S)$ (noting $\gamma > 0$ since $\varepsilon \ll \varepsilon'$). By Proposition 1.2(ii) we have

$$\mathbb{P}\left[|([n] \setminus f(S)) \cap [n]_p| < \left(1 - \frac{\mu(n, \mathcal{L}^d)}{n} - \frac{\varepsilon}{2}\right) np\right] \leqslant \mathbb{P}\left[|Y| < (1-\gamma)\mathbb{E}[|Y|]\right]$$
$$\leqslant 2e^{-\mathbb{E}[|Y|]\gamma^2/3} \leqslant e^{-2\delta np},$$

where the last inequality follows since $\delta \ll \varepsilon \ll \varepsilon'$. Note that since $|R| \leqslant (1-\frac{\mu(n,\mathcal{L}^d)}{n}-\varepsilon)np$

and $X \setminus f(S) = Y \setminus R$ we have

$$\mathbb{P}[|X \setminus f(S)| < \varepsilon np/2] \leqslant e^{-2\delta np}, \tag{6.2.2}$$

for all $S \in \mathcal{S}$. Also since $S \subseteq f(S)$, the events $S \subseteq [n]_p$ and $|X \setminus f(S)|$ being small are independent. Thus

$$\begin{aligned}
&\mathbb{P}[\text{There exists } S \in \mathcal{S} \text{ such that } S \subseteq [n]_p \text{ and } |X \setminus f(S)| < \varepsilon np/2] \\
&\leqslant \sum_{S \in \mathcal{S}} \mathbb{P}[S \subseteq [n]_p \text{ and } |X \setminus f(S)| < \varepsilon np/2] \\
&\leqslant \sum_{S \in \mathcal{S}} (\mathbb{P}[S \subseteq [n]_p] \cdot \mathbb{P}[|X \setminus f(S)| < \varepsilon np/2]) \overset{(6.2.2)}{\leqslant} e^{-2\delta np} \sum_{S \in \mathcal{S}} \mathbb{P}[S \subseteq [n]_p] \\
&\overset{(6.2.1)}{\leqslant} e^{-2\delta np} \cdot e^{\delta np} = e^{-\delta np},
\end{aligned}$$

which tends to zero as n tends to infinity, as required. □

□

6.3 Proof of Breaker's win in Theorem 6.3

The proof will follow a similar tactic to that used by Rödl and Ruciński [93] for their proof of Theorem 5.8. Recall that the goal of Rödl and Ruciński was to show that, given an irredundant partition regular matrix A, an integer $r \geqslant 2$, and an upper bound on the probability p, then w.h.p. there exists an r-colouring of $[n]_p$ such that there are no monochromatic k-distinct solutions to $Ax = 0$. The proof consisted of three parts:

(P1) **A reduction of the problem.** It is shown that it suffices to prove the result for the associated matrix $B(A)$. The problem is then rephrased to one about an *associated hypergraph.*

(P2) **A deterministic lemma.** It is shown that if all r-colourings of $[n]_p$ contain a monochromatic k-distinct solution to $Ax = 0$, then the associated hypergraph must contain a certain kind of connected subhypergraph.

(P3) **A probabilistic lemma.** It is shown that if $p < cn^{-1/m(A)}$, then w.h.p. the associated hypergraph does not contain the kind of subhypergraph given by the deterministic lemma.

Recall that our aim is to show that under the hypothesis of Theorem 6.3(ii), w.h.p. Breaker wins the $\mathcal{L}$-game on $[n]_p$. Our proof consists of the same three general parts, with appropriate amendments to the lemmas.

(Q1) **A reduction of the problem.** As (P1) above.

(Q2) **Two deterministic lemmas.** These together show that if Maker wins the $\mathcal{L}$-game on $[n]_p$, then the associated hypergraph must contain a certain kind of connected subhypergraph.

(Q3) **A probabilistic lemma.** It is shown that if $B(A)$ is an $\ell \times k$ matrix of full rank ℓ which satisfies ℓ divides $k - 1$, and $p < cn^{-1/m(A)}$, then w.h.p. the associated hypergraph does not contain the kind of subhypergraph given by the deterministic lemmas.

We will of course make this more rigorous as we get to each part of the proof. We will shortly compare the differences between Rödl and Ruciński's proof and ours, but in order to do this it first makes sense to reduce each problem to one about an associated hypergraph.

(Q1) A reduction of the problem. First we show that in order for Breaker to win the (A, b)-game on any set of integers X, its suffices to show that Breaker wins the

(B, b')-game on X, for some matrix $B := B(A)$ and vector $b' := b'(A, b)$. For a vector $x = (x_1, \ldots, x_k)$ and a non-empty set $W \subseteq [k]$, let $x_W := (x_i)_{i \in W}$.

Proposition 6.8 ([75], Corollary 4.3 and Lemma 4.2). *Let A be a fixed integer-valued matrix of dimension $\ell \times k$ and b a fixed integer-valued vector of dimension ℓ. Suppose the system of linear equations $\mathcal{L}$ is irredundant and A is irredundant and satisfies $(*)$. There exists a non-empty set $W \subseteq [k]$, a matrix B of full row-rank which is irredundant, with column-set W, satisfies $(*)$, and is strictly balanced, and a vector b' for which the pair (B, b') is irredundant, such that if $Ax = b$, then $Bx_W = b'$.*

Note that the homogeneous case for where b, b' are zero vectors is implicitly stated in [93]. We call the pair (B, b') above the *associated pair of* (A, b), and call $B = B(A)$ the *associated matrix of* A. The consequence for us of Proposition 6.8 is that if Maker wins the (A, b)-game, then Maker also wins the (B, b')-game (since a solution to $Ax = b$ always gives rise to a solution to $Bx' = b'$). Thus in order to prove Breaker wins the (A, b)-game, it suffices to prove that Breaker wins the (B, b')-game.

With any game $(X, \mathcal{F})$ there exists an associated hypergraph $H(X, \mathcal{F})$ with vertex set X and edge set $\mathcal{F}$. Write $H(X, B, b') := H(X, \mathcal{F})$ to represent the hypergraph where X is a set of integers, and $\mathcal{F}$ is the set of all k-distinct solutions to $Bx' = b'$ (assuming that B is an $\ell \times k$ matrix). Thus we may think of the game as one where Maker and Breaker take turns claiming a vertex of the k-uniform hypergraph $H([n]_p, B, b')$ and Maker's aim is to obtain an edge of $H([n]_p, B, b')$, and Breaker's aim is to prevent this. For the remainder of the proof we will assume that we have fixed A and b (and therefore B and b'), and set $H := H([n]_p, B, b')$. We will assume that B is an $\ell \times k$ matrix, and note that since B satisfies property $(*)$, by Proposition 5.18(iv) we have $k \geqslant 3$.

Before we continue with the proof, we give a quick impression of the task at hand by considering the following example. Let $\mathcal{L}$ be $x + y = z$ and consider the hypergraph H drawn in Figure 6.1, where $a_i \in [n]_p$ for $i \in [15]$. Suppose Maker and Breaker are

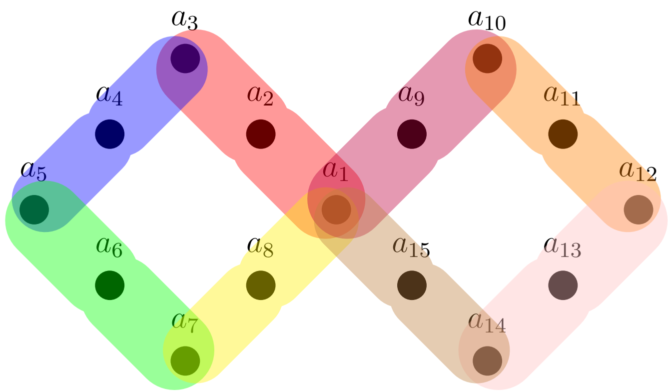


Figure 6.1: Example H: different colours are used to represent each edge.

playing the $\mathcal{L}$-game on H, (i.e. Maker must claim all of the vertices of an edge of H). Maker has a strategy to win here. First she picks a_1, then Breaker must pick from one of the two cycles; without loss of generality he picks a number from $\{a_9, \dots, a_{15}\}$. Now Maker picks a_3, enforcing Breaker to pick a_2. Next, Maker picks a_5 enforcing Breaker to pick a_4. Finally Maker picks a_7 ensuring that in her next turn she can win, since Breaker cannot select both of a_6 and a_8, meaning that Maker will obtain the sum $\{a_1, a_8, a_7\}$ or $\{a_5, a_6, a_7\}$. Since Maker has a winning strategy here, this hypergraph H should be in the list of connected subhypergraphs in **(Q2)**. We now give a sketch of why with high probability we can expect the associated hypergraph not to contain H.

We need to select numbers from $[n]_p$ to represent each of the a_i. We go through each edge in turn. For the first edge $\{a_1, a_2, a_3\}$ first pick a_1 and a_2 arbitrarily from $[n]_p$. Then note a_3 must form a solution to $x + y = z$ with a_1 and a_2 so there are at most 3 choices for it (corresponding to which role of x, y, z a_3 plays). Thus there are $O(n^2p^3)$ choices for this first edge (since each number is included with probability p). For the next edge $\{a_3, a_4, a_5\}$, note that a_3 has already been chosen. One can arbitrarily pick a_4, then there are at most 3 choices for a_5, so there are $O(np^2)$ choices for the second edge. Continuing in this manner, there is a total of $O(n^7p^{15})$ choices for H. Now observe that for $x+y=z$, the threshold is $n^{-1/2}$, so the expected number of choices for H is $o(1)$ as $n \to \infty$. Thus with high probability H does not appear in the associated hypergraph.

The overall aim of **(Q2)** and **(Q3)** is to ensure that any hypergraph for which Maker could win on does not appear in the associated hypergraph with high probability. As a comparison to the Rödl and Ruciński's proof of Theorem 5.8, note that if Breaker can win the $\mathcal{L}$-game on the associated hypergraph, then certainly there exists a 2-colouring of the vertices of the associated hypergraph such that there are no monochromatic edges. Thus finding such a colouring should be an easier task than finding a strategy for Breaker. Indeed, the deterministic lemma **(P2)** uses the fact that if a hypergraph has the property that however its vertices are 2-coloured there is always a monochromatic edge, then the hypergraph has chromatic number at least 3. Using this property it is then fairly easy to show that the hypergraph must contain one of a small list of certain connected subhypergraphs. Since the task of creating a Breaker strategy is harder, more work is required to do the analogous step **(Q2)**; in particular the list of certain subhypergraphs in **(P2)** is a proper subset of the list of certain subhypergraphs for **(Q2)**. To further demonstrate this point, note that the example in Figure 6.1 is a hypergraph for which there exists a 2-colouring such that there are no monochromatic edges (simply colour a_1, a_5, a_{12} red and the rest blue), while simultaneously being a win for Maker.

Hypergraph notation. We now introduce some notation which will be required for the deterministic and probabilistic lemmas. For the rest of Section 6.3, we assume that $k \geqslant 3$. For a k-uniform hypergraph H with edge set $E := E(H)$ and vertex set $V := V(H)$, let an *edge order* be an enumeration of the edges E. For a given edge order of E and edge $e \in E$, call a vertex $v \in e$ *new in* e if v did not appear in any edge which came before e in the edge order. Otherwise call v *old in* e. We call an edge e *good* if it has precisely one old vertex, *bad* if it has between two and $k-1$ old vertices, and k-*bad* if it has k old vertices. Note that we always consider edges to be good, bad, or k-bad with respect to a given edge order; similarly whether a vertex is new or old in a given edge also depends on the given edge order. So throughout we will make it clear which edge order we are referring to. Note that given an edge order, a vertex will always be new in precisely one edge (and old in every other edge it appears in). For ease of notation we may sometimes identify a hypergraph with an edge order of its edges, e.g. if we have $P := e_0, \ldots, e_t$, then we consider the hypergraph P to have $E(P) := \{e_0, \ldots, e_t\}$ and $V(P) := \{x \in e, e \in E(P)\}$.

Let $e_0, \ldots, e_t$ be an edge order. We call the edge order *allowed* if for all $i \in [t]$, e_i is good, bad or k-bad (that is, there is no edge e_i with $i \geqslant 1$ such that e_i is vertex-disjoint from all the edges $e_0, \ldots, e_{i-1}$). We call it *valid* if for all $i \in [t]$, e_i is good or bad. It is *simple* if for all $i \in [t]$, e_i is good. For a subset of edges $e_{f_1}, \ldots, e_{f_u}$ of $e_0, \ldots, e_t$ we do not assume $f_1 \leqslant \ldots \leqslant f_u$ unless otherwise stated. For two vertex-disjoint sets $X_1, X_2 \subseteq V(H)$, we define a *minimal path from* X_1 *to* X_2 *in* $\{e_0, \ldots, e_t\}$ to be a subset of edges $e_{f_1}, \ldots, e_{f_u}$ of $e_0, \ldots, e_t$ such that

(i) we have $X_1 \cap e_{f_1} \neq \emptyset$, and $x \notin e_{f_a}$ for any $a \geqslant 2$ and $x \in X_1$;

(ii) we have $X_2 \cap e_{f_u} \neq \emptyset$, and $x \notin e_{f_a}$ for any $a \leqslant u-1$ and $x \in X_2$;

(iii) for all $i, j \in [u]$ with $i < j$ we have $|e_{f_i} \cap e_{f_j}| \geqslant 1$ if $i = j-1$ and $|e_{f_i} \cap e_{f_j}| = 0$ otherwise.

We now give names to a variety of k-uniform hypergraphs which will appear in our deterministic and probabilistic lemmas. Suppose that $e_{f_1}, \ldots, e_{f_u}$ for some $u \in \mathbb{N}$ is a valid edge order, where $|e_{f_i} \cap e_{f_{i+1}}| \geqslant 1$ for all $i \in [u-1]$. We call $e_{f_1}, \ldots, e_{f_u}$:

- An *overlapping pair*, if $u = 2$ and $2 \leqslant |e_{f_1} \cap e_{f_2}| \leqslant k-1$;

- A *loose cycle*, if $u \geqslant 3$, and for all $i, j \in [u]$ with $i < j$ we have

$$|e_{f_i} \cap e_{f_j}| = \begin{cases} 1 \text{ if } i = j-1, \text{ or } i = 1 \text{ and } j = u; \\ 0 \text{ otherwise;} \end{cases}$$

- A *loose path*, if for all $i, j \in [u]$ with $i < j$ we have

$$|e_{f_i} \cap e_{f_j}| = \begin{cases} 1 \text{ if } i = j-1; \\ 0 \text{ otherwise;} \end{cases}$$

- A *spoiled cycle*, if $P_1 := e_{f_1}, e_{f_2}$ forms an overlapping pair, $P_2 := e_{f_3}, \dots, e_{f_u}$ forms a loose path, and P_1 and P_2 are vertex-disjoint except for two vertices $x \neq y$, where we have $x = (e_{f_2} \setminus e_{f_1}) \cap (e_{f_3} \setminus e_{f_z})$ (where $z = 4$ if $u \geqslant 4$, and $z = 1$ otherwise) and $y = (e_{f_1} \setminus e_{f_2}) \cap (e_{f_u} \setminus e_{f_{u-1}})$;

- A *double loose cycle*, if for some $v \leqslant u-2$, $P_1 := e_{f_1}, \dots, e_{f_v}$ forms a loose cycle, $P_2 := e_{f_{v+1}}, \dots, e_{f_u}$ forms a loose path, and P_1 and P_2 are vertex-disjoint except for two vertices $x \neq y$, where we have $x = (e_{f_{v+1}} \setminus e_{f_{v+2}}) \cap e_{f_v}$ and $y = (e_{f_u} \setminus e_{f_{u-1}}) \cap e_{f_a}$ for some $a \in [v]$;

- A *double overlapping pair*, if $u = 4$, e_{f_1}, e_{f_2} and e_{f_3}, e_{f_4} each form overlapping pairs, which are vertex-disjoint except for two vertices $x \neq y$, where we have $x = (e_{f_1} \setminus e_{f_2}) \cap (e_{f_4} \setminus e_{f_3})$ and $y = (e_{f_2} \setminus e_{f_1}) \cap (e_{f_3} \setminus e_{f_4})$, and $|e_{f_3} \cap e_{f_4}| \leqslant k-2$;

- An *overlapping pair with handle/loose cycle with handle*, if $e_{f_1}, \dots, e_{f_{u-1}}$ forms an overlapping pair/loose cycle and e_{f_u} is bad in the edge order $e_{f_1}, \dots, e_{f_u}$;

- An *overlapping pair/loose cycle to overlapping pair/loose cycle*, if for some $w \leqslant v < u$, $P_1 := e_{f_1}, \dots, e_{f_w}$ forms an overlapping pair or loose cycle, $P_2 := e_{f_{w+1}}, \dots, e_{f_v}$

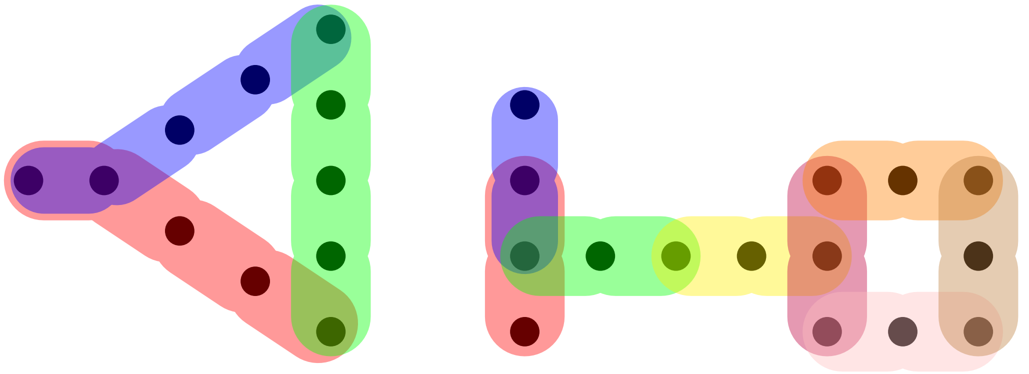
forms a loose path and $P_3 := e_{f_{v+1}}, \dots, e_{f_u}$ forms an overlapping pair or loose cycle; moreover if $w = v$ then $|V(P_1) \cap V(P_3)| = 1$; otherwise $|V(P_1) \cap V(P_2)| = 1$, $V(P_1) \cap V(P_3) = \emptyset$, $|V(P_2) \cap V(P_3)| = 1$, and additionally if $w \leqslant v - 2$, then $e_{f_{w+2}} \cap V(P_1) = \emptyset$ and $e_{f_{v-1}} \cap V(P_3) = \emptyset$.

Note that since we identify hypergraphs with one of their allowed edge orders, a hypergraph may fit the description of more than one of the above (e.g. a hypergraph could be both a spoiled cycle and an overlapping pair with handle, see Figure 6.2). We define a *bicycle* to be a hypergraph which is one of:

- a spoiled cycle;
- a double overlapping pair;
- a double loose cycle;
- an overlapping pair with handle;
- a loose cycle with handle;
- an overlapping pair to overlapping pair;
- an overlapping pair to loose cycle;
- a loose cycle to overlapping pair;
- a loose cycle to loose cycle.

Suppose that $e_{f_1}, \dots, e_{f_u}$ for some $u \in \mathbb{N}$ is an allowed edge order. We call $e_{f_1}, \dots, e_{f_u}$:

- A *Pasch configuration* if $k = 3$, $u = 4$, there are six vertices within the four edges, and each of these appear in precisely two of the edges (one vertex for each of the six pairs of edges); see Figure 6.3;



An overlapping pair with handle
Also a spoiled cycle

An overlapping pair to loose cycle

Figure 6.2: Examples of our hypergraphs: different colours are used to represent each edge.

- A *k-uniform loose u-star*, if the edges are completely disjoint except for all intersecting in one 'central vertex';

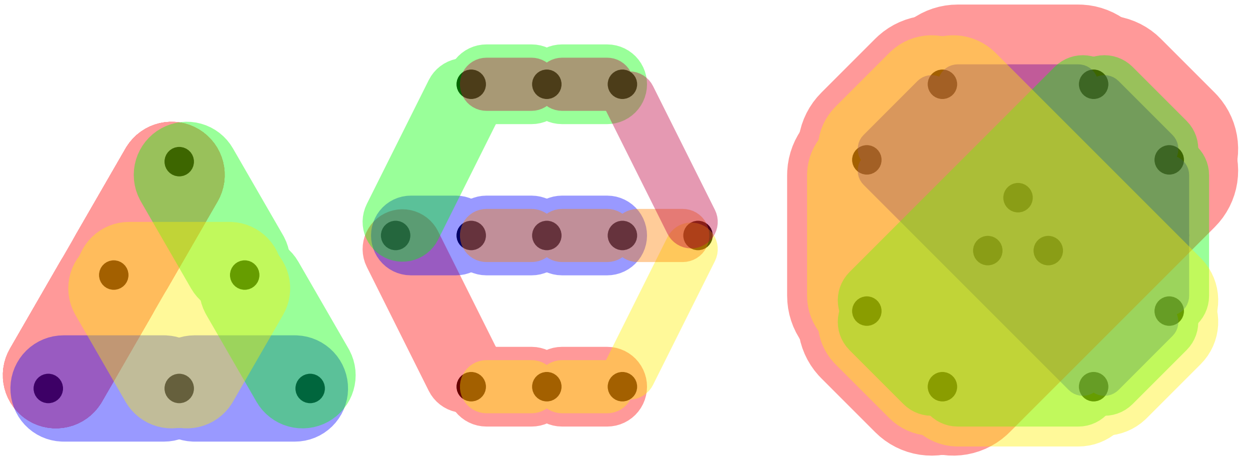
- A *$(k, u/2, 2)$-star*, if u is even and the edges form two k-uniform loose $(u/2)$-stars S_1 and S_2, and there is a bijection f between the edges of S_1 to those of S_2 such that e and $f(e)$ share all their vertices except for the two central vertices, for each edge e in S_1 (so a $(k, u, 2)$-star has one more vertex than a k-uniform loose u-star, but has twice as many edges);

- A *(k, u, a)-link*, if there are $k + a$ vertices within the u edges, and given any $i, j \in [u]$ with $i < j$ we have $|e_{f_i} \cup e_{f_j}| = k + a$ (i.e. any pair of the edges contain all $k + a$ vertices between them).

Observe that a (k, u, a)-link with $u \geqslant 3$ must have $a \leqslant \lfloor k/(u-1) \rfloor$, meanwhile a Pasch configuration is unique up to isomorphism.

Given S is any of our defined hypergraphs, we say that e_t *completes* S if the edge order $e_0, \ldots, e_{t-1}$ does not contain a copy of S, whereas $e_0, \ldots, e_t$ does.

Note that all bicycles have a valid edge order which contains at least two bad edges;



A Pasch configuration A $(4, 3, 2)$-star A $(9, 4, 2)$-link

Figure 6.3: Examples of our hypergraphs: different colours are used to represent each edge.

we will in fact show that any hypergraph which has a valid edge order with at least two bad edges must contain a bicycle (see Claim 6.12). Meanwhile Pasch configurations, $(k, u, 2)$-stars with $u \geqslant 2$ and (k, u, a)-links with $u \geqslant 3$ and $a \leqslant \lfloor k/(u-1) \rfloor$ all have the property that any allowed edge order contains at least one k-bad edge, and also precisely one bad edge; in particular these hypergraphs do not contain a bicycle. The roles of bicycles are crucial in our proof. The deterministic lemmas will imply that Breaker has a winning strategy for the game played on any component of H which does not contain a bicycle. The probabilistic lemma will show that w.h.p. H does not contain any bicycles.

(Q2) Two deterministic lemmas. Recall that we wish to show that if Maker wins the game on the associated hypergraph H, then H must contain a particular subhypergraph. (In particular, this subhypergraph will be a bicycle.) This section contains two deterministic lemmas, which together prove that the contrapositive statement holds; that is, if H does not contain a bicycle, then Breaker has a strategy to win the game on H.

Lemma 6.9. *Let H' be a connected component of H and suppose H' does not contain a* *bicycle. Then H' has an edge order $e_0, \dots, e_t$ with the property that there exists $a \in [0, t]$ such that e_i is good for all $i \in [a+1, t]$, and also precisely one of the following holds:*

(i) $a = 0$;

(ii) $a \geqslant 2$ *and* $e_0, \dots, e_a$ *forms a loose cycle;*

(iii) $a = 1$ *and* e_0, e_1 *forms an overlapping pair;*

(iv) $a = 3$, $k = 3$ *and* $e_0, \dots, e_3$ *forms a Pasch configuration;*

(v) $a \geqslant 3$ *is odd and* $e_0, \dots, e_a$ *forms a* $(k, (a+1)/2, 2)$*-star;*

(vi) $a \geqslant 2$, *and* $e_0, \dots, e_a$ *forms a* $(k, (a+1), d)$*-link, where* $d \leqslant \lfloor k/a \rfloor$.

Lemma 6.10. *Let H' be a component of H which is as described in Lemma 6.9. Breaker has a strategy for winning the Maker-Breaker game played on H'.*

Note that by Breaker always choosing a vertex from the same component as Maker if he can, these results imply that if H does not contain a bicycle, then Breaker can win the game played on H, and therefore the (B, b')-game on $[n]_p$.

We now prove four claims; the proof of Lemma 6.9 will follow easily from the statements of these claims.

Claim 6.11. *Suppose that $E_1 := e_0, \dots, e_t$ is an allowed edge order of the edges of a connected hypergraph J, for which e_i for some $i \in [t]$ is the first bad or k-bad edge. Then we have the following:*

(i) Either there exists $j \in [0, i-1]$ such that e_j, e_i forms an overlapping pair, or e_i completes a loose cycle.

(ii) Suppose that S is a connected subhypergraph of J with s edges, which contains the overlapping pair or loose cycle guaranteed by (i). Then there exists an allowed edge

order E_2 of $E(J)$ which starts with the overlapping pair or loose cycle, followed by the rest of the edges of S, followed by any remaining edges of J.

(iii) If E_1 is valid and if S in (ii) is a loose cycle or overlapping pair, then it is possible to construct E_2 in (ii) so that additionally it is valid.

Proof. For (i), if there exists $j \in [0, i-1]$ such that $|e_j \cap e_i| \geqslant 2$ then e_j, e_i forms an overlapping pair and we are done. So suppose:

(A1) For all $j \in [0, i-1]$ we have $|e_j \cap e_i| \leqslant 1$.

Also note that since e_i is the first bad or k-bad edge in E_1, we have:

(A2) For all $j \in [1, i-1]$, e_j is good in E_1.

Since e_i is bad or k-bad, it has $q \geqslant 2$ old vertices in the edge order E_1. Label these as $x_1, \dots, x_q$ and consider a minimal path $P_1 := e_{f_1}, \dots, e_{f_u}$ in $\{e_0, \dots, e_{i-1}\}$ from $X_1 := \{x_1\}$ to $X_2 := \{x_2, \dots, x_q\}$. By definition of P_1, (A1) and (A2) we have

- $|e_{f_j} \cap e_i| = \begin{cases} 1 \text{ if } j = 1 \text{ or } j = u; \\ 0 \text{ otherwise;} \end{cases}$
- $u \geqslant 2$;
- P_1 is a loose path.

It follows from these three facts that $e_{f_1}, \dots, e_{f_u}, e_i$ forms a loose cycle. By (A2) $e_0, \dots, e_{i-1}$ clearly does not contain a loose cycle, and hence e_i completes a loose cycle in $e_0, \dots, e_i$.

For (ii), such an allowed edge order E_2 exists since both S and J are connected; simply pick the overlapping pair or loose cycle first, then pick the remaining edges of S in any way so that each edge has non-empty intersection with the set of all previously chosen edges. Then pick the remaining edges of J in the same way.

For (iii), suppose E_1 is valid and S is an overlapping pair or loose cycle. Then by the definition of E_1, there are k new vertices in e_0, $k-1$ new vertices in e_a for each $a \in [i-1]$ and at least one new vertex in e_i. Thus the hypergraph $J' := e_0, \ldots, e_i$ satisfies

(A3) $|V(J')| \geqslant k + (k-1)(i-1) + 1$.

If there was an allowed edge order E_3 of $E(J')$ which contained a k-bad edge, we would have $|V(J')| \leqslant k + (k-1)(i-1)$ since E_3 has one initial edge, at most $i-1$ good or bad edges, and at least one k-bad edge. However this violates (A3) and thus:

(A4) All allowed edge orders of $E(J')$ are valid.

Now consider the edge order E_2 which starts with the loose cycle or overlapping pair, followed by the rest of the edges in $\{e_0, \ldots, e_i\}$ chosen so that each edge has non-empty intersection with the set of all previously chosen edges, then followed by $e_{i+1}, \ldots, e_t$ (in this order). First note that for any e_a with $a \in [i+1, t]$, the set of all previously chosen edges is $\{e_0, \ldots, e_{a-1}\}$ in both edge orders E_1 and E_2. Thus if e_a is good or bad in E_1, then it is also good or bad in E_2. Finally note that by (A4), the first $i+1$ edges in E_2 form a valid edge order of $E(J')$. We conclude that E_2 is the valid edge order of $E(J)$ required. $\square$

Claim 6.12. *A hypergraph J does not contain a bicycle if and only if any valid edge order of the edges of any connected subhypergraph J' of J has at most one bad edge.*

Proof. First note that if J contains a bicycle, then by considering the edge order $e_{f_1}, \ldots, e_{f_u}$ given in the definitions of each of the hypergraphs which the bicycle could be, we see immediately that J contains a connected subhypergraph which has a valid edge order with at least two bad edges.

Now we must show that if there exists a connected subhypergraph J' of J and a valid edge order of $E(J')$ with at least two bad edges, then J contains a bicycle.

So let J' be such a hypergraph, and let $E_1 := e_0, \dots, e_t$ be the valid edge order of $E(J')$ with at least two bad edges. By using all three parts of Claim 6.11, we may assume without loss of generality that there exists $i \in [t]$ such that we have precisely one of the following:

(B1) $i = 1$ and e_0, e_1 forms an overlapping pair;

(B2) $e_0, \dots, e_i$ forms a loose cycle (with edges ordered cyclically).

Let $P_1 := e_0, \dots, e_i$ and let $j \in [i+1, t]$ be such that e_j is the next bad edge in E_1 after e_i. We have precisely one of the following:

(B3) $2 \leqslant |e_j \cap V(P_1)| \leqslant k-1$;

(B4) There exists $a \in [i+1, j-1]$ such that $|e_j \cap e_a| \geqslant 2$ and $x = e_j \cap V(P_1)$ and $y = e_a \cap V(P_1)$ are distinct vertices;

(B5) There exists $a \in [i+1, j-1]$ such that $|e_j \cap e_a| \geqslant 2$ and $|(e_a \cup e_j) \cap V(P_1)| = 1$;

(B6) There exists $a \in [i+1, j-1]$ such that $|e_j \cap e_a| \geqslant 2$ and $(e_a \cup e_j) \cap V(P_1) = \emptyset$;

(B7) For all $a \in [0, j-1]$ we have $|e_j \cap e_a| \leqslant 1$, and $|e_j \cap V(P_1)| = 1$;

(B8) For all $a \in [0, j-1]$ we have $|e_j \cap e_a| \leqslant 1$, and $e_j \cap V(P_1) = \emptyset$.

For each case, it suffices to find a subhypergraph of J' which is a bicycle. Throughout we will make use of the following fact:

(B9) For all $a \in [j-1] \setminus \{i\}$, e_a is good in E_1.

Case 1: (B3) holds. Clearly $e_0, \dots, e_i, e_j$ forms an overlapping pair/loose cycle with handle.

Case 2: (B4) holds. We have precisely one of the following:

- There exists $d \in [0, i]$ such that $x, y \in e_d$; then e_j, e_a, e_d forms an overlapping pair with handle.

- There does not exist $d \in [0, i]$ such that $x, y \in e_d$; without loss of generality suppose that $x \in e_0 \setminus e_1$. If P_1 is an overlapping pair, then e_0, e_1, e_a, e_j forms a double overlapping pair. (Note that $|e_a \cap e_j| \leqslant k - 2$ since otherwise e_j would be k-bad in the edge order E_1.) If P_1 is a loose cycle, then let $d \in [i]$ be the smallest integer such that $y \in e_d$. If $d = i$, then e_a, e_j, e_0, e_i forms a spoiled cycle; otherwise $e_a, e_j, e_0, \dots, e_d$ forms a spoiled cycle.

Case 3: (B5) holds. Let $P_2 := e_a, e_j$ and note that $|V(P_2) \cap V(P_1)| = 1$. It follows that $e_0, \dots, e_i, e_a, e_j$ forms an overlapping pair/loose cycle to overlapping pair.

Case 4: (B6) holds. Again let $P_2 := e_a, e_j$ and note that we have $V(P_2) \cap V(P_1) = \emptyset$. So consider a minimal path $P_3 := e_{f_1}, \dots, e_{f_u}$ in $\{e_{i+1}, \dots, e_{j-1}\} \setminus e_a$ from $X_1 := V(P_1)$ to $X_2 := V(P_2)$. By (B9) P_3 is a loose path, moreover by definition of P_3, we have $|V(P_1) \cap V(P_3)| = 1$ and $|V(P_2) \cap V(P_3)| = 1$. Additionally if $u \geqslant 2$, then $e_{f_2} \cap V(P_1) = \emptyset$ and $e_{f_{u-1}} \cap V(P_2) = \emptyset$. Thus P_1, P_3 and P_2 together form an overlapping pair/loose cycle to overlapping pair.

Case 5: (B7) holds. Since e_j has at least one old vertex which is not in $e_0, \dots, e_i$, we may consider a minimal path $P_2 := e_{f_1}, \dots, e_{f_u}$ in $\{e_{i+1}, \dots, e_{j-1}\}$ from $X_1 := V(P_1)$ to $X_2 := e_j \setminus V(P_1)$. First note by (B9) that P_2 is a loose path. Now we have precisely one of the following:

- We have $(e_{f_1} \cap e_j \cap V(P_1)) \neq \emptyset$; then $P_3 := e_{f_1}, \dots, e_{f_u}, e_j$ forms a loose cycle. Now since $|V(P_1) \cap V(P_3)| = 1$, P_1 and P_3 together form an overlapping pair/loose cycle to loose cycle.

- There exists $d \in [0, i]$ such that $x = e_j \cap e_d$ and $y = e_{f_1} \cap e_d$ are distinct vertices; then $P_3 := e_d, e_{f_1}, \dots, e_{f_u}, e_j$ forms a loose cycle. If P_1 is an overlapping pair, then let

$d' \in \{0,1\}$ be such that $d' \neq d$. Then P_3 together with $e_{d'}$ forms a loose cycle with handle. If P_1 is a loose cycle, then let P_4 be the loose path $e_{f_1}, \ldots, e_{f_u}, e_j$ and define $z := j$ if $u = 1$ and $z := f_2$ otherwise. Observe that P_1 and P_4 are vertex-disjoint except for $x = (e_j \setminus e_{f_u}) \cap e_d$ and $y = (e_{f_1} \setminus e_z) \cap e_d$, and thus together form a double loose cycle.

- We have $x = e_j \cap V(P_1)$ and $y = e_{f_1} \cap V(P_1)$ are not together in any edge of P_1; without loss of generality suppose that $x \in e_0 \setminus e_1$. If P_1 is an overlapping pair, then $e_0, e_1, e_{f_1}, \ldots, e_{f_u}, e_j$ forms a spoiled cycle. If P_1 is a loose cycle, then define P_4, z as in the previous bullet point. Then observe that P_1 and P_4 are vertex-disjoint except for $x = (e_j \setminus e_{f_u}) \cap e_0$ and $y = (e_{f_1} \setminus e_z) \cap e_d$ for some $d \in [i]$, and thus as before, form a double loose cycle.

Case 6: (B8) holds. Label the $q \geqslant 2$ old vertices of e_j as $x_1, \ldots, x_q$ and consider a minimal path $P_2 := e_{f_1}, \ldots, e_{f_u}$ in $\{e_{i+1}, \ldots, e_{j-1}\}$ from $X_1 := V(P_1)$ to $X_2 := \{x_1, \ldots, x_q\}$. Let $x_a := e_{f_u} \cap e_j$ and consider a minimal path $P_3 := e_{f_{u+1}}, \ldots, e_{f_v}$ in $\{e_{i+1}, \ldots, e_{j-1}\} \setminus E(P_2)$ from $X_3 := X_2 \setminus x_a$ to $X_4 := V(P_1) \cup V(P_2)$. By (B9) both P_2 and P_3 are loose paths. Now we have precisely one of the following:

- We have $e_{f_v} \cap V(P_1) = \emptyset$; let $d \in [u]$ be the largest integer such that $e_{f_d} \cap e_{f_v} \neq \emptyset$. Then $P_4 := e_{f_d}, \ldots, e_{f_u}, e_j, e_{f_{u+1}}, \ldots, e_{f_v}$ forms a loose cycle. If $d = 1$, then since $|V(P_1) \cap V(P_4)| = 1$, we have that P_1 and P_4 together form an overlapping pair/loose cycle to loose cycle. Otherwise let $P_5 := e_{f_1}, \ldots, e_{f_{d-1}}$. Then we have $|V(P_1) \cap V(P_5)| = 1$, $|V(P_4) \cap V(P_5)| = 1$ and $V(P_1) \cap V(P_4) = \emptyset$. Additionally, if $d \geqslant 3$, then we see that $e_{f_2} \cap V(P_1) = \emptyset$ and $e_{f_{d-2}} \cap V(P_4) = \emptyset$. Thus P_1, P_4, P_5 form an overlapping pair/loose cycle to loose cycle.

- We have $e_{f_v} \cap V(P_1) \neq \emptyset$; the properties of the hypergraph

$$e_0, \dots, e_i, e_{f_1}, \dots, e_{f_u}, e_j, e_{f_{u+1}}, \dots, e_{f_v}$$

are identical to that the hypergraph $e_0, \dots, e_i, e_{f_1}, \dots, e_{f_u}, e_j$ found in Case 5 (up to the labelling of the edges), so a similar case study yields a bicycle.

$\square$

For the remainer of the proof, we shall call a valid edge order which contains at least two bad edges a *bad edge order*. If a hypergraph J does not contain a bicycle, then by Claim 6.12, the existence of a bad edge order of $E(J')$ where J' is a subhypergraph of J is a contradiction. In the claims which follow we will always assume that J does not contain a bicycle, and hence whenever some assumed condition of a case within a case analysis leads to the discovery of a bad edge order, we can immediately stop and move onto the next case.

Claim 6.13. *Let S be a hypergraph with s edges, which is an overlapping pair, a loose cycle, a $(k, s/2, 2)$-star with $s \geqslant 4$, a (k, s, a)-link with $s \geqslant 3$ and $a \in [\lfloor k/(s-1) \rfloor]$, or a Pasch configuration. Suppose J is a connected hypergraph which does not contain a bicycle, and does contain S. Then there exists an allowed edge order $E_1 := e_0, \dots, e_t$ of $E(J)$ such that*

(i) $e_0, \dots, e_i$ forms an overlapping pair or loose cycle, $e_0, \dots, e_{s-1}$ forms S, and every edge e_j for $j \in [s, t]$ is either good or k-bad;

(ii) For all $j \in [s+1, t]$, if e_j is k-bad, then either e_{j-1} is also k-bad, or there exists a vertex $x \in e_j$ which is new in e_{j-1}.

Proof. For (i), by Claim 6.11(ii) we can assume that the edge order starts with the overlapping pair or loose cycle, followed by the rest of the edges of S, and that e_i is bad.

If there exists another bad edge e_j for some $j \in [s,t]$, then the edge order E_2, found by deleting from $e_0, \ldots, e_j$ all k-bad edges, is bad. Thus for all $j \in [s,t]$, e_j must either be good or k-bad.

For (ii) suppose that $E_1 = e_0, \ldots, e_t$ does not satisfy the property stated in (ii). Then we have the following:

(C4) There exists $j \in [s+1,t]$ such that e_{j-1} is good and e_j is k-bad, and all vertices in e_j appeared in the edge order before e_{j-1}.

Now consider the edge order $E_2 := e_0, \ldots, e_{j-2}, e_j, e_{j-1}, e_{j+1}, \ldots, e_t$. In this order, e_j is still k-bad and e_{j-1} is still good; moreover E_2 still starts with the overlapping pair or loose cycle. Hence by continuously performing swaps whenever such a pair e_{j-1} and e_j exists (satisfying (C4)), we eventually reach an edge order E_p where no such pair exists. Thus in the final edge order E_p the property stated in (ii) holds. $\square$

Claim 6.14. *Let S be a hypergraph with s edges, which is an overlapping pair, a loose cycle, a $(k, s/2, 2)$-star with $s \geqslant 4$, a (k,s,a)-link with $s \geqslant 3$ and $a \in [\lfloor k/(s-1) \rfloor]$, or a Pasch configuration. Suppose J is a connected hypergraph which does not contain a bicycle, and does contain S. Finally suppose $E_1 := e_0, \ldots, e_t$ is the allowed edge order of $E(J)$ guaranteed by Claim 6.13, which starts with the edges of S, in particular with the overlapping pair or loose cycle $P_1 := e_0, \ldots, e_i$. Suppose that E_1 contains at least one k-bad edge amongst the edges $e_s, \ldots, e_t$. Then we have precisely one of the following:*

(i) We have that e_0, e_1, e_{j-1}, e_j forms a $(k,2,2)$-star for some $j \in [s+1,t]$. Moreover either S is an overlapping pair, or S is a $(k,s/2,2)$-star with $s \geqslant 4$ and $e_0, \ldots, e_{s-1}, e_{j-1}, e_j$ forms a $(k, s/2+1, 2)$-star.

(ii) We have that e_0, e_1, e_s forms a $(k,3,a)$-link, where $a = |e_0 \setminus e_1|$. Moreover either S is an overlapping pair, or S is a (k,s,a)-link with $s \geqslant 3$ and $e_0, \ldots, e_s$ forms a $(k, s+1, a)$-link.

(iii) *We have that S is a loose cycle with three edges, $k = 3$, and e_0, e_1, e_2, e_3 forms a Pasch configuration.*

Proof. Suppose that e_j is the first k-bad edge amongst the edges $e_s, \dots, e_t$ (so if $j > s$ then $e_s, \dots, e_{j-1}$ are good). Let $e_j := \{x_1, \dots, x_k\}$ and let $e_{f_1}, \dots, e_{f_k}$ be the respective edges in which each x_i is new in E_1, noting that without loss of generality we have $f_1 \leqslant \dots \leqslant f_k$.

We have precisely one of the following:

(D1) We have $j \in [s+1, t]$;

(D2) We have $j = s$ and $P_1 = e_0, e_1$ is an overlapping pair;

(D3) We have $j = s$ and $P_1 = e_0, \dots, e_i$ is a loose cycle.

We will go through each of these cases in turn and show that Claims 6.14(i), (ii) and (iii) hold respectively.

Case 1: (D1) holds. Without loss of generality, we have precisely one of the following:

(D4) We have $2 \leqslant |e_j \cap V(P_1)| \leqslant k-1$;

(D5) We have $|e_j \cap V(P_1)| \leqslant 1$ and for all $a \in [i+1, j-1]$ we have $|e_j \cap e_a| \leqslant 1$;

(D6) There exists $a \in [i+1, j-1]$ such that $|e_j \cap e_a| \geqslant 2$ and $|(e_j \cup e_a) \cap V(P_1)| \leqslant 1$;

(D7) There exists $a \in [i+1, j-1]$ such that $|e_j \cap e_a| \geqslant 2$ and $x = e_j \cap V(P_1)$ and $y = e_a \cap V(P_1)$ are distinct vertices.

Case 1a: (D4) holds. We have that $e_0, \dots, e_i, e_a$ forms an overlapping pair/loose cycle with handle, a contradiction to J not containing a bicycle.

Case 1b: (D5) holds. Since we have $|V(P_1) \cap e_j| \leqslant 1$ and for all $a \in [i+1, j-1]$ we have $|e_j \cap e_a| \leqslant 1$, we obtain

(D8) $f_2 > i$;

(D9) $f_1 < \cdots < f_k$.

Then consider the edge orders $E_2 := e_0, \ldots, e_{f_2}, e_j$ and E_3, which is formed by deleting from E_2 each edge which is k-bad. We will show that E_3 is a bad edge order. First note that clearly E_3 is a valid edge order since all k-bad edges were deleted. By (D8), E_2 and hence also E_3 both start with the edges of P_1, so have at least one bad edge. Further e_j is bad in E_2 since precisely two of the vertices in e_j are old in E_2, namely x_1 and x_2, which appear in e_{f_1} and e_{f_2} respectively. Since e_{f_1} and e_{f_2} both contain a new vertex, they are not k-bad, and so are both contained in E_3. Thus e_j is also bad in E_3, and so E_3 is indeed a bad edge order.

Case 1c: (D6) holds. Let $P_2 := e_a, e_j$ and note that if $|V(P_1) \cap V(P_2)| = 1$, then $V(P_1)$ and $V(P_2)$ together form an overlapping pair/loose cycle to overlapping pair, a contradiction to J not containing a bicycle. Otherwise we have $V(P_1) \cap V(P_2) = \emptyset$, so consider a minimal path $P_3 := e_{g_1}, \ldots, e_{g_u}$ in $\{e_d : d \in [i+1, j-1], d \neq a, e_d \text{ is good in } E_1\}$ from $X_1 := V(P_1)$ to $X_2 := V(P_2)$. By the choice of where the edges in P_3 are selected from, P_3 is a loose path. Additionally we have $|V(P_1) \cap V(P_3)| = 1$, $|V(P_2) \cap V(P_3)| = 1$, and if $u \geqslant 2$, then $V(P_1) \cap e_{g_2} = \emptyset$ and $V(P_2) \cap e_{g_{u-1}} = \emptyset$. Thus P_1, P_2, P_3 form an overlapping pair/loose cycle to overlapping pair, a contradiction to J not containing a bicycle.

Case 1d: (D7) holds. First suppose that P_1 is a loose cycle and without loss of generality that $x \in e_0 \setminus e_i$. If $y \in e_0$, then let $E_2 := e_a, e_j, e_0$. If $y \in e_i \setminus e_0$, then let $E_2 := e_a, e_j, e_0, e_i$. Otherwise let $E_2 := e_a, e_j, e_0, \ldots, e_i$. For each case e_j and the last edge are both bad edges in E_2. Further it is easy to see that in each case, E_2 is valid, and thus E_2 is bad.

Now suppose that P_1 is an overlapping pair. If x and y are both in e_d for $d = 0$ or $d = 1$, then e_j, e_a, e_d is a bad edge order. So suppose without loss of generality that

$x \in e_1 \setminus e_0$, and $y \in e_0 \setminus e_1$. If $|e_j \cap e_a| \leqslant k-2$, then e_0, e_1, e_j, e_a forms a double overlapping pair. Similarly if $|e_0 \cap e_1| \leqslant k-2$, then e_j, e_a, e_0, e_1 forms a double overlapping pair. Both would contradict J not containing a bicycle, and thus we have $|e_j \cap e_a| = |e_0 \cap e_1| = k-1$. Hence we also have $x = e_1 \setminus e_0 = e_j \setminus e_a$ and $y = e_0 \setminus e_1 = e_a \setminus e_j$ and so in particular, we have that e_0, e_1, e_j, e_a forms a $(k, 2, 2)$-star.

The conclusion of our case analysis is that there exists $a \in [2, j-1]$ such that e_0, e_1, e_a, e_j forms a $(k, 2, 2)$-star and that P_1 is an overlapping pair where $x = e_1 \setminus e_0$ and $y = e_0 \setminus e_1$.

Since E_1 is the edge order obtained from Claim 6.13, we have e_j comes immediately after the edge for which the last of the vertices of e_j are new, or following another k-bad edge. Therefore, since e_{j-1} is not k-bad and $e_j \subseteq (e_0 \cup e_1 \cup e_a)$, we conclude that $a = j-1$.

Note that S cannot be a loose cycle or Pasch configuration, since P_1 is an overlapping pair. If S is a (k, s, d)-link for some $s \geqslant 3$, then we have $e_2 \cap (e_{j-1} \cup e_j) = \{x, y\}$ and so the edge order e_{j-1}, e_j, e_2 is bad. If S is an overlapping pair, then there is nothing further to prove. Finally if S is a $(k, s/2, 2)$-star, then clearly $e_0, \ldots, e_{s-1}, e_{j-1}, e_j$ forms a $(k, s/2+1, 2)$-star (with central vertices x and y).

Case 2: (D2) holds. First note that S cannot be a loose cycle or Pasch configuration, since P_1 is an overlapping pair. We have precisely one of the following:

(D10) We have $|e_s \cap (e_0 \cup e_1)| = k$;

(D11) We have $2 \leqslant |e_s \cap (e_0 \cup e_1)| \leqslant k-1$;

(D12) We have $|e_s \cap (e_0 \cup e_1)| \leqslant 1$.

Case 2a: (D10) holds. First suppose that $|e_s \cap e_1| \leqslant 1$. Since e_0, e_1 is an overlapping pair, we have

$$k = |e_s| = |e_s \cap e_1| + |e_s \cap (e_0 \setminus e_1)| \leqslant k - 1,$$

a contradiction, and hence we must have $|e_s \cap e_1| \geqslant 2$. Similarly $|e_s \cap e_0| \geqslant 2$. Thus any permutation of the edges e_0, e_1, e_s must have that the second edge is bad and the third edge is bad or k-bad. In order to not obtain a bad edge oder, we must have that the third edge is k-bad in all of these permutations. Thus by definition e_0, e_1, e_s forms a $(k, 3, |e_0 \setminus e_1|)$-link. If S is an overlapping pair, then there is nothing further to prove. If S is a $(k, s/2, 2)$-star for some $s \geqslant 4$, then without loss of generality we have $x = e_0 \backslash e_1 = e_2 \backslash e_3$ and $y = e_1 \backslash e_0 = e_3 \backslash e_2$. But then since $x, y \in e_s$, the edge order e_2, e_3, e_s is bad. Finally suppose that S is a (k, s, a)-link. Then for all $d, d' \in [0, s-1]$ with $d < d'$, we have that $e_d, e_{d'}$ forms an overlapping pair, and $|e_d \setminus e'_d| = |e_0 \setminus e_1| = a$. By repeating the argument above for the permutations of e_0, e_1, e_s, we see that any permutation of $e_d, e_{d'}, e_s$ must have that the second edge is bad and the third is k-bad. Thus $e_0, \ldots, e_s$ forms a $(k, s+1, |e_0 \setminus e_1|)$-link.

Case 2b: (D11) holds. We have that e_0, e_1, e_s forms an overlapping pair with handle, a contradiction to J not containing a bicycle.

Case 2c: (D12) holds. First note that since e_s contains vertices outside of $e_0 \cup e_1$, we have that S cannot be an overlapping pair or a $(k, s, |e_0 \setminus e_1|)$-link, and thus S must be a $(k, s/2, 2)$-star. If there exists i, j such that $|e_i \cap e_j| = k - 1$ and $|e_s \cap (e_i \cup e_j)| \geqslant 2$, then repeat the argument from Case 2a or 2b with e_0 and e_1 replaced by e_i and e_j. For the remaining case we have that for all i, j such that $|e_i \cap e_j| = k - 1$, we have $|e_s \cap (e_i \cup e_j)| \leqslant 1$ and in particular, the central vertices of the two stars are not in e_s. Now without loss of generality, suppose e_2 and e_3 is an overlapping pair, $|e_s \cap (e_0 \cap e_1)| = 1$ and $|e_s \cap (e_2 \cap e_3)| = 1$. Then the edge order e_0, e_1, e_2, e_s is a bad edge order.

Case 3: (D3) holds. Since S contains a loose cycle, S cannot be an overlapping pair, $(k, s/2, 2)$-star or a (k, s, a)-link, and hence S is a loose cycle or Pasch configuration. If S is the latter, then we have $j = 4$, and by the symmetry of the Pasch configuration, we have without loss of generality that $|e_0 \cap e_4| = 2$ and $|e_1 \cap (e_0 \cup e_4)| = 2$. Hence e_0, e_4, e_1 is a bad edge order. Thus S must be a loose cycle and $j = i + 1$. Without loss of generality we have precisely one of the following:

- $|e_0 \cap e_{i+1}| \geqslant 2$ and $(e_i \setminus e_0) \cap e_{i+1} = \emptyset$; let $E_2 := e_{i+1}, e_0, \ldots, e_i$. For all $a \in [i]$, we have $|(e_a \setminus e_{a-1}) \cap e_{i+1}| \leqslant k - 2$, and thus

$$|e_a \cap (e_{a-1} \cup e_{i+1})| \leqslant |e_{a-1} \cap e_a| + |(e_a \setminus e_{a-1}) \cap e_{i+1}| \leqslant k - 1.$$

  Thus E_2 is a valid edge order. Further e_0 is bad in E_2. Since e_i completes the cycle $e_0, \ldots, e_i$, it is also bad in E_2, and thus E_2 is a bad edge order.

- $|e_0 \cap e_{i+1}| \geqslant 2$ and $(e_i \setminus e_0) \cap e_{i+1} \neq \emptyset$; let $E_2 := e_{i+1}, e_0, e_i$. Here we have

$$2 \leqslant |e_i \cap (e_{i+1} \cup e_0)| \leqslant k - 1$$

  and so E_2 is a bad edge order.

- For all $a \in [0, i]$, we have $|e_{i+1} \cap e_a| \leqslant 1$ and $(e_i \setminus (e_0 \cup e_{i-1})) \cap e_{i+1} = \emptyset$; We have that (D9) holds, and thus $E_2 := e_0, \ldots, e_{f_2}, e_{i+1}, e_{f_2+1}, \ldots, e_i$ is a bad edge order.

- For all $a \in [0, i]$, we have $|e_{i+1} \cap e_a| \leqslant 1$, $|(e_a \setminus \{e_d : d \in [0, i], d \neq a\}) \cap e_{i+1}| = 1$ and $k \geqslant 4$; Then $E_2 := e_0, e_1, e_{i+1}, e_2$ is a bad edge order.

- For all $a \in [0, i]$, we have $|e_{i+1} \cap e_a| \leqslant 1$, $|(e_a \setminus \{e_d : d \in [0, i], d \neq a\}) \cap e_{i+1}| = 1$ and $k = 3$. Then we have $i = 2$ and e_0, e_1, e_2, e_3 forms a Pasch configuration.

Only the last case does not produce a bad edge order, and thus we have that e_0, e_1, e_2, e_3 forms a Pasch configuration, as required. □

We are now ready to prove our two deterministic lemmas.

Proof of Lemma 6.9. If there exists a simple edge order of $E(H')$, then we have (i), so suppose that any edge order of $E(H')$ contains at least one bad or k-bad edge. Now by Claim 6.11 we may assume that $E_1 := e_0, \ldots, e_t$ is an allowed edge order of $E(H')$ which starts with an overlapping pair or loose cycle $P_1 := e_0, \ldots, e_i$. Then using Claim 6.13 applied with $S := P_1$, we have that every edge $e_{i+1}, \ldots, e_t$ is either k-bad or good. If all of these edges are good, then we have (ii) or (iii), so are done. So assume that there is at least one k-bad edge. Then using Claim 6.14 applied with $S := P_1$, we have that H' contains a $(k, 2, 2)$-star, a $(k, 3, |e_0 \setminus e_1|)$-link or a Pasch configuration. We can now repeatedly use Claims 6.13 and 6.14 as follows:

(a) Let S be the $(k, p, 2)$-star, a $(k, p+1, |e_0 \setminus e_1|)$-link or a Pasch configuration found previously (where $p \geqslant 2$). By Claim 6.13, there exists an allowed edge order of $E(H')$ which starts with all of the edges of S. If there are no further k-bad edges, then we have (iv), (v) or (vi). If there are, then move to step (b).

(b) By Claim 6.14, either S was a $(k, p, 2)$-star and H' contains a $(k, p+1, 2)$-star, or S was a $(k, p+1, |e_0 \setminus e_1|)$-link and H' contains a $(k, p+2, |e_0 \setminus e_1|)$-link. Now return to step (a).

Since H' is a finite hypergraph, this process must eventually stop, and hence we have (iv), (v) or (vi). $\square$

For the proof of Lemma 6.10, we simply find an explicit strategy for Breaker to win the game played on H'.

Proof of Lemma 6.10. By Lemma 6.9, H' may contain a subhypergraph S for which there exists an edge order which starts with all of the edges of S, and all subsequent edges are good: The subhypergraph S if it exists must be

Case 1: A $(k, u, 2)$-star for some integer $u \geqslant 2$;

Case 2: A (k,u,a)-link for some $u,a \in \mathbb{N}$ with $u \geqslant 3$ and $a \leqslant \lfloor k/(u-1) \rfloor$;

Case 3: A Pasch configuration;

Case 4: A loose cycle;

Case 5: An overlapping pair.

Let the edges of S be $e_0, \dots, e_{s-1}$, and the rest of the good edges $e_s, \dots, e_t$. (If S does not exist, then set $s = 0$.)

Breaker uses the following strategy.

- If Maker selects a vertex in S, then Breaker does the following, corresponding to the cases above for what S could be.

  Case 1: There are $2u$ edges; suppose without loss of generality that they are labelled so that e_i and e_{i+u} have intersection $k-1$ for each $i \in [0, u-1]$. Suppose Maker has selected a vertex in $e_j \cap e_{j+u}$ for $j \in [0, u-1]$. Then Breaker if he can, also selects such a vertex. Otherwise he selects an arbitrary vertex.

  Case 2: Breaker selects an arbitrary vertex in S if he can. Otherwise he selects an arbitrary vertex.

  Case 3: If Maker has two out of three vertices from one of the edges of S, Breaker chooses the final vertex from this edge. Otherwise Breaker chooses an arbitrary vertex in S if he can. If he cannot then he chooses an arbitrary vertex.

  Case 4: Assume without loss of generality that the edges are ordered cyclically $e_0, \dots, e_{s-1}$. If Maker has selected an element from $e_i \setminus e_j$ where $i \in [s-1]$ and $j = i-1$, or $i = 0$ and $j = s-1$, then Breaker if he can, also selects such a vertex. Otherwise he selects an arbitrary vertex.

  Case 5: These two edges are e_0 and e_1. If Maker has selected an element from $e_0 \cap e_1$, then Breaker if he can, also selects such a vertex. Otherwise he selects an arbitrary vertex.

- If Maker selects any other vertex, let i be such that e_i is the edge in which this vertex is new. If Breaker can, he also selects a vertex which is new in e_i. Otherwise he selects an arbitrary vertex.

We must now show that at the end of the game, Maker has failed to claim every vertex of any edge in H'. First observe that Maker has failed to claim any of the edges $e_s, \dots, e_t$. Indeed, let $i \in [s,t]$. In the edge order, e_i is good, and so there exists at least $k-1 \geqslant 2$ vertices x_i and y_i which are new in e_i (they do not appear in any edge e_j for $j < i$). By part two of the strategy above, Maker cannot claim all of the vertices of e_i since as soon as she tries to claim one of the at least two new vertices in e_i, Breaker will claim another new vertex in e_i. Thus if Maker has won the game, she must have claimed an edge from S. However, we will now run through each case, corresponding to the cases for S in Breaker's strategy above, showing that Maker has not claimed such an edge.

Case 1: Let $j \in [0, u-1]$ and suppose Maker is trying to claim e_j or e_{j+u}. There are $k-1 \geqslant 2$ vertices in $e_j \cap e_{j+u}$ and so Maker cannot claim all of the vertices of e_j or e_{j+u} since as soon as she tries to claim one of the vertices which lie in $e_j \cap e_{j+u}$, Breaker will claim another vertex in $e_j \cap e_{j+u}$. Since all edges of S are of this form, Maker cannot claim any edge of S.

Case 2: Note that a (k,u,a)-link has at most $2k-2$ vertices. Hence by Breaker always claiming any vertex in S whenever Maker does, he ensures that Maker can claim at most $\lceil (2k-2)/2 \rceil = k-1$ of the vertices in S, therefore does not have enough to claim a full edge of S.

Case 3: Breaker always tries to claims a vertex in S if Maker does, hence Maker claims at most three of the six vertices in S. Note that any pair of vertices in S lie together in at most one edge. Hence by Breaker selecting the third vertex of an edge if Maker has selected the first two, Maker is never able to claim all three vertices of an edge of S.

Case 4: Suppose Maker is trying to claim e_i for some $i \in [0, s-1]$. Let $j = i-1$ if $i \geqslant 1$, and let $j = s-1$ if $i = 0$. There are $k-1 \geqslant 2$ vertices in $e_i \setminus e_j$ and so Maker cannot claim all of the vertices of e_i since as soon as she tries to claim one of the vertices in $e_i \setminus e_j$, Breaker will claim another vertex in $e_i \setminus e_j$. Since e_i was arbitrary, Maker cannot claim any edge of S.

Case 5: There are at least two vertices in $e_0 \cap e_1$ and so Maker cannot claim all of the vertices in e_0 or e_1 since as soon as she tries to claim one of the vertices in $e_0 \cap e_1$, Breaker will claim another vertex in $e_0 \cap e_1$. □

(Q3) A probabilistic lemma.

Lemma 6.15. *Suppose B is a strictly balanced full rank $\ell \times k$ matrix and suppose ℓ divides $k-1$. Then there exists a positive constant c such that if $p < cn^{-1/m(A)}$, then with high probability H does not contain a bicycle.*

Proof. Let R_{b_t} be a random variable counting the number of bicycles which are in H. By Proposition 1.1, it suffices to show that the expectation of R_{b_t} converges to zero as n tends to infinity.

We let $R_{b_1}, \ldots, R_{b_8}$ respectively count the number of hypergraphs $J := e_{f_1}, \ldots, e_{f_u}$ in H with $u \leqslant \log n$, for which J corresponds to

(i) a spoiled cycle;

(ii) a double overlapping pair;

(iii) a double loose cycle;

(iv) an overlapping pair with handle;

(v) a loose cycle with handle;

(vi) a loose cycle to loose cycle;

(vii) an overlapping pair to overlapping pair;

(viii) an overlapping pair to loose cycle.

Note that each of these hypergraphs contain a loose path of length $u-2$; hence let R_{b_9} count the number of loose paths $e_{f_1}, \ldots, e_{f_u}$ in H with $u \geqslant (\log n) - 1$. Then we have $R_{b_t} \leqslant \sum_{i=1}^{9} R_{b_i}$ and hence it suffices to show $\mathbb{E}(R_{b_i}) = o(1)$ for each i. The cases for $i = 1, 5, 9$ were covered by Rödl and Ruciński's proof, however we will repeat them here for clarity.

Suppose that $J := e_{f_1}, \ldots, e_{f_u}$ is the valid edge order corresponding to one of the nine cases listed above given by the definitions earlier. When calculating an upper bound on the expected number of copies of some hypergraph J in H, we need to first bound the number of ways to *draw* J (i.e. bound the number of non-isomorphic hypergraphs which J could be - e.g. for a spoiled cycle $e_{f_1}, \ldots, e_{f_u}$ we need to choose the size of the intersection $e_{f_1} \cap e_{f_2}$, and also the number of edges u). Second, we should consider J as being drawn, and bound the number of ways to pick elements from $[n]_p$ to represent each vertex of J. Thus we are interested in bounding the number of ways of drawing each J and also the number of ways of choosing representatives from $[n]_p$ for each vertex of J.

Each hypergraph J which we wish to count can be written as a union of at most three hypergraphs P_1, P_2, P_3, for which each of these are one of an overlapping pair, loose cycle, or loose path. Further, if P_2 and P_3 exist, we have $|V(P_1) \cap V(P_2)| \leqslant k-1$, $|V(P_2) \cap V(P_3)| \leqslant k-1$ and $|V(P_1) \cap V(P_3)| = \emptyset$. Thus for each $i \in [2]$ we have at most $(|V(P_i)| \cdot |V(P_{i+1})|)^{k-1}$ choices for how to make P_i and P_{i+1} intersect. There is only one way to draw a loose cycle or loose path of given length, and at most $k-2$ ways to draw an overlapping pair. Further for each J in (i)–(viii), we have $|V(J)| \leqslant k \log n$. Thus the total number of ways of drawing each J in (i)–(viii) is at most polylogarithmic in n.

Recall that B is a strictly balanced matrix of dimension $\ell \times k$, and hence for every

$W \subseteq [k]$ for which $2 \leqslant |W| < k$ we have

$$\frac{|W|-1}{|W|-1+\mathrm{rank}(B_{\overline{W}})-\ell} < \frac{k-1}{k-1-\ell}. \tag{6.3.1}$$

Additionally we have $m(B) = \frac{k-1}{k-1-\ell}$. Now let $p < cn^{-1/m(B)} = cn^{-(k-\ell-1)/(k-1)}$ where we choose c to be a constant satisfying

$$c < 1/(ke^2). \tag{6.3.2}$$

Given $i \in [u]$ and $J = e_{f_1}, \ldots, e_{f_u}$ has been drawn, we wish to bound the expected number of ways of picking e_{f_i} to be an edge with q old elements. Such an edge represents a solution x to $Bx = b'$, where q of the x_i have already been chosen. Let these indices be W; we are now attempting to solve $B_{\overline{W}}x' = b''$ for some vector b'' of $k-q$ elements. Note also we must choose one of the $q!$ possible assignments of the q indices in W to the q old elements. Thus the expected number of ways, Y, of picking the $k-q$ new vertices for e_i, satisfies

$$Y \leqslant \sum_{\substack{W \subseteq [k] \\ |W|=q}} q! n^{k-q-\mathrm{rank}(B_{\overline{W}})} p^{k-q}.$$

We wish to bound this conditional expectation Y. By rearranging the inequality given by (6.3.1), we have (if $|W| \geqslant 2$)

$$\ell(k-|W|) - (k-1)\mathrm{rank}(B_{\overline{W}}) < 0.$$

In fact since all quantities above are integers and ℓ divides $k-1$, we must have

$$\ell(k-|W|) - (k-1)\mathrm{rank}(B_{\overline{W}}) \leqslant -\ell. \tag{6.3.3}$$

Thus we have

$$n^{k-|W|-\operatorname{rank}(B_{\overline{W}})}p^{k-|W|} \leqslant \begin{cases} c^k n^{\frac{\ell}{k-1}} & \text{if } |W| = 0; \\ c^{k-1} & \text{if } |W| = 1; \\ c n^{\frac{-\ell}{k-1}} & \text{if } 2 \leqslant |W| \leqslant k-1. \end{cases} \tag{6.3.4}$$

(Note that here we used Proposition 5.18(i) which states that $\operatorname{rank}(B_{\overline{W}}) = \ell$ if $|W| = 1$.)

For each hypergraph J in (i)–(viii), there is always precisely one initial edge, $u-3$ good edges, and two bad edges. Thus the number of choices we have for picking which element of $[n]_p$ to use for each vertex in J has expectation which is at most:

$$\begin{aligned} & n^{k-\ell}p^k \left(\sum_{\substack{W\subseteq[k] \\ |W|=1}} n^{k-|W|-\operatorname{rank}(B_{\overline{W}})}p^{k-|W|} \right)^{u-3} \left(\sum_{\substack{W\subseteq[k] \\ 2\leqslant|W|\leqslant k-1}} |W|! n^{k-|W|-\operatorname{rank}(B_{\overline{W}})}p^{k-|W|} \right)^2 \\ & \overset{(6.3.4)}{\leqslant} (k!)^2 \cdot (kc)^u \cdot n^{-\ell/(k-1)}. \end{aligned}$$

We conclude that for each $i \in [1,8]$ we have $\mathbb{E}(R_{b_i}) < O(n^{-\ell/(k-1)} \cdot \operatorname{polylog}(n)) = o(1)$.

Finally, note that a loose path with at most n vertices clearly has at most n edges. Further using Proposition 5.18(iv), we have $k \geqslant \ell + 2$. Thus we have

$$\begin{aligned} \mathbb{E}(R_{b_9}) &\leqslant O\left(\sum_{u\geqslant(\log n)-1}^{n} n^{k-\ell}p^k \prod_{i=1}^{u-1} \left(\sum_{\substack{W\subseteq[k] \\ |W|=1}} n^{k-|W|-\operatorname{rank}(B_{\overline{W}})}p^{k-|W|} \right)\right) \\ &\overset{(6.3.4)}{\leqslant} O\left(n^{\frac{\ell}{k-1}} \sum_{u\geqslant(\log n)-1}^{n} (kc)^u \right) \overset{(6.3.2)}{=} o(1), \end{aligned}$$

as required. $\square$

Putting the parts together. To reiterate the main points of the proof of Theorem 6.3(ii), we finish by showing that it follows easily from the lemmas in each of the parts (Q1)–(Q3).

Proof (summary) of Theorem 6.3(ii). Let A be a fixed integer-valued matrix of dimension $\ell' \times k'$ and b a fixed integer-valued vector of dimension ℓ', such that the system of linear equations $\mathcal{L}$ is irredundant, and A is irredundant and satisfies $(*)$. In order to prove w.h.p. Breaker wins the (A, b)-game on $[n]_p$, by Proposition 6.8, it suffices to show w.h.p. Breaker wins the (B, b')-game on $[n]_p$, where (B, b') is the associated pair of (A, b). We rephrase the problem to a game on the hypergraph $H := H([n]_p, B, b')$. We then show that if a component H' of H does not contain a bicycle, then it satisfies certain conditions stated in Lemma 6.9. Breaker wins the game played on such a component by Lemma 6.10. Supposing that B is an $\ell \times k$ matrix where ℓ divides $k-1$, then by Lemma 6.15, w.h.p. H (and therefore each component of H) does not contain a bicycle. Finally since Breaker can win the game on each component of H, he wins the game on H, and thus wins the (A, b)-game on $[n]_p$ w.h.p., as required. $\square$

6.4 Concluding remarks and Proof of Theorem 6.2

6.4.1 Improvements on Breaker's strategy

The strange fact that our proof of Theorem 6.3(ii) works when $B(A)$ is an $\ell \times k$ matrix such that ℓ divides $k-1$ follows precisely via inequality given by (6.3.3). We found an equivalence between bicycles and valid edge orders with two bad edges in Claim 6.12. Suppose we extended our language to *p-cycles* (corresponding to valid edge orders with p bad edges), and were able to find a winning strategy for Breaker playing the game on any component of H that does not contain a p-cycle. We could then obtain a proof for matrices A for which the associated matrix B of dimension $\ell \times k$ satisfies $\ell = p - 1$ (without the need for any divisibility conditions). However given the number of cases that arose from considering bicycles, it would seem unfeasible to attempt this.

Lemma 6.9 gives a precise description of hypergraphs with at most one bad edge and a fixed number of k-bad edges. What can be said of hypergraphs with at most p bad edges (for fixed p) and a fixed number of k-bad edges? Also note that our definition of a valid edge order (where every edge after the first one is either good or bad, i.e. there are zero k-bad edges) is a hypergraph generalisation of a tree. This is since trees have precisely this property in the graph case; a 2-bad edge here is an edge which completes a cycle. Thus it would be interesting to obtain a more detailed description of hypergraphs with a valid edge order.

Observe the following connection of this with Rödl and Ruciński's proof of Theorem 5.8. It is very easy to 2-colour a hypergraph with a valid edge order so that it has no monochromatic edges; simply go through the edges in order, colouring the (at least one) new vertex of an edge e_i the colour which was not assigned to one of the old vertices of e_i. Thus if the hypergraph associated to $[n]_p$ has a valid edge order, then $[n]_p$ can be 2-coloured so that there are no monochromatic solutions to $Ax = 0$.

6.4.2 Matrices which do not satisfy $(*)$

Matrices A which are irredundant and do not satisfy $(*)$ traditionally have not received as much attention. For such a matrix, $\mathbb{N}$ is not $(\mathcal{L}^d, r)$-Rado for any $r \geqslant 2$ since we can 2-colour $\mathbb{N}$ and avoid any monochromatic solutions to $Ax = 0$ (see Section 5.3.1). Also note that $m(A)$ is ill-defined in this case. Further in the bias version of the $\mathcal{L}$-game, recall that Theorem 6.1(ii) states that Breaker wins the (1:2) $\mathcal{L}$-game on $[n]$. All of these facts follow easily via use of the row of the matrix which (under Gaussian elimination) has at most two non-zero entries. We show through some examples that the threshold for the random $\mathcal{L}$-game is at least slightly less trivial.

Theorem 6.16. *Let A be a fixed integer-valued matrix of dimension $\ell \times k$ and b a fixed* *integer-valued vector. Given the system of linear equations $\mathcal{L}$ is irredundant and A is irredundant and does not satisfy $(*)$, we have the following:*

(i) If additionally $A = \begin{pmatrix} \alpha & -\beta \end{pmatrix}$ where α, β are non-equal positive integers, then Maker wins the $\mathcal{L}$-game on $[n]_p$ w.h.p. if $p \gg n^{-1/3}$.

(ii) Breaker wins the $\mathcal{L}$-game on $[n]_p$ w.h.p. if $p \ll n^{-1/3}$.

(iii) If additionally $A = \begin{pmatrix} \alpha & -\beta & 0 \\ 0 & \alpha & -\beta \end{pmatrix}$ where α, β are non-equal positive integers, then Breaker wins the $(A, 0)$-game on $[n]$ (i.e. the non-biased non-random game; thus with probability equal to one, Breaker wins the $(A, 0)$-game on $[n]_p$ for any $0 < p < 1$).

Proof. For the $\mathcal{L}$-game in (i), since $\mathcal{L}$ is irredundant, we have that there exists a solution to $\alpha x_1 - \beta x_2 = b$ in $\mathbb{N}$ with $x_1 \neq x_2$. Thus we have that $t := \gcd(\alpha, \beta)$ must divide b: hence we may assume without loss of generality that $\gcd(\alpha, \beta) = 1$.

Maker wins the $\mathcal{L}$-game in (i) if there exists a distinct triple $\{(\alpha x - b)/\beta, x, (\beta x + b)/\alpha\} \subseteq [n]_p$, since she can have the first pick and choose x. Then she can complete a solution by picking whichever of $(\alpha x - b)/\beta$ and $(\beta x + b)/\alpha$ remains unchosen after Breaker's turn.

Claim 6.17. *There exists a fixed $z \in [0, \alpha\beta - 1]$ (depending on α, β, b) such that whenever $x \equiv z \mod \alpha\beta$, the triple $\{(\alpha x - b)/\beta, x, (\beta x + b)/\alpha\}$ is contained in the integers.*

Proof. Let $x \in \mathbb{Z}$. Note that we have $(\beta x + b)/\alpha \in \mathbb{Z}$ whenever $\beta x \equiv -b \mod \alpha$. Since $\gcd(\alpha, \beta) = 1$, there exists $y \in [0, \alpha - 1]$ such that whenever $x \equiv y \mod \alpha$, we have $(\beta x + b)/\alpha \in \mathbb{Z}$. Similarly there exists $y' \in [0, \beta - 1]$ such that whenever $x \equiv y' \mod \beta$, we have $(\alpha x - b)/\beta \in \mathbb{Z}$. Combining these two facts, the Chinese remainder theorem implies that there exists $z \in [0, \alpha\beta - 1]$ such that whenever $x \equiv z \mod \alpha\beta$ we have $(\beta x + b)/\alpha \in \mathbb{Z}$ and $(\alpha x - b)/\beta \in \mathbb{Z}$. □

Call triples which satisfy the property in Claim 6.17 *good*. From the claim, for sufficiently large n we deduce that $[n]$ contains $n/(2\alpha^2\beta^2)$ good triples. Further, each $x \in [n]$ is in at most three good triples. Thus, there is a collection X of at least $n/(6\alpha^2\beta^2)$ good triples in $[n]$ that are all pairwise disjoint. The expected number of triples in X in $[n]_p$ is $\Theta(np^3)$. Hence if $p \gg n^{-1/3}$ then by Proposition 1.2(ii) w.h.p. there exists x such that $\{(\alpha x - b)/\beta, x, (\beta x + b)/\alpha\} \subseteq [n]_p$, so Maker wins as required.

If $p \ll n^{-1/3}$, then the expected number of triples is $o(1)$, so via Proposition 1.1 w.h.p. there are no triples of this form at all. For the game in (ii), since A is irredundant but does not satisfy $(*)$, under Gaussian elimination there exists one row of A which consists of $\alpha, -\beta$ (where α and β are non-equal positive integers) and zeroes, and thus any solution to $Ax = b$ contains the positive integers $(\beta z+c)/\alpha$ and $(\beta((\beta z+c)/\alpha)+c)/\alpha$ for some rational number z and fixed integer $c = c(A, b)$. Since w.h.p. there are no triples (replacing b with c and x with z), whenever we have the pair $(\beta z+c)/\alpha, (\beta((\beta z+c)/\alpha)+c)/\alpha \in [n]_p$, then $z, (\beta((\beta((\beta z+c)/\alpha)+c)/\alpha)+c)/\alpha \notin [n]_p$. Thus Breaker can devise a pairing strategy to win the game.

For (iii), as in (i) we may assume without loss of generality that $\gcd(\alpha, \beta) = 1$. Then every solution in $[n]$ is a triple of the form $\{\alpha^2 x, \alpha\beta x, \beta^2 x\}$ for some $x \in \mathbb{N}$. Every element of $[n]$ which could be in a solution is of the form $\alpha^i\beta^j y$ with $y \in \mathbb{N}$, where α and β do not divide y, and i, j are non-negative integers where at most one of i or j is zero. Using these facts, Breaker has a strategy to win the game. Indeed, Breaker can create a pairing strategy as follows: pair $\alpha^i\beta^j y$ with $\alpha^{i+1}\beta^{j-1}y$ whenever i is even and $j \geqslant 1$. Then observe that for any triple $\{\alpha^2 x, \alpha\beta x, \beta^2 x\}$ with $x \in \mathbb{N}$, the middle element is paired with one of the two end elements, so Maker cannot obtain a triple. $\square$

6.4.3 Proof of Theorem 6.2

We conclude the chapter by finishing the proof of Theorem 6.2.

Theorem 6.2. *Let A be a fixed non-zero-integer-valued matrix of dimension $1 \times k$ and b a fixed integer (i.e. $Ax = b$ corresponds to a single linear equation $a_1x_1 + \cdots + a_kx_k = b$ with the a_i non-zero integers).*

(i) If the system of linear equations $\mathcal{L}$ is irredundant and A is irredundant and satisfies $(*)$, then the $\mathcal{L}$-game on $[n]_p$ has a threshold probability of $\Theta(n^{-\frac{k-2}{k-1}})$;

(ii) If the system of linear equations $\mathcal{L}$ is irredundant and A is irredundant and does not satisfy $(*)$, then the $\mathcal{L}$-game on $[n]_p$ is Maker's win if $p \gg n^{-1/3}$ and Breaker's win if $p \ll n^{-1/3}$;

(iii) If the system of linear equations $\mathcal{L}$ is irredundant and A is not irredundant, then

(a) the $\mathcal{L}$-game on $[n]_p$ is Breaker's win w.h.p. for any $p = o(1)$ if the coefficients a_i are all positive or all negative;

(b) the $\mathcal{L}$-game on $[n]_p$ is Maker's win if $p \gg n^{-1/3}$ and Breaker's win if $p \ll n^{-1/3}$ otherwise;

(iv) If the system of linear equations $\mathcal{L}$ is not irredundant, then the $\mathcal{L}$-game on $[n]$ is (trivially) Breaker's win.

Proof.

(i) As discussed in the introduction, this follows immediately from Theorem 6.3.

(ii) We have $k \geqslant 3$ if and only if A satisfies $(*)$. Hence if A does not satisfy $(*)$ and is a linear equation, we must have $k = 2$. So write $A = \begin{pmatrix} \alpha & \beta \end{pmatrix}$, where $\alpha, \beta \in \mathbb{Z}$. Note that since A is irredundant there exist $x_1, x_2 \in \mathbb{N}$ such that $\alpha x_1 + \beta x_2 = 0$ where $x_1 \neq x_2$. Thus we must have $\alpha > 0$ and $\beta < 0$ or vice versa, and $\alpha \neq -\beta$. Thus the result follows from Theorem 6.16(i) and (ii).

(iii) Note that any linear equation $a_1x_1 + \dots a_kx_k = 0$ with $k \geqslant 3$ clearly has a k-distinct solution in $\mathbb{N}$ if there exists at least one positive a_i and at least one negative a_j, for some $i, j \in [k]$. The same holds for $k = 2$ unless if $a_1 = -a_2$. Thus, since A is not irredundant, we have one of the following:

(a) the a_i are all positive integers or all negative integers;

(b) we have $k = 2$ and $a_1 = -a_2$.

For (a), we may assume without loss of generality that $a_1, \dots, a_k$ and therefore b are positive integers. For such a game there are a finite number of k-distinct solutions in $\mathbb{N}$, all of which are contained in $[b]$. Thus for any $p = o(1)$, w.h.p. there are no solutions in $[n]_p$ by Proposition 1.1, so the game is Breaker's win. For (b), the existence of any triple $\{x - b/a_1, x, x + b/a_1\}$ leads to a win for Maker, meanwhile if no triples exist then Breaker can win by a pairing strategy. Since the number of such triples in $[n]_p$ is of order np^3, the result follows by a similar argument to that given for Theorems 6.16(i) and (ii).

(iv) The $\mathcal{L}$-game is trivially Breaker's win, since there are no winning sets in $\mathbb{N}$. □

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