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European Journal of Combinatorics 29 (2008) 712–725

European Journal of Combinatorics

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On two coloring problems in mixed graphs

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Received 22 November 2006; accepted 5 March 2007
Available online 1 April 2007

Abstract

We are interested in coloring the vertices of a mixed graph, i.e., a graph containing edges and arcs. We consider two different coloring problems: in the first one, we want adjacent vertices to have different colors and the tail of an arc to get a color strictly less than a color of the head of this arc; in the second problem, we also allow vertices linked by an arc to have the same color. For both cases, we present bounds on the mixed chromatic number and we give some complexity results which strengthen earlier results given in [B. Ries, Coloring some classes of mixed graphs, Discrete Applied Mathematics 155 (2007) 1–6].

1. Introduction

A mixed graph $G_M = (V, U, E)$ on vertex set $V = \{v_1, v_2, \dots, v_n\}$ is a graph containing arcs (set U) and edges (set E). We denote by $[v_i, v_j]$ an edge joining vertices v_i and v_j, and by (v_l, v_q) an arc oriented from v_l to v_q. Here, we consider only connected finite mixed graphs containing no multiple edges, no multiple arcs, and no loops. The number of vertices in a mixed graph $G_M = (V, U, E)$ will be denoted by $|V| = n$. Mixed graph coloring has been introduced for the first time in [15].

In this paper, we are interested in two coloring problems in mixed graphs. The first problem is called *strong mixed graph coloring problem*. A *strong mixed p-coloring* of a mixed graph G_M is a mapping $c : V \to \{0, 1, \dots, p-1\}$ such that, for each edge $[v_i, v_j] \in E$, $c(v_i) \neq c(v_j)$ and for each arc $(v_l, v_q) \in U$, $c(v_l) < c(v_q)$. Notice that such a coloring can exist if and only if the mixed graph G_M does not contain any directed circuit. We denote by $\gamma_M(G_M)$ the *strong mixed chromatic number* of G_M, that is the smallest integer p such that G_M admits a strong

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doi:10.1016/j.ejc.2007.03.006

mixed p-coloring. A mixed coloring of G_M with $\gamma_m(G_M)$ colors will be called optimal. We will generally consider the following problem: Given a mixed graph $G_M = (V, U, E)$ and a positive integer p, find out whether G_M admits a strong mixed p-coloring. This coloring problem has been studied in [4,6,12–14]. In [6], some upper bounds on the strong mixed chromatic number are given. An $O(n^2)$ time algorithm to color optimally mixed trees and a branch-and-bound algorithm are also developed. In [4], a linear time algorithm for mixed trees is given as well as an $O(n^{3.376} \log(n))$ time algorithm for series parallel graphs. In [12] it is proven that the strong mixed graph coloring problem is *NP*-complete even if the graph is planar bipartite or bipartite of maximum degree 3. Also some polynomially solvable cases are considered. Finally in [13,14], the unit-time job-shop problem is considered via strong mixed graph coloring. In that case, the partial graph $(V, \emptyset, E)$ is a disjoint union of cliques, and the graph $(V, U, \emptyset)$ is a disjoint union of directed paths. In [14], three branch-and-bound algorithms are presented and tested on randomly generated mixed graphs of order $n \leq 200$ for an exact solution, and of order $n \leq 900$ for an approximate solution. Also, some complexity results are given concerning this special class of mixed graphs.

The second problem which we consider in this paper is called the *weak mixed graph coloring problem*, which was introduced for the first time in [15]. A *weak mixed p-coloring* of a mixed graph G_M is a mapping $c : V \to \{0, 1, \ldots, p-1\}$ such that, for each edge $[v_i, v_j] \in E$, $c(v_i) \neq c(v_j)$ and for each arc $(v_l, v_q) \in U$, $c(v_l) \leq c(v_q)$. Notice that in such a coloring of a mixed graph, all vertices on a directed circuit must have the same color. We denote by $\chi_M(G_M)$ the *weak mixed chromatic number* of G_M, that is the smallest integer p such that G_M admits a weak mixed p-coloring. Given a mixed graph $G_M = (V, U, E)$ and a positive integer p, we are interested in finding out whether G_M admits a weak mixed p-coloring. The weak mixed graph coloring problem has been studied in [1,9,10,15,16]. In [16], some algorithms calculating the exact value of the weak mixed chromatic number of graphs of order $n \leq 40$, and upper bounds for graphs of order larger than 40 are presented.

This paper is organized as follows. In Section 2, we give some definitions and notations which will be used later. Section 3 deals with the strong mixed graph coloring problem. Some bounds on the strong mixed chromatic number are given, as well as some complexity results concerning special classes of graphs. In Section 4, the weak mixed graph coloring problem is considered, and bounds on the weak mixed chromatic number are given with some complexity results.

2. Preliminaries

For all graph theoretical terms not defined here, the reader is referred to [2].

Let $G_M = (V, U, E)$ be a mixed graph, and let V_o be the set of vertices which are incident to at least one arc in G_M. We denote by $G(V_o)$ the mixed subgraph of G_M induced by V_o, and by $G_M^o = (V_o, U, \emptyset)$ the directed partial graph of $G(V_o)$. $n(G_M^o)$ denotes the number of vertices on a longest directed path in G_M^o. Notice that the length of a longest directed path in G_M (i.e. the number of edges of a longest directed path in G_M) is equal to $n(G_M^o) - 1$.

Let P be a directed path in G_M^o. The number of vertices in P will be denoted by $|P|$.

Let v_i be a vertex in G_M. The *inrank* of v_i, denoted by $in(v_i)$, is the length of a longest directed path in G_M^o ending at vertex v_i. Similarly, we define the *outrank* of v_i, denoted by $out(v_i)$, as being the length of a longest directed path in G_M^o starting at vertex v_i. If v_i is not incident to any arc, then $in(v_i) = out(v_i) = 0$. Notice that the length of a longest directed path in G_M is given by $\max_{v_i \in V}(in(v_i) + out(v_i))$.

Notice that the parameters introduced above can only be defined if G_M^o has no directed circuit.

The *degree* of a vertex v in G_M, denoted by $d_{G_M}(v)$, is the number of edges and arcs incident to v. We shall simply write $d(v)$ if no confusion can occur.

3. Strong mixed graph coloring problem

In this section, we study the following problem, which we will call the *Strong Mixed Graph Coloring Problem*:
Instance: A mixed graph $G_M = (V, U, E)$, $E \neq \emptyset$, and an integer $p \geq n(G_M^o)$.
Question: Can the vertices of G_M be colored using at most p colors such that, for each edge $[v_i, v_j] \in E$, $c(v_i) \neq c(v_j)$ and for each arc $(v_l, v_q) \in U$, $c(v_l) < c(v_q)$?

We will refer to this problem as $S(G_M, p)$. Notice that in this problem, we can suppose w.l.o.g. that whenever $(v_l, v_q) \in U$, then $[v_l, v_q] \notin E$, since $(v_l, v_q) \in U$ implies that $c(v_l) < c(v_q)$ and thus $c(v_l) \neq c(v_q)$.

A necessary and sufficient condition for a mixed graph to admit a strong mixed coloring is that it does not contain any directed circuit. We will suppose for the rest of this section that it is satisfied.

3.1. Bounds on the strong mixed chromatic number

Upper bounds on the mixed chromatic number have been given in [6]. In particular, one of these bounds implies that for mixed bipartite graphs, we have $n(G_M^o) \leq \gamma_M(G_M) \leq n(G_M^o) + 1$. In this section, we will give some upper bounds for special classes of mixed graphs, and in some cases the exact value of the strong mixed chromatic number.

Lemma 1. *Let $G_M^o = (V_1 \cup V_2, U, \emptyset)$ be a mixed bipartite graph. Assume that all paths of length $n(G_M^o) - 1$ start in the same vertex set, say V_1. Then it is possible to find a strong mixed $n(G_M^o)$-coloring such that all vertices in V_1 have an even color, and all vertices in V_2 have an odd color.*

Proof. Since G_M^o has no circuit, we may decompose its set of vertices into subsets $C_0, C_1, \ldots, C_{n(G_M^o)-1}$, where C_i is the class of vertices having no predecessors when vertices in $C_0, C_1, \ldots, C_{i-1}$ have been removed.

So we start with the vertices in C_0, and give each vertex v color 0 if it is in V_1 or color 1 if it is in V_2 and we continue with the vertices in $C_1, C_2, \ldots$, by giving each vertex the smallest color which is larger than the color of all its predecessors.

This will give an odd color to vertices in V_2 and an even color to vertices in V_1 (since G_M^o is bipartite a vertex in V_1 (resp. V_2) has all its predecessors in V_2 (resp. V_1)). Clearly, we will have $c(v) < c(w)$ for each arc (v, w). Furthermore not more than $n(G_M^o)$ colors will be used (the longest paths starting in V_2 will have length less than $n(G_M^o)$, and therefore contain colors in $\{1, 2, \ldots, n(G_M^o) - 1\}$). □

Now using this Lemma, we obtain the following result.

Theorem 2. *Let $G_M = (V_1 \cup V_2, U, E)$ be a mixed bipartite graph. Assume that all paths of length $n(G_M^o) - 1$ start in the same vertex set, say V_1. Then, it is possible to find a strong mixed $n(G_M^o)$-coloring such that all vertices in V_1 have an even color, and all vertices in V_2 have an odd color.*

Proof. From Lemma 1 we know that the vertices of G_M^o can be colored using at most $n(G_M^o)$ colors, and such that all vertices in V_1 have an even color and all vertices in V_2 have an odd color. Notice that whenever there is an edge between two colored vertices v, w, we necessarily have that $c(v) \neq c(w)$, since if one color is even, then the second one is odd. By coloring the remaining uncolored vertices of V_1 with color 0 and the remaining uncolored vertices of V_2 with color 1, we obtain a strong mixed $n(G_M^o)$-coloring such that all vertices in V_1 have an even color and all vertices in V_2 have an odd color. □

Theorem 3. *Let $G_M = (V_1 \cup V_2, U, E)$ be a complete mixed bipartite graph. Then $\gamma_M(G_M) = n(G_M^o)$ if and only if all paths of length $n(G_M^o) - 1$ start in the same vertex set V_i, $i \in \{1, 2\}$.*

Proof. From Theorem 2, we know that if these paths start in the same vertex set, then $\gamma_M(G_M) = n(G_M^o)$. Now suppose that the strong mixed chromatic number is equal to $n(G_M^o)$. Assume there are two paths of length $n(G_M^o) - 1$ having their start-vertices not in the same vertex set V_i, $i \in \{1, 2\}$; these vertices are necessarily linked by an edge, since the graph is complete. But in this case, a proper strong mixed $n(G_M^o)$-coloring would clearly not be possible. So we conclude that all paths of length $n(G_M^o) - 1$ start in the same vertex set V_i, $i \in \{1, 2\}$. □

Theorem 4. *Let $G_M = (V, U, E)$ be a mixed graph such that $G(V_o)$ has strong mixed chromatic number $\gamma_M(G(V_o)) \leq n(G_M^o) + 1$. Suppose that we have $\max_{G' \subseteq G_M}(\min_{v \in G'}(d_{G'}(v))) \leq n(G_M^o)$, where G' is a subgraph of G_M containing V_o. Then $\gamma_M(G_M) \leq n(G_M^o) + 1$.*

Proof. Consider $G(V_o)$; it can be colored with at most $n(G_M^o) + 1$ colors. Now assume that the above condition holds. We can remove the vertices of set $V - V_o$ by taking, at each step, a vertex with minimum degree in the remaining graph (this is the Smallest Last Ordering of [11]); all these degrees will be at most $n(G_M^o)$, as we will now show. So, when reinserting the vertices in the opposite order, it will be possible to color the graph with at most $n(G_M^o) + 1$ colors (for each vertex there will be a color available among the $n(G_M^o) + 1$ colors).

Let us call $v_1, v_2, \ldots, v_q$ the vertices of $V - V_o$ in the order in which they are removed, and let us call G_i the subgraph of G remaining when vertices $v_1, \ldots, v_{i-1}$ have been removed; so $G_1 = G_M$. We denote by G' a subgraph of G_M containing V_o. We have $\max_{G' \subseteq G_M}(\min_{v \in G'}(d_{G'}(v))) \geq \max_{1 \leq i \leq q}(\min_{v \in G_i}(d_{G_i}(v))) = \max_{1 \leq i \leq q}(d_{G_i}(v_i))$, since in the left hand side all possible subgraphs G' of G_M containing V_o are considered, while in the right hand side, only $G_1, \ldots, G_q$ are considered.

We also have $\max_{G' \subseteq G_M}(\min_{v \in G'}(d_{G'}(v))) \leq \max_{1 \leq i \leq q}(d_{G_i}(v_i))$. In fact, let G'' be the subgraph for which the maximum on the left is attained. Let v_r be the first vertex of G'' which is removed in the above process. Then $\max_{G' \subseteq G_M}(\min_{v \in G'}(d_{G'}(v))) = \min_{v \in G''}(d_{G''}(v)) \leq d_{G''}(v_r) \leq d_{G_r}(v_r) \leq \max_{1 \leq i \leq q}(d_{G_i}(v_i))$. So the above inequality holds. It follows that $\max_{G' \subseteq G_M}(\min_{v \in G'}(d_{G'}(v))) = \max_{1 \leq i \leq q}(d_{G_i}(v_i)) \leq n(G_M^o)$. Hence, the coloring of G_M is possible with at most $n(G_M^o) + 1$ colors. □

As already mentioned at the beginning of this section, we know that for a mixed bipartite graph G_M, $\gamma_M(G_M) \leq n(G_M^o) + 1$, and so we obtain the following corollary.

Corollary 5. *Let G_M be a mixed graph such that $G(V_o)$ is mixed bipartite, and such that $\max_{G' \subseteq G_M}(\min_{v \in G'}(d_{G'}(v))) \leq n(G_M^o)$, where G' is a subgraph of G_M containing V_o. Then $\gamma_M(G_M) \leq n(G_M^o) + 1$.*

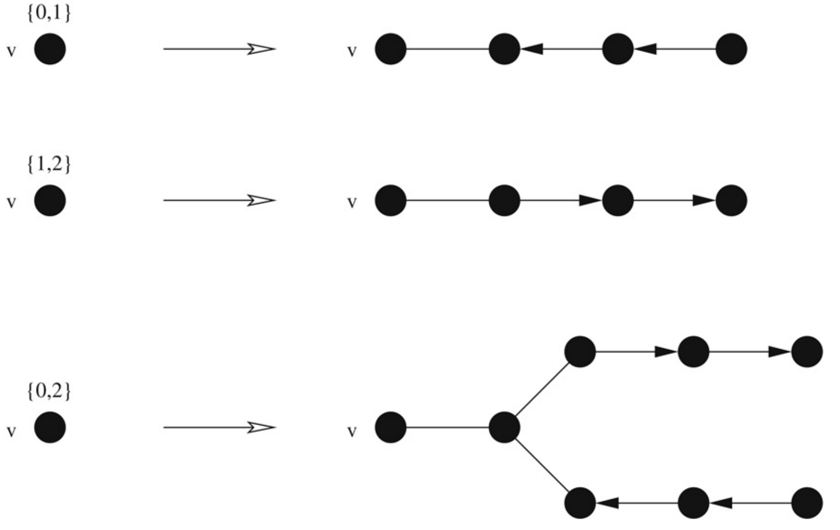

Fig. 1. Depending on the list $L(v)$, we add new vertices, edges and arcs.

Corollary 6. *Let G_M be a mixed graph such that each odd cycle C in G_M contains at least one vertex which is not incident to any arc, and such that* $\max_{G' \subseteq G_M}(\min_{v \in G'}(d_{G'}(v))) \leq n(G_M^o)$*, where G' is a subgraph of G_M containing V_o. Then $\gamma_M(G_M) \leq n(G_M^o) + 1$.*

Proof. Consider the mixed graph $G(V_o)$. Since each odd cycle in G_M contains at least one vertex which is not incident to any arc, $G(V_o)$ has no odd cycle, and hence is mixed bipartite. We conclude by using Corollary 5. □

3.2. Complexity results

In [12], it is shown that $S(G_M, 3)$ is *NP*-complete even if G_M is planar bipartite or bipartite with maximum degree 3. The following theorem strengthens the first result.

Theorem 7. *$S(G_M, 3)$ is NP-complete even if G_M is a planar bipartite graph with maximum degree* 4 *and each vertex incident to an arc has maximum degree* 2*.*

Proof. We use a reduction from the List Coloring problem (LiCol) which is defined as follows:
Instance: An undirected graph $G = (V, E)$ together with sets of feasible colors $L(v)$ for all vertices $v \in V$.
Question: Does there exist a proper vertex coloring of G with colors from $L = \bigcup_{v \in V} L(v)$ such that every vertex v is colored with a feasible color from $L(v)$?

This problem is shown to be *NP*-complete even if G is a 3-regular planar bipartite graph and the total number of colors is 3 and each list $L(v)$ contains 2 or 3 colors (see [3]).

Let G be a 3-regular planar bipartite graph. Suppose that each vertex v is given a list $L(v)$ with feasible colors such that $2 \leq |L(v)| \leq 3$, and such that the total number of colors is 3 (colors 0, 1, 2). For each vertex v in G such that $|L(v)| = 2$, introduce new vertices as shown in Fig. 1 depending on the list $L(v)$. The mixed graph G_M we thereby obtain is clearly planar and bipartite with $\Delta(G_M) \leq 4$; each vertex incident to an arc has maximum degree 2 and $n(G_M^o) = 3$.

Suppose now, that LiCol(G) has a positive answer. Denote by c the coloring corresponding to the solution. Then in G_M, color each vertex v which is also in G with the color $c(v)$. It is easy to see that the remaining uncolored vertices (those which were added) can be colored using colors 0, 1, 2 such that all the constraints are satisfied. Conversely, if $S(G_M, 3)$ has a solution, each original vertex gets necessarily a color from its list $L(v)$ in G, and hence we obtain a solution of LiCol(G) in G by removing in G_M the new vertices added at the beginning. □

We will now give some polynomially solvable cases in special classes of graphs. First, let us introduce the Precoloring Extension problem (PrExt) which is defined as follows:
Instance: An unoriented graph $G = (V, E)$ and some vertices of V are precolored properly using at most q colors.
Question: Can this precoloring of G be extended to a proper coloring of G using at most q colors?

This problem was shown to be polynomially solvable in special classes of graphs like split graphs [7], cographs [8], complements of bipartite graphs [7], or graphs of maximum degree 3 [3].

Theorem 8. *$S(G_M, n(G^o_M))$ is polynomially solvable if every vertex in G^o_M is on a path of length $n(G^o_M)-1$, and if the Precoloring Extension problem on the graph G with at most $n(G^o_M)$ colors, obtained by transforming each arc of G_M into an edge, is polynomially solvable.*

Proof. Let G_M be a mixed graph with G^o_M satisfying the above hypothesis, and such that PrExt(G) is polynomially solvable. Notice that if there exists a strong mixed $n(G^o_M)$-coloring c of G_M, then each vertex v belonging to G^o_M must get color $c(v) = in(v)$. So we color each vertex v incident to an arc with the color $c(v) = in(v)$. If a conflict occurs, i.e. if there are two adjacent vertices which get the same color, then no solution exists. Otherwise, consider all arcs as edges. We get an undirected graph G with some precolored vertices. Thus we get an instance of the Precoloring Extension problem in G. We know that PrExt(G) is polynomially solvable. It is easy to see that the two problems are equivalent. Thus, our problem is polynomially solvable. □

We denote by n_i the number of vertices on a longest directed path P in G_M containing vertex v_i (if v_i is not incident to any arc, $n_i = 1$ and $P = \{v_i\}$). Notice that $n_i = in(v_i) + out(v_i) + 1$. Let $h \geq |P|$ be an integer. We define S_i as the set of possible colors for v_i such that whenever v_i has a color $c(v_i) \in S_i$, there exists a coloring c of G_M (with an arbitrary number of colors) with $c(v) \leq h - 1$, for any $v \in P$. We have the following result:

Proposition 9. *Let $P = \{v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{n_i}\}$ be a longest directed path in G_M containing v_i, and let $h \geq |P|$ be an integer. Then $S_i = \{in(v_i), in(v_i)+1, \ldots, h-(out(v_i)+1)\}$.*

Proof. It is easy to see that the smallest feasible color for v_i is $in(v_i)$. Suppose that $c(v_i) = in(v_i) + q$, where $q \geq 0$. We can color the vertices $v_1, v_2, \ldots, v_{i-1}$ with colors $c(v_1) = in(v_1), c(v_2) = in(v_2), \ldots, c(v_{i-1}) = in(v_{i-1})$, and vertices $v_{i+1}, \ldots, v_{n_i}$ with colors $c(v_{i+1}) = in(v_i)+q+1, \ldots, c(v_{n_i}) = in(v_i)+q+n_i-i$. Notice that $n_i - i = out(v_i)$ since P is a longest directed path containing v_i. Thus, $c(v_{n_i}) = in(v_i) + q + out(v_i)$. This way we get a feasible coloring c of G_M (the vertices of G_M not belonging to P can easily be colored properly), and since the condition $c(v) \leq h - 1$ must hold for any $v \in P$, we have that $in(v_i) + q + out(v_i) \leq h-1$, i.e. $q \leq h-(in(v_i)+out(v_i)+1)$. Thus $S_i = \{in(v_i), in(v_i)+1, \ldots, h-(out(v_i)+1)\}$. □

We will now focus on a special class of graphs: partial k-trees. A *k-tree* is a graph defined recursively as follows: a k-tree on k vertices consists of a k-clique; given any k-tree T_n on n vertices, we construct a k-tree on $n + 1$ vertices by adjoining a new vertex v_{n+1} to T_n, which

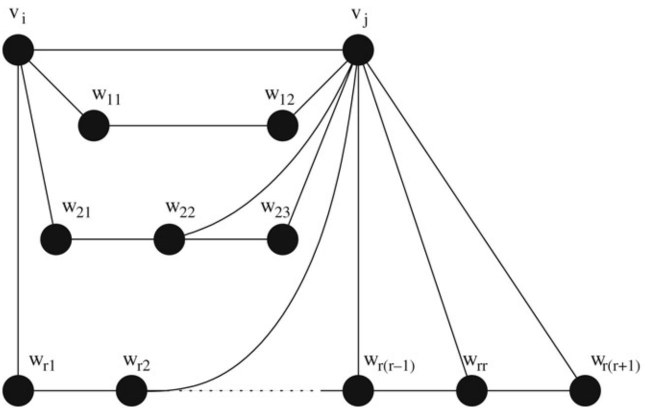

Fig. 2. New vertices and edges added to $[v_i, v_j]$.

is made adjacent to each vertex of some k-clique of T_n and nonadjacent to the remaining $n - k$ vertices. A *partial k-tree* is a subgraph of a k-tree.

Consider now an undirected partial k-tree $G = (V, E)$. Suppose that for some edges $[v_i, v_j] \in E$, we add new vertices and edges as shown in Fig. 2. Denote by G' the graph we obtain.

We have the following result.

Proposition 10. *Let G be a partial k-tree. Then G′ is also a partial k-tree.*

Proof. Since G is a partial k-tree, it is the subgraph of a k-tree T_k. Notice that $[v_i, v_j] \in K_{ij}$ in T_k, where K_{ij} is a $(k+1)$-clique. Consider T', which is the graph obtained by adding to G' all the edges and vertices of T_k which are not in G. In order to show that G' is a partial k-tree, we just need to show how edges can be added to T' to make it become a k-tree T^*.

For each new vertex w_{s1}, $s = 1, \dots, r$, make it adjacent to v_j and to $k - 2$ vertices in $K_{ij} - \{v_i, v_j\}$. We obtain, for each s, a $(k+1)$-clique K_{s1} containing w_{s1}. Each new vertex w_{st}, $s = 1, \dots, r$, $t = 2, \dots, r+1$, is linked to $k-2$ vertices in $K_{(s-1)t} - \{w_{(s-1)t}, v_j\}$. We obtain, then, for each s and t, $t \neq 1$, a $(k+1)$-clique K_{st} containing w_{st}. Clearly, the resulting graph is a k-tree, and thus G' is a partial k-tree. □

In [4], it is shown that $S(G_M, p)$ is polynomially solvable for series parallel graphs, i.e., partial 2-trees, by giving an exact algorithm which has complexity $O(n^{3.376} \log(n))$. In [12], a special case in bipartite partial k-trees is shown to be polynomially solvable. The following Theorem will strengthen the result of [12].

Theorem 11. *$S(G_M, p)$ is polynomially solvable if $G_M = (V, U, E)$ is a partial k-tree for fixed k.*

Proof. We use a transfomation to the LiCol problem, which is known to be solvable in $O(n^{k+2})$ time for partial k-trees (see [8]).

For each $v_l \in V$ which is not incident to any arc, we set $L(v_l) = \{0, 1, \dots, p-1\}$. For each vertex $v_i \in V$ which is incident to at least one arc, we set $L(v_i) = \{in(v_i), in(v_i)+1, \dots, p - (out(v_i)+1)\}$. For each arc $(v_i, v_j) \in U$ such that $p - (out(v_i)+1) > in(v_j)$, we introduce new vertices and edges as shown in Fig. 2, with $r = p - (out(v_i) + in(v_j) + 1)$. For the new vertices

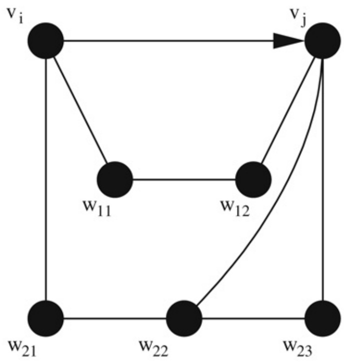

Fig. 3. Example of how new edges and vertices are introduced in the case of $L(v_i) = \{3, 4, 5, 6\}$ and $L(v_j) = \{4, 5, 6, 7\}$.

we set:

$$L(w_{st}) = \begin{cases} \{in(v_j) + s, in(v_j) + s + 1\} & \text{if } 1 \leq s \leq r \text{ and } t = 1, \\ \{in(v_j) + s + 1, in(v_j)\} & \text{if } 1 \leq s \leq r \text{ and } t = 2, \\ \{in(v_j) + t - 3, in(v_j) + t - 2\} & \text{if } 1 \leq s \leq r \text{ and } 3 \leq t \leq r + 1. \end{cases}$$

Fig. 3 shows a case where we have $L(v_i) = \{3, 4, 5, 6\}$, $L(v_j) = \{4, 5, 6, 7\}$ and $p = 8$. For the new vertices, we set $L(w_{11}) = \{5, 6\}$, $L(w_{12}) = \{6, 4\}$, $L(w_{21}) = \{6, 7\}$, $L(w_{22}) = \{7, 4\}$ and $L(w_{23}) = \{4, 5\}$. This way we do not allow vertex v_j to get a color less than the color of vertex v_i.

By considering all arcs as edges, we obtain a new undirected graph G' which is still a partial k-tree (see Proposition 10). Furthermore we associate to each vertex v in G' a list $L(v)$ of integers such that $L(v) \subseteq \{0, 1, \ldots, p-1\}$. Thus, we get an instance of the LiCol problem with p colors in a partial k-tree G', where k is fixed.

Suppose that an instance of the LiCol(G') problem has answer 'yes', and denote by c the corresponding list-coloring. We will show that c restricted to G_M is also a feasible coloring for $S(G_M, p)$. Clearly, for each edge $[v_i, v_j]$ in G', we have that $c(v_i) \neq c(v_j)$, and so $c(v_i) \neq c(v_j)$ for each $[v_i, v_j]$ or (v_i, v_j) in G_M. Consider now an arc (v_i, v_j) in G_M. We have to verify that $c(v_i) < c(v_j)$. If $c(v_i) \leq in(v_j)$, we clearly have that $c(v_i) < c(v_j)$. So suppose that $c(v_i) = in(v_j) + q$, $q > 0$. In that case, vertex w_{q1} necessarily has color $in(v_j) + q + 1$, and vertices $w_{q2}, \ldots, w_{q(q+1)}$ must have colors $in(v_j), \ldots, in(v_j) + q - 1$, due to their lists. Since these vertices are adjacent to v_j, $c(v_j) > in(v_j) + q$, and hence $c(v_i) < c(v_j)$. We conclude that $S(G_M, p)$ has answer 'yes'.

Conversely, suppose now that an instance of $S(G_M, p)$ has answer 'yes', and denote by c' the corresponding strong mixed p-coloring. Then clearly, each vertex v in G_M has a color which belongs to the corresponding list $L(v)$ in G', i.e., $c'(v) \in L(v)$. In fact, for each vertex v_l not incident to any arc in G_M, we have $L(v_l) = \{0, 1, \ldots, p-1\}$, and for each vertex v_i which is incident to at least one arc, we have $L(v_i) = \{in(v_i), in(v_i) + 1, \ldots, p - (out(v_i) + 1)\}$. By Proposition 9, we know that these colors are the only ones possible if P_{v_i} (a longest directed path containing v_i) is colored properly and $c'(v) < p$, for any $v \in P_{v_i}$. Furthermore, it is not difficult to verify that coloring c' can easily be extended in G' by coloring the new vertices w_{st} (using the colors in their associated lists), and so we get a feasible coloring for the LiCol problem in G'. Thus, the LiCol problem on G' has answer ‘yes’.

Clearly G' can be obtained from G_M in polynomial time, since n_i and $in(v_i)$ can be computed in polynomial time for each vertex v_i in G_M. The number of new vertices is restricted by $O(n^2m)$, where m is the number of arcs, and thus $S(G_M, p)$ can be solved in time $O(n^{2k+4}m^{k+2})$ if G_M is a partial k-tree, with fixed k. □

4. Weak mixed graph coloring problem

In this section, we study the following problem which we will call the *Weak Mixed Graph Coloring Problem*:

Instance: A mixed graph $G_M = (V, U, E)$, $E \neq \emptyset$, and an integer $p > 1$.

Question: Can the vertices of G_M be colored using at most p colors such that for each edge $[v_i, v_j] \in E$, $c(v_i) \neq c(v_j)$ and for each arc $(v_l, v_q) \in U$, $c(v_l) \leq c(v_q)$?

We will refer to this problem as $W(G_M, p)$. Notice that we clearly have $\chi_M(G_M) \leq \gamma_M(G_M)$.

Necessary and sufficient conditions for a mixed graph to admit a weak mixed coloring have been given:

Theorem 12 (*See for instance [15,16]*). *For the existence of a weak mixed coloring of a mixed graph $G_M = (V, U, E)$, it is necessary and sufficient that graph $(V, \emptyset, E)$ does not have loops and that G_M does not contain any directed circuit with a chord.*

In the rest of this section, we will suppose that these conditions are satisfied. Notice that in the case of weak mixed coloring, we may have $(v_l, v_q) \in U$ and $[v_l, v_q] \in E$. Then, in any proper weak mixed coloring c, we must have $c(v_l) < c(v_q)$. So the strong mixed graph coloring problem $S(G_M, p)$ is the special case of $W(G_M, p)$ where, for each arc $(v_l, v_q) \in U$, we have $[v_l, v_q] \in E$.

4.1. Bounds on the weak mixed chromatic number

We will start with a few observations which will allow us to simplify the original mixed graph G_M (see also [17], where a similar merging operation is designed for vertices belonging to the same strongly connected component).

Lemma 13. *Let $G_M = (V, U, E)$ be a mixed graph, and let C be a strongly connected component of G_M^o. Then, in any feasible weak mixed coloring c of G_M, $c(v_i) = c(v_j)$ $\forall v_i, v_j \in C$.*

Proof. Let c be a feasible coloring of G_M. Suppose that $c(v_i) < c(v_j)$ for some v_i, v_j in C. Since there is a directed path from v_j to v_i contained in C, we obtain a contradiction, because we should have $c(v_j) \leq c(v_i)$. □

Consider a mixed graph G_M and let $\mathcal{D} = \{D_1, \ldots, D_t\}$ be a set of disjoint directed partial graphs of G_M. Let $G_M/\mathcal{D}$ be the mixed graph obtained by deleting the arcs of $\bigcup_{l=1}^{t} D_l$, and by replacing the vertices of each graph D_l by a single vertex v_l. $G_M/\mathcal{D}$ may have multiple edges or arcs, in which case we delete them. We say that D_l has been *contracted* to a single vertex v_l, for all $l = 1, \ldots, t$. Then we have the following result.

Lemma 14. *Let $G_M = (V, U, E)$ be a mixed graph and let $\mathcal{C} = \{C_1, C_2, \ldots, C_q\}$ be the set of strongly connected components in G_M^o such that $\forall v, w \in C_k$, $k \in \{1, 2, \ldots, q\}$, we have $[v, w] \notin E$. Then $\chi_M(G_M) = \chi_M(G_M/\mathcal{C})$.*

Proof. Let c be an optimal coloring of $G_M/\mathcal{C}$. Let v_i be the vertex in $G_M/\mathcal{C}$ representing component C_i, and let $c(v_i)$ be its color, for $i = 1, 2, \ldots, q$. Consider now G_M, and color each vertex $w \notin C_i$, $i = 1, 2, \ldots, q$ in G_M with the same color as in $G_M/\mathcal{C}$. Color each vertex in C_i with color $c(v_i)$. Clearly, we obtain a feasible coloring of G_M. Furthermore, this coloring is optimal. In fact, suppose that G_M can be colored with $\chi_M(G_M) < \chi_M(G_M/\mathcal{C})$ colors. By Lemma 13, we know that all vertices of C_i, $i = 1, 2, \ldots, q$, necessarily have the same color c_i. Contracting each component C_i to a single vertex v_i and coloring it with color c_i, we obtain a feasible coloring of $G_M/\mathcal{C}$ with $\chi_M(G_M) < \chi_M(G_M/\mathcal{C})$ colors, which is a contradiction. □

Consider a mixed graph $G_M = (V, U, E)$. As we have seen in Lemma 14, all strongly connected components of G_M^o such that no two vertices of a same component are linked by an edge can be contracted to single vertices without changing the weak mixed chromatic number of the original graph. So, from now on, we suppose that in G_M, all these strongly connected components have been contracted to single vertices. Let v be a vertex of G_M which is not incident to any edge. Denote by $Pred(v)$ the set of its neighbors w such that $(w, v) \in U$, and by $Succ(v)$ the set of its neighbors u such that $(v, u) \in U$. Delete vertex v from G_M, and introduce arcs (w, u) for all $w \in Pred(v)$ and $u \in Succ(v)$. Suppose we perform this operation as long as there is a vertex v which is not incident to any edge. Let $G_M^* = (V^*, U^*, E)$ be the mixed graph obtained. Then we have the following result.

Theorem 15. *Let $G_M = (V, U, E)$ be a mixed graph. Then $\chi_M(G_M) = \chi_M(G_M^*)$.*

Proof. Consider an optimal weak mixed coloring of G_M^*. This coloring, c, can be extended to an optimal weak mixed coloring of G_M. In fact, consider the mixed graph G_M and color each vertex v, which is incident to at least one edge with color $c(v)$. Now color each remaining uncolored vertex v (incident to no edge) with color $c(v) = \max_{w \in Pred(v)}(c(w))$. We clearly obtain a feasible weak mixed coloring of G_M. Furthermore this coloring is optimal. Suppose that it is possible to color G_M with p colors, $p < \chi(G_M^*)$. Then by transforming G_M into G_M^*, we obtain a feasible coloring of G_M^* with at most p colors, which is a contradiction. Thus $\chi_M(G_M) = \chi_M(G_M^*)$. □

So from now on, we can also suppose that G_M does not contain any vertex incident only to arcs.

Let us consider the set $\mathcal{DP}$ of all maximal directed paths in G_M. Let $P = (v_1, \ldots, v_r)$ be a maximal directed path, and $E_P = \{[v_i, v_j] | 0 < i < j \le r\}$ be the set of edges linking each a pair of vertices of P. We denote by $E_P^1, \ldots, E_P^t$ the subsets of E_P such that, if $[v_i, v_j], [v_k, v_l] \in E_P^s$, then $\max(i, j) \le \min(k, l)$, for $s = 1, \ldots, t$. If $e_P = \max_{s=1,\ldots,t}(|E_P^s|)$, then we obtain the following lower bound on the weak mixed chromatic number.

Theorem 16. *Let G_M be a mixed graph. Then* $\max_{P \in \mathcal{DP}}(e_P + 1) \le \chi_M(G_M)$.

Proof. Let $P' = (v'_1, \ldots, v'_q)$ be a maximal directed path such that $P' = \arg\max_{P \in \mathcal{DP}}(e_P + 1)$. Suppose that $e_{P'} = |E_{P'}^f|$ for a certain integer f, and $E_{P'}^f = \{[v'_{i_1}, v'_{i_2}], [v'_{i_3}, v'_{i_4}], \ldots, [v'_{i_{r-1}}, v'_{i_r}]\}$, $0 < i_1 < i_2 \le i_3 < \cdots \le i_{r-1} < i_r \le q$.

If we want to construct a weak mixed graph coloring c, we clearly must have $c(v'_{i_j}) < c(v'_{i_{j+1}})$ for $j = 1, 3, \ldots, r-1$, since there is a directed path from v'_{i_j} to $v'_{i_{j+1}}$ and there is an edge $[v'_{i_j}, v'_{i_{j+1}}]$ for all $j = 1, 3, \ldots, r-1$. Furthermore we can color each vertex v'_{i_k} with the same color as $v'_{i_{k-1}}$ for $k = 3, 5, \ldots, r-1$. In fact there cannot be any edge between two vertices v'_{i_h}, v'_{i_g}, $i_{k-1} \le i_h < i_g \le i_k$ for $k = 3, 5, \ldots, r-1$, as otherwise $|E^f_{P'}|$ would not be maximal. Thus, we use at least $e_{P'} + 1$ colors. □

Remark 17. The lower bound given in Theorem 16 is tight. Indeed, if for all edges $[v_i, v_j] \in E$ we have $(v_i, v_j) \in U$ or $(v_j, v_i) \in U$, then $\max_{P \in \mathcal{DP}}(e_P + 1) = \chi_M(G_M)$.

We will give now two very simple classes of graphs for which we can determine the exact value of the weak mixed chromatic number.

Theorem 18. *Let $T_M = (V, U, E)$ be a mixed tree, $E \neq \emptyset$. Then $\chi_M(T_M) = 2$.*

Proof. Choose a root r in T_M. Color it with color $c(r) \in \{0, 1\}$. As long as there is an uncolored vertex, choose such a vertex v having one colored neighbor w (it is easy to see that this is always possible). If $[v, w] \in E$, color v with color $c(v) = 1 - c(w)$, and if (v, w) or $(w, v) \in U$, color it with $c(v) = c(w)$.

Clearly, we will only use two colors and $\forall [v, w] \in E$, $c(v) \neq c(w)$ and $\forall (v, w) \in U$, $c(v) = c(w)$ and hence the conditions are satisfied. We conclude that $\chi_M(T_M) = 2$. □

Theorem 19. *Let $C_M = (V, U, E)$ be a mixed chordless cycle. Then $\chi_M(C_M) = 2$.*

Proof. We distinguish two cases:

(1) if $|E|$ is even

We contract each arc (v, w) to a single vertex vw. We get an undirected even cycle which we can color with 2 colors. A feasible 2-coloring of C_M is obtained by expanding each vertex vw, and by coloring the vertices of the corresponding arc with the same color as vertex vw.

(2) if $|E|$ is odd

We choose an arc (v, w). Contract all arcs (v', w') to single vertices $v'w'$ except arc (v, w). We get an even cycle containing a single arc (v, w), which we can color properly using exactly two colors. A feasible 2-coloring of C_M is obtained by expanding each vertex $v'w'$ and by coloring the vertices of the corresponding arc with the same color as vertex $v'w'$. □

4.2. Complexity results

In this section, we will give some complexity results concerning the weak mixed graph coloring problem for some special classes of graphs.

Theorem 20. *$W(G_M, 3)$ is NP-complete even if G_M is planar bipartite with maximum degree* 4.

Proof. We use a reduction from $S(G_M, 3)$, which we have shown to be *NP*-complete even if G_M is planar bipartite with maximum degree 4 and each vertex incident to an arc has maximum degree 2. Let G_M be such a mixed graph. We replace each arc (v, w) by a path (v, u, z, w), where u and z are new vertices, and we introduce an edge $[v, w]$. Clearly, the mixed graph G'_M obtained is planar bipartite and has maximum degree 4.

Suppose that $S(G_M, 3)$ has a positive answer. Then by keeping this coloring c in G'_M, and by coloring the new vertices u, z with color $c(v)$, we obtain a solution for our problem. Conversely if $W(G_M, 3)$ has a positive answer, then we color in G_M each vertex v with the same color it gets in G'_M. Clearly, we obtain a solution for $S(G_M, 3)$. □

Remark 21. Notice that in the mixed graph G'_M, vertices which are incident to an arc may have a degree greater than two.

If we consider a mixed graph G_M such as was constructed in the proof of Theorem 7, then our problem $W(G_M, p)$ is trivial: we can color G_M using only two colors. In fact, the initial undirected planar cubic bipartite graph G is 2-colorable, and it is easy to see that the added vertices can be properly colored (with respect to the weak mixed graph coloring problem) using the same two colors. Hence for this particular class of planar bipartite graphs, $S(G_M, 3)$ is *NP*-complete while $W(G_M, p)$ is trivial, for any $p > 1$.

Theorem 22. *$W(G_M, 3)$ is NP-complete even if G_M is bipartite with maximum degree* 3.

Proof. We use a reduction from $S(G_M, 3)$, which has been shown to be *NP*-complete if G_M is bipartite with maximum degree 3 (see [12]). In G_M, replace each arc (v, w) by a directed path $(v, u_1, u_2, u_3, u_4, w)$, and add a new edge $[u_1, u_4]$. The resulting graph G'_M is clearly bipartite with maximum degree 3.

Suppose that $W(G'_M, 3)$ has a positive answer. Denote the coloring by c. Then, for each pair of vertices v, w such that $(v, w) \in G_M$, we must have $c(v) < c(w)$ because of the edge $[u_1, u_4]$. Thus, by replacing again the directed path by the arc (v, w) and by keeping the coloring c for the vertices of G_M, we obtain a solution for $S(G_M, 3)$. Similarly, if $S(G_M, 3)$ has a positive answer, denote by c' the coloring. Consider the mixed graph G'_M and keep the coloring c' for the vertices of G'_M, which are also vertices of G_M. By coloring the new vertices u_1, u_2, u_3 with color $c'(v)$ and vertex u_4 with color $c'(w)$, we clearly obtain a solution for $W(G'_M, 3)$. □

Theorem 23. *$W(G_M, 2)$ is polynomially solvable.*

Proof. We shall transform our problem into a 2*SAT* problem, which is known to be polynomially solvable (see [5]). Consider a mixed graph G_M. For each vertex x in G_M, we introduce two variables x_0, x_1 as well as two clauses $(x_0 \vee x_1)$ and $(\bar{x}_0 \vee \bar{x}_1)$. For each edge $[x, y] \in E$, we introduce two clauses $(\bar{x}_0 \vee \bar{y}_0)$ and $(\bar{x}_1 \vee \bar{y}_1)$. Finally, for each arc $(x, y) \in U$ we introduce a clause $(\bar{x}_1 \vee \bar{y}_0)$. Thus, we get an instance of 2*SAT*.

Suppose that the 2*SAT* instance is 'true'. Then by coloring each vertex x with color 0 if x_0 is true, and with color 1 if x_1 is true, we get a feasible 2-coloring of G_M. Conversely, if G_M admits a feasible 2-coloring, then by setting variable x_i to true if x has color i, $i \in \{0, 1\}$, we get a positive answer for the 2*SAT* instance. □

Theorem 24. *$W(G_M, p)$ is polynomially solvable if G_M is a partial k-tree, for fixed k.*

Proof. We will use a similar proof as for the case of strong mixed graph coloring in partial k-trees. Let $G_M = (V, U, E)$ be a mixed partial k-tree, for some fixed k. To each vertex $v \in V$, we associate a list $L(v) = \{0, 1, \ldots, p-1\}$ of possible colors. Notice that each list contains all possible colors $0, 1, \ldots, p-1$. Now for each arc $(v_i, v_j) \in U$, we introduce new vertices and edges as shown in Fig. 2 with $r = p - 1$. For these new vertices we set:

$$L(w_{st}) = \begin{cases} \{s, s+1\} & \text{if } 1 \le s \le r \text{ and } t = 1, \\ \{s+1, 0\} & \text{if } 1 \le s \le r \text{ and } t = 2, \\ \{t-3, t-2\} & \text{if } 1 \le s \le r \text{ and } 3 \le t \le r+1. \end{cases}$$

Remember that the graph we obtain (by considering all the arcs as edges) is also a partial k-tree for the same fixed k (see Proposition 10). Clearly, by deleting the arcs we still have a partial k-tree. So consider the partial k-tree G' obtained by deleting the arcs. Because for each vertex in G' we have associated a list of possible colors, we get an instance of the LiCol problem, which is polynomially solvable in partial k-trees, for fixed k [8]. By using similar arguments as in Theorem 11 one can easily prove that $W(G_M, p)$ and LiCol(G') are equivalent, and thus $W(G_M, p)$ is polynomially solvable. □

5. Conclusion

We considered two coloring problems in mixed graphs. In the first one, we were interested in coloring the vertices of the graph such that two adjacent vertices get different colors and the tail of an arc must get a color which is strictly smaller than the color of the head of the arc. We gave some bounds on the minimum number of colors necessary to color the vertices of special classes of graphs, as well as some complexity results. In particular, we showed that the strong mixed graph coloring problem is *NP*-complete, even if the mixed graph is planar bipartite of maximum degree 4 and each vertex incident to an arc has maximum degree 2. This strengthens a result of [12]. Furthermore we proved that the problem is polynomially solvable in partial k-trees, for fixed k, which extends a result of [12].

In the second problem, we were interested in coloring the vertices of the graph such that two adjacent vertices get different colors and the tail of an arc must not get a color larger than the head of the arc. Again, we gave some bounds on the minimum number of colors necessary to color the vertices, together with some complexity results. In particular, we showed that this problem is polynomially solvable in partial k-trees, for fixed k.

The results presented here concerned special classes of graphs. Further research is needed to extend these results to other classes of graphs. In particular, it would be interesting to know the complexity of the two problems in planar cubic bipartite graphs.

Acknowledgements

We would like to thank Prof. M.C. Golumbic and Prof. F. Werner for helpful discussions and remarks as well as two anonymous referees for their comments which improved the presentation of the paper.

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