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Hyperbolic Geometry and Binocular Visual Space

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HYPERBOLIC GEOMETRY AND BINOCULAR VISUAL SPACE

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Abstract

Euclidean geometry is widely accepted as the model for our physical space; however, there is not a consistent model for our visual space. There is evidence that our eyes work to make pictures and images of the physical space using a hyperbolic model. In this paper we are going to explore hyperbolic geometry and hyperbolic models of binocular visual space. In hyperbolic geometry, all of the axioms of Euclidean geometry hold except the parallel postulate. The models of hyperbolic parallel lines explain how we perceive parallel lines as curved, such as how railroad tracks going off in the distance appear to converge. We will show that binocular visual space may indeed be best explained by a hyperbolic model.

1 Introduction

Janos Bolyai was the first mathematician to attempt to publish ideas and findings on non-Euclidean geometry. He published his findings in the appendix of his fathers book; his father tried to advise him that this was not a good idea, but Janos was very confident in his findings and insisted on publishing.

His father was good friends with the famous mathematician Carl Friedrich Gauss, and they were in constant correspondence [1]. Bolyai's father sent his article to Gauss; Bolyai looked up to Gauss and was hoping he would approve of his findings. Gauss responded negatively to Bolyai's work. Gauss claimed that he had already made these discoveries, so Bolyai was not the first. Gauss said that he had never published his ideas because they are incomplete. Bolyai was devastated, and fell into a deep depression. After his life work was thrown away, he never published again [1].

Gauss claimed that he had been working on ideas of non-Euclidean geometry since he was fifteen years old. Gauss mostly wrote about the discovery of triangles whose sum of the angle measures can be less than 180 degrees. However, Gauss was also a perfectionist and would only publish his works once he felt they were totally complete. He was scared to publish such revolutionary ideas [1].

After Nikolai Lobachevsky published his account of non-Euclidean geometry in 1829, this area of mathematics began to gain popularity. Lobachevsky's work did not receive much attention when he published in his home country of Russia; he even lost his long time career at his university. He called his geometry "imaginary geometry" and later "pangeometry." However, his ideas were highly under appreciated until he published in Germany and Gauss praised his work. Even then mathematicians did not approve of his findings until after his death [1].

These ideas were once controversial, but now they are the basis of much modern mathematics around the world. At the beginning of the nineteenth century, visualizing a non-Euclidean geometry became possible for mathematicians. Mathematicians work with the idea of replacing Euclid's postulates and exploring the implications [2]. These are the ideas that we will be exploring throughout this paper.

2 Hyperbolic Geometry

Euclidean geometry is the geometry commonly taught in schools, and is the most familiar geometry. Euclidean geometry is based on the axioms of neutral geometry along with Euclid's parallel postulate [2]. This parallel postulate states that for every line l and point P not on l, there exists an unique line through P parallel to l. Euclidean geometry is modeled on the Cartesian coordinate system, and was commonly thought to model the physical world.

One form of non-Euclidean geometry is elliptic geometry which follows the axioms of neutral geometry and replaces the parallel postulate with the elliptic parallel postulate. The elliptic parallel postulate states that for every line l and point P not on l, there exists no line through P parallel to l. Elliptic geometry is modeled on the sphere, where lines and points are determined by great circles. This model will be discussed later in this section.

We will focus on another form of non-Euclidean geometry, called hyperbolic geometry. In hyperbolic geometry, we assume all of the axioms of neutral geometry but replace the parallel postulate with the hyperbolic parallel postulate. The hyperbolic parallel postulate states that for every line l and point P not on l, there pass through P at least two distinct parallels to l [1]. In hyperbolic geometry, there are at least two, and possibly infinitely many, lines through P parallel to l.

Lines in geometry are defined as the shortest distance between two points. As shown in Figure 1, hyperbolic lines take the form of either a circle centered on the x-axis or a straight line perpendicular to the x-axis [1]. Thus most lines in hyperbolic geometry take the form of arcs.

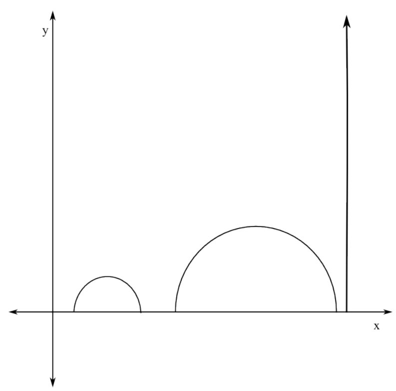

Figure 1: Hyperbolic lines on the cartesian plane

2.1 Properties of Parallel Lines

There are two types of parallel lines in hyperbolic geometry: *diverging parallel lines* and *limiting parallel rays.* If you have a line l and point P, not on l, then there are an infinite number of diverging parallel lines on this point parallel to line l [1].

We need two important theorems before we can define diverging parallel lines [1].

Theorem 2.1. *If l and l' are any distinct parallel lines, then any set of points on l equidistant from l' contains at most two points.*

In other words, there are at most two points on the line l that are a given distance from line l', as shown in Figure 2. For example, there is only one point on l' at distance d_1 from l.

From this picture, we can see that there are no points on l' with a distance less than d_1 from l and exactly two points at a distance d from l, when $d > d_1$.

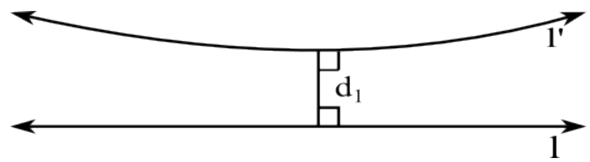

Figure 2: Diverging parallel lines with a common perpendicular

Theorem 2.2. *If l and l' are parallel lines for which there exist a pair of points A and B on l equidistant from l', then l and l' have a common perpendicular segment.*

The common perpendicular is the shortest segment from line l to line l', as shown in Figure 2. There is one unique common perpendicular segment between two diverging parallel lines [1]. Thus, diverging parallel lines are two lines that admit a common perpendicular and have at most two points of equal distance [1].

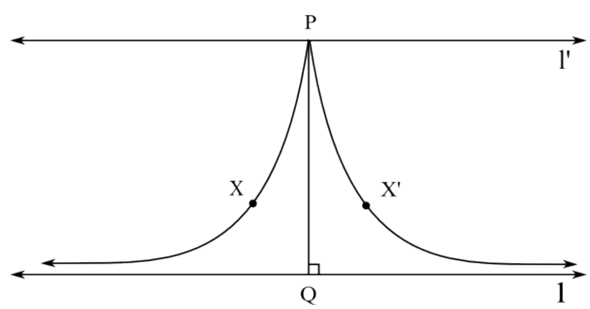

Figure 3: Limiting parallel rays

The second type of parallel lines are *asymptotic parallel lines*. Asymptotic parallel lines are two lines such that any line between them intersects one of the two lines [2]. As shown in Figure 3, any ray between l' and $\overrightarrow{PX}$ or $\overrightarrow{PX'}$ will not intersect the line l. However, any ray between $\overrightarrow{PX}$ or $\overrightarrow{PX'}$ and the perpendicular $\overleftrightarrow{PQ}$ will intersect the line l. There are exactly two limiting parallel rays to line l from point P, but there are infinitely many diverging parallel lines from this point.

3 Binocular Visual Space

3.1 The Curved World

Perception of parallel lines has been a source of controversy and challenge for artists. Artists use a variety of theories on how parallel lines should be perceived in pieces of artwork. We have been trained for centuries to ignore what we do not see as straight. We have to be aware of these curves and their properties to see them correctly. We have been taught for a long time to see these straight lines, thus we must use a lot of effort to fight against this.

The most famous and commonly accepted methods for depictions of parallel lines in art are the properties of Leon Battista Alberti. He dictated that in art there should be no distortion of straight lines or objects. He states that all horizontal and vertical lines parallel to the picture plane must be parallel and perpendicular in the picture [3].

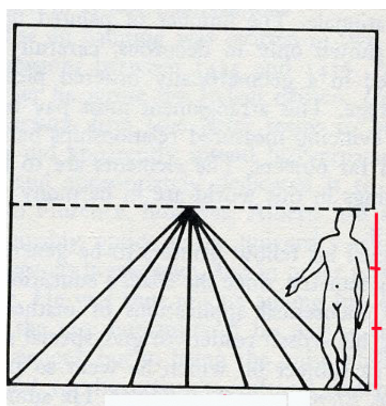

Figure 4: Alberti's Method for Artwork [4]

As shown in Figure 4, Alberti started his paintings with a rectangle. He would then draw his first subject, or the man shown to the right of the figure. He would then divide that man into three parts, which he called a *braccia*. Using the braccia, Alberti would then divide the base of his rectangle. Alberti would then pick a point at the center line of the rectangle, which he called the *centric point*, and must be at the height of the subject or higher. Once he had picked the centric point he would connect all the baseline division points to this centric point. Alberti claims that this would allow all objects he painted here to appear on the same plane as the subject [4].

Leonardo De Vinci is the only known mathematician to have critically argued against

Alberti. According to De Vinci, instead of a flat surface we should be imagining more of a concave spherical surface. Robert Hansen states in his article, “The Curving World: Hyperbolic Linear Perspective”, that artists are not trying to consider other diagrams. He reminds us that artists paintings are private interpretations of the real world, their perspective is going to be different than others [3].

Hansen claims that the closer we get to a line, or edge, the more curved we perceive that line to be. Curves and distortions in lines are easier to see the longer the lines become and the farther they move out towards our peripheral vision, as shown in Figure 5. The lines or curves also appear more distorted or curved when we move past them or they move past us. Hansen claims that the segment of the line closer to your eye will appear to have the most curvature [3].

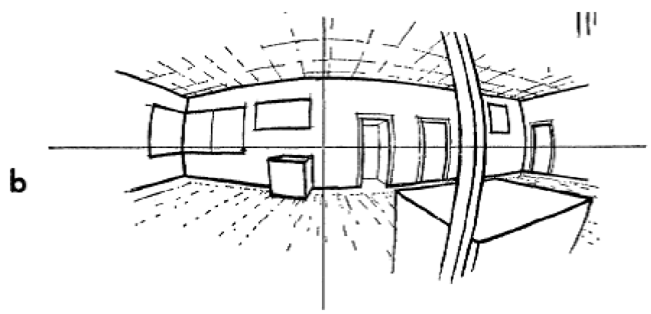

Figure 5: How we might see distortions in a wall [3]

According to Hansen, we see straight lines as hyperbolas; however, this is a difficult thing to prove. He states that the visible world can be metaphorically located at any moment on a concave surface of all points equidistant from your eye. We can imagine that we have a glass globe around our heads and at any moment we can draw what is in our visual space onto this globe. Hansen claims that this is why wide-angle views or reflections in concave mirrors are only a magnification, they do not distort the image in any other way [3].

In Hansens model, there are six vanishing points in our visual space: above us, below us, to each side, right in front of us, and one right behind our heads. This is apparent when looking at a wall. The wall appears the biggest directly in front of us, but appears smaller closer to the vanishing points around the edges. Hansen claims that vertical lines are only exactly perpendicular at the horizon bend as the line reaches the upper and lower vanishing points, as we can see from Figure 6 [3].

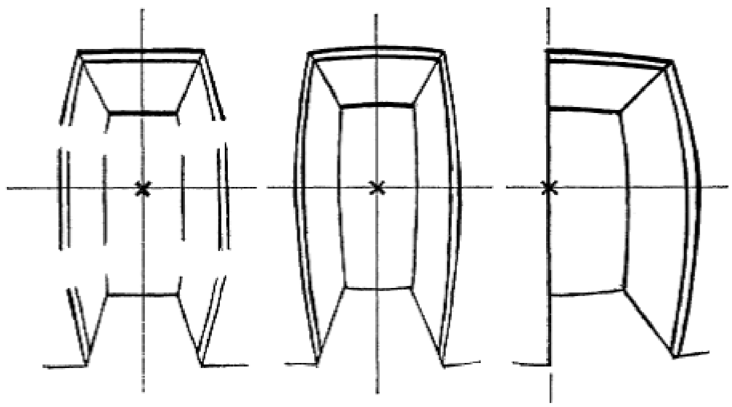

Figure 6: How we might see distortions in a door frame [3]

3.2 The Alley Experiments

Hillebrand and Blumenfeld conducted two separate experiments, called the alley experiments, to investigate how we perceive parallel lines as curved. The most famous example of our misperception of parallel lines is railroad tracks appearing to converge at the horizon. Hillebrand and Blumenfeld both put their subjects in a dark room and told them to arrange different lights into lines that appear parallel.

In Hillebrand's experiment, the participants lined up lights into lines that appeared parallel. The subjects' heads were fixed so that they could not move, but they could freely move their eyes. As shown in Figure 7, the points L and R are the subjects' left and right eyes, respectively. The curves represent the lines of lights arranged by the subjects. These curves are neither straight, nor parallel [5].

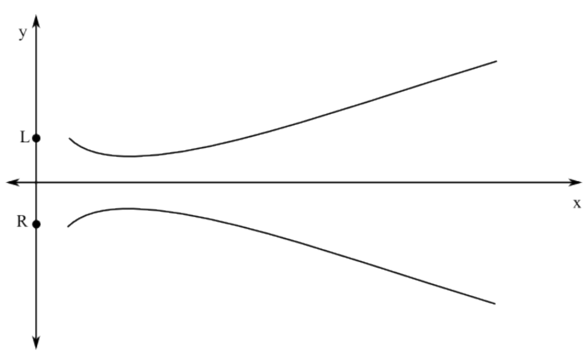

Figure 7: Hillebrand's experiment results [5]

Blumenfeld conducted a similar experiment. Instead of having all of the lights on at once, the lights appeared in sequence. First, he told the subjects to adjust the lights to appear straight and parallel. He called this the parallel alley with results shown by line p in Figure 8 [5]. Next, he told the participants to arrange the pairs so that the lateral distance was constant [6]. He called these curves the distance alleys, and they are shown by line d in Figure 8. He found that the lines were neither straight nor parallel, as you can see from Figure 8. The fact that the parallel alleys fell inside of the distance alleys was an important result, as we will discuss later.

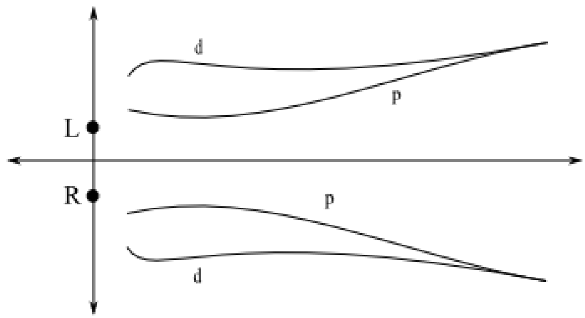

Figure 8: Blumenfeld's experiment results [5]

Mathematicians can use these results to show that our binocular vision allows people to perceive parallel lines as curved. They can also use these findings to show that our binocular vision does not function according to the rules of Euclidean geometry – raising the questions of if there exist forms of geometry that better describe these functions.

However, some researchers claim these findings may not be as accurate as they seem. Robert French claims in his article, "The Geometry of Visual Space," that the alley experiments can be analyzed in terms of size constancy. The size constancy tendency is the tendency for an object in visual space to appear the same size when it is seen at different depths. French claims that the experimenters would have gotten different results with different instructions. If the experimenters change the wording from "actual size" to "projective size" they could get very different results. Actual size refers to the measured distance between two lights, whereas the projected distance is the distance that you perceive between the two lights. If the participant was asked to judge actual size, the results would show no difference in size as depth increases. However, if the participant is asked to judge on projective size, results would show a decrease in distance as depth increases. French also claims that the subjects were told to avoid the appearance of the alleys converging. This could lead to them unconsciously curving the lines away from each other [7].

3.3 Hyperbolic Models

In Zage's article about the alley experiments, he proves that the results of the alley experiments are best modeled by hyperbolic geometry [5]. He uses elliptic, Euclidean, and hyperbolic geometry to model the results of the alley experiments. He eventually proves that results can only be truly modeled by hyperbolic geometry. The parallel alley lying inside the distance alley is an important detail of these results.

Zage begins by assuming that visual space has constant curvature, because this gives us free mobility [5]. On a surface of constant curvature, any two points are connected by a unique arc of minimal length. When this arc is extended in both directions, it becomes a curve of minimal arc length, called a geodesic. In each model, the parallel alleys, p, from the experiments follow these geodesics. This follows from Zage's assumption that the parallel alleys are the shortest lines from the fixed points to the participant's eyes. The distance alleys, d, represent the path to maintain the same distance from each other in each model.

In each model that we examine, the points P and Q are the fixed lights in Blumenfeld's experiments. The line m is the median line, from the middle of the participant's eyes straight out in front of them. The points X and Y are the participants left and right eyes. The line f is the simply the y-axis.

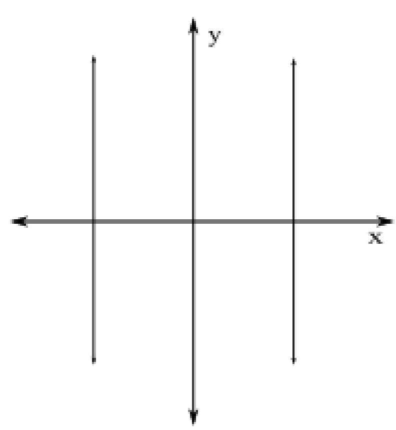

Figure 9: Geodesics in the Euclidean Model

The first model we will examine is the Euclidean model. As shown in Figure 9, the geodesics in the Euclidean model are simply straight lines. In Figure 10, we can see that the distance alleys, d, and the parallel alleys, p, are the same lines. In Euclidean geometry, there is a unique perpendicular line from a point to a given line; thus the parallel lines and the distance lines are the same.

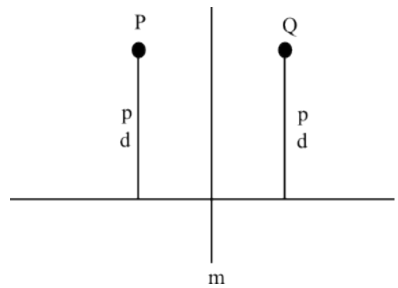

Figure 10: Euclidean Model of the Alley Experiments

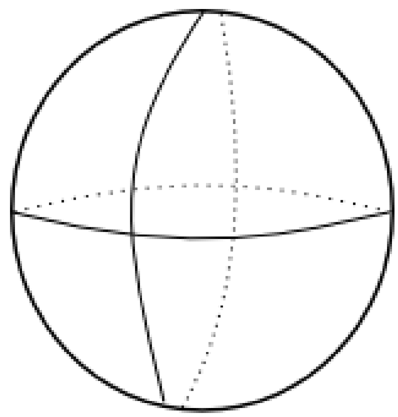

Figure 11: Geodesics in the Elliptic Model

Next we will examine the elliptic model of geometry. This geometry is modeled with a sphere. The geodesics of the sphere are called great circles. A great circle is the intersection of the sphere with a plane containing the center of the sphere. An example of a great circle would be the equator. As shown in Figure 12, the parallel lines follow the geodesics of elliptic geometry. Thus, the parallel lines are going to fall outside of the distance lines.

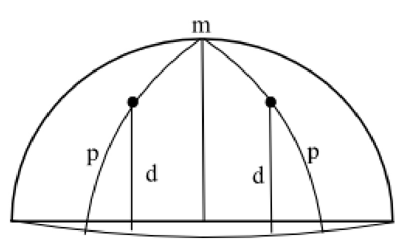

Figure 12: Elliptic Model of the Alley Experiments [5]

Finally, we examine the hyperbolic model of geometry. Hyperbolic geometry is modeled

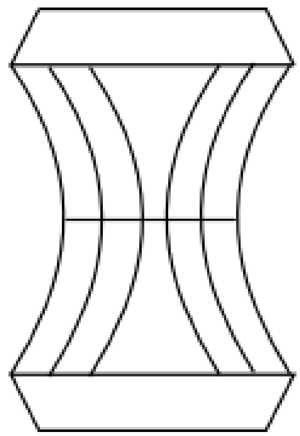

Figure 13: Geodesics in the Hyperbolic Pseudosphere Model

using a *pseudosphere*, as shown in Figure 13. There are two types of geodesics in this figure – the vertical curves and the horizontal line around the "waist".

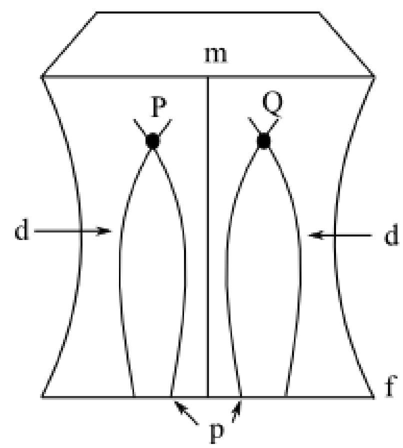

Figure 14: Hyperbolic Model of the Alley Experiments [5]

As illustrated in Figure 14, the parallel lines in the hyperbolic model fall inside the distance lines. This is the only model which is consistent with the results from the alley experiment. Zage concludes that hyperbolic geometry is the only possible model for the experiments. Thus, our visual space, and our perception of parallel lines, can best be described by hyperbolic geometry [5].

Robert Hansen agrees that hyperbolic curves have certain advantages in modeling the visual space. He believes that it models the way the world appears to him and to everyone else. The hyperbola would also permit all straight lines to share the same visual form [3].

Both Zage's and Hansen's arguments depend upon the assumption that visual space is three-dimensional. However, Robert French claims that visual space can be modeled in two-dimensions instead of three. French states that our immediate visual awareness consists of a phenomenal field of colors. If we define visual space in this way, it can be divided into two regions: a region of one homogeneous color, and another surrounding that region of a sharply contrasting color [7].

French states that we need to use the analysis of dimension where we suppose the minimum number of dimensions required of a space in order for that space to "cut" or give boundaries to the space whose dimensionality is being tested. Using this terms of analysis for visual space, we can bound a region of color by a one-dimensional space – the line constituted by the boundary between the given color and its background color [7]. Therefore visual space must be two dimensional.

According to French, in reality we only see the surfaces of objects and not their interiors. Even for transparent objects we only see the surface, unless we focus on a point in the interior. Thus, our visual space is truly two-dimensional instead of three-dimensional as proposed by Hansen and Zage [7].

French also disagrees with Hansen and Zage in claiming that visual space is better modeled by spherical geometry. French states that visual space may approach a Euclidean metric the closer an individual is to the object. The closer an individual is to an object, the more properties of the physical space it holds. He claims that this is consistent with the fact that visual space holds a spherical metric [7].

French uses the example of wide-angle photography. Objects at the edges of the pictures on a flat plane are much bigger than the objects in the center. When these images are imposed onto a spherical plane, there are no distortions. Since we do not experience these distortions in our vision in real life, French claims that visual space has a spherical metric.

A metric is a function to measure distances [2]. The function must be defined so that the distance from a point a to another point b is the same as the distance from b to a. Furthermore, the distance between two points is zero if and only if the points are the same. The Euclidean metric between two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

A spherical metric is the length of the shortest arc of a great circle containing the two points [2]. The greatest distance between two points on a sphere would be between poles. In the case that the two points are not at opposite ends of the sphere, they lie on a unique great circle. The distance is measured by the subarc, of which the two points are the endpoints, of the great circle that they lay on.

French claims this explains why in panoramic pictures, the film is curved at the back of the camera. Furthermore, when movies are shown on a very large screen, the screen is usually curved. Another common example is the distortion of Mercator projection of a globe onto a two-dimensional map. Projecting a Cartesian coordinate system onto the surface of a sphere without extreme distortion is impossible [7].

French doubts that lines are truly being seen as straight. If visual space really does correspond to the surface of the sphere, we are not truly seeing straight lines. We are seeing lines that are as straight as they possibly could be. He claims that straighter lines do not exist in visual space [7].

4 Conclusion

We have seen that our visual space is not purely Euclidean, as was believed prior to the alley experiments. There is a difference between our visual perceptions and the physical world. We explored the works of mathematicians who have investigated these differences in terms of different types of geometries. The applications that come from the alley experiments show that hyperbolic geometry is the best fit to model our binocular visual space. Even though hyperbolic geometry was initially controversial, it is becoming a standard piece of modern mathematics.

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