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Author: H. Bessaih, Rafał Kapica, T. Szarek

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RESEARCH ARTICLE

Criterion on stability for Markov processes applied to a model with jumps

H. Bessaih · R. Kapica · T. Szarek

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Abstract We formulate and prove a new criterion for stability of e-processes. In particular we show that any e-process which is averagely bounded and concentrating is asymptotically stable. This general result is applied to a stochastic process with jumps that is a continuous counterpart of the chain considered in Szarek (Ann. Probab. 34:1849–1863, 2006).

Keywords Ergodicity of Markov families · Invariant measures · Dynamical systems with jumps

1 Introduction

In this paper we will present a new criterion for the stability of Markov semigroups and apply it to a stochastic model with jumps proving the existence of a unique invariant measure and its stability. More specifically, we prove that any averagely bounded

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H. Bessaih
Department of Mathematics, University of Wyoming, Laramie, WY, USA
e-mail: Bessaih@uwyo.edu

R. Kapica
Department of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland
e-mail: rkapica@math.us.edu.pl

T. Szarek (✉)
Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland
e-mail: szarek@intertele.pl

Markov semigroup with the e-property concentrating at some point admits a unique invariant measure. Moreover the semigroup is stable. The criterion generalizes results obtained in [19], where under more restrictive conditions on a semigroup we managed to prove existence and uniqueness. We were not able to show stability unless we assumed the tightness of the semigroup. The proof is based on the lower bound technique introduced by A. Lasota and J. Yorke in [12], where the authors showed the existence of an absolutely continuous invariant measure for the Frobenius–Perron operator corresponding to piecewise monotonic transformations. Since then, the technique has been generalized first for Markov semigroups acting on densities (see [10]), subsequently for general Markov semigroups defined on arbitrary Borel measures in finite dimensions (see [13]) and finally it has been extended to infinite dimensional spaces (see [17]). Generally speaking the method relies on an easy observation that two regular trajectories starting at different measures which visit some small set with positive, bounded from below, probability converge in the weak topology. Additionally, if we assume that every neighborhood of some point is visited infinitely many times, then we may show that the process admits an invariant measure. The *e-process property* is a slight generalization of the *e-chain property* introduced in [14] (see [8, 11, 21]). It is a more general concept than the asymptotic strong Feller property introduced by J. Mattingly and M. Hairer see [5].

In the second part of our paper we are concerned with a dynamical system with jumps. To be precise we study a general flow on some Polish spaces disturbed by an iterated function system at an exponentially distributed random time. The assumptions on the flow and the iterated function system are quite general. In particular, the iterated function system is contracting on the average only and its probability distributions depend upon position. Our proof is based on the asymptotic stability of the system. The proof of stability was given in [2] in the case of finitely dimensional spaces (see also [12]) and in [17] when the iterated system is defined on an arbitrary Polish space. There is a huge literature of models with jumps partly due to a large class of possible applications in physics or biology, partly due to their purely mathematical properties. It is worth mentioning here that our very general model is closely related to such objects as some stochastic differential equations with Poisson noise (see [15]), randomly forced PDE's (see [20]), random dynamical systems based on skew product flows [1] and piecewise-deterministic Markov processes introduced by Davies in [4].

The paper is organized as follows. In Sect. 2 we introduce the concepts of the e-property, averagely bounded and concentrating at a point. We also prove (Proposition 1) the main result about asymptotic stability for Markov processes. In Sect. 3 we introduce a model based on an iterated functions system for which we apply our results of Sect. 2. Indeed we show that it satisfies the e-property, the average boundedness and the concentrating property and hence we obtain its stability.

2 Criterion on stability

Let (X, ρ) be a Polish space. By $B_b(X)$ we denote the space of all bounded Borel-measurable functions equipped with the supremum norm $\|\cdot\|_\infty$. Let $(P_t)_{t\geq 0}$ be a

Markovian semigroup defined on $B_b(X)$. For each $t \geq 0$ we have $P_t \mathbf{1}_X = \mathbf{1}_X$ and $P_t\varphi \geq 0$ if $\varphi \geq 0$. Throughout this paper we shall assume that the semigroup is *Feller*, i.e. $P_t(C_b(X)) \subset C_b(X)$ for all $t > 0$. Here and in the sequel $C_b(X)$ is the subspace of all bounded continuous functions with the supremum norm. By $L_b(X)$ we will denote the subspace of all bounded Lipschitz functions. We shall also assume that $(P_t)_{t\geq 0}$ is *stochastically continuous*, which implies that $\lim_{t\to 0^+} P_t\varphi(x) = \varphi(x)$ for all $x \in X$ and $\varphi \in C_b(X)$.

Let $\mathcal{M}_1$ stands for the space of all Borel probability measures on X. Denote by $\mathcal{M}_1^W$, $W \subset X$, the subspace of all Borel probability measures supported in W, i.e. $\{x \in X : \mu(B(x,r)) > 0 \text{ for any } r > 0\} \subset W$, where $B(x,r)$ denotes the ball in X with center at x and radius r. For $\varphi \in B_b(X)$ and $\mu \in \mathcal{M}_1$ we will use the notation $\langle \varphi, \mu\rangle = \int_X \varphi(x)\mu(\mathrm{d}x)$. Recall that the *total variation norm* of a finite signed measure $\mu \in \mathcal{M}_1 - \mathcal{M}_1$ is given by $\|\mu\|_{TV} = \mu^+(X) + \mu^-(X)$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ.

We say that $\mu_* \in \mathcal{M}_1$ is *invariant* for $(P_t)_{t\geq 0}$ if $\langle P_t\varphi, \mu_*\rangle = \langle \varphi, \mu_*\rangle$ for every $\varphi \in B_b(X)$ and $t \geq 0$. Alternatively, we can say that $P_t^*\mu_* = \mu_*$ for all $t \geq 0$, where $(P_t^*)_{t\geq 0}$ denotes the semigroup dual to $(P_t)_{t\geq 0}$, i.e. for a given Borel measure μ, Borel subset A of X, and $t \geq 0$ we set

$$P_t^*\mu(A) := \langle P_t \mathbf{1}_A, \mu\rangle.$$

A semigroup $(P_t)_{t\geq 0}$ is said to be *asymptotically stable* if there exists a unique invariant measure $\mu_* \in \mathcal{M}_1$ such that $(P_t^*\mu)_{t\geq 0}$ converges weakly to μ_* as $t \to +\infty$ for every $\mu \in \mathcal{M}_1$. Recall that the sequence $(P_t^*\mu)_{t\geq 0}$ converges weakly to μ_* if the following condition holds:

$$\lim_{t\to\infty} \langle \varphi, P_t^*\mu\rangle = \langle \varphi, \mu_*\rangle \quad \text{for all } \varphi \in C_b(X).$$

Definition 2.1 We say that a semigroup $(P_t)_{t\geq 0}$ has *the e-property* if the family of functions $(P_t\varphi)_{t\geq 0}$ is equicontinuous at every point x of X for any bounded and Lipschitz function φ, i.e. for arbitrary $\varphi \in L_b(X)$ and $x \in X$ we have

$$\lim_{y\to x} \sup_{t\geq 0} \left| P_t\varphi(y) - P_t\varphi(x) \right| = 0.$$

Remark One can show (see [7]) that to obtain the e-property in the case when X is a Hilbert space, it is enough to verify the above condition for every function with bounded Fréchet derivative.

Definition 2.2 A semigroup $(P_t)_{t\geq 0}$ is called *averagely bounded* if for any $\varepsilon > 0$ and bounded set $A \subset X$ there is a bounded Borel set $B \subset X$ such that

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T P_s^*\mu(B)\mathrm{d}s > 1 - \varepsilon \quad \text{for } \mu \in \mathcal{M}_1^A.$$

(Note that from the stochastic continuity, the integrand in the above integral is Borel-measurable with respect to s.)

Definition 2.3 A semigroup $(P_t)_{t\geq 0}$ is *concentrating* at z if for any $\varepsilon > 0$ and bounded set $A \subset X$ there exists $\alpha > 0$ such that for any two measures $\mu_1, \mu_2 \in \mathcal{M}_1^A$

$$P_t^* \mu_i\big(B(z,\varepsilon)\big) \geq \alpha \quad \text{for } i = 1, 2 \text{ and some } t > 0.$$

Proposition 1 *Let $(P_t)_{t\geq 0}$ be averagely bounded and concentrating at some $z \in X$. If $(P_t)_{t\geq 0}$ satisfies the e-property, then for any $\varphi \in L_b(X)$ and $\mu_1, \mu_2 \in \mathcal{M}_1$ we have*

$$\lim_{t\to\infty} \big|\langle \varphi, P_t^* \mu_1\rangle - \langle \varphi, P_t^* \mu_2\rangle\big| = 0. \tag{2.1}$$

Proof First observe that to finish the proof it is enough to show that condition (2.1) holds for arbitrary Borel probability measures with bounded support. Indeed, the set of all probability measures with bounded support is dense in the space $(\mathcal{M}_1, \|\cdot\|_{TV})$. Moreover, P_t^*, $t \geq 0$, is nonexpansive with respect to the total variation norm.

Fix $\varphi \in L_b(X)$, $x_0 \in X$ and $\varepsilon \in (0, 1/2)$. Let $\mu_1, \mu_2 \in \mathcal{M}_1^{B(x_0,r_0)}$ for some $r_0 > 0$. Choose $\delta > 0$ such that

$$\sup_{t\geq 0} \big|P_t\varphi(x) - P_t\varphi(y)\big| < \varepsilon/2 \tag{2.2}$$

for $x, y \in B(z,\delta)$, by the e-property.

Since $(P_t)_{t\geq 0}$ is averagely bounded we may find $R_0 > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^* \mu\big(B(x_0, R_0)\big)\mathrm{d}s > 1 - \varepsilon^2\big/\big(4\|\varphi\|_\infty\big) \tag{2.3}$$

for any $\mu \in \mathcal{M}_1^{B(x_0,r_0)}$. Let $R > \max\{R_0, r_0\}$ satisfy

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^* \mu\big(B(x_0, R)\big)\mathrm{d}s > 3/4 \tag{2.4}$$

for any $\mu \in \mathcal{M}_1^{B(x_0,R_0)}$. Since $(P_t)_{t\geq 0}$ is concentrating at z we may choose $\alpha > 0$ such that for any $\nu_1, \nu_2 \in \mathcal{M}_1^{B(x_0,R)}$ there exists $t > 0$ and the condition

$$P_t^* \nu_i\big(B(z,\delta)\big) \geq \alpha \quad \text{for } i = 1, 2 \tag{2.5}$$

holds.

Set $\gamma := \alpha\varepsilon/2 > 0$. Let k be the minimal integer such that $4(1-\gamma)^k\|\varphi\|_\infty \leq \varepsilon$.

We will show by induction that for every $l \leq k$, $l \in \mathbb{N}$, there exist $t_1, \ldots, t_l > 0$ and $\nu_1^i, \ldots, \nu_l^i, \mu_l^i \in \mathcal{M}_1$ such that $\nu_j^i \in \mathcal{M}_1^{B(z,\delta)}$ for $j = 1, \ldots, l$ and

$$\begin{aligned} P_{t_1+\cdots+t_l}^* \mu_i = {} & \gamma P_{t_2+\cdots+t_l}^* \nu_1^i + \gamma(1-\gamma) P_{t_3+\cdots+t_l}^* \nu_2^i \\ & + \cdots + \gamma(1-\gamma)^{l-1}\nu_l^i + (1-\gamma)^l \mu_l^i \quad \text{for } i = 1, 2. \end{aligned} \tag{2.6}$$

Indeed, let $t_1 > 0$ be such that

$$P_{t_1}^* \mu_i\big(B(z,\delta)\big) \geq \alpha > \gamma \quad \text{for } i = 1, 2.$$

Set

$$\nu_1^i = \frac{P_{t_1}^* \mu_i(\cdot \cap B(z,\delta))}{P_{t_1}^* \mu_i(B(z,\delta))}, \qquad \mu_1^i = (1-\gamma)^{-1}\big(P_{t_1}^* \mu_i - \gamma \nu_1^i\big) \quad \text{for } i = 1,2 \tag{2.7}$$

and observe that $\mu_1^i \in \mathcal{M}_1$ and $\nu_1^i \in \mathcal{M}_1^{B(z,\delta)}$ for $i = 1,2$. Then condition (2.6) holds for $l = 1$.

Now assume that we have done it for some l and $4(1-\gamma)^l \|\varphi\|_\infty > \varepsilon$. Then there exist $s_i > 0$ for $i = 1,2$ such that

$$P_{t_1+\cdots+t_l+s_i}^* \mu_i\big(X \setminus B(x_0,R_0)\big) < \varepsilon^2/\big(4\|\varphi\|_\infty\big)$$

for $i = 1,2$, by (2.3). Since $(1-\gamma)^l > \varepsilon/(4\|\varphi\|_\infty)$, from the linearity of $P_{s_i}^*$ we obtain that

$$P_{s_i}^* \mu_l^i\big(B(x_0,R_0)\big) > \varepsilon \quad \text{for } i = 1,2.$$

Indeed, if it does not hold, then

$$P_{s_i}^* \mu_l^i\big(X \setminus B(x_0,R_0)\big) \geq \varepsilon \quad \text{for } i = 1,2$$

and

$$P_{t_1+\cdots+t_l+s_i}^* \mu_i\big(X \setminus B(x_0,R_0)\big) \geq (1-\gamma)^l P_{s_i}^* \mu_l^i\big(X \setminus B(x_0,R_0)\big) \geq \varepsilon^2\big/\big(4\|\varphi\|_\infty\big).$$

Thus we may find two measures $\tilde{\mu}_l^1, \tilde{\mu}_l^2 \in \mathcal{M}_1^{B(x_0,R_0)}$ such that

$$P_{s_i}^* \mu_l^i \geq \varepsilon \tilde{\mu}_l^i. \tag{2.8}$$

These measures may be defined as restriction of $P_{s_i}^* \mu_l^i$ to $B(x_0,R_0)$ suitably normed (see formula (2.7)). Further, from (2.4) it follows that

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\int_0^T \big[P_{s+s_2}^*\big(\tilde{\mu}_l^1/2\big)\big(B(x_0,R)\big) + P_{s+s_1}^*\big(\tilde{\mu}_l^2/2\big)\big(B(x_0,R)\big)\big]\mathrm{d}s \\
&= \limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^*\big(\tilde{\mu}_l^1/2 + \tilde{\mu}_l^2/2\big)\big(B(x_0,R)\big)\mathrm{d}s > 3/4,
\end{aligned}$$

by the fact that $\tilde{\mu}_l^1/2 + \tilde{\mu}_l^2/2 \in \mathcal{M}_1^{B(x_0,R_0)}$. Consequently, for some $s > 0$ we have

$$P_{s+s_2}^* \tilde{\mu}_l^1\big(B(x_0,R)\big) \geq 1/2 \quad \text{and} \quad P_{s+s_1}^* \tilde{\mu}_l^2\big(B(x_0,R)\big) \geq 1/2.$$

Comparing (2.8) and the above we obtain

$$P_{s+s_1+s_2}^* \mu_l^i \geq (\varepsilon/2)\hat{\mu}_l^i$$

for some $\hat{\mu}_l^i \in \mathcal{M}_1^{B(x_0,R)}$, $i = 1,2$, by argument similar to that in (2.8). Using it once again and taking into consideration (2.5) we obtain that there exists $t > 0$ such that

$$P_{t+s+s_1+s_2}^* \mu_l^i \geq (\alpha\varepsilon/2)\nu_{l+1}^i = \gamma \nu_{l+1}^i$$

for some $\nu_{l+1}^i \in \mathcal{M}_1^{B(z,\delta)}$ for $i = 1, 2$. Therefore, setting $t_{l+1} = t + s + s_1 + s_2$ we obtain

$$P^*_{t_1+\cdots+t_l+t_{l+1}}\mu_i = \gamma P^*_{t_2+\cdots+t_{l+1}}\nu_1^i + \gamma(1-\gamma)P^*_{t_3+\cdots+t_{l+1}}\nu_2^i + \cdots + \gamma(1-\gamma)^{l-1}P^*_{t_{l+1}}\nu_l^i + \gamma(1-\gamma)^l\nu_{l+1}^i + (1-\gamma)^{l+1}\mu_{l+1}^i,$$

where

$$\mu_{l+1}^i = (1-\gamma)^{-1}\big(P^*_{t_{l+1}}\mu_l^i - \gamma\nu_{l+1}^i\big) \quad \text{for } i = 1, 2.$$

This completes the proof of condition (2.6). In turn, this and (2.2) give for $t \geq t_1 + \cdots + t_k$

$$\begin{aligned}
&\big|\langle\varphi, P_t^*\mu_1\rangle - \langle\varphi, P_t^*\mu_2\rangle\big| \\
&\quad = \big|\big\langle P_{t-(t_1+\cdots+t_k)}\varphi, P^*_{t_1+\cdots+t_k}\mu_1\big\rangle - \big\langle P_{t-(t_1+\cdots t_k)}\varphi, P^*_{t_1+\cdots+t_k}\mu_2\big\rangle\big| \\
&\quad \leq \gamma\big|\big\langle P_{t-t_1}\varphi, \nu_1^1 - \nu_1^2\big\rangle\big| + \gamma(1-\gamma)\big|\big\langle P_{t-(t_1+t_2)}\varphi, \nu_2^1 - \nu_2^2\big\rangle\big| + \cdots \\
&\qquad + \gamma(1-\gamma)^{k-1}\big|\big\langle P_{t-(t_1+\cdots+t_k)}\varphi, \nu_k^1 - \nu_k^2\big\rangle\big| + 2(1-\gamma)^k\|\varphi\|_\infty \\
&\quad \leq \big(\gamma + \gamma(1-\gamma) + \cdots + \gamma(1-\gamma)^{k-1}\big)\sup_{t\geq 0,\, x,y\in B(z,\delta)}\big|P_t\varphi(x) - P_t\varphi(y)\big| \\
&\qquad + \varepsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

Proposition 2 *Assume that there exists $z \in X$ such that for any $\varepsilon > 0$*

$$\limsup_{T\to\infty}\sup_{\mu\in\mathcal{M}_1}\frac{1}{T}\int_0^T P_s^*\mu\big(B(z,\varepsilon)\big)\mathrm{d}s > 0. \tag{2.9}$$

If $(P_t)_{t\geq 0}$ satisfies the e-property, then it admits an invariant measure.

Remark Observe that our result generalizes Theorem 7.4.4 in [21]. Indeed, condition (2.9) is weaker than the condition therein, here we take supremum over all probability measures instead of a concrete measure.

Proof Assume, contrary to our claim, that $(P_t)_{t\geq 0}$ does not possess any invariant measure. From Step I of Theorem 3.1 in [11] it follows that there exists an $\varepsilon > 0$, a sequence of compact sets $(K_i)_{i\geq 1}$, and an increasing sequence of positive reals $(q_i)_{i\geq 1}$, $q_i \to \infty$, satisfying

$$P^*_{q_i}\delta_z(K_i) \geq \varepsilon \quad \text{for } i \in \mathbb{N}$$

and

$$\min\big\{\rho(x,y) : x \in K_i,\, y \in K_j\big\} \geq \varepsilon \quad \text{for } i \neq j,\ i, j \in \mathbb{N}.$$

We will show that for every open neighborhood U of z and every $i_0 \in \mathbb{N}$ there exists $y \in U$ and $i \geq i_0$, $i \in \mathbb{N}$, such that

$$P^*_{q_i}\delta_y\big(K_i^{\varepsilon/3}\big) < \varepsilon/2,$$

where $K_i^{\varepsilon/3} = \{y \in X : \inf_{v \in K_i} \rho(y, v) < \varepsilon/3\}$.

On the contrary, suppose that there exists an open neighbourhood U of z and $i_0 \in \mathbb{N}$ such that

$$\inf\big\{P^*_{q_i}\delta_y\big(K_i^{\varepsilon/3}\big) : y \in U, i \geq i_0\big\} \geq \varepsilon/2. \tag{2.10}$$

Clearly

$$\limsup_{T\to\infty} \sup_{\mu\in\mathcal{M}_1} \frac{1}{T}\int_0^T P^*_s\mu(U)\mathrm{d}s > \alpha \tag{2.11}$$

for some $\alpha > 0$. Further, let $N \in \mathbb{N}$ satisfy $(N - i_0 + 1)\alpha\varepsilon > 2$. Choose $\gamma \in (0, \alpha\varepsilon/2)$ such that

$$(N - i_0 + 1)(\alpha\varepsilon - 2\gamma) > 2.$$

It easily follows that there exists $T_0 > 0$ such that for any $\mu \in \mathcal{M}_1$ and $T \geq T_0$ we have

$$\max_{i\leq N}\left\|\frac{1}{T}\int_0^T P^*_s\mu\,\mathrm{d}s - \frac{1}{T}\int_0^T P^*_{s+q_i}\mu\,\mathrm{d}s\right\|_{TV} < \gamma.$$

Choose $T \geq T_0$ and $\mu \in \mathcal{M}_1$ such that

$$\frac{1}{T}\int_0^T P^*_s\mu(U)\mathrm{d}s \geq \alpha, \tag{2.12}$$

by (2.11). From (2.10) and the Markov property it follows that

$$P^*_{s+q_i}\mu\big(K_i^{\varepsilon/3}\big) = \int_X P^*_{q_i}\delta_y\big(K_i^{\varepsilon/3}\big)P^*_s\mu(\mathrm{d}y) \geq \int_U P^*_{q_i}\delta_y\big(K_i^{\varepsilon/3}\big)P^*_s\mu(\mathrm{d}y) \geq \frac{\varepsilon}{2}P^*_s\mu(U)$$

for $i \geq i_0$ and $s \geq 0$. Consequently, we have for $i_0 \leq i \leq N$

$$\begin{aligned}\frac{1}{T}\int_0^T P^*_s\mu\big(K_i^{\varepsilon/3}\big)\mathrm{d}s &\geq \frac{1}{T}\int_0^T P^*_{s+q_i}\mu\big(K_i^{\varepsilon/3}\big)\mathrm{d}s - \gamma\\ &\geq \frac{\varepsilon}{2}\frac{1}{T}\int_0^T P^*_s\mu(U)\mathrm{d}s - \gamma \geq \frac{\varepsilon}{2}\alpha - \gamma,\end{aligned}$$

by (2.12). From this and the fact that $K_i^{\varepsilon/3} \cap K_j^{\varepsilon/3} = \emptyset$ for $i \neq j$ we obtain

$$\begin{aligned}\frac{1}{T}\int_0^T P^*_s\mu\left(\bigcup_{i=i_0}^N K_i^{\varepsilon/3}\right)\mathrm{d}s &= \sum_{i=i_0}^N \frac{1}{T}\int_0^T P^*_s\mu\big(K_i^{\varepsilon/3}\big)\mathrm{d}s\\ &\geq (N - i_0 + 1)(\varepsilon\alpha - 2\gamma)/2 > 1,\end{aligned}$$

which is impossible.

Now analogously as in the proof of Theorem 3.1 in [11], Step III, we define a sequence of Lipschitzian functions $(f_n)_{n\geq 1}$, a sequence of points $(y_n)_{n\geq 1}$, $y_n \to z$ as $n \to \infty$, two increasing sequences of integers $(i_n)_{n\geq 1}$, $(k_n)_{n\geq 1}$, $i_n < k_n < i_{n+1}$ for $n \in \mathbb{N}$, and a sequence of reals $(p_n)_{n\geq 1}$ such that

$$f_n|_{K_{i_n}} = 1, \qquad 0 \leq f_n \leq \mathbf{1}_{K_{i_n}^{\varepsilon/3}}, \qquad \mathrm{Lip}\ f_n \leq 3/\varepsilon, \tag{2.13}$$

$$\left| P_{p_n}\left(\sum_{i=1}^{n} f_i\right)(z) - P_{p_n}\left(\sum_{i=1}^{n} f_i\right)(y_n)\right| > \frac{\varepsilon}{4}, \tag{2.14}$$

$$P_{p_n}^{*}\delta_u\left(\bigcup_{i=k_n}^{\infty} K_i^{\varepsilon/3}\right) < \frac{\varepsilon}{16} \quad \text{for } u \in \{z, y_n\} \tag{2.15}$$

for every $n \in \mathbb{N}$. From (2.13)–(2.15) it follows (see the proof of Theorem 3.1 in [11], Step III, once again) that

$$\left|P_{p_n} f(z) - P_{p_n} f(y_n)\right| > \frac{\varepsilon}{8}$$

for $n \in \mathbb{N}$ and $f := \sum_{n=1}^{\infty} f_n \in L_b(X)$. Since $y_n \to z$ as $n \to \infty$, this contradicts the assumption that the family $\{P_t f : t \geq 0\}$ is equicontinuous in z. The proof is complete. □

Theorem 1 *Let $(P_t)_{t\geq 0}$ be averagely bounded and concentrating at some $z \in X$. If $(P_t)_{t\geq 0}$ satisfies the e-property, then it is asymptotically stable.*

Proof Fix $x \in X$. Since $(P_t)_{t\geq 0}$ is averagely bounded there is $R > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P_s^{*}\delta_x\big(B(x,R)\big)\mathrm{d}s > \frac{1}{2}.$$

Let $(T_n)_{n\geq 1}$ be an increasing sequence of reals such that $T_n \to \infty$ as $n \to \infty$ and

$$\frac{1}{T_n}\int_0^{T_n} P_s^{*}\delta_x\big(B(x,R)\big)\mathrm{d}s > \frac{1}{2} \quad \text{for } n \in \mathbb{N}.$$

Set $\mu_n = \frac{1}{T_n}\int_0^{T_n} P_s^{*}\delta_x\,\mathrm{d}s$, $n \in \mathbb{N}$, and observe that there are $\mu_n^R \in \mathcal{M}_1^{B(x,R)}$ such that

$$\mu_n \geq \frac{1}{2}\mu_n^R \quad \text{for } n \in \mathbb{N}.$$

Indeed, we may define μ_n^R by the formula $\mu_n^R = \mu_n(\cdot \cap B(x,R))/\mu_n(B(x,R))$ for $n \in \mathbb{N}$. Further, observe that, by concentrating at z, for fixed $\varepsilon > 0$ there is $\alpha > 0$ such that we have

$$P_{s_n}^{*}\mu_n^R\big(B(z,\varepsilon)\big) \geq \alpha$$

for some $s_n > 0$, $n \in \mathbb{N}$. Hence

$$P^*_{s_n}\mu_n\big(B(z,\varepsilon)\big) \geq \frac{1}{2}\alpha \quad \text{for } n \in \mathbb{N},$$

by linearity of $(P^*_t)_{t\geq 0}$. Consequently,

$$\sup_{\mu\in\mathcal{M}_1} \frac{1}{T_n}\int_0^{T_n} P^*_s\mu\big(B(z,\varepsilon)\big)\mathrm{d}s \geq \frac{1}{T_n}\int_0^{T_n} P^*_s\big(P^*_{s_n}\delta_x\big)\big(B(z,\varepsilon)\big)\mathrm{d}s \geq \frac{1}{2}\alpha \quad \text{for } n \in \mathbb{N},$$

and condition (2.9) in Proposition 2 is satisfied. Now Proposition 2 implies the existence of an invariant measure. Further, from Proposition 1 it follows that for any $\varphi \in L_b(X)$ and $\mu \in \mathcal{M}_1$

$$\big\langle \varphi, P^*_t\mu\big\rangle \to \langle \varphi, \mu_*\rangle$$

as t tends to $+\infty$. Application of the Alexandrov theorem finishes the proof (see [3]). □

3 A model

We are concerned with a jump process connected with an iterated function system. A large class of applications of such models, both in physics and biology, is worth mentioning here: the short noise, the photoconductive detectors, the growth of the size of structural population, the motion of relativistic particles, both fermions and bosons, and many others (see [6] and references therein). Similar processes appeared in [16], where the authors analyzed large scale phenomena of some transport equations. Our process generalizes also iterated function systems that are considered mainly because of their close connection to fractals and semifractals (see [9]).

Let (X,ρ) be a Polish space and let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let $(\tau_n)_{n\geq 0}$ be a sequence of random variables $\tau_n : \Omega \to \mathbb{R}_+$ with $\tau_0 = 0$ and such that $\Delta\tau_n = \tau_n - \tau_{n-1}$, $n \geq 1$, are independent and have the same density $\lambda e^{-\lambda t}$. Let $(S(t))_{t\geq 0}$ be a continuous semigroup on X. We have also given a sequence of Lipschitz functions $w_i : X \to X$, $i = 1,\ldots,N$, and a probabilistic vector $(p_1(x),\ldots,p_N(x))$, $p_i(x) \geq 0$, $\sum_{i=1}^N p_i(x) = 1$ for $x \in X$. The pair $(w_1,\ldots,w_N; p_1,\ldots,p_N)$ is called an iterated function system.

Now we define the X-valued Markov process $\Phi = (\Phi(t))_{t\geq 0}$ in the following way. Let $x \in X$ and $\xi_1 = S(\tau_1)(x)$. We randomly select from the set $\{1,\ldots,N\}$ an integer i_1 and the probability that $i_1 = k$ is equal to $p_k(\xi_1)$. Set $\Phi^x_1 = w_{i_1}(\xi_1)$.

Let $\Phi^x_1,\ldots,\Phi^x_{n-1}$, $n \geq 2$, be given. Assuming that $\Delta\tau_n = \tau_n - \tau_{n-1}$ is independent upon $\Phi^x_1,\ldots,\Phi^x_{n-1}$, we define $\xi_n = S(\Delta\tau_n)(\Phi^x_{n-1})$. Further, we randomly choose i_n from the set $\{1,\ldots,N\}$ in such a way that the probability of the event $\{i_n = k\}$ is equal to $p_k(\xi_n)$. Then we define $\Phi^x_n = w_{i_n}(\xi_n)$. Finally we set $\Phi^x(t) = S(t-\tau_n)(\Phi^x_n)$ if $\tau_n \leq t < \tau_{n+1}$ for $n \geq 0$.

In [18] we considered the Markov chain $(\Phi_n)_{n\geq 1}$ proving the existence of its invariant distribution. Now we are aimed at showing that the Markov process Φ is

asymptotically stable. In [19] we provided a criterion on stability of Markov processes under some additional condition. The condition is applicable when the dynamical system is non-degenerate, i.e. the support of its invariant measure is the entire phase space. This is not the case in the studied system.

Denote by $(P_t)_{t\geq 0}$ its semigroup, i.e.

$$P_t\varphi(x) = \mathbb{E}\varphi\big(\Phi^x(t)\big) \quad \text{for } \varphi \in B_b(X).$$

We will assume that there exists $r \in (0, 1)$ such that

$$\sum_{i=1}^{N} p_i(x)\rho\big(w_i(x), w_i(y)\big) \leq r\rho(x, y) \quad \text{for } x, y \in X. \tag{3.1}$$

Moreover, there exist a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{N} \big|p_i(x) - p_i(y)\big| \leq \omega\big(\rho(x, y)\big) \quad \text{for } x, y \in X \tag{3.2}$$

and ω satisfies the *Dini condition*, i.e. ω is a nondecreasing and concave function with

$$\int_0^{\varepsilon} \frac{\omega(t)}{t}\mathrm{d}t < \infty \quad \text{for some } \varepsilon > 0,$$

and $\alpha \geq 0$ such that

$$\rho\big(S(t)(x), S(t)(y)\big) \leq e^{\alpha t}\rho(x, y) \quad \text{for } x, y \in X \text{ and } t \geq 0. \tag{3.3}$$

We will assume that the semigroup $(S(t))_{t\geq 0}$ admits a global attractor. Recall that a compact set $\mathcal{K} \subset X$ is called a *global attractor* if it is invariant and attracting for $(S(t))_{t\geq 0}$, i.e. $S(t)\mathcal{K} = \mathcal{K}$ for every $t \geq 0$ and for every bounded set B and open set U, $\mathcal{K} \subset U$, there exists $t_* > 0$ such that $S(t)B \subset U$ for $t \geq t_*$.

Proposition 3 *Assume that conditions* (3.1)–(3.3) *hold and*

$$r + \alpha/\lambda < 1. \tag{3.4}$$

If $(S(t))_{t\geq 0}$ has a global attractor, then the semigroup $(P_t)_{t\geq 0}$ corresponding to Φ is asymptotically stable.

For abbreviation we shall write

$$\begin{aligned}
&\mathbf{x}_1(\Delta\tau_1; x) = S(\Delta\tau_1)(x),\\
&\mathbf{x}_2(\Delta\tau_1, \Delta\tau_2; i_1; x) = S(\Delta\tau_2)\big(w_{i_1}\big(\mathbf{x}_1(\Delta\tau_1; x)\big)\big) \quad \text{and}\\
&\mathbf{x}_{n+1}(\Delta\tau_1, \ldots, \Delta\tau_{n+1}; i_1, \ldots, i_n; x)\\
&\quad = S(\Delta\tau_{n+1})\big(w_{i_n}\big(\mathbf{x}_n(\Delta\tau_1, \ldots, \Delta\tau_n; i_1, \ldots, i_{n-1}; x)\big)\big)
\end{aligned}$$

for $n \geq 2$. Moreover, for $n \geq 1$ we set

$$\wp_n(\Delta\tau_1, \ldots, \Delta\tau_n; i_1, \ldots, i_n; x) := p_{i_1}\big(\mathbf{x}_1(\Delta\tau_1; x)\big) p_{i_2}\big(\mathbf{x}_2(\Delta\tau_1, \Delta\tau_2; i_1; x)\big) \cdots p_{i_n}\big(\mathbf{x}_n(\Delta\tau_1, \ldots, \Delta\tau_n; i_1, \ldots, i_{n-1}; x)\big).$$

Before we come to the proof of the announced theorem we would like to make some useful computation. Indeed, fix $t > 0$ and let

$$\Omega_n(t) = \big\{\omega : \tau_n(\omega) \leq t \ \& \ \tau_{n+1}(\omega) > t\big\}.$$

Then we have

$$\begin{aligned}
&\int_{\bigcup_{n\geq k}\Omega_n(t)} e^{\alpha\tau_k}\mathrm{d}\mathbb{P} \\
&= \int_{\bigcup_{n\geq k}\Omega_n(t)} e^{\alpha(\Delta\tau_1+\cdots+\Delta\tau_k)}\mathrm{d}\mathbb{P} \\
&\leq \int\cdots\int_{\{(u_1,\ldots,u_k)\in\mathbb{R}^k : u_1+\cdots+u_k\leq t\}} \lambda^k e^{(\alpha-\lambda)u_1}\cdots e^{(\alpha-\lambda)u_k}\mathrm{d}u_1\cdots\mathrm{d}u_k \\
&\leq \frac{\lambda^k}{(\lambda-\alpha)^k}.
\end{aligned} \tag{3.5}$$

Further, from conditions (3.1)–(3.3) and the fact that ω is a nondecreasing and concave function it follows the following estimation

$$\begin{aligned}
&\sum_{i_1,\ldots,i_n=1}^{N} \big|\wp_n(\Delta\tau_1, \ldots, \Delta\tau_n; i_1, \ldots, i_n; x) - \wp_n(\Delta\tau_1, \ldots, \Delta\tau_n; i_1, \ldots, i_n; y)\big| \\
&\leq \sum_{i_1,\ldots,i_{n-1}=1}^{N} \big(\wp_{n-1}(\Delta\tau_1, \ldots, \Delta\tau_{n-1}; i_1, \ldots, i_{n-1}; x) \\
&\quad \times \omega\big(\rho\big(\mathbf{x}_n(\Delta\tau_1, \ldots, \Delta\tau_n; i_1, \ldots, i_{n-1}; x), \mathbf{x}_n(\Delta\tau_1, \ldots, \Delta\tau_n; i_1, \ldots, i_{n-1}; y)\big)\big)\big) \\
&\quad + \sum_{i_1,\ldots,i_{n-1}=1}^{N} \big|\wp_{n-1}(\Delta\tau_1, \ldots, \Delta\tau_{n-1}; i_1, \ldots, i_{n-1}; x) \\
&\quad - \wp_{n-1}(\Delta\tau_1, \ldots, \Delta\tau_{n-1}; i_1, \ldots, i_{n-1}; y)\big| \\
&\leq \omega\Bigg(\sum_{i_1,\ldots,i_{n-1}=1}^{N} \big(\wp_{n-1}(\Delta\tau_1, \ldots, \Delta\tau_{n-1}; i_1, \ldots, i_{n-1}; x) \\
&\quad \times \rho\big(\mathbf{x}_n(\Delta\tau_1, \ldots, \Delta\tau_n; i_1, \ldots, i_{n-1}; x), \mathbf{x}_n(\Delta\tau_1, \ldots, \Delta\tau_n; i_1, \ldots, i_{n-1}; y)\big)\big)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i_1,\dots,i_{n-1}=1}^{N} \Big|\wp_{n-1}(\Delta\tau_1,\dots,\Delta\tau_{n-1}; i_1,\dots,i_{n-1}; x) \\
&- \wp_{n-1}(\Delta\tau_1,\dots,\Delta\tau_{n-1}; i_1,\dots,i_{n-1}; y)\Big| \\
&\le \omega\big(r^{n-1}e^{\alpha\tau_n}\rho(x,y)\big) + \omega\big(r^{n-2}e^{\alpha\tau_{n-1}}\rho(x,y)\big) + \dots + \omega\big(e^{\alpha\tau_1}\rho(x,y)\big) \\
&= \sum_{k=1}^{n} \omega\big(r^{k-1}e^{\alpha\tau_k}\rho(x,y)\big), \qquad (3.6)
\end{aligned}
$$

by induction on n.

We split the proof of Proposition 3 into several lemmas devoted to verification of the e-property, boundedness in probability and concentrating at some $z \in X$ of the semigroup $(P_t)_{t\ge 0}$, respectively.

Lemma 1 *If conditions* (3.1)–(3.4) *hold, then the semigroup* $(P_t)_{t\ge 0}$ *corresponding to* Φ *satisfies the e-property.*

Proof Fix $\psi \in L_b(X)$ with the Lipschitz constant L and let $t \ge 0$. Then we have

$$
\big|P_t\psi(x) - P_t\psi(y)\big| \le \sum_{n=0}^{\infty} \mathbb{E}\big(\mathbf{1}_{\Omega_n(t)}\big|\psi\big(\Phi^x(t)\big) - \psi\big(\Phi^y(t)\big)\big|\big).
$$

We are going to evaluate the term $\mathbb{E}(\mathbf{1}_{\Omega_n(t)}|\psi(\Phi^x(t)) - \psi(\Phi^y(t))|)$. By (3.5) and (3.6) we have

$$
\begin{aligned}
&\mathbb{E}\big(\mathbf{1}_{\Omega_n(t)}\big|\psi\big(\Phi^x(t)\big) - \psi\big(\Phi^y(t)\big)\big|\big) \\
&= \int_{\Omega_n(t)} \Bigg| \sum_{i_1,\dots,i_n=1}^{N} \wp_n(\Delta\tau_1,\dots,\Delta\tau_n; i_1,\dots,i_n; x) \\
&\quad \times \psi\big(\mathbf{x}_{n+1}(\Delta\tau_1,\dots,\Delta\tau_n, t-\tau_n; i_1,\dots,i_n; x)\big) \\
&\quad - \sum_{i_1,\dots,i_n=1}^{N} \wp_n(\Delta\tau_1,\dots,\Delta\tau_n; i_1,\dots,i_n; y) \\
&\quad \times \psi\big(\mathbf{x}_{n+1}(\Delta\tau_1,\dots,\Delta\tau_n, t-\tau_n; i_1,\dots,i_n; y)\big)\Bigg| \mathrm{d}\mathbb{P} \\
&\le \|\psi\|_\infty \int_{\Omega_n(t)} \sum_{i_1,\dots,i_n=1}^{N} \big|\wp_n(\Delta\tau_1,\dots,\Delta\tau_n; i_1,\dots,i_n; x) \\
&\quad - \wp_n(\Delta\tau_1,\dots,\Delta\tau_n; i_1,\dots,i_n; y)\big| \mathrm{d}\mathbb{P} \\
&\quad + L\int_{\Omega_n(t)} \Bigg(\sum_{i_1,\dots,i_n=1}^{N} \wp_n(\Delta\tau_1,\dots,\Delta\tau_n; i_1,\dots,i_n; x)
\end{aligned}
$$

$$\times \rho\big(\mathbf{x}_{n+1}(\Delta\tau_1,\dots,\Delta\tau_n,t-\tau_n;i_1,\dots,i_n;x),\mathbf{x}_{n+1}(\Delta\tau_1,\dots,\Delta\tau_n,t-\tau_n;i_1,\dots,i_n;y)\big)\Bigg)\mathrm{d}\mathbb{P}$$

$$\le \|\psi\|_\infty \sum_{k=1}^{n}\int_{\Omega_n(t)} \omega\big(r^{k-1}\rho(x,y)e^{\alpha\tau_k}\big)\mathrm{d}\mathbb{P} + Lr^n\rho(x,y)e^{\alpha t}\frac{(\lambda t)^n}{n!}e^{-\lambda t}.$$

Consequently, we obtain

$$\big|P_t\psi(x)-P_t\psi(y)\big| \le \sum_{n=0}^{\infty}\mathbb{E}\big(\mathbf{1}_{\Omega_n(t)}\big|\psi\big(\Phi^x(t)\big)-\psi\big(\Phi^y(t)\big)\big|\big)$$

$$\le \|\psi\|_\infty \sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_{\Omega_n(t)} \omega\big(r^{k-1}\rho(x,y)e^{\alpha\tau_k}\big)\mathrm{d}\mathbb{P} + \sum_{n=0}^{\infty} L\rho(x,y)r^n e^{\alpha t}\frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

$$\le \|\psi\|_\infty \sum_{k=1}^{\infty}\omega\Bigg(r^{k-1}\rho(x,y)\int_{\bigcup_{n\ge k}\Omega_n(t)} e^{\alpha\tau_k}\mathrm{d}\mathbb{P}\Bigg) + Le^{(\alpha-\lambda+r\lambda)t}\rho(x,y)$$

$$\le \|\psi\|_\infty \sum_{k=1}^{\infty}\omega\big(\big(r\lambda/(\lambda-\alpha)\big)^{k-1}\big(\lambda/(\lambda-\alpha)\big)\rho(x,y)\big) + Le^{(\alpha-\lambda+r\lambda)t}\rho(x,y).$$

Set

$$\gamma := r\lambda/(\lambda-\alpha) < 1,$$

the last inequality by (3.4). Since

$$\sum_{k=1}^{\infty}\omega\big(\big(r\lambda/(\lambda-\alpha)\big)^{k-1}\big(\lambda/(\lambda-\alpha)\big)\rho(x,y)\big)$$

$$= \sum_{k=1}^{\infty}\omega\big(\gamma^k\rho(x,y)/r\big)$$

$$= \gamma/(1-\gamma)\sum_{k=1}^{\infty}\big(\gamma^{k+1}\rho(x,y)/r\big)^{-1}\omega\big(\big(\gamma^k\rho(x,y)/r\big)\big)\big(\gamma^k-\gamma^{k+1}\big)\rho(x,y)/r$$

$$\le \gamma/(1-\gamma)\int_0^{\rho(x,y)/r}\frac{\omega(t)}{t}\mathrm{d}t,$$

we obtain

$$\lim_{y\to x}\sup_{t\ge 0}\big|P_t\psi(y)-P_t\psi(x)\big| = 0$$

and we are done. □

Lemma 2 *If conditions* (3.1)–(3.4) *hold, then for any* $x_0 \in X$, $R > 0$ *and* $T > 0$

$$\sup_{x \in B(x_0,R)} \sup_{t \in [0,T]} P_t \rho(x, x_0) < +\infty. \tag{3.7}$$

Moreover, for any $T > 0$ *there exists* $\theta \in (0, 1)$ *and* $\Gamma > 0$ *such that*

$$P_T \rho(x, x_0) \le \theta \rho(x, x_0) + \Gamma \tag{3.8}$$

and consequently the semigroup $(P_t)_{t \ge 0}$ *is averagely bounded.*

Proof Fix $x_0 \in X$ and $R > 0$. From (3.3) and the fact that w_i's are Lipschitzean, by induction, we easily show that there exists $\Upsilon > 0$ such that for any $n \ge 1$, $x \in B(x_0, R)$, $i_1, \dots, i_n \in \{1, \dots, N\}$ and $\Delta\tau_1 + \cdots + \Delta\tau_n < t \le T$

$$\rho\big(\mathbf{x}_{n+1}(\Delta\tau_1, \dots, \Delta\tau_n; t - \tau_n; i_1, \dots, i_n; x), x_0\big) \le \Upsilon^n.$$

Since

$$\sum_{n=1}^{\infty} \Upsilon^n \mathbb{P}\big(\Omega_n(t)\big) = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda \Upsilon t)^n}{n!} = e^{\lambda t(\Upsilon - 1)} < \infty,$$

we obtain $\sup_{x \in B(x_0,R)} \sup_{t \in [0,T]} P_t \rho(x, x_0) < +\infty$.

Further, for given $T > 0$ we have

$$P_T \rho(x, x_0) = \sum_{n=0}^{\infty} \mathbb{E}\big(\mathbf{1}_{\Omega_n(T)} \rho\big(\Phi^x(T), x_0\big)\big).$$

On the other hand, we have

$$\begin{aligned}
&\mathbb{E}\big(\mathbf{1}_{\Omega_n(T)} \rho\big(\Phi^x(T), x_0\big)\big) \\
&= \int_{\Omega_n(T)} \sum_{i_1,\dots,i_n=1}^{N} \wp_n(\Delta\tau_1, \dots, \Delta\tau_n; i_1, \dots, i_n; x) \\
&\quad \times \rho\big(\mathbf{x}_{n+1}(\Delta\tau_1, \dots, \Delta\tau_n, T - \tau_n; i_1, \dots, i_n; x), x_0\big) \mathrm{d}\mathbb{P} \\
&\le \int_{\Omega_n(T)} \Bigg(\sum_{i_1,\dots,i_n=1}^{N} \wp_n(\Delta\tau_1, \dots, \Delta\tau_n; i_1, \dots, i_n; x) \\
&\quad \times \rho\big(\mathbf{x}_{n+1}(\Delta\tau_1, \dots, \Delta\tau_n, T - \tau_n; i_1, \dots, i_n; x), \\
&\quad \times x\mathbf{x}_{n+1}(\Delta\tau_1, \dots, \Delta\tau_n, T - \tau_n; i_1, \dots, i_n; x_0)\big)\Bigg) \\
&\quad + \int_{\Omega_n(T)} \sum_{i_1,\dots,i_n=1}^{N} \wp_n(\Delta\tau_1, \dots, \Delta\tau_n; i_1, \dots, i_n; x) \\
&\quad \times \rho\big(\mathbf{x}_{n+1}(\Delta\tau_1, \dots, \Delta\tau_n; T - \tau_n; i_1, \dots, i_n; x_0), x_0\big) \\
&\le r^n e^{\alpha T} \frac{(\lambda T)^n}{n!} e^{-\lambda T} \rho(x, x_0) + \Upsilon^n e^{-\lambda T} \frac{(\lambda T)^n}{n!},
\end{aligned}$$

which gives

$$P_T\rho(x,x_0) \le \theta\rho(x,x_0) + \Gamma$$

with $\theta := e^{(\alpha-\lambda+r\lambda)T} < 1$ and $\Gamma := e^{\lambda T(\Upsilon-1)}$.

Iterating condition (3.7) and taking into account condition (3.8) we obtain that for any $R > 0$

$$\sup_{x\in B(x_0,R)} \sup_{t\ge 0} P_t\rho(x,x_0) < \infty.$$

To prove that $(P_t)_{t\ge 0}$ is averagely bounded fix an $\varepsilon > 0$ and let $R > 0$ be given. Let $r > \sup_{x\in B(x_0,R)} \sup_{t\ge 0} P_t\rho(x,x_0)/\varepsilon$. If $\mu \in \mathcal{M}_1^{B(x_0,R)}$, then

$$\begin{aligned}
&\frac{1}{T}\int_0^T P_s^*\mu\big(X\setminus B(x_0,r)\big)\mathrm{d}s \\
&= \frac{1}{T}\int_X\int_0^T P_s^*\delta_x\big(X\setminus B(x_0,r)\big)\mathrm{d}s\mu(\mathrm{d}x) \\
&= \frac{1}{T}\int_X\int_0^T \mathbb{P}\big(\Phi^x(s)\notin B(x_0,r)\big)\mathrm{d}s\mu(\mathrm{d}x) \le \frac{1}{T}\int_X\int_0^T \frac{\mathbb{E}\rho(\Phi^x(s),x_0)}{r}\mathrm{d}s\mu(\mathrm{d}x) \\
&= \frac{1}{T}\int_X\int_0^T \frac{P_s\rho(x,x_0)}{r}\mathrm{d}s\mu(\mathrm{d}x) \le \frac{\sup_{x\in B(x_0,R)}\sup_{t\ge 0} P_t\rho(x,x_0)}{r} < \varepsilon.
\end{aligned}$$

The proof is complete. □

Lemma 3 *If conditions* (3.1)–(3.4) *hold, then there is* $z \in X$ *such that the semigroup* $(P_t)_{t\ge 0}$ *is concentrating at it.*

Proof Consider the iterated function system $(w,p) = (w_1,\dots,w_N; p_1,\dots,p_N)$. In [17] it was proved that under conditions (3.1)–(3.2) it is asymptotically stable. Denote by μ_0 its invariant distribution. Choose $z \in \operatorname{supp}\mu_0$. Fix an $\varepsilon > 0$. Since the iterated function system (w,p) is asymptotically stable, for any $x \in X$ we may find $n_x \ge 1$ such that

$$P^n\delta_x\big(B(z,\varepsilon/3)\big) \ge \mu_0\big(B(z,\varepsilon/3)\big)/2 := \beta \quad \text{for any } n \ge n_x,$$

where P is the Markov operator given by the formula $P\mu(\cdot) = \sum_{i=1}^N \int_{w_i^{-1}(\cdot)} p_i(x) \times \mu(\mathrm{d}x)$. It was proved that the Markov operator P is nonexpansive with respect to some Wasserstein metric (see [17]). To be precise, we proved that there exists a metric $\tilde\rho$ in X equivalent to the metric ρ (equivalence means that any sequence converges in ρ iff it is convergent in $\tilde\rho$) such that

$$\|P\mu - P\nu\|_W \le \|\mu-\nu\|_W \quad \text{for any } \mu,\nu \in \mathcal{M}_1,$$

where

$$\|\mu-\nu\|_W = \sup\big\{\big|\langle\varphi,\mu\rangle - \langle\varphi,\nu\rangle\big| : \varphi : X\to\mathbb{R}, \|\varphi\|_\infty \le 1, \big|\varphi(x)-\varphi(y)\big| \le \tilde\rho(x,y)\big\}.$$

From Lemma 3.1 in [17] it follows that we may find $n_0 \geq 1$ such that

$$P^{n_0}\delta_x\big(B(z, 2\varepsilon/3)\big) \geq \beta/2 \quad \text{for any } x \text{ in some open neighbourhood } O \text{ of } \mathcal{K}.$$

Denote by $\mathbb{P}_{\Omega_{n_0}(t_0)}(\cdot) := \mathbb{P}(\cdot \cap \Omega_{n_0}(t_0))/\mathbb{P}(\Omega_{n_0}(t_0))$ for some $t_0 > 0$ and let $\mathbb{E}_{\Omega_{n_0}(t_0)}$ denote the expectation with respect to the probability $\mathbb{P}_{\Omega_{n_0}(t_0)}$. Diminishing O if necessary and taking t_0 small enough we obtain

$$\mathbb{E}_{\Omega_{n_0}(t_0))}\mathbf{1}_{B(z,\varepsilon)}\big(\Phi^x(t_0)\big) \geq P^{n_0}\delta_x\big(B(z, 2\varepsilon/3)\big) \geq \beta/2 \quad \text{for } x \in O$$

and consequently

$$\mathbb{E}\mathbf{1}_{B(z,\varepsilon)}\big(\Phi^x(t_0)\big) \geq \beta(\lambda t_0)^{n_0}e^{-\lambda t_0}/(2n_0!) \quad \text{for } x \in O. \tag{3.9}$$

Fix $x_0 \in X$. Let $A \subset X$ be a bounded Borel set. From the fact that $\mathcal{K}$ is an attractor for $(S(t))_{t\geq 0}$ there is $t_1 > 0$ such that $S(t_1)(x) \in O$ for any $x \in B(x_0, R)$. Consequently,

$$\mathbb{E}\mathbf{1}_O\big(\Phi^x(t_1)\big) = \mathbb{P}\big(\Phi^x(t_1) \in O\big) \geq e^{-\lambda t_1} \quad \text{for any } x \in A. \tag{3.10}$$

Set

$$\hat{\alpha} := \beta(\lambda t_0)^{n_0}e^{-\lambda(t_0+t_1)}/(2n_0!).$$

From conditions (3.9), (3.10) and the Chapman–Kolmogorov equation we obtain for arbitrary $\mu \in \mathcal{M}_1^A$

$$\begin{aligned} P^*_{t_0+t_1}\mu\big(B(z,\varepsilon)\big) &= \int_A \mathbb{E}\mathbf{1}_{B(z,\varepsilon)}\big(\Phi^x(t_0+t_1)\big)\mu(\mathrm{d}x) \\ &\geq \int_A \mathbb{E}\mathbf{1}_{B(z,\varepsilon)}\big(\Phi^x(t_0)\big)\mathbb{E}\mathbf{1}_O\big(\Phi^x(t_1)\big)\mu(\mathrm{d}x) \geq \hat{\alpha}, \end{aligned}$$

which finishes the proof that the semigroup $(P_t)_{t\geq 0}$ is concentrating at z. □

Proof of Proposition 3 From Lemma 1 it follows that the semigroup $(P_t)_{t\geq 0}$ satisfies the e-property. It is also averagely bounded and concentrating at some z, by Lemmas 2 and 3. Application of Theorem 1 finishes the proof. □

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