GEORGIA INSTITUTE OF TECHNOLOGY OFFICE OF CONTRACT ADMINISTRATION

PROJECT ADMINISTRATION DATA SHEET

E-20-657

[X] ORIGINAL [ ] REVISION NO. ______

Project No. ~~E-23-621~~ (R5965-0A0) GTRC/~~GIT~~ DATE 8 / 15 / 85

Project Director: Gerald A. Wempner School/~~Lab~~ ESM

Sponsor: National Science Foundation

Type Agreement: Grant MSM-8411757

Award Period: From 7/1/85 To 12/31/87* (Performance) 3/31/87 (Reports)

Sponsor Amount:

	This Change	Total to Date
Estimated:	$	$ 60,429
Funded:	$	$ 60,429

Cost Sharing Amount: $ 3,021 Cost Sharing No: E-23-316

Title: An Investigation of Simple Finite Elements Via the Hu-Washizu Theorem

ADMINISTRATIVE DATA OCA Contact John Schonk x4820

1) Sponsor Technical Contact:

Gifford H. Albright
Program Office
National Science Foundation
Washington, DC 20550

202/ 357-9542

2) Sponsor Admin/Contractual Matters:

Altie H. Metcalf
Grants Official
National Science Foundation
Washington, DC 20550

202/ 357-9602

Defense Priority Rating: N/A

Military Security Classification: N/A
(or) Company/Industrial Proprietary: N/A

RESTRICTIONS

See Attached NSF Supplemental Information Sheet for Additional Requirements.

Travel: Foreign travel must have prior approval – Contact OCA in each case. Domestic travel requires sponsor approval where total will exceed greater of $500 or 125% of approved proposal budget category.

Equipment: Title vests with GIT

COMMENTS:

*Includes 6 month unfunded flexibility period.

Pre-award costs beginning 6/15/85 are specifically approved through OPAS.

AUG 1985 RECEIVED Research Reports Office

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Project Director
Research Administrative Network
Research Property Management
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Procurement/GTRI Supply Services
~~Research Security Services~~
Reports Coordinator (OCA)
~~Research Communications (2)~~
GTRC
Library
Project File
Other A. Jones

12/88

GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

NOTICE OF PROJECT CLOSEOUT

Date 8/24/89

oject No. E-20-657 Center No. R5965-0A1

oject Director G. A. Wempner School/Lab CE

onsor National Science Foundation

ntract/Grant No. MSM-8411757 GTRC XX GIT ___

me Contract No. N/A

le An Investigation of Simple Finite Elements via the Huwashizu Theorem

ective Completion Date 12/31/87 (Performance) 3/31/88 (Reports)

oseout Actions Required:

- _ None
- _ Final Invoice or Copy of Last Invoice
- _ Final Report of Inventions and/or Subcontracts- Patent questionnaire sent to P/I.
- _ Government Property Inventory & Related Certificate
- _ Classified Material Certificate
- _ Release and Assignment
- _ Other ___

ludes Subproject No(s). ___

project Under Main Project No. ___

tinues Project No. ___ Continued by Project No. ___

tribution:

- Project Director
- Administrative Network
- Accounting
- Procurement/GTRI Supply Services
- Research Property Management
- Research Security Services
- X Reports Coordinator (OCA)
- X GTRC
- X Project File
- 2 Contract Support Division (OCA)
- ___ Other ___

Hydraulic Model Study of Dilution Characteristics of Boston Wastewater Outfall

Georgia Institute of Technology

Monthly Progress Report

Period Ending 5/31/89

The Principal Investigator, Dr Philip Roberts, travelled to Boston on 27 March for a meeting to discuss the model study with Metcalf & Eddy and PBQ&D personnel. At this meeting, the initial modeling parameters were agreed upon. These were transmitted to the Fluid Modeling Facility in Research Triangle Park, and the model diffusers were constructed.

Hydraulic Model Study of Dilution Characteristics of Boston Wastewater Outfall

Georgia Institute of Technology

Monthly Progress Report

Period Ending 6/30/89

Model testing began in Research Triangle Park under the direction of Dr. Philip Roberts on 12 June. Extensive testing of the dilution characteristics of various numbers of risers was undertaken for the remainder of this period. The diffusers tested consisted of 28, 33, 48, 64, 80 , and 111 risers. Prototype conditions tested included flows of 390, 620, and 1270 mgd, current speeds of 0 and 12 cm/s perpendicular and parallel to the diffuser, and late summer and neutral stratifications. Most tests were conducted with the late summer stratification as this is the most critical for design purposes. Dilutions and wastefield rise heights were measured, and photographs and videotapes of the flow fields obtained. A demonstration of the model testing was held on 29 June for personnel from M&E, PBQ&D, and MWRA. The results obtained to date were reviewed, and plans for the remainder of Phase I testing were discussed and agreed upon.

EXHIBIT A.1

WEEKLY LEVEL OF EFFORT REPORT

LEAD DESIGN ENGINEERING SERVICES
FOR
MASSACHUSETTS WATER RESOURCES AUTHORITY

Name of Subcontractor: Georgia Institute of Technology

EMPLOYEE	WEEK ENDING	HOURS
Philip J.W. Roberts	4/24/89	10
Philip J.W. Roberts	5/05/89	10
Philip J.W. Roberts	5/12/89	20
Philip J.W. Roberts	5/19/89	20
Philip J.W. Roberts	5/26/89	20
Philip J.W. Roberts	6/02/89	20
Philip J.W. Roberts	6/09/89	20
Philip J.W. Roberts	6/16/89	40
Philip J.W. Roberts	6/23/89	40
Philip J.W. Roberts	6/30/89	40
Philip J.W. Roberts	7/07/89	40
Philip J.W. Roberts	7/14/89	40
Philip J.W. Roberts	7/21/89	40
Philip J.W. Roberts	7/28/89	40

Georgia Institute of Technology E-20-65?
Atlanta, Georgia 30332

College of Engineering
School of Engineering Science and Mechanics

October 22, 1986

Dr. Gifford H. Albright
Program Director
Structures and Building-Systems Program
Division of Mechanics, Structures and
Materials Engineering
National Science Foundation
Washington, D.C. 20550

Dear Gifford:

Please find the enclosed report of progress on our research, Grant No. MSM-8411757. Your comments or suggestions will be appreciated. Let me know if you want any additional information.

I have tentative plans to be in the District during the week of December 14 and hope that I can meet with you at that time.

Looking forward to seeing you, I send my best regards,

Sincerely,

Gerald Wempner

GW:ds

An Investigation of Simple Finite Elements
Via the Hu-Washizu Theorem

NSF Grant No. MSM-8411757

Progress Report 10-20-86

This investigation has proceeded along two avenues: One involves the exploration of simple triangular and quadrilateral elements of shells. Here the intent is the consistent and effective approximation, in accordance with the stationary conditions of Hu-Washizu. Various alternative approximations of the stresses and strains are under investigation. In each case the displacements are expressed by the linear and bilinear approximations which provide compatible assemblies of triangles and quadrilaterals, respectively. The formulations follow the concepts of the earlier work [Wempner, Talaslidis, Hwang, J. Appl. Mech., 1982]. It remains to provide additional comparisons, to implement the discrete constraints [Wempner, 1968] and to establish suitable guidelines for application.

The other course of investigation has focused upon the foundations of a theory which describes the shell in a manner most amenable to approximation by finite elements. This study has produced a theory [J. Appl. Mech., to appear] which incorporates the desired attributes: Transverse shear and extensional strains are included to admit the simple elements. The approximation of displacement is linear through the thickness, so that a transition to more layers, or three-dimensional assemblies, is readily accomplished. The strains and rotations are decomposed to provide the means to develop the equations which describe geometrical nonlinearities; finite rotations and strains are admissible. Finally, this theory is cast in terms of the potential, the complementary functional, the Hu-Washizu and the Reissner functionals. Accordingly, one can formulate the elements, and assemblies, based upon any of the stationary (or extremal) theorems, i.e. mixed or hybrid forms.

The investigation of the simple elements is being conducted in collaboration with Mr. William Dorris, graduate student. Additionl studies on quadrilateral elements are being conducted by Prof. D. Talaslidis.

The general theory [attached preprint] provided the basis for a presentation at the Euromech Symposium 197, Warsaw, 1985. The latter appears in the proceedings and includes a discussion of application to the finite elements of shells and assemblies which exhibit finite rotations and the associated nonlinearities.

Complementary Potentials for Finite

Deformations of Shells

Gerald Wempner

A general expression of complementary energy was given by Fraeijs de Veubeke (1972). Here, the complementary properties of the various functionals are examined. Two-dimensional formulations are derived for a shell with finite deformations. These forms incorporate measures of stress and strain which admit precise definitions of complementary energies. The theory is founded upon the one underlying approximation: Normals remain straight.

The generality of the theory and the alternative functionals provide bases for approximations, specificially, various finite elements of mixed type.

A General Theory of Shells and the Complementary Potentials

Abstract

This theory incorporates the attributes which are essential to the approximation of shells by finite elements. It is limited only by one assumption: Displacement is a linear function of distance along the normal to a reference surface. Deformation is decomposed into rotation and strain; the rotation carries elements of the reference surface to the same surface in any subsequent state. Transverse-shear deformations accommodate simple elements.

The theory is couched in the potential P_V and in the complementary potential P_C; these have the property, $P_V + P_C = 0$ for all admissible (equilibrated) states. The theory is also cast in the complementary functional $\overline{P}_C$ of stress and displacement, and the functional $\overline{P}_V$ of displacement, strain and stress; $\overline{P}_C$ and $\overline{P}_V$ are akin to the functionals of Hellinger-Reissner and Hu-Washizu. These alternate functionals provide the means to develop various hybrid elements.

A General Theory of Shells

and the

Complementary Potentials

by

Gerald Wempner
Georgia Institute of Technology
Atlanta, GA 30332

Introduction

The role of thin shells in modern structures is evident.Increasingly, we turn to numerical methods, often based on finite elements, to predict the response of shells. When the strains are small, then the analysis of an element entails only small relative rotations; large rotations in the assembly are accommodated by the decomposition of rigid rotation and strain. To avoid complicated elemental approximations, the theory of the shell must admit transverse shear strains; then kinks are admissible along the contiguous edges of elements. The foregoing observations and a theory for "finite elements, finite rotations and small strains" were presented previously (Wempner, 1969).

Before and after the earlier work (Wempner, 1969) many contributions have been made to the subject of shells. Intrinsically, most theories admit finite rotations. These include the important works of Koiter (1960, 1966, 1973), Sanders (1963), Leonard (1961), Naghdi (1972) and Reissner (1974). Simmonds and Danielson (1970) and Pietraszkiewicz (1980) have explicitly addressed the decomposition and alternative representations of the finite rotation. The works of Reissner (1974), Pietraszkiewicz (1980), Libai and Simmonds (1983) also accommodate transverse shear deformations. The literature is vast; the works cited include many additional references, beginning with the early work of Aron (1874) and Love (1927).

The foundations of structural mechanics were recently fortified by Fraeijs de Veubeke's formulation (1972) of the complementary potential. Independently, Koiter (1973) arrived at similar results: These demonstrate the roles of the rotation and the use of the tensors of stretch, engineering strain and the associated stress in the formulation of general which reveal the complementary character of those potentials and also the physical significance of the associated stress.

Here we present a general theory which is drawn from the three-dimensional theory with one underlying assumption: The displacement varies linearly through the thickness. Our decomposition of rotation and strain differs from the usual decomposition of three-dimensions: Our rotation is more natural for shells because it carries elements of the reference surface to the same surface in any subsequent state. With no restrictions upon the magnitudes of rotations or strains, the theory is expressed by the complementary functionals which are analogous to the functionals of three dimensions. Some basic equations (kinematics and dynamics) apply to any continuous shell; all results apply to any continuous elastic shell.

Since our theory is given by any of four functionals, accommodates finite deformations and transverse shear strain, it provides a vehicle for a variety of approximations and, specifically, hybrid elements.

Three-Dimensional Theory

In a previous paper (Wempner, 1980) we began with a primitive functional P of a stress vector $\underset{\sim}{T}^i$ and the position vector $\underset{\sim}{R}$ of a deformed state

$$P \equiv \int_v [\underset{\sim}{T}^i \cdot \underset{\sim}{R}_{,i} - \underset{\sim}{f} \cdot \underset{\sim}{R}]\, dv \tag{1}$$

$$- \int_a \underset{\sim}{t} \cdot \underset{\sim}{R}\, da - \int_{a_v} \underset{\sim}{t} \cdot (\underset{\sim}{R} - \tilde{\underset{\sim}{R}}) da$$

Here $\underset{\sim}{f}$ is body force (per unit of initial volume v), $\underset{\sim}{t}$ is the traction (per unit of the initial bounding surface a) and $\tilde{\underset{\sim}{R}}$ is the prescribed position on a portion a_v of the boundary surface. The variation of $\underset{\sim}{R}$ in v and on surface a_t (where tractions are prescribed) provides the equilibrium conditions for the stress $\underset{\sim}{T}^i$ in v and on a_t. The variations of $\underset{\sim}{T}^i$ in v and $\underset{\sim}{t}$ on a are subject to the conditions of equilibrium; then the variation of the functional vanishes for kinematically admissible displacements. In short, the functional includes the potential $P_v(\underset{\sim}{R})$ and the complementary functional $P_c(\underset{\sim}{T}^i)$. To appreciate this fully we introduce the strain, stress and complementary energy densities:

As before (Wempner, 1980), let $r^j_{.i}$ denote components of rigid rotation which carries the initial tangent vectors, $g_i \equiv \underset{\sim}{r}_{,i}$ to an intermediate system

$$g_i' \equiv r^j_{.i}\, g_j \tag{2}$$

A stretch with components C^j_i carries the intermediate triad g_i' to the current system:

$$\underset{\sim}{R}_{,i} \equiv \underset{\sim}{G}_i = C^j_i\, g_j' \tag{3}$$

Here the component of stretch, $C_{ij} = C_{ji}$, is related to the component of engineering strain:

$$h_{ij} = h_{ji} = C_{ij} - g_{ij}\ , \tag{4}$$

where

$$g_{ij} = g_i \cdot g_j = g_i' \cdot g_j'$$

The internal power is

$$\dot{W} = \underset{\sim}{T}^i \cdot \dot{\underset{\sim}{R}}_{,i} = \underset{\sim}{T}^i \cdot (C^j_i\, g_i')^{\boldsymbol{\cdot}} \tag{5a}$$

$$\dot{W} = \underset{\sim}{T}^i \cdot (\dot{C}_i^j \, g_j' + C_i^j \, \dot{\underset{\sim}{\Omega}} \times g_j') \tag{5b}$$

Here $\dot{\underset{\sim}{\Omega}}$ is the spin of the triad g_i' and $\dot{C}_i^j = \dot{h}_i^j$. In another form,

$$\dot{W} = (\underset{\sim}{T}^i \cdot g'^j) \, \dot{h}_{ij} + (T^{ik} \, C_i^j) \, (g_j' \times g_k') \cdot \dot{\underset{\sim}{\Omega}} \tag{5c}$$

The final sum of (5c) is the power expended in the rigid spin; it must vanish. The first term is the work expended in strain:

$$\dot{W} = T^{\langle ij \rangle} \, \dot{h}_{ij} \tag{5d}$$

where $T^{\langle ij \rangle}$ signifies the symmetric part of the stress component

$$T^{ij} = g'^j \cdot \underset{\sim}{T}^i \tag{6}$$

If $W_v(h_{ij})$ and $W_c(T^{\langle ij \rangle})$ denote the complementary densities, then

$$\underset{\sim}{T}^i \cdot \underset{\sim}{R}_{,i} = T^{ij}(h_{ij} + g_{ij}) \tag{7a}$$

$$= W_v + W_c + T^{ij} \, g_{ij} \tag{7b}$$

Upon substituting (7b) into (1), we obtain

$$P = P_v + P_c \tag{8}$$

where

$$P_v \equiv \int_v [W_v - \underset{\sim}{f} \cdot \underset{\sim}{R}] \, dv - \int_{a_t} \underset{\sim}{t} \cdot \underset{\sim}{R} \, da \tag{9a}$$

$$P_c \equiv \int_v [W_c + T^{ij} \, g_{ij}] \, dv - \int_{a_v} \underset{\sim}{t} \cdot \underset{\sim}{R} \, da \tag{9b}$$

$$- \int_{a_v} \underset{\sim}{t} \cdot (\underset{\sim}{R} - \tilde{\underset{\sim}{R}}) \, da$$

$P_v(\underset{\sim}{R})$ is the potential when $\underset{\sim}{f}$ and $\underset{\sim}{t}$ are dead loads. $P_c(\underset{\sim}{T}^i)$ is subject to variations $\dot{\underset{\sim}{T}}^i$ which fulfill equilibrium and $\dot{\underset{\sim}{t}} = 0$ on a_t; therefore, in (9b)

$$\int_{a_v} \dot{\underset{\sim}{t}} \cdot \underset{\sim}{R}\, da = \int_a \dot{\underset{\sim}{t}} \cdot \underset{\sim}{R}\, da = \int_v \dot{\underset{\sim}{T}}^i \cdot \underset{\sim}{R}_{,i}\, dv$$

In view of the foregoing, functional P_c can be rewritten:

$$\bar{P}_c = \int_v \left[W_c - T^{ij}\, g_j' \cdot (\underset{\sim}{G}_i - g_i')\right] dv - \int_{a_v} \underset{\sim}{t} \cdot (\underset{\sim}{R} - \tilde{\underset{\sim}{R}})\, da \tag{9c}$$

It is important to note that the variation $\dot{\underset{\sim}{T}}^i$ requires the variation of the components $\dot{T}^{ij}$, and the vector $\dot{g}'$, i.e. the rotation $\dot{\underset{\sim}{\Omega}}$, which leads to the conditions for equilibrium of moment:

$$(T^{ik}\, C_i^j)\, (\epsilon_{jk\ell})\, g'^{\ell} = 0 \tag{10}$$

Reduction to Two-Dimensions

Our theory of the shell is founded upon the assumption:

$$\underset{\sim}{R} = \underset{\sim}{R}^o + \theta^3\, \underset{\sim}{A}_3 \tag{11}$$

Here we follow the conventions: θ^α denotes an arbitrary coordinate of the reference surface ($\alpha = 1,2$) and θ^3 denotes distance along the initial normal $\underset{\sim}{n}$. Also,

$$\underset{\sim}{A}_\alpha \equiv \underset{\sim}{R}^o_{,\alpha}$$

Top and bottom surfaces lie at $\theta^3 = h_+, -h_-$; s denotes the reference surface; c denotes the bounding edge. If h and k denote the mean and gaussian curvatures of the initial reference surface, then

$$dv = \mu(\theta^3)\, ds$$

$$\mu \equiv 1 - 2h\theta^3 + k(\theta^3)^2$$

With these notations and the assumption (11), the functional (1) is integrated with respect to the coordinate θ^3, to obtain

$$P = \int_s [\underset{\sim}{N}^\alpha \cdot \underset{\sim}{R}^o_{,\alpha} - \underset{\sim}{F} \cdot \underset{\sim}{R}^o + \underset{\sim}{M}^\alpha \cdot \underset{\sim}{A}_{3,\alpha} + \underset{\sim}{T} \cdot \underset{\sim}{A}_3 - \underset{\sim}{C} \cdot \underset{\sim}{A}_3] \, ds \qquad (12)$$

$$- \int_c \underset{\sim}{N} \cdot \underset{\sim}{R}^o \, dc - \int_c \underset{\sim}{M} \cdot \underset{\sim}{A}_3 \, dc$$

$$- \int_{c_v} \underset{\sim}{N} \cdot (\underset{\sim}{R}^o - \tilde{\underset{\sim}{R}}) \, dc - \int_{c_v} \underset{\sim}{M} \cdot (\underset{\sim}{A}_3 - \tilde{\underset{\sim}{A}}_3) \, dc$$

Instead of the one vector $\underset{\sim}{R}(\theta^1, \theta^2, \theta^3)$ of three dimensions, we have two vectors, $\underset{\sim}{R}^o(\theta^1, \theta^2)$ and $\underset{\sim}{A}_3(\theta^1, \theta^2)$, which fully define the configuration. $\underset{\sim}{F}$ and $\underset{\sim}{C}$ are net external force and "couple", which include body force and surface tractions. The "stresses" are

$$\underset{\sim}{N}^\alpha \equiv \int_{-h_-}^{h_+} \underset{\sim}{T}^\alpha \, \mu d\theta^3 \qquad (13a)$$

$$\underset{\sim}{M}^\alpha \equiv \int_{-h_-}^{h_+} \underset{\sim}{T}^\alpha \, \theta^3 \mu d\theta^3 \qquad (13b)$$

$$\underset{\sim}{T} \equiv \int_{-h_-}^{h_+} \underset{\sim}{T}^3 \, \mu d\theta^3 \qquad (13c)$$

$\underset{\sim}{N}$ and $\underset{\sim}{M}$ are the edge tractions (force and "couple"); $\tilde{\underset{\sim}{N}}$ and $\tilde{\underset{\sim}{M}}$ are prescribed on part c_t of the edge.

The variation of vectors $\underset{\sim}{R}^o$ and $\underset{\sim}{A}_3$ provides a variation of P (the virtual work) and the stationary conditions are the equilibrium equations in s and on c_t. With the customary notation, $ds = \sqrt{a}\, d\theta^1\, d\theta^2$, and the usual integration-by-parts, we obtain

$$(\sqrt{a}\, \underset{\sim}{N}^{\alpha})_{,\alpha} + \underset{\sim}{F} = 0 \qquad \text{in } s \tag{14a}$$

$$(\sqrt{a}\, \underset{\sim}{M}^{\alpha})_{,\alpha} - \underset{\sim}{T} + \underset{\sim}{C} = 0 \qquad \text{in } s \tag{14b}$$

$$\underset{\sim}{N}^{\alpha} n_{\alpha} = \tilde{N} \quad , \quad \underset{\sim}{M}^{\alpha} n_{\alpha} = \tilde{M} \quad \text{on } c_t \tag{15a,b}$$

Again, the variation of stresses $\underset{\sim}{N}^{\alpha}$, $\underset{\sim}{M}^{\alpha}$ and $\underset{\sim}{T}$ produces meaningful results only when their components are referred to a suitable basis.

Natural Basis for the Shell

With the presence of transverse shear strain the usual rotation (which carries g_i to g_i') would rotate the (initial) tangent vector ($\underset{\sim}{a}_{\alpha} \equiv \underset{\sim}{r}^{o}_{,\alpha}$) out of the (deformed) surface. Therefore, it is more natural to employ a rotation which carries the initial triad ($\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$, $\underset{\sim}{a}_3 \equiv \underset{\sim}{n}$) to a triad ($\underset{\sim}{b}_1$, $\underset{\sim}{b}_2$, $\underset{\sim}{b}_3 \equiv \underset{\sim}{N}$) such that $\underset{\sim}{b}_{\alpha}$ are tangent and $\underset{\sim}{N}$ is normal to the deformed reference surface. With this new meaning, we have

$$\underset{\sim}{b}_i = r^{j}_{.i}\, \underset{\sim}{a}_j \tag{16}$$

The orientation of $\underset{\sim}{b}_{\alpha}$ is such that the stretch of the surface is given by a symmetrical tensor:

$$C_{\alpha\beta} \equiv \underset{\sim}{b}_{\alpha} \cdot \underset{\sim}{A}_{\beta} = \underset{\sim}{b}_{\beta} \cdot \underset{\sim}{A}_{\alpha} \tag{17a}$$

$$\underset{\sim}{A}_{\alpha} = \underset{\sim}{R}^{o}_{,\alpha}$$

The deformation also carries the vector $\underset{\sim}{N}$ to the vector $\underset{\sim}{A}_3$; components of the stretch are

$$C_{3i} = \underset{\sim}{b}_i \cdot \underset{\sim}{A}_3 \tag{17b}$$

Note that $\underset{\sim}{b}_3 \cdot \underset{\sim}{A}_\alpha = 0$.

All components of stress are referred to the natural basis:

$$N^{\alpha i} \equiv \underset{\sim}{b}^i \cdot \underset{\sim}{N}^\alpha \ , \ M^{\alpha i} \equiv \underset{\sim}{b}^i \cdot \underset{\sim}{M}^\alpha \tag{18a,b}$$

$$T^i \equiv \underset{\sim}{b}^i \cdot \underset{\sim}{T}$$

To illustrate the basis, the initial triad ($\underset{\sim}{a}_i$), the reference triad ($\underset{\sim}{b}_i$) and the current (deformed) triad ($\underset{\sim}{A}_i$) are depicted in Fig. 1.

<u>Internal Power</u>

The internal power of the stresses (per unit area) is

$$\dot{W} = \underset{\sim}{N}^\alpha \cdot \dot{\underset{\sim}{R}}^o_{,\alpha} + \underset{\sim}{M}^\alpha \cdot \dot{\underset{\sim}{A}}_{3,\alpha} + \underset{\sim}{T} \cdot \dot{\underset{\sim}{A}}_3 \tag{19a}$$

After much algebra, we obtain

$$\begin{aligned}\dot{W} = {} & N^{\alpha\beta} \dot{C}_{\alpha\beta} + M^{\alpha\beta} \dot{D}_{\alpha\beta} \\ & + T^\alpha \dot{C}_{3\alpha} + T^3 \dot{C}_{33} + M^{\alpha 3} \dot{D}_{\alpha 3} \\ & + [(N^{\alpha 3} - K^\alpha_n M^{n3}) C^\nu_\alpha - T^\nu C_{33} + T^3 C^\nu_3 - M^{\alpha\nu} D_{\alpha 3}] \dot{\omega}_{3\nu} \\ & + [(N^{\alpha\gamma} - K^\alpha_\beta M^{\beta\gamma}) C^\nu_\alpha + T^\gamma C^\nu_3] \dot{\omega}_{\gamma\nu}\end{aligned} \tag{19b}$$

Here a component of spin $\dot{\underset{\sim}{\Omega}}$ is expressed by

$$\dot{\Omega}^i \equiv \underset{\sim}{b}^i \cdot \dot{\underset{\sim}{\Omega}} = \frac{1}{2} \epsilon^{ikj} \dot{\omega}_{jk}$$

Spin components about the normal $\underset{\sim}{b}^3 \equiv \underset{\sim}{N}$, tangents $\underset{\sim}{b}^1$ and $\underset{\sim}{b}^2$ are, respectively, $\dot{\omega}_{21}/\ a$, $\dot{\omega}_{32}/\ a$, and $\dot{\omega}_{13}/\ a$. The flexure K^α_β is defined as follows:

$$K^\alpha_\beta \equiv - \underset{\sim}{A}^\alpha \cdot \underset{\sim}{A}_{3,\beta} \qquad (20a)$$

$$= C^3_3\ B^\alpha_\beta - \left(C^n_3\ c^\alpha_n\right) \big|_\beta \qquad (20b)$$

Here B^α_β and $B_{\alpha\beta}$ are components of curvature:

$$B^\alpha_\beta \equiv \underset{\sim}{A}^\alpha_{,\beta} \cdot \underset{\sim}{N} \quad , \quad B_{\alpha\beta} = \underset{\sim}{A}_{\alpha,\beta} \cdot \underset{\sim}{N} \qquad (21a,\ b)$$

Also, in (19b),

$$D_{\alpha\beta} \equiv \underset{\sim}{b}_\beta \cdot \underset{\sim}{A}_{3,\alpha} = - K^\nu_\alpha\ C_{\nu\beta} \qquad (21a,b)$$

$$D_{\alpha 3} \equiv \underset{\sim}{N} \cdot \underset{\sim}{A}_{3,\alpha} = C^3_{3,\alpha} + C^\beta_3\ c^\nu_\beta\ B_{\nu\alpha} \qquad (22a,b)$$

The expression (19b) serves to identify the strains associated with each of the stresses, $N^{\alpha\beta}$, $M^{\alpha i}$, T^α and T^3, respectively:

$$h_{\alpha\beta} \equiv C_{\alpha\beta} - a_{\alpha\beta} \qquad (23a)$$

$$k_{\alpha\beta} \equiv D_{\alpha\beta} + b_{\alpha\beta}, \quad k_{\alpha 3} = D_{\alpha 3} \qquad (23b,c)$$

$$h_{3\alpha} \equiv C_{3\alpha} \quad , \quad h_{33} = C_{33} - 1 \qquad (23d,e)$$

Since the stretch ($C_{\alpha\beta}$) and strain ($h_{\alpha\beta}$) of the surface are symmetric, only the symmetrical part of the membrane stress ($N^{\alpha\beta}$) plays a role in the power, in a potential or dissipation. Also, the shear stresses $N^{\alpha 3}$ are merely reactive.

In addition, the power (19b) serves to identify three conditions of equilibrium: Since no power is expended in the spin, each bracketed term vanishes. These three equations serve to determine, or eliminate, the skew-symmetric part of the stress $N^{\alpha\beta}$ and the reactive stresses $N^{\alpha 3}$.

Complementary Potentials of the Shell

With the identification of strains and the associated stresses, we can formulate the two-dimensional counterparts of the potentials (9a) and (9b). The potential is analogous to (9a) and follows from (12).

$$P_V = \int_S [W_V(h_{\alpha\beta}, k_{\alpha\beta}, h_{3i}, k_{\alpha 3}) - \underset{\sim}{F} \cdot \underset{\sim}{R}^o - \underset{\sim}{C} \cdot \underset{\sim}{A}_3]\, ds - \int_{c_t} [\underset{\sim}{N} \cdot \underset{\sim}{R}^o + \underset{\sim}{M} \cdot \underset{\sim}{A}_3]\, dc \tag{24}$$

Here, the strains are implicit functions of the displacements ($\underset{\sim}{R}_o$ and $\underset{\sim}{A}_3$), so that the potential (24) is implicitly a functional of displacement.

The complementary "potential" is analogous to (9b) and follows from (12) and (24); $P_c = P - P_V$:

$$P_c = \int_S [W_c(N^{\alpha\beta}, M^{\alpha\beta}, T^i . , M^{\alpha 3}) + N^{\alpha\beta}\, a_{\alpha\beta} - M^{\alpha\beta}\, b_{\alpha\beta} + T^3]\, ds - \int_{c_V} [\underset{\sim}{N} \cdot \underset{\sim}{R}^o + \underset{\sim}{M} \cdot \underset{\sim}{A}_3]\, dc - \int_{c_V} [\underset{\sim}{N} \cdot (\underset{\sim}{B}^o - \bar{\underset{\sim}{B}}^o) + \underset{\sim}{M} \cdot (\underset{\sim}{A}_3 - \bar{\underset{\sim}{A}}_3)]\, dc \tag{25}$$

The complementary density is defined, as in (7b), by the Legendre transformation:

$$W_c = N^{\alpha\beta} h_{\alpha\beta} + M^{\alpha\beta} k_{\alpha\beta} + M^{\alpha 3} k_{\alpha 3} + T^i h_{3i} - W_v \tag{26}$$

The sum of (24) and (25) is the functional (12):

$$P = P_v + P_c \tag{27}$$

Verification requires the definition of the complementary density (26), the stresses (18) and strains (23).

The stationary conditions for P_v ($\underset{\sim}{R}^o$, $\underset{\sim}{A}_3$) provide six equilibrium equations (14a,b) and edge conditions (15a,b), consistent with the potential W_v (dependent upon the elasticity).

The functional P_c of (25) depends on the stresses ($N^{\alpha i}$, $M^{\alpha i}$, T^i) <u>and</u> rotation of the triad ($\underset{\sim}{b}_i$) just as it's three-dimensional counterpart (9b). Admissible variations of stress must satisfy the equilibrium equations; in particular, variations vanish on c_t. Therefore, enforcing (14a,b) in P_c of (25), we obtain

$$\int_{c_v} [\underset{\sim}{N} \cdot \underset{\sim}{R}^o + \underset{\sim}{M} \cdot \underset{\sim}{A}_3]\, dc = \int_c [\quad]\, dc - \int_{c_t} [\quad]\, dc \tag{28a}$$

$$= \int_s [N^{\alpha i}\, \underset{\sim}{b}_i \cdot \underset{\sim}{R}^o_{,\alpha} + M^{\alpha i}\, \underset{\sim}{b}_i \cdot \underset{\sim}{A}_{3,\alpha} + \underset{\sim}{T} \cdot \underset{\sim}{A}_3 \tag{28b}$$

$$- \underset{\sim}{F} \cdot \underset{\sim}{R}^o - \underset{\sim}{C} \cdot \underset{\sim}{A}_3]\, ds - \int_{c_t} [\quad]\, dc$$

Then, by employing (28b) in (25), we obtain the two-dimensional counterpart of (9c):

$$\overline{P}_c = \int_s [W_c - N^{\alpha\beta}(\underset{\sim}{b}_\beta \cdot \underset{\sim}{R}^o_{,\alpha} - a_{\alpha\beta}) - N^{\alpha 3}\ \underset{\sim}{N} \cdot R^o_{,\alpha} \tag{29}$$

$$- M^{\alpha\beta}(\underset{\sim}{b}_\beta \cdot \underset{\sim}{A}_{3,\alpha} + b_{\alpha\beta}) - M^{\alpha 3}\ \underset{\sim}{N} \cdot \underset{\sim}{A}_{3,\alpha}$$

$$- T^\alpha\ \underset{\sim}{b}_\alpha \cdot \underset{\sim}{A}_3 - T^3\ \underset{\sim}{N} \cdot (\underset{\sim}{A}_3 - \underset{\sim}{N}) + \underset{\sim}{F} \cdot \underset{\sim}{R}^o + \underset{\sim}{C} \cdot \underset{\sim}{A}_3]\ ds$$

$$- \int_{c_v} [\underset{\sim}{N} \cdot (\underset{\sim}{R}^o - \tilde{\underset{\sim}{R}}^o) + \underset{\sim}{M} \cdot (\underset{\sim}{A}_3 - \tilde{\underset{\sim}{A}}_3)]\ dc + \int_{c_t} [\underset{\sim}{N} \cdot \underset{\sim}{R}^o + \underset{\sim}{M} \cdot \underset{\sim}{A}_3]\ dc$$

The latter form of P_c is akin to the Hellinger-Reissner functional (Hellinger, 1914 - Reissner, 1950). The functional is stationary under variations of stress provided that the displacement-stress conditions are satisfied, e.g.

$$\underset{\sim}{b}_\beta \cdot (\underset{\sim}{R}^o_{,\alpha} - \underset{\sim}{b}_\alpha) = \frac{\partial W_c}{\partial N^{\alpha\beta}}$$

In addition, the functional (29) can be regarded as a functional of displacemnts ($\underset{\sim}{R}_o$, $\underset{\sim}{A}_3$). The functional is stationary under variations of displacement provided that the equilibrium equations (14a,b) are satisfied in 's' and (15a,b) are satisfied on 'c_t'. Finally, the functional (29), like (25) is dependent on the rotation of the triad ($\underset{\sim}{b}_i$). Both are stationary under variations of rotation provided that the three conditions of equilibrium (of moments) are satisfied; these are the conditions that the bracketed terms of (19b) vanish.

If we employ the transformation (26) to eliminate W_c in (29), we obtain

$$\bar{P}_v(= - \bar{P}_c) =$$

$$\bar{P}_v = \int_s \{W_v - N^{\alpha\beta}[h_{\alpha\beta} - \underset{\sim}{b}_\beta \cdot (\underset{\sim}{R}^o_{,\alpha} - \underset{\sim}{b}_\alpha)] \tag{30}$$

$$- M^{\alpha\beta}[k_{\alpha\beta} - \underset{\sim}{b}_\beta \cdot (\underset{\sim}{A}_{3,\alpha} + b^\mu_\alpha \underset{\sim}{b}_\mu)]$$

$$+ N^{\alpha 3}\, [\underset{\sim}{N} \cdot R^o_{,\alpha}] - M^{\alpha 3}\, [k_{\alpha 3} - \underset{\sim}{N} \cdot \underset{\sim}{A}_{3,\alpha}] - T^\alpha[h_{3\alpha} - \underset{\sim}{b}_\alpha \cdot \underset{\sim}{A}_3]$$

$$- T^3[h_{33} - \underset{\sim}{N} \cdot (\underset{\sim}{A}_3 - \underset{\sim}{N})] - \underset{\sim}{F} \cdot \underset{\sim}{R}^o - \underset{\sim}{C} \cdot \underset{\sim}{A}_3\}\, ds$$

$$+ \int_{c_v} [\underset{\sim}{N} \cdot (\underset{\sim}{R}^o - \tilde{\underset{\sim}{R}}^o) + \underset{\sim}{M} \cdot (\underset{\sim}{A}_3 - \underset{\sim}{A}_3)]\, dc$$

$$- \int_{c_t} [\underset{\sim}{N} \cdot \underset{\sim}{R}^o + \underset{\sim}{M} \cdot \underset{\sim}{A}\ ]\, dc$$

The functional $\bar{P}_V$ is dependent on all variables, displacements ($\underset{\sim}{R}_o, \underset{\sim}{A}_3$), strains ($h_{ij}$, $k_{\alpha i}$), stresses ($N^{\alpha i}$, T^i, $M^{\alpha i}$) <u>and</u> rotations (of $\underset{\sim}{b}_i$). The latter is a two-dimensional counterpart of the Hu-Washizu functional (Hu, 1955 - Washizu, 1955) cast in terms of the rotated system ($\underset{\sim}{b}_i$) and the engineering strains (h_{ij}, $k_{\alpha i}$). The stationary conditions are all equilibrium conditions, stress-strain relations and the strain-displacement relations.

<u>Correlation with Classical Theory</u>

Alternative choices of strains and stresses are always possible. From (20a), we could adopt the flexural strain

$$\kappa^\alpha_\beta \equiv \underset{\sim}{A}^\alpha \cdot \underset{\sim}{A}_{3,\beta} + b^\alpha_\beta = - K^\alpha_\beta + b^\alpha_\beta$$

Then, from (21b)

$$\dot{D}_{\alpha\beta} = + \dot{k}^{\nu}_{\alpha} C_{\nu\beta} - K^{\nu}_{\alpha} \dot{C}_{\nu\beta}$$

The first terms of (19b) assume the form:

$$\dot{W} = (N^{\alpha\beta} - K^{\alpha}_{\nu} M^{\nu\beta}) \dot{C}_{\alpha\beta} + M^{\alpha\nu} C_{\beta\nu} \dot{k}^{\beta}_{\alpha}$$

This suggests that we adopt, as membrane and flexural stresses, respectively,

$$n^{\alpha\beta} = N^{\alpha\beta} - K^{\alpha}_{\nu} M^{\nu\beta}$$

$$m^{\alpha}_{\cdot\beta} = M^{\alpha\nu} C_{\beta\nu}$$

The latter are the usual choices (cf. Koiter, 1966, 1973; Sanders, 1963; Leonard, 1961; Naghdi, 1972). Under the Kirchhoff-Love hypothesis, $K^{\alpha}_{\beta} = B^{\alpha}_{\beta}$. If products of strains and stresses are also dismissed, then

$$n^{\alpha\beta} \doteq N^{\alpha\beta} - b^{\alpha}_{\nu} M^{\nu\beta}$$

$$m^{\alpha}_{\cdot\beta} \doteq M^{\alpha}_{\cdot\beta}$$

Under these circumstances the latter choices pose no difficulties; however in the general nonlinear theory, we need the separation of stresses and strains, and the unambiquous transformation (26). Though unconventional, our strains ($h_{\alpha i}$, $k_{\alpha i}$) and stresses ($N^{\alpha\beta}$, $M^{\alpha i}$, T^{i}) provide a precise theory under the one hypothesis (11).

If transverse strains are neglected, and surface strains are small, then (19b) assumes the usual form:

$$\dot{W} = n^{\alpha\beta} \dot{h}_{\alpha\beta} + m^{\alpha\beta} \dot{k}_{\alpha\beta} \tag{31}$$

$$+ (N^{\alpha 3} - T^{\alpha}) \dot{\omega}_{3\alpha} + n^{\alpha\beta} \dot{\omega}_{\beta\alpha}$$

From (31) we can draw the anticipated conclusions: Since $k_{\alpha\beta} = k_{\beta\alpha}$, only the symmetrical part of $m^{\alpha\beta}$ plays a role. Equilibrium requires that the stress $T^{\alpha} = N^{\alpha 3}$, the transverse shear force. Also, we note the equilibrium requirement $n^{\alpha\beta} = n^{\beta\alpha}$.

On Application of the Nonlinear Theory

In general, 'solutions' (actually approximations) of the nonlinear equations (differential equations of the continuous shell or algebraic equations of a discrete model) must be obtained by successive solutions of linear systems which govern increments (Wempner, 1971). In particular, we record the linear relations between incremental rotations ($\dot{\omega}_{ji} = b_j \cdot \dot{b}_i$), strains ($\dot{C}_{i\alpha}$) and displacements ($\dot{\underset{\sim}{R}}^{o}, \dot{\underset{\sim}{A}}_3$):

$$\dot{\omega}_{i\alpha} = c_{\alpha}^{\phi}(\underset{\sim}{b}_i \cdot \underset{\sim}{A}_{\phi}^{\cdot} - \dot{C}_{\phi i})$$

$$\dot{C}_{ij} = \underset{\sim}{b}_j \cdot \underset{\sim}{A}_i^{\cdot} - \dot{\omega}_{jk}\, C_i^k$$

Recall that $C_{\alpha 3} = C_{\alpha}^{3} = 0$ and

$$\underset{\sim}{A}_{\phi} \equiv \frac{\partial \underset{\sim}{R}^{o}}{\partial \theta^{\phi}} .$$

Note that the rotation ($\dot{\omega}_{i\alpha}$) is determined entirely by the displacement ($\underset{\sim}{R}^{o}$) of the reference surface. Also, increments of the rotation tensor are given by

$$\dot{r}^{j}_{.i} = \dot{\omega}^{k}_{.i}\, r^{j}_{.k}$$

The displacement of the 'normal' ($\underset{\sim}{A}_3$) enters only in the determination of transverse shear ($\dot{C}_{3\alpha}$).

Conclusion

A theory of shells is founded on the one assumption: The normal remains straight or, equivalently, the displacement is a linear function of the normal coordinate. The theory is otherwise general: Finite rotations, finite strains and transverse shear strains are admitted without additional approximations. The theory is expressed by the potential and the complementary potential, in the manner of Fraeijs de Veubeke (1972). These functionals are expressed in terms of a rotated system and engineering components of strain. The theory is also expressed by a functional of displacement and stress in the manner of Reissner (1950) and by a functional of displacement, strain and stress in the manner of Washizu (1955). All are precisely consistent with the one underlying assumption. The theory encompasses the more restrictive versions based upon the hypothesis of Kirchhoff-Love; all incorporating the decomposition of rotation and strain. This provides a general basis for the approximations, via finite elements, without the limitations of earlier work (Wempner, 1969). As noted then, approximations of small rotations within discrete elements involve only small rotations relative to the rotated basis ($\underset{\sim}{b}_i$); such elemental models are nonetheless applicable to finite rotations in the assembly.

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Appendix - Some Differential Geometry and Notations:

Where possible, minuscules signify variables of the initial state and majuscules signify variables of the current (deformed state). Unless specifically noted, components are associated via the metric of the initial state.

The basis of the initial state is the triad ($\underset{\sim}{a}_i$) and reciprocal triad ($\underset{\sim}{a}^i$):

$$\underset{\sim}{a}_\alpha \equiv \underset{\sim}{r}^o_{,\alpha} \ , \ \underset{\sim}{a}_3 \equiv \underset{\sim}{n} \ , \ \underset{\sim}{a}^i \cdot \underset{\sim}{a}_j = \delta^i_j$$

The rigidly rotated triad ($\underset{\sim}{b}_i$) and reciprocal triad ($\underset{\sim}{b}^i$) also form the components of the initial metric:

$$a_{\alpha\beta} \equiv \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta = \underset{\sim}{b}_\alpha \cdot \underset{\sim}{b}_\beta$$

$$a^{\alpha\beta} \equiv \underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}^\beta = \underset{\sim}{b}^\alpha \cdot \underset{\sim}{b}^\beta$$

The triad ($\underset{\sim}{A}_i$) and the reciprocal triad ($\underset{\sim}{A}^i$) are defined by the equations:

$$\underset{\sim}{A}_\alpha = \underset{\sim}{R}^o_{,\alpha} \ , \quad \underset{\sim}{A}_3 = \underset{\sim}{R}_{,3} \ (\theta^1,\theta^2,0), \quad \underset{\sim}{A}_i \cdot \underset{\sim}{A}^j = \delta^j_i$$

The stretch is defined by (17a); the inverse (or contraction) is denoted by the minuscule c^α_β and defined by

$$c^\alpha_\beta \ C^\beta_\gamma = \delta^\alpha_\gamma \ ,$$

Relations between the triads, ($\underset{\sim}{b}_i$, $\underset{\sim}{b}^j$) and ($\underset{\sim}{A}_i$, $\underset{\sim}{A}^j$) follow:

$$\underset{\sim}{A}_\alpha = C_\alpha^\beta \, \underset{\sim}{b}_\beta = C_{\alpha\beta} \, \underset{\sim}{b}^\beta$$

$$\underset{\sim}{A}^\alpha = c_\beta^\alpha \, \underset{\sim}{b}^\beta$$

$$\underset{\sim}{b}_\alpha = c_\alpha^\beta \, \underset{\sim}{A}_\beta = C_{\alpha\beta} \, \underset{\sim}{A}^\beta$$

$$\underset{\sim}{b}^\alpha = C_\beta^\alpha \, \underset{\sim}{A}^\beta$$

The shear is defined by (17b); the mixed components follow:

$$C_3^i = \underset{\sim}{b}^i \cdot \underset{\sim}{A}_3 = a^{ij} \, C_{3j}$$

Mechanics and finite elements of shells

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Mechanics and finite elements of shells

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This article begins with a brief review of the foundations: The classical theory of Love is described with attention to the underlying hypothesis and consequent limitations. A more general theory is described which removes the constraints of Love; the inclusion of transverse strains admits simpler finite elements, accommodates the thick shell via layers and even a transition to the three-dimensional approximation.

The concept of the finite element is reviewed in the context of the discrete approximation of shells. Specific attention is given to those problems which are peculiar to shells: The predominant roles of flexural and extensional deformations, the lesser role of transverse shear, can lead to excessive stiffness ("locking"). Origins and procedures are described to circumvent these problems.

The review is intended to bridge some chasms between the mechanics of the continua and the discrete models of finite elements. As such, the emphasis is upon those mechanical attributes of shells and elements which play key roles in forming practical models. Since the limitations of space, time and the author's knowledge, preclude a full expose, the review includes only commentaries on some topics, such as inelasticity, nonlinearity and instability. Citations include original sources and some recent works which provide entree to contemporary developments.

CONTENTS

INTRODUCTION

The mechanics of shells is a mature subject. Indeed, the classical theory of thin hookean shells was well developed a century ago. In a paraphrase from Koiter, all one needs is Love, Love, Love-[Love, 1888]. To be sure, the underlying theory of Love has been refined [Koiter, 1960; Leonard, 1961; Sanders, 1963] and extended to accommodate transverse shear [Reissner, 1945] and other effects as might occur in thick shells. Through the intervening years, much activity has been directed toward reexamination and simplification of the governing equations, to achieve closed-form solutions. The activities and developments were dictated by the availability of tools, mathematical and computational. A recasting of the differential equations, a change of variables, a form of asymptotic expansion, a deletion of incidental terms, all might produce more useful theory or solution. With the advent and rapid development of digital computers our attention shifts to a reassessment of the subject. Refinements or simplifications, which are important to the closed-form solutions, may be unimportant, even detrimental, to a formulation for numerical approximation; other changes can improve and simplify an approximation by finite elements. These few observations provide a backdrop for this review. In particular, we wish to identify features which are most relevant, or irrelevant, to present tools and, if necessary, recast our description in forms most amenable to formulations of discrete models for computational purposes. Two different approaches can be taken to devise approximations of a continuous shell: One can employ entirely mathematical tools, finite-difference approximations of differential equations, or approximations of functionals. Alternatively one can perceive a physical model of the shell as an assembly of finite pieces. Such assembly of finite elements can benefit greatly by an intimate view of the underly-

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ing mechanics. It is this aspect of the subject which commands our attention.

An extensive review of the vast literature is not intended. As an introduction, the reader can consult the standard texts of Timoshenko and Woinowsky-Krieger [1959] and Flügge [1962]; both provide much practical information and important historical commentary. Some recent works have compiled excellent bibliographies: The book by Calladine [1983] provides a good introduction and key references to earlier works; it offers a variety of practical information in a traditional way. The book by Basar and Krätzig [1985] gives a thorough presentation of the technical theory, important practical applications and results, and includes numerous citations. The recent monographs by Libai and Simmonds [1983; 1988] present the mathematical theory and extensive bibliographies. The definitive work of Naghdi [1972] presents the mathematical foundations and important references. A review of some contemporary topics is given by Wempner [1980b]. Recent work on approximation by finite elements is given in books edited by Hughes and Hinton [1986].

Thin shells play an ever-increasing role in structures, as the engineer strives to achieve strength and stiffness with minimal weight. The greater efficiency of the shell is attributed to the curvature; the price is the greater susceptibility to buckling. Buckling and postbuckling deformations, and at times the prebuckling deformations, are characterized by nonlinear equations. Often such deformations occur with small strains, and even hookean behavior. Then the nonlinearities are attributed to the occurrence of large, even moderate, rotations; the attendant curvatures and large membrane forces are coupled to produce crucial nonlinear terms. The nonlinear theories for thin hookean shells have been the subject of intense investigation. Most are founded on the hypothesis of Kirchhoff and Love; transverse shear strain is neglected. Nonlinearities are geometrical in origin. These aspects of the theory are presented in an article by Koiter [1956], the monograph by Naghdi [1972], the works of Reissner [1974] and Koiter and Simmonds [1973], and the recent monograph by Libai and Simmonds [1983]. The role of finite rotation and its representation has been the subject of papers by Wempner [1969b], Simmonds and Danielson [1970], and Pietraszkiewicz [1980]. These citations are incomplete, but provide the essentials as well as access to a wealth of additional references.

THEORY OF SHELLS—PAST AND PRESENT

By definition a shell is a thin body enclosed by two neighboring surfaces which are interrupted by edges, boundaries, or holes. The common attribute of all shells is the proximity of these surfaces. It is this thinness which justifies approximations through the thickness and a two-dimensional description, a "theory" of the shell. From a practical viewpoint, we need a "theory" which is suited to subsequent approximations in the remaining two dimensions, to provide the discrete model, the algebraic equations, and basis for numerical computation. Also, from a realistic viewpoint we want a theory which is suited to refinements through the thickness. In any case, a "theory" is but one step in the approximation of the real shell. The first approximation [Love, 1888] implies that a normal to a surface remains normal. Higher-order theories are less restrictive, but also incorporate approximations which render the description in two dimensions. In former times, a primary motivation was the simplification of the governing differential equations in order to obtain solutions. The desire to simplify for purposes of solution led to various special theories, such as the theories for shallow shells and Donnell's equations for cylindrical shells [Donnell, 1934].

The underlying approximations always impose some limitations. Basically, the theory precludes any refinements in the description of deformations or stress distributions through the thickness. A practical example is the question of stress at the juncture of two shells. At the fillet of such intersections the gradients can be large; the actual stress can be quite different than the distribution implicit in the two-dimensional theory. Stated otherwise, the actual behavior at the juncture, yielding or fracture, is a phenomenon which requires a three-dimensional description.

In the past, investigators addressed questions of thick shells, concentrated loadings, and other extraordinary questions by refined theories. Such theories were founded upon better approximations with respect to the thickness; for example, the displacement might be given by an approximation which included polynomials of higher degree. Invariably such theories tend toward more complicated equations. The reader may gain insights from the recent article by Reissner [1985].

The common feature of classical theories, first or higher approximations, is representation by continuous variables. Indeed, with the exception of isolated points or lines, continuity extends to derivatives of all orders. These features are all quite understandable. The attributes, the various simplifications or refinements, were intended to produce results by the traditional means of solving the boundary-value problems or by approximations via continuous functions throughout the shell.

The remarkable capacities and computational capabilities of electronic computers enable us to adopt quite different approaches to the description and approximation of shells. To be sure, phenomena which depend upon the limits, ie, fracture and stress concentrations, cannot be described by finite elements or differences, unless appropriate singularities are specifically incorporated. Otherwise, we can utilize these new-found capabilities and concentrate attention on the discrete approximation of the three- or two-dimensional shell.

If one seeks an assembly of finite elements, then the role of the theory is quite different. It is no longer important that the differential equations be simplified, but it is desirable that the theory be amenable to simple elemental approximations. It is also important that the theory be cast in alternative potentials and functionals which provide the means to formulate consistent elements based upon approximation of displacement, and/or stress and/or strain, so-called "mixed" elements.

The essence of approximation via finite elements is the introduction of nodal values with intermediate interpolation. In this spirit, a refined approximation of the shell is accomplished by introducing "nodes" through the thickness. Stated otherwise, the theory of the thin shell progresses to the theory of a thick shell via more layers. From this viewpoint, the underlying theory ought to be amenable to such transition; in its primitive form it ought to include transverse extension and shear. Such theory opens the door to a complete transition from the simplest thin membrane, to the thickest shell and to the three-dimensional model. Unfortunately, the traditional first approximation omits the transverse strains; hence, it does not provide the requisite basis for such transition. Nonetheless, our wealth of knowledge about this classical theory warrants our attention.

THE FIRST APPROXIMATION

It is fitting that any review of shells sketch the classical theory which originated with Aron [1874], Love [1888], and Kirchhoff [1850]. Our intent here is to present the essentials as simply as possible. The deformation of a thin shell is essentially described by the deformation of a surface, often the middle surface. The differential geometry of the surface is characterized by two second-order symmetrical tensors; they are the coefficients in the fundamental forms. The differential of length s on the surface is expressed by a quadratic form in the differentials of the surface coordinates θ^α ($\alpha = 1,2$)

$$ds^2 = a_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta. \tag{1}$$

If $\mathbf{r}$ is the position vector to a point of the surface and $\mathbf{n}$ the unit normal, the second fundamental form is

$$d\mathbf{r}\cdot d\mathbf{n} = -b_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta. \tag{2}$$

The coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the symmetrical components of the metric and curvature tensors, respectively. If the surface is deformed, then the position is given by the vector $\mathbf{R}$, the normal by $\mathbf{N}$; the surface is stretched and bent. The fundamental forms of the deformed surface are

$$dS^2 = A_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta, \quad d\mathbf{R}\cdot d\mathbf{N} = -B_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta. \tag{3}$$

The lines of the coordinates are now convected, stretched, and bent; eg, the measure of distance along the line of θ^1 is changed from a_{11} to A_{11} and the measure of curvature from b_{11} to B_{11}. Two symmetrical strain tensors characterize the deformation, an extensional strain $\gamma_{\alpha\beta}$ and flexural strain $\kappa_{\alpha\beta}$; simple measures are

$$\gamma_{\alpha\beta} \equiv \tfrac{1}{2}\left(A_{\alpha\beta} - a_{\alpha\beta}\right), \tag{4a}$$

$$\kappa_{\alpha\beta} \equiv \left(B_{\alpha\beta} - b_{\alpha\beta}\right). \tag{4b}$$

The internal power (per unit of area) in any deformation assumes the form:

$$\dot{w} = n^{\alpha\beta}\dot{\gamma}_{\alpha\beta} + m^{\alpha\beta}\dot{\kappa}_{\alpha\beta}. \tag{5}$$

In short, the theory of the surface (first approximation of the thin shell) calls for two symmetrical stresses, $n^{\alpha\beta}$ and $m^{\alpha\beta}$. To ascribe physical meaning to these stresses, one can invoke the hypothesis of Kirchhoff and Love: The initial *normal* $\mathbf{n}$ turns to the *normal* $\mathbf{N}$ during deformation. Work is done by forces and couples acting upon a section (θ^α = const). Then the stresses $n^{\alpha\beta}$ and $m^{\alpha\beta}$ are identified with components of the force and couple.

Several features of the theory are noteworthy: Transverse shear strain is nonexistent. Transverse shear force is merely reactive and transverse normal stress is also workless; neither contribute to the internal energy w. If the shell is elastic, then an internal potential w must depend upon the strain, $w = w(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$, and the stress-strain relations follow:

$$n^{\alpha\beta} = \frac{\partial w}{\partial \gamma_{\alpha\beta}}, \tag{6a}$$

$$m^{\alpha\beta} = \frac{\partial w}{\partial \kappa_{\alpha\beta}}. \tag{6b}$$

If the shell obeys the theory of classical plasticity, then a yield function $Y(n^{\alpha\beta}, m^{\alpha\beta})$ exists and incremental plastic strains follow:

$$\dot{\gamma}^P_{\alpha\beta} = \dot{\lambda}\,\frac{\partial Y}{\partial n^{\alpha\beta}}, \tag{7a}$$

$$\dot{\kappa}^P_{\alpha\beta} = \dot{\lambda}\,\frac{\partial Y}{\partial m^{\alpha\beta}}. \tag{7b}$$

Now, the first approximation of elastic shells and, especially hookean shells, has attained a most definitive form. Edge conditions, constitutive equations, kinematical approximations for small strains, finite and moderate rotations, numerous approximations, and solutions are available. The reader can consult the works of Koiter [1956], Naghdi [1972], and Libai and Simmonds [1983]. These give references to notable contributions by Reissner, Vlasov, Sanders, Leonard, and Goldenveizer, to name but a few.

First approximations of inelastic shells remain quite obscure and pose many unresolved questions. Some recent work is given by Bieniek and Funaro [1976], Kutt and Bieniek [1988], and Eggers and Kröplin [1978]. Derived theories and approximations are given by Wempner [1977; 1980a]. Additional references on specific theories are contained in the author's earlier review [1980b].

GENERAL THEORY FOR FINITE ELEMENTS

In our opening remarks, we have noted certain limitations of the first approximation and have alluded to higher approximations, derived from refined descriptions of the displacement, strain and/or stress distributions through the thickness. With a view toward approximation by finite elements, such refinements, and even a progression to a three-dimensional description, can be achieved by a succession of layers, *provided* only that our basic theory of the shell (the layer) includes the six components of strain (and stress); specifically, the theory must then include transverse shear and extensional strain which are lacking in the first approximation. We can achieve the requisite theory by approximating the position in the shell (or layer) as a linear function of the thickness coordinate (θ^3) in the current configuration ($\mathbf{R}$) as in the initial configuration ($\mathbf{r}$):

$$\mathbf{r} = \mathbf{r}_0(\theta^1, \theta^2) + \theta^3\mathbf{n}(\theta^1, \theta^2), \tag{8a}$$

$$\mathbf{R} = \mathbf{R}_0(\theta^1, \theta^2) + \theta^3\mathbf{A}_3(\theta^1, \theta^2). \tag{8b}$$

Here, θ^3 denotes distance along the initial normal $\mathbf{n}$ to a reference surface: Vector $\mathbf{A}_3$ differs from the unit normal $\mathbf{N}$ of the deformed surface by virtue of a transverse shear and extensional strain. The geometrical approximation is expressed above in the symbolism of shell theories, ie, positions of a surface, $\mathbf{r}_0$ and $\mathbf{R}_0$, initially and subsequently, and distance from the surface θ^3. Instead of the vectors $\mathbf{R}_0$ and $\mathbf{A}_3$, the approximation can be expressed in terms of positions $\mathbf{R}_-$ and $\mathbf{R}_+$, on the inner ($\theta^3 = -h$) and outer ($\theta^3 = +h$) surfaces, respectively:

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{R}_+ + \mathbf{R}_-) + \tfrac{1}{2}(\mathbf{R}_+ - \mathbf{R}_-)\theta^3/h.$$

The vectors $\mathbf{R}_0$ and $\mathbf{A}_3$, and the alternatives $\mathbf{R}_+$ and $\mathbf{R}_-$, are shown in Fig. 1. In the parlance of finite elements the position is approximated linearly between "nodal" values $\mathbf{R}_-$ and $\mathbf{R}_+$, at the inner ($-$) and outer ($+$) surfaces. If the vectors $\mathbf{R}_0$ and $\mathbf{A}_3$ (or $\mathbf{R}_-$ and $\mathbf{R}_+$) are approximated in a two-dimensional element of the surface by a *bi*linear form in the coordinates (θ^1, θ^2), then the hexahedral element of the shell is approximated by a *tri*linear form; it is a familiar *conforming* element in three dimensions. Usually, a thin shell requires but one element through the thickness. As need arises, near an edge, a discontinuity or a concentrated load, transition can be made from the one element to more elements.

To facilitate studies of finite deformations, especially the finite rotations of very thin shells, the motions can be decomposed into rigid motions and strains. In theories of three-dimensional continua the so-called "polar decomposition" separates the strain from the rigid rotation of principal lines; the strain is expressed by a symmetrical tensor. Since the foregoing theory admits transverse shear strain, the rotation of the "polar decomposition" does not carry the reference surface to its final orientation, but rotates tangent lines out of the deformed sur-

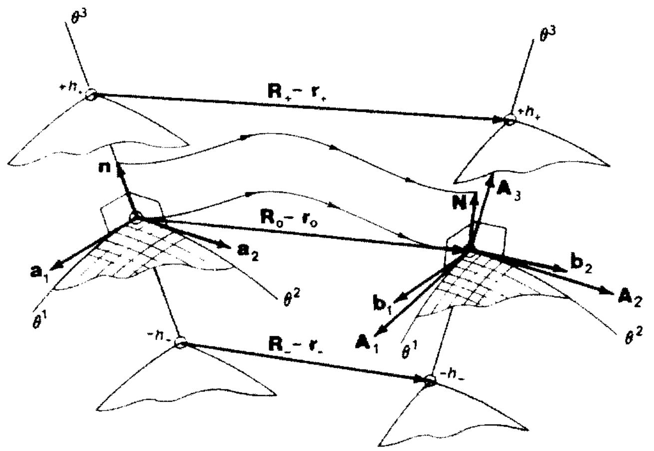

FIG. 1. Kinematics of the shell.

face. Accordingly, it is more natural to employ the rotation which carries the initial normal **n** to the current normal **N**, and the initial tangent $\mathbf{a}_\alpha$ ($\mathbf{a}_\alpha \equiv \mathbf{r}_{0,\alpha}$) to the tangent $\mathbf{b}_\alpha$. Thus the initial triad ($\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}$) is *rigidly* transported to a similar triad ($\mathbf{b}_1, \mathbf{b}_2, \mathbf{N}$) wherein the $\mathbf{b}_\alpha$ are tangent to the current (deformed) surface and **N** is normal. That rigid motion is depicted in Fig. 1. The deformation carries the rotated triad ($\mathbf{b}_1, \mathbf{b}_2, \mathbf{N}$) to the triad ($\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$) ($\mathbf{A}_\alpha \equiv \mathbf{R}_{0,\alpha}$), also shown in Fig. 1. The rotation is unique since the orientation of the intermediate triad is determined by the condition $\mathbf{A}_\alpha \cdot \mathbf{b}_\beta = \mathbf{A}_\beta \cdot \mathbf{b}_\alpha$. Complete details of the decomposition, the rotation and strain are contained in a recent article [Wempner, 1986]. That article gives also the potential, modified potential (a functional of displacements, strains and stresses), the complementary potential, and the modified complementary functional (dependent on displacements, rotations, and stresses). Accordingly, the theory provides a basis for approximations of finite deformations by various "mixed" elements. Finally, if the transverse strains are neglected, the foregoing theory assumes a simpler form, like the classical first approximation.

TWO FORMS OF APPROXIMATION

Most approximations of a function are expressed as a linear combination of prescribed functions; for example, a function $f(x)$ of coordinate x might be approximated by a linear combination of functions $g_N(x)$:

$$f(x) = \sum A_N g_N(x), \quad N = 1, 2, \cdots.$$

The function g_N can be of two distinct types. It may have global support or local support. The former has nonzero values throughout the region (the body). The latter is nonzero only in a subregion. An example of the former is the trigonometric function $\sin N\pi x/l$; a combination serves to approximate the deflection of a beam, simply supported at the ends ($x = 0, l$). An example of the latter is a "pyramid," nonzero in the subinterval $x_{N-1} < x_n < x_{N+1}$:

$$g_N = \frac{x - x_{N-1}}{x_N - x_{N-1}}, \quad x_{N-1} \le x \le x_N,$$

$$g_N = \frac{x_{N+1} - x}{x_{N+1} - x_N}, \quad x_N \le x \le x_{N+1}.$$

This latter form is no more than a linear interpolation between adjoining nodal values A_N (at x_N). The functions of global support have long been the bases for the approximation of solid bodies and have proven particularly effective when the deformations throughout the body are strongly coupled. The function of our example ($\sin N\pi x/l$) is a natural mode of vibration and also a buckled mode of equilibrium. The functions of local support are typical of the approximations via finite elements; they are often termed "shape" functions and described in books on finite elements. The region of support may be one element or, more often, the region of elements adjoining the node. Some pertinent comments are given in a previous note [Wempner, 1971b].

Traditionally, theories of shells are founded upon approximations in the thickness (coordinate θ^3) by functions which are nonzero (usually continuous of all orders) throughout the thickness. For example, our theory [Wempner, 1986] assumes a displacement which is linear through the thickness. Higher-order theories employ polynomials of higher degree to describe the displacement and/or strain and/or stress throughout the thickness [cf. Reissner, 1985]. Such theories involve no sublayers and no functions of local support (through the thickness).

Here, our focus is on approximations via finite elements with respect to a thin *or* moderately thick three-dimensional body. From this viewpoint, it is essential that the underlying theory of the shell incorporate all attributes (strains) of a three-dimensional deformation. But in the spirit of the finite elements, high-order functions are not needed through the thickness, since refinement is anticipated by increasing the numbers of layers. Stated otherwise, most traditional theories constitute a description which is one element in the thickness. Now, we take the view that the description can be simple since more elements can be employed; however, the description must possess those features (specifically, the six strains), which admit the transition to the three-dimensional body.

CONCEPT OF FINITE ELEMENTS

The approximation of solids by finite elements is an effective means to utilize modern computational capabilities. The idea is quite simple: The body is subdivided into small finite elements. The variables (usually the variables which govern the theory of the continuum) are approximated within the elements. The approximations must be such that the assembly approaches the continuum as the size of the elements diminishes. At least the position must be approximated by some form of interpolation between the adjoining elements. Indeed, if the approximation of position is continuous and if the algebraic equations of the discrete assembly are drawn from the principle of minimum potential, then a convergence is assured [Johnson and McLay, 1968]. One very important feature of such approximations of an elastic body deserves particular emphasis: The principle of minimum potential asserts that any valid approximation of a stable configuration possesses greater potential than the actual configuration (the "exact solution"). In practice, this means that the approximation of the body (eg, the assembly of elements) exhibits excessive stiffness. Very bad approximations often exhibit such unacceptable stiffness that they are said to "lock." The very thinness of shells and the associated flexibility makes them most susceptible to inappropriate approximations and "locking."

The use of finite elements has special implications in the theory and approximation of shells: A traditional theory of shells may be viewed as a first step toward the complete approximation via three-dimensional elements since the theory is founded upon approximations of the continuous variables (displacement and strain and/or stress) with respect to one

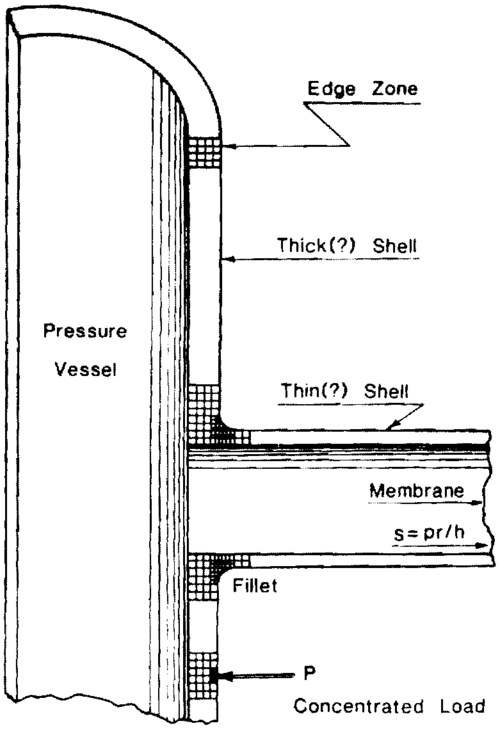

FIG. 2. Some realities of shells.

coordinate, distance along the normal. The subsequent, or simultaneous, approximation in the remaining two dimensions provides the fully discrete model. Our studies show, however, that traditional theories are not necessarily well suited to such approximation. Specifically, the theory of Kirchhoff and Love requires a high order of interpolation so that the normals remain continuous at the contiguous edges of adjoining elements. Stated otherwise, the reference surface cannot have kinks; the normals must possess continuous derivatives along the interelement lines. To use the theory of Kirchhoff and Love and also to insure such conformity of adjoining elements requires hermitian interpolation [see, eg, Zienkiewicz, 1977, p. 206]; then, 16 nodal values are needed to approximate a function in a quadrilateral element of a surface. Also, since assemblies of elements are intended to accommodate unknown configurations, many elements are needed; then computational efficiency and limited storage call for simple elements. To admit simpler elements, the basic theory must forsake Kirchhoff's hypothesis and admit the relative rotation of the normal (transverse shear strain).

In the context of approximation via finite elements, the shell is most conveniently viewed as a three-dimensional, albeit thin body. This viewpoint has important practical consequences: some are illustrated by Fig. 2. Here we note that parts of the structure are quite adequately described by the simple theory of membranes: The thin cylindrical pipe expands under internal pressure; the only significant stress is the "hoop" stress ($S = pr/h$). Other regions are adequately described by the first approximation of Love, which accounts for bending, but discounts transverse shear strains. Now, important practical questions (yielding and failure) arise at the junctures and at points of loading. Such questions entail a three-dimensional description or, at least, a higher-order theory with respect to the third dimension. In a discrete model of elements, it is most natural to introduce transitions from the shell (one layer) to additional layers and/or to the three-dimensional assembly. Accordingly a basic theory of the shell need not incorporate higher-order approximations through the thickness, but rather an approximation which is readily adaptable to the transition.

Two further considerations are basic to the discrete assembly of elements and to the effective numerical approximation. First, the theory of the continuum ought to be couched in the alternative functionals and the associated variational theorems: The principle of minimum potential, or the stationary theorems of Reissner [1950] or Hu and Washizu [1955]. These alternatives are needed as means to develop so-called "mixed models" or "hybrid elements," which are devised by approximation of stresses and/or strains, as well as displacement, and often admit discontinuities at interfaces [see texts by Zienkiewicz, 1977 and Gallagher, 1974]. Second, the effective treatment of geometrical nonlinearities, particularly finite rotations, calls for the decomposition of strains and rotations.

APPROXIMATIONS OF SHELLS BY FINITE ELEMENTS

From the practical viewpoint it is well to remember that a shell is actually a three-dimensional body, albeit one which is thin in one dimension. As such, concepts which apply generally are also applicable to shells. Additionally, one tries to exploit the thinness and to reduce the theory of three dimensions to a theory of two. Practically, one must remain wary that a given shell is thin enough and that the particular phenomena, deformations, and other responses are amenable to the inherent limitations of such two-dimensional theory. With this in mind, one ought to view each shell as a candidate for a higher-order approximation or even a three-dimensional theory, as circumstances warrant. Examples were cited previously and depicted in Fig. 2.

The engineer can find some comfort in an approximation which is derived, in a consistent manner, by means of a variational theorem: The basis is a potential, or functional, of the three-dimensional variables; the stationary conditions are the governing equations of the theory. Here, the term "consistent" is the key. In a very general way, consistent approximations must possess the mathematical attributes which insure convergence to the solution. From a practical viewpoint, the approximations must provide an adequate description without excessive computational costs.

Certainly the most reliable basis for the approximation of an elastic body is the principle of minimum potential energy. According to this principle, a stable equilibrium configuration is characterized by a local minimum of the potential. Any consistent approximation of the displacement increases the potential. Physically such approximation has the effect of constraining the body which then exhibits excessive stiffness. A simple example serves to illustrate our point: The simple beam of Fig. 3(a) is subject to opposing couples at the ends. The beam can be approximated by dividing it into a finite number of elements (five are depicted). If the displacement is approximated linearly in each segment, then the *best* approximation carries each rectangular element into a similar trapezoid. Under the usual assumptions of linear elasticity the potential is calculated and the result follows:

$$M = \frac{Eh^3}{12}\left(1 + 2\frac{G}{E}\frac{l^2}{h^2}\right)\frac{\Delta\phi}{l}.$$

In the limit, as the length of the element diminishes, $l \to 0$,

$$\Delta\phi/l \to \kappa \text{ (curvature)}.$$

$l/h \to 0$, and the exact solution follows: $M = (Eh^3/12)\kappa$. In the words of Bruce Irons, the elemental approximation is "legal";

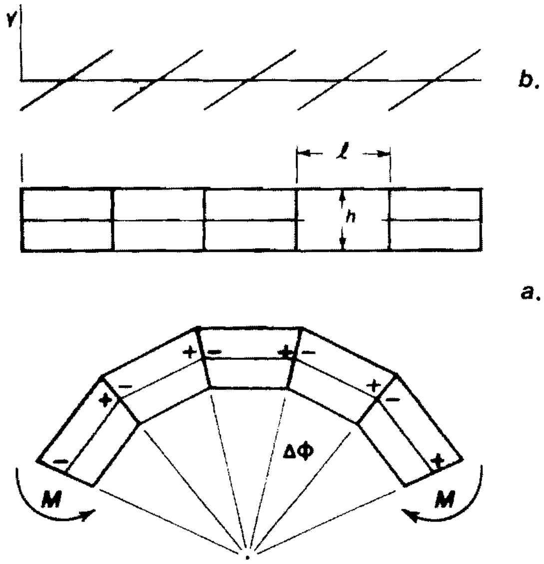

FIG. 3. Simple bending of a beam.

however, the approximation is *bad* from a practical viewpoint. It is bad, because the approximation of the shear strain is bad. The latter has the sawtooth plot of Fig. 3(b). The approximation is strictly consistent and, indeed, converges to the solution, but illustrates a need to examine the physical implications and consequences of an approximation. The foregoing example illustrates one pitfall of finite elements, a bad approximation which causes excessive internal energy and stiffness, so-called "shear locking." If one were to exclude the energy of the shear strain, then the element is entirely unconstrained from a mode of simple shear. The practical consequence of such "spurious" modes (zero energy) is worse: An assembly of elements may buckle; mathematically, the stiffness matrix may be singular. The challenge is to device consistent elements, which insure convergence and exhibit neither unwarranted "locking" nor "spurious" modes. One must examine the mechanics of simple elements with a view toward consistently simulating all motions, rigid and deformational modes, with attention to the resistance to all deformational modes.

To devise any approximation, we ought to draw upon our years of experience with thin shells: We have ample evidence that Love's approximation is indeed quite good for thin shells: transverse shear strain is usually negligible. Clearly, the approximation can not be used with simple approximations such as the piecewise linear form of Fig. 3. No kinks are admissible at the intersections of contiguous elements. To utilize the Love approximation, many earlier elements were derived from higher-order polynomials which provide the requisite continuity. An alternative was proposed by the author [1968], a discrete counterpart of Love's hypothesis: Admit transverse shear strain, impose *discrete* kinematical constraints and delete the energy of shear such that the model (the assembly of elements) converges to Love's approximation in the limit, as the size of the elements diminishes. In a manner of speaking, one can make Love simply, but one must do so discretely. The incidence of such affairs is quite common; the precise form of the constraint varies in keeping with the formulation.

The discrete form of Love's hypothesis is not the sole means to avoid the excessive stiffness caused by shear. Indeed, the flaw in the model of Fig. 3 is not the approximation of the displacement, but the consequent approximation of the shear strain (the sawteeth). Note that the mean value of the latter is zero, the correct value. A consistent and rational procedure is needed to achieve a better approximation of the strain, one which assures convergence to the correct relation of strain and displacement, but also inhibits any spurious modes. One vehicle is the modified form of the potential as given by Hu and Washizu [1955]. The latter has been applied to formulate simple, yet effective elements [Wempner, 1982; 1983] and is illustrated in our subsequent discussion.

The example of Fig. 3 serves also to illustrate the notion of "reduced integration," as espoused by various writers, eg, Zienkiewicz et al [1971]: If the shear strain is approximated by the midpoint value, then one obtains the correct result; in general, that value approaches the mean value. However, the "one-point integration" fails in the case of two-dimensional (plate or shell) elements; the latter exhibit one or two additional shear modes (in triangles or quadrilaterals, respectively), which are not constrained by the one value. The unrestrained modes possess no strain energy; hence, they are "spurious." The simple elements of our subsequent discussion illustrate such additional shear modes.

The recognition of spurious modes led to alternative forms of "reduced integration," "two-point integration," etc. and then to "selective integration," ie, to procedures which select the locations and/or values for the approximation. Always, the goal is a better selection of the value(s) and the suppression of all deformational modes. The interested reader can consult articles by Zienkiewicz et al [1971], Hughes and Taylor [1977; Hughes and Hinton, 1986], and Belytschko [1985; 1986].

In general, thin bodies (beams, plates, and shells) have special attributes which greatly influence the character of an approximation: The beam or plate resists transverse loads by virtue of transverse shear stresses and bending stresses, which are accompanied by flexural strains. In marked contrast, the very thin shell (eg, an egg) can resist transverse loads by virtue of membrane forces, which are accompanied by extensional strains. However, most shells support loads by the actions of transverse shear, bending stresses *and* membrane forces. Moreover, these actions and the associated deformations are usually coupled; specifically, flexural strains (changes-of-curvature) are usually accompanied by extensional strains. A strong coupling accounts for the buckling and snap-buckling, ie, instability at a critical load. Coupling between extension and flexure can be aggravated by poor approximations; excessive stiffness, termed "membrane locking," is a consequence. The reader is referred to articles by Belytschko et al [1985; Belytschko, 1986].

Usually the dimensions of a finite element are much smaller than a radius of curvature. Also, the dimensions are necessarily much less than any characteristic length of the deformational pattern. (Otherwise, the shape functions must anticipate the deformation.) It follows that the concepts and approximations of shallow shells may be applicable to the *individual* finite element. Discussions of shallow shells are given by Koiter [1956] and Libai and Simmonds [1983]. Applications to finite elements are given by Connor and Brebbia [1967], Cowper, Lindberg, and Olson [1970; 1971], Mote [1971], Morris [1973], and Dawe [1974].

The deformations of thin bodies are peculiar in another respect: Elements can undergo finite rotations although strains are small; curvatures and changes-of-curvatures can be relatively larger than strains. These peculiarities have far-reaching consequences, in the theories of continuous shells and also in the approximations of finite elements. The mathematical theories and the approximations must include geometrical nonlinearities which are traceable to the rotations and curvatures, though strains may be very small. On the other hand, small strains imply small *relative* rotations of neighboring lines. Consequently, *if* the element is small enough, then relative rotations are small *within* the element. Therefore, it is possible to employ a linear theory of a hookean element to approximate certain problems of finite rotation, but, then, the nonlinearities must be introduced in the assembly [Wempner, 1968; 1969a, b].

A very insightful account of the roles of extension and flexure (membrane and bending actions) is given in the recent article by Morley and Mould [1987]. Here, the authors note the various behaviors of shells, "pure" membrane action, inextensional bending, and edge effects. The authors cite the classical theory wherein extensional and membrane energies may be uncoupled and employ the simple triangular element as the vehicle for their investigation.

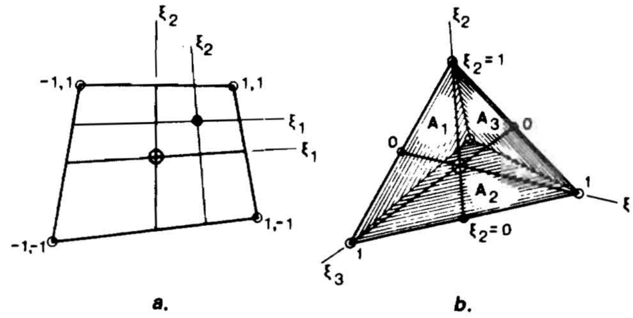

FIG. 5. Quadrilateral and triangle—local coordinates.

SIMPLE CONFORMING ELEMENTS

Any surface can be approximated by discrete "nodal" points and straight interconnecting lines which delineate quadrilateral and/or triangular elements, as shown on the surfaces of the shell in Fig. 4. The simplest quadrilateral elements are hyperbolic paraboloids; the simplest triangular elements are plane. A discrete model of the shell is customarily founded on such approximations of an intermediate surface *and* an approximation of the normal. In any subsequent state the configuration of the shell can be approximated, in the manner of the reference state: The new position of the nodes provides the approximation of the reference surface. The configuration of the shell requires also the motion (rotation and stretch) of the normal. The latter embodies the essential attributes and limitations of the shell theory. In the spirit of finite elements, we anticipate that additional layers could serve to accommodate thick shells, localized loads, and discontinuities in shape. Then the description of the individual layer need only include the rigid rotation and uniform extension of the normal. This is the theory of Fig. 1. Now, our attention is focused upon the individual element and the approximation of the element; however, the selection of nodes, the shape and size of the elements, are no less important in the approximation of the entire shell. Specifically, one must anticipate that irregular shapes are likely to cause poor approximations and that large elements are less accurate, particularly near loads and edges.

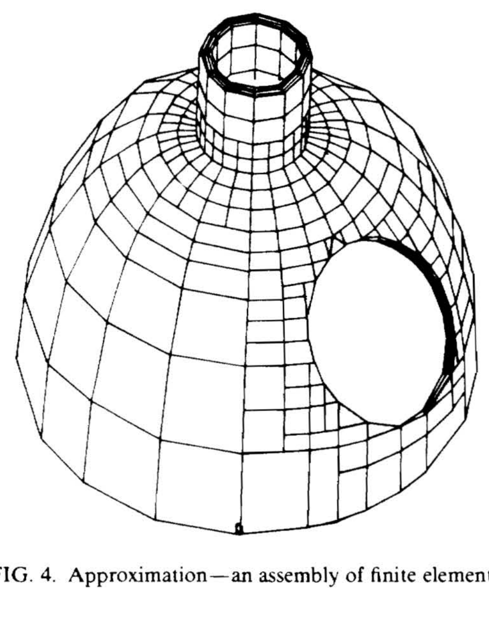

FIG. 4. Approximation—an assembly of finite elements.

With the foregoing geometrical description, the approximations of initial and subsequent configurations are similar. This is consistent because both configurations play equivalent roles in a description of the strained state. (This is an "isoparametric" model.) The simple mathematical description of the vectors which define position ($\mathbf{R}_0$) and deformed normal ($\mathbf{A}_3$) is conveniently expressed in a normalized system of coordinates. For the quadrilateral element, the origin is at the midpoint, the intersection of the straight lines which bisect opposing edges in Fig. 5(a): On the straight edges the normalized coordinates are $\xi_\alpha = \pm 1$ ($\alpha = 1, 2$). The simplest conforming quadrilateral is defined by vectors ($\mathbf{r}_0, \mathbf{n}; \mathbf{R}_0, \mathbf{A}_3$) in the bilinear form, eg,

$$\mathbf{R}_0 = \bar{\mathbf{R}} + \mathbf{R}_1\xi_1 + \mathbf{R}_2\xi_2 + \mathbf{R}_{12}\xi_1\xi_2. \tag{9a}$$

Alternatively the four discrete vectors ($\bar{\mathbf{R}}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_{12}$) can be expressed in terms of the four nodal positions. A feature of such approximation is that edges conform to those of the contiguous element (since both vary linearly and both are given by the common nodal values). One natural system of coordinates for the triangle are three so-called areal coordinates:

$$\xi_1 = A_1/A, \quad \xi_2 = A_2/A, \quad \xi_3 = A_3/A,$$

where A_1, A_2, and A_3 are the areas shown in Fig. 5(b) and $A = A_1 + A_2 + A_3$ is the entire area. Clearly, these coordinates are not independent ($\xi_1 + \xi_2 + \xi_3 = 1$), but are natural in the sense that they exhibit no geometrical bias. It is especially *noteworthy* that the coordinate $\xi_\alpha = 1$ at the node α and vanishes at the opposing side; along any line parallel to that side the coordinate is constant. The counterpart of the approximation (9a) has the linear form

$$\mathbf{R}_0 = \mathbf{R}_1\xi_1 + \mathbf{R}_2\xi_2 + \mathbf{R}_3\xi_3. \tag{9b}$$

It is important to identify the motions in each case, rigid motion, homogeneous strain, and any higher modes of deformation: First, the quadrilateral has, in accordance with the bilinear approximations of vectors $\mathbf{R}_0$ and $\mathbf{A}_3$, 24 degrees-of-freedom: six represent rigid-body motion and those remaining are deformational modes. For the sake of simplicity and practicality, let us exclude extensional strain in the transverse (θ_3) direction; it is usually negligible. Then four deformational modes are excluded, and 14 deformational modes remain. It is important to note that three modes of deformation are linear in the thickness (θ_3): these are not higher modes in a shell, but represent homogeneous states of bending. To be explicit, the element has eight dominant strains; these are the midpoint values of the membrane strains ($\bar{\epsilon}_{\alpha\beta}$), flexural strains ($\bar{\kappa}_{\alpha\beta}$) and transverse shear ($\bar{\gamma}_\alpha$). In the case of a rectangular element, the six remaining terms are linear; two are extensional, two are flexural, and two are shear modes, termed torsional and warping, as depicted graphically in Fig. 6. To accept all terms in the strain, which are compatible with the displacements ($\mathbf{R}_0 - \mathbf{r}_0$ and $\mathbf{A}_3 - \mathbf{n}$), pro-

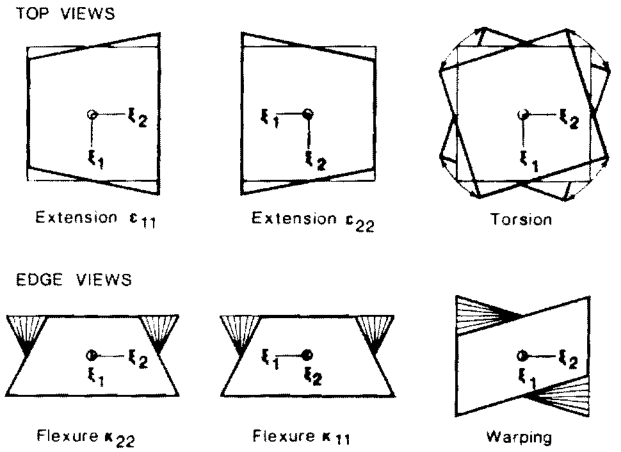

FIG. 6. Higher deformational modes of the quadrilateral.

duces locking, as in the beam of Fig. 3. Somehow, one must introduce better, yet consistent approximations of the strains; better approximations avoid such locking but still inhibit the higher modes.

The motion of the plane triangular element has but 18 degrees-of-freedom. Again, let us neglect the transverse extension and thereby reduce the freedom to 15 degrees, six rigid modes, three homogeneous modes of extension, three of flexure, and two transverse shears. Evidently there is but one higher deformational mode, the torsional mode of Fig. 7(b). Here too, the retention of all terms in the strains, compatible with the displacement (9b), produces the unacceptable shear energy and stiffness. Again, an approximation is required which avoids such "locking"; yet inhibits all deformational modes, particularly the transverse shear modes. To approach these questions in a natural way [Argyris and Scharpf, 1969; Argyris et al, 1982], the nine deformational modes can be described in terms of three flexural strains, three extensional strains, and three transverse shear strains in the three directions of the sides. The latter encompass the two transverse shears *and* the higher-order torsional mode.

The "shear locking," as illustrated by the simple beam of Fig. 3, must be avoided in any case. To this end, one can call upon Love and impose appropriate geometrical constraints against the transverse shear. Conceptually, the simplest are the discrete constraints, as proposed by the author [1968]: It is

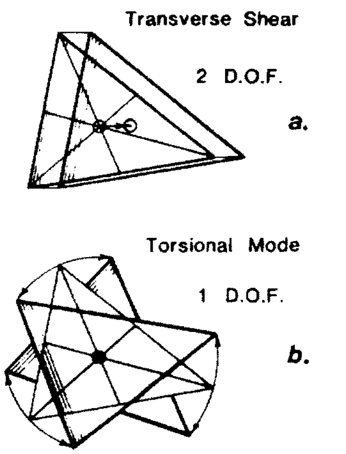

FIG. 7. Shear modes of the triangle.

sufficient to constrain the normal at prescribed points of the element. For example, one can impose constraints such that the tangential component of transverse shear vanishes at midpoints of each edge. The four shear modes of the quadrilateral and the three shear modes of the triangle are thereby constrained, the shear energy is omitted, and the discrete model emulates the lovable shell. This approach is workable, but has shortcomings; in particular, such models cannot exhibit the effects of transverse shear and, therefore, are not suited to thick shells.

One means to accommodate the transverse shear and avoid the excessive shear resistance is to employ the functional and stationary theorem of Hu and Washizu [1955]. The functional is a modified form of the potential wherein the strain energy is expressed in terms of the strains, and strain-displacement conditions are enforced via lagrangian multipliers; the multipliers are the associated stresses. In short, strain, stress, and displacement are independently variable, so that better approximations of the strains (and stresses) are admissible, albeit not pointwise compatible. For example, the shear strain can be approximated by a constant within each element of a beam; such approximation is better than the compatible sawteeth of Fig. 3(b), avoids the "locking," yet inhibits the shear. (The constants are zero under the simple bending.) An approximation for the quadrilateral element was given in an earlier article [Wempner et al, 1982]. To appreciate that approach, consider the strains associated with the higher flexural mode κ_{11}; the flexural and shear strains, which derive from the bilinear approximation (9a), include the following terms:

$$\kappa_{11} = \bar{\kappa}_{11} + \underline{\tilde{\kappa}_{11}\xi^2}, \tag{10a}$$

$$\gamma_1 = \bar{\gamma}_1 + \tilde{\gamma}_1\xi^2 + \underline{\tfrac{1}{2}\tilde{\kappa}_{11}\xi^1}. \tag{10b}$$

The final term of the shear (10b) accounts for the unwarranted stiffness, just as the strain in the beam of Fig. 3. However, that term is not needed, since the mode is inhibited if the approximation of the flexural strain κ_{11} includes the final term of (10a). Further details are given in the earlier work by the author [Wempner et al, 1982]. Suffice it to say that the strains are approximated independently. Such approximation must include the mean value, but need only include such additional terms as are needed to inhibit the higher modes. The procedure provides a ready device for the enforcement of a discrete shear constraint so that the model approaches the continuum of Kirchhoff and Love: One need only delete transverse shear strain in the functional. The precise form of the constraints depend upon the approximation of the associated stresses (multipliers). An example is the formulation by Dvorkin and Bathe [1984], who chose multipliers in the form of Dirac-delta functions at the midpoints of the sides; then the discrete constraints are those suggested previously [Wempner, 1968].

The triangular element also admits transverse shear strain. To formulate that model in a natural way, the strain energies ought to be expressed in terms of natural components, extensions, flexures, and shears in the directions of the edges ($\epsilon_i, \kappa_i, \gamma_i$, $i = 1, 2, 3$). Specifically, the energy of the isotropic hookean element is the quadratic form $u = u_e + u_k + u_s$, the sum of the extensional, flexural, and shear energies, u_e, u_k, and u_s, respectively. In particular,

$$u_s = \frac{Eh}{12(1+\nu)} D^{ij}\gamma_i\gamma_j,$$

$$D^{11} \equiv 1 + \frac{\cos^2\alpha_1}{\sin^2\alpha_2} + \frac{\cos^2\alpha_1}{\sin^2\alpha_3},$$

$$D^{12} \equiv -\frac{\cos\alpha_1\cos\alpha_2}{\sin^2\alpha_3}, \quad \text{etc.}$$

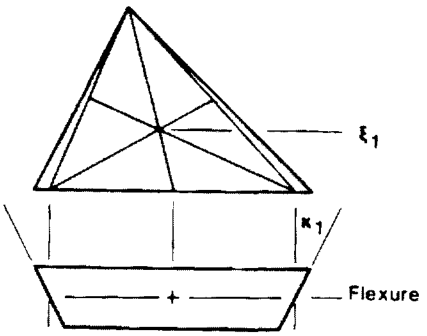

FIG. 8. Flexure of the triangle.

Again, the approximation of the strain must suppress the three shear modes of Fig. 7, but admit flexural modes without shear. For example, a homogeneous flexure in the direction of the coordinate ξ_1 corresponds to the *relative* rotation (actually, a shear γ_1)

$$\kappa_1 = \bar{\kappa}(\xi_2 - \xi_3). \tag{11}$$

Such flexure is shown graphically in Fig. 8. Though the areal coordinates are natural, they are less familiar. Integrals of powers are given in numerous texts [see, eg, Zienkiewicz, 1977]. One can readily verify that the flexural mode (11) is orthogonal to a homogeneous transverse shear, and to a simple torsional mode (also transverse shear), as depicted in Figs. 7(a) and 7(b), respectively. Again, if the shear strains γ_i are suppressed in the functional, then three kinematical constraints follow. These constraints serve to suppress the shear modes and provide a discrete counterpart of the Kirchhoff–Love theory.

Whether the elements are simple or complicated, conforming or nonconforming, the forms of the approximations are certainly not limited to polynomials, nor even to continuous functions. Mathematically, convergence requires that the discrete values approach the mean values of the continuous functions and that the difference equations approach the differential equations. Examples of some piecewise constant, and discontinuous, approximations of the strains and stresses are given in earlier articles [Wempner et al, 1982; 1983].

MORE ON FINITE ELEMENTS

The foregoing review is but a synopsis of the mechanics of shells and an introduction to approximation via finite elements. The reader should have little difficulty in researching the well-established theories of shells. Approximation via finite elements is another matter. The latter approach is relatively recent, a by-product of electronic computation. In just two decades, the profusion of literature contains some important concepts, a few well-developed theories, many useful methods, but also many ill-conceived by-products of numerical experimentation. In time the fertile seeds will sprout from the chaff. Meanwhile, practitioners are well advised to accept computational programs with caution. Here, rudimentary notions are introduced and illustrated by the simplest elements. The vehicles for these formulations, the potential and modified potential of Hu and Washizu, are specifically mentioned because they are well founded and insightful. Many alternative methods and nuances are available. Now, we cite some works, which set forth particular viewpoints, and some others, which provide an entree to the literature:

In addition to the potential and the Hu–Washizu functional, one can employ the functional of Hellinger [1914] and Reissner [1950] which admits approximations of displacement and stress. The application to "hybrid" elements was pioneered by Pian [1964; Pian et al, 1986] and elaborated by Spilker [1980; Spilker and Jakobs, 1986] and others. In the "hybrid" element the additional variables (discrete parameters), eg, stresses, are eliminated from the description of the element, prior to the assembly. The complementary functional, wherein stress is the variable, has been employed, particularly in approximations of linear problems; then, the potential and the complementary functional provide upper and lower bounds, as discussed by Fraeijs de Veubeke [1965; Fraeijs de Veubeke and Zienkiewicz, 1967]. We are unaware of a similar application of the complementary functional [Fraeijs de Veubeke, 1972] for finite deformations.

The term "mixed" signifies approximations of stresses and/or strains *and* displacements; often such "mixed" approximations are founded upon variational theorems of Hu and Washizu or Hellinger and Reissner. An early example is the plate element of Hermann [1967]. More recent work was reported at a conference on mixed and hybrid elements [Proceedings edited by Atluri and Zienkiewicz, 1982]. Our simple examples are conforming elements, wherein the displacement is continuous across interelement edges; such continuity is not essential to the formulation of effective approximations. Indeed, discontinuities (nonconforming elements) are admissible provided that appropriate conditions are satisfied along interfaces of discontinuity. Such conditions are needed to insure the convergence of the discrete model; they can be auxiliary conditions of a variational theorem, as described by Prager [1967]. An advantage of approximations founded upon the modified potential and complementary functional (Hu–Washizu and Hellinger–Reissner functionals) is the freedom to employ simple, even discontinuous, approximations of strains and/or stresses. The piecewise constant approximations of our earlier article [Wempner, 1983] are examples. "Reduced integration" is another device which also can improve the discrete model; for example, the use of the midpoint shear in the beam of Fig. 3. Again, the practitioner must take care to insure that such approximations include the essential mean values and also inhibit all deformational modes of the finite element. The notion of "reduced integration" can be traced to the articles by Zienkiewicz, Taylor, and Too [1971] and Pawsey and Clough [1971]. The selection of the terms to "integrate" in this manner and the values to employ has led to "selective integration." Discussion of the latter is given in the article by Malkus and Hughes [1978].

As noted earlier, strict enforcement of the Kirchhoff hypothesis implies the absence of transverse shear and requires the higher-order approximations of curved elements. Many of the early elements were founded upon such theory and, consequently, possess many degrees of freedom; for example, the reader can consult the works of Argyris [1965], Argyris and Sharpf [1968, 1969], Cantin and Clough [1968], Dupius and Goel [1970], Morley [1972], Ashwell and Sabir [1972], and Dawe [1976]. The inclusion of transverse shear was suggested by Fraeijs de Veubeke [1965], who also recognized the need to accommodate flexure without shear strain, so-called "Kirchhoff modes." An account of the many elements, curved or flat, is beyond the scope of this review. The reader will find abundant references in the text of Zienkiewicz [1977], the collection of articles edited by Ashwell and Gallagher [1976], and the recent collection edited by Hughes and Hinton [1986].

Many investigators have influenced the evolution of the topic: The form of approximation, ie, shape functions of local support, was known in the mathematical community at an early date [Courant, 1943]. Argyris and Kelsey [1960] recognized the role of discrete approximations to harness the new-found capabilities of electronic computers. Clough [1960] also envisaged the assembly of finite elements as a means to approximate continuous bodies. Important steps in such approximation of shells were made by Clough [1964] and by Argyris and Scharpf [1968], but many others have contributed. Here, we will attempt to highlight early contributions:

The isoparametric approximation of quadrilateral and triangular elements can be traced to articles by Taig and Kerr [1964] and Irons [1966]. Mathematical features and elaborations are given in most texts [eg, Zienkiewicz, 1977].

Approximations which are not founded upon the theorem of minimum potential require some assurance of convergence. To this end, Irons and Razzaque [1972] advanced the "patch test" as a necessary condition. The test assures that the discrete assembly accommodates homogeneous states. This clearly applies equally to small extensional and flexural deformations of plates; applicability to shells is less apparent since these modes are generally coupled. The coupling of extension and flexure may impart excessive stiffness, termed "membrane locking"; see articles by Morris [1973; 1976], Morley [1984], and Belytschko et al [1985]. Approximations, which uncouple extension and flexure, are reported by Morley [1982].

Finite elements of shells are usually based upon quadrilateral or triangular elements of the surface. It is quite natural and effective to place the four corners of a quadrilateral on lines of curvature; then the dominant deformational modes of the element are associated with the familiar orthogonal components of strain in the continuum. Moreover, the algebraic equations of the discrete system can be associated with corresponding equations which govern the continuum. Representation of the surface by flat elements, usually triangular, can be traced to the works of Clough and Tocher [1964], Melosh [1966], and Zienkiewicz et al [1968]. Other effective elements were devised by Thomas and Gallagher [1976], Argyris et al [1977], and Horrigmoe and Bergan [1978]. More recently, acceptance of transverse shear has led to renewed interest in simpler elements, flat triangular and quadrilateral elements. These necessarily incorporate approximations of the shear strain, which are not compatible with displacement; such approximations may be couched in the variational theorem of Hu and Washizu, as described previously, or devised by some alternative such as the "reduced integrations." A good approximation of the transverse shear strain provides the mean value, and also inhibits higher modes. Mechanically, all modes must possess strain energy; mathematically, the matrix must be nonsingular. The system is then stable. Methods to insure such stability are discussed by Belytschko [Belytschko et al, 1981; Belytschko and Tsay, 1983]. Various methods and forms of approximation are used to accommodate the transverse shear: Some are given by Hughes et al [1977], MacNeal [1978], Wempner et al [1982], Tesslar and Hughes [1983], and Dvorkin and Bathe [1984]. The shear modes can always be constrained by a discrete counterpart of the Kirchhoff hypothesis, discrete kinematical constraints. Examples of the latter are found in the articles by Wempner [1968], Baldwin et al [1973], Dhatt [1970], Batoz et al [1980; Batoz and Dhatt, 1972] and Irons [1976].

GEOMETRICAL NONLINEARITIES AND BUCKLING

Finite strains are always governed by equations which contain nonlinearities of geometrical and, usually, physical origins. Thin structures, beams, plates and shells, seldom experience finite strain but can undergo large rotations. Indeed, the flexibility of such thin bodies admits large rotations and relatively large changes of curvature, though the strains remain small.

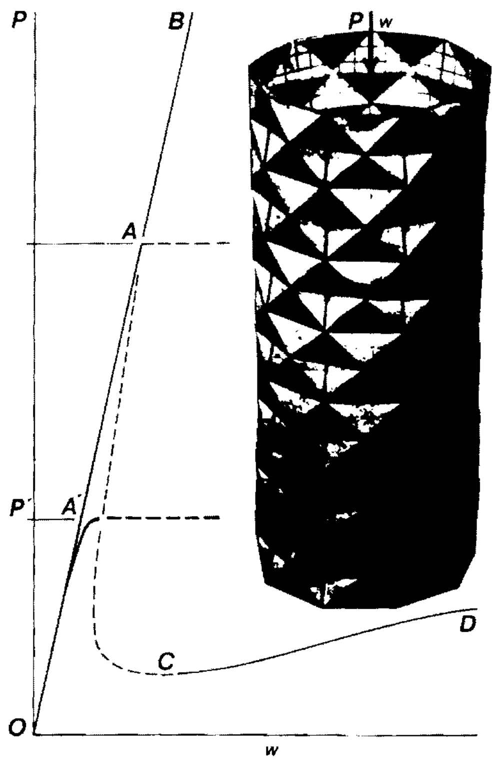

FIG. 9. Buckled cylinder.

Often the behavior remains elastic and nearly hookean. Then the nonlinearities are entirely geometrical. Most significant are the couplings between membrane forces and curvatures which are associated with the buckling and postbuckling behavior. The subject has received much attention from theoretical, experimental, and practical viewpoints. The reader can find theoretical foundations and references in the text of Libai and Simmonds [1983]. Valuable information on stability is given in the *Collected papers on instability of shell structures* [NASA TN D-1510, 1962] and later articles by Budiansky and Hutchinson [1979] and Simitses [1986].

Two classic examples illustrate forms of nonlinearity and instability which are typical of shells. The response of a thin cylindrical shell under uniform axial loading is illustrated by a plot of load versus mean axial displacement. A nearly perfect shell is very stiff as depicted by the steep initial slope OB of Fig. 9. At a critical load the equilibrium states exhibit a bifurcation A; states along branches AB and AC are unstable. The severely deformed equilibrium states of CD are theoretically stable, although practically unattainable because of the severity of the deformation. Indeed, the critical load is also unattainable, since imperfections promote flexure and premature buckling [see Koiter, 1945]. Theoretically the shell would exhibit the abrupt "snap buckling" from the bifurcation point A. Practically the plot would trace another path OA', as flexural deformations develop; actual buckling occurs at the lower load P'. To appreciate the low resistance of a postbuckled state C, one need only observe the severely deformed cylinder of Fig. 9: Here, the membrane resistance is nearly absent and the cylinder collapses like an accordian. Severe deformation is attributed to the bending which is confined to the narrow folds. These deformed, but inextensional, patterns were identified by Yoshimura [1955].

Another example, which illustrates a different response, is the shallow cone of Fig. 10. The plot of load vs deflection traces

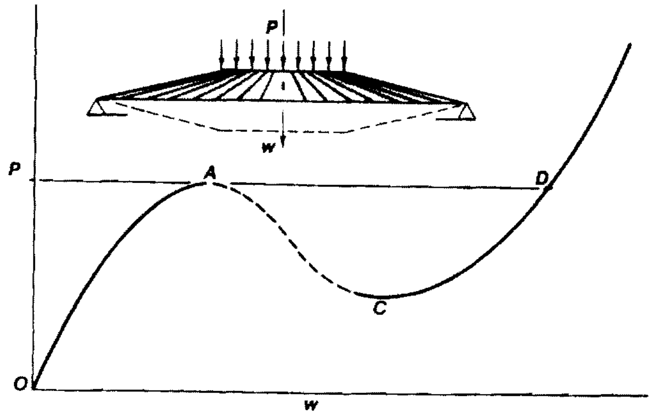

FIG. 10. Compressed shallow cone.

the nonlinear curve *OACD*. The initial stiffness can be traced to the circumferential extension in the outer portion, compression near the hole, and the attendant membrane forces. As the shell reaches a nearly flat configuration, the plot reaches the "limit point" *A*, and the shell "snaps through" to the stable configuration *D* (a nearly inverted form).

General concepts and methods of finite elements apply as well to the nonlinear problems. Indeed, most nonlinear problems require some form of discrete approximation and also computational procedures which accommodate the nonlinear equations. Usually the procedures trace the nonlinear path(s) in incremental steps; each step is governed by linear equations which must be revised with each successive step. Such schemes require special attention to bifurcation and limit points. Information about incremental procedures are given by Wempner [1971a], recent articles by Thurston et al [1986], Padovan [1985], and Bushnell [1985].

INELASTICITY OF SHELLS

Theories of inelastic shells are much less prevalent than theories of elastic, and, especially, Hookean shells. This is quite understandable: First, one encounters the basic difficulties of nonconservative systems, specifically, the need to adopt an incremental approach. Secondly, even thin shells exhibit significant gradients of strain and stress through the thickness; accordingly, inelastic deformations are usually initiated at a surface and progress through the thickness. Because of this evolution of the inelastic regions, it is particularly difficult to devise theories which are expressed in terms of the two-dimensional fields, ie, bona fide shell theories. In all likelihood, workable theories of thin inelastic shells can be founded upon the Kirchhoff–Love hypothesis. Then, the strains and associated stresses follow [see Eq (5)] and, according to the concepts of classical plasticity, a yield condition and "flow law" must follow [see Eqs (7a) and (7b). The *initial* form of the yield condition is readily obtained [Robinson, 1971; Crisfield, 1974; Bieniek and Funaro, 1976]. However, the evolution of the yield condition poses a great challenge: In the space of the *shell* stresses (forces and couples), even a shell of perfectly plastic material exhibits a form of "strain hardening" [Bieniek and Funaro, 1976; Atkatsh et al, 1983; Bank and Bieniek, 1988; Kutt and Bieniek, 1988]. Two avenues are possible: The direct approach begins with the two-dimensional variables; then the evolution of the yield condition and flow is a matter of insightful speculation and, ultimately, correlation with experimental or computational data. The latter can be obtained by a series of numerical experiments wherein the model is a shell with many layers; essentially, the results are obtained by a three-dimensional treatment of the shell under selected loadings. Such formulation of the yield condition is described by Eggers and Kröplin [1978, Eqs (13) and (14)]. An alternative approach to the inelastic shell is a derivation which begins with a three-dimensional description, assumptions with respect to the distributions through the thickness and, finally, reduction to a two-dimensional theory. An attempt at such derived theory is described by Wempner [1977].

As in any nonconservative mechanical system, the theory of an inelastic shell, or a discrete model, must be given in incremental form; it may be rate dependent as well [Atkatsh et al, 1983]. In a classical approach, the strain is expressed as a polynomial through the thickness (θ^3); then virtual work provides a hierarchy of stresses [Wempner, 1972]. In that spirit, Kollmann and Mukherjee [1985] set forth the bases of an inelastic theory which incorporates "rates of inelastic pseudo-resultants." In their conclusion, "the inelastic pseudo-resultants have to be computed by numerical integration" [through the thickness]. Hence, the evolution of the constitutive equations for the shell are a matter of step-by-step numerical evaluation during the progressive loading.

In the spirit of finite elements, it is logical to approximate the inelastic shell by layers. This seems to be the prevailing approach. Then the constitutive equations of the medium are applied to each layer; in this respect, the shell is treated as a three-dimensional body. The requisite storage, and the repetitive steps of incremental procedures, make the computations lengthy and expensive. Clearly, the theory and approximation of inelastic shells pose intellectual and practical challenges for current investigators and engineers.

The concept of finite elements is applied to *large* plastic deformations in a unique manner by Lukasiewicz [1987]: His approach is based on the observation that such large deformations are largely inextensional; for example, the deformational patterns might be formed by bending along folds, as the cylinder of Fig. 9. It appears that this approach is a practical means to study instability and collapse.

CONCLUDING REMARKS

The foregoing overview is an attempt to integrate the burgeoning, often disconcerting, methodologies for finite elements of shells with the generally accepted concepts and methods of the continuum theories. It is hoped that the focus upon simple forms of approximation will best serve the interests of the uninitiated and provide a sound basis for further study. The references are necessarily limited: Hopefully, they will enable the reader to trace historical developments and pursue further research.

Lest the forest be lost behind the trees, the reader is reminded of a few specific peculiarities: The shell (plate or beam) is a body which can exhibit large rotations with small strain. These are accompanied by changes of curvature and attendant gradients of strain through the thickness. The occurrence of large changes of curvature and rotations, coupled with the dominant role of membrane forces, accounts for geometrical nonlinearities, buckling and snapbuckling of shells. The strain gradients through the thickness, the flexural strains, play an essential, even crucial, role in shells. Hence, they pose problems which are less evident in bulky bodies: In the latter such gradients are merely higher-order contributions to the energy and to the stiffness of a finite element. In a shell, these gradients are paramount; moreover, they are accompanied by transverse shear of the element. Additional approximations are needed, namely, the hypothesis of Kirchhoff and Love, a discrete counterpart for finite elements, or the omission of the parasitic shear via additional approximations. These gradients of strain, and stress, through the thickness are also the sources of much difficulty in rendering a two-dimensional theory of plasticity for shells. Of course, these and other special problems of shells are avoided, theoretically, if one treats them as three-dimensional and accepts the much larger computational effort. Eventually, this course may prove feasible. Meanwhile efficient means are needed and intellectual challenges persist.

Undoubtedly, time will cast a heavy shadow on this effort, as most technical writings, but, hopefully, it will shed some needed light upon the current state of the art.

ACKNOWLEDGMENTS

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