DESINGULARIZING b^m-SYMPLECTIC STRUCTURES

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ABSTRACT. A Poisson manifold (M^{2n}, Π) is said to be b^m-symplectic if it is symplectic on the complement of a hypersurface Z and has a simple Darboux canonical form at points of Z (which we will describe below). In this paper we will discuss a desingularization procedure which, for m even, converts Π into a family of symplectic forms ω_ϵ having the property that ω_ϵ is equal to the b^m-symplectic form dual to Π outside an ϵ-neighborhood of Z and, in addition, converges to this form as ϵ tends to zero in a sense that will be made precise in the theorem below. We will then use this construction to show that a number of somewhat mysterious properties of b^m-manifolds can be more clearly understood by viewing them as limits of analogous properties of the ω_ϵ's. We will also prove versions of these results for m odd; however, in the odd case the family ω_ϵ has to be replaced by a family of "folded"symplectic forms.

1. Introduction

A b-symplectic manifold is an oriented Poisson manifold (M, Π) which has the property that the map $\Pi^n : M \longrightarrow \Lambda^{2n}(TM)$ intersects the zero section of $\Lambda^{2n}(TM)$ transversally in a codimension one submanifold $Z \subset M$. For such a Poisson manifold the dual to the bivector field Π is a generalized De Rham form of b-type and defines a "b-symplectic"structure on M. These structures and applications of them have been the topic of a number of recent articles (see [GMP],[GMP2], [GL], [MO1], [FMM], [Ca], [MO2], [GMPS], [GLPR] [KMS], [GMPS2]) and generalizations of these structures in which one no longer requires the transversality assumption above have also been considered. In this paper we will be concerned with one such generalization, due to Geoffrey Scott [S] in which the transversality assumption is replaced by the assumption that away from Z, M is symplectic while at Z the Poisson structure has a simple Darboux canonical form. These structures are known as b^m structures (for reasons that will be clear below) and our goal in this paper will be the "desingularization"of these structures: Where m is even, we will construct in a more or less canonical way a family of symplectic forms on M, depending on a parameter ϵ, and having the property that as ϵ tends to zero these forms tend in the limit to the b^m form that is the dual object to the b^m Poisson bivector field Π. Where m is odd, we prove an analogous result, but with the family of symplectic forms replaced by a family of folded symplectic forms. More explicitly we prove (Theorems 4.1 and 6.1):

Theorem. *Given a b^m-symplectic structure ω on a compact manifold M^{2n} let Z be its critical hypersurface.*

- *If m is even, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*

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- *If m is odd, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

Our goal in introducing these families is that in attempting to define b^m analogues of a number of basic invariants of symplectic and folded symplectic manifolds such as symplectic volume and, for Hamiltonian G manifolds, moment polytopes and Duistermaat-Heckman measures, one encounters a number of frustrating "infinities"that are hard to interpret or eliminate. However, we believe (and will verify below in a couple of important cases) that desingularization is an effective tool for getting around this problem.

2. Preliminaries

Let M be a compact manifold and let $Z \subset M$ a hypersurface in M. In [GMP] a b-symplectic form was defined as being a 2-form in the complex of b-forms. In order to define this complex we first considered the b-tangent bundle ${}^bT(M)$ (whose sections are defined as vector fields tangent to the critical hypersurface Z) and its dual ${}^bT^*(M)$. The complex of b-forms was introduced à la De Rham as sections of the bundles $\Lambda^k({}^bT^*(M))$.

In [S] a similar description was obtained for b^m-symplectic forms. Let M be a compact oriented manifold and let $Z \subset M$ be a hypersurface in M, along with a choice of function $x \in C^\infty(M)$ such that 0 is a regular value of x and $x^{-1}(0) = Z$. Given such a triple (M, Z, x), the fibers of the b^m-(co)tangent bundle are given by

$$ {}^{b^m}T_pM \cong \begin{cases} T_pZ + < x^m \frac{\partial}{\partial x} > & \text{if } p \in Z \\ T_pM & \text{if } p \notin Z \end{cases} $$

$$ {}^{b^m}T_p^*M \cong \begin{cases} T_p^*Z + < \frac{dx}{x^m} > & \text{if } p \in Z \\ T_p^*M & \text{if } p \notin Z \end{cases} $$

As in the case of b-manifolds, these fibres combine to form a bundle; a b^m-manifold is a triple (M, Z, x), along with these bundles.[1]

We then define

Definition 2.1. A **symplectic b^m-manifold** is a b^m-manifold (M, Z) with a closed b^m-two form ω which has maximal rank at every $p \in M$.

To describe the properties of such forms we will need the following definitions and propositions (see [S]).

Definition 2.2. A **Laurent Series** of a closed b^m-form ω is a decomposition of ω in a tubular neighbourhood U of Z of the form

$$ \omega = \frac{dx}{x^m} \wedge (\sum_{i=0}^{m-1} \pi^* \alpha_i x^i) + \beta \qquad (1) $$

where $\pi : U \to Z$ is the projection, where each α_i is a closed smooth De Rham form on Z, and β is a De Rham form on M.

Proposition 2.3 (Scott)**.** *In a collar neighbourhood of Z, every closed b^m-form ω can be written in a Laurent form of type (1).*

[1]By abuse of notation, we denote a b^m-manifold by (M, Z), suppressing the function x. Note that Scott [S]'s definition of a b^m-manifold differs from ours by allowing *local* defining functions for Z.

3. Symplectic foliations and normal forms for b^m-symplectic manifolds

We begin by studying the symplectic foliation of the Poisson structure induced by a b^m-symplectic form on the critical hypersurface Z.

Proposition 3.1. *Given a symplectic b^m-structure with b^m symplectic form ω, the closed one-form α_0 in the Laurent decomposition*

$$\omega = \frac{dx}{x^m} \wedge (\sum_{i=0}^{m-1} \pi^*(\alpha_i)x^i) + \beta$$

defines the codimension-one symplectic foliation $\mathcal{F}$ of the regular Poisson structure induced by the dual b^m-Poisson structure on the critical hypersurface Z. In addition one can find a Poisson vector field v on Z transverse to this foliation.

See [GMP] and [GMP2] for the proof of this in the $m = 1$ case. (For $m > 1$ the proof is essentially the same).

Since $i_L^*(d\alpha_0) = d(i_L^*\alpha_0) = 0$ for all leaves $L \in \mathcal{F}$, we have

$$d\alpha_0 = \beta \wedge \alpha_0 \text{ for some } \beta \in \Omega^1(Z). \tag{2}$$

As a consequence, the complex $\alpha_0 \wedge \Omega(Z)$ is a sub-complex of $\Omega(Z)$ and we have the following short exact sequence of complexes

$$0 \longrightarrow \alpha_0 \wedge \Omega(Z) \longrightarrow \Omega(Z) \stackrel{j}{\longrightarrow} \Omega(Z)/(\alpha_0 \wedge \Omega(Z)) \longrightarrow 0.$$

Thus, even though the form β is not unique for a fixed choice of α_0, the projection $j\beta$ is unique and $d(j\beta) = 0$. Thus the **first obstruction class** $c_{\mathcal{F}} \in H^1(\Omega(Z)/(\alpha_0 \wedge \Omega(Z)))$ is defined to be $c_{\mathcal{F}} = [j\beta]$.

The following is theorem 4 in [GMP]:

Theorem 3.2. *The first obstruction class $c_{\mathcal{F}}$ vanishes identically if and only if we can choose the defining one-form α_0 of the foliation $\mathcal{F}$ to be closed.*

Hence by Theorem 13 in [GMP] one gets

Theorem 3.3. *If $\mathcal{F}$ contains a compact leaf L, then every leaf of $\mathcal{F}$ is diffeomorphic to L. Furthermore, Z is the total space of a fibration $f : M \to \mathbb{S}^1$ with fiber L, and $\mathcal{F}$ is the fiber foliation $\{f^{-1}(\theta) | \theta \in \mathbb{S}^1\}$.*

In addition Corollary 14 in [GMP] implies:

Corollary 3.4. *If $c_{\mathcal{F}} = 0$, and if the foliation contains a compact leaf L, then, the manifold Z is the mapping torus of the map $\phi : L \to L$ given by the holonomy map of the fibration over $\mathbb{S}^1$, $\frac{L\times[0,1]}{(x,0)\sim(\phi(x),1)}$.*

(Recall that ϕ is the first return map of $\exp tv$, where v is the unique vector field v satisfying the equations

$$\begin{cases} \iota_v\alpha_0 &= 1 \\ \iota_v\omega &= 0 \end{cases}$$

where α_0 is the defining one-form for the foliation $\mathcal{F}$ and ω a closed 2-form on Z that restricts to the symplectic form on every leaf of $\mathcal{F}$.)

Remark 3.5. In the papers [GMP] and [GMP2] these results are, strictly speaking, only proved for $m = 1$, but for m arbitrary the proofs are identical.

3.1. **b^m-versions of the Moser and Darboux theorems.** For $m = 1$ the statement and proof of these results can be found in [GMP2]. The proof in [GMP2] is based on the Moser path method for b-symplectic structures; however, the Moser path method also works for b^m-symplectic structures (see [S]), so the results apply to b^m-symplectic manifolds as well. The first of these theorems asserts

Theorem 3.6. *If ω_0, ω_1 are symplectic b^m-forms on (M^{2n}, Z) with Z compact and $\omega_0|_Z = \omega_1|_Z$, then there are neighbourhoods U_0, U_1 of Z and a b^m-symplectomorphism $\varphi : (U_0, Z, \omega_0) \to (U_1, Z, \omega_1)$ such that $\varphi|_Z = Id$.*

(For the proof see [GMP2], Theorem 6.5). Consider now the decomposition of a b^m form

$$\omega = \alpha \wedge \frac{dx}{x^m} + \beta, \text{ with } \alpha \in \Omega^1(M) \text{ and } \beta \in \Omega^2(M). \tag{3}$$

To prove the b^m-version of the Darboux theorem we will need

Proposition 3.7. *(See Proposition 10 in* [GMP]*) Let $\tilde{\alpha} = i^*\alpha$ and $\tilde{\beta} = i^*\beta$, where $i : Z \hookrightarrow M$ denotes the inclusion. Then the forms $\tilde{\alpha}$ and $\tilde{\beta}$ are closed. Furthermore,*

1. *The form $\tilde{\alpha}$ is nowhere vanishing and intrinsically defined in the sense that it does not depend on the splitting (3). In particular, the codimension-one foliation of Z defined by $\tilde{\alpha}$ is intrinsically defined.*
2. *For each leaf $L \overset{i_L}{\hookrightarrow} Z$ of this foliation, the form $i_L^*\tilde{\beta}$ is intrinsically defined, and is a symplectic form on L.*
3. *In (3) we can assume without loss of generality that:*
   - *The forms α and β are closed.*
   - *The form $\alpha \wedge \beta^{n-1} \wedge df$ is nowhere vanishing.*
   - *And, in particular, the form $i^*(\alpha \wedge \beta^{n-1})$ is nowhere vanishing.*

We will now show

Theorem 3.8 (b^m-Darboux theorem)**.** *Let ω be a b^m-symplectic form on (M, Z) and $p \in Z$. Then we can find a coordinate chart $(U, x_1, y_1, \ldots, x_n, y_n)$ centered at p such that on U the hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = dx_1 \wedge \frac{dy_1}{y_1^m} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Proof. Write $\omega = \alpha \wedge \frac{dx}{x^m} + \beta$, and $\tilde{\alpha} = i^*\alpha$ and $\tilde{\beta} = i^*\beta$, with $i : Z \hookrightarrow M$ the inclusion. From Proposition 3.7, for all $p \in Z$, we have $\tilde{\alpha}_p$ non-vanishing. Thus $\tilde{\alpha}_p \wedge \tilde{\beta}_p \neq 0$ and $\tilde{\beta}_p \in \Lambda^2(T_p^*Z)$ has rank $n - 1$. Thus we can assume

$$\omega|_Z = (dx_1 \wedge \frac{dy_1}{y_1^m} + \sum_{i=2}^{n} dx_i \wedge dy_i)|_Z.$$

and the assertion above follows from Theorem 3.6. □

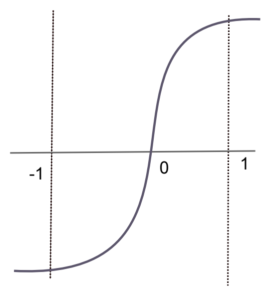
4. Desingularizing b^{2k}-symplectic structures

Consider a manifold M equipped with a b^{2k}-symplectic structure given by a b^{2k}-symplectic form ω. In view of the Laurent decomposition given in Proposition 2.3, we have in a tubular neighbourhood U of Z

$$\omega = \frac{dx}{x^{2k}} \wedge (\sum_{i=0}^{2k-1} \alpha_i x^i) + \beta \tag{4}$$

where $\alpha_i = \pi^*(\widehat{\alpha_i})$ with $\widehat{\alpha_i}$ closed one-forms on Z and $\pi : U \to Z$ denoting the projection.

Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an odd smooth function satisfying $f'(x) > 0$ for all $x \in [-1, 1]$ as shown below,



and satisfying

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1 \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1 \end{cases}$$

outside the interval $[-1, 1]$.

Now we scale the function f to construct a new function

$$f_\epsilon(x) := \frac{1}{\epsilon^{2k-1}} f\left(\frac{x}{\epsilon}\right). \tag{5}$$

Thus outside the interval $[-\epsilon, \epsilon]$,

$$f_\epsilon(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x < -\epsilon \\ \frac{-1}{(2k-1)x^{2k-1}} + \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x > \epsilon \end{cases}$$

We replace $\frac{dx}{x^{2k}}$ by df_ϵ in the expansion (4) an ϵ-neighborhood and obtain a differential form

$$\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} \alpha_i x^i) + \beta$$

Since ω_ϵ agrees with ω outside an ϵ neighborhood of Z, it extends to a differential form on all of M. We denote this extension (by abuse of notation) by ω_ϵ, the f_ϵ-**desingularization**[2] of the b^{2k}-symplectic structure ω.

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[2]Or **deblogging**

Theorem 4.1. *The f_ϵ-**desingularization** ω_ϵ is symplectic. The family ω_ϵ coincides with the b^{2k}-symplectic form ω outside an ϵ-neighbourhood. The family of bivector fields ω_ϵ^{-1} converges to the Poisson structure ω^{-1} in the C^{2k-1}-topology as $\epsilon \to 0$.*

As a consequence of this theorem we obtain,

Theorem 4.2. *A manifold admitting a b^{2k}-symplectic structure also admits a symplectic structure.*

In particular the topological constraints that apply for symplectic structures also apply for b^{2k}-symplectic structures.

This point of view in the study of b^m-symplectic forms yields several consequences. In this paper we concentrate on a couple of them concerning volume forms and Hamiltonian actions.

We now prove Theorem 4.1.

Proof. Clearly for all ϵ, the form $\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} \alpha_i x^i) + \beta$ is closed since all the one forms α_i are closed.

Let us check that it is symplectic. Outside U, ω_ϵ coincides with ω. In U but away from Z,

$$\omega_\epsilon^n = \frac{df_\epsilon}{dx} x^{-2k} \omega^n$$

which is nowhere vanishing. To check that ω_ϵ is symplectic at Z, observe that

$$\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} x^i \alpha_i) + \beta = \epsilon^{-2k} \frac{df}{dx}\left(\frac{x}{\epsilon}\right) dx \wedge (\sum_{i=0}^{2k-1} x^i \alpha_i) + \beta$$

which on the interval $|x| < \epsilon$ is equal to

$$\epsilon^{-2k} (\frac{df}{dx}\left(\frac{x}{\epsilon}\right) dx \wedge \alpha_0) + \beta + \mathcal{O}(\epsilon)$$

and hence

$${\omega_\epsilon}^n = \epsilon^{-2k} (\frac{df}{dx}\left(\frac{x}{\epsilon}\right) dx \wedge \alpha_0 \wedge \beta^{n-1} + \mathcal{O}(\epsilon))$$

which is non-vanishing for ϵ sufficiently small because of Proposition 3.7 applied to the original b^{2k}-symplectic form and the definition of f. This proves that ω_ϵ is symplectic.

Let us now prove that the family of bivector fields ω_ϵ^{-1} converges to ω^{-1} when $\epsilon \to 0$ in the C^{2k-1}-topology.

Consider the form ω and the family ω_ϵ. Then in b^{2k}-Darboux coordinates (Theorem 3.8),

$$\omega_\epsilon = \epsilon^{-2k} f'\left(\frac{x}{\epsilon}\right) dx \wedge dy + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n$$

and

$$\omega = \frac{1}{x^{2k}} dx \wedge dy + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n$$

We wish to verify that the family ω_ϵ^{-1} of bivector fields given by

$$\omega_\epsilon^{-1} = \epsilon^{2k} g\left(\frac{x}{\epsilon}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} + \cdots + \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n} \tag{6}$$

where $g(x) = \frac{1}{f'(x)}$, converges to

$$\omega^{-1} = x^{2k} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} + \cdots + \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n} \tag{7}$$

as ϵ tends to zero.

Let $h(x) = (\frac{d}{dx})^{2k-1}g(x)$. Then ω_ϵ^{-1} converges to ω^{-1} in the C^{2k-1} topology if $\epsilon h\big(\frac{x}{\epsilon}\big)$ converges in the uniform norm to $2kx$. But $x^{2k} = \epsilon^{2k} g\big(\frac{x}{\epsilon}\big)$ for $|x| > \epsilon$, so for $\epsilon < |x|$, $\epsilon h\big(\frac{x}{\epsilon}\big)$ is equal to $2kx$, and for $\epsilon > |x|$ both functions are bounded by a constant multiple of ϵ.

Hence $\epsilon h\big(\frac{x}{\epsilon}\big)$ converges in the uniform norm to $2kx$ when $\epsilon \to 0$ and this gives the C^{2k-1}-convergence of (6) to (7), thus finishing the proof of the theorem.

□

5. Desingularization and volume formulae

5.1. **Volume formulae for b^{2k}-symplectic manifolds.** We recall from section 5.1 in [S] the following construction which relates the volume with the Laurent decomposition of a b^m-symplectic structure.

On a tubular neighbourhood $\mathcal{U} = Z \times (-1, 1)$, $\omega = \frac{dx}{x^{2k}} \wedge (\sum_{i=0}^{2k-1} x^i \alpha_i) + \beta$. Hence for $\mathcal{U}_\epsilon = Z \times (-\epsilon, \epsilon)$, the symplectic volume of $M \setminus \mathcal{U}_\epsilon$ is, up to a bounded error term, given by

$$\sum_{i=0}^{2k-1} \int_{\mathcal{U}-\mathcal{U}_\epsilon} \frac{dx}{x^{2k-i}} \wedge \alpha_i \wedge \beta^{n-1} \tag{8}$$

Furthermore,

$$\beta = dx \wedge \gamma + \sum_{j=0}^{2k-1} x^i \pi^*(\beta_j) + \mathcal{O}(x^{2k}) \tag{9}$$

where β_j are 2-forms on Z. Plugging equation (9) into equation (8) we get,

$$2 \sum_{i=0}^{2k-1} \int_{I_\epsilon} \frac{dx}{x^{2k-i}} (\int_Z \alpha_i \wedge (\sum_{j=0}^{2k-1} x^i \beta_j)^{n-1}) + \mathcal{O}(1) \tag{10}$$

where $I_\epsilon = (-1, -\epsilon) \cup (\epsilon, 1)$. Thus,

$$\int_{M \setminus \mathcal{U}_\epsilon} \omega^n = \sum_{i=1}^{k} c_i \epsilon^{-2i-1} + \mathcal{O}(1) \tag{11}$$

where the c_i are linear combinations of the integrals

$$\int_Z \alpha_{j_1} \wedge \beta_{j_2} \wedge \cdots \wedge \beta_{j_n}$$

5.2. **The desingularized version of this result.** Let us compute the symplectic volume of M with respect to the symplectic form

$$\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} x^i \alpha_i) + \beta \tag{12}$$

Outside the tubular neighbourhood, $\mathcal{U}_\epsilon$, ω_ϵ coincides with ω, so we get, for the integral of ${\omega_\epsilon}^n$ over the complement of this tube neighbourhood, the result described above. What about the integral on the tube neighbourhood?

Recall that

$$f_\epsilon(x) = \epsilon^{-(2k-1)} f\left(\frac{x}{\epsilon}\right)$$

where f is the function defined in (5) and thus the integral of ${\omega_\epsilon}^n$ over $\mathcal{U}_\epsilon$ is given by

$$\int_{\mathcal{U}_\epsilon} df_\epsilon \wedge (\sum_{i=0}^{2k-1} x^i \alpha_i) \wedge \beta^{n-1}, \tag{13}$$

which by equation (9) can be rewritten as

$$\sum_{i=1}^{2k-1} b_i \int_{-\epsilon}^{\epsilon} \frac{df_\epsilon}{dx} x^i dx \tag{14}$$

plus a bounded error term where the coefficients b_i like the c_i are linear combinations of the integrals

$$\int_Z \alpha_{j_1} \wedge \beta_{j_2} \wedge \cdots \wedge \beta_{j_n}.$$

To evaluate the integrals

$$\int_{-\epsilon}^{\epsilon} \frac{df_\epsilon}{dx} x^i dx$$

we make the change of coordinates $x = \epsilon y$ which converts the integral above into

$$\epsilon^{-(2k-1)+i} \int_{-1}^{1} \frac{df}{dy}(y) y^i dx. \tag{15}$$

Therefore since $f(y) = -f(y)$ this integral is zero for i odd and equal to a positive constant multiple of $\epsilon^{-(2k-1)+i}$ for i even. Thus,

$$\int_{\mathcal{U}_\epsilon} \omega_\epsilon^n = \sum_{i=1}^{k} a_i \epsilon^{-(2i-1)} \tag{16}$$

where the a_i's like the b_i's and c_i's are linear combinations of integrals of type

$$\int_Z \alpha_{j_1} \wedge \beta_{j_2} \wedge \cdots \wedge \beta_{j_n}.$$

Finally combining equations (16) and equation (11) this proves

Theorem 5.1. *The volume determined by the desingularized symplectic form ω_ϵ is given by a formula of type*

$$\int_M {\omega_\epsilon}^n = \sum_{i=1}^{k} (a_i + c_i) \epsilon^{-(2n-1)} + \mathcal{O}(1)$$

where the coefficients a_i's and c_i's are linear combinations of integrals of type $\int_Z \alpha_{j_1} \wedge \beta_{j_2} \wedge \cdots \wedge \beta_{j_n}$.

5.3. **Leading terms.** The leading term in the asymptotic expansion given by formula (11) is

$$\frac{2}{2k-1}\epsilon^{-(2k-1)}\int_Z \alpha_0\wedge\beta^{n-1} \tag{17}$$

and the leading term in the asymptotic expansion inside $\mathcal{U}_\epsilon$ is

$$4\epsilon^{-(2k-1)}\int_Z \alpha_0\wedge\beta^{n-1} \tag{18}$$

so adding equations (17) and (18) we obtain the following asymptotic result for the symplectic volume of M with respect to ω_ϵ :

Theorem 5.2 (Asymptotics for the symplectic volume)**.**

$$\int {\omega_\epsilon}^n \sim 2(2+\frac{1}{2k-1})\epsilon^{-(2k-1)}\int_Z \alpha_0\wedge\beta^{n-1}$$

6. Desingularizing b^{2k+1}-symplectic structures

Let M be a b^{2k+1}-symplectic manifold. In view of the Laurent decomposition given in Proposition 2.3 in an ϵ-neighbourhood of Z the b-symplectic form has the decomposition, in local coordinates

$$\omega = \frac{dx}{x^{2k+1}}\wedge(\sum_{i=0}^{2k}\pi^*(\alpha_i)x^i)+\beta$$

Let $f\in\mathcal{C}^\infty(\mathbb{R})$ satisfy

- $f>0$.
- $f(x)=f(-x)$.
- $f'(x)>0$ if $x<0$.
- $f(x)=-x^2+2$ if $x\in[-1,1]$.
- $f(x)=\log(|x|)$ if $k=0$, $x\in\mathbb{R}\setminus[-2,2]$.
- $f(x)=\frac{-1}{(2k+2)x^{2k+2}}$ if $k>0$, $x\in\mathbb{R}\setminus[-2,2]$.

Now define

$$f_\epsilon(x) := \frac{1}{\epsilon^{2k}}f\left(\frac{x}{\epsilon}\right) \tag{19}$$

and, as in the even case, let

$$\omega_\epsilon = df_\epsilon\wedge(\sum_{i=0}^{2k}\pi^*(\alpha_i)x^i)+\beta \tag{20}$$

We can prove the following

Theorem 6.1. *The 2-form ω_ϵ is a folded symplectic form which coincides with ω outside an ϵ-neighbourhood of Z.*

Proof. By definition of the function f_ϵ, ω_ϵ coincides with ω outside an ϵ-neighbourhood of the critical hypersurface Z. As in the proof of Theorem 4.1, it is easy to see that ω_ϵ is symplectic away from Z. In order to check that ω_ϵ is a folded symplectic structure, we need to check that ${\omega_\epsilon}^n$ is transverse to the zero section of the bundle $\Lambda^n(T^*M)$.

Let us denote by $\alpha_i = \pi^*(\widehat{\alpha_i})$ with $\widehat{\alpha_i}$ one-forms on Z. Since

$$\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k} x^i \alpha_i) + \beta = \epsilon^{-(2k+1)} \frac{df}{dx}\left(\frac{x}{\epsilon}\right) dx \wedge (\sum_{i=0}^{2k} x^i \alpha_i) + \beta,$$

on the interval $|x| < \epsilon$ we have

$$\omega_\epsilon = \epsilon^{-(2k+1)}(\frac{df}{dx}\left(\frac{x}{\epsilon}\right) dx \wedge \alpha_0 + \mathcal{O}(\epsilon)) + \beta.$$

Thus,

$${\omega_\epsilon}^n = \epsilon^{-(2k+1)}(\frac{df}{dx}\left(\frac{x}{\epsilon}\right) dx \wedge \alpha_0 \wedge \beta^{n-1} + \mathcal{O}(\epsilon)).$$

By Proposition 3.7 the form $\alpha_0 \wedge \beta^{n-1}$ is nondegenerate. Since df vanishes only at zero, $\frac{df}{dx} dx \wedge \alpha_0 \wedge \beta^{n-1}$ does not vanish away from Z. Hence for ϵ sufficiently small, neither does ω_ϵ^n.

From the construction, $\frac{df}{dx}$ vanishes linearly at $x = 0$; so ${\omega_\epsilon}^n$ intersects the zero section of $\Lambda^n(T^*M)$ transversely. Thus ω_ϵ is a folded symplectic structure.

□

As a consequence of this fact we obtain the following theorem which generalizes some of the results contained in Section 3 in [FMM] for b-symplectic manifolds:

Theorem 6.2. *A manifold admitting a b^{2k+1}-symplectic structure also admits a folded symplectic structure.*

7. Group actions and desingularization

We conclude by briefly mentioning some applications of desingularization which we propose to explore in detail in a sequel to this paper.

In the papers [GMPS] and [GMPS2] it was shown that two classical theorems in equivariant symplectic geometry, the Delzant theorem and the Atiyah-Guillemin-Sternberg convexity theorem, have analogs for b-symplectic manifolds. We will show that these theorems also have analogs for for b^k-manifolds (except for the assertion in Delzant's theorem that "the moment image of M determines M up to symplectomorphism").

In addition we will use the desingularization procedure to prove b^k-versions of the Kirwan convexity theorem and of the Duistermaat-Heckman theorem (concerning the latter the main ingredient in our proof will be the observation that in the vicinity of the critical hypersurface Z, the desingularized Duistermaat-Heckman measure can be easily computed and its behavior as ϵ tends to zero easily described using Theorems 4.1 and 6.1.)

Finally we note that the complexities that are required to keep track of the "infinities" occurring in the b^k-versions of the theorems above can largely be avoided by viewing these infinities as coming from the desingularization process as ϵ tends to zero.

7.1. A convexity result for b^m-symplectic manifolds.

In what follows we will assume for simplicity that the hypersurface Z is connected; this assumption can readily be removed. As we did in [GMPS] for b-symplectic manifolds, we can define Hamiltonian actions in the b^m-setting.

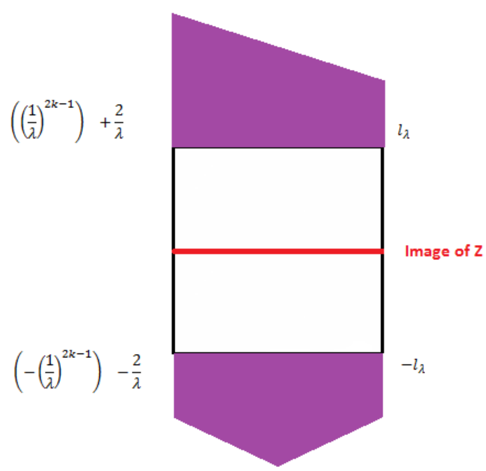
Definition 7.1. An action of a torus $G = \mathbb{T}^n$ on the b^m-symplectic manifold (M, ω) is called **Hamiltonian** if it preserves ω and $\iota_{X^\#}\omega$ is b^m-exact for any $X \in \mathfrak{g}$.

Given such a Hamiltonian action on a b^m-manifold M, this action is also Hamiltonian with respect to the desingularized forms. Hence if m is even, we obtain a family of symplectic forms and a family of Hamiltonian actions on the pairs (M, ω_ϵ). Observe in this case the desingularized forms are symplectic and we can invoke the Atiyah-Guillemin-Sternberg convexity theorem for the moment map ([At], [GS]).

Let us denote by F_ϵ the associated family of moment maps. Then the image of these moment maps are convex polytopes. To describe those polytopes, there are two cases to consider[3]:

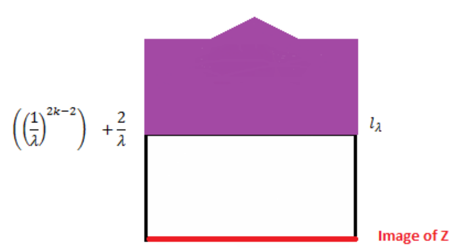
Case 1. *The image of the moment map coincides with the image of the moment map induced in the symplectic foliation in the critical set* [4]. In this case the moment polytope for M coincides with the image of the moment map on one of the symplectic leaves in Z.

Case 2. *The function f_ϵ is one of the components of the moment map*[5]. These polytopes are as depicted in the picture below:



In this figure the region above l_λ and the region below $-l_\lambda$ are independent of ϵ for $\epsilon < \lambda$.

Finally for b^{2k+1}-symplectic manifolds the desingularization gives us folded symplectic manifolds and for these the moment polytopes are the folded versions of the polytopes above, as depicted in the figure below:



[3]By analogs of the results in [GMPS], these are the only two cases that occur, even when the number of connected components of Z is greater than 1.

[4]In this case all the connected components for the initial action have zero modular weight. Cf. [GMPS].

[5]In this case, all the connected components for the initial action have non-zero modular weight. Cf. [GMPS].

We will give a more detailed and rigorous account of these results in a future paper. Similarly we will prove analogues of the Duistermaat-Heckman theorems and the Delzant theorem using these methods.

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