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Random mechanism design on multidimensional domains ☆

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Abstract

We study random mechanism design in an environment where the set of alternatives has a Cartesian product structure. We first show that all generalized random dictatorships are sd-strategy-proof on a minimally rich domain if and only if all preferences are *top-separable*. We call a domain satisfying top-separability a *multidimensional domain*, and furthermore generalize the notion of connectedness (Monjardet, 2009) to a broad class of multidimensional domains: *connected*$^+$ *domains*. We show that in the class of minimally rich and connected$^+$ domains, the *multidimensional single-peakedness* restriction is necessary and sufficient for the design of a *flexible* random social choice function that is unanimous and sd-strategy-proof. Such a flexible random social choice function allows for a systematic notion of compromise. We prove an analogous result for deterministic social choice functions satisfying anonymity. Our characterization remains valid for a problem of voting under constraints where not all alternatives are feasible (Barberà et al., 1997).

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1. Introduction

Multidimensional models arise very naturally in economic environments as it is often the case that the object of choice consists of several attributes or components (commodities in consumer theory, positions in political economy, different levels of provision of public goods, etc), with no dependence across choices in different components.[1] The set of alternatives thus has the structure of a Cartesian product set, i.e., $A = \times_{s \in M} A^s$, where M collects all components, $s \in M$ is a component, and A^s is the corresponding component set.[2] The underlying Cartesian product structure on the set of alternatives allows for a richer description of available alternatives and introduces furthermore the possibility of defining domains of restricted preferences which take cognizance of the multidimensional structure, and allow positive results for aggregation and economic design. We explore the theoretical underpinnings of such multidimensional preference domains from the perspective of mechanism design. We first identify that a particular condition, *top-separability* (introduced by Le Breton and Weymark, 1999), is fundamental in formulating multidimensional preferences that admit new possibilities for mechanism design.[3] Our principal finding is that within the class of top-separable preferences, *multidimensional single-peaked domains* (introduced by Barberà et al., 1993), a particular generalization of single-peakedness to a multidimensional setting, emerge as the unique preference domains that allow for the design of attractive random mechanisms. Thus the notion of single-peakedness, which is well-studied and prominent in aggregation theory, voting theory and political economy, turns out to be a particularly distinguished one in the context of multidimensional random mechanism design.

We focus on probabilistic mechanisms in multidimensional settings in the absence of monetary transfers where the set of alternatives is assumed to be finite.[4] We impose a strong version of the incentive compatibility requirement by requiring that truth-telling first-order stochastically dominate every possible manipulation of preferences. We thus study Random Social Choice Functions (RSCFs) that satisfy the ordinal version of strategy-proofness formulated by Gibbard (1977), which we henceforth term *sd-strategy-proofness*.[5] We also impose the mild condition that the RSCFs satisfy unanimity, which says that if an alternative is top ranked for every agent at a preference profile, then it receives probability one under the RSCFs at that profile.

1 See for instance, legislative, political and club-member elections (e.g., Border and Jordan, 1983; Barberà et al., 1991, 1993, 1999, 2005; Le Breton and Sen, 1999; Le Breton and Weymark, 1999; Aswal et al., 2003; Bahel and Sprumont, 2018) and public goods location and provision problems (e.g., Zhou, 1991; Peters et al., 1992; Chichilnisky and Heal, 1997; Ehlers, 2002; Svensson and Torstensson, 2008; Reffgen and Svensson, 2012).

2 We pick an element in each component set, and assemble the selected elements to form an alternative.

3 In a top-separable preference, when we compare two alternatives which disagree on exactly one component, the alternative that inherits the element from the top ranked alternative in that disagreed component is always preferred. Throughout the paper, the term *multidimensional preference* refers to a preference satisfying top-separability (and possibly some other restrictions).

4 We focus on the classic voting model which we hope will be useful in formulating more general models where some of the dimensions include private goods or monetary transfers. Recent work (e.g., Morimoto and Serizawa, 2015; Kazumura et al., 2017) studies formulations with monetary transfers under non-quasi-linear preferences.

5 This is equivalent to requiring that the expected utility of truth-telling be at least as large as the expected utility of manipulating, for *every* possible utility representation of the primitive ordinal preference.

An important class of RSCFs is the class of *random dictatorships*. These are defined by fixing a probability distribution over agents; the probability assigned to an alternative at a preference profile is then the sum of the weights of the agents who have this particular alternative as their top ranked alternative. Random dictatorships are sd-strategy-proof and ex-post efficient (a strengthening of unanimity), and allow for a equitable distribution of power among agents which is precluded by a deterministic dictatorship. These are however not entirely satisfactory from the design point of view as they *lack flexibility*; indeed *any* alternative that is not top ranked for some agent at the profile in question can never get strictly positive probability. In particular, such an alternative may be second ranked for all agents in a profile where agents disagree on peaks; we refer to such an alternative as a *compromise alternative*, and suggest that it is desirable to design RSCFs that have the flexibility to give positive probability to such an alternative.[6]

Under a Cartesian product structure, random dictatorships can be naturally generalized to accord with the multidimensional setting in the following way. Instead of fixing a probability distribution over agents, we fix a probability on each *voter sequences*, which is an $|M|$-tuple of agents, and associates each component with an agent who can be viewed as the dictator of that component (note that one agent can be associated to multiple components). At a preference profile, according to one voter sequence, we can assemble a unique alternative whose kth component is the kth component of the corresponding component dictator's preference peak. The probability assigned to an alternative at a preference profile is then the sum of the weights of the voter sequences which can assemble this alternative. These random mechanisms are called *generalized random dictatorships*, and were introduced by Chatterji et al. (2012). Generalized random dictatorships recognize the Cartesian product structure and allow for greater flexibility than do random dictatorships as at some preference profiles: Some non-peak alternatives can be assembled and receive strictly positive probability. In contrast to random dictatorships, certain preference restrictions must however be imposed to ensure sd-strategy-proofness of a generalized random dictatorship. We show in Proposition 1 that top-separability is necessary and sufficient for sd-strategy-proofness of all generalized random dictatorships. However, due to the somewhat limited assembling capability of voter sequences, generalized random dictatorships sometimes ignore compromise alternatives.

This paper examines restricted domains of multidimensional preferences that allow us to construct sd-strategy-proof RSCFs which are flexible in that they systematically admit compromise. The preference domains we study satisfy a particular "richness" property that is based on the idea of connectedness initially proposed by Grandmont (1978) and Monjardet (2009), and has been recently adopted to explore various issues which include the equivalence of local sd-strategy-proofness and sd-strategy-proofness (e.g., Carroll, 2012; Sato, 2013; Cho, 2016; Mishra, 2016), the extent to which RSCFs can depend on agents' preferences (Chatterji and Zeng, 2018), and the characterization of preference restrictions that allow one to design attractive RSCFs (Chatterji et al., 2016). The notion of connectedness requires that one be able to reconcile the differences between two preferences via a sequence of preferences in the domain where each successive pair involves one "local switch" of two contiguously ranked alternatives. This richness condition

[6] Gibbard (1977) proved that on the domain of unrestricted preferences, the only sd-strategy-proof and unanimous RSCFs are random dictatorships. Recent literature has examined restricted preference domains where one may design more flexible RSCFs that are sd-strategy-proof and unanimous. See for instance, almost random dictatorships (Chatterji et al., 2014), fixed probabilistic ballots rules (Ehlers et al., 2002) and probabilistic generalized median voter schemes (Peters et al., 2014).

restricts the probabilities received by alternatives that do not switch across two successive preferences, and plays a fundamental methodological role in deriving the results mentioned above.

This notion of connectedness however does not apply to domains of multidimensional preferences, e.g., the top-separable domain, as it is often the case that multiple pairs of alternatives have to be switched simultaneously across two successive preferences. We introduce a new notion of a connectedness which permits the requisite simultaneous local switches, and allows us to investigate systematically domains of multidimensional preferences that permit the design of nice sd-strategy-proof RSCFs. The domains we consider are termed *connected$^+$ domains*; these are subsets of the top-separable domain, and include the well studied instances of separable preferences (Barberà et al., 1991; Le Breton and Sen, 1999), multidimensional single-peaked preferences (Barberà et al., 1993), and their intersection and unions. Connected$^+$ domains also possess the requisite generality and structure that would in principle allow one to investigate other issues being studied in the literature (like the equivalence of local sd-strategy-proofness and sd-strategy-proofness, etc, alluded to above), and can presumably be exploited beyond this paper.

In the class of connected$^+$ domains, multidimensional single-peaked domains are an important and well studied class. These are a particular generalization of the idea of single-peaked preferences to a multidimensional setting using the Cartesian product structure and the city block metric. Our first theorem characterizes multidimensional single-peaked domains as the unique domains that permit the design of sd-strategy-proof and unanimous RSCFs departing from random dictatorships/generalized random dictatorships systematically, in that they admit compromises, wherein the compromise alternatives necessarily receive strictly positive probabilities whenever they appear (see Theorem 1). Our version of multidimensional single-peaked domains allows elements of each component set to be arranged on a tree which is a generalization of multidimensional single-peakedness initiated by Barberà et al. (1993).[7] In the special case where the connected$^+$ domain contains two complete reversal preferences, we refine the domain characterization to the more familiar formulation of Barberà et al. (1993). We next provide a characterization result for multidimensional single-peaked domains using deterministic social choice functions (see Theorem 2). We do so by replacing the compromise property by the familiar axiom of anonymity.[8]

We finally turn to the setup of voting under constraints originally proposed by Barberà et al. (1997). Here, not all alternatives in the underlying Cartesian product structure are feasible. We investigate what structure on the set of feasible alternatives and preferences (applicable now only to the restriction of the original preferences to the feasible alternatives) would allow us to define RSCFs which satisfy our requirements of unanimity, sd-strategy-proofness and compromise on connected$^+$ domains. We deduce that the set of feasible alternatives must be factorizable as a Cartesian product of trees, and the preferences must satisfy a particular version of multidimensional single-peakedness w.r.t. the feasible alternatives (see Theorem 3). Our results are therefore robust to voting under constraints.

The rest of the paper is organized as follows. The remainder of the Introduction explains in greater detail the relation of this paper to the literature. Section 2 describes the model, introduces generalized random dictatorships, establishes the domain richness condition, and specifies the formal notion of compromise. Section 3 presents the domain characterization results for multidi-

[7] In the formulation of multidimensional single-peakedness of Barberà et al. (1993), all elements of each component set are located on a line.

[8] Anonymity implies that the social outcome is immune to the identities of agents.

mensional single-peaked preferences, while Section 4 concludes. The Appendix gathers proofs, examples and verifications that are not included in the main text.

1.1. Related literature

Much of the literature on multidimensional models has focused on deterministic social choice functions (DSCFs). The early literature proved impossibility results for various generalizations of single-peakedness to cases where the set of alternatives is a convex subset of $\mathbb{R}^{|M|}$ (e.g., Border and Jordan, 1983; Bordes et al., 1990; Zhou, 1991; Peters et al., 1992). The case of separable/top-separable preferences over a convex subset of alternatives was analyzed by Le Breton and Weymark (1999) while general results on formulations where the set of alternatives is a subset of a metric space were presented by Weymark (2008). Barberà et al. (1991) provided a possibility result for *voting by committees* when the number of elements in each component set is two, while in the general case of finitely many elements in each component set studied by Le Breton and Sen (1999), sd-strategy-proof DSCFs degenerate to *generalized dictatorships* which are the deterministic counterparts of generalized random dictatorships. Positive characterization results for *generalized median voter schemes* have been introduced by Barberà et al. (1993) who proposed the restriction of multidimensional single-peakedness, and by Barberà et al. (1997) who introduced the intersection property on generalized median voter schemes to accord with voting under constraints. Two comprehensive surveys of these results are provided by Sprumont (1995) and Barberà (2010). Besides the characterizations of sd-strategy-proof DSCFs on multidimensional domains, several papers also verify the necessity of separable preferences (see Hatsumi et al., 2014), and variants of multidimensional single-peaked preferences for the existence of particular sd-strategy-proof generalized median voter schemes (e.g., Barberà et al., 1993, 1999) or general sd-strategy-proof DSCFs satisfying different well-behavedness criteria (e.g., neutrality and anonymity in Nehring and Puppe, 2007, and the tops-only property and anonymity in Chatterji and Massó, 2018).[9]

The literature on random mechanism design on restricted domains arising from multidimensional models is not as large. An early paper by Dutta et al. (2002) studied lotteries defined on a convex subset of $\mathbb{R}^{|M|}$ where preferences are convex, continuous and single-peaked, and established a random dictatorship result. Subsequently, Chatterji et al. (2012) characterized generalized random dictatorships on the lexicographically separable domain, which is a particular subset of the separable domain, while Chatterji and Zeng (2018) characterize random dictatorships using sd-strategy-proofness and ex-post efficiency on the multidimensional single-peaked domain. A more directly related to this paper is the paper by Chatterji et al. (2016) which characterized single-peaked preferences on a tree in a class of connected domains. The characterization of the multidimensional single-peaked domain in this paper differs from their result in two important ways. First, Chatterji et al. (2016) used an extra tops-only axiom on the RSCFs, and secondly, as mentioned earlier, their connectedness assumption excludes the multidimensional domains studied in this paper. In the present paper, the tops-only property emerges *endogenously* (Proposition 2) from our richness condition. Our richness condition is a strengthening of the "Interior and Exterior" properties of Chatterji and Zeng (2018) that was shown to induce the tops-only property. Their results do not apply for the connected^{+} domains in this paper. We

[9] Neutrality implies that the social outcome is immune to relabelings of alternatives. The tops-only property implies that across two preference profiles, if each agent has the same preference peak, the social outcomes remain unchanged.

extend their tops-only result to our setting by postulating the existence of sufficiently many separable preferences that allow the sort of multiple switches of alternatives we alluded to earlier. This strengthening is critical for establishing that the alternatives in the Cartesian product structure can be embedded in a product of trees, and that preferences be multidimensional single-peaked (as stated in Theorem 1). Similarly, our characterization result for multidimensional single-peaked domains using deterministic social choice functions extends the analysis of Chatterji et al. (2013) to multidimensional domains (which were excluded by their hypothesis of connected domains), and does so by endogenizing the tops-only property (which was an exogenous axiom in their paper). For the set up of voting under constraints, Barberà et al. (1997) characterized all unanimous and sd-strategy-proof DSCFs on the multidimensional single-peaked domain for arbitrary feasible sets. We investigate and provide an answer to the converse question: What can be inferred about the structure of the set of the feasible alternatives and the preferences from the existence of a well-behaved sd-strategy-proof RSCF satisfying the property of compromise on a connected^{+} domain?

2. Model

Let A be a *finite* set of alternatives with $|A| \geq 4$. We assume that the alternative set can be represented as a **Cartesian product** of a finite number of sets, each of which contains finitely many elements. Formally, throughout the paper, we fix $A = \times_{s \in M} A^s$ where $M = \{1, 2, \ldots, m\}$, $m \geq 2$ is an integer; and $|A^s| \geq 2$ is an integer for each $s \in M$.[10] Each $s \in M$ is called a **component**; A^s is referred to as a **component set**, and an element in A^s is denoted as a^s. Accordingly, an alternative is represented by a m-tuple, i.e., $a \equiv (a^1, a^2, \ldots, a^m) \equiv (a^s)_{s \in M}$. Given a nonempty strict subset $S \subset M$, let $A^S = \times_{s \in S} A^s$, $a^S \equiv (a^s)_{s \in S} \in A^S$; $A^{-S} \equiv \times_{s \notin S} A^s$ and $a^{-S} \equiv (a^s)_{s \notin S} \in A^{-S}$.[11] Therefore, we also write alternative $a \equiv (a^s, a^{-s}) \equiv (a^S, a^{-S})$. In particular, we say that a pair of alternatives $a, b \in A$ is **similar** if they disagree on *exactly one* component, i.e., $a^s \neq b^s$ and $a^{-s} = b^{-s}$ for some $s \in M$. For notational convenience, given a non-empty strict subset $S \subset M$, $X^S \subseteq A^S$ and $Y^{-S} \subseteq A^{-S}$, let $(X^S, Y^{-S}) = \{a \in A : a^S \in X^S \text{ and } a^{-S} \in Y^{-S}\}$.[12] Let $\Delta(A)$ denote the space of lotteries/probability distributions over A. In particular, $e_a \in \Delta(A)$ is a degenerate lottery where a is chosen with probability one.

Let $I = \{1, \ldots, N\}$ be a *finite* set of voters with $N \geq 2$. Each voter i has a preference order P_i over A which is complete, antisymmetric and transitive, i.e., a *linear order*. For any $a, b \in A$, $a P_i b$ is interpreted as "a is *strictly* preferred to b according to P_i".[13] Moreover, we use $\boldsymbol{a P_i ! b}$ to denote that a is *contiguously* ranked above b in P_i, i.e., $a P_i b$ and there exists no $c \in A$ such that $a P_i c$ and $c P_i b$. Two preferences P_i, P_i' are **complete reversals** if $[a P_i b] \Leftrightarrow [b P_i' a]$ for all $a, b \in A$. Given a preference P_i and a strict subset $B \subset A$, let $P_{i|B}$ denote the induced preference over B which preserves the relative rankings of all alternatives of B in preference P_i. Given a preference P_i, let $r_k(P_i)$ denote the kth ranked alternative in P_i, $1 \leq k \leq |A|$. Let $\mathbb{P}$ denote the set containing *all* linear orders over A. The set of all admissible preferences is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as a **preference domain**. We call $\mathbb{P}$ *the complete domain*, and refer to $\mathbb{D}$ as a **restricted domain**

[10] To make sure all components indispensable, we assume $|A^s| \geq 2$ for all $s \in M$.

[11] In this paper, $\subseteq$ and $\subset$ denote the weak and strict inclusion relations respectively.

[12] We henceforth frequently use $(x^s, A^{-s}) = \{a \in A : a^s = x^s\}$ and $(A^s, x^{-s}) = \{a \in A : a^{-s} = x^{-s}\}$.

[13] In a table, we specify a preference "vertically", while in a sentence, we specify a preference "horizontally". For instance, preference $P_i: a \rightharpoonup b \rightharpoonup c \rightharpoonup \cdots$ is one where a is at the top, b is the second best, c is the third ranked alternative while the remaining alternatives are arbitrarily ranked.

when $\mathbb{D} \neq \mathbb{P}$. For notational convenience, given $a \in A$, let $\mathbb{D}^a = \{P_i \in \mathbb{D} : r_1(P_i) = a\}$ denote a subdomain where each preference's peak is a. Correspondingly, a domain $\mathbb{D}$ is **minimally rich** if $\mathbb{D}^a \neq \emptyset$ for every $a \in A$. Throughout the paper, we restrict attention to minimally rich domains.

Each voter reports a preference, and all reported preferences are collected to formulate a preference profile $P \equiv (P_1, P_2, \ldots, P_N) \equiv (P_i, P_{-i}) \in \mathbb{D}^N$. A **Random Social Choice Function** (or RSCF) is a map $\varphi : \mathbb{D}^N \to \Delta(A)$, which associates to each profile $P \in \mathbb{D}^N$, a "socially desirable" lottery $\varphi(P)$. For any alternative $a \in A$, $\varphi_a(P)$ is the probability with which a will be chosen in $\varphi(P)$. Thus, $\varphi_a(P) \geq 0$ for all $a \in A$ and $\sum_{a \in A} \varphi_a(P) = 1$. A **Deterministic Social Choice Function** (or DSCF) is a particular RSCF where a degenerate lottery is specified at each preference profile, i.e., $\varphi(P) = e_a$ for some $a \in A$ at profile P.[14] First, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is **unanimous** if it assigns probability one to an alternative that is top ranked in a profile by all voters, i.e., $[r_1(P_i) = a$ for all $i \in I] \Rightarrow [\varphi_a(P) = 1]$ for all $a \in A$ and $P \in \mathbb{D}^N$. Next, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is **sd-strategy-proof** if for all $i \in I$, $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$, the lottery $\varphi(P_i, P_{-i})$ first-order stochastically dominates $\varphi(P_i', P_{-i})$ according to P_i, i.e., $\sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i, P_{-i}) \geq \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i', P_{-i})$ for all $t = 1, \ldots, |A|$.[15]

A prominent class of unanimous and sd-strategy-proof RSCFs is the class of random dictatorships (Gibbard, 1977). Each voter first is assigned a non-negative weight such that the sum of all weights equals one. In a random dictatorship, at each preference profile, the probability received by an alternative is determined by the set of voters who prefer this alternative the most, and equals the sum of these voters' weights. Formally, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a **random dictatorship** if there exists $\varepsilon_i \geq 0$ for each $i \in I$ with $\sum_{i \in I} \varepsilon_i = 1$ such that for all $P \in \mathbb{D}^N$ and $a \in A$, $\varphi_a(P) = \sum_{i \in I : r_1(P_i) = a} \varepsilon_i$.[16] Note that a random dictatorship is sd-strategy-proof on any arbitrary preference domain.

2.1. *Generalized random dictatorships*

Under the Cartesian product setting, one may consider the following generalization of a random dictatorship. We associate each component $s \in M$ with a voter $i^s \in I$, and construct an m-tuple of voters $\underline{i} = (i^s)_{s \in M} \in I^m$ to form a *voter sequence*. A voter sequence can be viewed as a combination of m dictators (note that one voter may appear multiple times); on each component $s \in M$, voter i^s is the dictator over the component set A^s. Given a profile $P \in \mathbb{D}^N$, for notational convenience, assume $r_1(P_i) \equiv (x_i^s)_{s \in M}$, $i \in I$. We say that an alternative $a \equiv (a^s)_{s \in M}$ is *assembled* by a voter sequence $\underline{i} \equiv (i^s)_{s \in M}$ at profile P if $a^s = x_{i^s}^s \equiv r_1(P_{i^s})^s$ for all $s \in M$. Analogously to random dictatorships, we associate a non-negative weight to each voter sequence $\underline{i} \in I^m$, denoted $\gamma(\underline{i}) \geq 0$, and let $\sum_{\underline{i} \in I^m} \gamma(\underline{i}) = 1$. Last, at each preference profile, the probability assigned to an alternative is determined by the set of voter sequences which can assemble this alternative. Such an RSCF is referred to as a generalized random dictatorship (Chatterji et al., 2012). Formally, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a **generalized random dictatorship** if there exists $\gamma(\underline{i}) \geq 0$ for each $\underline{i} \in I^m$ with $\sum_{\underline{i} \in I^m} \gamma(\underline{i}) = 1$ such that for all $P \in \mathbb{D}^N$ and $a \in A$,

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14 We sometimes simply write a DSCF as $f : \mathbb{D}^N \to A$.

15 To avoid confusion, we also use the term "sd-strategy-proofness" for DSCFs. One can easily infer that the definition of sd-strategy-proofness for a DSCF $f : \mathbb{D}^N \to A$ degenerates to that for all $i \in I$, $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$, we have $f(P_i, P_{-i}) = f(P_i', P_{-i})$ or $f(P_i, P_{-i}) P_i f(P_i', P_{-i})$. Note that a **mixture** (equivalently, a convex combination) of finitely many sd-strategy-proof DSCFs is an sd-strategy-proof RSCF.

16 In particular, if $\varepsilon_i = 1$ for some $i \in I$, the random dictatorship degenerates to a **dictatorship**.

$\varphi_a(P) = \sum_{\underline{i}\equiv(i^s)_{s\in M}\in I^m:\, a=\left(r_1(P_{i^s})^s\right)_{s\in M}} \gamma(\underline{i})$. In particular, if $\gamma(\underline{i}) = 1$ for some $\underline{i} \in I^m$, the generalized random dictatorship degenerates to a **generalized dictatorship**. Therefore, each generalized random dictatorship is a mixture of generalized dictatorships.

Evidently, every generalized random dictatorship satisfies unanimity. On the one hand, if only *constant voter sequences* (i.e., one voter dictates all components) receive positive weights, the generalized random dictatorship degenerates to a random dictatorship. On the other hand, if every voter sequence receives a strictly positive weight, the generalized random dictatorship prescribes a maximal support for the social lottery at each preference profile compared to others. The characterization of random dictatorships (Gibbard, 1977) implies that to restore sd-strategy-proofness on generalized random dictatorships, especially those associating strictly positive weights to non-constant voter sequences, we must impose some preference restriction. This preference restriction turns out to be *top-separability*; it was initially introduced by Le Breton and Weymark (1999) on continuous preferences over a product of first-countable Tychonoff space.[17] To formulate a top-separable preference, we first fix exactly one "acceptable" element in each component set. Evidently, the most preferred alternative must be the one assembled by all acceptable elements. Furthermore, when comparing a pair of similar alternatives, the one which inherits the acceptable element in the disagreed component is always preferred.

Definition 1. A preference P_i, say $r_1(P_i) \equiv (x^s)_{s\in M}$, is **top-separable** if for all similar alternatives $a, b \in A$, say $a^s \neq b^s$ and $a^{-s} = b^{-s}$, we have $\left[a^s = x^s\right] \Rightarrow [a\, P_i\, b]$.

Let $\mathbb{D}_{\text{TS}}$ denote **the top-separable domain** containing all top-separable preferences. The proposition below shows that under minimal richness, top-separability is necessary and sufficient for sd-strategy-proofness of all generalized random dictatorships, in particular, those that assign strictly positive weights to all voter sequences.

Proposition 1. *Let $\mathbb{D}$ be a minimally rich domain. All generalized random dictatorships are sd-strategy-proof if and only if all preferences are top-separable.*

The proof of Proposition 1 is available in Appendix A.

The Cartesian product structure would be redundant, for instance it could be simply viewed as a relabeling of alternatives, if it is not involved in establishing preference restrictions. The restriction of top-separability however respects the Cartesian product structure, systematically restores sd-strategy-proofness in a class of RSCFs which is significantly more flexible than random dictatorships, and therefore distinguishes us from the models in the one-dimensional setting (e.g., Gibbard, 1977). Henceforth, we restrict attention to domains of top-separable preferences.

Definition 2. A domain is called a **multidimensional domain** if all preferences are top-separable.

In the literature, two well studied preference restrictions, *separability* (Le Breton and Sen, 1999) and *multidimensional single-peakedness* (Barberà et al., 1993) naturally meet the requirement of top-separability, and refine top-separable preferences by imposing additional restrictions.

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[17] Theorem 5 of Le Breton and Weymark (1999) showed that a DSCF with a full range is sd-strategy-proof on their top-separable domain if and only if it is a generalized dictatorship.

In order to formulate a separable preference, not just a unique acceptable element is fixed in advance, but a linear order, referred to as a *marginal preference*, is first fixed over all elements of each component set. Separability then requires that between each pair of similar alternatives, the one endowed with a better element in the disagreed component is always preferred.

Definition 3. A preference P_i is **separable** if there exists a (unique) marginal preference $[P_i]^s$ over A^s for each $s \in M$ such that for all similar alternatives $a, b \in A$, say $a^s \neq b^s$ and $a^{-s} = b^{-s}$, we have $\left[a^s [P_i]^s b^s\right] \Rightarrow \left[a P_i b\right]$.

In a separable preference P_i, the relative ranking of two similar alternatives are determined independently of their agreed components, i.e., for all $s \in M$, $a^s, b^s \in A^s$ and $x^{-s}, y^{-s} \in A^{-s}$, $\left[(a^s, x^{-s}) P_i (b^s, x^{-s})\right] \Leftrightarrow \left[(a^s, y^{-s}) P_i (b^s, y^{-s})\right]$. The domain including all separable preferences is referred to as **the separable domain**, denoted $\mathbb{D}_{\mathrm{S}}$. Evidently, $\mathbb{D}_{\mathrm{S}} = \mathbb{D}_{\mathrm{TS}}$ if $|A^s| = 2$ for all $s \in M$, and $\mathbb{D}_{\mathrm{S}} \subset \mathbb{D}_{\mathrm{TS}}$ if $|A^s| > 2$ for some $s \in M$. For more detailed studies on separable preferences, please refer to Barberà et al. (1991), Le Breton and Sen (1999), Barberà et al. (2005) and Reffgen and Svensson (2012).

Alternatively, multidimensional single-peakedness adopts a particular "grid" to measure the geometric distance between alternatives, and then requires one alternative be preferred to another when it stands "closer" to the preference peak. Formally, for each $s \in M$, all elements of A^s are located on a *tree*, denoted $G(A^s)$.[18] Let $\langle a^s, b^s \rangle$ denote the *unique* graph path between a^s and b^s in $G(A^s)$.[19] Combining all trees $G(A^s)$, we generate a **product of trees** $\times_{s \in M} G(A^s)$ where the set of vertices is A, and two distinct alternatives a and b form an edge if and only if a and b are similar, say $a^{-s} = b^{-s}$ for some $s \in M$, and moreover, a^s and b^s form an edge in $G(A^s)$. Given $a, b \in A$, for notational convenience, let $\langle a, b \rangle = \{x \in A : x^s \in \langle a^s, b^s \rangle \text{ for each } s \in M\}$ denote the **minimal box** containing *all* alternatives located between a and b in each component. Thus, in a multidimensional single-peaked preference, if one alternative is in the minimal box formed by the preference peak and another alternative, it is naturally closer to the preference peak, and hence is ranked relatively higher.

Definition 4. A preference P_i is **multidimensional single-peaked** on a product of trees $\times_{s \in M} G(A^s)$ if for all distinct $a, b \in A$, we have $\left[a \in \langle r_1(P_i), b \rangle\right] \Rightarrow [a P_i b]$.

Therefore, a domain is multidimensional single-peaked if there exists a product of trees on which every preference is multidimensional single-peaked. Given a product of trees $\times_{s \in M} G(A^s)$, let $\mathbb{D}_{\mathrm{MSP}}$ denote **the multidimensional single-peaked domain** which contains all corresponding multidimensional single-peaked preferences.[20] For more details on multidimensional single-peaked preferences, please refer to Barberà et al. (1993) and Sprumont (1995).

Remark 1. In the multidimensional single-peaked domain, some preferences are separable while some are not separable. Note that a separable preference P_i, say $r_1(P_i) = x \equiv (x^s)_{s \in M}$, is multi-

[18] A *graph* is a combination of vertices and edges. A *graph path* is a sequence of vertices with each consecutive pair forming an edge. A *tree* is a graph where each pair of vertices is connected via a unique graph path.

[19] If $a^s = b^s$, $\langle a^s, b^s \rangle = \{a^s\}$ is a singleton set.

[20] The version of multidimensional single-peakedness we derive is a generalization of the one studied by Barberà et al. (1993), since they restrict attention to a product of lines. Throughout this paper, any strict subset of the multidimensional single-peaked domain is just referred to as "a multidimensional single-peaked domain". Two distinct product graphs of trees always induce two distinct multidimensional single-peaked domains.

dimensional single-peaked on a product of trees $\times_{s\in M} G(A^s)$ if and only if for every $s \in M$, the marginal preference $[P_i]^s$ is **single-peaked** on $G(A^s)$, i.e., for all distinct $a^s, b^s \in A^s$, we have $\left[a^s \in \langle x^s, b^s\rangle\right] \Rightarrow \left[a^s [P_i]^s b^s\right]$.[21] $\square$

2.2. *Connected*$^+$ *domains*

To execute our investigation, we restrict attention to a class of multidimensional domains that satisfies a particular richness condition, *connectedness*$^+$.

To establish connectedness$^+$, we first introduce two notions to address the relation between two preferences which are sufficiently close to each other in the Kemeny distance (Kemeny, 1959). Let $\Gamma(P_i, P_i') = \left\{\{a, b\} \in A^2 \middle| a P_i b \text{ and } b P_i' a\right\}$ denote the set collecting all pairs of alternatives that are oppositely ranked across P_i and P_i'. Correspondingly, the Kemeny distance between P_i and P_i' equals $\left|\Gamma(P_i, P_i')\right|$. Henceforth, to avoid confusion, whenever we write $\{a, b\} \in \Gamma(P_i, P_i')$, we also presume $a P_i b$ and $b P_i' a$. First, the notion of *adjacency* links two distinct preferences with Kemeny distance 1, which thereby disagree on the relative ranking of *exactly* one pair of alternatives. The second notion, *adjacency*$^+$, is customized for two separable preferences which happen to disagree on the relative ranking of some pair of similar alternatives, and meanwhile have a “minimum” Kemeny distance.[22] To see why this is a natural notion of adjacency between two separable preferences, notice that if separable preferences P_i and P_i' disagree on the relative ranking of two similar alternatives, say $\{(a^s, z^{-s}), (b^s, z^{-s})\} \in \Gamma(P_i, P_i')$, then separability implies $(a^s, z^{-s}) P_i (b^s, z^{-s})$ and $(b^s, z^{-s}) P_i' (a^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. Hence, the minimum Kemeny distance between P_i and P_i' is $|A^{-s}|$, and can only be reached at $\Gamma(P_i, P_i') = \left\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\right\}_{z^{-s}\in A^{-s}}$.

Definition 5. Preferences P_i and P_i' are **adjacent**, denoted $P_i \sim P_i'$, if $\Gamma(P_i, P_i') = \left\{\{a, b\}\right\}$ for some $a, b \in A$. Preferences P_i and P_i' are **adjacent$^+$**, denoted $P_i \sim^+ P_i'$, if they are separable preferences, and $\Gamma(P_i, P_i') = \left\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\right\}_{z^{-s}\in A^{-s}}$ for some $s \in M$ and $a^s, b^s \in A^s$.

Observe that across two adjacent preferences P_i and P_i', the pair $\{a, b\}$ is *locally switched*, referred to as a **local switching pair**, while every other alternative is identically ranked. Hence, $a P_i !b$, $b P_i' !a$, and for every $c \notin \{a, b\}$, $c = r_k(P_i) = r_k(P_i')$ for some $1 \le k \le |A|$. Similarly, across two adjacent$^+$ preferences P_i and P_i', all pairs $\{(a^s, z^{-s}), (b^s, z^{-s})\}$, $z^{-s} \in A^{-s}$, are locally switched simultaneously, and every other alternative is identically ranked. Hence, $(a^s, z^{-s}) P_i !(b^s, z^{-s})$, $(b^s, z^{-s}) P_i' !(a^s, z^{-s})$ for all $z^{-s} \in A^{-s}$, and for every $c \in A$ with $c^s \notin \{a^s, b^s\}$, $c = r_k(P_i) = r_k(P_i')$ for some $1 \le k \le |A|$.

In an sd-strategy-proof RSCF, if one voter *unilaterally* changes her sincere preference to an adjacent or adjacent$^+$ preference, the probability associated to the alternative in a local switching pair whose ranking is lifted up from the sincere preference to the other one, might increase, while the sum of two probabilities in each local switching pair and the probability received by every alternative excluded from the local switching pair(s) remain fixed (see Lemma 8 of Appendix B).

[21] To show the necessity part, given $s \in M$, $a^s \in \langle x^s, b^s\rangle$ first implies $(a^s, x^{-s}) \in \langle x, (b^s, x^{-s})\rangle$, and then multidimensional single-peakedness implies $(a^s, x^{-s}) P_i (b^s, x^{-s})$. Furthermore, separability implies $a^s [P_i]^s b^s$. For the sufficiency part, given distinct $a, b \in A$ with $a \in \langle x, b\rangle$, we first know $a^s \in \langle x^s, b^s\rangle$ for every $s \in M$. Hence, by single-peakedness of the marginal preferences, we have $a^s [P_i]^s b^s$ for all $s \in M$ with $a^s \neq b^s$ which furthermore implies $a P_i b$ by separability, as required.

[22] See the coexistence of adjacency and adjacency$^+$ in the separable domain of Example 8 of Appendix E.2.

This makes the variation of the two corresponding social lotteries in the sd-strategy-proof RSCF more tractable.

Given two distinct preferences P_i and P_i', a sequence of preferences $\{P_i^k\}_{k=1}^t$, $t \geq 2$, which is required to contain no repetition, is referred to as a **path** connecting P_i and P_i' if for all $1 \leq k \leq t-1$, either $P_i^k \sim P_i^{k+1}$ or $P_i^k \sim^+ P_i^{k+1}$. This indicates that the differences between P_i and P_i' can be reconciled via a sequence of one-pair or multiple-pair local switchings. In particular, if every consecutive pair of preferences in a path is adjacent, this path is referred to as an **adjacency path**.

As the difference between two preferences may be reconciled via multiple paths, the length of the path matters. We impose two properties: the Interior$^+$ property and the Exterior$^+$ property, which ensure that for each pair of distinct preferences, a sufficiently short path in the domain can be used to reconcile the difference between the two. First, we partition the domain into several subdomains of preferences according to the preference peaks. The Interior$^+$ property is established on each subdomain, and requires two preferences in one subdomain be connected via a path in this subdomain. The Exterior$^+$ property imposes conditions on two preferences in two distinct subdomains. When these two preferences share the same relative ranking of a pair of alternatives, we can construct a path in the domain connecting them, and meanwhile preserve the relative ranking of this particular pair of alternatives along the path. In particular, when these two preferences have similar peaks, say (a^s, z^{-s}) and (b^s, z^{-s}), an additional condition is imposed so that the peak of each preference in the path lies in the set (A^s, z^{-s}).

Definition 6. Domain $\mathbb{D}$ satisfies the **Interior$^+$ property** if given distinct $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = r_1(P_i') \equiv a$, there exists a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}^a$ connecting P_i and P_i'.

Definition 7. Domain $\mathbb{D}$ satisfies the **Exterior$^+$ property** if given $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P_i')$, and $a, b \in A$ with $a P_i b$ and $a P_i' b$, there exists a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P_i' such that $a P_i^k b$ for all $1 \leq k \leq q$. In addition, when $r_1(P_i)$ and $r_1(P_i')$ are similar, say $r_1(P_i) = (a^s, z^{-s})$ and $r_1(P_i') = (b^s, z^{-s})$, the path $\{P_i^k\}_{k=1}^q$ satisfies the **no-detour property**, i.e., $r_1(P_i^k) \in (A^s, z^{-s})$ for all $1 \leq k \leq q$.[23]

Throughout the paper, a multidimensional domain satisfying the Interior$^+$ property and the Exterior$^+$ property is referred to as a **connected$^+$ domain**.

Remark 2. The top-separable domain, the separable domain, multidimensional single-peaked domains, their intersection and unions are all included in the class of connected$^+$ domains. The detailed verifications are available in Appendices E.2 - E.6. Drawing on recent work (Chatterji et al., 2018), we introduce a new multidimensional domain by generalizing their preference restriction, *eventual-single-peakedness*, from the one-dimensional setting to the multidimensional setting. This *multidimensional eventually-single-peaked domain* can be applied to models that seek to allocate multiple public facilities (e.g., Bochet and Gordon, 2012). The details of this formulation are available in Appendix E.7. The lexicographically separable domain of Chatterji et al. (2012) fails connectedness$^+$ due to the non-existence of preferences delivering

[23] Appendix E.1 provides an example of top-separable domain which violates the no-detour property but satisfies the Interior$^+$ property and the remainder of the Exterior$^+$ property.

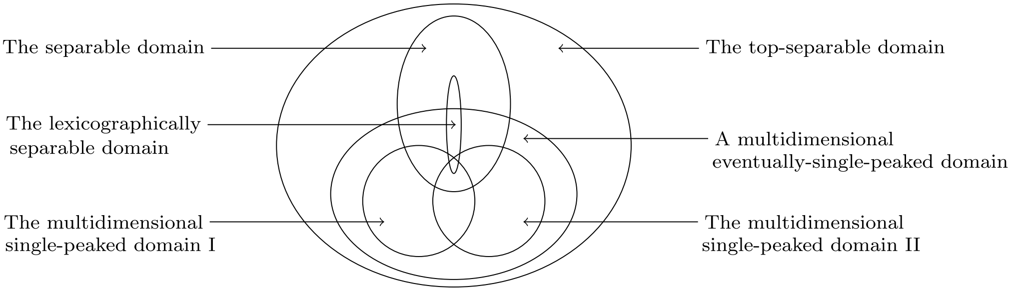

Fig. 1. The relations among several multidimensional domains.

adjacency.[24] We use Fig. 1 to summarize the relations among these multidimensional domains. The class of connected$^+$ domains also excludes domains studied in the one-dimensional setting (e.g., Gibbard, 1977; Moulin, 1980; Demange, 1982; Saporiti, 2009).[25] In particular, if we disregard the no-detour property, our Interior$^+$ and Exterior$^+$ properties generalize the connected domains of Sato (2013) which only adopt adjacency paths. $\square$

We next turn to an important property of unanimous and sd-strategy-proof RSCFs on connected$^+$ domains which plays a critical role in the subsequent analysis: The social lottery at every preference profile depends only on voters' preference peaks. We say that such an RSCF satisfies the tops-only property. Formally, an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies **the tops-only property** if for all $P, P' \in \mathbb{D}^N$, we have $[r_1(P_i) = r_1(P'_i)$ for all $i \in I] \Rightarrow [\varphi(P) = \varphi(P')]$.

Proposition 2. *Every unanimous and sd-strategy-proof RSCF on a connected$^+$ domain satisfies the tops-only property.*

The proof of Proposition 2 is available in Appendix B.

Remark 3. We add the superscript "+" to highlight the role of adjacency$^+$ in our two properties, and distinguish our two properties from the Interior and Exterior properties of Chatterji and Zeng (2018) which also endogenize the tops-only property in all unanimous and sd-strategy-proof RSCFs. The connected$^+$ domains considered here fail to satisfy their Interior and Exterior properties: The Interior property only adopts adjacency paths, and cannot be applied to the separable domain (see Example 8 of Appendix E.2), while the Exterior property is significantly weaker than the Exterior$^+$ property as it is defined by using the notion of *isolation* which is weaker than both adjacency and adjacency$^+$. The verification of Proposition 2 is similar to the proof of the Theorem of Chatterji and Zeng (2018), but requires an additional step (Lemma 11 of

[24] A separable preference P_i is **lexicographically separable** if there exists a *lexicographic order* (a linear order) $\succ$ over M such that for all $x, y \in A$, we have $\left[x^s [P_i]^s y^s \text{ and } x^\tau = y^\tau \text{ for all } \tau \in M \text{ with } \tau \succ s\right] \Rightarrow [x\, P_i\, y]$. Let $\mathbb{D}_{\text{LS}}$ denote **the lexicographically separable domain** which contains all lexicographically separable preferences. Evidently, $\mathbb{D}_{\text{LS}} \subseteq \mathbb{D}_{\text{S}}$. In the lexicographically separable domain, we know that (i) there exists no pair of adjacent preferences when $|M| > 2$ or $|A^s| > 2$ for some $s \in M$, and (ii) even though adjacency$^+$ exists, every pair of adjacent$^+$ preferences shares the same lexicographic order over M. Therefore, the difference of two lexicographically separable preferences which have two distinct lexicographic orders can never be reconciled via a path in the lexicographically separable domain.

[25] The complete domain satisfies both the Interior$^+$ and Exterior$^+$ properties, but fails top-separability, while the single-peaked domain and the single-crossing domain violate both top-separability and the no-detour property.

Appendix B) that specifically applies to adjacent$^+$ preferences. Finally, we note that Proposition 2 still holds even when the no-detour property fails. □

We believe that Proposition 2 is of some independent interest for the study of RSCFs. For instance, we use Proposition 2 to generalize an existing characterization result of generalized random dictatorships on all connected$^+$ supersets of the lexicographically separable domain (recall footnote 24), like the separable domain and the top-separable domain.

Corollary 1. *Let $|A^s| \geq 3$ for each $s \in M$, and $\mathbb{D}$ be a connected$^+$ domain that includes the lexicographically separable domain. A unanimous RSCF is sd-strategy-proof if and only if it is a generalized random dictatorship.*

Proof. The sufficiency part is implied by Proposition 1. We show the necessity part. First, recall Theorem 3 of Chatterji et al. (2012) which shows that every unanimous and sd-strategy-proof RSCF on the lexicographically separable domain is a generalized random dictatorship. Next, by Proposition 2, we know that every unanimous and sd-strategy-proof RSCF on $\mathbb{D}$ satisfies the tops-only property. Last, since the lexicographically separable domain is included in $\mathbb{D}$, the tops-only property implies that every unanimous and sd-strategy-proof RSCF on $\mathbb{D}$ is a generalized random dictatorship. □

We conclude this section with an important implication of Proposition 2. Since the tops-only property emerges endogenously, every unanimous and sd-strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ degenerates to a *random voting rule* $\varphi : A^N \to \Delta(A)$. We hence simplify the notation of a preference profile $(P_1, \ldots, P_N)$ to $(x_1, \ldots, x_N)$, where $r_1(P_i) = x_i$, $i = 1, \ldots, N$. We also mix the notation of alternatives and preferences, e.g., (a, P_j) represents a two-voter preference profile where voter i's preference peak is a and voter j's preference is P_j. More importantly, due to the tops-only property, we henceforth can simply focus on the peak alternatives in each pair of adjacent$^+$ preferences which disagree on peaks. Accordingly, we induce an adjacency$^+$ relation between alternatives from the adjacency$^+$ relation between preferences. We say that a pair of distinct *alternatives* $a, b \in A$ is **adjacent$^+$**, denoted $a \sim^+ b$, if there exist $P_i \in \mathbb{D}^a$ and $P'_i \in \mathbb{D}^b$ such that $P_i \sim^+ P'_i$. Given distinct $a, b \in A$, let $\{x_k\}_{k=1}^t$ denote an **adjacent$^+$ sequence** (of alternatives) connecting a and b such that $x_1 = a$, $x_t = b$ and $x_k \sim^+ x_{k+1}$ for all $k = 1, \ldots, t-1$. Similar to the definition of a path, we do not allow repetition in an adjacent$^+$ sequence. Consequently, we can specify a geometric relation on all alternatives that will be useful in the subsequent analysis.

2.3. *The compromise property*

Random dictatorships never admit compromise as probabilities are assigned only to the peak alternatives at every preference profile. Generalized random dictatorships improve upon random dictatorships in this respect by diversifying social lotteries. However, they do not *systematically* admit compromise, and only recognize the "compromise alternative" at a preference profile which can be assembled via some voter sequence. For instance, let $A = A^1 \times A^2$, $A^1 = \{a^1, b^1, c^1\}$ and $A^2 = \{a^2, b^2\}$. Two voters may disagree on each other's most preferred alternatives but may nonetheless have a common second best alternative which is naturally viewed as the compromise alternative. Given two groups of three alternatives (1) $x \equiv (a^1, a^2)$, $y \equiv (b^1, b^2)$ and $z \equiv (a^1, b^2)$, and (2) $x' \equiv (a^1, a^2)$, $y' \equiv (b^1, a^2)$ and $z' \equiv (c^1, a^2)$, we identify two profiles of top-separable preferences: (1) (P_i, P_j) where $r_1(P_i) = x$, $r_1(P_j) = y$ and

$r_2(P_i) = r_2(P_j) = z$, and (2) (P'_i, P'_j) where $r_1(P'_i) = x'$, $r_1(P'_j) = y'$ and $r_2(P'_i) = r_2(P'_j) = z'$. First, the compromise alternatives z and z' receive zero probability at (P_i, P_j) and (P'_i, P'_j) in a random dictatorship. Second, consider a generalized random dictatorship φ where each voter sequence is associated to a strictly positive weight. At profile (P_i, P_j), since z can be assembled by the voter sequence (i, j), it is recognized by φ, i.e., $\varphi_z(P_i, P_j) > 0$. However, the compromise alternative z' cannot be assembled by any voter sequence at (P'_i, P'_j), and therefore receives zero probability, i.e., $\varphi_{z'}(P'_i, P'_j) = 0$.

We are interested in identifying a class of unanimous and sd-strategy-proof RSCFs which differ from random dictatorships/generalized random dictatorships in a "minimal" but significant degree by systematically admitting compromise. Our formulation of the compromise property guarantees that a non-assemblable compromise alternative arising out of a particular preference profile must get strictly positive probability in the social lottery. First, we pick two preferences P_i and P_j with similar peaks, say (x^s, a^{-s}) and (y^s, a^{-s}), and with a common second best alternative which is also similar to *both* peaks, say (z^s, a^{-s}). We treat (z^s, a^{-s}) as a natural compromise alternative. Next, we consider a preference profile where the voters are separated into two groups of approximately equal size, the voters of the first group have preference P_i and the remaining voters have preference P_j. Our compromise property insists that the RSCF assign a strictly positive probability to the compromise alternative in every such situation. Note that our compromise property is formulated on profiles where the two peaks and the compromise alternative are pairwise similar; this makes our version of compromise weaker than the version introduced by Chatterji et al. (2016) which considers profiles where this pairwise similarity is not required.

Definition 8. An RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies **the compromise property** if there exists $\hat{I} \subseteq I$ with $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd, such that given $P_i, P_j \in \mathbb{D}$, we have[26]

$$\begin{bmatrix} r_1(P_i) \equiv (x^s, a^{-s}) \neq (y^s, a^{-s}) \equiv r_1(P_j) \text{ and} \\ r_2(P_i) = r_2(P_j) \equiv (z^s, a^{-s}) \text{ where } z^s \notin \{x^s, y^s\} \end{bmatrix} \Rightarrow \left[\varphi_{(z^s, a^{-s})}\Big(\frac{P_i}{\hat{I}}, \frac{P_j}{I \backslash \hat{I}}\Big) > 0 \right].$$

Remark 4. The compromise effect has been widely studied in the literature on bargaining theory (e.g., Kıbrıs and Sertel, 2007) and choice theory (e.g., De Clippel and Eliaz, 2012). Specifically, in the two-agent bargaining model studied by both Kıbrıs and Sertel (2007) and De Clippel and Eliaz (2012), given a bargaining problem (B, P_1, P_2), where $B \subseteq A$ contains at least three elements, if $x P_1 z P_1 y$ and $y P_2 z P_2 x$, then z is recognized as a compromise for this bargaining problem, and the compromise effect requires that either x or y be excluded from the bargaining solution. They then show that the *fallback bargaining solution* is uniquely characterized by the compromise effect in conjunction with other bargaining axioms. Börgers (1991) studied the game-form mechanism which consists of all agents' strategy spaces and a consequence function associating each strategy profile to an alternative, and restricted attention to the solution concept of undominated strategies in each induced normal-form game. Börgers introduced a stronger compromise notion which at each preference profile captures all non-peak and Pareto efficient alternatives, established a compromise effect which requires that at some preference profile, a

[26] The notation $\big(\frac{P_i}{\hat{I}}, \frac{P_j}{I \backslash \hat{I}}\big)$ denotes a preference profile where all voters of $\hat{I}$ report preference P_i, while all voters not in $\hat{I}$ report preference P_j.

compromise alternative is enforced at each profile of undominated strategies, but showed that every game-form mechanism that delivers Pareto efficient alternative at each profile of undominated strategies (relative to the preference profile) never enforces compromise. Our compromise alternative notion is relatively weaker as we only concern ourselves with the non-assemblable and commonly second ranked alternative at a preference profile. Our compromise property differs from the two alluded compromise effects as we always include the compromise alternative in the social lottery with a strictly positive probability. □

3. Main results

We ask what multidimensional domains admit unanimous and sd-strategy-proof RSCFs satisfying the compromise property. In this section, we show that the existence of a unanimous and sd-strategy-proof RSCF satisfying the compromise property on a connected$^+$ domain implies that all preferences must be multidimensional single-peaked, and conversely, we construct a particular RSCF, a mixed multidimensional projection rule, satisfying unanimity, sd-strategy-proofness and the compromise property on a multidimensional single-peaked domain. Next, we switch our model to the deterministic setting, and establish an analogous characterization result by replacing the compromise property by anonymity. Finally, we generalize our analysis to the case of voting under constraints where some alternatives are not feasible in social lotteries.

3.1. A characterization of multidimensional single-peaked preferences

In this section, we prove that if a minimally rich and connected$^+$ domain admits an RSCF which satisfies the aforementioned properties: unanimity, sd-strategy-proofness and the compromise property, then it is a multidimensional single-peaked domain. Furthermore, we use several counter examples to illustrate the indispensability of each axiom and domain condition in our characterization result.

Now, we formally state the main result.

Theorem 1. *Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. If it admits a unanimous and sd-strategy-proof RSCF satisfying the compromise property, it is multidimensional single-peaked. Conversely, a multidimensional single-peaked domain admits a unanimous and sd-strategy-proof RSCF satisfying the compromise property.*

Proof. We start from the verification of the necessity part. If $|A^s| = 2$ for each $s \in M$, top-separability implies $\mathbb{D} \subseteq \mathbb{D}_{\text{TS}} = \mathbb{D}_{\text{S}}$. For each $s \in M$, since $|A^s| = 2$, we naturally construct a line $G(A^s)$ to connect the two elements of A^s. Thus, we assemble a product of lines $\times_{s\in M} G(A^s)$, and generate the multidimensional single-peaked domain $\mathbb{D}_{\text{MSP}}$. Since $\mathbb{D}_{\text{S}} = \mathbb{D}_{\text{MSP}}$ when $|A^s| = 2$ for each $s \in M$, we have $\mathbb{D} \subseteq \mathbb{D}_{\text{MSP}}$. Henceforth, we assume $|A^s| > 2$ for some $s \in M$. Let $\phi : \mathbb{D}^N \to \Delta(A)$ be a unanimous and sd-strategy-proof RSCF satisfying the compromise property. First, Proposition 2 implies that ϕ satisfies the tops-only property. Since ϕ satisfies the compromise property, we accordingly separate voters into two groups $\hat{I}$ and $I\backslash\hat{I}$ such that $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd. We induce a two-voter RSCF: $\varphi(P_i, P_j) = \phi\big(\frac{P_i}{\hat{I}}, \frac{P_j}{I\backslash\hat{I}}\big)$ for all $P_i, P_j \in \mathbb{D}$. It is easy to verify that φ is unanimous, tops-only and sd-strategy-proof, and satisfies the compromise property.

In Lemma 1 below, we show that no two preferences with similar peaks are complete reversals, and induce that every pair of similar alternatives (a^s, x^{-s}) and (b^s, x^{-s}) is connected via an adjacent$^+$ sequence in (A^s, x^{-s}).

Lemma 1. *Given $s \in M$, $a^s, b^s \in A^s$ and $x^{-s} \in A^{-s}$, there exists an adjacent$^+$ sequence $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ connecting (a^s, x^{-s}) and (b^s, x^{-s}).*

Proof. Since $\mathbb{D}^{(a^s,x^{-s})} \neq \emptyset$ and $\mathbb{D}^{(b^s,x^{-s})} \neq \emptyset$ by minimal richness, there are two situations: (i) There exist $P_i \in \mathbb{D}^{(a^s,x^{-s})}$ and $P_i' \in \mathbb{D}^{(b^s,x^{-s})}$ such that they agree on the relative ranking of some pair of alternatives, and (ii) both $\mathbb{D}^{(a^s,x^{-s})}$ and $\mathbb{D}^{(b^s,x^{-s})}$ are singleton sets, say $\mathbb{D}^{(a^s,x^{-s})} = \{P_i\}$ and $\mathbb{D}^{(b^s,x^{-s})} = \{P_i'\}$, and P_i and P_i' are complete reversals.

In the first situation, the no-detour property implies that there exists a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P_i' such that $r_1(P_i^k) \in (A^s, x^{-s})$ for all $1 \leq k \leq q$. We partition the path $\{P_i^k\}_{k=1}^q$ (without any rearrangement) according to the preference peaks, and then elicit a sequence of preference peaks which starts from (a^s, x^{-s}) and ends at (b^s, x^{-s}):

$$\left\{\underbrace{P_i \equiv P_i^1, \ldots, P_i^{k_1}}_{\text{the same peak } x_1}, \underbrace{P_i^{k_1+1}, \ldots, P_i^{k_2}}_{\text{the same peak } x_2}, \ldots, \underbrace{P_i^{k_{t-1}+1}, \ldots, P_i^{k_t} \equiv P_i'}_{\text{the same peak } x_t}\right\}$$
$$\xrightarrow{\text{Elicit peaks}} \{x_1, x_2, \ldots, x_t\}.$$

Note that the sequence $\{x_1, x_2, \ldots\ldots, x_t\}$ may contain repetition, and every pair of consecutive alternatives is distinct. First, since two preferences with distinct peaks are never adjacent by Lemma 9 of Appendix B, we know that each consecutive pair of $\{x_1, x_2, \ldots\ldots, x_t\}$ is adjacent$^+$, i.e., $x_k \sim^+ x_{k+1}$ for all $k = 1, \ldots, t-1$. Second, whenever a repetition of one alternative appears in the sequence, we remove all alternatives strictly between the repetition and one alternative of the repetition. For instance, if $x_k = x_l$ where $1 \leq k < l \leq t$, we remove $x_k, x_{k+1}, \ldots, x_{l-1}$, and refine the sequence to $\{x_1, \ldots, x_{k-1}, x_l, \ldots, x_t\}$. Thus, by repeatedly eliminating repetitions, we finally elicit an adjacent$^+$ sequence $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ such that $x_1 = (a^s, x^{-s})$, $x_q = (b^s, x^{-s})$ and $x_k \sim^+ x_{k+1}$ for all $k = 1, \ldots, q-1$.

We next show that the second situation is invalid. Suppose that the second situation occurs. Consequently, (b^s, x^{-s}) must be bottom ranked in P_i. Pick arbitrary $\tau \in M \backslash \{s\}$ and $z^\tau \in A^\tau \backslash \{x^\tau\}$, and assemble the alternative $(b^s, z^\tau, x^{-\{s,\tau\}})$. Since $r_1(P_i) = (a^s, x^{-s}) = (a^s, x^\tau, x^{-\{s,\tau\}})$, top-separability implies $(b^s, x^\tau, x^{-\{s,\tau\}}) P_i (b^s, z^\tau, x^{-\{s,\tau\}})$, which contradicts the hypothesis that (b^s, x^{-s}) is the worst alternative in P_i. $\square$

Lemma 2. *Given $s \in M$ and $x^{-s} \in A^{-s}$, let $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$, $q > 2$, be an adjacent$^+$ sequence. There exist $0 \leq \alpha_1 < \cdots < \alpha_{q-1} \leq 1$ such that for all $1 \leq k < k' \leq q$, we have*

$$\varphi(x_k, x_{k'}) = \alpha_k e_{x_k} + \sum\nolimits_{l=k+1}^{k'-1} (\alpha_l - \alpha_{l-1}) e_{x_l} + (1 - \alpha_{k'-1}) e_{x_{k'}}.$$

Moreover, for every $P_i \in \mathbb{D}^{x_1}$, we have $x_k P_i x_{k+1}$ for all $k = 1, \ldots, q-1$.

Proof. Given $1 \leq k \leq q-1$, since $x_k \sim^+ x_{k+1}$, we have $P_i \in \mathbb{D}^{x_k}$ and $P_j \in \mathbb{D}^{x_{k+1}}$ with $P_i \sim^+ P_j$. Thus, $r_1(P_i) = r_2(P_j) = x_k$ and $r_2(P_i) = r_1(P_j) = x_{k+1}$. Then, by tops-onlyness, item 2(ii) of Lemma 8 of Appendix B and unanimity, we have $\varphi_{x_k}(x_k, x_{k+1}) + \varphi_{x_{k+1}}(x_k, x_{k+1}) = \varphi_{x_k}(P_i, P_j) + \varphi_{x_{k+1}}(P_i, P_j) = \varphi_{x_k}(P_i, P_i) + \varphi_{x_{k+1}}(P_i, P_i) = \varphi_{x_k}(P_i, P_i) = 1$. Let $\varphi_{x_k}(x_k, x_{k+1}) =$

α_k and $\varphi_{x_{k+1}}(x_k, x_{k+1}) = 1 - \alpha_k$ where $0 \le \alpha_k \le 1$. Thus, $\varphi(x_k, x_{k+1}) = \alpha_k e_{x_k} + (1-\alpha_k) e_{x_{k+1}}$. We adopt an induction argument.

Induction Hypothesis: Given $l \ge 2$, for all $1 \le k < k' \le q$ with $0 < k' - k < l$, we have $\varphi(x_k, x_{k'}) = \alpha_k e_{x_k} + \sum_{\nu=k+1}^{k'-1}(\alpha_\nu - \alpha_{\nu-1}) e_{x_\nu} + (1-\alpha_{k'-1}) e_{x_{k'}}$.

Let $k' - k = l$. We show $\varphi(x_k, x_{k'}) = \alpha_k e_{x_k} + \sum_{\nu=k+1}^{k'-1}(\alpha_\nu - \alpha_{\nu-1}) e_{x_\nu} + (1-\alpha_{k'-1}) e_{x_{k'}}$. Since $x_k \sim^+ x_{k+1}$, we have $P_i \in \mathbb{D}^{x_k}$ and $P_i' \in \mathbb{D}^{x_{k+1}}$ with $P_i \sim^+ P_i'$. Then, by the induction hypothesis and items 2(ii) and 2(iii) of Lemma 8 of Appendix B, the following three equalities hold:

(i) $\varphi_{x_k}(P_i, x_{k'}) + \varphi_{x_{k+1}}(P_i, x_{k'}) = \varphi_{x_k}(P_i', x_{k'}) + \varphi_{x_{k+1}}(P_i', x_{k'}) = \alpha_{k+1}$,
(ii) $\varphi_{x_\nu}(P_i, x_{k'}) = \varphi_{x_\nu}(P_i', x_{k'}) = \alpha_\nu - \alpha_{\nu-1}$ for all $\nu = k+2, \dots, k'-1$, and
(iii) $\varphi_{x_{k'}}(P_i, x_{k'}) = \varphi_{x_{k'}}(P_i', x_{k'}) = 1 - \alpha_{k'-1}$.

Similarly, since $x_{k'} \sim^+ x_{k'-1}$, we have $P_j \in \mathbb{D}^{x_{k'}}$ and $P_j' \in \mathbb{D}^{x_{k'-1}}$ such that $P_j \sim^+ P_j'$. Then, item 2(iii) of Lemma 8 and the induction hypothesis imply $\varphi_{x_k}(x_k, P_j) = \varphi_{x_k}(x_k, P_j') = \alpha_k$. Thus, $\varphi_{x_{k+1}}(x_k, x_{k'}) = \varphi_{x_k}(P_i, x_{k'}) + \varphi_{x_{k+1}}(P_i, x_{k'}) - \varphi_{x_k}(x_k, P_j) = \alpha_{k+1} - \alpha_k$. Therefore, $\varphi(x_k, x_{k'}) = \alpha_k e_{x_k} + \sum_{\nu=k+1}^{k'-1}(\alpha_\nu - \alpha_{\nu-1}) e_{x_\nu} + (1-\alpha_{k'-1}) e_{x_{k'}}$. This completes the verification of the induction hypothesis.

Next, we show $\alpha_k < \alpha_{k+1}$ for all $k = 1, \dots, q-2$. Given $1 \le k \le q-2$, since $x_k \sim^+ x_{k+1}$ and $x_{k+1} \sim^+ x_{k+2}$, we have $P_i \in \mathbb{D}^{x_k}$ and $P_j \in \mathbb{D}^{x_{k+2}}$ with $r_2(P_i) = r_2(P_j) = x_{k+1}$ and $x_k, x_{k+2}, x_{k+1} \in (A^s, x^{-s})$. Then, the compromise property implies $\alpha_{k+1} - \alpha_k = \varphi_{x_{k+1}}(P_i, P_j) > 0$.

Last, given $P_i \in \mathbb{D}^{x_1}$, we show $x_k P_i x_{k+1}$ for all $k = 1, \dots, q-1$. Given $1 \le k \le q-1$, suppose $x_{k+1} P_i x_k$. Evidently, $1 < k < q$. At the profile (P_i, x_{k+1}), we have $\varphi_{x_k}(P_i, x_{k+1}) = \alpha_k - \alpha_{k-1} > 0$. Assume $x_{k+1} = r_\eta(P_i)$ for some $1 < \eta < |A|$. We then have $\sum_{t=1}^{\eta} \varphi_{r_t(P_i)}(P_i, x_{k+1}) \le 1 - \varphi_{x_k}(P_i, x_{k+1}) < 1 = \varphi_{x_{k+1}}(x_{k+1}, x_{k+1}) = \sum_{t=1}^{\eta} \varphi_{r_t(P_i)}(x_{k+1}, x_{k+1})$. Consequently, voter i will manipulate at (P_i, x_{k+1}) via a preference with peak x_{k+1}. Therefore, $x_k P_i x_{k+1}$ for all $k = 1, \dots, q-1$. $\square$

Given $s \in M$ and $x^{-s} \in A^{-s}$, we induce a graph $G_{\sim^+}\big((A^s, x^{-s})\big)$ where (A^s, x^{-s}) is the set of vertices, and two distinct alternatives form an edge if and only if they are adjacent$^+$.

Lemma 3. *Given $s \in M$ and $x^{-s} \in A^{-s}$, $G_{\sim^+}\big((A^s, x^{-s})\big)$ is a tree.*

Proof. By Lemma 1, we first know that every pair of distinct alternatives of (A^s, x^{-s}) is connected via an adjacent$^+$ sequence of (A^s, x^{-s}). Suppose that $G_{\sim^+}\big((A^s, x^{-s})\big)$ is not a tree. Then, there must exist a cycle $\{x_k\}_{k=1}^t \subseteq (A^s, x^{-s})$, $t \ge 3$, such that $x_k \sim^+ x_{k+1}$ for all $k = 1, \dots, t$, where $x_{t+1} = x_1$. According to the sequence $\{x_k\}_{k=1}^t$, Lemma 2 implies $\varphi_{x_1}(x_1, x_t) + \varphi_{x_t}(x_1, x_t) < 1$. However, $x_1 \sim^+ x_t$ implies $\varphi_{x_1}(x_1, x_t) + \varphi_{x_t}(x_1, x_t) = 1$. Contradiction! $\square$

We are going to show that two trees $G_{\sim^+}\big((A^s, x^{-s})\big)$ and $G_{\sim^+}\big((A^s, y^{-s})\big)$ are "identical" in the sense that for all $a^s, b^s \in A^s$, (a^s, x^{-s}) and (b^s, x^{-s}) form an edge in $G_{\sim^+}\big((A^s, x^{-s})\big)$ if and only if (a^s, y^{-s}) and (b^s, y^{-s}) form an edge in $G_{\sim^+}\big((A^s, y^{-s})\big)$. With this result, we can generate a tree $G(A^s)$ on the component set A^s.

For the next lemma, we fix the following four alternatives: $a = (x^s, x^\tau, z^{-\{s,\tau\}})$, $b = (y^s, y^\tau, z^{-\{s,\tau\}})$, $c = (x^s, y^\tau, z^{-\{s,\tau\}})$ and $d = (y^s, x^\tau, z^{-\{s,\tau\}})$ where $x^s \neq y^s$ and $x^\tau \neq y^\tau$.

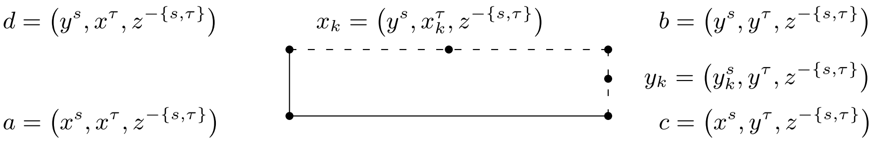

Fig. 2. The geometric relations among a, b, c and d. The dash line represents an adjacent$^+$ sequence connecting two alternatives.

Lemma 4. *If $a \sim^+ c$ and $a \sim^+ d$, then $b \sim^+ c$ and $b \sim^+ d$.*

Proof. Since $b, d \in (y^s, A^s, z^{-\{s,\tau\}})$ and $b, c \in (A^s, y^\tau, z^{-\{s,\tau\}})$, Lemma 3 implies that there exists a unique adjacent$^+$ sequence $\{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$ connecting b and d, and a unique adjacent$^+$ sequence $\{y_k\}_{k=1}^q \subseteq (A^s, y^\tau, z^{-\{s,\tau\}})$ connecting b and c. We use Fig. 2 to illustrate the geometric relations among a, b, c and d.

To verify this lemma, we show $q = 2$ and $p = 2$ (equivalently, $b \sim^+ c$ and $b \sim^+ d$). Suppose not, i.e., either $q > 2$ or $p > 2$. Assume w.l.o.g. that $q > 2$. The verification related to $p > 2$ is symmetric, and we hence omit it. Thus, $y_2 \equiv (y_2^s, y^\tau, z^{-\{s,\tau\}})$, $y_2 \notin \{b, c\}$ and $y_2^s \notin \{x^s, y^s\}$.

Since $a \sim^+ c$, we have $P_i \in \mathbb{D}^a$ and $P_i' \in \mathbb{D}^c$ with $P_i \sim^+ P_i'$. According to $\{y_k\}_{k=1}^q$, $\varphi_{y_2}(P_i', b) = \varphi_{y_2}(y_q, y_1) > 0$ by Lemma 2. Let $z_2 \equiv (y_2^s, x^\tau, z^{-\{s,\tau\}})$. Thus, $\{z_2, y_2\}$ is a local switching pair of P_i and P_i', and hence item 2(ii) of Lemma 8 of Appendix B implies $\varphi_{z_2}(P_i, b) + \varphi_{y_2}(P_i, b) = \varphi_{z_2}(P_i', b) + \varphi_{y_2}(P_i', b) > 0$. On the other hand, since $a \sim^+ d$, we have $\bar{P}_i \in \mathbb{D}^a$ and $\bar{P}_i' \in \mathbb{D}^d$ with $\bar{P}_i \sim^+ \bar{P}_i'$. Since $y_2, z_2 \notin \{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$, Lemma 2 implies $\varphi_{y_2}(\bar{P}_i', b) = \varphi_{y_2}(x_p, x_1) = 0$ and $\varphi_{z_2}(\bar{P}_i', b) = \varphi_{z_2}(x_p, x_1) = 0$. Furthermore, since $y_2, z_2 \notin (x^s, A^{-s}) \cup (y^s, A^{-s})$, item 2(iii) of Lemma 8 of Appendix B implies $\varphi_{y_2}(\bar{P}_i, b) = \varphi_{y_2}(\bar{P}_i', b) = 0$ and $\varphi_{z_2}(\bar{P}_i, b) = \varphi_{z_2}(\bar{P}_i', b) = 0$. Thus, $\varphi_{z_2}(\bar{P}_i, b) + \varphi_{y_2}(\bar{P}_i, b) = 0$. Consequently, $\varphi(P_i, b) \neq \varphi(\bar{P}_i, b)$ which contradicts the tops-only property. Therefore, $q = 2$. By a similar argument, $p = 2$. $\square$

Lemma 5. *Given $s \in M$ and $a^s, b^s \in A^s$, if $(a^s, x^{-s}) \sim^+ (b^s, x^{-s})$ for some $x^{-s} \in A^{-s}$, then $(a^s, y^{-s}) \sim^+ (b^s, y^{-s})$ for all $y^{-s} \in A^{-s}$.*

Proof. Given $y^{-s} \in A^{-s} \backslash \{x^{-s}\}$ and $\tau \in M \backslash \{s\}$ with $x^\tau \neq y^\tau$, we show $(a^s, y^\tau, x^{-\{s,\tau\}}) \sim^+ (b^s, y^\tau, x^{-\{s,\tau\}})$. By switching $x^{-\{s,\tau\}}$ to $y^{-\{s,\tau\}}$ component by component and applying the symmetric argument, we can complete the verification of the lemma.

Since $G_{\sim^+}\big((a^s, A^\tau, x^{-\{s,\tau\}})\big)$ is a tree, there exists a unique adjacent$^+$ sequence $\{a_k\}_{k=1}^q \subseteq (a^s, A^\tau, x^{-\{s,\tau\}})$ such that $a_1 = (a^s, x^\tau, x^{-\{s,\tau\}})$, $a_q = (a^s, y^\tau, x^{-\{s,\tau\}})$ and $a_k \sim^+ a_{k+1}$ for all $k = 1, \ldots, q-1$. For each $k = 1, \ldots, q-1$, we switch the element a^s of the alternative a_k to b^s, and construct an alternative $b_k = (b^s, a_k^\tau, x^{-\{s,\tau\}})$. Thus, we have a sequence $\{b_k\}_{k=1}^q \subseteq (b^s, A^\tau, x^{-\{s,\tau\}})$, $b_1 = (b^s, x^\tau, x^{-\{s,\tau\}})$ and $b_q = (b^s, y^\tau, x^{-\{s,\tau\}})$ (see Fig. 3(1)). Note that $\{b_k\}_{k=1}^q$ is *not* necessarily an adjacent$^+$ sequence.

Since $a_1 = (a^s, x^\tau, x^{-\{s,\tau\}}) = (a^s, x^{-s}) \sim^+ (b^s, x^{-s}) = (b^s, x^\tau, x^{-\{s,\tau\}}) = b_1$ by the hypothesis, and $a_1 \sim^+ a_2$, we note that $\{a_1, b_2, b_1, a_2\}$ are analogous to $\{a, b, c, d\}$ of Lemma 4. Hence, Lemma 4 implies $b_2 \sim^+ b_1$ and $b_2 \sim^+ a_2$ (see Fig. 3(2)). Following the adjacent$^+$ sequence $\{a_k\}_{k=1}^t$ and repeatedly applying Lemma 4, we have $b_k \sim^+ b_{k-1}$ and $b_k \sim^+ a_k$ for all $k = 2, \ldots, q$ (see Fig. 3(3)). Eventually, we have $(a^s, y^\tau, x^{-\{s,\tau\}}) = a_q \sim^+ b_q = (b^s, y^\tau, x^{-\{s,\tau\}})$, as required. $\square$

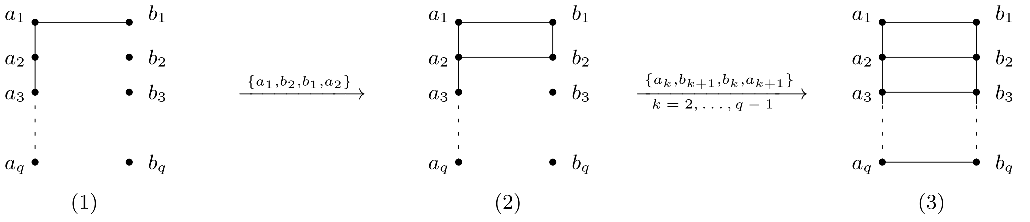

Fig. 3. The graphic illustration of the proof of Lemma 5.

Now, by Lemma 5, we know that for all $s \in M$ and $x^{-s}, y^{-s} \in A^{-s}$, both trees $G_{\sim^+}\big((A^s, x^{-s})\big)$ and $G_{\sim^+}\big((A^s, y^{-s})\big)$ induced by Lemma 3 coincide: $[(a^s, x^{-s}) \sim^+ (b^s, x^{-s})] \Leftrightarrow [(a^s, y^{-s}) \sim^+ (b^s, y^{-s})]$. Therefore, for each $s \in M$, we induce a tree over A^s, denoted $G(A^s)$, such that $a^s, b^s \in A^s$ form an edge in $G(A^s)$ if and only if $(a^s, x^{-s}) \sim^+ (b^s, x^{-s})$ for *all* $x^{-s} \in A^{-s}$. Then, combining all these trees, we have a product of trees $\times_{s\in M} G(A^s)$. Thus, we know that in the product of trees $\times_{s\in M} G(A^s)$, a pair of distinct alternatives a and b forms an edge if and only if they are similar, i.e., $a^{-s} = b^{-s}$ for some $s \in M$, and a^s and b^s form an edge in $G(A^s)$.

Lemma 6. *Given a separable preference $P_i \in \mathbb{D}$, it is multidimensional single-peaked on $\times_{s\in M} G(A^s)$.*

Proof. Assume $r_1(P_i) = a \equiv (a^s)_{s\in M}$. To verify this lemma, it suffices to show that for every $s \in M$, the marginal preference $[P_i]^s$ is single-peaked on the tree $G(A^s)$ (recall Remark 1).

Given $s \in M$ and distinct $x^s, y^s \in A^s$ such that $x^s \in \langle a^s, y^s\rangle$, we show $x^s [P_i]^s y^s$. If $x^s = a^s$, the result holds evidently. Assume $x^s \neq a^s$. Let $\langle a^s, y^s\rangle = \{x_k^s\}_{k=1}^q$ where $x_1^s = a^s$ and $x_q^s = y^s$. Thus, $x^s = x_l^s$ for some $1 < l < q$. Let $x_k = (x_k^s, a^{-s})$ for all $k = 1, \ldots, q$. Thus, $x_1 = a = r_1(P_i)$ and $\{x_k\}_{k=1}^q \subseteq (A^s, a^{-s})$ is an adjacent$^+$ sequence. Then, Lemma 2 implies $(x^s, a^{-s}) = x_l P_i x_q = (y^s, a^{-s})$. Furthermore, by separability, we have $x^s [P_i]^s y^s$, as required. Therefore, P_i is multidimensional single-peaked on $\times_{s\in M} G(A^s)$. $\square$

Lemma 7. *Domain $\mathbb{D}$ is multidimensional single-peaked on $\times_{s\in M} G(A^s)$.*

Proof. Given $P_i \in \mathbb{D}$, say $r_1(P_i) = a \equiv (a^s)_{s\in M}$, suppose that it is not multidimensional single-peaked on $\times_{s\in M} G(A^s)$. Thus, there exist distinct $x, y \in A$ such that $x \in \langle a, y\rangle$ but $y P_i x$. Evidently, $a \neq y$. Since $\mathbb{D}$ is minimally rich, we have $P_i' \in \mathbb{D}^y$. Thus, P_i and P_i' differ on peaks, but agree on the relative ranking of y and x, i.e., $y P_i x$ and $y P_i' x$. Then, the Exterior$^+$ property implies that there exists a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P_i' such that $y P_i^k x$ for all $k = 1, \ldots, q$. Note that since $r_1(P_i^1) = a \neq y = r_1(P_i^q)$, there must exist $1 \leq k < q$ such that $r_1(P_i^k) = a \neq r_1(P_i^{k+1})$. Consequently, Lemma 9 of Appendix B implies $P_i^k \sim^+ P_i^{k+1}$. Hence, P_i^k is a separable preference, and hence multidimensional single-peaked on $\times_{s\in M} G(A^s)$ by Lemma 6. Consequently, $x \in \langle a, y\rangle$ implies $x P_i^k y$. Contradiction! This proves the lemma, and completes the verification of the necessity part of Theorem 1. $\square$

Now, we turn to the sufficiency part of Theorem 1. Given a product of trees $\times_{s\in M} G(A^s)$, let $\mathbb{D}_{\mathrm{MSP}}$ be the multidimensional single-peaked domain, and $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{MSP}}$. Thus, to complete the verification of the sufficiency part, it suffices to construct an RSCF on $\mathbb{D}_{\mathrm{MSP}}$ which satisfies unanimity, sd-strategy-proofness and the compromise property. The construction consists of 3 steps and 4 claims below.

If $|A^s| = 2$ for each $s \in M$, then every generalized random dictatorship is unanimous and sd-strategy-proof, and satisfies the compromise property vacuously. Henceforth, we consider the case that $|A^s| > 2$ for some $s \in M$. For notational convenience, let $\bar{\mathbb{D}}_{\text{MSP}} = \mathbb{D}_{\text{S}} \cap \mathbb{D}_{\text{MSP}}$ denote the intersection of the separable domain and the multidimensional single-peaked domain, and $[\bar{\mathbb{D}}_{\text{MSP}}]^s = \{[P_i]^s : P_i \in \bar{\mathbb{D}}_{\text{MSP}}\}$ denote the induced marginal domain over A^s for each $s \in M$. Evidently, for each $s \in M$, $[\bar{\mathbb{D}}_{\text{MSP}}]^s$ is the single-peaked (marginal) domain on the tree $G(A^s)$. We construct an RSCF on $\mathbb{D}_{\text{MSP}}$ in three steps.

STEP 1. We introduce a class of DSCFs on each marginal domain. Fix $s \in M$. Given a N-tuple $(x_1^s, \ldots, x_N^s) \in [A^s]^N$, let $G(x_1^s, \ldots, x_N^s)$ denote the *minimal subgraph* of $G(A^s)$ containing $x_1^s, \ldots, x_N^s$ as vertices.[27] Fixing $a^s \in A^s$, we have the *projection* of a^s on $G(x_1^s, \ldots, x_N^s)$, denoted $\pi^s\big(a^s, G(x_1^s, \ldots, x_N^s)\big)$, which is unique.[28] Thus, we have a particular marginal function $g^{a^s} : [A^s]^N \to A^s$ such that $g^{a^s}(x_1^s, \ldots, x_N^s) = \pi^s\big(a^s, G(x_1^s, \ldots, x_N^s)\big)$ for all $(x_1^s, \ldots, x_N^s) \in [A^s]^N$.

STEP 2. Fixing $a \equiv (a^s)_{s\in M} \in A$, we assemble all marginal functions $(g^{a^s})_{s\in M}$ to construct a DSCF on $\mathbb{D}_{\text{MSP}}$: Given $(P_1, \ldots, P_N) \in \mathbb{D}_{\text{MSP}}^N$, say for notational convenience $r_1(P_i) = x_i \equiv (x_i^s)_{s\in M}$ for each $i \in I$, let $f^a(P_1, \ldots, P_N) = \big(g^{a^s}(x_1^s, \ldots, x_N^s)\big)_{s\in M}$. DSCF f^a is called a **multidimensional projection rule**, and alternative $a \equiv (a^s)_{s\in M}$ is referred to as the **projector**.

STEP 3. Last, we construct an RSCF on $\mathbb{D}_{\text{MSP}}$ by a mixture of all multidimensional projection rules. We associate each projector $a \in A$ with a strictly positive weight $\lambda_a > 0$, and let $\sum_{a\in A} \lambda_a = 1$. We then construct an RSCF $\varphi(P) = \sum_{a\in A} \lambda_a f^a(P)$ for all $P \in \mathbb{D}_{\text{MSP}}^N$. RSCF φ is called a **mixed multidimensional projection rule**.

Evidently, φ is well-defined, and satisfies unanimity. Also note that since only preference peaks are used in constructing the multidimensional projection rule in Step 2, it is true that all multidimensional projection rules and the mixed multidimensional projection rule φ satisfy the tops-only property.

CLAIM 1: Each multidimensional projection rule f^a is *decomposable* on $\bar{\mathbb{D}}_{\text{MSP}}$, i.e., for each $s \in M$, there exists a marginal DSCF $f^s : \big[[\bar{\mathbb{D}}_{\text{MSP}}]^s\big]^N \to A^s$ such that for all $P \in \bar{\mathbb{D}}_{\text{MSP}}^N$, we have $f^a(P) = \big(f^s([P_1]^s, \ldots, [P_N]^s)\big)_{s\in M}$.

By the construction of f^a at Step 2, for each $s \in M$, we can construct a marginal DSCF: For all $\big([P_1]^s, \ldots, [P_N]^s\big) \in \big[[\bar{\mathbb{D}}_{\text{MSP}}]^s\big]^N$, $f^s\big([P_1]^s, \ldots, [P_N]^s\big) = g^{a^s}\big(r_1([P_1]^s), \ldots, r_1([P_N]^s)\big)$. Therefore, f^a is decomposable on $\bar{\mathbb{D}}_{\text{MSP}}$. This completes the verification of the claim.

CLAIM 2: Each multidimensional projection rule f^a is sd-strategy-proof on $\bar{\mathbb{D}}_{\text{MSP}}$.

[27] For details of minimal subgraph, please refer to Chatterji et al. (2013).

[28] Fix a tree G, a subtree $G' \subseteq G$ and a vertex a. If a belongs to the vertex set of G', the projection of a on G' is a itself. If a does not belong to the vertex set of G', there exists an *unique* vertex a' in G' which lies in every path connecting a and every vertex of G'. Thus, a' is referred to as the projection of a on G'.

First, note that $\bar{\mathbb{D}}_{\text{MSP}}$ satisfies Properties A and B of Le Breton and Sen (1999).[29] Then, by Theorem 4.1 of Le Breton and Sen (1999), we know that to prove the claim, it suffices to show that for each $s \in M$, the marginal DSCF f^s of Claim 1 is sd-strategy-proof. Last, following exactly the proof of the sufficiency part of the Theorem of Chatterji et al. (2013), we obtain that f^s is sd-strategy-proof on $[\bar{\mathbb{D}}_{\text{MSP}}]^s$. This completes the verification of the claim.

CLAIM 3: RSCF φ is sd-strategy-proof on $\mathbb{D}_{\text{MSP}}$.

First, since φ is a mixture of all multidimensional projection rules, Claim 2 implies that φ is sd-strategy-proof on $\bar{\mathbb{D}}_{\text{MSP}}$. Now, given $i \in I$, $P_i, P_i' \in \mathbb{D}_{\text{MSP}}$ and $P_{-i} \in \mathbb{D}_{\text{MSP}}^{N-1}$, we show that for each $a \in A$, either $f^a(P_i, P_{-i}) = f^a(P_i', P_{-i})$ or $f^a(P_i, P_{-i}) P_i f^a(P_i', P_{-i})$. Suppose not, i.e., there exists $a \in A$ such that $f^a(P_i', P_{-i}) \equiv y P_i x \equiv f^a(P_i, P_{-i})$. For notational convenience, assume $r_1(P_i) = z$. Next, by the definition of multidimensional single-peakedness, $y P_i x$ implies $x \notin \langle z, y \rangle$. By minimal richness of $\bar{\mathbb{D}}_{\text{MSP}}$, we identify two arbitrary preferences $\bar{P}_i, \bar{P}_i' \in \bar{\mathbb{D}}_{\text{MSP}}$ and $\bar{P}_{-i} \in \bar{\mathbb{D}}_{\text{MSP}}^{N-1}$ such that $r_1(\bar{P}_i) = r_1(P_i)$, $r_1(\bar{P}_i') = r_1(P_i')$ and $r_1(\bar{P}_j) = r_1(P_j)$ for all $j \neq i$. By the construction at Step 2, we know $f^a(\bar{P}_i, \bar{P}_{-i}) = f^a(P_i, P_{-i}) = x$ and $f^a(\bar{P}_i', \bar{P}_{-i}) = f^a(P_i', P_{-i}) = y$. Then, Claim 2 implies $x \bar{P}_i y$. Since we choose $\bar{P}_i$ arbitrarily, it is also true that $x \bar{P}_i y$ for all $\bar{P}_i \in \bar{\mathbb{D}}_{\text{MSP}}$ with $r_1(\bar{P}_i) = z$. Then, the definition of $\bar{\mathbb{D}}_{\text{MSP}}$ implies $x \in \langle z, y \rangle$.[30] Contradiction! Therefore, for each $a \in A$, the multidimensional projection rule f^a is sd-strategy-proof on $\mathbb{D}_{\text{MSP}}$. Consequently, as a mixture of all multidimensional projection rules, RSCF φ is sd-strategy-proof on $\mathbb{D}_{\text{MSP}}$. This completes the verification of the claim.

CLAIM 4: RSCF φ satisfies the compromise property.

Let $\hat{I} \subseteq I$ be a subset of voters with $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd. Given $P_i, P_j \in \mathbb{D}_{\text{MSP}}$, assume $r_1(P_i) = (x^s, a^{-s}) \neq (y^s, a^{-s}) = r_1(P_j)$ and $r_2(P_i) = r_2(P_j) = (z^s, a^{-s})$. According to the product of trees $\times_{s \in M} G(A^s)$, one can easily tell $z^s \in \langle x^s, y^s \rangle$. Thus, $(z^s, a^{-s}) \in \langle (x^s, a^{-s}), (y^s, a^{-s}) \rangle$, and hence, $f^{(z^s, a^{-s})}\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I \backslash \hat{I}}\right) = (z^s, a^{-s})$. Consequently, $\varphi_{(z^s, a^{-s})}\left(\frac{P_i}{\hat{I}}, \frac{P_j}{I \backslash \hat{I}}\right) \geq \lambda_{(z^s, a^{-s})} > 0$. This completes the verification of the claim, and hence proves the sufficiency part of Theorem 1. □

Remark 5. The Theorem of Chatterji et al. (2016) shows that in the class of minimally rich and *path-connected* domains, the existence of a unanimous, tops-only and sd-strategy-proof

[29] Both Properties A and B are specified on a domain of separable preferences, say $\mathbb{D} \subseteq \mathbb{D}_{\text{S}}$. Property A requires each marginal domain be minimally rich which is evidently satisfied by $\bar{\mathbb{D}}_{\text{MSP}}$. Property B contains two parts. Fixing marginal preferences $\left([P_i]^\tau\right)_{\tau \in M}$, the first part requires that for each $s \in M$, there exists a preference $\bar{P}_i \in \mathbb{D}$ such that $[\bar{P}_i]^\tau = [P_i]^\tau$ for all $\tau \in M$, and component s is lexicographically dominant, i.e., $\left[x^s [P_i]^s y^s\right] \Rightarrow [(x^s, a^{-s}) \bar{P}_i (y^s, b^{-s})$ for all $a^{-s}, b^{-s} \in A^{-s}]$, while the second part requires that for each $s \in M$, there exists a preference $\underline{P}_i \in \mathbb{D}$ such that $[\underline{P}_i]^\tau = [P_i]^\tau$ for all $\tau \in M$, and component s is lexicographically dominated by all other components, i.e., $\left[x \underline{P}_i y \text{ and } x^s [P_i]^s y^s\right] \Rightarrow$ [either $x^{-s} = y^{-s}$, or there exists $\tau \in M \backslash \{s\}$ such that $x^\tau [P_i]^\tau y^\tau$]. Given single-peaked marginal preferences $\left([P_i]^\tau\right)_{\tau \in M}$, for each $s \in M$, we pick two lexicographic orders $\succ$ and $\succ'$ such that $s \succ \tau$ and $\tau \succ' s$ for all $\tau \neq s$, and then assemble two preferences of $\bar{\mathbb{D}}_{\text{MSP}}$ which meet the two parts of Property B. Therefore, $\bar{\mathbb{D}}_{\text{MSP}}$ satisfies Property B.

[30] Suppose $x \notin \langle z, y \rangle$. There must exist $s \in M$ such that $x^s \notin \langle z^s, y^s \rangle$. Consequently, there exists $[\bar{P}_i]^s \in [\bar{\mathbb{D}}_{\text{MSP}}]^s$ such that $r_1([\bar{P}_i]^s) = z^s$ and $y^s [\bar{P}_i]^s x^s$. Next, for each $\tau \neq s$, pick $[\bar{P}_i]^\tau \in [\bar{\mathbb{D}}_{\text{MSP}}]^\tau$ with $r_1([\bar{P}_i]^\tau) = z^\tau$. Last, we fix a lexicographic order $\succ$ where component s is lexicographically dominant, i.e., $s \succ \tau$ for all $\tau \neq s$, and assemble all marginal preferences according to $\succ$. Thus, we have $\bar{P}_i \in \bar{\mathbb{D}}_{\text{MSP}}$ and $y \bar{P}_i x$. Contradiction!

RSCF satisfying a stronger version of the compromise property implies that the domain must be single-peaked on a tree.[31] Theorem 1 significantly generalizes their result in three ways. First, all multidimensional domains studied here are excluded by their domain richness condition. Second, we endogenize the tops-only property. Third, the necessity part of Theorem 1 does not require the full power of their compromise property, and the mixed multidimensional projection rule constructed for the sufficiency part of Theorem 1 outperforms their compromise property: Whenever a common second best alternative appears in a preference profile where some two voters disagree on peaks, it receives a strictly positive probability.[32] Next, we observe that even though the Interior and Exterior properties of Chatterji and Zeng (2018) include multidimensional single-peaked domains, the characterization in Theorem 1 cannot be achieved in their model. In their setup, the boundary that distinguishes one-dimensional models from multidimensional models is not clear. On the contrary, the key notion of this paper, adjacency$^+$, brings sufficiently many separable preferences into consideration, which not only clearly separate the domains in question from the one-dimensional setting, but also create the basis for embodying the restriction of multidimensional single-peakedness (see Lemma 6) and furthermore spreading the restriction to other preferences (see the proof of Lemma 7). More importantly, in a connected$^+$ domain, we utilize the notion of adjacency$^+$ to induce a general geometric relation among alternatives which is eventually refined (via Lemmas 4 and 5) to a product of trees, a necessary step for establishing multidimensional single-peakedness. □

If two complete reversal preferences happen to be included in the domain in question, we refine the necessity part of Theorem 1 to the multidimensional single-peakedness of Barberà et al. (1993).[33]

Corollary 2. *Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. If it contains two complete reversal preferences and admits a unanimous and sd-strategy-proof RSCF satisfying the compromise property, it is multidimensional single-peaked on a product of lines.*

Proof. By Theorem 1, we know that domain $\mathbb{D}$ is multidimensional single-peaked on a product of trees $\times_{s\in M} G(A^s)$. Let $\underline{P}_i, \overline{P}_i \in \mathbb{D}$ be two complete reversal preferences. Assume $r_1(\underline{P}_i) = \underline{x}$ and $r_1(\overline{P}_i) = \overline{x}$. Evidently, $\underline{x} \neq \overline{x}$. We show that $\times_{s\in M} G(A^s)$ is a product of lines. Suppose that it is not true. Then, it must be the case that $\langle \underline{x}, \overline{x}\rangle \neq A$. Thus, there exists $a \notin \langle \underline{x}, \overline{x}\rangle$. For each $s \in M$, let $\hat{a}^s$ be the projection of a^s on $\langle \underline{x}^s, \overline{x}^s\rangle$. Let $\hat{a} \equiv (\hat{a}^s)_{s\in M}$. Since $a \notin \langle \underline{x}, \overline{x}\rangle$ and $\hat{a} \in \langle \underline{x}, \overline{x}\rangle$, it is evident that $\hat{a} \neq a$. Since $\hat{a} \in \langle \underline{x}, a\rangle$ and $\hat{a} \in \langle \overline{x}, a\rangle$, multidimensional single-peakedness implies $\hat{a}\underline{P}_i a$ and $\hat{a}\overline{P}_i a$. Contradiction! □

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31 Given a domain $\mathbb{D}$, a pair of distinct alternatives $a, b \in A$ is *adjacent* if there exist $P_i \in \mathbb{D}^a$ and $P'_i \in \mathbb{D}^b$ such that $P_i \sim P'_i$. Then, a domain is said **path-connected** if every pair of distinct alternatives is connected via a sequence of alternatives which are consecutively adjacent. Note that this notion of adjacency between alternatives is stronger than the notion of adjacency$^+$ between alternatives specified in the end of Section 2.2.

32 Given a preference profile $(P_1, \ldots, P_N) \in \mathbb{D}_{\text{MSP}}$, assume $r_1(P_i) = x_i \equiv (x_i^s)_{s\in M}$ for all $i \in I$, $x_1 \neq x_2$ and $r_2(P_1) = \cdots = r_2(P_N) = z \equiv (z^s)_{s\in M}$. It is easy to show that for each $s \in M$, z^s is included in the minimal subgraph induced by $(x_1^s, \ldots, x_N^s)$. Consequently, according to the multidimensional projection rule f^z, we have $f^z(P_1, \ldots, P_N) = z$, and hence $\varphi(P_1, \ldots, P_N) = \sum_{a\in A} \lambda_a f^a(P_1, \ldots, P_N) \geq \lambda_z > 0$.

33 Let $\times_{s\in M} G(A^s)$ be a product of lines, and $\mathbb{D}_{\text{MSP}}$ be the corresponding multidimensional single-peaked domain. Given $s \in M$, according to the line $G(A^s)$, we can arrange all elements in A^s on a linear order $>^s$, and identify $\underline{x}^s, \overline{x}^s \in A^s$ such that $\overline{x}^s >^s x^s >^s \underline{x}^s$ for all $x^s \in A^s \backslash \{\underline{x}^s, \overline{x}^s\}$. We then find $P_i \in \mathbb{D}_{\text{MSP}}$ with $r_1(P_i) = (\underline{x}_s)_{s\in M}$ and $P'_i \in \mathbb{D}_{\text{MSP}}$ with $r_1(P'_i) = (\overline{x}_s)_{s\in M}$ which are complete reversals.

3.1.1. Indispensability

In this subsection, we demonstrate the indispensability of each axiom of the RSCF and each domain condition in the characterization result of Theorem 1. For each case, we drop or weaken an axiom or a domain condition, keep all other axioms and domain conditions fixed, and construct a domain that diverges from multidimensional single-peakedness.

Example 1 *(Indispensability of unanimity).* Consider the top-separable domain $\mathbb{D}_{\text{TS}}$. We refer to Barberà (1979), and construct a two-voter *point voting scheme* $\varphi : \mathbb{D}^2_{\text{TS}} \to \Delta(A)$. First, we specify a vector $(\alpha_1, \alpha_2, \ldots, \alpha_{|A|}) \in \mathbb{R}^{|A|}_+$ such that $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 = \frac{1}{2}$ and $\alpha_k = 0$ for all $k = 3, \ldots, |A|$. Next, given $(P_i, P_j) \in \mathbb{D}^2_{\text{TS}}$ and $a \in A$, if $a = r_s(P_i)$ and $a = r_t(P_j)$, then φ gives a probability $\alpha_s + \alpha_t$. Therefore, a social lottery is well defined at each preference profile. Theorem 1 of Barberà (1979) implied that φ is sd-strategy-proof. We assert that φ also satisfies the compromise property since $\alpha_2 > 0$. However, φ fails unanimity as at each preference profile, each voter's second best alternative receives a strictly positive probability. □

Example 2 *(Indispensability of sd-strategy-proofness).* Consider the top-separable domain $\mathbb{D}_{\text{TS}}$. We construct the following two-voter DSCF $f : \mathbb{D}^2_{\text{TS}} \to A$, which chooses the common second best alternative whenever it appears in a preference profile with distinct peaks, and chooses voter i's preference peak otherwise, i.e.,

$$f(P_i, P_j) = \begin{cases} a & \text{if } r_1(P_i) \neq r_1(P_j) \text{ and } r_2(P_i) = r_2(P_j) \equiv a, \text{ and} \\ r_1(P_i) & \text{otherwise.} \end{cases}$$

DSCF f satisfies unanimity and the compromise property. However, by Corollary 1, it fails sd-strategy-proofness since it is never a generalized random dictatorship. □

We have explained the violation of the compromise property in all generalized random dictatorships at the beginning of Section 2.3. This demonstrates the indispensability of the compromise property in Theorem 1. The next example shows that if we impose more restrictions on recognizing a compromise alternative, the corresponding compromise property becomes too weak to diversify the social lottery's support, and consequently, can be vacuously satisfied by generalized random dictatorships.

Example 3 *(Indispensability of the compromise property[34]).* An alternative x is called a *modified compromise alternative* at P if all preference peaks are pairwise distinct, and x is the common second best alternative. We then say that an RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies the *modified compromise property* if we have [x is a modified compromise alternative at P] $\Rightarrow$ [$\varphi_x(P) > 0$].

Now, let $I = \{1, 2, 3\}$, $A = A^1 \times A^2$, $A^1 = \{0, 1, 2\}$ and $A^2 = \{0, 1\}$, and consider the top-separable domain $\mathbb{D}_{\text{TS}}$. Note that whenever a profile $P \in \mathbb{D}^3_{\text{TS}}$ has a compromise alternative, this modified compromise alternative can be assembled by the three distinct peaks. Therefore, a generalized random dictatorship where all voter sequences have strictly positive weights is unanimous and sd-strategy-proof, and satisfies the modified compromise property. □

Example 4 *(Indispensability of top-separability).* Let $A = A^1 \times A^2$ and $A^1 = A^2 = \{0, 1\}$. Consider the complete domain $\mathbb{P}$ which satisfies the Interior$^+$ and the Exterior$^+$ properties, but fails

[34] We are grateful to an anonymous referee for suggesting this example.

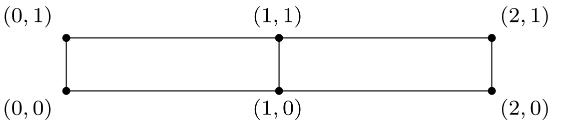

Fig. 4. The product of lines $G(A^1) \times G(A^2)$.

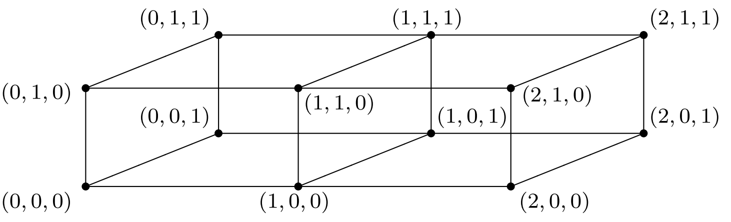

Fig. 5. The product of lines $G(A^1) \times G(A^2) \times G(A^3)$.

top-separability. An arbitrary random dictatorship is unanimous and sd-strategy-proof, and satisfies the compromise property vacuously since $|A^1| = |A^2| = 2$. $\square$

Example 5 *(Indispensability of minimal richness).* Let $A = A^1 \times A^2$, $A^1 = \{0, 1, 2\}$ and $A^2 = \{0, 1\}$. Let $\mathbb{D}_{\text{MSP}}$ be the multidimensional single-peaked domain on the product of two lines $G(A^1) \times G(A^2)$ specified in Fig. 4.

We first pick the subdomain of $\mathbb{D}_{\text{MSP}}$ containing every preference whose peak is neither $(2, 0)$ nor $(2, 1)$, i.e., $\hat{\mathbb{D}} = \{P_i \in \mathbb{D}_{\text{MSP}} : r_1(P_i) \neq (2, 0) \text{ and } r_1(P_i) \neq (2, 1)\}$. Meanwhile, we specify one particular top-separable preference $\bar{P}_i :\ (0, 0) \rightharpoonup (0, 1) \rightharpoonup (1, 0) \rightharpoonup (2, 0) \rightharpoonup (2, 1) \rightharpoonup (1, 1)$. Note that $\bar{P}_i$ is excluded from $\mathbb{D}_{\text{MSP}}$ since $(1, 1) \in \langle r_1(\bar{P}_i), (2, 1)\rangle$ but $(2, 1)\bar{P}_i(1, 1)$. Finally, we construct the domain $\mathbb{D} = \hat{\mathbb{D}} \cup \{\bar{P}_i\}$.

Evidently, $\mathbb{D}$ is a top-separable domain, and violates minimal richness. It is also true that $\mathbb{D}$ satisfies the Interior$^+$ and Exterior$^+$ properties (the detailed verification is available in Appendix E.8). However, $\mathbb{D}$ is never a multidimensional single-peaked domain. We refer to an arbitrary random dictatorship which is unanimous and sd-strategy-proof on $\mathbb{D}$. Moreover, since no preference of $\mathbb{D}$ has peak $(2, 0)$ or $(2, 1)$, no preference profile has a compromise alternative (i.e., the nonassemblable common second best alternative). Consequently, the random dictatorship satisfies the compromise property vacuously. $\square$

Example 6 *(Indispensability of paths in connectedness$^+$).* Let $A = A^1 \times A^2 \times A^3$, $A^1 = \{0, 1, 2\}$, $A^2 = \{0, 1\}$ and $A^3 = \{0, 1\}$. Let $\mathbb{D}_{\text{MSP}}$ be the multidimensional single-peaked domain on the product of three lines $G(A^1) \times G(A^2) \times G(A^3)$ specified in Fig. 5.

We specify a particular top-separable preference:

$$\begin{aligned}\bar{P}_i : {} & (0, 0, 0) \rightharpoonup (1, 0, 0) \rightharpoonup (2, 0, 0) \rightharpoonup (0, 1, 0) \rightharpoonup (1, 1, 0) \rightharpoonup (2, 1, 0) \rightharpoonup (0, 0, 1) \rightharpoonup \\ & (1, 0, 1) \rightharpoonup (2, 0, 1) \rightharpoonup (0, 1, 1) \rightharpoonup (2, 1, 1) \rightharpoonup (1, 1, 1).\end{aligned}$$

Note that $\bar{P}_i$ is excluded from $\mathbb{D}_{\text{MSP}}$ since $(1, 1, 1) \in \langle r_1(\bar{P}_i), (2, 1, 1)\rangle$ but $(2, 1, 1)\bar{P}_i(1, 1, 1)$. Thus, $\mathbb{D} = \mathbb{D}_{\text{MSP}} \cup \{\bar{P}_i\}$ is a minimally rich top-separable domain, but never a multidimensional single-peaked domain. Next, domain $\mathbb{D}$ satisfies the Interior$^+$ property, but violates the Exterior$^+$ property since there exists no path in $\mathbb{D}$ which reconciles the difference between $\bar{P}_i$

and a preference with peak $(2, 1, 1)$, and meanwhile keeps $(2, 1, 1)$ ranked above $(1, 1, 1)$ in every involved preference. However, a two-voter mixed multidimensional projection rule which only associates strictly positive weights to projectors $A \backslash \{(1, 1, 1), (2, 1, 1)\}$ satisfies unanimity, sd-strategy-proofness and the compromise property. All detailed verifications are put in Appendix E.9. □

We conclude this subsection by mentioning some issues that remain unresolved. First, the example in Appendix E.1 suggests that in some special cases, the full extent of the no-detour property may not be required for establishing the characterization result of Theorem 1. However, in the general setting, in particular, when the number of dimensions M is large, and each dimension includes sufficiently many elements, we are unable to establish the result of Theorem 1 without using the no-detour property.[35] Second, we have also been unable to construct an example to illustrate the indispensability of the Interior$^+$ property, since a violation of the Interior$^+$ property usually implies a failure of the Exterior$^+$ property.[36] It is also inconclusive whether the Interior$^+$ property is redundant for establishing Theorem 1 since we have been unable to show that the Exterior$^+$ property is by itself sufficient for the tops-onlyness result of Proposition 2.

3.2. Deterministic voting

In this section, we provide a characterization of multidimensional single-peaked domains using deterministic social choice functions. Unlike the random setting where anonymity can be satisfied by the random dictatorship that gives equal weights to all voters, the axiom of anonymity is appropriate for distinguishing deterministic social choice functions from dictatorships. We replace the compromise property by anonymity, and obtain a characterization result that is analogous to Theorem 1.

Formally, a DSCF $f : \mathbb{D}^N \to A$ is **anonymous** if for all $(P_1, \ldots, P_N) \in \mathbb{D}^N$ and every permutation $\sigma : I \to I$, we have $f(P_1, \ldots, P_N) = f\big(P_{\sigma(1)}, \ldots, P_{\sigma(N)}\big)$.

Theorem 2. *Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. If it admits a unanimous, anonymous and sd-strategy-proof DSCF, it is multidimensional single-peaked. Conversely, a multidimensional single-peaked domain admits a unanimous, anonymous and sd-strategy-proof DSCF.*

The proof of Theorem 2 is available in Appendix C.

Remark 6. The verification of the indispensability of axioms and domain conditions in Theorem 2 is similar to the examples of Section 3.1.1. First, fixing the top-separable domain, we can illustrate the indispensability of unanimity via a constant rule which is anonymous and sd-strategy-proof; the indispensability of anonymity via a generalized dictatorship which is unanimous and sd-strategy-proof, and the indispensability of sd-strategy-proofness via a plurality rule with a fixed tie-breaking order over alternatives which is unanimous and anonymous. Second, we use the domain of Example 5, and show the indispensability of minimal richness via a multidimensional projection rule which has the projector $(0, 0)$, or $(1, 0)$, or $(0, 1)$, or $(1, 1)$. Third, we

35 For instance, the proof of Lemma 4 fails without the path connecting b and d, and the path connecting b and c.

36 Recall the example in Appendix E.1. If we drop P_4, then the Interior$^+$ property fails. Meanwhile, one can tell that the Exterior$^+$ property is also violated, e.g., the path $\{P_3, P_2, P_1, \bar{P}_1, \bar{P}_2, \bar{P}_3\}$ becomes the unique path connecting P_3 and $\bar{P}_3$, and however, $(1, 0)$ does not always rank above $(0, 0)$ along the path.

adopt both the domain and an arbitrary multidimensional projection rule of Example 6 to show the indispensability of the path in the domain richness condition of connectedness^{+}. Last, unlike Example 4, we are unable to show the indispensability of top-separability. However, we refer to the single-peaked domain and a median voter rule to show the indispensability of the *conjunction* of top-separability and the no-detour property. □

Remark 7. The characterization of multidimensional single-peaked preferences in Theorem 2 can be interpreted as further evidence in favor of the Gul conjecture (see Section 6.5.2 of Barberà, 2010). Our work is related to earlier work by Chatterji et al. (2013) and Chatterji and Massó (2018). The paper by Chatterji et al. (2013) excludes multidimensional models. They derive a weaker version of single-peakedness, *semi-single-peakedness*, assuming tops-onlyness and an even number of voters. Chatterji and Massó (2018), using the same additional assumptions on the DSCF, derive another weaker version of single-peakedness, *semilattice-single-peakedness*, in a more general setup which allows A to be infinite, and includes multidimensional models. Our Theorem 2 recovers full multidimensional single-peakedness without the tops-onlyness assumption and the restriction on the number of voters. We get a sharper result here because our richness condition is more elaborate. In particular, the Exterior^{+} property plays a key role in recovering full multidimensional single-peakedness.[37] □

3.3. Voting under constraints

Barberà et al. (1997) first studied the model where not all alternatives are feasible. The set of feasible alternatives then becomes a strict subset of the Cartesian product structure. In such a setup, our result is not valid. In particular, the necessity part of Theorem 1 fails once invalid alternatives appear,[38] while the sufficiency part of Theorem 1 may not hold as the multidimensional projection rule may select infeasible alternatives.

In this section, we adapt our model to accord with the infeasible alternatives problem in the following three ways: (1) modify RSCFs to constrained RSCFs which only assign probabilities to feasible alternatives, (2) adjust the axioms of unanimity and the compromise property w.r.t. feasible alternatives, and (3) restrict attention to the class of feasible alternative sets such that two preferences with the same peak also share the same best feasible alternative. We then show that without any change in the domain condition: minimal richness and connectedness^{+}, the existence of a unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF satisfying the compromise property (w.r.t. feasibility) implies that the domain must be multidimensional single-peaked w.r.t. feasible alternatives, i.e., (i) the set of feasible alternatives is factorizable (in other words, the feasible set itself is a Cartesian product), and located on a product of trees, and (ii) for each preference over A, the induced preference over the feasible alternatives is multidimensional single-peaked on the product of trees consisting of feasible alternatives. With these modifications, every multidimensional projection rule that has a projector of a feasible alternative (recall the proof of the sufficiency part of Theorem 1) is well-defined, and the mixture of

[37] If we drop the Interior^{+} property, keep the Exterior^{+} property, and add the top-only property on the DSCF to cover the invalidation of Proposition 2, the characterization result of Theorem 2 still holds. If we further weaken the Exterior^{+} property to a richness condition stated in Lemma 1, a significantly weaker version of multidimensional single-peakedness can be elicited, *multidimensional semi-single-peakedness*, which is an extension of semi-single-peakedness of Chatterji et al. (2013) to the multidimensional setting (see Appendix E.9).

[38] For instance, the verification of Lemma 4 relies on the feasibility of the four alternatives a, b, c and d.

these multidimensional projection rules satisfies the requirements of unanimity (w.r.t. feasibility), sd-strategy-proofness and the compromise property (w.r.t. feasibility). This indicates that our characterization of multidimensional single-peaked preferences is robust to voting under constraints.

Let $\bar{A} \subseteq A \equiv \times_{s\in M} A^s$ denote the set of **feasible** alternatives. Note that if there exists $s \in M$ such that $a^s = b^s$ for all $a, b \in \bar{A}$, then the component set s becomes redundant, and hence can be eliminated. For simplicity, we impose an assumption to make all components indispensable.

Assumption 1. For each $s \in M$, there exist $a, b \in \bar{A}$ such that $a^s \neq b^s$.

Under Assumption 1, we say that the feasible set $\bar{A}$ is **factorizable** if there exists $\bar{A}^s \subseteq A^s$ for each $s \in M$ such that $\bar{A} = \times_{s\in M} \bar{A}^s$.

Given a preference P_i over A, let $P_{i|\bar{A}}$ denote the *induced preference* over $\bar{A}$ which preserves the relative rankings of feasible alternatives in preference P_i. Accordingly, let $\mathbb{D}_{|\bar{A}} = \{P_{i|\bar{A}} : P_i \in \mathbb{D}\}$ denote the domain of induced preferences over $\bar{A}$ and $\Delta(\bar{A}) = \{\lambda \in \Delta(A) : \sum_{a\in\bar{A}} \lambda_a = 1\}$ denote the constrained lottery space where each lottery over A assigns zero probability to infeasible alternatives.[39] A **constrained** RSCF is a map $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$. We modify the axioms of unanimity and the compromise property to accord with feasibility. Formally, a constrained RSCF $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$ is **unanimous** (w.r.t. feasibility) if for all $a \in \bar{A}$ and $P \in \mathbb{D}^N$, $[r_1(P_{i|\bar{A}}) = a \text{ for all } i \in I] \Rightarrow [\varphi_a(P) = 1]$. Next, a constrained RSCF $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$ satisfies **the compromise property** (w.r.t. feasibility) if there exists $\hat{I} \subseteq I$ with $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd, such that given $P_i, P_j \in \mathbb{D}$, we have

$$\begin{bmatrix} r_1(P_i) \equiv (x^s, a^{-s}) \neq (y^s, a^{-s}) \equiv r_1(P_j) \text{ and} \\ r_2(P_i) = r_2(P_j) \equiv (z^s, a^{-s}) \in \bar{A} \text{ where } z^s \notin \{x^s, y^s\} \end{bmatrix} \Rightarrow \left[\varphi_{(z^s,a^{-s})}\Big(\frac{P_i}{\hat{I}}, \frac{P_j}{I\backslash\hat{I}}\Big) > 0\right].$$

Note that in the definition of compromise property (w.r.t. feasibility), the peaks of preferences P_i and P_j need not be feasible. The definition of sd-strategy-proofness is not affected by the feasibility issue. For voting under constraints, the definition of multidimensional single-peaked domain is modified as follows.

Definition 9. A domain $\mathbb{D}$ is **multidimensional single-peaked w.r.t.** $\bar{A}$ if the following two conditions are satisfied:

(i) The feasible set $\bar{A}$ is factorizable, i.e., $\bar{A} = \times_{s\in M} \bar{A}^s$.
(ii) There exists a product of trees $\times_{s\in M} G(\bar{A}^s)$ such that every $P_i \in \mathbb{D}$ is multidimensional single-peaked w.r.t. $\bar{A}$, i.e., given distinct $x, y \in \bar{A}$, $\left[x \in \langle r_1(P_{i|\bar{A}}), y\rangle\right] \Rightarrow [x\, P_i\, y]$.[40]

We shall continue to restrict attention to the class of minimally rich and connected$^+$ domains. However, without additional conditions imposed on the feasible set or domains, the factorization of the feasible set cannot be elicited, and the characterization of multidimensional single-peakedness over feasible alternatives eventually fails. We provide an example to illustrate.

39 For mathematical consistency and notational convenience, we define each element of $\Delta(\bar{A})$ as a lottery over A, not over $\bar{A}$. Thus, $\Delta(\bar{A}) \subset \Delta(A)$, and, for instance, Lemma 8 of Appendix B still holds in voting under constraints.

40 For each $s \in M$, graph $G(\bar{A}^s)$ is a tree over $\bar{A}^s$. The vertex set of $\times_{s\in M} G(\bar{A}^s)$ is $\bar{A} = \times_{s\in M} \bar{A}^s$.

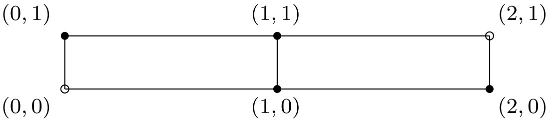

Fig. 6. The product of lines $G(A^1) \times G(A^2)$.

Example 7. Let $A = A^1 \times A^2$, $A^1 = \{0, 1, 2\}$ and $A^2 = \{0, 1\}$. Let $\mathbb{D}_{\text{MSP}}$ be the multidimensional single-peaked domain on the product of two lines $G(A^1) \times G(A^2)$ in Fig. 6. It is evident that $\mathbb{D}_{\text{MSP}}$ is a connected⁺ domain.

Let $\bar{A} = A \backslash \{(0, 0), (2, 1)\}$. Thus, $\bar{A}$ is not factorizable, and $\mathbb{D}_{\text{MSP}}$ is not multidimensional single-peaked w.r.t. $\bar{A}$. We can construct a unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF $\varphi : \mathbb{D}^2_{\text{MSP}} \to \Delta(\bar{A})$ which satisfies the compromise property (w.r.t. feasibility): For all $P_i, P_j \in \mathbb{D}_{\text{MSP}}$, $\varphi(P_i, P_j) = \frac{1}{2} e_{r_1(P_{i|\bar{A}})} + \frac{1}{2} e_{r_1(P_{j|\bar{A}})}$. In fact, φ is a *constrained random dictatorship*,[41] and therefore is naturally unanimous (w.r.t. feasibility) and sd-strategy-proof. Note that a preference profile $(P_i, P_j) \in \mathbb{D}^2_{\text{MSP}}$ has a non-assemblable feasible compromise alternative if and only if one of the following two cases occurs:

(i) $\{r_1(P_i), r_1(P_j)\} = \{(0, 0), (2, 0)\}$ and $r_2(P_i) = r_2(P_j) = (1, 0)$, and
(ii) $\{r_1(P_i), r_1(P_j)\} = \{(0, 1), (2, 1)\}$ and $r_2(P_i) = r_2(P_j) = (1, 1)$.

When all alternatives are feasible, these two compromise alternatives never receive any probability from any random dictatorship or generalized random dictatorship. However, when $(0, 0)$ becomes infeasible, alternative $(1, 0)$ becomes the feasible peak of some voter's preference in a preference profile of case (i), and then inherits probability $\frac{1}{2}$ in φ. The same argument holds for case (ii) as well. Therefore, φ satisfies the compromise property (w.r.t. feasibility). □

Example 7 indicates that under the feasibility constraint, the compromise property is weakened so that it is no longer incompatible with constrained random dictatorships. In general, we are unable to elicit any meaningful preference restrictions over the feasible alternatives from a constrained random dictatorship. Therefore, without further conditions, our characterization of multidimensional single-peakedness in Theorem 1 fails in voting under constraints. We observe that in Example 7, there are two preferences of $\mathbb{D}_{\text{MSP}}$ which share the same the infeasible alternative $(0, 0)$ as the peak, but have two distinct feasible peaks $(1, 0)$ and $(0, 1)$, e.g.,

$$\begin{array}{l} P_i : (0,0) \succ (1,0) \succ (0,1) \succ (1,1) \succ (2,0) \succ (2,1) \\ P'_i : (0,0) \succ (0,1) \succ (1,0) \succ (1,1) \succ (2,0) \succ (2,1) \end{array} \Rightarrow \begin{array}{l} P_{i|\bar{A}} : (1,0) \succ (0,1) \succ (1,1) \succ (2,0) \\ P'_{i|\bar{A}} : (0,1) \succ (1,0) \succ (1,1) \succ (2,0) \end{array}.$$

We impose an additional condition which excludes the alluded observation by ensuring that two preferences with the same peak over A always share the same feasible peak.

Assumption 2. For all $P_i, P'_i \in \mathbb{D}$, we have $[r_1(P_i) = r_1(P'_i)] \Rightarrow \left[r_1(P_{i|\bar{A}}) = r_1(P'_{i|\bar{A}})\right]$.

---

41 A constrained RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a **constrained random dictatorship** if there exists $\varepsilon_i \geq 0$ for each $i \in I$ with $\sum_{i \in I} \varepsilon_i = 1$ such that for all $P \in \mathbb{D}^N$ and $a \in \bar{A}$, $\varphi_a(P) = \sum_{i \in I : r_1(P_{i|\bar{A}}) = a} \varepsilon_i$. A constrained random dictatorship is unanimous (w.r.t. feasibility) and sd-strategy-proof on an arbitrary domain.

If $\bar{A} = A$, then both Assumptions 1 and 2 are automatically satisfied. In the framework of Example 7, there are three ways to satisfy Assumptions 1 and 2: Set $\bar{A} = \{0, 1, 2\} \times \{0, 1\}$, or $\bar{A} = \{0, 1\} \times \{0, 1\}$, or $\bar{A} = \{1, 2\} \times \{0, 1\}$. In either case, the feasible set $\bar{A}$ is factorizable, and one can immediately infer that $\mathbb{D}_{\text{MSP}}$ is multidimensional single-peaked w.r.t. $\bar{A}$. Note that if we make $\bar{A} = \{0, 2\} \times \{0, 1\}$, the feasible set $\bar{A}$ is factorizable, domain $\mathbb{D}_{\text{MSP}}$ remains multidimensional single-peaked w.r.t. $\bar{A}$, but Assumption 2 fails since two preferences with the same peak $(1, 0)$ may own two distinct feasible peaks $(0, 0)$ and $(2, 0)$ respectively.

Now, we present the result.

Theorem 3. *Fix Assumptions 1 and 2. The following two statements hold:*

(i) *Let $\mathbb{D}$ be a minimally rich and connected^{+} domain. If it admits a unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF $\phi : \mathbb{D}^N \to \Delta(\bar{A})$ satisfying the compromise property (w.r.t. feasibility), then it is multidimensional single-peaked w.r.t. $\bar{A}$.*

(ii) *Conversely, a multidimensional single-peaked domain $\mathbb{D}$ w.r.t. $\bar{A}$ admits a unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$ satisfying the compromise property (w.r.t. feasibility).*

The proof of Theorem 3 is available in Appendix D.

Remark 8. Example 7 demonstrates that Assumption 2 is needed for Theorem 3(i), as we provide a multidimensional single-peaked domain which is connected^{+} but violates Assumption 2, construct a unanimous (w.r.t. feasibility) and sd-strategy-proof constrained random dictatorship which satisfies the compromise property (w.r.t. feasibility), but we are unable to elicit the multidimensional single-peakedness restriction on all feasible alternatives. More specifically, Assumption 2 is critical for establishing that the sd-strategy-proofness and unanimity (w.r.t. feasibility) of a constrained RSCF on a connected^{+} domain imply that the constrained RSCF satisfies the tops-only property (see Proposition 3 in the proof of Theorem 3 in Appendix D). This step commences the proof of Theorem 3, and its role is analogous to the one played by Proposition 2 in the proofs of Theorems 1 and 2 respectively. Assumption 2 is strong, but we are unable to proceed without it. Indeed if one considers a constrained voting scenario where Assumption 2 does not hold, there exists no sd-strategy-proof and unanimous (w.r.t. feasibility) constrained RSCF that satisfies the tops-only property, and then it may not be possible to make any inferences about the structure of the domain. Example 7 is a case in point.[42] Proposition 3 summarizes the role of Assumption 2 with regards to the tops-only property. $\square$

Remark 9. We briefly discuss the indispensability of our assumptions. Assumption 1 is introduced to simplify the analysis, and can be dispensed with by refining the Cartesian product structure appropriately. The indispensability of Assumption 2 has been shown in Example 7. Examples 1 - 6 of Section 3.1.1 can be appropriately adapted to show the indispensability of our axioms and domain conditions of Theorem 3. The details are available in Appendix E.10. $\square$

We conclude this section by discussing the relation of Theorem 3 to the literature.

[42] Note that the constrained random dictatorship in Example 7 violates the tops-only property.

Remark 10. Barberà et al. (1997) studied a deterministic constrained voting model on the multidimensional single-peaked domain $\mathbb{D}_{\text{MSP}}$ of Barberà et al. (1993) where A is located on a product of lines, and $\bar{A}$ is an arbitrary subset of A. They characterized the class of unanimous (w.r.t. feasibility) and sd-strategy-proof DSCFs that map to $\bar{A}$: These are feasible generalized median voter schemes satisfying the intersection property.[43] The structure of the feasible set $\bar{A}$ determines the size of the class of feasible generalized median voter schemes satisfying the intersection property. On the one hand, if $\bar{A}$ is factorizable, then all feasible generalized median voter schemes satisfy the intersection property automatically, and $\mathbb{D}_{\text{MSP}}$ is multidimensional single-peaked w.r.t. $\bar{A}$. Furthermore, one can construct a multidimensional projection rule on $\mathbb{D}_{\text{MSP}|\bar{A}}$ using a projector of a feasible alternative, and then extend it to a feasible generalized median voter scheme on $\mathbb{D}_{\text{MSP}}$ which satisfies unanimity (w.r.t. feasibility), anonymity and sd-strategy-proofness. On the other hand, if $\bar{A}$ is not factorizable, see for instance, Example 7, Section 4 of Aswal et al. (2003) and Theorem 2 of Barberà et al. (2005), every feasible generalized median voter schemes satisfying the intersection property degenerates to a constrained dictatorship. We ask what structure on $\bar{A}$ is implied by the existence of a "well-behaved" sd-strategy-proof RSCF, and shows that the existence of a unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF satisfying the compromise property (w.r.t. feasibility) implies that $\bar{A}$ must be factorizable. Moreover, in contrast to the model of Barberà et al. (1997) where domain $\mathbb{D}_{\text{MSP}}$ was the primitive and automatically multidimensional single-peaked w.r.t. every factorizable feasible set, our characterization analysis (i) takes a more general class of domains as the primitive, connected$^+$ domains, (ii) endogenously establishes the factorizability of $\bar{A} = \times_{s \in M} \bar{A}^s$ and induces a product of trees $\times_{s \in M} G(\bar{A}^s)$, and (iii) elicits the embedded restriction of multidimensional single-peakedness w.r.t. feasibility. Barberà et al. (1999) considered the same model of Barberà et al. (1997), fixed a feasible generalized median voter scheme satisfying the intersection property, and induced preference restrictions to retrieve sd-strategy-proofness of the fixed generalized median voter scheme. However, their induced preference restrictions depend on the specific form of the primitive generalized median voter scheme. On the contrary, our analysis only takes a general unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF as the primitive, and more importantly, our notion of multidimensional single-peaked preferences is independent of the primitive RSCF. □

4. Conclusion

We have proposed a class of multidimensional domains, connected$^+$ domains. We first prove that multidimensional single-peakedness is necessary and sufficient in the class of minimally rich and connected$^+$ domain for the existence of a unanimous and sd-strategy-proof RSCF satisfying the compromise property. We also present an analogous result for DSCFs using anonymity instead of the compromise property. Last, we show that our characterization is robust to voting under constraints. The results for multidimensional models presented here are in the spirit of earlier results (e.g., Bogomolnaia, 1998; Nehring and Puppe, 2007; Chatterji et al., 2013, 2016; Chatterji and Massó, 2018) that indicate that some form of single-peakedness is inherent in preference domains that allow the construction of "well-behaved" sd-strategy-proof rules.

[43] A feasible generalized median voter scheme is a generalized median voter scheme which always chooses a feasible alternative at each preference profile. The formal definition of the intersection property can be found in Definition 9 of Barberà et al. (1997). An alternative formulation of the intersection property can be found in Section 3.3 of Nehring and Puppe (2007).

We suggest that connected$^+$ domains may be useful in resolving other open issues; one such issue is the equivalence of sd-strategy-proofness and local sd-strategy-proofness where the latter is formulated by requiring that *only* a manipulation via a preference adjacent or adjacent$^+$ to the sincere one is forbidden from being profitable.

The characterization of all well-behaved sd-strategy-proof RSCFs on connected$^+$ domains is not attempted in this paper, and is left for future work. It would also be of interest to extend the analysis to situations where some of the dimensions include private goods or monetary transfers.

Appendix A. Proof of Proposition 1

Given $\underline{i} \equiv (i^s)_{s\in M}$, let $f^{\underline{i}}: \mathbb{D}^N \to A$ denote a generalized dictatorship. Recall the definition of a generalized random dictatorship, given $\gamma(\underline{i}) \geq 0$ for each $\underline{i} \in I^m$ and $\sum_{\underline{i}\in I^m} \gamma(\underline{i}) = 1$, $\varphi_a(P) = \sum_{\underline{i}\equiv(i^s)_{s\in M}\in I^m:\, a=\left(r_1(P_{i^s})^s\right)_{s\in M}} \gamma(\underline{i})$ for all $a \in A$ and $P \in \mathbb{D}^N$. We know that φ can be rewritten as a mixture of generalized dictatorships, i.e., $\varphi(P) = \sum_{\underline{i}\in I^m} \gamma(\underline{i})\, f^{\underline{i}}(P)$ for all $P \in \mathbb{D}^N$. Therefore, to verify the sufficiency part of Proposition 1, it suffices to show that every generalized dictatorship is sd-strategy-proof.

Fix $\underline{i} \equiv (i^s)_{s\in M}$ and $i \in I$. Given $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$, assume $f^{\underline{i}}(P_i, P_{-i}) = x \equiv (x^s)_{s\in M}$ and $f^{\underline{i}}(P_i', P_{-i}) = y \equiv (y^s)_{s\in M}$. We show either $x = y$ or $x P_i y$. Assume $r_1(P_i) = a \equiv (a^s)_{s\in M}$ and $r_1(P_i') = b \equiv (b^s)_{s\in M}$. In the voter sequence $\underline{i} \equiv (i^s)_{s\in M}$, we identify $S \subseteq M$ such that $i^s = i$ for all $s \in S$ and $i^\tau \neq i$ for all $\tau \notin S$. Thus, we know $x^s = a^s$ and $y^s = b^s$ for all $s \in S$, and $x^\tau = y^\tau$ for all $\tau \notin S$. Evidently, if $S = \emptyset$, then $x = y$. Similarly, if $S \neq \emptyset$ and $a^s = b^s$ for all $s \in S$, we also have $x = y$. Last, we assume that $S \neq \emptyset$, and there exists a non-empty $S^+ \subseteq S$ such that $a^s \neq b^s$ for all $s \in S^+$ and $a^\tau = b^\tau$ for all $\tau \in S\backslash S^+$. For notational simplicity, assume $S^+ = \{1, \ldots, s\}$. Thus, we know $x^k = a^k \neq b^k = y^k$ for all $k = 1, \ldots, s$, and $x^\tau = y^\tau \equiv z^\tau$ for all $\tau = s+1, \ldots, m$, and write $x = (a^1, \ldots, a^s, z^{s+1}, \ldots, z^m)$ and $y = (b^1, \ldots, b^s, z^{s+1}, \ldots, z^m)$. We identify alternatives $a_k = (a^1, \ldots, a^k, b^{k+1}, \ldots, b^s, z^{s+1}, \ldots, z^m)$ for all $k = 0, 1, \ldots, s$. Evidently, $a_0 = y$ and $a_s = x$. Since $r_1(P_i) = a$, top-separability implies $a_k P_i a_{k-1}$ for all $k = 1, \ldots, s$. Consequently, we have $x P_i y$ by transitivity. This completes the verification of sd-strategy-proofness of $f^{\underline{i}}$, as required.

Conversely, to verify the necessity part of Proposition 1, we consider a particular generalized random dictatorship $\varphi: \mathbb{D}^N \to \Delta(A)$ where $\gamma(\underline{i}) > 0$ for all $\underline{i} \in I^m$. We show that all preferences of $\mathbb{D}$ are top-separable. Suppose not, i.e., there exists $\bar{P}_i \in \mathbb{D}$, say $r_1(\bar{P}_i) = a \equiv (a^s)_{s\in M}$, such that $(b^s, z^{-s})\bar{P}_i(a^s, z^{-s})$ for some $s \in M$, $b^s \in A^s\backslash\{a^s\}$ and $z^{-s} \in A^{-s}$. Let $(b^s, z^{-s}) = r_\eta(\bar{P}_i)$ for some $1 < \eta < |A|$. We construct a particular preference profile $(\bar{P}_i, P_{-i})$ where $r_1(P_j) = (b^s, z^{-s})$ for all $j \neq i$. We know that (a^s, z^{-s}) can be assembled by a voter sequence $\underline{i}$ such that $i^s = i$ and $i^\tau \neq i$ for all $\tau \neq s$. Hence, $\varphi_{(a^s, z^{-s})}(\bar{P}_i, P_{-i}) \geq \gamma(\underline{i}) > 0$. Given $P_i' \in \mathbb{D}^{(b^s, z^{-s})}$ by minimal richness, it is evident that $\varphi_{(b^s, z^{-s})}(P_i', P_{-i}) = 1$. Consequently, we have $\sum_{t=1}^{\eta} \varphi_{r_t(\bar{P}_i)}(\bar{P}_i, P_{-i}) < 1 = \sum_{t=1}^{\eta} \varphi_{r_t(\bar{P}_i)}(P_i', P_{-i})$, and hence voter i will manipulate at $(\bar{P}_i, P_{-i})$ via P_i'. Therefore, all preferences must be top-separable.

Appendix B. Proof of Proposition 2

We first provide four general results which will be repeatedly applied. Let $\mathbb{D}$ be a connected$^+$ domain and $\varphi: \mathbb{D}^N \to \Delta(A)$ be an sd-strategy-proof RSCF.

Lemma 8. *Fix $i \in I$, $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$. The following two statements hold:*

1. *If $P_i \sim P_i'$ and $\Gamma(P_i, P_i') = \big\{\{a, b\}\big\}$,*[44] *then we have*
   (i) $\varphi_a(P_i, P_{-i}) \geq \varphi_a(P_i', P_{-i})$ *and* $\varphi_b(P_i, P_{-i}) \leq \varphi_b(P_i', P_{-i})$;
   (ii) $\varphi_a(P_i, P_{-i}) + \varphi_b(P_i, P_{-i}) = \varphi_a(P_i', P_{-i}) + \varphi_b(P_i', P_{-i})$;
   (iii) $\varphi_z(P_i, P_{-i}) = \varphi_z(P_i', P_{-i})$ *for all* $z \notin \{a, b\}$.
2. *If $P_i \sim^+ P_i'$ and $\Gamma(P_i, P_i') = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s} \in A^{-s}}$, then we have*
   (i) $\varphi_{(a^s, z^{-s})}(P_i, P_{-i}) \geq \varphi_{(a^s, z^{-s})}(P_i', P_{-i})$ *and* $\varphi_{(b^s, z^{-s})}(P_i, P_{-i}) \leq \varphi_{(b^s, z^{-s})}(P_i', P_{-i})$ *for all* $z^{-s} \in A^{-s}$;
   (ii) $\varphi_{(a^s, z^{-s})}(P_i, P_{-i}) + \varphi_{(b^s, z^{-s})}(P_i, P_{-i}) = \varphi_{(a^s, z^{-s})}(P_i', P_{-i}) + \varphi_{(b^s, z^{-s})}(P_i', P_{-i})$ *for all* $z^{-s} \in A^{-s}$;
   (iii) $\varphi_c(P_i, P_{-i}) = \varphi_c(P_i', P_{-i})$ *for all* $c \in A$ *with* $c^s \notin \{a^s, b^s\}$.

The verification of Lemma 8 is routine, and we hence omit it.

Lemma 9. *Two preferences with distinct peaks are never adjacent.*

Proof. Suppose not, i.e., there exist two preferences $P_i, P_i' \in \mathbb{D}$ such that $r_1(P_i) \equiv a \neq b \equiv r_1(P_i')$ and $P_i \sim P_i'$. Alternatives a and b must disagree on some component, say $a^s \neq b^s$. Given the Cartesian product structure, we can identify $x, y \in A \backslash \{a, b\}$ such that $x^s = a^s$, $y^s = b^s$ and $x^{-s} = y^{-s}$. Thus, x and y are similar, and $P_i \sim P_i'$ implies either $x P_i y$ and $x P_i' y$, or $y P_i x$ and $y P_i' x$. However, since P_i and P_i' are top-separable preferences, $r_1(P_i) = a$ implies $x P_i y$ while $r_1(P_i') = b$ implies $y P_i' x$. Contradiction! □

Lemma 10. *Let $P_i \sim P_i'$ and $\Gamma(P_i, P_i') = \big\{\{a, b\}\big\}$. Let $P_j \sim P_j'$ or $P_j \sim^+ P_j'$. Assume either $a P_j b$ and $a P_j' b$, or $b P_j a$ and $b P_j' a$. We have*

$$\big[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})\big] \Rightarrow \big[\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})\big].$$

Proof. Since $P_j \sim P_j'$ or $P_j \sim^+ P_j'$, and P_j and P_j' agree on the relative ranking of a and b, we can identify an integer $1 \leq t \leq |A|$ such that P_j and P_j' have the same set of top-t ranked alternatives which either includes a and excludes b, or includes b and excludes a, i.e., either $a \in \{r_k(P_j)\}_{k=1}^t = \{r_k(P_j')\}_{k=1}^t \not\ni b$ (if $a P_j b$ and $a P_j' b$), or $a \notin \{r_k(P_j)\}_{k=1}^t = \{r_k(P_j')\}_{k=1}^t \ni b$ (if $b P_j a$ and $b P_j' a$). Thus, we assert in the terminology of Chatterji and Zeng (2018): Alternatives a and b are *isolated* in P_j and P_j'. Then, the verification of this lemma follows exactly from Lemma 1 of Chatterji and Zeng (2018). □

Lemma 11. *Let $P_i \sim^+ P_i'$ and $\Gamma(P_i, P_i') = \big\{\{(x^s, z^{-s}), (y^s, z^{-s})\}\big\}_{z^{-s} \in A^{-s}}$. Assume $P_j \sim^+ P_j'$, and either $(x^s, z^{-s}) P_j (y^s, z^{-s})$ and $(x^s, z^{-s}) P_j' (y^s, z^{-s})$ for all $z^{-s} \in A^{-s}$, or $(y^s, z^{-s}) \times P_j (x^s, z^{-s})$ and $(y^s, z^{-s}) P_j' (x^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. We have*

$$\big[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})\big] \Rightarrow \big[\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})\big].$$

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[44] Recall that whenever we write $\{a, b\} \in \Gamma(P_i, P_i')$, we presume $a P_i b$ and $b P_i' a$.

Proof. According to items 2(ii) and 2(iii) of Lemma 8, to verify this lemma, it suffices to show that given $z^{-s} \in A^{-s}$, $\varphi_{(a^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$ for some $a^s \in \{x^s, y^s\}$.

We assume $(x^s, z^{-s})P_j(y^s, z^{-s})$ and $(x^s, z^{-s})P'_j(y^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. The verification related to the other case is symmetric, and we hence omit it. Since $P_j \sim^+ P'_j$, we know that P_j and P'_j are separable preferences, and $\Gamma(P_j, P'_j) = \big\{\{(\bar{x}^\tau, z^{-\tau}), (\bar{y}^\tau, z^{-\tau})\}\big\}_{z^{-\tau}\in A^{-\tau}}$ for some $\tau \in M$ and $\bar{x}^\tau, \bar{y}^\tau \in A^\tau$. We consider two situations: $\tau = s$ and $\tau \neq s$.

Assume $\tau = s$. Given $z^{-s} \in A^{-s}$, since $(x^s, z^{-s})P_j(y^s, z^{-s})$ and $(x^s, z^{-s})P'_j(y^s, z^{-s})$, it is true that there exists $a^s \in \{x^s, y^s\}$ such that $a^s \notin \{\bar{x}^s, \bar{y}^s\}$. Therefore, item 2(iii) of Lemma 8 and the hypothesis imply $\varphi_{(a^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P_i, P_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P'_i, P_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$, as required.

Next, assume $\tau \neq s$. Given $z^{-s} \in A^{-s}$, either one of two cases occurs: (i) There exists $a^s \in \{x^s, y^s\}$ such that $(a^s, z^{-s}) \notin (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$, or (ii) (x^s, z^{-s}), $(y^s, z^{-s}) \in (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$. In the first case, item 2(iii) of Lemma 8 and the hypothesis imply $\varphi_{(a^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P_i, P_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P'_i, P_j, P_{-\{i,j\}}) = \varphi_{(a^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$, as required. If the second case occurs, it must be either $(x^s, z^{-s}) = (x^s, \bar{x}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{x}^\tau, z^{-\{s,\tau\}})$, or $(x^s, z^{-s}) = (x^s, \bar{y}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{y}^\tau, z^{-\{s,\tau\}})$.

Given $(x^s, z^{-s}) = (x^s, \bar{x}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{x}^\tau, z^{-\{s,\tau\}})$, by item 2(ii) of Lemma 8 and the hypothesis, we have

$$\begin{aligned}\sum\nolimits_{\bar{a}^\tau\in\{\bar{x}^\tau,\bar{y}^\tau\}} \varphi_{(x^s,\bar{a}^\tau,z^{-\{s,\tau\}})}\big(P_i, P'_j, P_{-\{i,j\}}\big) &= \sum\nolimits_{\bar{a}^\tau\in\{\bar{x}^\tau,\bar{y}^\tau\}} \varphi_{(x^s,\bar{a}^\tau,z^{-\{s,\tau\}})}\big(P_i, P_j, P_{-\{i,j\}}\big)\\ &= \sum\nolimits_{\bar{a}^\tau\in\{\bar{x}^\tau,\bar{y}^\tau\}} \varphi_{(x^s,\bar{a}^\tau,z^{-\{s,\tau\}})}\big(P'_i, P_j, P_{-\{i,j\}}\big)\\ &= \sum\nolimits_{\bar{a}^\tau\in\{\bar{x}^\tau,\bar{y}^\tau\}} \varphi_{(x^s,\bar{a}^\tau,z^{-\{s,\tau\}})}\big(P'_i, P'_j, P_{-\{i,j\}}\big).\end{aligned}$$

Furthermore, since item 2(i) of Lemma 8 implies

$$\begin{aligned}&\varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i, P'_j, P_{-\{i,j\}}) \geq \varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P'_i, P'_j, P_{-\{i,j\}}), \text{ and}\\ &\varphi_{(x^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P_i, P'_j, P_{-\{i,j\}}) \geq \varphi_{(x^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P'_i, P'_j, P_{-\{i,j\}}),\end{aligned}$$

we have $\varphi_{(x^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) \equiv \varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P'_i, P'_j, P_{-\{i,j\}}) \equiv \varphi_{(x^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$, as required.

Given $(x^s, z^{-s}) = (x^s, \bar{y}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{y}^\tau, z^{-\{s,\tau\}})$, by item 2(ii) of Lemma 8 and the hypothesis, we have

$$\begin{aligned}\sum\nolimits_{\bar{a}^\tau\in\{\bar{x}^\tau,\bar{y}^\tau\}} \varphi_{(y^s,\bar{a}^\tau,z^{-\{s,\tau\}})}\big(P_i, P'_j, P_{-\{i,j\}}\big) &= \sum\nolimits_{\bar{a}^\tau\in\{\bar{x}^\tau,\bar{y}^\tau\}} \varphi_{(y^s,\bar{a}^\tau,z^{-\{s,\tau\}})}\big(P_i, P_j, P_{-\{i,j\}}\big)\\ &= \sum\nolimits_{\bar{a}^\tau\in\{\bar{x}^\tau,\bar{y}^\tau\}} \varphi_{(y^s,\bar{a}^\tau,z^{-\{s,\tau\}})}\big(P'_i, P_j, P_{-\{i,j\}}\big)\\ &= \sum\nolimits_{\bar{a}^\tau\in\{\bar{x}^\tau,\bar{y}^\tau\}} \varphi_{(y^s,\bar{a}^\tau,z^{-\{s,\tau\}})}\big(P'_i, P'_j, P_{-\{i,j\}}\big).\end{aligned}$$

Furthermore, since item 2(i) of Lemma 8 implies

$$\begin{aligned}&\varphi_{(y^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i, P'_j, P_{-\{i,j\}}) \leq \varphi_{(y^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P'_i, P'_j, P_{-\{i,j\}}), \text{ and}\\ &\varphi_{(y^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P_i, P'_j, P_{-\{i,j\}}) \leq \varphi_{(y^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P'_i, P'_j, P_{-\{i,j\}}),\end{aligned}$$

we have $\varphi_{(y^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) \equiv \varphi_{(y^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{(y^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P'_i, P'_j, P_{-\{i,j\}})$ $\equiv \varphi_{(y^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$, as required. $\square$

Now, we start to prove Proposition 2. Let domain $\mathbb{D}$ be connected$^+$. If $N = 1$, it is evident that unanimity implies the tops-only property.[45] Next, we provide an induction argument on the number of voters.

Induction Hypothesis: Given $N \geq 2$, every unanimous and sd-strategy-proof RSCF $\phi : \mathbb{D}^n \to \Delta(A)$ with $1 \leq n < N$ satisfies the tops-only property.

Given a unanimous and sd-strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$, we show that φ satisfies the tops-only property. According to the Interior$^+$ property, it suffices to show that **fixing** $i \in I$, for all $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i)$ and either $P_i \sim P'_i$ or $P_i \sim^+ P'_i$ and $P_{-i} \in \mathbb{D}^{N-1}$, we have $\varphi(P_i, P_{-i}) = \varphi(P'_i, P_{-i})$.

We first induce an $(N-1)$-voter RSCF. Fixing $j \in I\backslash\{i\}$, let $\phi(P_i, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$ for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. It is evident that ϕ is a well-defined RSCF satisfying unanimity and sd-strategy-proofness. Hence, the induction hypothesis implies that ϕ satisfies the tops-only property. Henceforth, we fix $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i) \equiv x^*$ and either $P_i \sim P'_i$ or $P_i \sim^+ P'_i$, and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. We show $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$ for all $P_j \in \mathbb{D}$.

The lemma below implies that if $r_1(P_j) = x^*$, then $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$.

Lemma 12. *Given $P_j, P'_j \in \mathbb{D}$ with $r_1(P_j) = r_1(P'_j) = x^*$, we have*

(i) $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$ *and* $\varphi(P_i, P'_j, P_{-\{i,j\}}) = \varphi(P'_i, P'_j, P_{-\{i,j\}})$;
(ii) $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i, P'_j, P_{-\{i,j\}})$ *and* $\varphi(P'_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P'_j, P_{-\{i,j\}})$.

Proof. Given $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, by sd-strategy-proofness, we have that for every $1 \leq l \leq |A|$,

$$\left.\begin{aligned}\textstyle\sum_{k=1}^{l} \varphi_{r_k(P_i)}(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P_i)}(P_i, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P_i)}(P_i, P_i, P_{-\{i,j\}}),\\ \textstyle\sum_{k=1}^{l} \varphi_{r_k(P'_i)}(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P'_i)}(P'_i, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P'_i)}(P'_i, P'_i, P_{-\{i,j\}}),\end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned}\textstyle\sum_{k=1}^{l} \varphi_{r_k(P_i)}(P'_j, P'_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P_i)}(P_i, P'_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P_i)}(P_i, P_i, P_{-\{i,j\}}),\\ \textstyle\sum_{k=1}^{l} \varphi_{r_k(P'_i)}(P'_j, P'_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P'_i)}(P'_i, P'_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P'_i)}(P'_i, P'_i, P_{-\{i,j\}}),\end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}\textstyle\sum_{k=1}^{l} \varphi_{r_k(P_j)}(P_i, P_i, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P_j)}(P_i, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P_j)}(P_j, P_j, P_{-\{i,j\}}),\\ \textstyle\sum_{k=1}^{l} \varphi_{r_k(P'_j)}(P_i, P_i, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P'_j)}(P_i, P'_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P'_j)}(P'_j, P'_j, P_{-\{i,j\}}),\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned}\textstyle\sum_{k=1}^{l} \varphi_{r_k(P_j)}(P'_i, P'_i, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P_j)}(P'_i, P_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P_j)}(P_j, P_j, P_{-\{i,j\}}),\\ \textstyle\sum_{k=1}^{l} \varphi_{r_k(P'_j)}(P'_i, P'_i, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P'_j)}(P'_i, P'_j, P_{-\{i,j\}}) \leq \sum_{k=1}^{l} \varphi_{r_k(P'_j)}(P'_j, P'_j, P_{-\{i,j\}}).\end{aligned}\right\} \tag{4}$$

[45] We follow Chatterji and Sen (2011) and add the case $N = 1$ just to simplify the proof.

In Inequalities (1), since $\varphi(P_j, P_j, P_{-\{i,j\}}) \equiv \phi(P_j, P_{-\{i,j\}}) = \phi(P_i, P_{-\{i,j\}}) \equiv \varphi(P_i, P_i, P_{-\{i,j\}})$ and $\varphi(P_j, P_j, P_{-\{i,j\}}) \equiv \phi(P_j, P_{-\{i,j\}}) = \phi(P_i', P_{-\{i,j\}}) \equiv \varphi(P_i', P_i', P_{-\{i,j\}})$ by the induction hypothesis, it is true that $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_j, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$. This proves the first part of item (i). Symmetrically, by Inequalities (2), (3), (4) and the induction hypothesis, we complete the verification of the lemma. $\square$

Henceforth, we fix $P_j \in \mathbb{D}$ with $r_1(P_j) \neq x^*$. The lemma below considers the situation $P_i \sim P_i'$.

Lemma 13. *Let $P_i \sim P_i'$ and $\Gamma(P_i, P_i') = \big\{\{a, b\}\big\}$. Given $P_j \in \mathbb{D}$ with $r_1(P_j) \neq x^*$, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$.*

Proof. We assume w.l.o.g. that $a P_j b$. The verification related to the case $b P_j a$ is symmetric and we hence omit it. Now, by the Exterior$^+$ property, we have a path $\{P_j^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_j such that $a P_j^k b$ for all $1 \leq k \leq t$.[46] Since $\varphi(P_i, P_j^1, P_{-\{i,j\}}) = \varphi(P_i', P_j^1, P_{-\{i,j\}})$ by item (i) of Lemma 12, following the path from P_j^1 to $P_j^t = P_j$, and repeatedly applying Lemma 10, we eventually have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$. $\square$

Now, to complete the verification, we consider the situation $P_i \sim^+ P_i'$.

Lemma 14. *Let $P_i \sim^+ P_i'$ and $\Gamma(P_i, P_i') = \big\{\{(x^s, z^{-s}), (y^s, z^{-s})\}\big\}_{z^{-s} \in A^{-s}}$. Given $P_j \in \mathbb{D}$ with $r_1(P_j) \neq x^*$, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$.*

Proof. Given an arbitrary $z^{-s} \in A^{-s}$, we assume w.l.o.g. that $(x^s, z^{-s}) P_j (y^s, z^{-s})$. The verification related to the case $(y^s, z^{-s}) P_j (x^s, z^{-s})$ is symmetric, and we hence omit it. According to the Exterior$^+$ property, we have a path $\{P_j^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_j such that $(x^s, z^{-s}) P_j^k (y^s, z^{-s})$ for all $1 \leq k \leq t$.[47] Evidently, $\varphi(P_i, P_j^1, P_{-\{i,j\}}) = \varphi(P_i', P_j^1, P_{-\{i,j\}})$ by item (i) of Lemma 12. We introduce another induction argument.

The Secondary Induction Hypothesis: Given $1 < k \leq t$, for all $1 \leq k' < k$, we have $\varphi(P_i, P_j^{k'}, P_{-\{i,j\}}) = \varphi(P_i', P_j^{k'}, P_{-\{i,j\}})$.

We show $\varphi(P_i, P_j^k, P_{-\{i,j\}}) = \varphi(P_i', P_j^k, P_{-\{i,j\}})$. First, we know either $P_j^{k-1} \sim^+ P_j^k$ or $P_j^{k-1} \sim P_j^k$. Assume $P_j^{k-1} \sim^+ P_j^k$. Thus, P_j^{k-1} and P_j^k are separable preferences. Since $(x^s, z^{-s}) P_j^{k-1} (y^s, z^{-s})$ and $(x^s, z^{-s}) P_j^k (y^s, z^{-s})$, separability implies $(x^s, z^{-s}) P_j^{k-1} (y^s, z^{-s})$ and $(x^s, z^{-s}) P_j^k (y^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. Consequently, by Lemma 11, $\varphi(P_i, P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P_i', P_j^{k-1}, P_{-\{i,j\}})$ implies $\varphi(P_i, P_j^k, P_{-\{i,j\}}) = \varphi(P_i', P_j^k, P_{-\{i,j\}})$, as required.

Next, assume $P_j^{k-1} \sim P_j^k$ and $\Gamma(P_j^{k-1}, P_j^k) = \big\{\{a, b\}\big\}$ (note that $a P_j^{k-1} !b$ and $b P_j^k !a$). Thus, Lemma 9 implies $r_1(P_j^{k-1}) = r_1(P_j^k) \equiv x$ and $x \notin \{a, b\}$. We consider two situations: $x = x^*$ and $x \neq x^*$. First, if $x = x^*$, then item (i) of Lemma 12 implies $\varphi(P_i, P_j^k, P_{-\{i,j\}}) =$

[46] If $b P_j a$, we have a path $\{P_j^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i' and P_j such that $b P_j^k a$ for all $1 \leq k \leq t$.

[47] If $(y^s, z^{-s}) P_j (x^s, z^{-s})$, we have a path $\{P_j^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i' and P_j such that $(y^s, z^{-s}) P_j^k (x^s, z^{-s})$ for all $1 \leq k \leq t$.

$\varphi(P_i', P_j^k, P_{-\{i,j\}})$. Second, assume $x \neq x^*$. Assume w.l.o.g. that aP_ib. The verification related to the case bP_ia is symmetric, and we hence omit it. Thus, the Exterior$^+$ property implies that there exists a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_j^{k-1} and P_i such that aP_i^kb for all $1 \leq k \leq q$.[48] Since $\varphi(P_i^1, P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P_i^1, P_j^k, P_{-\{i,j\}})$ by item (ii) of Lemma 12, following the path $\{P_i^k\}_{k=1}^q$ from P_i^1 to $P_i^q = P_i$, and repeatedly applying Lemma 10, we eventually have $\varphi(P_i, P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P_i, P_j^k, P_{-\{i,j\}})$.[49] Analogously, we also have $\varphi(P_i', P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P_i', P_j^k, P_{-\{i,j\}})$.[50] Last, by the secondary induction hypothesis, $\varphi(P_i, P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P_i', P_j^{k-1}, P_{-\{i,j\}})$ implies $\varphi(P_i, P_j^k, P_{-\{i,j\}}) = \varphi(P_i', P_j^k, P_{-\{i,j\}})$. This completes the verification of the secondary induction hypothesis. Therefore, $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$. $\square$

Finally, by Lemmas 13 and 14, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$ for all $P_j \in \mathbb{D}$. This completes the verification of the induction hypothesis and hence proves Proposition 2.

Appendix C. Proof of Theorem 2

First, by the verification of the sufficiency part of Theorem 1, we know that all multidimensional projection rules are unanimous, anonymous and sd-strategy-proof on the multidimensional single-peaked domain. Therefore, we focus on the necessity part of Theorem 2.

Let $\mathbb{D}$ be a minimally rich and connected$^+$ domain. Let $\bar{f} : \mathbb{D}^N \to A$ be a unanimous, anonymous and sd-strategy-proof DSCF. First, Proposition 2 implies that $\bar{f}$ satisfies the tops-only property. Next, note that we establish all Lemmas 1, 5, 6 and 7 in the proof of the necessity part of Theorem 1 without referring to any RSCFs. These lemmata therefore remain valid for the DSCF $\bar{f}$. Therefore, to complete the verification, we only need to use $\bar{f}$ and its induced DSCFs to prove the results of Lemmas 3 and 4.

There are two cases: N is an even integer, and N is an odd integer. If N is an even integer, we separate I into $\hat{I} = \{1, \ldots, \frac{N}{2}\}$ and $\bar{I} = \{\frac{N}{2} + 1, \ldots, N\}$, and similar to the proof of the necessity part of Theorem 1, we induce a two-voter DSCF $\hat{f} : \mathbb{D}^2 \to A$ which is unanimous, anonymous, tops-only and sd-strategy-proof. If N is an odd integer, we separate I into three subgroups $\hat{I} = \{1, 2, \ldots, \frac{N-1}{2}\}$, $\bar{I} = \{\frac{N+1}{2}, \ldots, N-1\}$ and $I \backslash [\hat{I} \cup \bar{I}] = \{N\}$, and induce the following three DSCFs: For all $P_i, P_j, P_N \in \mathbb{D}$,

$$f(P_i, P_j, P_N) = \bar{f}\Big(\frac{P_i}{\hat{I}}, \frac{P_j}{\bar{I}}, P_N\Big),$$
$$g(P_i, P_j) = f(P_i, P_j, P_j) \text{ and } h(P_i, P_N) = f(P_i, P_i, P_N).$$

Note that f is unanimous, tops-only and sd-strategy-proof, and satisfies **constrained anonymity**: $f(P_i, P_j, P_N) = f(P_j, P_i, P_N)$ for all $P_i, P_j, P_N \in \mathbb{D}$, while both DSCFs g and h are unani-

48 If bP_ia, we have a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_j^k and P_i such that bP_i^ka for all $1 \leq k \leq q$.

49 To apply Lemma 10 here, we need to make a notational change on the expression of Lemma 10 by switching voters i and j: Let $P_j \sim P_j'$ and $\Gamma(P_j, P_j') = \big\{\{a, b\}\big\}$. Let $P_i \sim P_i'$ or $P_i \sim^+ P_i'$. Assume that either aP_ib and $aP_i'b$, or bP_ia and $bP_i'a$. We have $\big[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i, P_j', P_{-\{i,j\}})\big] \Rightarrow \big[\varphi(P_i', P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})\big]$.

50 If $aP_i'b$, then by the Exterior$^+$ property, we have a path connecting P_j^{k-1} and P_i' along which each preference ranks a above b. If $bP_i'a$, then by the Exterior$^+$ property, we have a path connecting P_j^k and P_i' along which each preference ranks b above a. Then, by a similar argument, we can show $\varphi(P_i', P_j^{k-1}, P_{-\{i,j\}}) = \varphi(P_i', P_j^k, P_{-\{i,j\}})$.

mous, tops-only and sd-strategy-proof. The verification for the case of an even number of voters is significantly simpler. Henceforth, we assume that N is an odd integer, and establish lemmas accordingly. To make clear that our proof applies to the case of an even number of voters as well, we provide a paragraph at the end of each lemma to explain how the proof can be adapted to the two-voter DSCF $\hat{f}$.

We first provide an intermediate step which will be repeatedly applied in the subsequent verification.

Lemma 15. *The following three statements hold:*

1. *Given $a, x, y, z \in A$, if $f(x, y, a) = y$, $f(y, z, a) = z$ and $y \sim^+ z$, then $f(x, z, a) = z$.*
2. *Let $\{x_k\}_{k=1}^t \subseteq (A^s, x^{-s})$ be an adjacent$^+$ sequence. Given $a, y \in A$, if $f(x_1, y, a) = (x_1^s, z^{-s})$ for some $z^{-s} \in A^{-s}$, then $f(x_l, y, a) \in \{(x_k^s, z^{-s})\}_{k=1}^l$ for all $l = 1, \ldots, t$.*
3. *Let $\{x_k\}_{k=1}^t \subseteq (A^s, z^{-s})$ be an adjacent$^+$ sequence. Given $a \in A$, if $f(x_1, x_1, a) = x_1$ and $f(x_t, x_t, a) = x_t$, then $f(x_k, x_k, a) = x_k$ for all $k = 1, \ldots, t$.*

Proof. Since $z \sim^+ y$ and $f(x, y, a) = y$, sd-strategy-proofness implies $f(x, z, a) \in \{y, z\}$. If $f(x, z, a) = y$, sd-strategy-proofness implies $f(y, z, a) = y$ which contradicts the hypothesis $f(y, z, a) = z$. Therefore, $f(x, z, a) = z$. This completes the verification of the first statement.

Since $f(x_1, y, a) = (x_1^s, z^{-s})$, we adopt an induction hypothesis: Given $1 < k \leq t$, for all $1 \leq k' < k$, we have $f(x_{k'}, y, a) \in \{(x_1^s, z^{-s}), (x_2^s, z^{-s}), \ldots, (x_{k'}^s, z^{-s})\}$. We show $f(x_k, y, a) \in \{(x_1^s, z^{-s}), (x_2^s, z^{-s}), \ldots, (x_k^s, z^{-s})\}$. Since $x_{k-1} \sim^+ x_k$, we have $P_i \in \mathbb{D}^{x_{k-1}}$ and $P_i' \in \mathbb{D}^{x_k}$ with $P_i \sim^+ P_i'$. Note that $\{(x_{k-1}^s, z^{-s}), (x_k^s, z^{-s})\} \in \Gamma(P_i, P_i')$. If $f(P_i, y, a) = (x_l^s, z^{-s})$ where $l < k-1$ by the induction hypothesis, item 2(iii) of Lemma 8 implies $f(x_k, y, a) = f(P_i', y, a) = f(P_i, y, a) = (x_l^s, z^{-s})$. If $f(P_i, y, a) = (x_{k-1}^s, z^{-s})$ by the induction hypothesis, item 2(ii) of Lemma 8 implies $f(x_k, y, a) = f(P_i', y, a) \in \{(x_{k-1}^s, z^{-s}), (x_k^s, z^{-s})\}$. This completes the verification of the induction hypothesis, and hence proves the second statement.

To verify the third statement, we refer to DSCF h, and have $h(x_1, a) = f(x_1, x_1, a) = x_1$ and $h(x_t, a) = f(x_t, x_t, a) = x_t$. Similar to the verification of Statement 2, we have $h(x_k, a) \in \{x_1, x_2, \ldots, x_k\}$ for all $k = 1, \ldots, t$. Given $1 < k < t$, suppose $h(x_k, a) \neq x_k$. Thus, $h(x_k, a) = x_l$ for some $1 \leq l < k$. Following the sequence $\{x_k\}_{k=1}^t$ from x_k to x_t, since $x_l^s \notin \{x_\nu^s, x_{\nu+1}^s\}$ for all $\nu = k, \ldots, t-1$, by repeatedly applying item 2(iii) of Lemma 8, we eventually have $h(x_t, a) = x_l$. Contradiction! Therefore, $f(x_k, x_k, a) = h(x_k, a) = x_k$. This proves the third statement. $\square$

In order to adapt the proof to the case of an even number of voters, we erase alternative a in each preference profile in Lemma 15; it is evident that the first two statements of Lemma 15 still hold under $\hat{f}$ while the third statement follows directly from the unanimity of $\hat{f}$.

Next, we replicate the conclusion of Lemma 3.

Lemma 16. *Given $s \in M$ and $x^{-s} \in A^{-s}$, $G_{\sim^+}\big((A^s, x^{-s})\big)$ is a tree.*

Proof. Suppose not, i.e., by Lemma 1, there exists a cycle $\{x_k\}_{k=1}^t \subseteq (A^s, x^{-s})$, $t \geq 3$, such that $x_k \sim^+ x_{k+1}$ for all $k = 1, \ldots, t$, where $x_{t+1} = x_1$. Start from profile (x_1, x_2). First, by unanimity and sd-strategy-proofness, $x_1 \sim^+ x_2$ implies $f(x_1, x_2, x_2) \in \{x_1, x_2\}$. We consider two cases separately: $f(x_1, x_2, x_2) = x_1$ and $f(x_1, x_2, x_2) = x_2$, and induce a contradiction in each case.

First, assume $f(x_1, x_2, x_2) = x_1$. We recall DSCF g. Thus, $g(x_1, x_2) = f(x_1, x_2, x_2) = x_1$. We claim $f(x_k, x_{k'}, x_{k'}) = g(x_k, x_{k'}) = x_k$ for all $1 \leq k < k' \leq t$. We introduce an induction hypothesis: Given $2 < l \leq t$, for all $1 \leq k < k' < l$, we have $g(x_k, x_{k'}) = x_k$. To verify the induction hypothesis, it suffices to show $g(x_k, x_l) = x_k$ for all $1 \leq k < l$. We consider two situations: (1) $k < l-1$ and (2) $k = l-1$. In situation (1), the induction hypothesis implies $g(x_k, x_{l-1}) = x_k$. Furthermore, since $x_l \sim^+ x_{l-1}$ and $x_k^s \notin \{x_{l-1}^s, x_l^s\}$, item 2(iii) of Lemma 8 implies $g(x_k, x_l) = x_k$. Next, assume that situation (2) occurs. Since $g(x_{l-2}, x_l) = x_{l-2}$ by situation (1) and $x_{l-1} \sim^+ x_{l-2}$, sd-strategy-proofness implies $g(x_{l-1}, x_l) \in \{x_{l-2}, x_{l-1}\}$. Meanwhile, since $x_{l-1} \sim^+ x_l$, unanimity and sd-strategy-proofness imply $g(x_{l-1}, x_l) \in \{x_{l-1}, x_l\}$. Therefore, it is true that $g(x_{l-1}, x_l) = x_{l-1}$. This completes the verification of the induction hypothesis. Thus, $f(x_k, x_{k'}, x_{k'}) = x_k$ for all $1 \leq k < k' \leq t$. Now, we have $f(x_t, x_2, x_t) = f(x_2, x_t, x_t) = x_2$ by constrained anonymity. Then, sd-strategy-proofness implies $f(x_t, x_2, x_2) = x_2$. Furthermore, since $x_t \sim^+ x_1$ and $x_2^s \notin \{x_t^s, x_1^s\}$, item 2(iii) of Lemma 8 implies $f(x_1, x_2, x_2) = x_2$. This contradicts the hypothesis $f(x_1, x_2, x_2) = x_1$. Therefore, $G_{\sim^+}\big((A^s, x^{-s})\big)$ must be a tree in the case $f(x_1, x_2, x_2) = x_1$.

Next, assume $f(x_1, x_2, x_2) = x_2$. Since $x_1 \sim^+ x_2$, by unanimity and sd-strategy-proofness, we also know $f(x_2, x_1, x_1) \in \{x_1, x_2\}$. If $f(x_2, x_1, x_1) = x_2$, we can relabel the cycle as $\{z_k\}_{k=1}^t$ such that $z_1 = x_2$, $z_2 = x_1$ and $z_k = x_{t+3-k}$ for all $k = 3, \ldots, t$. Thus, $f(z_1, z_2, z_2) = f(x_2, x_1, x_1) = x_2 = z_1$. Then, by the same argument in the second paragraph, we induce a contradiction.

Therefore, we further assume $f(x_2, x_1, x_1) = x_1$. We recall DSCF g. Thus, $g(x_1, x_2) = x_2$ and $g(x_2, x_1) = x_1$. We claim $g(x_{k'}, x_k) = x_k$ for all $1 \leq k, k' \leq t$. We introduce an induction hypothesis: Given $2 < l \leq t$, for all $1 \leq k, k' < l$, we have $g(x_k, x_{k'}) = x_{k'}$. To verify the induction hypothesis, it suffices to show $g(x_k, x_l) = x_l$ and $g(x_l, x_k) = x_k$ for all $1 \leq k \leq l$. If $k = l$, the result follows from the axiom of unanimity. Next, assume $k < l-1$, the induction hypothesis implies $g(x_k, x_{l-1}) = x_{l-1}$ and $g(x_{l-1}, x_k) = x_k$. According to $g(x_{l-1}, x_k) = x_k$, since $x_l \sim^+ x_{l-1}$ and $x_k^s \notin \{x_{l-1}^s, x_l^s\}$, item 2(iii) of Lemma 8 implies $g(x_l, x_k) = x_k$. According to $g(x_k, x_{l-1}) = x_{l-1}$, since $x_l \sim^+ x_{l-1}$, sd-strategy-proofness implies $g(x_k, x_l) \in \{x_{l-1}, x_l\}$. Suppose $g(x_k, x_l) = x_{l-1}$. We refer to the counter clockwise sequence $\{x_k, x_{k-1}, \ldots, x_1, x_t, x_{t-1}, \ldots, x_{l+1}, x_l\}$ from x_k to x_l which excludes x_{l-1}. For all x_η and x_κ in the counter clockwise sequence with $x_\eta \sim^+ x_\kappa$, we know $x_{l-1}^s \notin \{x_\eta^s, x_\kappa^s\}$. Therefore, by repeatedly applying item 2(iii) of Lemma 8, we have $x_{l-1} = g(x_k, x_l) = g(x_{k-1}, x_l) = \cdots = g(x_1, x_l) = g(x_t, x_l) = g(x_{t-1}, x_l) = \cdots = g(x_{l+1}, x_l) = g(x_l, x_l)$ which contradicts unanimity. Therefore, $g(x_k, x_l) = x_l$. Last, assume $k = l-1$. Since $l > 2$, we know $g(x_{l-2}, x_l) = x_l$ and $g(x_l, x_{l-2}) = x_{l-2}$. According to $g(x_{l-2}, x_l) = x_l$, since $x_{l-1} \sim^+ x_{l-2}$ and $x_l^s \notin \{x_{l-1}^s, x_{l-2}^s\}$, item 2(iii) of Lemma 8 implies $g(x_{l-1}, x_l) = x_l$. According to $g(x_l, x_{l-2}) = x_{l-2}$, since $x_{l-1} \sim^+ x_{l-2}$, sd-strategy-proofness implies $g(x_l, x_{l-1}) \in \{x_{l-2}, x_{l-1}\}$. Meanwhile, since $x_{l-1} \sim^+ x_l$, unanimity and sd-strategy-proofness imply $g(x_l, x_{l-1}) \in \{x_{l-1}, x_l\}$. Therefore, we have $g(x_l, x_{l-1}) = x_{l-1}$. This completes the verification of the induction hypothesis. Hence, $g(x_{k'}, x_k) = x_k$ for all $1 \leq k, k' \leq t$.

Now, we have $f(x_t, x_2, x_2) = g(x_t, x_2) = x_2$. Since $x_1 \sim^+ x_2$, sd-strategy-proofness implies $f(x_t, x_2, x_1) \in \{x_1, x_2\}$. We show that both cases $f(x_t, x_2, x_1) = x_1$ and $f(x_t, x_2, x_1) = x_2$ are invalid. Suppose $f(x_t, x_2, x_1) = x_2$. Since $x_t \sim^+ x_1$ and $x_2^s \notin \{x_1^s, x_t^s\}$, constrained anonymity and item 2(iii) of Lemma 8 imply $f(x_2, x_1, x_1) = f(x_1, x_2, x_1) = x_2$, which contradicts the hypothesis $f(x_2, x_1, x_1) = x_1$. Suppose $f(x_t, x_2, x_1) = x_1$. On the one hand, we refer to the counter clockwise sequence $\{x_t, x_{t-1}, \ldots, x_3, x_2\}$ from x_t to x_2 which excludes x_1. For all x_η and x_κ in the counter clockwise sequence with $x_\eta \sim^+ x_\kappa$, we know $x_1^s \notin \{x_\eta^s, x_\kappa^s\}$. Then, by repeatedly applying item 2(iii) of Lemma 8, we have $x_1 = f(x_t, x_2, x_1) = f(x_{t-1}, x_2, x_1) = \cdots = f(x_3, x_2, x_1) = f(x_2, x_2, x_1)$. Thus, $f(x_2, x_2, x_1) = x_1$. On the other hand, by anonymity of $\bar{f}$ and the hypothesis $f(x_1, x_2, x_2) = x_2$, we have $\bar{f}\big(\frac{x_1}{\bar{I}\backslash\{i\}}, x_2, \frac{x_2}{\bar{I}}, x_1\big) =$

$\bar{f}\big(\frac{x_1}{\hat{I}\backslash\{i\}}, x_1, \frac{x_2}{\bar{I}}, x_2\big) \equiv \bar{f}\big(\frac{x_1}{\hat{I}}, \frac{x_2}{\bar{I}}, x_2\big) = f(x_1, x_2, x_2) = x_2$. Then, sd-strategy-proofness implies $f(x_2, x_2, x_1) \equiv \bar{f}\big(\frac{x_2}{\hat{I}\backslash\{i\}}, x_2, \frac{x_2}{\bar{I}}, x_1\big) = x_2$. Contradiction! Therefore, $G_{\sim^+}\big((A^s, x^{-s})\big)$ must be a tree in the case $f(x_1, x_2, x_2) = x_2$. This completes the verification of the lemma. □

For the case of an even number of voters, we prove the result of Lemma 16 simply by erasing the third element in each preference profile of f, and changing f and g to $\hat{f}$ in the first three paragraphs of the proof of Lemma 16. The remaining paragraphs in the proof of Lemma 16 are omitted.

Lemma 17. *Given an adjacent*$^+$ *sequence* $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ *and* $P_i \in \mathbb{D}^{x_1}$, *we have* $x_k P_i x_{k+1}$ *for all* $k = 1, \ldots, q-1$.

Proof. Suppose $x_{k+1} P_i x_k$ for some $1 \leq k < q$. It is evident $1 < k < q$. Pick an arbitrary $P_i' \in \mathbb{D}^{x_{k+1}}$ by minimal richness. By the no-detour property, we have a path $\{P_i^l\}_{l=1}^p \subseteq \mathbb{D}^{(A^s, x^{-s})}$ connecting P_i and P_i' such that $x_{k+1} P_i^l x_k$ for all $l = 1, \ldots, p$. Evidently, $r_1(P_i^l) \neq x_k$ for all $1 \leq l \leq p$. Then, by the proof of Lemma 1, we elicit an adjacent$^+$ sequence connecting x_1 and x_{k+1} from $\{P_i^l\}_{l=1}^p$ which excludes x_k. This contradicts Lemma 16. Therefore, $x_k P_i x_{k+1}$ for all $k = 1, \ldots, q-1$. □

The verification of Lemma 17 has nothing related to the number of voters, and applies to both the odd and even cases.

Before proceeding further with the proof, we note that the order of Lemmas 2 and 3 is opposite to the order of Lemmas 16 and 17, which arises mainly from the difference between the random setting and the deterministic one. In the random setting, the preference restriction in Lemma 2 is simply induced from the compromise property of the RSCF, and Lemma 3 is proved by the RSCF characterization result in Lemma 2. In the deterministic case, Lemma 16 (identical to Lemma 3) is proved using mainly the anonymity of the DSCF, and the preference restriction in Lemma 17 (the counterpart of Lemma 2) is elicited from the result of Lemma 16 and the richness condition of connectedness$^+$.

We fix four alternatives: $a = (x^s, x^\tau, z^{-\{s,\tau\}})$, $b = (y^s, y^\tau, z^{-\{s,\tau\}})$, $c = (x^s, y^\tau, z^{-\{s,\tau\}})$ and $d = (y^s, x^\tau, z^{-\{s,\tau\}})$ where $x^s \neq y^s$ and $x^\tau \neq y^\tau$. Assume $a \sim^+ c$ and $a \sim^+ d$. Let $\{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$ denote the adjacent$^+$ sequence connecting $b \equiv x_1$ and $d \equiv x_p$, and $\{y_k\}_{k=1}^q \subseteq (A^s, y^\tau, z^{-\{s,\tau\}})$ denote the adjacent$^+$ sequence connecting $b \equiv y_1$ and $c \equiv y_q$ (recall Fig. 2).

Lemma 18. *Given* $a \sim^+ c$ *and* $a \sim^+ d$, *we have* $[b \sim^+ d] \Leftrightarrow [b \sim^+ c]$.

Proof. Assume $b \sim^+ d$. We show $b \sim^+ c$. Suppose not, i.e., $q > 2$. Fixing an arbitrary $1 < k < q$, Lemma 17 implies $y_k P_i c$ for all $P_i \in \mathbb{D}^b$ and $y_k \hat{P}_i b$ for all $\hat{P}_i \in \mathbb{D}^c$. Since $y_k^s \notin \{x^s, y^s\}$, we induce an alternative $x^* \equiv (y_k^s, x^\tau, z^{-\{s,\tau\}}) \in A\backslash\{a, d\}$. Thus, $a, d, x^* \in (A^s, x^\tau, z^{-\{s,\tau\}})$. Since $a \sim^+ d$, by Lemma 16, there exists an adjacent$^+$ sequence $\{z_k^k\}_{k=1}^\eta \subseteq (A^s, x^\tau, z^{-\{s,\tau\}})$ such that one of the following two cases occurs:

(1) $\{z_k^k\}_{k=1}^\eta$ connects d and x^*, and includes a (see Fig. 7(1)).

(2) $\{z_k^k\}_{k=1}^\eta$ connects d and x^*, and excludes d (see Fig. 7(2)).

In each case, we induce a contradiction.

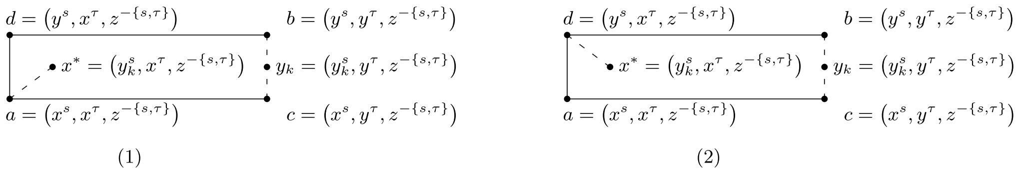

Fig. 7. Two cases for the adjacent$^+$ sequence $\{z_k^k\}_{k=1}^{\eta}$ connecting d and x^*.

Fig. 8. Four situations. Taking the first diagram of Fig. 8 as an example, the arrow "$a \to c$" represents that $a \sim^+ c$ and $f(a, c, a) = f(c, a, a) = c$.

In case (1), Lemma 17 implies $aP_i'x^*$ for all $P_i' \in \mathbb{D}^d$. Since $b \sim^+ d$, we have two separable preferences $\bar{P}_i \in \mathbb{D}^b$ and $\bar{P}_i' \in \mathbb{D}^d$ with $P_i \sim^+ P_i'$. Thus, $\Gamma(\bar{P}_i, \bar{P}_i') = \big\{\{(y^\tau, z^{-\tau}), (x^\tau, z^{-\tau})\}\big\}_{z^{-\tau} \in A^{-\tau}}$. Note that $y_k \bar{P}_i c$ and $a \bar{P}_i' x^*$. Since $y_k \equiv (y_k^s, y^\tau, z^{-\{s,\tau\}}) \bar{P}_i (x^s, y^\tau, z^{-\{s,\tau\}}) \equiv c$, separability implies $x^* \equiv (y_k^s, x^\tau, z^{-\{s,\tau\}}) \bar{P}_i (x^s, x^\tau, z^{-\{s,\tau\}}) \equiv a$. Thus, $x^* \bar{P}_i a$ and $a \bar{P}_i' x^*$. Hence, $\big\{(y_k^s, x^\tau, z^{-\{s,\tau\}}), (x^s, x^\tau, z^{-\{s,\tau\}})\big\} \in \Gamma(\bar{P}_i, \bar{P}_i')$. Contradiction! Therefore, $b \sim^+ c$.

In case (2), Lemma 17 implies $dP_i'x^*$ for all $P_i' \in \mathbb{D}^a$. Since $c \sim^+ a$, we have two separable preferences $\bar{P}_i \in \mathbb{D}^c$ and $\bar{P}_i' \in \mathbb{D}^a$ with $P_i \sim^+ P_i'$. Thus, $\Gamma(\bar{P}_i, \bar{P}_i') = \big\{\{(y^\tau, z^{-\tau}), (x^\tau, z^{-\tau})\}\big\}_{z^{-\tau} \in A^{-\tau}}$. Note that $y_k \bar{P}_i b$ and $d \bar{P}_i' x^*$. Since $y_k \equiv (y_k^s, y^\tau, z^{-\{s,\tau\}}) \bar{P}_i (y^s, y^\tau, z^{-\{s,\tau\}}) \equiv b$, separability implies $x^* \equiv (y_k^s, x^\tau, z^{-\{s,\tau\}}) \bar{P}_i (y^s, x^\tau, z^{-\{s,\tau\}}) \equiv d$. Thus, $x^* \bar{P}_i d$ and $d \bar{P}_i' x^*$. Hence, $\big\{(y_k^s, x^\tau, z^{-\{s,\tau\}}), (y^s, x^\tau, z^{-\{s,\tau\}})\big\} \in \Gamma(\bar{P}_i, \bar{P}_i')$. Contradiction! Therefore, $b \sim^+ c$.

Therefore, we conclude $[b \sim^+ d] \Rightarrow [b \sim^+ c]$. Symmetrically, we can show $[b \sim^+ d] \Leftarrow [b \sim^+ c]$. $\square$

The verification of Lemma 18 has nothing related to the number of voters, and applies to both the odd and even cases.

Now, we are ready to prove the equivalent of Lemma 4. Since $a \sim^+ c$ and $a \sim^+ d$, by sd-strategy-proofness and constrained anonymity of f, we have $f(a, c, a) = f(c, a, a) \in \{c, a\}$ and $f(a, d, a) = f(d, a, a) \in \{a, d\}$. Therefore, there are four situations (also see Fig. 8):

Situation 1. $f(a, c, a) = f(c, a, a) = c$ and $f(a, d, a) = f(d, a, a) = a$.
Situation 2. $f(a, c, a) = f(c, a, a) = c$ and $f(a, d, a) = f(d, a, a) = d$.
Situation 3. $f(a, c, a) = f(c, a, a) = a$ and $f(a, d, a) = f(d, a, a) = d$.
Situation 4. $f(a, c, a) = f(c, a, a) = a$ and $f(a, d, a) = f(d, a, a) = a$.

Note that Situations 1 and 3 are analogous. Therefore, we only consider Situations 1, 2 and 4. We show that in each situation, $b \sim^+ c$ and $b \sim^+ d$. After removing the third element in each preference profile of f and switching notation f to $\hat{f}$, we establish the counterpart 4 situations in the case of an even number of voters.

Lemma 19. *In Situation 1, $b \sim^+ c$ and $b \sim^+ d$.*

Proof. Since $f(d,a,a)=a$, $f(a,c,a)=c$ and $c\sim^+ a$, Statement 1 of Lemma 15 implies $f(d,c,a)=c$. By Lemma 18, it suffices to show $b\sim^+ d$. Suppose not, i.e., $p>2$. Thus, we have $x_{p-1}\equiv(y^s,x^\tau_{p-1},z^{-\{s,t\}})$ and $x^\tau_{p-1}\notin\{x^\tau,y^\tau\}$.

CLAIM 1: $f(x_p,x_{p-1},a)=f(x_{p-1},x_p,a)=x_p$.

According to $f(x_p,c,a)\equiv f(d,c,a)=c$, since $x_p\sim^+ x_{p-1}$ and $c^\tau=y^\tau\notin\{x^\tau_p,x^\tau_{p-1}\}$, item 2(iii) of Lemma 8 implies $f(x_{p-1},c,a)=c$. Next, since $a\sim^+ c$, sd-strategy-proofness implies $f(x_{p-1},a,a)\in\{c,a\}$. Suppose $f(x_{p-1},a,a)=c$. Since $x_p\sim^+ x_{p-1}$ and $c^\tau\equiv y^\tau\notin\{x^\tau_p,x^\tau_{p-1}\}$, item 2(iii) of Lemma 8 implies $f(d,a,a)\equiv f(x_p,a,a)=f(x_{p-1},a,a)=c$ which contradicts the hypothesis of Situation 1. Therefore, $f(x_{p-1},a,a)=a$. Furthermore, since $x_p\sim^+ a$, sd-strategy-proofness implies $f(x_{p-1},x_p,a)\in\{x_p,a\}$. Suppose $f(x_{p-1},d,a)\equiv f(x_{p-1},x_p,a)=a$. On the one hand, following the sequence $\{x_k\}_{k=1}^p$ from x_{p-1} to $x_1\equiv b$, since $a^\tau\notin\{x^\tau_k,x^\tau_{k-1}\}$ for all $k=p-1,\ldots,2$, by repeatedly applying item 2(iii) of Lemma 8, we eventually have $f(b,d,a)=a$. On the other hand, recall $f(y_q,d,a)\equiv f(c,d,a)=f(d,c,a)=c\equiv y_q$ and the sequence $\{y_k\}_{k=1}^q$ (from y_q to y_1). Statement 2 of Lemma 15 implies $f(y_k,d,a)\in\{y_l\}_{l=k}^q$ for all $k=1,\ldots,q$. Consequently, $f(b,d,a)\equiv f(y_1,d,a)\neq a$. Contradiction! Therefore, it must be the case $f(x_{p-1},x_p,a)=x_p$. This completes the verification of the claim.

Next, we show $f(b,d,a)=d$. Since $f(x_{p-1},x_p,a)=x_p$ and $x^\tau_p\notin\{x^\tau_k,x^\tau_{k-1}\}$ for all $k=p-1,\ldots,2$, following the sequence $\{x_k\}_{k=1}^p$ from x_{p-1} to $x_1\equiv b$, by repeatedly applying item 2(iii) of Lemma 8, we eventually have $f(b,d,a)\equiv f(x_1,x_p,a)=x_p\equiv d$. Furthermore, since $d\sim^+ a$ and $f(d,a,a)=a$ by the hypothesis of Situation 1, Statement 1 of Lemma 15 implies $f(b,a,a)=a$. Pick an arbitrary $\hat{P}_i\in\mathbb{D}^b$. Since $r_1(\hat{P}_i)=b\equiv(y^s,y^\tau,z^{-\{s,\tau\}})$, top-separability implies $c\equiv(x^s,y^\tau,z^{-\{s,\tau\}})\hat{P}_i(x^s,x^\tau,z^{-\{s,\tau\}})\equiv a$. Consequently, by the hypothesis of Situation 1, we have $f(c,a,a)=c\hat{P}_i a=f(\hat{P}_i,a,a)$, and hence voter i will manipulate at $(\hat{P}_i,a,a)$ via $P'_i\in\mathbb{D}^c$. Therefore, it must be the case $b\sim^+ d$, as required. $\square$

In order to adapt the proof of Lemma 19 to the case of an even number of voters, we remove alternative a in each preference profile of f, and switch f to $\hat{f}$.

Lemma 20. *In Situation 2, $b\sim^+ c$ and $b\sim^+ d$.*

Proof. Suppose not, i.e., Lemma 18 implies $p>2$ and $q>2$.

CLAIM 1: According to sequence $\{x_k\}_{k=1}^p$, we have $f(a,b,a)=f(b,a,a)=d$.

Since $x_p\sim^+ x_{p-1}$, unanimity, constrained anonymity and sd-strategy-proofness imply $f(x_p,x_{p-1},x_p)=f(x_{p-1},x_p,x_p)\in\{x_{p-1},x_p\}$. Suppose $f(x_p,x_{p-1},x_p)=f(x_{p-1},x_p,x_p)=x_{p-1}$. Meanwhile, since $f(a,x_p,a)=f(x_p,a,a)=x_p$ by the hypothesis of Situation 2, sd-strategy-proofness implies $f(a,x_p,x_p)=f(x_p,a,x_p)=x_p$. According to alternatives x_p,x_{p-1} and a, we induce another alternative $x^*=(x^s,x^\tau_{p-1},z^{-\{s,\tau\}})$. Thus, $x_p\sim^+ a$, $x_p\sim^+ x_{p-1}$, $f(x_p,a,x_p)=f(a,x_p,x_p)=x_p$ and $f(x_p,x_{p-1},x_p)=f(x_{p-1},x_p,x_p)=x_{p-1}$ which together formulate an analogy of Situation 1 (see Fig. 9). Consequently, $x^*\sim^+ a$ and $x^*\sim^+ x_{p-1}$ by Lemma 19.

We next show $f(a,x_{p-1},a)=f(x_{p-1},a,a)=x_{p-1}$. Since $f(a,x_p,a)=x_p$ by the hypothesis of Situation 2 and $x_{p-1}\sim^+ x_p$, sd-strategy-proofness implies $f(a,x_{p-1},a)\in\{x_{p-1},x_p\}$. Suppose $f(a,x_{p-1},a)=x_p$. Then, sd-strategy-proofness implies $f(x_p,x_{p-1},x_p)=x_p$, which contradicts the hypothesis $f(x_p,x_{p-1},x_p)=x_{p-1}$. Therefore, $f(a,x_{p-1},a)=f(x_{p-1},a,a)=x_{p-1}$.

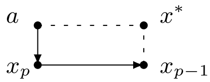

Fig. 9. The analogy of Situation 1 on $\{x_p, x^*, x_{p-1}, a\}$.

We next show $f(a, x^*, a) = f(x^*, a, a) = x^*$. Since $a \sim^+ x^*$, unanimity and sd-strategy-proofness imply $f(a, x^*, a) \in \{a, x^*\}$. Meanwhile, since $f(a, x_{p-1}, a) = x_{p-1}$ and $x^* \sim^+ x_{p-1}$, sd-strategy-proofness implies $f(a, x^*, a) \in \{x_{p-1}, x^*\}$. Therefore, $f(a, x^*, a) \in \{a, x^*\} \cap \{x_{p-1}, x^*\} = \{x^*\}$, and hence $f(a, x^*, a) = x^*$.

Note that $a, c, x^* \in (x^s, A^\tau, z^{-\{s,\tau\}})$, $x^* \sim^+ a$ and $a \sim^+ c$. According to $f(a, c, a) = c$ by the hypothesis of Situation 2, since $a \sim^+ x^*$ and $c^\tau \notin \{a^\tau, x^{*\tau}\}$, item 2(iii) of Lemma 8 implies $f(x^*, c, a) = c$. Furthermore, since $c \sim^+ a$, sd-strategy-proofness implies $f(x^*, a, a) \in \{a, c\}$ which contradicts $f(x^*, a, a) = x^*$. Therefore, it must be the case $f(x_p, x_{p-1}, x_p) = f(x_{p-1}, x_p, x_p) = x_p$. Equivalently, $f(x_{p-1}, d, d) = d$.

For notational convenience, recall DSCF g. We know $g(x_{p-1}, d) = f(x_{p-1}, d, d) = d$. Next, since $a \sim^+ d$, sd-strategy-proofness implies $g(x_{p-1}, a) \in \{a, d\}$. Suppose $g(x_{p-1}, a) = a$. Since $x_{p-1} \sim^+ x_p$, we have $P_i \in \mathbb{D}^{x_{p-1}}$ and $P_i' \in \mathbb{D}^{x_p}$ with $P_i \sim^+ P_i'$. Note that $\{x^*, a\} \in \Gamma(P_i, P_i')$, i.e., $x^* P_i !a$ and $a P_i' !x^*$. Since $g(P_i, a) = a$, item 2(i) of Lemma 8 implies $g(P_i', a) = a$. Thus, $f(d, a, a) = g(d, a) = a$ which contradicts the hypothesis of Situation 2. Therefore, $f(x_{p-1}, a, a) = g(x_{p-1}, a) = d$. Last, following the sequence $\{x_k\}_{k=1}^p$ from x_{p-1} to $x_1 \equiv b$, since $d^\tau \notin \{x_k^\tau, x_{k-1}^\tau\}$ for all $k = p-1, \ldots, 3, 2$, by repeatedly applying item 2(iii) of Lemma 8, we eventually have $f(b, a, a) = d$. This completes the verification of the claim.

Symmetric to Claim 1, according to sequence $\{y_k\}_{k=1}^q$ from $y_q = c$ to $y_1 = b$, we can show $f(a, b, a) = f(b, a, a) = c$, which contradicts Claim 1. Therefore, it must be the case $b \sim^+ c$ and $b \sim^+ d$. $\square$

In order to adapt the proof of Lemma 20 to the case of an even number of voters, we remove the third element in each preference profile of f, and replace f and g by $\hat{f}$.

The verification related to Situation 4 is more complicated. First, since $a \sim^+ d$, unanimity and sd-strategy-proofness imply $f(a, d, d) = f(d, a, d) \in \{a, d\}$. We then have to consider two separated cases under Situation 4:

(i) DSCF f is *invariant* at profiles (a, d, d) and (a, d, a), i.e., $f(a, d, d) = a = f(a, d, a)$, and
(ii) DSCF f is *variant* at profiles (a, d, d) and (a, d, a), i.e., $f(a, d, d) = d \neq a = f(a, d, a)$.

Note that in the case of an even number of voters, the hypothesis of Situation 4 coincides with the invariant case after removing the third element of each preference profile of f and replacing f by $\hat{f}$. Therefore, the invariant case arises for both cases of the odd and even number of voters, while the variant case only occurs for the case of an odd number of voters.

Lemma 21. *In Situation 4, if f is invariant at (a, d, d) and (a, d, a), i.e., $f(a, d, d) = a = f(a, d, a)$, then $b \sim^+ c$ and $b \sim^+ d$.*

Proof. By Lemma 18, it suffices to show $b \sim^+ d$. Suppose not, i.e., $p > 2$. According to the adjacent$^+$ sequence $\{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$, we replace the element y^s in each x_k by x^s, and then construct another sequence of alternatives $\{\bar{x}_k\}_{k=1}^p = \{(x^s, x_k^{-s})\}_{k=1}^p \equiv$

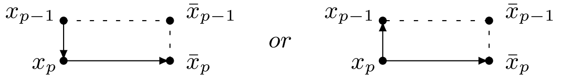

Fig. 10. The analogy of Situation 1 or 2 on $\{x_p, \bar{x}_{p-1}, \bar{x}_p, x_{p-1}\}$.

$\{(x^s, x_k^\tau, z^{-\{s,\tau\}})\}_{k=1}^p \subseteq (x^s, A^\tau, z^{-\{s,\tau\}})$. Note that $\bar{x}_p \equiv (x^s, x_p^\tau, z^{-\{s,\tau\}}) = (x^s, x^\tau, z^{-\{s,\tau\}}) \equiv a$, $\bar{x}_1 \equiv (x^s, x_1^\tau, z^{-\{s,\tau\}}) = (x^s, y^\tau, z^{-\{s,\tau\}}) \equiv c$, and $\{\bar{x}_k\}_{k=1}^p$ may *not* be an adjacent$^+$ sequence.

We start from $x_p, \bar{x}_{p-1}, \bar{x}_p$ and x_{p-1}. First, note that $x_p \sim^+ x_{p-1}$, $x_p \sim^+ \bar{x}_p$ and $f(x_p, \bar{x}_p, x_p) = f(\bar{x}_p, x_p, x_p) = \bar{x}_p$ by the invariance hypothesis. Second, since $x_p \sim^+ x_{p-1}$, unanimity and sd-strategy-proofness imply $f(x_p, x_{p-1}, x_p) = f(x_{p-1}, x_p, x_p) \in \{x_p, x_{p-1}\}$. Therefore, if $f(x_p, x_{p-1}, x_p) = f(x_{p-1}, x_p, x_p) = x_p$, then $\{x_p, \bar{x}_{p-1}, \bar{x}_p, x_{p-1}\}$ formulate an analogy of Situation 1, and if $f(x_p, x_{p-1}, x_p) = f(x_{p-1}, x_p, x_p) = x_{p-1}$, then $\{x_p, \bar{x}_{p-1}, \bar{x}_p, x_{p-1}\}$ formulate an analogy of Situation 2 (see Fig. 10). Then, by Lemma 19 or 20, we have $\bar{x}_{p-1} \sim^+ \bar{x}_p$ and $\bar{x}_{p-1} \sim^+ x_{p-1}$.

Furthermore, we recall DSCF g and claim $f(x_{p-1}, \bar{x}_{p-1}, x_{p-1}) = f(\bar{x}_{p-1}, x_{p-1}, x_{p-1}) = g(\bar{x}_{p-1}, x_{p-1}) = \bar{x}_{p-1}$. Since $x_{p-1} \sim^+ \bar{x}_{p-1}$, unanimity and sd-strategy-proofness imply $g(\bar{x}_{p-1}, x_{p-1}) \in \{x_{p-1}, \bar{x}_{p-1}\}$. Suppose $g(\bar{x}_{p-1}, x_{p-1}) = x_{p-1}$. On the one hand, since $x_p \sim^+ x_{p-1}$, sd-strategy-proofness implies $g(\bar{x}_{p-1}, x_p) \in \{x_{p-1}, x_p\}$. On the other hand, by the invariance hypothesis, we know $g(\bar{x}_p, x_p) = f(\bar{x}_p, x_p, x_p) \equiv f(a, d, d) = a \equiv \bar{x}_p$. Furthermore, since $\bar{x}_{p-1} \sim^+ \bar{x}_p$, sd-strategy-proofness implies $g(\bar{x}_{p-1}, x_p) \in \{\bar{x}_p, \bar{x}_{p-1}\}$. Contradiction! Therefore, $f(x_{p-1}, \bar{x}_{p-1}, x_{p-1}) = f(\bar{x}_{p-1}, x_{p-1}, x_{p-1}) = g(\bar{x}_{p-1}, x_{p-1}) = \bar{x}_{p-1}$.

Following the adjacent$^+$ sequence $\{x_k\}_{k=1}^p$ from x_{p-1} to $x_1 \equiv b$, we consecutively consider $\{x_k, \bar{x}_{k-1}, \bar{x}_k, x_{k-1}\}$ from $k = p-1$ to $k = 2$, and by repeatedly applying the similar argument in the two paragraphs right above, we have $\bar{x}_{k-1} \sim^+ \bar{x}_k$, $\bar{x}_{k-1} \sim^+ x_{k-1}$ and $f(x_{k-1}, \bar{x}_{k-1}, x_{k-1}) = f(\bar{x}_{k-1}, x_{k-1}, x_{k-1}) = \bar{x}_{k-1}$ for all $k = p-1, \ldots, 2$. Eventually, we have $c = \bar{x}_1 \sim^+ x_1 = b$ which furthermore implies $b \sim^+ d$ by Lemma 18. This contradicts the hypothesis $p > 2$. Therefore, it must be the case $b \sim^+ d$, as required. $\square$

In order to adapt the result of Lemma 21 to the case of an even number of voters, we remove the third element in each preference profile of f, and replace f and g by $\hat{f}$. Recall that the variant case cannot occur in the case of an even number of voters. Therefore, we have completed the verification of the necessity part of Theorem 2 for the case of an even number of voters. The remainder of the proof only applies to the case of an odd number of voters.

Lemma 22. *In Situation 4, if f is variant at (a, d, d) and (a, d, a), i.e., $f(a, d, d) = d \neq a = f(a, d, a)$, then $b \sim^+ c$ and $b \sim^+ d$.*

Proof. Suppose not, i.e., Lemma 18 implies $p > 2$ and $q > 2$.

CLAIM 1: $f(a, c, c) = f(c, a, c) = c$.

Since $a \sim^+ c$, unanimity and sd-strategy-proofness imply $f(c, a, c) = f(a, c, c) \in \{a, c\}$. Suppose $f(c, a, c) = f(a, c, c) = a$. Then, by the hypothesis of Situation 4, we have $f(a, c, c) = a = f(a, c, a)$. Thus, f is invariant at (a, c, c) and (a, c, a). Consequently, analogous to the proof of Lemma 21, we can eventually induce a contradiction! Therefore, $f(c, a, c) = f(a, c, c) = c$. This completes the verification of the claim.

CLAIM 2: $f(d, c, a) = f(c, d, a) = a$.

Since $a \sim^+ c$ and $f(d,a,a) = a$ by the hypothesis of Situation 4, sd-strategy-proofness implies $f(d,c,a) \in \{a,c\}$. Symmetrically, since $d \sim^+ a$ and $f(a,c,a) = a$ by the hypothesis of Situation 4, sd-strategy-proofness implies $f(d,c,a) \in \{a,d\}$. Therefore, it is true that $f(d,c,a) = a$. This completes the verification of the claim.

CLAIM 3: $f(d,d,a) = d$ and $f(c,c,a) = c$.

First, by anonymity of $\bar{f}$ and the variance hypothesis, we have $\bar{f}\left(\frac{d}{\hat{I}}, \frac{a}{\bar{I}\backslash\{j\}}, d, a\right) = \bar{f}\left(\frac{d}{\hat{I}}, \frac{a}{\bar{I}\backslash\{j\}}, a, d\right) \equiv \bar{f}\left(\frac{d}{\hat{I}}, \frac{a}{\bar{I}}, d\right) = f(d,a,d) = d$. Then, sd-strategy-proofness implies $f(d,d,a) \equiv \bar{f}\left(\frac{d}{\hat{I}}, \frac{d}{\bar{I}}, a\right) = d$. Symmetrically, by anonymity of $\bar{f}$ and Claim 1, we have $\bar{f}\left(\frac{c}{\hat{I}}, \frac{a}{\bar{I}\backslash\{j\}}, c, a\right) = \bar{f}\left(\frac{c}{\hat{I}}, \frac{a}{\bar{I}\backslash\{j\}}, a, c\right) \equiv \bar{f}\left(\frac{c}{\hat{I}}, \frac{a}{\bar{I}}, c\right) = f(c,a,c) = c$. Then, sd-strategy-proofness implies $f(c,c,a) \equiv \bar{f}\left(\frac{c}{\hat{I}}, \frac{c}{\bar{I}}, a\right) = c$. This completes the verification of the claim.

According to the adjacent⁺ sequence $\{x_k\}_{k=1}^{p} \subseteq (y^s, A^\tau, z^{-\{s,\tau\}})$, we replace the element y^s in each x_k by x^s, and then construct another sequence of alternatives $\{\bar{x}_k\}_{k=1}^{p} = \{(x^s, x_k^{-s})\}_{k=1}^{p} \subseteq (x^s, A^\tau, z^{-\{s,\tau\}})$. Note that $\bar{x}_p \equiv (x^s, x_p^\tau, z^{-\{s,\tau\}}) = (x^s, x^\tau, z^{-\{s,\tau\}}) \equiv a$, $\bar{x}_1 \equiv (x^s, x_1^\tau, z^{-\{s,\tau\}}) = (x^s, y^\tau, z^{-\{s,\tau\}}) \equiv c$, and $\{\bar{x}_k\}_{k=1}^{p}$ may not be an adjacent⁺ sequence. Symmetrically, according to the adjacent⁺ sequence $\{y_k\}_{k=1}^{q} \subseteq (A^s, y^\tau, z^{-\{s,\tau\}})$, we replace the element y^τ in each y_k by x^τ, and then construct another sequence of alternatives $\{\bar{y}_k\}_{k=1}^{q} = \{(y_k^s, x^\tau, z^{-\{s,\tau\}})\}_{k=1}^{p} \subseteq (A^s, x^\tau, z^{-\{s,\tau\}})$. Note that $\bar{y}_q \equiv (y_q^s, x^\tau, z^{-\{s,\tau\}}) = (x^s, x^\tau, z^{-\{s,\tau\}}) \equiv a$, $\bar{y}_1 \equiv (y_1^s, x^\tau, z^{-\{s,\tau\}}) = (y^s, x^\tau, z^{-\{s,\tau\}}) \equiv d$, and $\{\bar{y}_k\}_{k=1}^{q}$ may not be an adjacent⁺ sequence. Note that $\{\bar{x}_k\}_{k=1}^{p} \cap \{\bar{y}_k\}_{k=1}^{q} = \{a\}$.

CLAIM 4: $f(a, x_k, a) = f(x_k, a, a) = a$ for all $k = 1, \ldots, p$.

By the hypothesis of Situation 4, we first have $f(a, x_p, a) \equiv f(a,d,a) = a$. We next show $f(a, x_1, a) \equiv f(a,b,a) = a$. Since $f(a,d,a) = a$, following the sequence $\{x_k\}_{k=1}^{p}$ from $x_p = d$ to $x_1 = b$, Statement 2 of Lemma 15 implies $f(a,b,a) \in \{\bar{x}_k\}_{k=1}^{p}$. Symmetrically, since $f(a,c,a) = a$ by the hypothesis of Situation 4, following the sequence $\{y_k\}_{k=1}^{q}$ from $y_q = c$ to $y_1 = b$, Statement 2 of Lemma 15 implies $f(a,b,a) \in \{\bar{y}_k\}_{k=1}^{q}$. Thus, $f(a,b,a) \in \{\bar{x}_k\}_{k=1}^{p} \cap \{\bar{y}_k\}_{k=1}^{q} = \{a\}$, and hence $f(a, x_1, a) \equiv f(a,b,a) = a$. Furthermore, following the sequence $\{x_k\}_{k=1}^{p}$ from $x_1 \equiv b$ to x_{p-1}, since $a^\tau \notin \{x_k^\tau, x_{k+1}^\tau\}$ for all $k = 1, \ldots, p-3, p-2$, by repeatedly applying item 2(iii) of Lemma 8, we have $f(a, x_k, a) = a$ for all $k = 2, \ldots, p-1$. This completes the verification of the claim.

CLAIM 5: $f(x_k, x_k, a) = x_k$ for all $k = 1, \ldots, p$.

Recall DSCF h. We first show $h(b,a) = b$. On the one hand, according to the adjacent⁺ sequence $\{x_k\}_{k=1}^{p}$ from $x_p = d$ to $x_1 = b$, since $h(d,a) = f(d,d,a) = d$ by Claim 3, by a similar proof of Statement 2 of Lemma 15, we know $h(b,a) \in \{x_k\}_{k=1}^{p}$. On the other hand, according to the sequence $\{y_k\}_{k=1}^{q}$ from $y_q = c$ to $y_1 = b$, since $h(c,a) = f(c,c,a) = c$ by Claim 3, by a similar proof of Statement 2 of Lemma 15, we know $h(b,a) \in \{y_k\}_{k=1}^{q}$. Therefore, $h(b,a) \in \{x_k\}_{k=1}^{p} \cap \{y_k\}_{k=1}^{q} = \{b\}$, and hence $f(b,b,a) = h(b,a) = b$. Last, since $f(x_1, x_1, a) = f(b,b,a) = b = x_1$ and $f(x_p, x_p, a) = f(d,d,a) = d = x_p$ by Claim 3, Statement 3 of Lemma 15 implies $f(x_k, x_k, a) = x_k$ for all $k = 1, \ldots, p$. This completes the verification of the claim.

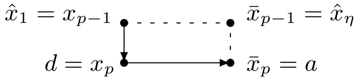

Fig. 11. The geometric relation among $x_p, \bar{x}_{p-1}, \bar{x}_p$ and x_{p-1}.

Claim 6: $f(x_k, x_{k-1}, a) = f(x_{k-1}, x_k, a) = x_k$ for all $k = 2, \dots, p$.

We first show $f(d, b, a) = d$. Since $f(x_p, x_p, a) = x_p$ by Claim 5, Statement 2 of Lemma 15 implies $f(d, b, a) \equiv f(x_p, x_1, a) \in \{x_k\}_{k=1}^p$. Next, since $d \sim^+ a$ and $f(a, b, a) = a$ by Claim 4, sd-strategy-proofness implies $f(d, b, a) \in \{a, d\}$. Thus, $f(d, b, a) \in \{x_k\}_{k=1}^p \cap \{a, d\} = \{d\}$, and hence $f(d, b, a) = d \equiv x_p$.

Now, given $1 < k \leq p$, since $x_k \sim^+ x_{k-1}$ and $f(x_k, x_k, a) = x_k$ by Claim 5, sd-strategy-proofness implies $f(x_k, x_{k-1}, a) \in \{x_k, x_{k-1}\}$. Suppose $f(x_k, x_{k-1}, a) = x_{k-1}$. First, if $k = p$, we have $f(x_p, x_{k-1}, a) = x_{k-1}$. Second, assume $k < p$. Following the sequence $\{x_k\}_{k=1}^p$ from x_k to $x_p = d$, since $x_{k-1}^s \notin \{x_l^s, x_{l+1}^s\}$ for all $k \leq l < p$, by repeatedly applying item 2(iii) of Lemma 8, we have $f(x_p, x_{k-1}, a) = x_{k-1}$. In conclusion, given $k \leq p$, we have $f(x_p, x_{k-1}, a) = x_{k-1}$. Next, following the sequence $\{x_k\}_{k=1}^p$ from x_{k-1} to $x_1 = b$, Statement 2 of Lemma 15 implies $f(d, b, a) \equiv f(x_p, x_1, a) \in \{x_{k-1}, \dots, x_2, x_1\}$. Contradiction! Therefore, $f(x_k, x_{k-1}, a) = x_k$ for all $k = 2, \dots, p$. This completes the verification of the claim.

Claim 7: Given $1 \leq k < p$, let $\{\hat{x}_\nu\}_{\nu=1}^\eta \subseteq (A^s, x_k^\tau, z^{-\{s,\tau\}})$ be the adjacent⁺ sequence connecting $x_k \equiv \hat{x}_1$ and $\bar{x}_k \equiv \hat{x}_\eta$. Then, $f(\hat{x}_\nu, \hat{x}_\nu, a) = \hat{x}_\nu$ for all $\nu = 1, \dots, \eta$.

Recall DSCF h. We first show $h(\bar{x}_k, a) = \bar{x}_k$. Since $\bar{x}_k, a \in (x^s, A^\tau, z^{-\{s,t\}})$, by Lemma 1, there exists an adjacent⁺ sequence in $(x^s, A^\tau, z^{-\{s,t\}})$ connecting $\bar{x}_k$ and a. Furthermore, since $h(\bar{x}_k, \bar{x}_k) = \bar{x}_k \in (x^s, A^\tau, z^{-\{s,t\}})$ by unanimity, by the proof of Statement 2 of Lemma 15, following the adjacent⁺ sequence in $(x^s, A^\tau, z^{-\{s,t\}})$ from $\bar{x}_k$ to a, we know $h(\bar{x}_k, a) \in (x^s, A^\tau, z^{-\{s,t\}})$. Next, since $h(\hat{x}_1, a) \equiv f(x_k, x_k, a) = x_k \equiv \hat{x}_1$ by Claim 6, following the sequence $\{\hat{x}_\nu\}_{\nu=1}^\eta \subseteq (A^s, x_k^\tau, z^{-\{s,\tau\}})$ from $\hat{x}_1$ to $\hat{x}_\eta \equiv \bar{x}_k$, Statement 2 of Lemma 15 implies $h(\bar{x}_k, a) \equiv h(\hat{x}_\eta, a) \in \{\hat{x}_\nu\}_{\nu=1}^\eta \subseteq (A^s, x_k^\tau, z^{-\{s,\tau\}})$. Therefore, $h(\bar{x}_k, a) \in (x^s, A^\tau, z^{-\{s,t\}}) \cap (A^s, x_k^\tau, z^{-\{s,\tau\}}) = \{(x^s, x_k^\tau, z^{-\{s,\tau\}})\} = \{\bar{x}_k\}$, and hence, $f(\hat{x}_\eta, \hat{x}_\eta, a) \equiv f(\bar{x}_k, \bar{x}_k, a) = h(\bar{x}_k, a) = \bar{x}_k \equiv \hat{x}_\eta$. Last, since $f(\hat{x}_1, \hat{x}_1, a) \equiv f(x_k, x_k, a) = x_k \equiv \hat{x}_1$ by Claim 5, according to $\{\hat{x}_\nu\}_{\nu=1}^\eta$, Statement 3 of Lemma 15 implies $f(\hat{x}_\nu, \hat{x}_\nu, a) = \hat{x}_\nu$ for all $\nu = 1, \dots, \eta$. This completes the verification of the claim.

We start from $\{x_p, \bar{x}_{p-1}, \bar{x}_p, x_{p-1}\}$. Let $\{\hat{x}_\nu\}_{\nu=1}^\eta \subseteq (A^s, x_k^\tau, z^{-\{s,\tau\}})$, $\eta \geq 2$, be the adjacent⁺ sequence connecting $x_{p-1} \equiv \hat{x}_1$ and $\bar{x}_{p-1} \equiv \hat{x}_\eta$ (see Fig. 11). Note that $f(x_p, \bar{x}_p, a) = f(\bar{x}_p, x_p, a) = a \equiv \bar{x}_p$ by the hypothesis of Situation 4, and $f(x_p, x_{p-1}, a) = f(x_{p-1}, x_p, a) = x_p$ by Claim 6 (see the two arrows in Fig. 11). We focus on preference profiles $(\cdot, \cdot, a)$ of f, where the first two elements belong to $\{x_p, \bar{x}_{p-1}, \bar{x}_p, x_{p-1}\}$, and the third element is fixed to be a. The next claim shows $\bar{x}_{p-1} \sim^+ x_{p-1}$.

Claim 8: $\bar{x}_{p-1} \sim^+ x_{p-1}$ and $\bar{x}_{p-1} \sim^+ \bar{x}_p$.

Suppose that $\bar{x}_{p-1}$ is not adjacent⁺ to x_{p-1}. Thus, $\eta > 2$, $\hat{x}_2^s \notin \{\hat{x}_1^s \equiv x_p^s, \hat{x}_\eta^s \equiv \bar{x}_p^s\}$ and $a^s \equiv \bar{x}_p^s \notin \{\hat{x}_1^s, \hat{x}_2^s\}$. We first show $f(\hat{x}_1, \hat{x}_2, a) = f(\hat{x}_2, \hat{x}_1, a) = \hat{x}_1$. According to $f(\hat{x}_1, a, a) \equiv f(x_{p-1}, a, a) = a$ by Claim 4, since $\hat{x}_1 \sim^+ \hat{x}_2$ and $a^s \notin \{\hat{x}_1^s, \hat{x}_2^s\}$, item 2(iii) of Lemma 8 implies $f(\hat{x}_2, a, a) = a$. Next, since $x_p \sim^+ a$, sd-strategy-proofness implies $f(\hat{x}_2, x_p, a) \in \{a, x_p\}$. Suppose $f(\hat{x}_2, x_p, a) = a$. Consequently, since $\hat{x}_1 \sim^+ \hat{x}_2$ and $a^s \notin \{\hat{x}_1^s, \hat{x}_2^s\}$, item

2(iii) of Lemma 8 implies $f(x_{p-1}, x_p, a) \equiv f(\hat{x}_1, x_p, a) = f(\hat{x}_2, x_p, a) = a$ which contradicts Claim 6. Therefore, $f(\hat{x}_2, x_p, a) = x_p$. Since $x_p \sim^+ x_{p-1}$, sd-strategy-proofness implies $f(\hat{x}_2, \hat{x}_1, a) \equiv f(\hat{x}_2, x_{p-1}, a) \in \{x_p, x_{p-1} \equiv \hat{x}_1\}$. Meanwhile, since $f(\hat{x}_1, \hat{x}_1, a) = \hat{x}_1$ by Claim 7 and $\hat{x}_2 \sim^+ \hat{x}_1$, sd-strategy-proofness implies $f(\hat{x}_2, \hat{x}_1, a) \in \{\hat{x}_1, \hat{x}_2\}$. Therefore, it is true that $f(\hat{x}_2, \hat{x}_1, a) = \hat{x}_1$. Now, following the adjacent⁺ sequence $\{\hat{x}_\nu\}_{\nu=1}^{\eta}$ from $\hat{x}_2$ to $\hat{x}_\eta \equiv \bar{x}_{p-1}$, since $\hat{x}_1^s \notin \{\hat{x}_\nu^s, \hat{x}_{\nu+1}^s\}$ for all $\nu = 2, \ldots, \eta - 1$, by repeatedly applying item 2(iii) of Lemma 8, we eventually have $f(\bar{x}_{p-1}, x_{p-1}, a) \equiv f(\hat{x}_\eta, \hat{x}_1, a) = \cdots = f(\hat{x}_2, \hat{x}_1, a) = \hat{x}_1 \equiv x_{p-1}$. Furthermore, since $f(x_{p-1}, x_p, a) = x_p$ by Claim 6 and $x_{p-1} \sim^+ x_p$, Statement 1 of Lemma 15 implies $f(\bar{x}_{p-1}, x_p, a) = x_p$. Last, by connectedness⁺ and minimal richness, we have a top-separable preference $P_i \in \mathbb{D}$ with $r_1(P_i) = \bar{x}_{p-1} \equiv (x^s, x_{p-1}^\tau, z^{-\{s,\tau\}})$. Then, top-separability implies $a \equiv (x^s, x_p^\tau, z^{-\{s,\tau\}}) P_i (y^s, x_p^\tau, z^{-\{s,\tau\}}) \equiv x_p$. Recall $f(a, x_p, a) = a$ by Claim 4. Thus, $f(a, x_p, a) = a P_i x_p = f(P_i, x_p, a)$, and consequently, voter i will manipulate at (P_i, x_p, a) via $P_i' \in \mathbb{D}^a$. Therefore, it must be the case $\bar{x}_{p-1} \sim^+ x_{p-1}$. Furthermore, Lemma 18 implies $\bar{x}_{p-1} \sim^+ \bar{x}_p$. This completes the verification of the claim.

Now, note that $f(x_p, \bar{x}_p, a) = f(\bar{x}_p, x_p, a) = \bar{x}_p$ by the hypothesis of Situation 4, and $\bar{x}_{p-1} \sim^+ x_{p-1}$ and $\bar{x}_{p-1} \sim^+ \bar{x}_p$ by Claim 8. Following the adjacent⁺ sequence $\{x_k\}_{k=1}^{p}$ from x_p to x_1, we adopt an induction argument.

Induction Hypothesis: Given $1 < k < p$, for all $k < k' \leq p$, we have

- $f(x_{k'}, \bar{x}_{k'}, a) = f(\bar{x}_{k'}, x_{k'}, a) = \bar{x}_{k'}$, and
- $\bar{x}_{k'-1} \sim^+ x_{k'-1}$ and $\bar{x}_{k'-1} \sim^+ \bar{x}_{k'}$.

To complete the verification of the induction hypothesis, we show $f(x_k, \bar{x}_k, a) = f(\bar{x}_k, x_k, a) = \bar{x}_k$, and $\bar{x}_{k-1} \sim^+ x_{k-1}$ and $\bar{x}_{k-1} \sim^+ \bar{x}_k$ in the following two claims.

Claim 9: $f(x_k, \bar{x}_k, a) = f(\bar{x}_k, x_k, a) = \bar{x}_k$.

First, we have $f(x_k, x_{k+1}, a) = x_{k+1}$ by Claim 6 and $f(x_{k+1}, \bar{x}_{k+1}, a) = \bar{x}_{k+1}$ by the induction hypothesis. Furthermore, if $k < p - 1$, we refer to $k' = k + 2$, and hence have $x_{k+1} \sim^+ \bar{x}_{k+1}$ by the induction hypothesis, and if $k = p - 1$, we refer to the hypothesis $d \sim^+ a$, and hence have $x_{k+1} = x_p \equiv d \sim^+ a \equiv \bar{x}_p = \bar{x}_{k+1}$. In conclusion, we have $x_{k+1} \sim^+ \bar{x}_{k+1}$. Then, Statement 1 of Lemma 15 implies $f(x_k, \bar{x}_{k+1}, a) = \bar{x}_{k+1}$. Furthermore, since $\bar{x}_k \sim^+ \bar{x}_{k+1}$ by the induction hypothesis, sd-strategy-proofness implies $f(x_k, \bar{x}_k, a) \in \{\bar{x}_{k+1}, \bar{x}_k\}$. Meanwhile, since $f(x_k, x_k, a) = x_k$ by Claim 5 and $\bar{x}_k \sim^+ x_k$ by the induction hypothesis, sd-strategy-proofness implies $f(x_k, \bar{x}_k, a) \in \{x_k, \bar{x}_k\}$. Therefore, it is true that $f(x_k, \bar{x}_k, a) = \bar{x}_k$. This completes the verification of the claim.

Claim 10: $\bar{x}_{k-1} \sim^+ x_{k-1}$ and $\bar{x}_{k-1} \sim^+ \bar{x}_k$.

Note that $f(x_k, \bar{x}_k, a) = f(\bar{x}_k, x_k, a) = \bar{x}_k$ by Claim 9, $f(x_k, x_{k-1}, a) = f(x_{k-1}, x_k, a) = x_k$ by Claim 6, $x_k \sim^+ \bar{x}_k$ by the induction hypothesis and $x_k \sim^+ x_{k-1}$ according to the adjacent⁺ sequence $\{x_k\}_{k=1}^{p}$. Thus, $\{x_k, \bar{x}_{k-1}, \bar{x}_k, x_{k-1}\}$ formulate an analogy of $\{x_p, \bar{x}_{p-1}, \bar{x}_p, x_{p-1}\}$ in Claim 8. Therefore, we immediately obtain $\bar{x}_{k-1} \sim^+ x_{k-1}$ and $\bar{x}_{k-1} \sim^+ \bar{x}_k$. This proves the claim, and hence completes the verification of the induction hypothesis.

Therefore, we eventually have $c = \bar{x}_1 \sim^+ x_1 = b$ which contradicts the hypothesis $q > 2$ in the beginning of the proof. Hence, $b \sim^+ c$ and $b \sim^+ d$. This completes the verification of Lemma 22. □

Now, by Lemmas 19 - 22, we have $[a \sim^+ c$ and $a \sim^+ d] \Rightarrow [b \sim^+ c$ and $b \sim^+ d]$ which is analogous to Lemma 4, as required. This completes the necessity part of Theorem 2.

Appendix D. Proof of Theorem 3

For simplicity, we first show part (ii) of Theorem 3. Let $\bar{A} = \times_{s\in M}\bar{A}^s$, where $\bar{A}^s \subseteq A^s$ for each $s \in M$, satisfy Assumption 1. Thus, it is true that $|\bar{A}^s| \geq 2$ for each $s \in M$. Let $G(\bar{A}^s)$ be a tree for each $s \in M$. Thus, $\bar{A}$ is located on a product of trees $\times_{s\in M}G(\bar{A}^s)$. Let $\mathbb{D}$ be multidimensional single-peaked w.r.t. $\bar{A}$, and satisfy Assumption 2. Let $\mathbb{D}_{|\bar{A}} = \{P_{i|\bar{A}} : P_i \in \mathbb{D}\}$. Thus, the induced domain $\mathbb{D}_{|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M}G(\bar{A}^s)$.

By the proof of the sufficiency part of Theorem 1, we construct the multidimensional projection rules $\left\{f^a : [\mathbb{D}_{|\bar{A}}]^N \to \bar{A}\right\}_{a\in\bar{A}}$, and assemble them as a mixed multidimensional projection rule $\phi : [\mathbb{D}_{|\bar{A}}]^N \to \Delta(\bar{A})$ such that $\phi(P_{1|\bar{A}}, \ldots, P_{N|\bar{A}}) = \sum_{a\in\bar{A}}\lambda_a f^a(P_{1|\bar{A}}, \ldots, P_{N|\bar{A}})$ for all $(P_{1|\bar{A}}, \ldots, P_{N|\bar{A}}) \in [\mathbb{D}_{|\bar{A}}]^N$, where $\lambda_a > 0$ for all $a \in \bar{A}$ and $\sum_{a\in\bar{A}}\lambda_a = 1$. Recalling the construction of the mixed multidimensional projection rule, we know that ϕ is unanimous and sd-strategy-proof, and satisfies the compromise property. Next, we extend ϕ to a constrained RSCF $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$ such that for all $(P_1, \ldots, P_N) \in \mathbb{D}^N$, $\varphi(P_1, \ldots, P_N) = \phi\big(P_{1|\bar{A}}, \ldots, P_{N|\bar{A}}\big)$. It is evident that φ is well defined, and satisfies unanimity (w.r.t. feasibility) and sd-strategy-proofness. Last, we show that φ satisfies the compromise property (w.r.t. feasibility). Given $\hat{I} \subseteq I$ with $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd, fix $P_i, P_j \in \mathbb{D}$ with $r_1(P_i) \equiv (x^s, a^{-s}) \neq (y^s, a^{-s}) \equiv r_1(P_j)$, $r_2(P_i) = r_2(P_j) \equiv (z^s, a^{-s}) \in \bar{A}$ and $z^s \notin \{x^s, y^s\}$. If either $x^s \notin \bar{A}^s$ or $y^s \notin \bar{A}^s$, we know either $r_1(P_{i|\bar{A}}) = (z^s, a^{-s})$ or $r_1(P_{j|\bar{A}}) = (z^s, a^{-s})$, and hence $f^{(z^s,a^{-s})}\big(\frac{P_{i|\bar{A}}}{\hat{I}}, \frac{P_{j|\bar{A}}}{I\backslash\hat{I}}\big) = (z^s, a^{-s})$. If $x^s, y^s \in \bar{A}^s$, by Claim 4 in the proof of the sufficiency part of Theorem 1, we know $(z^s, a^{-s}) \in \langle(x^s, a^{-s}), (y^s, a^{-s})\rangle$, and hence $f^{(z^s,a^{-s})}\big(\frac{P_{i|\bar{A}}}{\hat{I}}, \frac{P_{j|\bar{A}}}{I\backslash\hat{I}}\big) = (z^s, a^{-s})$. Therefore, $\varphi_{(z^s,a^{-s})}\big(\frac{P_i}{\hat{I}}, \frac{P_j}{I\backslash\hat{I}}\big) = \phi_{(z^s,a^{-s})}\big(\frac{P_{i|\bar{A}}}{\hat{I}}, \frac{P_{j|\bar{A}}}{I\backslash\hat{I}}\big) \geq \lambda_{(z^s,a^{-s})} > 0$. Hence, φ satisfies the compromise property (w.r.t. feasibility). This completes the verification of part (ii) of Theorem 3.

Now, we turn to part (i) of Theorem 3. Analogous to Proposition 2, we first show in Proposition 3 below that every unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF on a connected$^+$ domain satisfies the tops-only property. Meanwhile, to illustrate the key role of Assumption 2, we show that Assumption 2 is necessary and sufficient for the existence of a unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF that satisfies the tops-only property.

Proposition 3. *Under Assumption 2, every unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF on a connected$^+$ domain satisfies the tops-only property. Moreover, a unanimous (w.r.t. feasibility), tops-only and sd-strategy-proof constrained RSCF exists if and only if the domain satisfies Assumption 2.*

Proof. The verification of the first statement of Proposition 3 follows almost exactly from the proof of Proposition 2. Let $\mathbb{D}$ be a connected$^+$ domain, and satisfy Assumption 2. First, note that the notions of sd-strategy-proofness and connectedness$^+$ are unchanged in voting under constraints, and Lemmas 8 - 11 are established according to an arbitrary sd-strategy-proof RSCF which is either unconstrained or constrained, and is not required to satisfy unanimity. Therefore, Lemmas 8 - 11 remain valid for constrained RSCFs. Second, the initial argument in the proof of Proposition 2 (see the middle of page 33) is modified as follows: If $N = 1$, by unanimity (w.r.t. feasibility) and Assumption 2, we know that every constrained RSCF in question satisfies the tops-only property. Last, since the induction argument in the proof of Proposition 2 (Lemmas 12 - 14) completely relies on Lemmas 8 - 11, it is also applicable to the proof of Proposition 3. Therefore, we assert that every unanimous (w.r.t. feasibility) and sd-strategy-proof constrained RSCF satisfies the tops-only property.

We next show that Assumption 2 is necessary for the existence of a unanimous (w.r.t. feasibility), tops-only and sd-strategy-proof constrained RSCF. Given an arbitrary domain $\mathbb{D}$, let $\phi : \mathbb{D}^N \to \Delta(\bar{A})$ be a unanimous, tops-only and sd-strategy-proof constrained RSCF. We partition I into to groups $\hat{I}$ and $I\backslash\hat{I}$ such that $|\hat{I}| = \frac{N}{2}$ if N is even, and $|\hat{I}| = \frac{N+1}{2}$ if N is odd, and then induce a two-voter constrained RSCF $\varphi : \mathbb{D}^2 \to \Delta(\bar{A})$ such that $\varphi(P_i, P_j) = \phi(\frac{P_i}{\hat{I}}, \frac{P_j}{I\backslash\hat{I}})$ for all $P_i, P_j \in \mathbb{D}$. It is easy to show that φ is unanimous (w.r.t. feasibility), tops-only and sd-strategy-proof. Suppose that $\mathbb{D}$ violates Assumption 2. Thus, we have $P_i, P_i' \in \mathbb{D}$ such that $r_1(P_i) = r_1(P_i') \notin \bar{A}$ and $r_1(P_{i|\bar{A}}) \equiv a \neq b \equiv r_1(P'_{i|\bar{A}})$. We construct profiles (P_i, P_j) and (P_i', P_j') where $P_i = P_j$ and $P_i' = P_j'$. On the one hand, the tops-only property implies $\varphi(P_i, P_j) = \varphi(P_i', P_j')$. On the other hand, unanimity (w.r.t. feasibility) implies $\varphi_a(P_i, P_j) = 1$ and $\varphi_b(P_i', P_j') = 1$. Hence, $\varphi(P_i, P_j) \neq \varphi(P_i', P_j')$. Contradiction! Therefore, $\mathbb{D}$ satisfies Assumption 2.

Last, we show that Assumption 2 is sufficient for the existence of a unanimous (w.r.t. feasibility), tops-only and sd-strategy-proof constrained RSCF. Let $\mathbb{D}$ satisfy Assumption 2. We construct a constrained random dictatorship $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$ (recall footnote 41). It is evident that φ is unanimous (w.r.t. feasibility) and sd-strategy-proof. Last, we claim that φ satisfies the tops-only property. Given $P, P' \in \mathbb{D}^N$, let $r_1(P_i) = r_1(P_i')$ for all $i \in I$. Then, Assumption 2 implies $r_1(P_{i|\bar{A}}) = r_1(P'_{i|\bar{A}})$ for all $i \in I$. Consequently, by the construction of the constrained random dictatorship φ, we have $\varphi(P) = \varphi(P')$. This completes the verification of Proposition 3. □

Now, let $\bar{A}$ satisfy Assumption 1. Let $\mathbb{D}$ be a minimally rich connected$^+$ domain, and satisfy Assumption 2. Let $\phi : \mathbb{D}^N \to \Delta(\bar{A})$ be a constrained RSCF satisfying unanimity (w.r.t. feasibility), sd-strategy-proofness and the compromise property (w.r.t. feasibility). By Proposition 3, we know that ϕ satisfies the tops-only property. Then, similar to the proof of the necessity part of Theorem 1, we induce a two-voter unanimous (w.r.t. feasibility), tops-only and sd-strategy-proof constrained RSCF $\varphi : \mathbb{D}^2 \to \Delta(\bar{A})$ satisfying the compromise property (w.r.t. feasibility).

Before proceeding with the proof, we point out that the following proof is essentially analogous to the proof of Theorem 1. Lemmas 23, 24 and 25 replicate the conclusions of Lemmas 1, 2 and 3 respectively. Lemma 26 is a key result which identifies the best feasible alternative in a preference whose original peak is infeasible. Under Lemma 26, we establish Lemma 27 which is analogous to Lemma 4. However, the subsequent proof of Theorem 3 appears significantly more complicated in the following two aspects. First, due to the feasibility issue, Lemma 5 is not directly applicable to elicit a product of trees over the feasible alternatives. We therefore introduce an intermediate step, Lemma 28, which implies that the feasible set is factorizable. This eventually allows us to state Lemma 29 which is analogous to Lemma 5. Second, we can only use the results of Lemmas 6 and 7 to reveal the restriction of multidimensional single-peakedness embedded in preferences whose peaks are feasible alternatives. We have to establish additional Lemmas 30 - 32 to deal with preferences whose peaks are infeasible alternatives.

Lemma 23. *Given $s \in M$, $a^s, b^s \in A^s$ and $x^{-s} \in A^{-s}$, if $(a^s, x^{-s}), (b^s, x^{-s}) \in \bar{A}$, there exists an adjacent⁺ sequence $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s}) \cap \bar{A}$ connecting (a^s, x^{-s}) and (b^s, x^{-s}), i.e., $x_1 = (a^s, x^{-s})$, $x_q = (b^s, x^{-s})$ and $x_k \sim^+ x_{k+1}$ for all $k = 1, \dots, q-1$.*

Proof. First, by Lemma 1, we have an adjacent⁺ sequence $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ connecting (a^s, x^{-s}) and (b^s, x^{-s}). Suppose that $x_k \notin \bar{A}$ for some $1 < k < q$. Thus, we elicit an adjacent⁺ subpath $\{x_{\underline{k}}, x_{\underline{k}+1}, \dots, x_{\overline{k}-1}, x_{\overline{k}}\}$ such that $\overline{k} - \underline{k} \geq 2$, $x_{\underline{k}}, x_{\overline{k}} \in \bar{A}$ and $x_{\underline{k}+1}, \dots, x_{\overline{k}-1} \notin \bar{A}$.

Since $x_{\underline{k}} \sim^+ x_{\underline{k}+1}$, we have $P_i \in \mathbb{D}^{x_{\underline{k}}}$ and $P_j \in \mathbb{D}^{x_{\underline{k}+1}}$ with $P_i \sim^+ P_j$. Since $x_{\underline{k}} \in \bar{A}$ and $x_{\underline{k}+1} \notin \bar{A}$, the tops-only property and unanimity (w.r.t. feasibility) imply $\varphi_{x_{\underline{k}}}(x_{\underline{k}}, x_{\underline{k}+1}) = \varphi_{x_{\underline{k}}}(P_i, P_j) = 1$. Symmetrically, since $x_{\overline{k}} \sim^+ x_{\overline{k}-1}$, $x_{\overline{k}} \in \bar{A}$ and $x_{\overline{k}-1} \notin \bar{A}$, we have $\varphi_{x_{\overline{k}}}(x_{\overline{k}-1}, x_{\overline{k}}) = 1$. On the one hand, following the subsequence $\{x_{\underline{k}}, x_{\underline{k}+1}, \dots, x_{\overline{k}}\}$ from $x_{\underline{k}+1}$ to $x_{\overline{k}}$, since $x_{\underline{k}}^s \notin \{x_k^s, x_{k+1}^s\}$ for all $k = \underline{k}+1, \dots, \overline{k}-1$, by repeatedly applying item 2(iii) of Lemma 8, we have $1 = \varphi_{x_{\underline{k}}}(x_{\underline{k}}, x_{\underline{k}+1}) = \varphi_{x_{\underline{k}}}(x_{\underline{k}}, x_{\underline{k}+2}) = \cdots = \varphi_{x_{\underline{k}}}(x_{\underline{k}}, x_{\overline{k}})$. On the other hand, following the subsequence $\{x_{\underline{k}}, \dots, x_{\overline{k}-1}, x_{\overline{k}}\}$ from $x_{\overline{k}-1}$ to $x_{\underline{k}}$, since $x_{\overline{k}}^s \notin \{x_k^s, x_{k-1}^s\}$ for all $k = \overline{k}-1, \dots, \underline{k}+1$, by repeatedly applying item 2(iii) of Lemma 8, we have $1 = \varphi_{x_{\overline{k}}}(x_{\overline{k}-1}, x_{\overline{k}}) = \varphi_{x_{\overline{k}}}(x_{\overline{k}-2}, x_{\overline{k}}) = \cdots = \varphi_{x_{\overline{k}}}(x_{\underline{k}}, x_{\overline{k}})$. Contradiction! Hence, $\{x_k\}_{k=1}^q \subseteq \bar{A}$. $\square$

Lemma 24. *Given $s \in M$ and $x^{-s} \in A^{-s}$, let the adjacent⁺ sequence $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ be such that $x_1, \dots, x_{\bar{k}} \in \bar{A}$ and $x_{\bar{k}+1}, \dots, x_q \notin \bar{A}$, where $1 \leq \bar{k} \leq q$. The following statements hold:*

(i) *If $\bar{k} = 1$, then $\varphi(x_1, x_q) = e_{x_1}$.*
(ii) *If $1 < \bar{k} \leq q$, there exist $0 \leq \alpha_1 < \cdots < \alpha_{\bar{k}-1} \leq 1$ such that $\varphi(x_1, x_q) = \alpha_1 e_{x_1} + \sum_{k=2}^{\bar{k}-1}(\alpha_k - \alpha_{k-1})e_{x_k} + (1 - \alpha_{\bar{k}-1})e_{x_{\bar{k}}}$. Moreover, for every $P_i \in \mathbb{D}^{x_1}$, $x_k P_i x_{k+1}$ for all $k = 1, \dots, \bar{k}-1$.*

Proof. The verification is similar to Lemma 2. We omit the detailed proof. $\square$

Given $s \in M$ and $x^{-s} \in A^{-s}$ with $(A^s, x^{-s}) \cap \bar{A} \neq \emptyset$, let $G_{\sim^+}\big((A^s, x^{-s}) \cap \bar{A}\big)$ be a graph where the vertex set is $(A^s, x^{-s}) \cap \bar{A}$, and two distinct feasible alternatives form an edge if and only if they are adjacent⁺. The next lemma shows that $G_{\sim^+}\big((A^s, x^{-s}) \cap \bar{A}\big)$ is a tree.

Lemma 25. *Given $s \in M$ and $x^{-s} \in A^{-s}$, if $|(A^s, x^{-s}) \cap \bar{A}| \geq 2$, then $G_{\sim^+}\big((A^s, x^{-s}) \cap \bar{A}\big)$ is a tree. Furthermore, every two distinct feasible alternatives of (A^s, x^{-s}) are connected via a unique adjacent⁺ sequence in (A^s, x^{-s}), and all alternatives of this sequence are feasible.*

Proof. Suppose that $G_{\sim^+}\big((A^s, x^{-s}) \cap \bar{A}\big)$ is not a tree. By Lemma 23, we know that there must exist a cycle of feasible alternatives $\{x_k\}_{k=1}^t \subseteq (A^s, x^{-s}) \cap \bar{A}$, where $t \geq 3$, $x_k \sim^+ x_{k+1}$ for all $k = 1, \dots, t$, and $x_{t+1} = x_1$. Then, by the proof of Lemma 3, we induce a contradiction. Therefore, $G_{\sim^+}\big((A^s, x^{-s}) \cap \bar{A}\big)$ is a tree. Furthermore, by the proof of Lemma 23, we know that every adjacent⁺ sequence connecting two feasible alternatives of (A^s, x^{-s}) must consist of feasible alternatives. Therefore, the adjacent⁺ sequence connecting two feasible alternatives of (A^s, x^{-s}) is unique. $\square$

Lemma 26. *Fix $s \in M$, $x^{-s} \in A^{-s}$ and $a \in (A^s, x^{-s})$. If $(A^s, x^{-s}) \cap \bar{A} \neq \emptyset$, there exists $\bar{a} \in (A^s, x^{-s}) \cap \bar{A}$ such that $\bar{a} = r_1\big(P_{i|\bar{A}}\big)$ for all $P_i \in \mathbb{D}^a$.*

Proof. The lemma holds evidently if $a \in (A^s, x^{-s}) \cap \bar{A}$. Thus, we assume $a \in (A^s, x^{-s}) \backslash \bar{A}$. Pick an arbitrary $b \in (A^s, x^{-s}) \cap \bar{A}$. By Lemma 1, we have an adjacent$^+$ sequence $\{x_k\}_{k=1}^q \subseteq (A^s, x^{-s})$ connecting b and a. Since $x_1 \in \bar{A}$ and $x_q \notin \bar{A}$, there must exist $1 \leq \bar{k} < q$ such that $x_{\bar{k}} \in \bar{A}$ and $x_{\bar{k}+1}, \ldots, x_q \notin \bar{A}$. Moreover, we know either $\bar{k} = 1$, or $\bar{k} > 1$ and $x_1, x_2, \ldots, x_{\bar{k}} \in (A^s, x^{-s}) \cap \bar{A}$ by Lemma 25. Thus, the sequence $\{x_k\}_{k=1}^q$ is separated into two parts $\{x_1, x_2, \ldots, x_{\bar{k}}\}$ which are feasible, and $\{x_{\bar{k}+1}, \ldots, x_q\}$ which are not feasible. We show $x_{\bar{k}} = r_1\big(P_{i|\bar{A}}\big)$ for all $P_i \in \mathbb{D}^a$.

First, since $x_{\bar{k}+1} \sim^+ x_{\bar{k}}$, we have $P_i \in \mathbb{D}^{x_{\bar{k}+1}}$ such that $r_2(P_i) = x_{\bar{k}}$, which implies $r_1(P_{i|\bar{A}}) = x_{\bar{k}}$. Furthermore, Assumption 2 implies $r_1(P_{i|\bar{A}}) = x_{\bar{k}}$ for all $P_i \in \mathbb{D}^{x_{\bar{k}+1}}$. Next, we adopt an induction hypothesis: Given $\bar{k}+1 < l \leq q$, for all $\bar{k}+1 \leq l' < l$, we have $r_1(P_{i|\bar{A}}) = x_{\bar{k}}$ for all $P_i \in \mathbb{D}^{x_{l'}}$. We show $r_1(P_{i|\bar{A}}) = x_{\bar{k}}$ for all $P_i \in \mathbb{D}^{x_l}$. Since $x_l \sim^+ x_{l-1}$, we have $P_i \in \mathbb{D}^{x_l}$ and $P'_i \in \mathbb{D}^{x_{l-1}}$ with $P_i \sim^+ P'_i$. Since $x_{\bar{k}} = r_1(P'_{i|\bar{A}})$ by the induction hypothesis, we know that every alternative ranked above $x_{\bar{k}}$ in P'_i is infeasible. Since $x_{\bar{k}}$ is not involved in any local switching pair across P_i and P'_i, $P_i \sim^+ P'_i$ implies that all alternatives ranked above $x_{\bar{k}}$ in P_i are also infeasible. Hence, $x_{\bar{k}} = r_1(P_{i|\bar{A}})$. Furthermore, Assumption 2 implies $x_{\bar{k}} = r_1(P_{i|\bar{A}})$ for all $P_i \in \mathbb{D}^{x_l}$. This completes the verification of the induction hypothesis. Therefore, $x_{\bar{k}} = r_1\big(P_{i|\bar{A}}\big)$ for all $P_i \in \mathbb{D}^a$. $\square$

To establish the next lemma which is similar to Lemma 4, we fix four alternatives: $a = (x^s, x^\tau, z^{-\{s,\tau\}})$, $b = (y^s, y^\tau, z^{-\{s,\tau\}})$, $c = (x^s, y^\tau, z^{-\{s,\tau\}})$ and $d = (y^s, x^\tau, z^{-\{s,\tau\}})$ where $x^s \neq y^s$ and $x^\tau \neq y^\tau$.

Lemma 27. *If $a, c, d \in \bar{A}$, $a \sim^+ c$ and $a \sim^+ d$, then $b \in \bar{A}$, $b \sim^+ c$ and $b \sim^+ d$.*

Proof. We first show $b \in \bar{A}$. Suppose that it is not true. Since $b, d \in (y^s, A^\tau, z^{-\{s,\tau\}})$ and $d \in \bar{A}$, Lemma 26 implies that there exists $\bar{b} \in (y^s, A^\tau, z^{-\{s,\tau\}}) \cap \bar{A}$ such that $r_1(P_{i|\bar{A}}) = \bar{b}$ for all $P_i \in \mathbb{D}^b$. Symmetrically, since $b, c \in (A^s, y^\tau, z^{-\{s,\tau\}})$ and $c \in \bar{A}$, Lemma 26 implies that there exists $\underline{b} \in (A^s, y^\tau, z^{-\{s,\tau\}}) \cap \bar{A}$ such that $r_1(P_{i|\bar{A}}) = \underline{b}$ for all $P_i \in \mathbb{D}^b$. However, since $(y^s, A^\tau, z^{-\{s,\tau\}}) \cap (A^s, y^\tau, z^{-\{s,\tau\}}) = \{b\}$ and $b \notin \bar{A}$, we have $\bar{b} \neq \underline{b}$ which contradicts Assumption 2. Therefore, $b \in \bar{A}$.

Now, by Lemma 25, we have a unique adjacent$^+$ path $\{x_k\}_{k=1}^p \subseteq (y^s, A^\tau, z^{-\{s,\tau\}}) \cap \bar{A}$ connecting b and d, and a unique adjacent$^+$ path $\{y_k\}_{k=1}^q \subseteq (A^s, y^\tau, z^{-\{s,\tau\}}) \cap \bar{A}$ connecting b and c. Then, the rest of the lemma follows exactly from Lemma 4. $\square$

We introduce a new notion. Given distinct $c, d \in \bar{A}$ and nonempty subset $S \subseteq M$, let $c^s \neq d^s$ for every $s \in S$ and $c^{-S} = d^{-S} \equiv z^{-S}$. We say that c and d **formulate a feasible box** if the following two conditions are satisfied.

(i) For each $s \in S$, there exists a sequence $\{x_k^s\}_{k=1}^{q(s)} \subseteq A^s$ where $q(s) \geq 2$,[51] $x_1^s = c^s$ and $x_{q(s)}^s = d^s$ such that $B(c, d) \equiv \left(\left(\{x_k^s\}_{k=1}^{q(s)}\right)_{s \in S}, z^{-S}\right) \subseteq \bar{A}$.

(ii) For all $x, y \in B(c, d)$, we have

$$[x^s = x_k^s,\ y^s = x_{k+1}^s \text{ and } x^{-s} = y^{-s} \text{ for some } s \in S \text{ and some } 1 \leq k < q(s)] \Rightarrow [x \sim^+ y].$$

[51] The notation $q(s)$ implies that the length of the sequence may vary across all components of S.

We highlight the key role of the feasible box $B(c, d)$: We can always find an adjacent$^+$ sequence consisting of feasible alternatives in the feasible box $B(c, d)$ which connects c and d.

Lemma 28. *Every pair of distinct feasible alternatives formulate a feasible box.*

Proof. Evidently, Lemma 23 implies that every two similar feasible alternatives always formulate a feasible box. Next, we provide an induction argument to complete the verification.

Induction Hypothesis: Given an integer $2 \leq l \leq m$, for all $c, d \in \bar{A}$ which disagree on at least one component and at most $l - 1$ components, e.g., $S \subseteq M$, $1 \leq |S| < l$, $c^s \neq d^s$ for every $s \in S \subseteq M$ and $c^{-S} = d^{-S} \equiv z^{-S}$, we know that c and d formulate a feasible box.

Given $c, d \in \bar{A}$ and $S \subseteq M$, let $|S| = l$, $c^s \neq d^s$ for every $s \in S \subseteq M$ and $c^{-S} = d^{-S} \equiv z^{-S}$. We show that c and d formulate a feasible box. For notational convenience, let $S = \{1, 2, \dots, l\}$.

CLAIM 1: If there exists $s \in S$ such that $\hat{a} \equiv (c^1, \dots, c^{s-1}, d^s, c^{s+1}, \dots, c^l, z^{-S}) \in \bar{A}$, then c and d formulate a feasible box.

Assume w.l.o.g. that $s = 1$. Thus, $\hat{a} \equiv (d^1, c^2, \dots, c^l, z^{-S})$ and d disagree on $l - 1$ components, and induction hypothesis implies that $\hat{a}$ and d formulate a feasible box $B(\hat{a}, d)$. Specifically,

(i) for each $s \in \{2, \dots, l\}$, there exists a sequence $\{x_k^s\}_{k=1}^{q(s)} \subseteq A^s$ where $q(s) \geq 2$, $x_1^s = c^s$ and $x_{q(s)}^s = d^s$, such that $B(\hat{a}, d) \equiv \left(d^1, \{x_k^2\}_{k=1}^{q(2)}, \dots, \{x_k^l\}_{k=1}^{q(l)}, z^{-S}\right) \subseteq \bar{A}$, and
(ii) for all $x, y \in B(\hat{a}, d)$, we have

$$\left[x^s = x_k^s, y^s = x_{k+1}^s \text{ and } x^{-s} = y^{-s} \text{ for some } s \in \{2, \dots, l\} \text{ and some } 1 \leq k < q(s)\right]$$
$$\Rightarrow [x \sim^+ y].$$

Next, since $\hat{a}, c \in (A^1, c^2, \dots, c^l, z^{-S}) \cap \bar{A}$, we have a unique adjacent$^+$ sequence $\{x_k\}_{k=1}^q \equiv \left\{(x_k^1, c^2, \dots, c^l, z^{-S})\right\}_{k=1}^q \subseteq \bar{A}$ connecting $\hat{a}$ and c by Lemma 25.

Pick an arbitrary adjacent$^+$ sequence $\{y_k\}_{k=1}^p \subseteq B(\hat{a}, d)$ connecting $\hat{a}$ and d. Note that all alternatives of $\{y_k\}_{k=1}^p$ agree on d^1, and for every $1 \leq k < p$, y_k and y_{k+1} disagree on exactly one component of $\{2, \dots, l\}$. Since $\hat{a} = x_1$, we know $d^1 = x_1^1$, and rewrite $B(\hat{a}, d) \equiv \left(x_1^1, \{x_k^2\}_{k=1}^{q(2)}, \dots, \{x_k^l\}_{k=1}^{q(l)}, z^{-S}\right)$. Note that $d^1 = x_1^1 \neq x_2^1$. For each $k = 1, \dots, p$, we replace d^1 by x_2^1 in the alternative $y_k = (d^1, y_k^{-1})$, and hence construct the alternative $z_k \equiv (x_2^1, y_k^{-1})$. Note that $z_1 = (x_2^1, y_1^{-1}) \equiv (x_2^1, c^2, \dots, c^l, z^{-S})$ and $z_p \equiv (x_2^1, y_p^{-1}) \equiv (x_2^1, d^2, \dots, d^l, z^{-S})$. Thus, we have a sequence $\{z_k\}_{k=1}^p$ (see Fig. 12(1)). We next show that $\{z_k\}_{k=1}^p \subseteq \bar{A}$ and $\{z_k\}_{k=1}^p$ is an adjacent$^+$ sequence.

First, consider four alternatives $y_1 = (d^1, y_1^{-1})$, $z_2 = (x_2^1, y_2^{-1})$, $z_1 = (x_2^1, y_1^{-1})$ and $y_2 = (d^1, y_2^{-2})$. Note that y_1^{-1} and y_2^{-1} disagree on exactly one component of $\{2, \dots, l\}$, $y_1 = x_1 \sim^+ x_2 = z_1$ and $y_1 \sim^+ y_2$. Thus, y_1, z_2, z_1, y_2 are analogous to a, b, c, d of Lemma 27 respectively. Then, Lemma 27 implies $z_2 \in \bar{A}$ and $z_2 \sim^+ z_1$ and $z_2 \sim^+ y_2$ (see Fig. 12(2)).

Next, along the adjacent$^+$ sequence $\{y_k\}_{k=1}^p$ from y_1 to y_p, by repeatedly applying the argument above, we have $z_{k+1} \in \bar{A}$, $z_{k+1} \sim^+ z_k$ and $z_{k+1} \sim^+ y_{k+1}$ for all $k = 1, \dots, p - 1$. Thus, $\{z_k\}_{k=1}^p \subseteq \bar{A}$ and $\{z_k\}_{k=1}^p$ is an adjacent$^+$ sequence. Furthermore, since we choose the

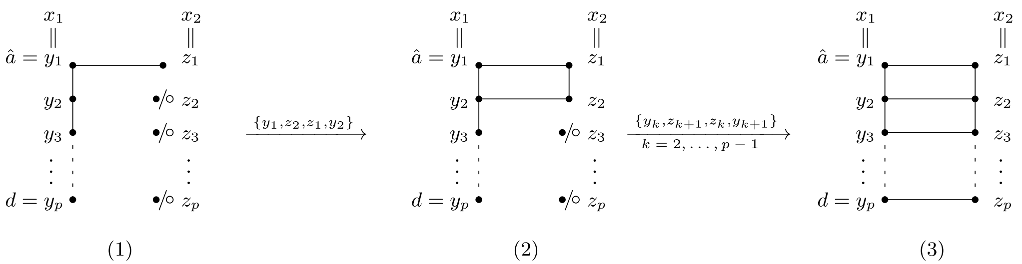

Fig. 12. The graphic illustration. In Fig. 12, "•" denotes a feasible alternative, "∘" denotes an infeasible alternative, while "•/∘" represents that we are uncertain on whether the alternative is feasible or not.

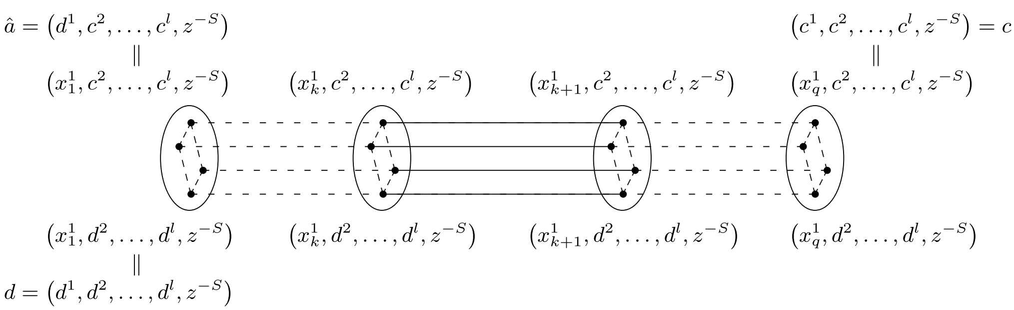

Fig. 13. Combine $B\left((x_k^1, c^2, \ldots, c^l, z^{-S}), (x_k^1, d^2, \ldots, d^l, z^{-S})\right)$ for all $k = 1, \ldots, q$ to formulate $B(c, d)$.

adjacent⁺ sequence $\{y_k\}_{k=1}^p$ arbitrarily, it is true that $z_1 \equiv (x_2^1, c^2, \ldots, c^m, z^{-S})$ and $z_p \equiv (x_2^1, d^2, \ldots, d^m, z^{-S})$ formulate a feasible box $B(z_1, z_p) = \left(x_2^1, \{x_k^2\}_{k=1}^{q(2)}, \ldots, \{x_k^l\}_{k=1}^{q(l)}, z^{-S}\right)$, and moreover, for all $y \in B(\hat{a}, d) \equiv B(y_1, y_p)$ and $z \in B(z_1, z_p)$, $[y^{-1} = z^{-1}] \Rightarrow [y \sim^+ z]$ (see Fig. 12(3)).

Along the adjacent⁺ sequence $\{x_k\}_{k=1}^q$, moving from x_2 to $x_q \equiv c$, by repeatedly applying the argument above, we know that the following two statements hold:

(i) Given $k = 1, \ldots, q$, $(x_k^1, c^2, \ldots, c^l, z^{-S})$ and $(x_k^1, d^2, \ldots, d^l, z^{-S})$ formulate a feasible box $B\left((x_k^1, c^2, \ldots, c^l, z^{-S}), (x_k^1, d^2, \ldots, d^l, z^{-S})\right)$. See the ellipses in Fig. 13.

(ii) Given $k = 1, \ldots, q-1$, $y \in B\left((x_k^1, c^2, \ldots, c^l, z^{-S}), (x_k^1, d^2, \ldots, d^l, z^{-S})\right)$ and $z \in B\left((x_{k+1}^1, c^2, \ldots, c^l, z^{-S}), (x_{k+1}^1, d^2, \ldots, d^l, z^{-S})\right)$, we have $[y^{-1} = z^{-1}] \Rightarrow [y \sim^+ z]$. See for instance the solid lines in Fig. 13.

Consequently, we assert that c and d formulate a feasible box $B(c, d) = \left(\{x_k^1\}_{k=1}^q, \{x_k^2\}_{k=1}^{q(2)}, \ldots, \{x_k^l\}_{k=1}^{q(l)}, z^{-S}\right)$. This completes the verification of Claim 1.

Claim 2: If there exists $s \in S$ such that $\hat{a} \equiv (d^1, \ldots, d^{s-1}, c^s, d^{s+1}, \ldots, d^l, z^{-S}) \in \bar{A}$, then c and d formulate a feasible box.

The verification of this claim is symmetric to the verification of Claim 1.

Claim 3: There exists $s \in S$ such that either $(d^1, \ldots, d^{s-1}, c^s, d^{s+1}, \ldots, d^l, z^{-S}) \in \bar{A}$ or $(c^1, \ldots, c^{s-1}, d^s, c^{s+1}, \ldots, c^l, z^{-S}) \in \bar{A}$.

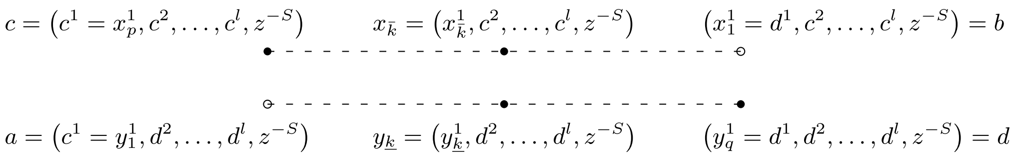

Fig. 14. The geometric relations among $a, b, c, d, x_{\bar{k}}$ and $y_{\underline{k}}$.

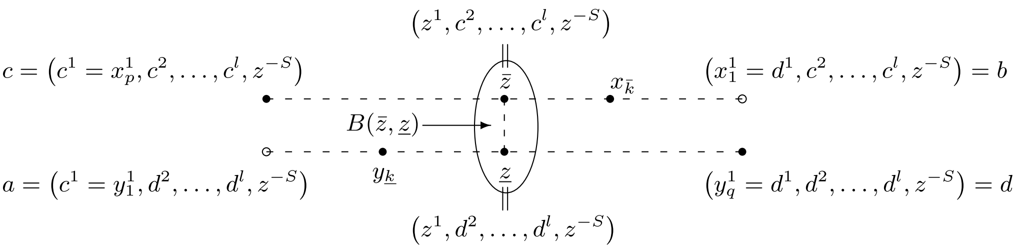

Fig. 15. The geometric relations among $a, b, c, d, x_{\bar{k}}, y_{\underline{k}}, \bar{z}, \underline{z}$ and $B(\bar{z}, \underline{z})$ in Case (1).

Suppose that it is not true. Thus, we have $a \equiv (c^1, d^2, \ldots, d^l, z^{-S}) \notin \bar{A}$ and $b \equiv (d^1, c^2, \ldots, c^l, z^{-S}) \notin \bar{A}$. Since $b, c \in (A^1, c^2, \ldots, c^l, z^{-S})$ and $c \in \bar{A}$, by Lemma 26 and its proof, we have an adjacent⁺ sequence $\{x_k\}_{k=1}^{p} \subseteq (A^1, c^2, \ldots, c^l, z^{-S})$ connecting b and c, and a particular $1 < \bar{k} \le p$, such that $x_1, \ldots, x_{\bar{k}-1} \notin \bar{A}$, $x_{\bar{k}}, x_{\bar{k}+1}, \ldots, x_p \in \bar{A}$ and $x_{\bar{k}} = r_1(P_{i|\bar{A}})$ for all $P_i \in \mathbb{D}^b$. Symmetrically, since $a, d \in (A^1, d^2, \ldots, d^l, z^{-S})$ and $d \in \bar{A}$, by Lemma 26 and its proof, we have an adjacent⁺ sequence $\{y_k\}_{k=1}^{q} \subseteq (A^1, d^2, \ldots, d^l, z^{-S})$ connecting a and d, and a particular $1 < \underline{k} \le q$, such that $y_1, \ldots, y_{\underline{k}-1} \notin \bar{A}$, $y_{\underline{k}}, y_{\underline{k}+1}, \ldots, y_q \in \bar{A}$ and $y_{\underline{k}} = r_1(P_{i|\bar{A}})$ for all $P_i \in \mathbb{D}^a$ (see Fig. 14).

There are two cases: (1) $\{x_{\bar{k}}^1, \ldots, x_p^1\} \cap \{y_{\underline{k}}^1, \ldots, y_q^1\} \neq \emptyset$ and (2) $\{x_{\bar{k}}^1, \ldots, x_p^1\} \cap \{y_{\underline{k}}^1, \ldots, y_q^1\} = \emptyset$. In each of these two cases, we induce a contradiction.

In the first case, assume $z^1 \in \{x_{\bar{k}}^1, \ldots, x_p^1\} \cap \{y_{\underline{k}}^1, \ldots, y_q^1\}$. Since $x_1^1 = y_q^1 = d^1$ and $x_p^1 = y_1^1 = c^1$, we know $z^1 \notin \{c^1, d^1\} = \{x_p^1, y_q^1\}$. Thus, we have two feasible alternatives $\bar{z} \equiv (z^1, c^2, \ldots, c^l, z^{-S}) \in \{x_{\bar{k}}, \ldots, x_{p-1}\}$ and $\underline{z} \equiv (z^1, d^2, \ldots, d^l, z^{-S}) \in \{y_{\underline{k}}, \ldots, y_{q-1}\}$ which disagree on $l-1$ components. Then, the induction hypothesis implies that $\bar{z}$ and $\underline{z}$ formulate a feasible box $B(\bar{z}, \underline{z})$ (see Fig. 15).

Note that c, $\underline{z}$ and $\bar{z}$ are pairwise distinct, and more importantly, are analogous to c, d and $\hat{a}$ in Claim 1. Then, by the proof of Claim 1, we know that c and $\underline{z}$ formulate a feasible box $B(c, \underline{z})$. Consequently, $a = (c^1, d^2, \ldots, d^l, z^{-S}) = (x_p^1, d^2, \ldots, d^l, z^{-S}) \in B(c, \underline{z}) \subseteq \bar{A}$. This contradicts the hypothesis $a \notin \bar{A}$.

Now, assume that the second case occurs. First, according to $a = (c^1, d^2, \ldots, d^l, z^{-S})$ and $c = (c^1, c^2, \ldots, c^l, z^{-S})$, we identify l alternatives $c_k = (c^1, \ldots, c^k, d^{k+1}, \ldots, d^l, z^{-s})$ for all $k = 1, \ldots, l$. Thus, $c_1 = a$ and $c_l = c$. For each $k = 1, \ldots, l-1$, by Lemma 1, we have an adjacent⁺ sequence in $(c^1, \ldots, c^k, A^{k+1}, d^{k+2}, \ldots, d^l, z^{-S})$ connecting $(c^1, \ldots, c^k, d^{k+1}, d^{k+2}, \ldots, d^l, z^{-S})$ and $(c^1, \ldots, c^k, c^{k+1}, d^{k+2}, \ldots, d^l, z^{-S})$. Combining all $l-1$ adjacent⁺ sequences, we have an adjacent⁺ sequence $\{z_k\}_{k=1}^{\eta} \subseteq (c^1, A^2, \ldots, A^l, z^{-S})$ connecting a and c (see Fig. 16). Note that all alternatives of $\{z_k\}_{k=1}^{\eta}$ agree on c^1, i.e., $z_k^1 = c^1$ for all $k = 1, \ldots, \eta$.

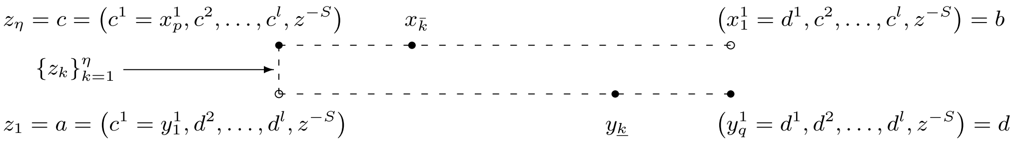

Fig. 16. The geometric relations among $a, b, c, d, x_{\bar{k}}, y_{\underline{k}}$ and $\{z_k\}_{k=1}^{\eta}$ in Case (2).

Second, since a and d are connected via the adjacent$^+$ sequence $\{y_k\}_{k=1}^{q}$, Lemma 24 implies $supp\big(\varphi(z_1, d)\big) \equiv \{x \in A : \varphi_x(z_1, d) \equiv \varphi_x(a, d) > 0\} \subseteq \{y_{\underline{k}}, \dots, y_q\} \subseteq \big(\{y_k^1\}_{k=\underline{k}}^{q}, A^{-1}\big)$. We adopt another induction argument.

The Secondary Introduction Hypothesis: Given $1 < \nu \leq \eta$, for all $1 \leq \nu' < \nu$, we have $supp\big(\varphi(z_{\nu'}, d)\big) \equiv \{x \in A : \varphi_x(z_{\nu'}, d) > 0\} \subseteq \big(\{y_k^1\}_{k=\underline{k}}^{q}, A^{-1}\big)$.

We show $supp\big(\varphi(z_\nu, d)\big) \subseteq \big(\{y_k^1\}_{k=\underline{k}}^{q}, A^{-1}\big)$. Assume $z_{\nu-1} = (c^1, \dots, c^k, \underline{z}^{k+1}, d^{k+2}, \dots, d^l, z^{-s})$ and $z_\nu = (c^1, \dots, c^k, \bar{z}^{k+1}, d^{k+2}, \dots, d^l, z^{-s})$ for some $1 < k < l$. Since $z_{\nu-1} \sim^+ z_\nu$, we have $P_i \in \mathbb{D}^{z_{\nu-1}}$ and $P_i' \in \mathbb{D}^{z_\nu}$ such that $P_i \sim^+ P_i'$. Thus, $\Gamma(P_i, P_i') = \big\{(\underline{z}^{k+1}, x^{-(k+1)}), (\bar{z}^{k+1}, x^{-(k+1)})\big\}_{x^{-(k+1)} \in A^{-(k+1)}}$. Suppose that there exists $y \equiv (y^s)_{s \in M} \in supp\big(\varphi(z_\nu, d)\big) \setminus \big(\{y_k^1\}_{k=\underline{k}}^{q}, A^{-1}\big)$. Thus, $\varphi_y(z_\nu, d) > 0$, $y \in \bar{A}$ and $y^1 \notin \{y_k^1\}_{k=\underline{k}}^{q}$. By the secondary induction hypothesis, $y^1 \notin \{y_k^1\}_{k=\underline{k}}^{q}$ implies $y \notin supp\big(\varphi(z_{\nu-1}, d)\big)$. Hence, $\varphi_y(z_{\nu-1}, d) = 0$. If y is not involved in any local switching pair of $\Gamma(P_i, P_i')$, by item 2(iii) of Lemma 8, we have $0 = \varphi_y(z_{\nu-1}, d) \equiv \varphi_y(P_i, d) = \varphi_y(P_i', d) \equiv \varphi_y(z_\nu, d) > 0$. Contradiction! If y is involved in a local switching pair of $\Gamma(P_i, P_i')$, we know either $y = (\underline{z}^{k+1}, y^{-(k+1)})$ or $y = (\bar{z}^{k+1}, y^{-(k+1)})$. Let $y = (\underline{z}^{k+1}, y^{-(k+1)})$. The verification related to the other case is symmetric, and we hence omit it. Then, let $\bar{y} = (\bar{z}^{k+1}, y^{-(k+1)})$. Thus, $\{y, \bar{y}\} \in \Gamma(P_i, P_i')$ and $\bar{y}^1 = y^1 \notin \{y_k^1\}_{k=\underline{k}}^{q}$. Hence, $\bar{y} \notin supp\big(\varphi(z_{\nu-1}, d)\big)$ by the secondary induction hypothesis, and equivalently, $\varphi_{\bar{y}}(z_{\nu-1}, d) = 0$. Consequently, item 2(ii) of Lemma 8 implies $0 = \varphi_y(z_{\nu-1}, d) + \varphi_{\bar{y}}(z_{\nu-1}, d) = \varphi_y(P_i, d) + \varphi_{\bar{y}}(P_i, d) = \varphi_y(P_i', d) + \varphi_{\bar{y}}(P_i', d) = \varphi_y(z_\nu, d) + \varphi_{\bar{y}}(z_\nu, d) > 0$. Contradiction! Therefore, $supp\big(\varphi(z_\nu, d)\big) \subseteq \big(\{y_k^1\}_{k=\underline{k}}^{q}, A^{-1}\big)$. This completes the verification of the second induction hypothesis. Therefore, $supp\big(\varphi(c, d)\big) \subseteq \big(\{y_k^1\}_{k=\underline{k}}^{q}, A^{-1}\big)$.

Third, symmetrically, according to $b = (d^1, c^2, \dots, c^l, z^{-S})$ and $d = (d^1, d^2, \dots, d^l, z^{-S})$, we can induce an adjacent$^+$ sequence in $(d^1, A^2, \dots, A^l, z^{-S})$ connecting b and d. According to the adjacent$^+$ sequence $\{x_k\}_{k=1}^{p}$ connecting b and c, Lemma 24 implies $supp\big(\varphi(c, b)\big) \subseteq \{x_{\bar{k}}, \dots, x_p\} \subseteq \Big(\{x_k^1\}_{k=\bar{k}}^{p}, A^{-1}\Big)$. Then, following the adjacent$^+$ sequence connecting b and d, by a similar argument, we have $supp\big(\varphi(c, d)\big) \subseteq \Big(\{x_k^1\}_{k=\bar{k}}^{p}, A^{-1}\Big)$.

Last, by the hypothesis of $\{x_{\bar{k}}^1, \dots, x_p^1\} \cap \{y_{\underline{k}}^1, \dots, y_q^1\} = \emptyset$, we have $\Big(\{x_k^1\}_{k=\bar{k}}^{p}, A^{-1}\Big) \cap \big(\{y_k^1\}_{k=\underline{k}}^{q}, A^{-1}\big) = \emptyset$. Therefore, we induce two mutually exclusive supports for the same social lottery $\varphi(c, d)$. Contradiction! This completes the verification of Claim 3.

Now, by Claims 1, 2 and 3, we know that c and d formulate a feasible box. This completes the verification of the induction hypothesis, and hence proves Lemma 28. $\square$

To establish the next lemma, we first introduce a graph. Let $G_{\sim^+}(\bar{A})$ be a graph where the vertex set is $\bar{A}$, and two distinct feasible alternatives form an edge if and only if they are adjacent$^+$.

Lemma 29. *The feasible set $\bar{A}$ is factorizable and graph $G_{\sim^+}(\bar{A})$ is a product of trees.*

Proof. Given $s \in M$, we first show that there exists $a^{-s} \in A^{-s}$ such that $\left|(A^s, a^{-s}) \cap \bar{A}\right| \geq 2$. First, since $\bar{A} \neq \emptyset$, there exists $a^{-s} \in A^{-s}$ such that $(A^s, a^{-s}) \cap \bar{A} \neq \emptyset$. Given $a \equiv (a^s, a^{-s}) \in \bar{A}$, by Assumption 1, we have $b \equiv (b^s, b^{-s}) \in \bar{A}$ such that $a^s \neq b^s$. Then, Lemma 28 implies that a and b formulate a feasible box $B(a, b)$. Thus, $(b^s, a^{-s}) \in B(a, b) \subseteq \bar{A}$. Hence, $\left|(A^s, x^{-s}) \cap \bar{A}\right| \geq 2$.

Next, fixing an arbitrary $s \in M$, we pick $x^{-s} \in A^{-s}$ such that $\left|(A^s, x^{-s}) \cap \bar{A}\right| \geq 2$. Furthermore, by Assumption 1, there must exist $y^{-s} \in A^{-s}\backslash\{x^{-s}\}$ such that $(A^s, y^{-s}) \cap \bar{A} \neq \emptyset$. Then, it must be the case $\left|(A^s, y^{-s}) \cap \bar{A}\right| \geq 2$. Furthermore, by Lemma 25, we induce two trees $G_{\sim^+}\big((A^s, x^{-s}) \cap \bar{A}\big)$ and $G_{\sim^+}\big((A^s, y^{-s}) \cap \bar{A}\big)$. Note that by Lemma 28, each alternative in $(A^s, x^{-s}) \cap \bar{A}$ and each alternative in $(A^s, y^{-s}) \cap \bar{A}$ formulate a feasible box. Hence, it must be the case $[(a^s, x^{-s}) \in \bar{A}] \Leftrightarrow [(a^s, y^{-s}) \in \bar{A}]$. Consequently, there exists $\bar{A}^s \subseteq A^s$ with $|\bar{A}^s| \geq 2$ such that $(A^s, x^{-s}) \cap \bar{A} = (\bar{A}^s, x^{-s})$ and $(A^s, y^{-s}) \cap \bar{A} = (\bar{A}^s, y^{-s})$, and hence, $G_{\sim^+}\big((\bar{A}^s, x^{-s})\big)$ and $G_{\sim^+}\big((\bar{A}^s, y^{-s})\big)$ coincide such that for all $a^s, b^s \in \bar{A}^s$, $\big[(a^s, x^{-s}) \sim^+ (b^s, x^{-s})\big] \Leftrightarrow \big[(a^s, y^{-s}) \sim^+ (b^s, y^{-s})\big]$. In conclusion, we can induce a tree $G(\bar{A}^s)$ such that a^s and b^s form an edge if and only if $(a^s, z^{-s}), (b^s, z^{-s}) \in \bar{A}$ and $(a^s, z^{-s}) \sim^+ (b^s, z^{-s})$ for all $z^{-s} \in A^{-s}$ with $(\bar{A}^s, z^{-s}) \equiv (A^s, z^{-s}) \cap \bar{A} \neq \emptyset$. Hence, we assert that $\bar{A}$ is factorizable, i.e., $\bar{A} = \times_{s\in M} \bar{A}^s$, and $G_{\sim^+}(\bar{A}) = \times_{s\in M} G(\bar{A}^s)$ is a product of trees. $\square$

Applying the proofs of Lemmas 6 and 7, we know that for all $P_i \in \mathbb{D}$, if $r_1(P_i) \in \bar{A}$, then $P_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. In the rest of the proof, we focus on preferences whose peaks are infeasible alternatives.

Lemma 30. *Fix $a \in A\backslash\bar{A}$ and $\bar{P}_i \in \mathbb{D}^a$. If $\bar{P}_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$, then every preference of $\mathbb{D}^a$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.*

Proof. Let $r_1(\bar{P}_{i|\bar{A}}) = \bar{a} \equiv (\bar{a}^s)_{s\in M}$. Then, Assumption 2 implies $r_1(P_{i|\bar{A}}) = \bar{a}$ for all $P_i \in \mathbb{D}^a$. Note that $\bar{a}^s \in \bar{A}^s$ for all $s \in M$ by the factorization $\bar{A} = \times_{s\in M} \bar{A}^s$.

Claim 1: Given $s \in M$ and $x^s \in \bar{A}^s$, $\varphi\left(P_i, (x^s, \bar{a}^{-s})\right) = \varphi\left(\bar{a}, (x^s, \bar{a}^{-s})\right)$ for all $P_i \in \mathbb{D}^a$.

If $x^s = \bar{a}^s$, the claim follows from unanimity (w.r.t. feasibility). We henceforth assume $x^s \neq \bar{a}^s$. Note that $(x^s, \bar{a}^{-s}) \in \bar{A}$. By the tops-only property, we know $\varphi\left(P_i, (x^s, \bar{a}^{-s})\right) = \varphi\left(\bar{P}_i, (x^s, \bar{a}^{-s})\right)$ for all $P_i \in \mathbb{D}^a$. Therefore, to complete the verification, it suffices to show $\varphi\left(\bar{P}_i, (x^s, \bar{a}^{-s})\right) = \varphi\left(\bar{a}, (x^s, \bar{a}^{-s})\right)$.

Given the interval $\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle$ in $G_{\sim^+}\left((\bar{A}^s, \bar{a}^{-s})\right)$, since $\bar{P}_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$ by the hypothesis of the lemma and $r_1(\bar{P}_{i|\bar{A}}) = \bar{a}$, we know that $\bar{P}_{i|\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle}$ is single-peaked on $\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle$. Next, pick arbitrary $P'_i \in \mathbb{D}^{\bar{a}}$ by minimal richness. Since $\bar{a} \in \bar{A}$, we know that P'_i is multidimensional single-peaked w.r.t. $\bar{A}$. Hence, $P'_{i|\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle}$ is single-peaked on $\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle$ as well. Since $r_1(\bar{P}_{i|\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle}) = r_1(P'_{i|\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle}) = \bar{a}$, single-peakedness on $\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle$ implies $\bar{P}_{i|\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle} = P'_{i|\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle}$. Moreover, since $\sum_{z\in\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle} \varphi_z\big(\bar{P}_i, (x^s, \bar{a}^{-s})\big) = 1$ and $\sum_{z\in\langle \bar{a}, (x^s, \bar{a}^{-s})\rangle} \varphi_z\big(P'_i, (x^s, \bar{a}^{-s})\big) = 1$ by Lemma 24, sd-strategy-proofness and the tops-only property imply $\varphi\big(\bar{P}_i, (x^s, \bar{a}^{-s})\big) = \varphi\big(P'_i, (x^s, \bar{a}^{-s})\big) = \varphi\big(\bar{a}, (x^s_t, \bar{a}^{-s})\big)$, as required. This completes the verification of the claim.

Before establishing the next claim, note that minimal richness and connectedness$^+$ imply that there exists a separable preference $P_i \in \mathbb{D}$ such that $r_1(P_i) = a$.

Claim 2: Given a separable preference $P_i \in \mathbb{D}^a$, the induced preference $P_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.

By Remark 1, it suffices to show that $[P_i]^s_{|\bar{A}^s}$ is single-peaked on $G(\bar{A}^s)$ for every $s \in M$. Given $s \in M$ and $x^s \in \bar{A}^s$, since $\varphi\left(P_i, (x^s, \bar{a}^{-s})\right) = \varphi\left(\bar{a}, (x^s, \bar{a}^{-s})\right)$ by Claim 1, the proof of Lemma 6 implies that $[P_i]^s_{|\bar{A}^s}$ is single-peaked on $G(\bar{A}^s)$, as required. This completes the verification of the claim.

Claim 3: Given $P_i \in \mathbb{D}^a$, the induced preference $P_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.

Suppose not, i.e., there exist distinct $x, y \in \bar{A}$ such that $x \in \langle \bar{a}, y\rangle$ and $yP_i x$. Pick arbitrary $P_i' \in \mathbb{D}^y$ by minimal richness. Evidently, $a \neq y$. Since $yP_i x$ and $yP_i' x$, by the Exterior$^+$ property, we have a path $\{P_i^k\}_{k=1}^q$ connecting P_i and P_i' such that $yP_i^k x$ for all $k = 1, \ldots, q$. Since $a \neq y$, there exists $1 \leq k^* < q$ such that $r_1(P_i^{k^*}) = a$ and $r_1(P_i^{k^*+1}) \neq a$. Thus, Lemma 9 implies $P_i^{k^*} \sim^+ P_i^{k^*+1}$, and hence $P_i^{k^*}$ is a separable preference. Then, Claim 2 implies that $P^{k^*}_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$, and hence $xP_i^{k^*} y$. Contradiction! This completes the verification of the claim and the lemma. □

Lemma 31. *Given $P_i, P_i' \in \mathbb{D}$ with $P_i \sim^+ P_i'$, if $P_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$, then $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.*

Proof. If $r_1(P_i) = r_1(P_i')$, the result follows from Lemma 30. If $r_1(P_i') \in \bar{A}$, it is evident that $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. Hence, we assume $r_1(P_i) \equiv a \neq b \equiv r_1(P_i')$ and $b \notin \bar{A}$. Since $P_i \sim^+ P_i'$, we know $a^s \neq b^s$ and $a^{-s} = b^{-s}$ for some $s \in M$, $\Gamma(P_i, P_i') = \left\{\left\{(a^s, z^{-s}), (b^s, z^{-s})\right\}\right\}_{z^{-s}\in A^{-s}}$, and both P_i and P_i' are separable preferences. Let $\bar{a} = r_1(P_{i|\bar{A}})$ and $\bar{b} = r_1(P'_{i|\bar{A}})$.

Claim 1: If $\bar{a} \neq \bar{b}$, then $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.

Since $\bar{a}P_i\bar{b}$, $\bar{b}P_i'\bar{a}$ and $P_i \sim^+ P_i'$, we know $\{\bar{a}, \bar{b}\} \in \Gamma(P_i, P_i')$, and hence, $\bar{a}P_i!\bar{b}$, $\bar{b}P_i'!\bar{a}$, $\bar{a}^s = a^s$, $\bar{b}^s = b^s$ and $\bar{a}^{-s} = \bar{b}^{-s} \equiv \bar{z}^{-s}$. Moreover, since $\bar{a}, \bar{b} \in \bar{A}$, we know $a^s = \bar{a}^s \in \bar{A}^s$, $b^s = \bar{b}^s \in \bar{A}^s$ and $\bar{z}^{-s} \in \bar{A}^{-s}$. Next, we show $\langle a^s, b^s\rangle = \{a^s, b^s\}$. Suppose not, i.e., there exists $c^s \in \langle a^s, b^s\rangle\backslash\{a^s, b^s\}$. Thus, $c^s \in \bar{A}^s$, $\bar{c} \equiv (c^s, \bar{z}^{-s}) \in \bar{A}$ and $(c^s, \bar{z}^{-s}) \in \langle(a^s, \bar{z}^{-s}), (b^s, \bar{z}^{-s})\rangle \equiv \langle \bar{a}, \bar{b}\rangle$. Consequently, since P_i is multidimensional single-peaked w.r.t. $\bar{A}$, we have $\bar{a}P_i\bar{c}$ and $\bar{c}P_i\bar{b}$. This contradicts $\bar{a}P_i!\bar{b}$. Therefore, $\langle a^s, b^s\rangle = \{a^s, b^s\}$.

Last, suppose that $P'_{i|\bar{A}}$ is not multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$, i.e., there exist distinct $x, y \in \bar{A}$ such that $x \in \langle \bar{b}, y\rangle$ and $yP_i' x$. Since $x \in \langle \bar{b}, y\rangle$, we have $x^s \in \langle \bar{b}^s, y^s\rangle \equiv \langle b^s, y^s\rangle$ and $x^{-s} \in \langle \bar{b}^{-s}, y^{-s}\rangle = \langle \bar{z}^{-s}, y^{-s}\rangle$. Since $\langle a^s, b^s\rangle = \{a^s, b^s\}$, $x^s \in \langle b^s, y^s\rangle$ implies $x^s \in \langle a^s, y^s\rangle$. Thus, $x = (x^s, x^{-s}) \in \langle(a^s, \bar{z}^{-s}), (y^s, y^{-s})\rangle = \langle \bar{a}, y\rangle$, and hence $xP_i y$ by multidimensional single-peakedness w.r.t. $\bar{A}$. Thus, $\{x, y\} \in \Gamma(P_i, P_i')$, and hence $x^s = a^s$, $y^s = b^s$ and $x^{-s} = y^{-s}$.

Consequently, $x^s \in \langle b^s, y^s\rangle = \langle b^s, b^s\rangle = \{b^s\}$ which contradicts $x^s = a^s$. This completes the verification of the claim.

CLAIM 2: If $\bar{a} = \bar{b}$, then $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$.

If $a^s \notin \bar{A}^s$, we show $P_{i|\bar{A}} = P'_{i|\bar{A}}$. Suppose not, i.e., there exist $x, y \in \bar{A}$ such that $x P_{i|\bar{A}} y$ and $y P'_{i|\bar{A}} x$. Thus, $x P_i y$ and $y P'_i x$. Consequently, $P_i \sim^+ P'_i$ implies $x^s = a^s$, $y^s = b^s$ and $x^{-s} = y^{-s}$. Hence, by factorization $\bar{A} = \times_{s\in M} \bar{A}^s$, $x^s = a^s \notin \bar{A}^s$ implies $x \notin \bar{A}$. This contradicts the hypothesis $x \in \bar{A}$. Therefore, $P_{i|\bar{A}} = P'_{i|\bar{A}}$. Symmetrically, if $b^s \notin \bar{A}^s$, we have $P_{i|\bar{A}} = P'_{i|\bar{A}}$.

Now, we show either $a^s \notin \bar{A}^s$ or $b^s \notin \bar{A}^s$. Suppose not, i.e., $a^s, b^s \in \bar{A}^s$. Since $\bar{a} = \bar{b} \in \bar{A}$, it is evident that $\bar{z}^{-s} \equiv \bar{a}^{-s} = \bar{b}^{-s} \in \bar{A}^{-s}$. Thus, $(a^s, \bar{z}^{-s}), (b^s, \bar{z}^{-s}) \in \bar{A}$. Recall that both P_i and P'_i are separable preferences. Since $r_1(P_i) = a$ and $r_1(P'_i) = b$, separability implies (i) either $(a^s, \bar{z}^{-s}) P_i (\bar{a}^s, \bar{z}^{-s}) \equiv \bar{a}$ or $a^s = \bar{a}^s$, and (ii) either $(b^s, \bar{z}^{-s}) P'_i (\bar{b}^s, \bar{z}^{-s}) \equiv \bar{b}$ or $b^s = \bar{b}^s$. Furthermore, since $\bar{a} = \bar{b}$ and $a^s \neq b^s$, it must be the case either $(a^s, \bar{z}^{-s}) P_i \bar{a}$ or $(b^s, \bar{z}^{-s}) P'_i \bar{b}$ which contradicts the hypothesis $\bar{a} = r_1(P_{i|\bar{A}})$ and $\bar{b} = r_1(P'_{i|\bar{A}})$. This completes the verification of the claim, and proves the lemma. □

Lemma 32. *Domain $\mathbb{D}$ is multidimensional single-peaked w.r.t. $\bar{A}$.*

Proof. Given an arbitrary $P_i \in \mathbb{D}$, let $r_1(P_i) = a \notin \bar{A}$ and $r_1(P_{i|\bar{a}}) = \bar{a}$. Pick an arbitrary $P'_i \in \mathbb{D}^{\bar{a}}$ by minimal richness. We know that $P'_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. We have a path $\{P_i^k\}_{k=1}^q$ connecting P'_i and P_i. We first consider preference P_i^2. If $P_i^2 \sim P_i^1 = P'_i$, then $r_1(P_i^2) = r_1(P_i^1) = a$ by Lemma 9, and then Lemma 30 implies that $P^2_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. If $P_i^2 \sim^+ P_i^1$, Lemma 31 implies that $P^2_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. Following the path $\{P_i^k\}_{k=1}^q$ from P_i^2 to $P_i^q = P_i$, by repeatedly applying the argument above, we eventually show that $P_{i|\bar{A}}$ is multidimensional single-peaked on $\times_{s\in M} G(\bar{A}^s)$. This completes the verification of the lemma, and hence proves part (i) of Theorem 3. □

Appendix E. Remaining verifications

E.1. A minimally rich domain satisfying all conditions of connectedness$^+$ except the no-detour property

Let $A = A^1 \times A^2$, $A^1 = \{0, 1, 2\}$ and $A^2 = \{0, 1\}$. Let $\mathbb{D}_{\text{MSP}}$ be the multidimensional single-peaked domain on the product of two lines $G(A^1) \times G(A^2)$ specified in Fig. 4 of Example 5. Let $\mathbb{D} \subset \mathbb{D}_{\text{MSP}}$ be a domain of 20 preferences specified in Table 1.

Note that there exist no $P_i \in \mathbb{D}^{(1,0)}$ and $P'_i \in \mathbb{D}^{(1,1)}$ such that $P_i \sim^+ P'_i$. This implies the violation of the no-detour property. We use a graph of adjacency and adjacency$^+$ over $\mathbb{D}$ to show the Interior$^+$ property and the remainder of the Exterior$^+$ property.

The diagram of Fig. 17 directly shows that the Interior$^+$ property is met by $\mathbb{D}$. To verify the Exterior$^+$ property (without the requirement of the no-detour property) on domain $\mathbb{D}$, it suffices to make two observations on Fig. 17: First, each pair of preferences is connected via two *independent* paths, and second, each local switching pair in Fig. 17 appears exactly twice.

Table 1
Domain $\mathbb{D}$.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
(0, 0)	(0, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(2, 0)	(2, 0)
(0, 1)	(1, 0)	(0, 0)	(0, 0)	(0, 0)	(2, 0)	(2, 0)	(2, 0)	(1, 0)	(2, 1)
(1, 0)	(0, 1)	(1, 1)	(1, 1)	(2, 0)	(0, 0)	(1, 1)	(1, 1)	(2, 1)	(1, 0)
(1, 1)	(1, 1)	(0, 1)	(2, 0)	(1, 1)	(1, 1)	(0, 0)	(2, 1)	(1, 1)	(1, 1)
(2, 0)	(2, 0)	(2, 0)	(0, 1)	(0, 1)	(2, 1)	(2, 1)	(0, 0)	(0, 0)	(0, 0)
(2, 1)	(2, 1)	(2, 1)	(2, 1)	(2, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$	$\bar{P}_5$	$\bar{P}_6$	$\bar{P}_7$	$\bar{P}_8$	$\bar{P}_9$	$\bar{P}_{10}$
(0, 1)	(0, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(2, 1)	(2, 1)
(0, 0)	(1, 1)	(0, 1)	(0, 1)	(0, 1)	(2, 1)	(2, 1)	(2, 1)	(1, 1)	(2, 0)
(1, 1)	(0, 0)	(1, 0)	(1, 0)	(2, 1)	(0, 1)	(1, 0)	(1, 0)	(2, 0)	(1, 1)
(1, 0)	(1, 0)	(0, 0)	(2, 1)	(1, 0)	(1, 0)	(0, 1)	(2, 0)	(1, 0)	(1, 0)
(2, 1)	(2, 1)	(2, 1)	(0, 0)	(0, 0)	(2, 0)	(2, 0)	(0, 1)	(0, 1)	(0, 1)
(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)

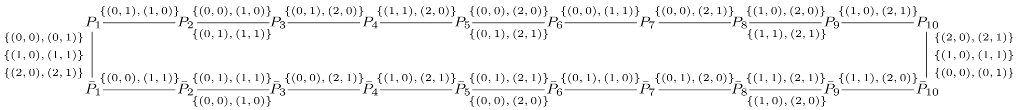

Fig. 17. The adjacency and adjacency$^+$ relations among preferences of $\mathbb{D}$. In Fig. 17, for instance, "$P_1 \xrightarrow{\{(0,1),(1,0)\}} P_2$" represents $P_1 \sim P_2$ and $\Gamma(P_1, P_2) = \{(0, 1), (1, 0)\}$, and "$P_2 \frac{\{(0,0),(1,0)\}}{\{(0,1),(1,1)\}} P_3$" represents $P_2 \sim^+ P_3$ and $\Gamma(P_2, P_3) = \{\{(0, 0), (1, 0)\}, \{(0, 1), (1, 1)\}\}$.

E.2. The separable domain is a connected$^+$ domain

We first provide an example to illustrate how to verify connectedness$^+$ on a particular separable domain.

Example 8. Let $A = A^1 \times A^2$, $A^1 = \{0, 1, 2\}$ and $A^2 = \{0, 1\}$. Fix two particular separable preferences:

$$P_i: \ (0,0) \rightharpoondown (0,1) \rightharpoondown (1,0) \rightharpoondown (1,1) \rightharpoondown (2,0) \rightharpoondown (2,1), \text{ and}$$
$$P_i': \ (2,1) \rightharpoondown (0,1) \rightharpoondown (2,0) \rightharpoondown (1,1) \rightharpoondown (0,0) \rightharpoondown (1,0).$$

Note that $(0, 1)\, P_i\, (1, 1)$ and $(0, 1)\, P_i'\, (1, 1)$.

First, we identify the transition of marginal preferences from P_i to P_i' below which consists of 3 steps.

$$\big([P_i]^1; [P_i]^2\big) \equiv (0 \rightharpoondown 1 \rightharpoondown 2; 0 \rightharpoondown 1) \xRightarrow{①} (0 \rightharpoondown 1 \rightharpoondown 2; 1 \rightharpoondown 0) \xRightarrow{②} (0 \rightharpoondown 2 \rightharpoondown 1; 1 \rightharpoondown 0)$$
$$\xRightarrow{③} (2 \rightharpoondown 0 \rightharpoondown 1; 1 \rightharpoondown 0) \equiv \big([P_i']^1; [P_i']^2\big).$$

For each transition step, we identify a pair of adjacent$^+$ preferences in Table 2.

Next, we make the following observations.

(i) $P_i = \bar{P}_i^1$.
(ii) $\hat{P}_i^1$ and $\bar{P}_i^2$ share the same marginal preferences, and $\hat{P}_i^1 \sim \bar{P}_i^2$.

Table 2
Three pairs of adjacent$^+$ preferences.

$\overset{①}{\Longrightarrow}$			$\overset{②}{\Longrightarrow}$			$\overset{③}{\Longrightarrow}$		
$\bar{P}_i^1$	$\sim^+$	$\hat{P}_i^1$	$\bar{P}_i^2$	$\sim^+$	$\hat{P}_i^2$	$\bar{P}_i^3$	$\sim^+$	$\hat{P}_i^3$
$(0,0)$		$(0,1)$	$(0,1)$		$(0,1)$	$(0,1)$		$(2,1)$
$(0,1)$		$(0,0)$	$(0,0)$		$(0,0)$	$(2,1)$		$(0,1)$
$(1,0)$		$(1,1)$	$(1,1)$		$(2,1)$	$(1,1)$		$(1,1)$
$(1,1)$		$(1,0)$	$(2,1)$		$(1,1)$	$(0,0)$		$(2,0)$
$(2,0)$		$(2,1)$	$(1,0)$		$(2,0)$	$(2,0)$		$(0,0)$
$(2,1)$		$(2,0)$	$(2,0)$		$(1,0)$	$(1,0)$		$(1,0)$

(iii) $\hat{P}_i^2$ and $\bar{P}_i^3$ share the same marginal preferences. We identify another separable preference $\tilde{P}_i$: $(0,1) \rightharpoonup (2,1) \rightharpoonup (0,0) \rightharpoonup (1,1) \rightharpoonup (2,0) \rightharpoonup (1,0)$, and construct an adjacency path $\{\hat{P}_i^2, \tilde{P}_i, \bar{P}_i^3\}$ along which $(0,1)$ always ranks above $(1,1)$.[52]

(iv) $\hat{P}_i^3$ and P_i' share the same marginal preferences, and $\hat{P}_i^3 \sim P_i'$.

Eventually, we construct a path connecting P_i and P_i' which meets the requirement of the Exterior$^+$ property: $P_i = \bar{P}_i^1 \sim^+ \hat{P}_i^1 \sim \bar{P}_i^2 \sim^+ \hat{P}_i^2 \sim \tilde{P}_i \sim \bar{P}_i^3 \sim^+ \hat{P}_i^3 \sim P_i'$. Note that $r_1(\hat{P}_i^1) = (0,1)$ and $r_1(P_i') = (2,1)$ are similar, and $\{\hat{P}_i^1, \bar{P}_i^2, \hat{P}_i^2, \tilde{P}_i, \bar{P}_i^3, \hat{P}_i^3, P_i'\}$ satisfies the requirement of the no-detour property. As an instance, $\{\hat{P}_i^1, \bar{P}_i^2, \hat{P}_i^2, \tilde{P}_i, \bar{P}_i^3\}$ meets the requirement of the Interior$^+$ property. $\square$

We make one observation on the separable domain $\mathbb{D}_S$.

Observation 1. If a pair of separable preferences is adjacent, they share the same marginal preference on each component. If a pair of separable preferences is adjacent$^+$, they have a pair of adjacent marginal preferences on one component, and share the same marginal preference on every other component. Fix a separable preference P_i. Given $a P_i! b$, if a and b disagree on at least two components, then by locally switching a and b in P_i, we generate a new separable preference P_i' such that $P_i \sim P_i'$ and $\Gamma(P_i, P_i') = \big\{\{a,b\}\big\}$. Similarly, if there exist $s \in M$ and $a^s, b^s \in A^s$ such that $(a^s, z^{-s}) P_i! (b^s, z^{-s})$ for all $z^{-s} \in A^{-s}$, then by locally switching (a^s, z^{-s}) and (b^s, z^{-s}) for all $z^{-s} \in A^{-s}$ simultaneously in P_i, we generate a new separable preference P_i' such that $P_i \sim^+ P_i'$ and $\Gamma(P_i, P_i') = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s} \in A^{-s}}$. $\square$

Now, we provide five general facts to show that the separable domain in general is a connected$^+$ domain. Fact 1 only serves for the proof of Fact 2, Facts 2 and 3 are step results for the proof of Fact 4, while Facts 4 and 5 are utilized to construct paths for the verification of the Interior$^+$ and Exterior$^+$ properties. We separately establish these five facts because they will be replicated for the verifications of connectedness$^+$ on other multidimensional domains. For notational convenience, let $[\mathbb{D}_S]^s$ denote the marginal domain over A^s which in fact is the complete domain of marginal preferences over A^s.

[52] Note that the existence of preference $\tilde{P}_i$ also illustrates the importance of the co-existence of adjacency and adjacency$^+$. If we forbid the presence of adjacency, then preference $\bar{P}_i^3$ cannot be obtained via the transition of preference $\hat{P}_i^2$, and consequently, the adjacency$^+$ relation between $\bar{P}_3$ and $\hat{P}_3$ disappears.

Fact 1. *Given $s \in M$ and distinct $[P_j]^s, [P'_j]^s \in [\mathbb{D}_S]^s$, there exist $[P''_j]^s \in [\mathbb{D}_S]^s$ and $a^s, b^s \in A^s$ such that $[P_j]^s \sim [P''_j]^s$, $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^s a^s$. (Note that it is possible $[P''_j]^s = [P'_j]^s$.)*

Proof. Since $[P_j]^s$ and $[P'_j]^s$ are distinct, there exist $a^s, b^s \in A^s$ such that $a^s[P_j]^s!b^s$ and $b^s[P'_j]^s a^s$. By locally switching a^s and b^s in $[P_j]^s$, we generate $[P''_j]^s \in [\mathbb{D}_S]^s$. Thus, $[P_j]^s \sim [P''_j]^s$ and $b^s[P''_j]^s!a^s$. □

Fact 2. *Fixing $P_j, P'_j \in \mathbb{D}_S$ with $[P_j]^q \neq [P'_j]^q$ for some $q \in M$, and $x, y \in A$ with xP_jy and xP'_jy, there exists $P''_j \in \mathbb{D}_S$ such that*

(i) *$[P''_j]^s \sim [P_j]^s$ for some $s \in M$, and $[P''_j]^\omega = [P_j]^\omega$ for all $\omega \neq s$,*
(ii) *$a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^s a^s$ for some $a^s, b^s \in A^s$, and*
(iii) *xP''_jy.*

Proof. Let $S = \{s \in M : [P_j]^s \neq [P'_j]^s\}$ and $T = \{\tau \in M : x^\tau \neq y^\tau\}$. Evidently, both S and T are nonempty. We consider two cases:

(1) There exists $\tau \in T$ such that $x^\tau[P_j]^\tau y^\tau$ and $x^\tau[P'_j]^\tau y^\tau$.
(2) For all $\tau \in T$, $\left[x^\tau[P_j]^\tau y^\tau\right] \Rightarrow \left[y^\tau[P'_j]^\tau x^\tau\right]$.

In case (1), we know either $\tau \in S$ or $\tau \notin S$. If $\tau \in S$, we know $[P_j]^\tau \neq [P'_j]^\tau$. Furthermore, by Fact 1, we have $[P''_j]^\tau \in [\mathbb{D}_S]^\tau$ and $a^\tau, b^\tau \in A^\tau$ such that $[P''_j]^\tau \sim [P_j]^\tau$, $a^\tau[P_j]^\tau!b^\tau$, $b^\tau[P''_j]^\tau!a^\tau$ and $b^\tau[P'_j]^\tau a^\tau$. Thus, $\{a^\tau, b^\tau\} \neq \{x^\tau, y^\tau\}$. Hence, $[P_j]^\tau \sim [P''_j]^\tau$ and $x^\tau[P_j]^\tau y^\tau$ imply $x^\tau[P''_j]^\tau y^\tau$. We then refer to a lexicographic order $\succ$ where τ is lexicographically dominant, assemble marginal preferences $[P''_j]^\tau$ and $[P_j]^\omega$ for all $\omega \neq \tau$, and form a lexicographically separable preference P''_j. Thus, P''_j must satisfy conditions (i) - (iii). If $\tau \notin S$, we pick an arbitrary $s \in S$. Since $[P_j]^s \neq [P'_j]^s$, by Fact 1, we have $[P''_j]^s \in [\mathbb{D}_S]^s$ and $a^s, b^s \in A^s$ such that $[P''_j]^s \sim [P_j]^s$, $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^s a^s$. We then refer to a lexicographic order $\succ$ where τ is lexicographically dominant, assemble marginal preferences $[P''_j]^s$ and $[P_j]^\omega$ for all $\omega \neq s$, and form a lexicographically separable preference P''_j. Thus, P''_j must satisfy conditions (i) - (iii).

Next, assume that case (2) occurs. Since xP'_jy, there exists $s \in T$ such that $x^s[P'_j]^s y^s$. Then, case (2) implies $y^s[P_j]^s x^s$, and hence $s \in S$. Then, by Fact 1, we have $[P''_j]^s \in [\mathbb{D}_S]^s$ and $a^s, b^s \in A^s$ such that $[P''_j]^s \sim [P_j]^s$, $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^s a^s$. Symmetrically, since xP_jy, there exists $\hat{\tau} \in T$ such that $x^{\hat{\tau}}[P_j]^{\hat{\tau}} y^{\hat{\tau}}$. Then, case (2) implies $y^{\hat{\tau}}[P'_j]^{\hat{\tau}} x^{\hat{\tau}}$. Since $x^{\hat{\tau}}[P_j]^{\hat{\tau}} y^{\hat{\tau}}$ and $y^s[P_j]^s x^s$, we know $\hat{\tau} \neq s$. Now, we refer to a lexicographic order $\succ$ where $\hat{\tau}$ is lexicographically dominant, assemble marginal preferences $[P''_j]^s$, $[P_j]^{\hat{\tau}}$ and $[P_j]^\omega$ for all $\omega \notin \{s, \hat{\tau}\}$, and form a lexicographically separable preference P''_j. Thus, P''_j must satisfy conditions (i) - (iii). □

Fact 3. *Given $P_j \in \mathbb{D}_S$, $s \in M$, $a^s, b^s \in A^s$ with $a^s[P_j]^s!b^s$, there exists $\bar{P}_j \in \mathbb{D}_S$ such that*

(i) *for all $x, y \in A \backslash (b^s, A^{-s})$, $[xP_jy] \Leftrightarrow [x\bar{P}_jy]$,*

(ii) *for all* $z^{-s} \in A^{-s}$, $(a^s, z^{-s})\bar{P}_j!(b^s, z^{-s})$, *and*
(iii) $[P_j]^\omega = [\bar{P}_j]^\omega$ *for all* $\omega \in M$.

Moreover, there exists $\hat{P}_j \in \mathbb{D}_S$ *such that* $\Gamma(\bar{P}_j, \hat{P}_j) = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s}\in A^{-s}}$.

Proof. We first construct preferences $\bar{P}_j$ satisfying conditions (i) and (ii). First, we remove all alternatives of (b^s, A^{-s}) from P_j, and induce preference $P_{j|A\backslash(b^s,A^{-s})}$ over $A\backslash(b^s, A^{-s})$. Next, we construct preference $\bar{P}_j$ over A by plugging all alternatives of (b^s, A^{-s}) back to $P_{j|A\backslash(b^s,A^{-s})}$ such that for all $z^{-s} \in A^{-s}$, (a^s, z^{-s}) contiguously ranks above (b^s, z^{-s}), i.e., $(a^s, z^{-s})\bar{P}_j!(b^s, z^{-s})$. Evidently, by the construction, $\bar{P}_j$ satisfies conditions (i) and (ii). Moreover, note that $\bar{P}_j$ satisfies condition (iii) if it is a separable preference.

Next, we show that $\bar{P}_j$ is a separable preference. Given $\tau \in M$, $\hat{a}^\tau, \hat{b}^\tau \in A^\tau$ and $\overline{z}^{-\tau}, \underline{z}^{-\tau} \in A^{-\tau}$, let $(\hat{a}^\tau, \overline{z}^{-\tau})\bar{P}_j(\hat{b}^\tau, \overline{z}^{-\tau})$. Suppose by contradiction $(\hat{b}^\tau, \underline{z}^{-\tau})\bar{P}_j(\hat{a}^\tau, \underline{z}^{-\tau})$. We consider the following four cases, and induce a contradiction in each case:

(1) $\tau \neq s$ and $\overline{z}^s \neq b^s$ or $\underline{z}^s \neq b^s$,
(2) $\tau \neq s$ and $\overline{z}^s = b^s = \underline{z}^s$,
(3) $\tau = s$ and $\hat{b}^\tau = b^s$, and
(4) $\tau = s$ and $\hat{b}^\tau \neq b^s$.

In case (1), we assume $\overline{z}^s \neq b^s$. The verification related to $\underline{z}^s \neq b^s$ is symmetric, and we hence omit it. Since $\tau \neq s$ and $\overline{z}^s \neq b^s$, we know $(\hat{a}^\tau, \overline{z}^{-\tau}), (\hat{b}^\tau, \overline{z}^{-\tau}) \notin (b^s, A^{-s})$. Then, by condition (i), the hypothesis $(\hat{a}^\tau, \overline{z}^{-\tau})\bar{P}_j(\hat{b}^\tau, \overline{z}^{-\tau})$ implies $(\hat{a}^\tau, \overline{z}^{-\tau})P_j(\hat{b}^\tau, \overline{z}^{-\tau})$. Furthermore, by separability, we have $(\hat{a}^\tau, \underline{z}^{-\tau})P_j(\hat{b}^\tau, \underline{z}^{-\tau})$ and $(\hat{a}^\tau, a^s, \underline{z}^{-\{\tau,s\}})P_j(\hat{b}^\tau, a^s, \underline{z}^{-\{\tau,s\}})$ which implies $(\hat{a}^\tau, a^s, \underline{z}^{-\{\tau,s\}})\bar{P}_j(\hat{b}^\tau, a^s, \underline{z}^{-\{\tau,s\}})$ by condition (i). If $\underline{z}^s \neq b^s$, condition (i) implies $(\hat{a}^\tau, \underline{z}^{-\tau})\bar{P}_j(\hat{b}^\tau, \underline{z}^{-\tau})$. This contradicts the hypothesis $(\hat{b}^\tau, \underline{z}^{-\tau})\bar{P}_j(\hat{a}^\tau, \underline{z}^{-\tau})$. Therefore, $\underline{z}^s = b^s$. Since $(\hat{a}^\tau, a^s, \underline{z}^{-\{\tau,s\}})\bar{P}_j!(\hat{a}^\tau, b^s, \underline{z}^{-\{\tau,s\}}) = (\hat{a}^\tau, \underline{z}^{-\tau})$ and $(\hat{b}^\tau, a^s, \underline{z}^{-\{\tau,s\}})\bar{P}_j!(\hat{b}^\tau, b^s, \underline{z}^{-\{\tau,s\}}) = (\hat{b}^\tau, \underline{z}^{-\tau})$ by condition (ii), $(\hat{a}^\tau, a^s, \underline{z}^{-\{\tau,s\}})\bar{P}_j(\hat{b}^\tau, a^s, \underline{z}^{-\{\tau,s\}})$ implies $(\hat{a}^\tau, \underline{z}^{-\tau})\bar{P}_j(\hat{b}^\tau, \underline{z}^{-\tau})$. Contradiction!

Assume that case (2) occurs. On the one hand, since $(\hat{a}^\tau, a^s, \overline{z}^{-\{\tau,s\}})\bar{P}_j!(\hat{a}^\tau, b^s, \overline{z}^{-\{\tau,s\}}) = (\hat{a}^\tau, \overline{z}^{-\tau})$ and $(\hat{b}^\tau, a^s, \overline{z}^{-\{\tau,s\}})\bar{P}_j!(\hat{b}^\tau, b^s, \overline{z}^{-\{\tau,s\}}) = (\hat{b}^\tau, \overline{z}^{-\tau})$ by condition (ii), the hypothesis $(\hat{a}^\tau, \overline{z}^{-\tau})\bar{P}_j(\hat{b}^\tau, \overline{z}^{-\tau})$ implies $(\hat{a}^\tau, a^s, \overline{z}^{-\{\tau,s\}})\bar{P}_j(\hat{b}^\tau, a^s, \overline{z}^{-\{\tau,s\}})$. Then, condition (i) implies $(\hat{a}^\tau, a^s, \overline{z}^{-\{\tau,s\}})P_j(\hat{b}^\tau, a^s, \overline{z}^{-\{\tau,s\}})$, and hence $\hat{a}^\tau[P_j]^\tau\hat{b}^\tau$ by separability. On the other hand, since $(\hat{a}^\tau, a^s, \underline{z}^{-\{\tau,s\}})\bar{P}_j!(\hat{a}^\tau, b^s, \underline{z}^{-\{\tau,s\}}) = (\hat{a}^\tau, \underline{z}^{-\tau})$ and $(\hat{b}^\tau, a^s, \underline{z}^{-\{\tau,s\}})\bar{P}_j!(\hat{b}^\tau, b^s, \underline{z}^{-\{\tau,s\}}) = (\hat{b}^\tau, \underline{z}^{-\tau})$ by condition (ii), the hypothesis $(\hat{b}^\tau, \underline{z}^{-\tau})\bar{P}_j(\hat{a}^\tau, \underline{z}^{-\tau})$ implies $(\hat{b}^\tau, a^s, \underline{z}^{-\{\tau,s\}}) \times \bar{P}_j(\hat{a}^\tau, a^s, \underline{z}^{-\{\tau,s\}})$. Then, condition (i) implies $(\hat{b}^\tau, a^s, \underline{z}^{-\{\tau,s\}})P_j(\hat{a}^\tau, a^s, \underline{z}^{-\{\tau,s\}})$, and hence $\hat{b}^\tau[P_j]^\tau\hat{a}^\tau$ by separability. Contradiction!

In case (3), since $(a^s, \underline{z}^{-s})\bar{P}_j!(b^s, \underline{z}^{-s}) = (\hat{b}^s, \underline{z}^{-s})$ by condition (ii) and $(\hat{b}^s, \underline{z}^{-s})\bar{P}_j(\hat{a}^s, \underline{z}^{-s})$ by the hypothesis, we know $(a^s, \underline{z}^{-s})\bar{P}_j(\hat{a}^s, \underline{z}^{-s})$ and hence $a^s \neq \hat{a}^s$. Then, condition (i) implies $(a^s, \underline{z}^{-s})P_j(\hat{a}^s, \underline{z}^{-s})$, and hence $(a^s, \overline{z}^{-s})P_j(\hat{a}^s, \overline{z}^{-s})$ by separability. Again, condition (i) implies $(a^s, \overline{z}^{-s})\bar{P}_j(\hat{a}^s, \overline{z}^{-s})$. Last, since $(a^s, \overline{z}^{-s})\bar{P}_j!(b^s, \overline{z}^{-s})$ by condition (ii), it must be the case $(b^s, \overline{z}^{-s})\bar{P}_j(\hat{a}^s, \overline{z}^{-s})$ which is equivalently $(\hat{b}^\tau, \overline{z}^{-\tau})\bar{P}_j(\hat{a}^\tau, \overline{z}^{-\tau})$. This contradicts the hypothesis $(\hat{a}^\tau, \overline{z}^{-\tau})\bar{P}_j(\hat{b}^\tau, \overline{z}^{-\tau})$.

In case (4), we first show $b^s = \hat{a}^s$. Suppose not, i.e., $b^s \neq \hat{a}^s$. By condition (i), the hypothesis $(\hat{a}^s, \overline{z}^{-s})\bar{P}_j(\hat{b}^s, \overline{z}^{-s})$ and $(\hat{b}^s, \underline{z}^{-s})\bar{P}_j(\hat{a}^s, \underline{z}^{-s})$ imply $(\hat{a}^s, \overline{z}^{-s})P_j(\hat{b}^s, \overline{z}^{-s})$ and $(\hat{b}^s, \underline{z}^{-s}) \times P_j(\hat{a}^s, \underline{z}^{-s})$ respectively, which contradict the separability of P_j. Therefore, $b^s = \hat{a}^s$. Next, since $(a^s, \overline{z}^{-s})\bar{P}_j!(b^s, \overline{z}^{-s})$ by condition (ii) and $(b^s, \overline{z}^{-s}) = (\hat{a}^s, \overline{z}^{-s})\bar{P}_j(\hat{b}^s, \overline{z}^{-s})$ by the hypothesis, transitivity implies $(a^s, \overline{z}^{-s})\bar{P}_j(\hat{b}^s, \overline{z}^{-s})$. Then, condition (i) implies $(a^s, \overline{z}^{-s})P_j(\hat{b}^s, \overline{z}^{-s})$, and hence $(a^s, \underline{z}^{-s})P_j(\hat{b}^s, \underline{z}^{-s})$ by separability. Furthermore, by condition (i), $(a^s, \underline{z}^{-s})P_j(\hat{b}^s, \underline{z}^{-s})$ implies $(a^s, \underline{z}^{-s})\bar{P}_j(\hat{b}^s, \underline{z}^{-s})$. Last, since $(a^s, \underline{z}^{-s})\bar{P}_j(\hat{b}^s, \underline{z}^{-s})$ and $(a^s, \underline{z}^{-s})\bar{P}_j!(b^s, \underline{z}^{-s}) = (\hat{a}^s, \underline{z}^{-s})$ by condition (ii), it must be the case $(\hat{a}^s, \underline{z}^{-s})\bar{P}_j(\hat{b}^s, \underline{z}^{-s})$ which contradicts the hypothesis $(\hat{b}^s, \underline{z}^{-s})\bar{P}_j(\hat{a}^s, \underline{z}^{-s})$. In conclusion, $\bar{P}_j$ is a separable preference. Hence, condition (iii) is satisfied by $\bar{P}_j$.

Last, according to Observation 1, by locally switching (a^s, z^{-s}) and (b^s, z^{-s}) for all $z^{-s} \in A^{-s}$ simultaneously in $\bar{P}_j$, we generate a separable preference $\hat{P}_j$ such that $\Gamma(\bar{P}_j, \hat{P}_j) = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s} \in A^{-s}}$. This completes the proof of Fact 3. □

Fact 4. *Fixing $P_j, P'_j \in \mathbb{D}_S$ with $[P_j]^s \neq [P'_j]^s$ for some $s \in M$, and $x, y \in A$ with xP_jy and xP'_jy, there exist $t \geq 1$ pair(s) $\{\bar{P}^k_j, \hat{P}^k_j : k = 1, \ldots, t\} \subseteq \mathbb{D}_S$ such that*

(i) $\bar{P}^k_j \sim^+ \hat{P}^k_j$ *for all* $k = 1, \ldots, t$,
(ii) $[P_j]^s = [\bar{P}^1_j]^s$ *for all* $s \in M$,
(iii) $[\hat{P}^k_j]^s = [\bar{P}^{k+1}_j]^s$ *for all* $s \in M$ *and* $k = 1, \ldots, t-1$,
(iv) $[\hat{P}^t_j]^s = [P'_j]^s$ *for all* $s \in M$, *and*
(v) $x\bar{P}^k_jy$ *and* $x\hat{P}^k_jy$ *for all* $k = 1, \ldots, t$.

In particular, if $r_1(P_j)$ and $r_1(P'_j)$ are similar, say $r_1(P_j) = (a^s, z^{-s})$ and $r_1(P'_j) = (b^s, z^{-s})$, then $r_1(\bar{P}^k_j), r_1(\hat{P}^k_j) \in (A^s, z^{-s})$ for all $k = 1, \ldots, t$.

Proof. Let $S = \{q \in M : [P_j]^q \neq [P'_j]^q\}$ and $T = \{\tau \in M : x^\tau \neq y^\tau\}$. Evidently, both S and T are nonempty. Since xP_jy, there exists $\hat{\tau} \in T$ such that $x^{\hat{\tau}}[P_j]^{\hat{\tau}}y^{\hat{\tau}}$. According to P_j and P'_j, we identify $P''_j \in \mathbb{D}_S$ satisfying conditions (i) - (iii) of Fact 2. Specifically, (i) $[P''_j]^s \sim [P_j]^s$ for some $s \in S$, and $[P''_j]^\omega = [P_j]^\omega$ for all $\omega \neq s$, (ii) $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^sa^s$ for some $a^s, b^s \in A^s$, and (iii) xP''_jy.

Claim 1: According to P_j and P''_j, there exist $\bar{P}_j, \hat{P}_j \in \mathbb{D}_S$ satisfying the following four conditions:

(i) $\bar{P}_j \sim^+ \hat{P}_j$ and $\Gamma(\bar{P}_j, \hat{P}_j) = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s} \in A^{-s}}$,
(ii) $[\bar{P}_j]^\omega = [P_j]^\omega$ for all $\omega \in M$,
(iii) $[\hat{P}_j]^\omega = [P''_j]^\omega$, and
(iv) $x\bar{P}_jy$ and $x\hat{P}_jy$.

According to the subset T and the component s in the first paragraph, we consider four cases:

(1) $s \notin T$.

(2) $T=\{s\}$.
(3) $|T|\geq 2$, $s\in T$, and there exists $\tau\in T\backslash\{s\}$ such that $x^\tau[P_j]^\tau y^\tau$.
(4) $|T|\geq 2$, $s\in T$, $x^s[P_j]^s y^s$ and $y^\tau[P_j]^\tau x^\tau$ for all $\tau\in T\backslash\{s\}$.

In case (1), component $\hat{\tau}$ in the first paragraph is distinct to s. Since $x^{\hat{\tau}}[P_j]^{\hat{\tau}}y^{\hat{\tau}}$ mentioned in the first paragraph and $[P''_j]^{\hat{\tau}}=[P_j]^{\hat{\tau}}$ by condition (i) of Fact 2, we have $x^{\hat{\tau}}[P''_j]^{\hat{\tau}}y^{\hat{\tau}}$. We fix a lexicographic order $\succ$ where $\hat{\tau}$ is lexicographically dominant, and s is lexicographically dominated. According to $\succ$, we assemble all marginal preferences of P_j, and form preference $\bar{P}_j\in\mathbb{D}_{\text{LS}}$. Symmetrically, according to $\succ$, we assemble all marginal preferences of P''_j, and form preference $\hat{P}_j\in\mathbb{D}_{\text{LS}}$. Evidently, $\bar{P}_j$ and $\hat{P}_j$ satisfy conditions (i) - (iv) of Claim 1.

In case (2), we know $x=(x^s,x^{-s})$ and $y=(y^s,y^{-s})$ where $x^s\neq y^s$ and $x^{-s}=y^{-s}$. Since xP_jy by the hypothesis of Fact 4 and xP''_jy by condition (iii) of Fact 2, separability implies $x^s[P_j]^sy^s$ and $x^s[P''_j]^sy^s$. We fix a lexicographic order $\succ$ where s is lexicographically dominated. According to $\succ$, we assemble all marginal preferences of P_j, and form preference $\bar{P}_j\in\mathbb{D}_{\text{LS}}$. Symmetrically, according to $\succ$, we assemble all marginal preferences of P''_j, and form preference $\hat{P}_j\in\mathbb{D}_{\text{LS}}$. Evidently, $\bar{P}_j$ and $\hat{P}_j$ satisfy conditions (i) - (iv) of Claim 1.

In case (3), since $x^\tau[P_j]^\tau y^\tau$ and $[P''_j]^\tau=[P_j]^\tau$ by condition (i) of Fact 2, we have $x^\tau[P''_j]^\tau y^\tau$. We fix a lexicographic order $\succ$ where τ is lexicographically dominant, and s is lexicographically dominated. According to $\succ$, we assemble all marginal preferences of P_j, and form preference $\bar{P}_j\in\mathbb{D}_{\text{LS}}$. According to $\succ$, we also assemble all marginal preferences of P''_j, and form preference $\hat{P}_j\in\mathbb{D}_{\text{LS}}$. Evidently, $\bar{P}_j$ and $\hat{P}_j$ satisfy conditions (i) - (iv) of Claim 1.

In case (4), since $y^\tau[P_j]^\tau x^\tau$ for all $\tau\in T\backslash\{s\}$, condition (i) of Fact 2 implies $y^\tau[P''_j]^\tau x^\tau$ for all $\tau\in T\backslash\{s\}$. We first show $x^s[P''_j]^sy^s$. Suppose not, i.e., $y^s[P''_j]^sx^s$. Thus, $y^\tau[P''_j]^\tau x^\tau$ for all $\tau\in T$, and consequently, yP''_jx which contradicts condition (iii) of Fact 2. Therefore, $x^s[P''_j]^sy^s$. Then, according to condition (ii) of Fact 2, we know $\{x^s,y^s\}\neq\{a^s,b^s\}$. We thus have two subcases: $y^s\neq b^s$ and $y^s=b^s$.

First, assume $y^s\neq b^s$. According to P_j, since $a^s[P_j]^s!b^s$ by condition (ii) of Fact 2, we construct $\bar{P}_j,\hat{P}_j\in\mathbb{D}_{\text{S}}$ satisfying Fact 3. Thus, conditions (i) and (ii) of Claim 1 are satisfied. Furthermore, by $\Gamma(\bar{P}_j,\hat{P}_j)=\big\{\{(a^s,z^{-s}),(b^s,z^{-s})\}\big\}_{z^{-s}\in A^{-s}}$ in Fact 3 and conditions (i) and (ii) of Fact 2, we know that condition (iii) of Claim 1 also holds. Last, we show condition (iv) of Claim 1. We consider two situations: $x^s\neq b^s$ and $x^s=b^s$. If $x^s\neq b^s$, condition (i) of Fact 3 implies $x\bar{P}_jy$. Furthermore, since $x^s\neq b^s$ and $y^s\neq b^s$, $\Gamma(\bar{P}_j,\hat{P}_j)=\big\{\{(a^s,z^{-s}),(b^s,z^{-s})\}\big\}_{z^{-s}\in A^{-s}}$ in Fact 3 implies that $\bar{P}_j$ and $\hat{P}_j$ share the same relative ranking over x and y. Therefore, $x\hat{P}_jy$. This proves condition (iv) of Claim 1 in the situation $x^s\neq b^s$. If $x^s=b^s$, $a^s[P_j]^s!b^s$ by condition (ii) of Fact 2 implies $(a^s,x^{-s})P_j(b^s,x^{-s})=x$. Hence, condition (ii) of Fact 3 implies $(a^s,x^{-s})\bar{P}_j!(b^s,x^{-s})=x$. Moreover, since xP_jy by the hypothesis of Fact 4, $(a^s,x^{-s})P_jx$ implies $(a^s,x^{-s})P_jy$. Then, condition (i) of Fact 3 again implies $(a^s,x^{-s})\bar{P}_jy$, and furthermore, $(a^s,x^{-s})\bar{P}_j!x$ implies $x\bar{P}_jy$. Last, since $x^s=b^s$, $y^s\neq b^s$ and $x\bar{P}_jy$, $\Gamma(\bar{P}_j,\hat{P}_j)=\big\{\{(a^s,z^{-s}),(b^s,z^{-s})\}\big\}_{z^{-s}\in A^{-s}}$ in Fact 3 implies that $\bar{P}_j$ and $\hat{P}_j$ share the same relative ranking over x and y. Therefore, $x\hat{P}_jy$. This proves condition (iv) of Claim 1 in the situation $x^s=b^s$.

Next, assume $y^s=b^s$. Since $\{x^s,y^s\}\neq\{a^s,b^s\}$, we know $x^s\notin\{a^s,b^s\}$. Since $a^s[P_j]^s!b^s$ by condition (ii) of Fact 2 and $x^s[P_j]^sy^s=b^s$ by the hypothesis of case (4), we know $x^s[P_j]^sa^s$. We first identify preference $\tilde{P}_j\in\mathbb{D}_{\text{LS}}$ such that $[\tilde{P}_j]^\omega=[P_j]^\omega$ for all $\omega\in M$, and component s is lexicographically dominant in the lexicographic order. Thus, $x\tilde{P}_j(a^s, y^{-s})$ and $(a^s, y^{-s})\tilde{P}_j(b^s, y^{-s}) = y$. According to $\tilde{P}_j$, since $a^s[P_j]^s!b^s$ by condition (ii) of Fact 2 and $[\tilde{P}_j]^s = [P_j]^s$, we construct $\bar{P}_j, \hat{P}_j \in \mathbb{D}_S$ satisfying Fact 3. Thus, condition (i) of Claim 1 is satisfied. Moreover, by condition (iii) of Fact 3, we have $[\bar{P}_j]^\omega = [\tilde{P}_j]^\omega = [P_j]^\omega$ for all $\omega \in M$ which verifies condition (ii) of Claim 1. Next, by $\Gamma(\bar{P}_j, \hat{P}_j) = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s}\in A^{-s}}$ in Fact 3 and conditions (i) and (ii) of Fact 2, we know that condition (iii) of Claim 1 also holds. Last, we show condition (iv) of Claim 1. According to the construction of $\bar{P}_j$, since $x\tilde{P}_j(a^s, y^{-s})$, condition (i) of Fact 3 implies $x\bar{P}_j(a^s, y^{-s})$, and since $(a^s, y^{-s})\tilde{P}_j(b^s, y^{-s}) = y$, condition (ii) of Fact 3 implies $(a^s, y^{-s})\bar{P}_j!(b^s, y^{-s}) = y$. Therefore, $x\bar{P}_j y$ by transitivity. Furthermore, since $x^s \notin \{a^s, b^s\}$ and $x\bar{P}_j y$, $\Gamma(\bar{P}_j, \hat{P}_j) = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s}\in A^{-s}}$ in Fact 3 implies that $\bar{P}_j$ and $\hat{P}_j$ share the same relative ranking over x and y. Therefore, $x\hat{P}_j y$. This completes the verification of Claim 1.

By Claim 1, we identify the pair $\{\bar{P}_j, \hat{P}_j\}$. Let $\bar{P}_j^1 = \bar{P}_j$ and $\hat{P}_j^1 = \hat{P}_j$. Note that $\hat{P}_j^1$ is one-step closer to P'_j than P_j in the sense $\Gamma([\hat{P}_j^1]^s, [P'_j]^s) = \Gamma([P_j]^s, [P'_j]^s)\backslash\big\{\{a^s, b^s\}\big\}$ and $\Gamma([\hat{P}_j^1]^\omega, [P'_j]^\omega) = \Gamma([P_j]^\omega, [P'_j]^\omega)$ for all $\omega \neq s$. Next, according to $\hat{P}_j^1$ and P'_j, we identify another separable preference P''_j satisfying condition (i) - (iii) of Fact 2. Then, according to $\hat{P}_j^1$ and P''_j, we identify the second pair $\{\bar{P}_j^2, \hat{P}_j^2\}$ satisfying conditions (i) - (iv) of Claim 1. Note that $\hat{P}_j^2$ is also one-step closer to P'_j than $\hat{P}_j^1$. Repeat this procedure until identifying the pair $\{\bar{P}_j^t, \hat{P}_j^t\}$ where $\hat{P}_j^t$ and P'_j share the same marginal preferences. Thus, we have the pair(s) $\{\bar{P}_j^k, \hat{P}_j^k : k = 1, \ldots, t\} \subseteq \mathbb{D}_S$ satisfying condition (i) - (v) of Fact 4. In particular, note that if $r_1(P_j)$ and $r_1(P'_j)$ are similar, say $r_1(P_j) = (a^s, z^{-s})$ and $r_1(P'_j) = (b^s, z^{-s})$, the construction of $\{\bar{P}_j^k, \hat{P}_j^k : k = 1, \ldots, t\}$ implies $r_1(\bar{P}_j^k), r_1(\hat{P}_j^k) \in (A^s, z^{-s})$ for all $k = 1, \ldots, t$. This completes the verification of Fact 4. $\square$

Fact 5. *Given two distinct $P_j, P'_j \in \mathbb{D}_S$ with $[P_j]^s = [P'_j]^s$ for all $s \in M$, there exists an adjacency path $\{P_j^k\}_{k=1}^q \subseteq \mathbb{D}_S$ connecting P_j and P'_j such that $[xP_j y$ and $xP'_j y] \Rightarrow [xP_j^k y$ for all $k = 1, \ldots, q]$.*

Proof. Since $[P_j]^s = [P'_j]^s$ for all $s \in M$, it is evident that $r_1(P_j) = r_1(P'_j)$. Searching from the top of P_j and P'_j down to the bottom, since $P_j \neq P'_j$, we identify $1 < k \leq |A|$ such that $r_l(P_j) = r_l(P'_j)$ for all $1 \leq l < k$ and $r_k(P_j) \neq r_k(P'_j)$. For notational convenience, let $r_k(P'_j) \equiv y$ and $y \equiv r_\nu(P_j)$. It is evident that $\nu > k$, and hence $\nu - 1 \geq k$. Let $x \equiv r_{\nu-1}(P_j)$. Since $r_l(P_j) = r_l(P'_j)$ for all $1 \leq l < k$, we know $yP'_j x$. Thus, $xP_j!y$ and $yP'_j x$. We construct another preference P''_j by locally switching x and y in P_j. Thus, $P_j \sim P''_j$ and $\Gamma(P_j, P''_j) = \big\{\{x, y\}\big\}$. Last, to show that P''_j is a separable preference, by Observation 1, it suffices to show that x and y disagree on at least two components. Since $x \neq y$, there exists $s \in M$ such that $x^s \neq y^s$. Suppose $x^\tau = y^\tau$ for all $\tau \in M\backslash\{s\}$. Consequently, separability implies $x^s[P_j]^s y^s$ and $y^s[P'_j]^s x^s$ which contradict the hypothesis $[P_j]^s = [P'_j]^s$. Therefore, P''_j is a separable preference.

Note that P''_j is closer to P'_j than P_j since $\Gamma(P''_j, P'_j) = \Gamma(P_j, P'_j)\backslash\big\{\{x, y\}\big\}$. By repeatedly applying the argument in the first paragraph, we eventually construct an adjacency path $\{P_j^k\}_{k=1}^q \subseteq \mathbb{D}_S$ connecting P_j and P'_j. By the construction, we know that for each $1 \leq k < q$,

$[xP_j^k!y$ and $yP_j^{k+1}!x] \Rightarrow [yP_j'x]$. Therefore, we have $[xP_jy$ and $xP_j'y] \Rightarrow [xP_j^ky$ for all $k = 1, \ldots, q]$. □

Now, we use Facts 4 and 5 to construct paths for the verification of the Interior$^+$ and Exterior$^+$ properties. Fix $P_j, P_j' \in \mathbb{D}_S$ and $x, y \in A$ with xP_jy and $xP_j'y$. If P_j and P_j' share the same marginal preferences, then $r_1(P_j) = r_1(P_j')$, and we have a path connecting P_j and P_j' which satisfies the requirement of the Interior$^+$ property by Fact 5. If P_j and P_j' have distinct marginal preferences, we first identify pair(s) $\{\bar{P}_j^k, \hat{P}_j^k : k = 1, \ldots, t\}$ satisfying condition (i) - (v) of Fact 4. Next, according to conditions (ii) - (iv) of Fact 4, we construct an adjacency path connecting P_j and $\bar{P}_j^1$ which satisfies Fact 5, an adjacency path connecting $\hat{P}_j^k$ and $\bar{P}_j^{k+1}$ which satisfies Fact 5 for each $k = 1, \ldots, t-1$, and an adjacency path connecting $\hat{P}_j^t$ and P_j' which satisfies Fact 5. Last, we combine all these adjacency paths via condition (i) of Fact 4, and generate a path $\{P_j^k\}_{k=1}^q \subseteq \mathbb{D}_S$ connecting P_j and P_j'. If $r_1(P_j) = r_1(P_j') \equiv a$, then the proof of Fact 4 implies $r_1(\bar{P}_j^k) = r_1(\hat{P}_j^k) = a$ for all $k = 1, \ldots, t$, and furthermore, Fact 5 implies $r_1(P_j^k) = a$ for all $k = 1, \ldots, q$. This meets the requirement of the Interior$^+$ property. If $r_1(P_j) \neq r_1(P_j')$, we know $x\bar{P}_j^ky$ and $x\hat{P}_j^ky$ by condition (v) of Fact 4 for all $k = 1, \ldots, t$, and furthermore, Fact 5 implies xP_j^ky for all $k = 1, \ldots, q$. This meets the requirement of the Exterior$^+$ property. In particular, if $r_1(P_j)$ and $r_1(P_j')$ are similar, say $r_1(P_j) = (a^s, z^{-s})$ and $r_1(P_j') = (b^s, z^{-s})$, we know $r_1(\bar{P}_j^k), r_1(\hat{P}_j^k) \in (A^s, z^{-s})$ by Fact 4 for all $k = 1, \ldots, t$, and furthermore, Fact 5 implies $r_1(P_j^k) \in (A^s, z^{-s})$ for all $k = 1, \ldots, q$. This meets the requirement of no-detour property.

E.3. The top-separable domain is a connected$^+$ domain

We first provide Fact 6 which links $\mathbb{D}_{TS}$ to the connectedness$^+$ of $\mathbb{D}_S$.

Fact 6. *Given $P_j \in \mathbb{D}_{TS} \backslash \mathbb{D}_S$ and $a, b \in A$ with aP_jb, there exists $\bar{P}_j \in \mathbb{D}_S$ such that $r_1(\bar{P}_j) = r_1(P_j) \equiv \bar{a}$ and $a\bar{P}_jb$. Furthermore, there exists an adjacency path $\{P_j^k\}_{k=1}^t \subseteq \mathbb{D}_{TS}^{\bar{a}}$ connecting P_j and $\bar{P}_j$ such that $[xP_jy$ and $x\bar{P}_jy] \Rightarrow [xP_j^ky$ for all $k = 1, \ldots, t]$.*

Proof. Let $S = \{s \in M : a^s \neq b^s\}$. Evidently, $S \neq \emptyset$. Since aP_jb, there exists $s \in S$ such that $b^s \neq \bar{a}^s$ by top-separability. We then fix a marginal preference $[\bar{P}_j]^s$ such that $r_1([\bar{P}_j]^s) = \bar{a}^s$ and $a^s[\bar{P}_j]^sb^s$, and a marginal preference $[\bar{P}_j]^\tau$ with $r_1([\bar{P}_j]^\tau) = \bar{a}^\tau$ for each $\tau \neq s$. Last, we fix a lexicographic order $\succ$ such that component s is lexicographically dominant, and assemble all marginal preferences above to generate a preference $\bar{P}_j \in \mathbb{D}_{LS}$. Hence, $r_1(\bar{P}_j) = \bar{a} = r_1(P_j)$ and $a\bar{P}_jb$.

Recalling the proof of Fact 5, according to P_j and $\bar{P}_j$, we identify $x, y \in A$ such that $x \neq \bar{a}$, $y \neq \bar{a}$, $xP_j!y$ and $y\bar{P}_jx$. Let nonempty $T \subseteq M$ be such that $x^\tau \neq y^\tau$ for all $\tau \in T$ and $x^{-T} = y^{-T}$. Since $P_j, \bar{P}_j \in \mathbb{D}_{TS}$, $xP_j!y$ implies that there exists $\tau \in T$ such that $y^\tau \neq \bar{a}^\tau$, and $y\bar{P}_jx$ implies that there exists $\tau' \in T$ such that $x^{\tau'} \neq \bar{a}^{\tau'}$. Note that either $\tau = \tau'$ or $\tau \neq \tau'$. By locally switching x and y in P_j, we generate a preference P_j''. Thus, $r_1(P_j'') = \bar{a}$, $P_j \sim P_j''$, $xP_j!y$, $yP_j''!x$ and $y\bar{P}_jx$. Then, $y^\tau \neq \bar{a}^\tau$ and $x^{\tau'} \neq \bar{a}^{\tau'}$ imply $P_j'' \in \mathbb{D}_{TS}$. Note that P_j'' is closer to $\bar{P}_j$ than P_j since $\Gamma(P_j'', \bar{P}_j) = \Gamma(P_j, \bar{P}_j) \backslash \big\{\{x, y\}\big\}$. By repeatedly applying the argument above, we eventually generate an adjacency path $\{P_j^k\}_{k=1}^t \subseteq \mathbb{D}_{TS}^{\bar{a}}$ such that for each $1 \leq k < t$,

$[xP_j^k!y$ and $yP_j^{k+1}!x] \Rightarrow [y\bar{P}_jx]$. Therefore, we have $[xP_jy$ and $x\bar{P}_jy] \Rightarrow [xP_j^ky$ for all $k = 1,\dots,t]$. $\square$

To verify connectedness$^+$ of $\mathbb{D}_{\mathrm{TS}}$, we fix distinct $P_j, P'_j \in \mathbb{D}_{\mathrm{TS}}$, and consider the following three cases: (i) $P_j \in \mathbb{D}_{\mathrm{S}}$ and $P'_j \in \mathbb{D}_{\mathrm{S}}$, (ii) $P_j \in \mathbb{D}_{\mathrm{TS}}\backslash\mathbb{D}_{\mathrm{S}}$ and $P'_j \in \mathbb{D}_{\mathrm{S}}$, or $P_j \in \mathbb{D}_{\mathrm{S}}$ and $P'_j \in \mathbb{D}_{\mathrm{TS}}\backslash\mathbb{D}_{\mathrm{S}}$, and (iii) $P_j, P'_j \in \mathbb{D}_{\mathrm{TS}}\backslash\mathbb{D}_{\mathrm{S}}$. In each case, we construct a path of $\mathbb{D}_{\mathrm{TS}}$ connecting P_j and P'_j which satisfies the requirements of the Interior$^+$ and Exterior$^+$ properties.

Case (i) is covered by Section E.2. In case (ii), assume w.l.o.g. that $P_j \in \mathbb{D}_{\mathrm{TS}}\backslash\mathbb{D}_{\mathrm{S}}$ and $P'_j \in \mathbb{D}_{\mathrm{S}}$. If $r_1(P_j) = r_1(P'_j) \equiv a$, we first identify $\bar{P}_j \in \mathbb{D}_{\mathrm{S}}$ with $r_1(\bar{P}_j) = a$, and construct an adjacency path of $\mathbb{D}^a_{\mathrm{TS}}$ connecting P_j and $\bar{P}_j$ by Fact 6. Next, by Section E.2, we have a path of $\mathbb{D}^a_{\mathrm{S}}$ connecting $\bar{P}_j$ and P'_j. Last, combining these two paths, we have a path of $\mathbb{D}^a_{\mathrm{TS}}$ connecting P_j and P'_j. If $r_1(P_j) \neq r_1(P'_j)$, xP_jy and xP'_jy for some $x, y \in A$, we first identify $\bar{P}_j \in \mathbb{D}_{\mathrm{S}}$ with $r_1(\bar{P}_j) = r_1(P_j)$ and $x\bar{P}_jy$, and construct an adjacency path of $\mathbb{D}_{\mathrm{TS}}$ connecting P_j and $\bar{P}_j$ along which x ranks above y by Fact 6. Next, by Section E.2, we have a path of $\mathbb{D}_{\mathrm{S}}$ connecting $\bar{P}_j$ and P'_j along which x ranks above y. Last, combining these two paths, we have a path of $\mathbb{D}_{\mathrm{TS}}$ connecting P_j and P'_j along which x ranks above y. The verification of case (iii) is similar to that of the second case. Therefore, $\mathbb{D}_{\mathrm{TS}}$ is a connected$^+$ domain.

E.4. The intersection of the separable domain and the multidimensional single-peaked domain is a connected$^+$ domain

We fix a product of trees $\times_{s\in M} G(A^s)$ and the multidimensional single-peaked domain $\mathbb{D}_{\mathrm{MSP}}$. Let $\bar{\mathbb{D}}_{\mathrm{MSP}} = \mathbb{D}_{\mathrm{S}} \cap \mathbb{D}_{\mathrm{MSP}}$. Recall the notation $[\bar{\mathbb{D}}_{\mathrm{MSP}}]^s$ which denotes the single-peaked (marginal) domain on $G(A^s)$. Similar to Section E.2, we first provide three step results, Facts 7, 8 and 9, which are respectively analogous to Facts 1, 2 and 3. We next establish Facts 10 and 11, which are respectively analogous to Facts 4 and 5, and will be utilized to verify connectedness$^+$ of $\bar{\mathbb{D}}_{\mathrm{MSP}}$.

Fact 7. *Given $s \in M$ and distinct $[P_j]^s, [P'_j]^s \in [\bar{\mathbb{D}}_{\mathrm{MSP}}]^s$, there exist $[P''_j]^s \in [\bar{\mathbb{D}}_{\mathrm{MSP}}]^s$ and $a^s, b^s \in A^s$ such that $[P_j]^s \sim [P''_j]^s$, $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^sa^s$. (Note that it is possible $[P''_j]^s = [P'_j]^s$.)*

Proof. By the proof of Proposition 4.2 of Sato (2013), we have an adjacency path $\left\{[P_j^k]^s\right\}_{k=1}^q \subseteq [\bar{\mathbb{D}}_{\mathrm{MSP}}]^s$ connecting $[P_j]^s$ and $[P'_j]^s$, i.e., $[P_j^1]^s = [P_j]^s$, $[P_j^q]^s = [P'_j]^s$ and $[P_j^k]^s \sim [P_j^{k+1}]^s$ for all $k = 1,\dots,q-1$, such that $\left[x^s[P_j]^sy^s \text{ and } x^s[P'_j]^sy^s\right] \Rightarrow \left[x^s[P_j^k]^sy^s, k = 1,\dots,q\right]$. Thus, given $[P''_j]^s \equiv [P_j^2]^s$, we have $[P_j]^s \sim [P''_j]^s$, $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^sa^s$ for some $a^s, b^s \in A^s$. $\square$

Fact 8. *Fixing $P_j, P'_j \in \bar{\mathbb{D}}_{\mathrm{MSP}}$ with $[P_j]^q \neq [P'_j]^q$ for some $q \in M$, and $x, y \in A$ with xP_jy and xP'_jy, there exists $P''_j \in \bar{\mathbb{D}}_{\mathrm{MSP}}$ such that*

(i) *$[P''_j]^s \sim [P_j]^s$ for some $s \in S$, and $[P''_j]^\omega = [P_j]^\omega$ for all $\omega \neq s$,*
(ii) *$a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^sa^s$ for some $a^s, b^s \in A^s$, and*

(iii) $x P''_j y$.

Proof. After replacing the reference of Fact 1 and the notation $[\mathbb{D}_S]^s$ in the proof of Fact 2 by the reference of Fact 7 and the notation $[\bar{\mathbb{D}}_{MSP}]^s$ respectively, the modified proof of Fact 2 remains applicable for the verification of Fact 8. □

Fact 9. *Given $P_j \in \bar{\mathbb{D}}_{MSP}$, $s \in M$, $a^s, b^s \in A^s$ with $a^s[P_j]^s!b^s$, there exists $\bar{P}_j \in \bar{\mathbb{D}}_{MSP}$ such that*

(i) *for all $x, y \in A\backslash(b^s, A^{-s})$, $[x P_j y] \Leftrightarrow [x \bar{P}_j y]$,*
(ii) *for all $z^{-s} \in A^{-s}$, $(a^s, z^{-s})\bar{P}_j!(b^s, z^{-s})$, and*
(iii) *$[P_j]^\omega = [\bar{P}_j]^\omega$ for all $\omega \in M$.*

Moreover, if there exists $[P''_j]^s \in [\bar{\mathbb{D}}_{MSP}]^s$ such that $[P_j]^s \sim [P''_j]^s$ and $\Gamma([P_j]^s, [P''_j]^s) = \big\{\{a^s, b^s\}\big\}$, then there exists $\hat{P}_j \in \bar{\mathbb{D}}_{MSP}$ such that $\Gamma(\bar{P}_j, \hat{P}_j) = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s}\in A^{-s}}$.

Proof. By the proof of Fact 3, we have $\bar{P}_j \in \mathbb{D}_S$ satisfying condition (i) - (iii). Furthermore, by condition (iii), we know $[\bar{P}_j]^\omega = [P_j]^\omega \in [\bar{\mathbb{D}}_{MSP}]^\omega$ for all $s \in M$. Therefore, $\bar{P}_j \in \bar{\mathbb{D}}_{MSP}$ by Remark 1.

Next, by the proof of Fact 3, we have $\hat{P}_j \in \mathbb{D}_S$ such that $\Gamma(\bar{P}_j, \hat{P}_j) = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s}\in A^{-s}}$. Thus, $[\bar{P}_j]^s \sim [\hat{P}_j]^s$, $\Gamma([\bar{P}_j]^s, [\hat{P}_j]^s) = \big\{\{a^s, b^s\}\big\}$ and $[\bar{P}_j]^\omega = [\hat{P}_j]^\omega$ for all $\omega \in M\backslash\{s\}$. We show $\hat{P}_j \in \bar{\mathbb{D}}_{MSP}$. First, for every $\omega \in M\backslash\{s\}$, we know $[\hat{P}_j]^\omega = [\bar{P}_j]^\omega \in [\bar{\mathbb{D}}_{MSP}]^\omega$. Second, given $[\bar{P}_j]^s = [P_j]^s$ and $\Gamma([\bar{P}_j]^s, [\hat{P}_j]^s) = \big\{\{a^s, b^s\}\big\}$, the hypothesis $\Gamma([P_j]^s, [P''_j]^s) = \big\{\{a^s, b^s\}\big\}$ and $[P''_j]^s \in [\bar{\mathbb{D}}_{MSP}]^s$ imply $[\hat{P}_j]^s = [P''_j]^s \in [\bar{\mathbb{D}}_{MSP}]^s$. Therefore, we have $\hat{P}_j \in \bar{\mathbb{D}}_{MSP}$ by Remark 1. □

Fact 10. *Given $P_j, P'_j \in \bar{\mathbb{D}}_{MSP}$ with $[P_j]^s \neq [P'_j]^s$ for some $s \in M$, and $x, y \in A$ with $x P_j y$ and $x P'_j y$, there exist $t \geq 1$ pair(s) $\{\bar{P}^k_j, \hat{P}^k_j : k = 1, \ldots, t\} \subseteq \bar{\mathbb{D}}_{MSP}$ such that*

(i) *$\bar{P}^k_j \sim^+ \hat{P}^k_j$ for all $k = 1, \ldots, t$,*
(ii) *$[P_j]^s = [\bar{P}^1_j]^s$ for all $s \in M$,*
(iii) *$[\hat{P}^k_j]^s = [\bar{P}^{k+1}_j]^s$ for all $s \in M$ and $k = 1, \ldots, t-1$,*
(iv) *$[\hat{P}^t_j]^s = [P'_j]^s$ for all $s \in M$, and*
(v) *$x \bar{P}^k_j y$ and $x \hat{P}^k_j y$ for all $k = 1, \ldots, t$.*

In particular, if $r_1(P_j)$ and $r_1(P'_j)$ are similar, say $r_1(P_j) = (a^s, z^{-s})$ and $r_1(P'_j) = (b^s, z^{-s})$, then $r_1(\bar{P}^k_j), r_1(\hat{P}^k_j) \in (A^s, z^{-s})$ for all $k = 1, \ldots, t$.

Proof. Let $S = \{q \in M : [P_j]^q \neq [P'_j]^q\}$ and $T = \{\tau \in M : x^\tau \neq y^\tau\}$. Evidently, both S and T are nonempty. Since $x P_j y$, there exists $\hat{\tau} \in T$ such that $x^{\hat{\tau}}[P_j]^{\hat{\tau}} y^{\hat{\tau}}$. According to P_j and P'_j, we first identify $P''_j \in \bar{\mathbb{D}}_{MSP}$ satisfying conditions (i) - (iii) of Fact 8. Specifically, we have (i) $[P''_j]^s \sim [P_j]^s$ for some $s \in S$, and $[P''_j]^\omega = [P_j]^\omega$ for all $\omega \neq s$, (ii) $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^s a^s$ for some $a^s, b^s \in A^s$, and (iii) $x P''_j y$. Note that $P''_j \in \bar{\mathbb{D}}_{MSP}$ implies $[P''_j]^s \in [\bar{\mathbb{D}}_{MSP}]^s$.

Moreover, since $[P_j]^s \sim [P''_j]^s$ and $\Gamma([P_j]^s, [P''_j]^s) = \{\{a^s, b^s\}\}$, Fact 9 becomes applicable here. Then, after replacing the references of Facts 2, 3 and 4 and the notation $\mathbb{D}_{\text{LS}}$ in the proof of Fact 4 by the references of Facts 8, 9 and 10 and the notation $\mathbb{D}_{\text{LS}} \cap \mathbb{D}_{\text{MSP}}$ respectively, the modified proof of Fact 4 (from Claim 1 to the end) remains valid for the verification of Fact 10. □

Fact 11. *Given two distinct $P_j, P'_j \in \bar{\mathbb{D}}_{\text{MSP}}$ with $[P_j]^s = [P'_j]^s$ for all $s \in M$, there exists an adjacency path $\{P^k_j\}_{k=1}^q \subseteq \bar{\mathbb{D}}_{\text{MSP}}$ connecting P_j and P'_j such that $[xP_jy$ and $xP'_jy] \Rightarrow [xP^k_jy$ for all $k = 1, \dots, q]$.*

Proof. Recall the construction of $P''_j \in \mathbb{D}_{\text{S}}$ in the proof of Fact 5. If we show $[P''_j]^s \in [\bar{\mathbb{D}}_{\text{MSP}}]^s$ for all $s \in M$, then we have $P''_j \in \bar{\mathbb{D}}_{\text{MSP}}$, and the rest proof of Fact 5 on the construction of the adjacency path remains applicable for the verification of Fact 11. Since both P_j and P''_j are separable preferences, $P_j \sim P''_j$ implies that P_j and P''_j share the same marginal preferences by Observation 1. Therefore, $[P''_j]^s = [P_j]^s \in [\bar{\mathbb{D}}_{\text{MSP}}]^s$ for all $s \in M$, as required. □

Now, similar to the last paragraph of Section E.2, we use Facts 10 and 11 to construct paths satisfying the requirements of the Interior$^+$ and Exterior$^+$ properties.

E.5. The multidimensional single-peaked domain is a connected$^+$ domain

We fix a product of trees $\times_{s\in M} G(A^s)$ and the multidimensional single-peaked domain $\mathbb{D}_{\text{MSP}}$. We first provide Fact 12 which links $\mathbb{D}_{\text{MSP}}$ to the connectedness$^+$ of $\bar{\mathbb{D}}_{\text{MSP}}$.

Fact 12. *Given $P_j \in \mathbb{D}_{\text{MSP}} \backslash \bar{\mathbb{D}}_{\text{MSP}}$ and $a, b \in A$ with aP_jb, there exists $\bar{P}_j \in \bar{\mathbb{D}}_{\text{MSP}}$ such that $r_1(\bar{P}_j) = r_1(P_j) \equiv \bar{a}$ and $a\bar{P}_jb$. Furthermore, there exists an adjacency path $\{P^k_j\}_{k=1}^q \subseteq \mathbb{D}^{\bar{a}}_{\text{MSP}}$ connecting P_j and $\bar{P}_j$ such that $[xP_jy$ and $x\bar{P}_jy] \Rightarrow [xP^k_jy$ for all $k = 1, \dots, q]$.*

Proof. Let $S = \{s \in M : a^s \neq b^s\}$. Evidently, $S \neq \emptyset$. Since aP_jb, we know $b \notin \langle \bar{a}, a\rangle$, and hence, there exists $s \in S$ such that $b^s \notin \langle \bar{a}^s, a^s\rangle$. We then fix a marginal preference $[\bar{P}_j]^s \in [\bar{\mathbb{D}}_{\text{MSP}}]^s$ such that $r_1([\bar{P}_j]^s) = \bar{a}^s$ and $a^s[\bar{P}_j]^s b^s$, and a marginal preference $[\bar{P}_j]^\tau \in [\bar{\mathbb{D}}_{\text{MSP}}]^\tau$ with $r_1([\bar{P}_j]^\tau) = \bar{a}^\tau$ for each $\tau \neq s$. Last, we fix a lexicographic order $\succ$ such that component s is lexicographically dominant, and assemble all marginal preferences to generate a preference $\bar{P}_j \in \mathbb{D}_{\text{LS}} \cap \mathbb{D}_{\text{MSP}} \subseteq \bar{\mathbb{D}}_{\text{MSP}}$. Hence, $r_1(\bar{P}_j) = \bar{a} = r_1(P_j)$ and $a\bar{P}_jb$. This proves the first part of Fact 12. The second part of Fact 12 follows exactly from Lemma 8 of Chatterji and Zeng (2018). □

Now, similar to the last two paragraphs of Section E.3, by applying the connectedness$^+$ of $\bar{\mathbb{D}}_{\text{MSP}}$ and Fact 12, we assert that $\mathbb{D}_{\text{MSP}}$ is a connected$^+$ domain.

E.6. The union of the separable domain and several multidimensional single-peaked domains is a connected$^+$ domain

It suffices to consider the union $\mathbb{D}_{\text{U}} \equiv \mathbb{D}_{\text{S}} \cup \mathbb{D}_{\text{MSP}} \cup \mathbb{D}'_{\text{MSP}}$ where $\mathbb{D}_{\text{MSP}}$ is the multidimensional single-peaked domain on a product of trees $\times_{s\in M} G(A^s)$, and $\mathbb{D}'_{\text{MSP}}$ is the multidimensional single-peaked domain on another product of trees $\times_{s\in M} G'(A^s)$. Note that there exists at least one component $s \in M$ such that $G(A^s)$ and $G'(A^s)$ disagree on some edges.

To verify connectedness$^+$ of $\mathbb{D}_{\text{U}}$, we fix distinct $P_j, P'_j \in \mathbb{D}_{\text{U}}$. If $P_j, P'_j \in \mathbb{D}_{\text{S}}$, or $P_j, P'_j \in \mathbb{D}_{\text{MSP}}$, or $P_j, P'_j \in \mathbb{D}'_{\text{MSP}}$, then by Section E.2 or Section E.5, we have a path of $\mathbb{D}_{\text{U}}$ connecting P_j and P'_j which satisfies the requirements of connectedness$^+$. Next, we consider the following two cases:

(i) $P_j \in \mathbb{D}_{\text{MSP}}\backslash\mathbb{D}_{\text{S}}$ and $P'_j \in \mathbb{D}_{\text{S}}\backslash\mathbb{D}_{\text{MSP}}$, (symmetrically, $P_j \in \mathbb{D}'_{\text{MSP}}\backslash\mathbb{D}_{\text{S}}$ and $P'_j \in \mathbb{D}_{\text{S}}\backslash\mathbb{D}'_{\text{MSP}}$), and

(ii) $P_j \in \mathbb{D}_{\text{MSP}}\backslash\mathbb{D}_{\text{S}}$ and $P'_j \in \mathbb{D}'_{\text{MSP}}\backslash\mathbb{D}_{\text{S}}$.

In each case, we construct two paths of $\mathbb{D}_{\text{U}}$ connecting P_j and P'_j which satisfy the requirements of the Interior$^+$ and Exterior$^+$ properties respectively.

In case (i), assume w.l.o.g. that $P_j \in \mathbb{D}_{\text{MSP}}\backslash\mathbb{D}_{\text{S}}$ and $P'_j \in \mathbb{D}_{\text{S}}\backslash\mathbb{D}_{\text{MSP}}$. If $r_1(P_j) = r_1(P'_j) \equiv a$, we first identify $\bar{P}_j \in \mathbb{D}_{\text{S}} \cap \mathbb{D}_{\text{MSP}}$ with $r_1(\bar{P}_j) = a$, and construct an adjacency path of $\mathbb{D}^a_{\text{MSP}}$ connecting P_j and $\bar{P}_j$ by Fact 12. Next, by Section E.2, we have a path of $\mathbb{D}^a_{\text{S}}$ connecting $\bar{P}_j$ and P'_j. Last, combining these two paths, we have a path of $\mathbb{D}^a_{\text{U}}$ connecting P_j and P'_j. If $r_1(P_j) \neq r_1(P'_j)$, xP_jy and xP'_jy for some $x, y \in A$, we first identify $\bar{P}_j \in \mathbb{D}_{\text{S}} \cap \mathbb{D}_{\text{MSP}}$ with $r_1(\bar{P}_j) = r_1(P_j)$ and $x\bar{P}_jy$, and construct an adjacency path of $\mathbb{D}_{\text{MSP}}$ connecting P_j and $\bar{P}_j$ along which x ranks above y by Fact 12. Next, by Section E.2, we have a path of $\mathbb{D}_{\text{S}}$ connecting $\bar{P}_j$ and P'_j along which x ranks above y. Last, combining these two paths, we have a path of $\mathbb{D}_{\text{U}}$ connecting P_j and P'_j along which x ranks above y. The verification of case (ii) is similar to that of the first case. Therefore, $\mathbb{D}_{\text{U}}$ is a connected$^+$ domain.

E.7. The multidimensional eventually-single-peaked domain is a connected$^+$ domain

To motivate a multidimensional eventually-single-peaked domain, we consider the allocation of multiple public facilities to a region. Consider a region which has a central urban area and a large remote area surrounding the central urban area. There is a railway in the region which goes through the urban area. Along the railway, there are several stations $\Omega = \{l_1, l_2, \ldots, l_t\}$, $t \geq 2$, each of which is viewed as a *location*. A location in the urban area is referred to as an urban location, while a location in the remote area is called a remote location. To depict the locations' geometric relations on the railway, we use a linear order $l_1 < l_2 < \cdots < l_t$. We assume that there are least two urban locations, and all urban locations cluster, i.e., any location between two urban locations on the railway must also be an urban location. We then identify two particular urban locations $l_{\underline{k}}$ and $l_{\overline{k}}$, $1 \leq \underline{k} < \overline{k} \leq t$, which separate Ω into three disjoint subsets: the set of left remote locations $\mathcal{L} = \{l_1, l_2, \ldots, l_{\underline{k}-1}\}$, the set of urban locations $\mathcal{M} = \{l_{\underline{k}}, l_{\underline{k}+1}, \ldots, l_{\overline{k}}\}$, and the set of right remote locations $\mathcal{R} = \{l_{\overline{k}+1}, \ldots, l_{t-1}, l_t\}$. It is natural to postulate that the central urban area has a more advanced transportation than the remote area. We assume that all urban locations are pairwise connected by urban roads, while all left (respectively, right) remote locations are simply linked via the railway. Then, the railway and the urban roads together form the transportation system of the region.[53] Note that $l_{\underline{k}}$ is connected to all other urban locations via urban roads, and is the gate to all left remote locations on the railway. Therefore, it can be viewed as the left transportation hub. Symmetrically, $l_{\overline{k}}$ is referred to as the right transportation

[53] Mathematically speaking, the transportation system is a graph where the vertex set is Ω, and two distinct locations form an edge if they are directly connected by the railway or an urban road.

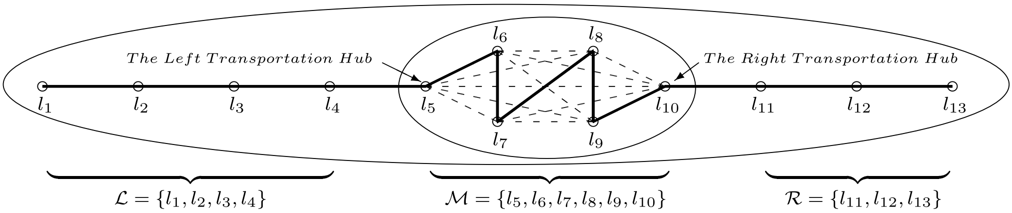

Fig. 18. The transportation system in the region.

hub. We use Fig. 18 to illustrate the transportation system of the region, where the space in the bigger oval represents the region, the space in the smaller oval represents the central urban area, the space between the two ovals therefore denotes the remote area, the bold line represents the railway $<$, each hollow node represents a location, and the dash lines denote the urban roads.

Fixing three locations $l_o, l_p, l_q \in \Omega$, let $l_o \leq l_p \leq l_q$ and $l_o \neq l_q$.[54] We know that l_p is between l_o and l_q on the railway, and hence l_p is closer to l_o than l_q on the railway. However, if the three locations are in the central urban area, their *geometric relations on the transportation system* change to pairwise connections due to urban roads. Consequently, we are no longer able to infer that l_p is closer to l_o than l_q. We make two more observations. First, if $l_p, l_q \in \mathcal{L} \cup \{l_{\underline{k}}\}$, then $l_o \leq l_p \leq l_q$ implies $l_o \in \mathcal{L}$. Then, the unique railway connection on $\mathcal{L} \cup \{l_{\underline{k}}\}$ implies that l_p is closer to l_o than l_q on the transportation system. Similarly, if $l_p, l_q \in \mathcal{R} \cup \{l_{\overline{k}}\}$, according to $l_o \leq l_p \leq l_q$, we know that l_p is between l_o and l_q on every route of the transportation system (either the railway, or the combination of the railway and the urban roads). Therefore, l_p is closer to l_o than l_q on the transportation system. Second, symmetrically, if $l_o, l_p \in \mathcal{L} \cup \{l_{\underline{k}}\}$ (respectively, $l_o, l_p \in \mathcal{R} \cup \{l_{\overline{k}}\}$), it is also true that l_p is closer to l_o than l_q on the transportation system. We introduce a ternary relation to summarize the two observations. Given $l_o, l_p, l_q \in A^s$ with $l_o \neq l_q$, let $\overline{(l_o, l_p, l_q)}$ denote a **ternary relation** such that l_p is between l_o and l_q on the railway $<$, i.e., $l_p \in \langle l_o, l_q \rangle$ (equivalently, either $l_o \leq l_p \leq l_q$ or $l_q \leq l_p \leq l_o$), and one of the following two additional conditions is satisfied:

(i) $l_p = l_q$, or $l_p, l_q \in \mathcal{L} \cup \{l_{\underline{k}}\}$, or $l_p, l_q \in \mathcal{R} \cup \{l_{\overline{k}}\}$, and
(ii) $l_p = l_o$, or $l_p, l_o \in \mathcal{L} \cup \{l_{\underline{k}}\}$, or $l_p, l_o \in \mathcal{R} \cup \{l_{\overline{k}}\}$.[55]

Multiple admissible public facilities $M = \{1, 2, \ldots, m\}$, $m \geq 2$, are to be constructed in the region, and each facility will be built at some location of Ω. For each facility $s \in M$, some locations are available for its construction while some locations are not available. We let $A^s \subseteq \Omega$ denote the set of locations that are available for the construction of the facility s. We normally write a location of A^s as $x^s \in A^s$. For simplicity, we assume that A^s always contains both transportation hubs. Let $A = \times_{s \in M} A^s$ be the set of alternatives. Note that each alternative $a \equiv (a^1, a^2, \ldots, a^m) \in A$ is an m-tuple which consists of m locations of Ω. Therefore, we sometimes also call an alternative a location bundle. Each agent has a preference over A, a linear order. To construct our restricted preferences over A, we first need to extend the ternary

[54] For notational convenience, let $l_o \leq l_p$ denote either $l_o < l_p$ or $l_o = l_p$.

[55] By the definition, the ternary relation is *symmetric*, i.e., $\left[\overline{(l_o, l_p, l_q)}\right] \Leftrightarrow \left[\overline{(l_q, l_p, l_o)}\right]$, and *transitive*, i.e., $\left[\overline{(l_o, l_p, l_q)} \text{ and } \overline{(l_o, l_q, l_r)}\right] \Leftrightarrow \left[\overline{(l_o, l_p, l_r)}\right]$.

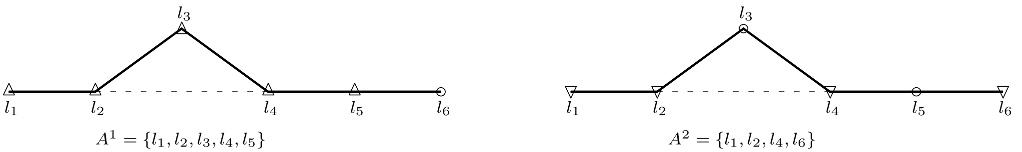

Fig. 19. Available location sets A^1 and A^2 on the transportation system.

relation on three locations to a ternary relation on three location bundles. Given $x, a, y \in A$ with $x \neq y$, let $\overline{(x, a, y)}$ be a **multidimensional ternary relation** such that for all $s \in M$, $[x^s \neq y^s] \Rightarrow \left[\overline{(x^s, a^s, y^s)}\right]$ and $[x^s = y^s] \Rightarrow [a^s = x^s = y^s]$. Geometrically, $\overline{(x, a, y)}$ indicates that the location bundle a is component-wise closer to x than y according to the transportation system, provided $a \neq y$. Since all public facilities are admissible, it is natural that an agent, who prefers x the best, would like all public facilities be constructed at the location bundle a rather than at y.

Definition 10. Given a linear order $<$ over $\Omega = \{l_1, l_2, \ldots, l_t\}$ and $l_{\underline{k}}, l_{\overline{k}} \in \Omega$ with $1 \leq \underline{k} < \overline{k} \leq t$, a preference P_i over $A = \times_{s\in M} A^s$ is **multidimensional eventually-single-peaked** if for all distinct $a, y \in A$, we have $\left[\overline{\big(r_1(P_i), a, y\big)}\right] \Rightarrow [a P_i y]$. Let $\mathbb{D}_{\text{MESP}}$ denote **the multidimensional eventually-single-peaked domain** which contains all admissible preferences.

If there are no urban roads, the railway $<$ simply describes the transportation system of the region, and then the closeness relations among location bundles can be simply recognized from the product of lines $\times_{s\in M}(<, A^s)$ where for each $s \in M$, $(<, A^s)$ is the linear order over A^s induced from the railway $<$. Intuitively, when urban roads are involved, some closeness relations recognized by $\times_{s\in M}(<, A^s)$ are destroyed. We identify the remaining closeness relations by the multidimensional ternary relations, and then apply the single-peakedness restriction accordingly to form a multidimensional eventually-single-peaked preference. Therefore, multidimensional eventually-single-peakedness is close to, but less restrictive than multidimensional single-peakedness on $\times_{s\in M}(<, A^s)$. We provide one example of a multidimensional eventually-single-peaked domain below to illustrate.

Example 9. Let $\Omega = \{l_1, l_2, l_3, l_4, l_5, l_6\}$, and l_2 and l_4 be the left and right transportation hubs respectively. Thus, $\mathcal{L} = \{l_1\}$, $\mathcal{M} = \{l_2, l_3, l_4\}$ and $\mathcal{R} = \{l_5, l_6\}$. Two admissible public facilities $M = \{1, 2\}$ will be constructed. The available location sets are $A^1 = \{l_1, l_2, l_3, l_4, l_5\}$ and $A^2 = \{l_1, l_2, l_4, l_6\}$. See the two diagrams of Fig. 19 where $\triangle$ represents a location of A^1 and $\triangledown$ represents a location of A^2.

We construct a multidimensional eventually-single-peaked preference P_j with peak (l_1, l_1). We first induce all corresponding multidimensional ternary relations:

$$\left\{\overline{\big((l_1, l_1), a, y\big)}\,\middle|\, y \neq (l_1, l_1), a^1 \in \langle l_1, y^1\rangle,\ a^2 \in \langle l_1, y^2\rangle, \text{ and } [y^1 \in \{l_4, l_5\}] \Rightarrow [a^1 \neq l_3]\right\}.$$

We notice that P_j still follows the restriction of multidimensional single-peakedness on $\{l_1, l_2, l_3\} \times \{l_1, l_2, l_4, l_6\}$ and $\{l_1, l_2, l_4, l_5\} \times \{l_1, l_2, l_4, l_6\}$ according to the product of lines $(<, A^1) \times (<, A^2)$. More importantly, as suggested by the last restriction of the multidimensional ternary relation $\overline{\big((l_1, l_1), a, y\big)}$, we have neither $\overline{(l_1, l_3, l_4)}$ nor $\overline{(l_1, l_3, l_5)}$, and therefore, $a = (l_3, l_1)$ is no longer closer to (l_1, l_1) than any location bundle $y \in \{l_4, l_5\} \times \{l_1, l_2, l_4, l_6\}$.

Therefore, the relative ranking between $a = (l_3, l_1)$ and $y \in \{l_4, l_5\} \times \{l_1, l_2, l_4, l_6\}$ is arbitrary in P_i, whereas $a = (l_3, l_1)$ ranks above all $y \in \{l_4, l_5\} \times \{l_1, l_2, l_4, l_6\}$ in every multidimensional single-peaked preference on $(<, A^1) \times (<, A^2)$ which has the peak (l_1, l_1). $\square$

Remark 11. First, we always induce the multidimensional ternary relation $\overline{(x, a, y)}$ such that $x^s = a^s \neq y^s$ and $a^{-s} = y^{-s}$ for some $s \in M$. Therefore, it is true that all preferences of $\mathbb{D}_{\text{MESP}}$ are top-separable. Second, since $\overline{(x, a, y)}$ implies $a^s \in \langle x^s, y^s \rangle$ for all $s \in M$, it is also true that every multidimensional single-peaked preference on $\times_{s\in M}(<, A^s)$ is multidimensional eventually-single-peaked. Therefore, $\mathbb{D}_{\text{MESP}}$ must contain many non-separable preferences. In one extreme case, if $A^s \subseteq \mathcal{M}$ for all $s \in M$ (in other words, for each public facility, all feasible locations are urban locations), then we induce all multidimensional ternary relations $\left\{\overline{(x, a, y)} : x \neq y, a^S = x^S \text{ and } a^{-S} = y^{-S} \text{ for some } S \subseteq M\right\}$.[56] Consequently, $\mathbb{D}_{\text{MESP}} = \mathbb{D}_{\text{TS}}$. In another extreme case, if $A^s \cap \mathcal{M} = \{\underline{x}^s, \overline{x}^s\}$ for all $s \in M$ (in other words, for each public facility, there is no feasible urban location other than the two transportation hubs), then we induce all multidimensional ternary relations $\left\{\overline{(x, a, y)} : x \neq y \text{ and } a^s \in \langle x^s, y^s \rangle \text{ for all } s \in M\right\}$.[57] Consequently, $\mathbb{D}_{\text{MESP}}$ is identical to the multidimensional single-peaked domain $\mathbb{D}_{\text{MSP}}$ on $\times_{s\in M}(<, A^s)$. $\square$

Remark 12. Chatterji et al. (2018) introduce a new preference restriction, *eventually-single-peakedness*, in the one-dimensional setting. Formally, given an linear order $<$ over $\Omega = \{l_1, l_2, \ldots, l_t\}$ and $l_{\underline{k}}, l_{\overline{k}} \in \Omega$ with $1 \leq \underline{k} < \overline{k} \leq t$, a preference P_j over Ω is **eventually-single-peaked** if it satisfies the following two conditions:

(i) For all distinct $l_o, l_q \in \mathcal{L} \cup \{l_{\underline{k}}\}$ or $l_o, l_q \in \mathcal{R} \cup \{l_{\overline{k}}\}$, $[l_o \in \langle r_1(P_j), l_q \rangle] \Rightarrow \left[l_o P_j l_q\right]$.
(ii) If $r_1(P_j) \in \mathcal{L}$, $\max(P_j, \mathcal{M}) = l_{\underline{k}}$. Symmetrically, if $r_1(P_j) \in \mathcal{R}$, $\max(P_j, \mathcal{M}) = l_{\overline{k}}$.

As suggested by its name, an eventually-single-peaked preference follows the single-peakedness restriction on both $\mathcal{L} \cup \{l_{\underline{k}}\}$ and $\mathcal{R} \cup \{l_{\overline{k}}\}$, but has no restriction on the relative rankings of elements in $\mathcal{M} \backslash \{l_{\underline{k}}, l_{\overline{k}}\}$. We unify the two conditions above using ternary relations, and then establish Definition 10 to generalize the restriction of eventually-single-peakedness to the multidimensional setting. $\square$

Now, we start to verify that $\mathbb{D}_{\text{MESP}}$ is a connected⁺ domain. For each $s \in M$, recall that A^s includes the two transportation hubs $l_{\underline{k}}$ and $l_{\overline{k}}$, and we for the notational convenience let $\underline{x}^s \equiv l_{\underline{k}}$ and $\overline{x}^s \equiv l_{\overline{k}}$. We first investigate the intersection $\bar{\mathbb{D}}_{\text{MESP}} = \mathbb{D}_{\text{S}} \cap \mathbb{D}_{\text{MESP}}$. According to $\bar{\mathbb{D}}_{\text{MESP}}$, for each $s \in M$, we induce the domain of marginal preferences, denoted $[\mathbb{D}_{\text{ESP}}]^s$. Note that $[\mathbb{D}_{\text{ESP}}]^s$ includes every marginal preference $[P_j]^s$ such that for all distinct $a^s, y^s \in A^s$, $\left[\overline{\left(r_1([P_i]^s), a^s, y^s\right)}\right] \Rightarrow \left[a^s [P_j]^s y^s\right]$. Therefore, all marginal preferences of $[\mathbb{D}_{\text{ESP}}]^s$ are

[56] When $A^s \subseteq \mathcal{M}$, we only have the ternary relations $\overline{(x^s, x^s, y^s)}$ and $\overline{(x^s, y^s, y^s)}$ for all distinct $x^s, y^s \in A^s$. Consequently, we have $\overline{(x, a, y)}$ if and only if $x \neq y$, and $a^s = x^s$ or $a^s = y^s$ for each $s \in M$ which implies $a^S = x^S$ and $a^{-S} = y^{-S}$ for some $S \subseteq M$.

[57] When $A^s \cap \mathcal{M} = \{\underline{x}^s, \overline{x}^s\}$, the two additional conditions in the definition of the ternary relation $\overline{(x^s, a^s, y^s)}$ is implied by the first condition $a^s \in \langle x^s, y^s \rangle$. Therefore, we have the ternary relation $\overline{(x^s, a^s, y^s)}$ if and only if $x^s \neq y^s$ and $a^s \in \langle x^s, y^s \rangle$. Correspondingly, we have $\overline{(x, a, y)}$ if and only if $x \neq y$ and $a^s \in \langle x^s, y^s \rangle$ for all $s \in M$.

eventually-single-peaked according to $\prec$, $\underline{x}^s$ and $\overline{x}^s$ (recall Remark 12 above). We show that $\bar{\mathbb{D}}_{\text{MESP}}$ is a connected$^+$ domain. The proof consists of five facts, Facts 13 - 17, which are respectively analogous to Facts 1 - 5 of Section E.2. Last, we establish Fact 18 which links $\mathbb{D}_{\text{MESP}}$ to the connectedness$^+$ of $\bar{\mathbb{D}}_{\text{MESP}}$.

Fact 13. *Given $s \in M$ and distinct $[P_j]^s, [P'_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$, there exist $[P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$ and $a^s, b^s \in A^s$ such that $[P_j]^s \sim [P''_j]^s$, $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^s a^s$. (Note that it is possible $[P''_j]^s = [P'_j]^s$.)*

Proof. Let $r_1([P_j]^s) = x^s$ and $r_1([P'_j]^s) = y^s$. If $x^s = y^s \equiv \hat{x}^s$, by a similar argument in the first paragraph of the proof of Fact 5, we identify $a^s, b^s \in A^s$ such that $a^s \neq \hat{x}^s$, $b^s \neq \hat{x}^s$, $a^s[P_j]^s!b^s$ and $b^s[P'_j]^s a^s$. Then, by locally switching a^s and b^s in $[P_j]^s$, we generate a marginal preference $[P''_j]^s$. Thus, $r_1([P''_j]^s) = \hat{x}^s$, $[P_j]^s \sim [P''_j]^s$, $a^s[P_j]^s!b^s$ and $b^s[P''_j]^s!a^s$. We show $[P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$. Suppose not, i.e., $[P''_j]^s \notin [\mathbb{D}_{\text{ESP}}]^s$. Then, we must have $\overline{(\hat{x}^s, a^s, b^s)}$ which consequently implies $a^s[P'_j]^s b^s$. This contradicts the hypothesis $b^s[P'_j]^s a^s$! This proves Fact 13 in the case $x^s = y^s$.

Henceforth, let $x^s \neq y^s$. We assume w.l.o.g. that $x^s \prec y^s$. The verification related to $y^s \prec x^s$ is symmetric, and we hence omit it. We consider the following four cases:

(1) $x^s \prec \underline{x}^s$,
(2) $\overline{x}^s \preceq x^s$,
(3) $\underline{x}^s \preceq x^s \prec \overline{x}^s \preceq y^s$, and
(4) $\underline{x}^s \preceq x^s \prec y^s \prec \overline{x}^s$.

In case (1), we identify the element b^s which is *contiguously* located behind x^s on $\prec$, i.e., $x^s \prec b^s$, and there exists no $c^s \in A^s$ such that $x^s \prec c^s \prec b^s$. Thus, $x^s \prec b^s \preceq \underline{x}^s$ and $x^s \prec b^s \preceq y^s$. Let $r_k([P_j]^s) = b^s$ for some $1 < k \leq |A^s|$. Meanwhile, identify $a^s \equiv r_{k-1}([P_j]^s)$. Thus, $a^s[P_j]^s!b^s$. Since $r_1([P_j]^s) = x^s \in \mathcal{L}$ and $[P_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$, $a^s[P_j]^s b^s$ implies $a^s \preceq x^s$. Hence, $a^s \preceq x^s \prec b^s \preceq \underline{x}^s$ and $a^s \preceq x^s \prec b^s \preceq y^s$, which imply $\overline{(y^s, b^s, a^s)}$ and hence, $b^s[P'_j]^s a^s$. Thus, we have $a^s[P_j]^s!b^s$ and $b^s[P'_j]^s a^s$. Now, by locally switching a^s and b^s in $[P_j]^s$, we generate a marginal preference $[P''_j]^s$. Thus, $[P_j]^s \sim [P''_j]^s$, $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^s a^s$. We last show $[P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$. We know either $r_1([P''_j]^s) = r_1([P_j]^s)$ or $r_1([P''_j]^s) \neq r_1([P_j]^s)$. If $r_1([P''_j]^s) = r_1([P_j]^s) = x^s$, then $b^s[P''_j]^s!a^s$ implies $a^s \neq x^s$. Hence, we have $a^s \prec x^s \prec b^s \preceq \underline{x}^s$ which implies neither $\overline{(x^s, a^s, b^s)}$ nor $\overline{(x^s, b^s, a^s)}$. Therefore, it is true that $[P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$. If $r_1([P''_j]^s) \neq r_1([P_j]^s) = x^s$, it must be the case $r_1([P_j]^s) = r_2([P''_j]^s) = x^s = a^s$ and $r_2([P_j]^s) = r_1([P''_j]^s) = b^s$. Since $r_1([P_j]^s) = x^s$ and $r_2([P_j]^s) = b^s$, it must be the case that $x^s, b^s \in \mathcal{L} \cup \{\underline{x}^s\}$ must form an edge on $\prec$. Suppose $[P''_j]^s \notin [\mathbb{D}_{\text{ESP}}]^s$, i.e., there exist $z^s, \hat{z}^s \in A^s$ such that $\overline{(b^s, z^s, \hat{z}^s)}$ and $\hat{z}^s[P''_j]^s z^s$. Since $x^s, b^s \in \mathcal{L} \cup \{\underline{x}^s\}$ form an edge on $\prec$, $\overline{(b^s, z^s, \hat{z}^s)}$ implies $\overline{(x^s, z^s, \hat{z}^s)}$, and hence $z^s[P_j]^s \hat{z}^s$. Since $\Gamma\big([P_j]^s, [P''_j]^s\big) = \big\{\{x^s, b^s\}\big\}$, we infer $z^s = x^s$ and $\hat{z}^s = b^s$. Thus, we have $\overline{(b^s, z^s, \hat{z}^s)} = \overline{(b^s, x^s, b^s)}$ which contradicts the definition of the ternary relation. Therefore, $[P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$. This completes the verification of Fact 13 in case (1).

In case (2), we identify the element b^s which is *contiguously* located behind x^s on $\prec$, i.e., $x^s \prec b^s$, and there exists no $c^s \in A^s$ such that $x^s \prec c^s \prec b^s$. Thus, $\overline{x}^s \preceq x^s \prec b^s \preceq y^s$. Let

$r_k([P_j]^s) = b^s$ for some $1 < k \le |A^s|$. Meanwhile, identify $a^s \equiv r_{k-1}([P_j]^s)$. Thus, $a^s[P_j]^s!b^s$. Since $r_1([P_j]^s) = x^s \in \mathcal{R} \cup \{\overline{x}^s\}$ and $[P_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$, $a^s[P_j]^s b^s$ implies $a^s \le x^s$. Consequently, we know either $a^s < \overline{x}^s \le x^s < b^s \le y^s$, or $\overline{x}^s \le a^s \le x^s < b^s \le y^s$, which implies $\overline{(y^s, b^s, a^s)}$, and hence $b^s[P_j']^s a^s$. Thus, we have $a^s[P_j]^s!b^s$ and $b^s[P_j']^s a^s$. Now, by locally switching a^s and b^s in $[P_j]^s$, we generate a marginal preference $[P_j'']^s$. Thus, $[P_j]^s \sim [P_j'']^s$, $a^s[P_j]^s!b^s$, $b^s[P_j'']^s!a^s$ and $b^s[P_j']^s a^s$. We last show $[P_j'']^s \in [\mathbb{D}_{\text{ESP}}]^s$. We know either $r_1([P_j'']^s) = r_1([P_j]^s)$ or $r_1([P_j'']^s) \ne r_1([P_j]^s)$. If $r_1([P_j'']^s) = r_1([P_j]^s) = x^s$, then $b^s[P_j'']^s!a^s$ implies $a^s \ne x^s$. Hence, we know either $a^s < \overline{x}^s \le x^s < b^s$ or $\overline{x}^s \le a^s < x^s < b^s$, which implies neither $\overline{(x^s, a^s, b^s)}$ nor $\overline{(x^s, b^s, a^s)}$. Therefore, it is true that $[P_j'']^s \in [\mathbb{D}_{\text{ESP}}]^s$. If $r_1([P_j'']^s) \ne r_1([P_j]^s) = x^s$, it must be the case $r_1([P_j]^s) = r_2([P_j'']^s) = x^s = a^s$ and $r_2([P_j]^s) = r_1([P_j'']^s) = b^s$. Since $r_1([P_j]^s) = x^s$ and $r_2([P_j]^s) = b^s$, $x^s, b^s \in \mathcal{R} \cup \{\overline{x}^s\}$ must form an edge on $<$. Similar to the verification in case (1), after locally switching of x^s and b^s in $[P_j]^s$ (the top-two elements of $[P_j]^s$) to obtain $[P_j'']^s$, it is true that $[P_j'']^s \in [\mathbb{D}_{\text{ESP}}]^s$. This completes the verification of Fact 13 in case (2).

In case (3), let $\overline{x}^s = r_k([P_j]^s)$ for some $1 < k \le |A^s|$. Meanwhile, identify $a^s \equiv r_{k-1}([P_j]^s)$. Thus, $a^s[P_j]^s!\overline{x}^s$. Since $\underline{x}^s \le x^s < \overline{x}^s$ and $[P_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$, $a^s[P_j]^s\overline{x}^s$ implies $a^s < \overline{x}^s$. Thus, we know either $a^s < \underline{x}^s < \overline{x}^s \le y^s$, or $\underline{x}^s \le a^s < \overline{x}^s \le y^s$, which implies $\overline{(y^s, \overline{x}^s, a^s)}$, and hence $\overline{x}^s[P_j']^s a^s$. Thus, we have $a^s[P_j]^s!\overline{x}^s$ and $\overline{x}^s[P_j']^s a^s$. Now, by locally switching a^s and $\overline{x}^s$ in $[P_j]^s$, we generate a marginal preference $[P_j'']^s$. Thus, $[P_j]^s \sim [P_j'']^s$, $a^s[P_j]^s!\overline{x}^s$, $\overline{x}^s[P_j'']^s!a^s$ and $\overline{x}^s[P_j']^s a^s$. We last show $[P_j'']^s \in [\mathbb{D}_{\text{ESP}}]^s$. We know either $r_1([P_j'']^s) = r_1([P_j]^s)$ or $r_1([P_j'']^s) \ne r_1([P_j]^s)$. If $r_1([P_j'']^s) = r_1([P_j]^s) = x^s$, we know $a^s \ne x^s$. Since $a^s < \overline{x}^s$ and $\underline{x}^s \le x^s < \overline{x}^s$ imply neither $\overline{(x^s, a^s, \overline{x}^s)}$ nor $\overline{(x^s, \overline{x}^s, a^s)}$, it is true that $[P_j'']^s \in [\mathbb{D}_{\text{ESP}}]^s$. If $r_1([P_j'']^s) \ne r_1([P_j]^s) = x^s$, it must be the case $r_1([P_j]^s) = r_2([P_j'']^s) = x^s = a^s$ and $r_2([P_j]^s) = r_1([P_j'']^s) = \overline{x}^s$. Suppose $[P_j'']^s \notin [\mathbb{D}_{\text{ESP}}]^s$, i.e., there exist $z^s, \hat{z}^s \in A^s$ such that $\overline{(\overline{x}^s, z^s, \hat{z}^s)}$ and $\hat{z}^s[P_j'']^s z^s$. Thus, $\hat{z}^s \ne z^s$ and $\overline{x}^s \ne z^s$. Furthermore, according to $\overline{(\overline{x}^s, z^s, \hat{z}^s)}$, we know $z^s \in \langle \overline{x}^s, \hat{z}^s \rangle$, and either $\overline{x}^s, z^s \in \mathcal{R} \cup \{\overline{x}^s\}$ or $z^s, \hat{z}^s \in \mathcal{R} \cup \{\overline{x}^s\}$ which further implies $\overline{x}^s < z^s < \hat{z}^s$, or $z^s, \hat{z}^s \in \mathcal{L} \cup \{\underline{x}^s\}$ which further implies $\hat{z}^s < z^s \le \underline{x}^s < \overline{x}^s$. Since $\underline{x}^s \le x^s < \overline{x}^s$, we know either $\underline{x}^s \le x^s < \overline{x}^s < z^s < \hat{z}^s$, or $\hat{z}^s < z^s \le \underline{x}^s \le x^s < \overline{x}^s$ which implies $\overline{(x^s, z^s, \hat{z}^s)}$, and hence $z^s[P_j]^s\hat{z}^s$. Since $\Gamma\big([P_j]^s, [P_j'']^s\big) = \big\{\{x^s, \overline{x}^s\}\big\}$, we infer $z^s = x^s$ and $\hat{z}^s = \overline{x}^s$. Thus, we have $\overline{(\overline{x}^s, z^s, \hat{z}^s)} = \overline{(\overline{x}^s, x^s, \overline{x}^s)}$ which contradicts the definition of the ternary relation. Therefore, $[P_j'']^s \in [\mathbb{D}_{\text{ESP}}]^s$. This completes the verification of Fact 13 in case (3).

In case (4), let $y^s = r_k([P_j]^s)$ for some $1 < k \le |A^s|$. Meanwhile, identify $a^s \equiv r_{k-1}([P_j]^s)$. Thus, $a^s[P_j]^s!y^s$ and $y^s[P_j']^s a^s$ (recall that $r_1([P_j']^s) = y^s$). Now, by locally switching a^s and y^s in $[P_j]^s$, we generate a marginal preference $[P_j'']^s$. Thus, $[P_j]^s \sim [P_j'']^s$, $a^s[P_j]^s!y^s$, $y^s[P_j'']^s!a^s$ and $y^s[P_j']^s a^s$. We last show $[P_j'']^s \in [\mathbb{D}_{\text{ESP}}]^s$. Since $y^s \notin \mathcal{L} \cup \{\underline{x}^s\}$ and $y^s \notin \mathcal{R} \cup \{\overline{x}^s\}$, $\Gamma\big([P_j]^s, [P_j'']^s\big) = \big\{\{a^s, y^s\}\big\}$ implies that $[P_j]^s$ and $[P_j'']^s$ share the same relative rankings on the elements of $\mathcal{L} \cup \{\underline{x}^s\}$ and the elements of $\mathcal{R} \cup \{\overline{x}^s\}$ respectively. Consequently, if $r_1([P_j'']^s) = r_1([P_j]^s) = x^s$, it is evident that $[P_j'']^s \in [\mathbb{D}_{\text{ESP}}]^s$. Next, assume $r_1([P_j'']^s) \ne r_1([P_j]^s) = x^s$. Then, it must be the case $r_1([P_j]^s) = r_2([P_j'']^s) = x^s = a^s$ and $r_2([P_j]^s) = r_1([P_j'']^s) = y^s$. Similar to the verification in case (3), since $\underline{x}^s \le x^s < y^s < \overline{x}^s$, after locally switching of x^s and y^s in $[P_j]^s$ (the top-two elements of $[P_j]^s$) to obtain $[P_j'']^s$, it is true that $[P_j'']^s \in [\mathbb{D}_{\text{ESP}}]^s$. This completes the verification of Fact 13 in case (4), and hence proves Fact 13. □

Fact 14. *Fixing $P_j, P'_j \in \bar{\mathbb{D}}_{\text{MESP}}$ with $[P_j]^q \neq [P'_j]^q$ for some $q \in M$, and $x, y \in A$ with $x P_j y$ and $x P'_j y$, there exists $P''_j \in \bar{\mathbb{D}}_{\text{MESP}}$ such that*

(i) *$[P''_j]^s \sim [P_j]^s$ for some $s \in S$, and $[P''_j]^\omega = [P_j]^\omega$ for all $\omega \neq s$,*
(ii) *$a^s [P_j]^s ! b^s$, $b^s [P''_j]^s ! a^s$ and $b^s [P'_j]^s a^s$ for some $a^s, b^s \in A^s$, and*
(iii) *$x P''_j y$.*

Proof. After replacing the reference of Fact 1 and the notation $[\mathbb{D}_{\text{S}}]^\tau$ and $[\mathbb{D}_{\text{S}}]^s$ in the proof of Fact 2 by the reference of Fact 13 and the notation $[\mathbb{D}_{\text{ESP}}]^\tau$ and $[\mathbb{D}_{\text{ESP}}]^s$ respectively, the modified proof of Fact 2 remains applicable for the verification of Fact 14. □

Fact 15. *Given $P_j \in \bar{\mathbb{D}}_{\text{MESP}}$, $s \in M$, $a^s, b^s \in A^s$ with $a^s [P_j]^s ! b^s$, there exists $\bar{P}_j \in \bar{\mathbb{D}}_{\text{MESP}}$ such that*

(i) *for all $x, y \in A \backslash (b^s, A^{-s})$, $[x P_j y] \Leftrightarrow [x \bar{P}_j y]$,*
(ii) *for all $z^{-s} \in A^{-s}$, $(a^s, z^{-s}) \bar{P}_j ! (b^s, z^{-s})$, and*
(iii) *$[P_j]^\omega = [\bar{P}_j]^\omega$ for all $\omega \in M$.*

Moreover, if there exists $[P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$ such that $[P_j]^s \sim [P''_j]^s$ and $\Gamma([P_j]^s, [P''_j]^s) = \big\{\{a^s, b^s\}\big\}$, then there exists $\hat{P}_j \in \bar{\mathbb{D}}_{\text{MESP}}$ such that $\Gamma(\bar{P}_j, \hat{P}_j) = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s} \in A^{-s}}$.

Proof. By the proof of Fact 3, we have $\bar{P}_j \in \mathbb{D}_{\text{S}}$ satisfying condition (i) - (iii). Furthermore, by condition (iii), we have $[\bar{P}_j]^\omega = [P_j]^\omega \in [\mathbb{D}_{\text{ESP}}]^\omega$ for all $\omega \in M$. Therefore, $\bar{P}_j \in \bar{\mathbb{D}}_{\text{MESP}}$.

Next, by the proof of Fact 3, we have $\hat{P}_j \in \mathbb{D}_{\text{S}}$ such that $\Gamma(\bar{P}_j, \hat{P}_j) = \big\{\{(a^s, z^{-s}), (b^s, z^{-s})\}\big\}_{z^{-s} \in A^{-s}}$. Thus, $[\bar{P}_j]^s \sim [\hat{P}_j]^s$, $\Gamma([\bar{P}_j]^s, [\hat{P}_j]^s) = \big\{\{a^s, b^s\}\big\}$ and $[\bar{P}_j]^\omega = [\hat{P}_j]^\omega$ for all $\omega \in M \backslash \{s\}$. We show $\hat{P}_j \in \bar{\mathbb{D}}_{\text{MESP}}$. First, for every $\omega \in M \backslash \{s\}$, we know $[\hat{P}_j]^\omega = [\bar{P}_j]^\omega \in [\mathbb{D}_{\text{ESP}}]^\omega$. Second, given $[\bar{P}_j]^s = [P_j]^s$ and $\Gamma([\bar{P}_j]^s, [\hat{P}_j]^s) = \big\{\{a^s, b^s\}\big\}$, the hypothesis $\Gamma([P_j]^s, [P''_j]^s) = \big\{\{a^s, b^s\}\big\}$ and $[P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$ imply $[\hat{P}_j]^s = [P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$. Therefore, $\hat{P}_j \in \bar{\mathbb{D}}_{\text{MESP}}$. □

Fact 16. *Fixing $P_j, P'_j \in \bar{\mathbb{D}}_{\text{MESP}}$ with $[P_j]^s \neq [P'_j]^s$ for some $s \in M$, and $x, y \in A$ with $x P_j y$ and $x P'_j y$, there exist $t \geq 1$ pair(s) $\{\bar{P}^k_j, \hat{P}^k_j : k = 1, \ldots, t\} \subseteq \bar{\mathbb{D}}_{\text{MESP}}$ such that*

(i) *$\bar{P}^k_j \sim^+ \hat{P}^k_j$ for all $k = 1, \ldots, t$,*
(ii) *$[P_j]^s = [\bar{P}^1_j]^s$ for all $s \in M$,*
(iii) *$[\hat{P}^k_j]^s = [\bar{P}^{k+1}_j]^s$ for all $s \in M$ and $k = 1, \ldots, t-1$,*
(iv) *$[\hat{P}^t_j]^s = [P'_j]^s$ for all $s \in M$, and*
(v) *$x \bar{P}^k_j y$ and $x \hat{P}^k_j y$ for all $k = 1, \ldots, t$.*

In particular, if $r_1(P_j)$ and $r_1(P'_j)$ are similar, say $r_1(P_j) = (a^s, z^{-s})$ and $r_1(P'_j) = (b^s, z^{-s})$, then $r_1(\bar{P}^k_j), r_1(\hat{P}^k_j) \in (A^s, z^{-s})$ for all $k = 1, \ldots, t$.

Proof. Let $S = \{q \in M : [P_j]^q \neq [P'_j]^q\}$ and $T = \{\tau \in M : x^\tau \neq y^\tau\}$. Evidently, both S and T are nonempty. Since xP_jy, there exists $\hat{\tau} \in T$ such that $x^{\hat{\tau}}[P_j]^{\hat{\tau}}y^{\hat{\tau}}$. According to P_j and P'_j, we first identify $P''_j \in \bar{\mathbb{D}}_{\text{MESP}}$ satisfying conditions (i) - (iii) of Fact 14. Specifically, (i) $[P''_j]^s \sim [P_j]^s$ for some $s \in S$, and $[P''_j]^\omega = [P_j]^\omega$ for all $\omega \neq s$, (ii) $a^s[P_j]^s!b^s$, $b^s[P''_j]^s!a^s$ and $b^s[P'_j]^s a^s$ for some $a^s, b^s \in A^s$, and (iii) xP''_jy. Note that $P''_j \in \bar{\mathbb{D}}_{\text{MESP}}$ implies $[P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$. Moreover, since $[P_j]^s \sim [P''_j]^s$ and $\Gamma([P_j]^s, [P''_j]^s) = \big\{\{a^s, b^s\}\big\}$, Fact 15 becomes applicable here. Then, after replacing the references of Facts 2, 3 and 4 and the notation $\mathbb{D}_{\text{LS}}$ in the proof of Fact 4 by the references of Facts 14, 15 and 16 and the notation $\mathbb{D}_{\text{LS}} \cap \mathbb{D}_{\text{MESP}}$ respectively, the modified proof of Fact 4 (from Claim 1 to the end) remains valid for the verification of Fact 16. □

Fact 17. *Given two distinct $P_j, P'_j \in \bar{\mathbb{D}}_{\text{MESP}}$ with $[P_j]^s = [P'_j]^s$ for all $s \in M$, there exists an adjacency path $\{P^k_j\}^q_{k=1} \subseteq \bar{\mathbb{D}}_{\text{MESP}}$ connecting P_j and P'_j such that $[xP_jy$ and $xP'_jy] \Rightarrow [xP^k_jy$ for all $k = 1, \ldots, q]$.*

Proof. Recall the construction of preference $P''_j \in \mathbb{D}_{\text{S}}$ in the proof of Fact 5. If we show $[P''_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$ for all $s \in M$, then we have $P''_j \in \bar{\mathbb{D}}_{\text{MESP}}$, and the rest proof of Fact 5 on the construction of the adjacency path remains applicable for the verification of Fact 17. Since both P_j and P''_j are separable preferences, $P_j \sim P''_j$ implies that P_j and P''_j share the same marginal preferences by Observation 1. Therefore, $[P''_j]^s = [P_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$ for all $s \in M$, as required. □

Now, similar to the last paragraph of Section E.2, we use Facts 16 and 17 to construct paths in $\bar{\mathbb{D}}_{\text{MESP}}$ satisfying the requirements of the Interior$^+$ and Exterior$^+$ properties. Therefore, $\bar{\mathbb{D}}_{\text{MESP}}$ is a connected$^+$ domain. Next, we establish a fact to link $\mathbb{D}_{\text{MESP}}$ to $\bar{\mathbb{D}}_{\text{MESP}}$.

Fact 18. *Given $P_j \in \mathbb{D}_{\text{MESP}} \backslash \bar{\mathbb{D}}_{\text{MESP}}$ and $a, b \in A$ with aP_jb, there exists $\bar{P}_j \in \bar{\mathbb{D}}_{\text{MESP}}$ such that $r_1(\bar{P}_j) = r_1(P_j) \equiv \bar{a}$ and $a\bar{P}_jb$. Furthermore, there exists an adjacency path $\{P^k_j\}^q_{k=1} \subseteq \mathbb{D}^{\bar{a}}_{\text{MESP}}$ connecting P_j and $\bar{P}_j$ such that $[xP_jy$ and $x\bar{P}_jy] \Rightarrow [xP^k_jy$ for all $k = 1, \ldots, q]$.*

Proof. Since aP_jb, we do not have $\overline{(\bar{a}, b, a)}$. Therefore, there must exist $s \in M$ such that either $\bar{a}^s \neq a^s$ and $\overline{(\bar{a}^s, b^s, a^s)}$ does not exist, or $\bar{a}^s = a^s$ and $b^s \neq \bar{a}^s = a^s$. Thus, $b^s \neq a^s$, and there exists $[\bar{P}_j]^s \in [\mathbb{D}_{\text{ESP}}]^s$ such that $r_1([\bar{P}_j]^s) = \bar{a}^s$ and $a^s[\bar{P}_j]^s b^s$. For each $\tau \neq s$, we fix a marginal preference $[\bar{P}_j]^\tau \in [\mathbb{D}_{\text{ESP}}]^\tau$ with $r_1([\bar{P}_j]^\tau) = \bar{a}^\tau$. Last, we fix a lexicographic order $\succ$ such that component s is lexicographically dominant, and assemble all alluded marginal preferences to generate a preference $\bar{P}_j \in \mathbb{D}_{\text{LS}} \cap \mathbb{D}_{\text{MESP}} \subseteq \bar{\mathbb{D}}_{\text{MESP}}$. Hence, $r_1(\bar{P}_j) = \bar{a} = r_1(P_j)$ and $a\bar{P}_jb$.

Recalling the proof of Fact 5, according to P_j and $\bar{P}_j$, we identify $x, y \in A$ such that $x \neq \bar{a}$, $y \neq \bar{a}$, $xP_j!y$ and $y\bar{P}_jx$. By locally switching x and y in P_j, we generate a preference P''_j. Thus, $r_1(P''_j) = \bar{a}$, $P_j \sim P''_j$, $xP_j!y$, $yP''_j!x$ and $y\bar{P}_jx$. We show $P''_j \in \mathbb{D}_{\text{MESP}}$. Suppose not, i.e., $P''_j \notin \mathbb{D}_{\text{MESP}}$. Then, we must have $\overline{(\bar{a}, x, y)}$, which consequently implies $x\bar{P}_jy$. This contradicts the hypothesis $y\bar{P}_jx$. Therefore, $P''_j \in \mathbb{D}_{\text{MESP}}$. Note that P''_j is closer to $\bar{P}_j$ than P_j since $\Gamma(P''_j, \bar{P}_j) = \Gamma(P_j, \bar{P}_j) \backslash \big\{\{x, y\}\big\}$. By repeatedly applying the argument above, we eventually generate an adjacency path $\{P^k_j\}^q_{k=1} \subseteq \mathbb{D}^{\bar{a}}_{\text{MESP}}$ connecting P_j and $\bar{P}_j$, such that for each $1 \leq k < q$, $\Gamma(P^{k+1}_j, \bar{P}_j) = \Gamma(P^k_j, \bar{P}_j) \backslash \big\{\{x, y\}\big\}$ for some $x, y \in A$ with $xP^k_j!y$, $yP^{k+1}_j!x$ and $y\bar{P}_jx$. Therefore, we have $[xP_jy$ and $x\bar{P}_jy] \Rightarrow [xP^k_jy$ for all $k = 1, \ldots, q]$. □

Last, similar to the last two paragraphs of Section E.3, by applying the connectedness$^+$ of $\bar{\mathbb{D}}_{\text{MESP}}$ and Fact 18, we assert that $\mathbb{D}_{\text{MESP}}$ is a connected$^+$ domain.

E.8. Detailed verification for Example 5

To show that $\mathbb{D}$ is a connected$^+$ domain, we first provide two observations on the subdomain $\hat{\mathbb{D}}$: (i) subdomain $\hat{\mathbb{D}}$ is connected$^+$, and (ii) every preference of $\hat{\mathbb{D}}$ ranks $(1,1)$ above $(2,1)$. Therefore, with respect to the pair $(2,1)$ and $(1,1)$, we need not to construct a path in $\mathbb{D}$ to reconcile the difference between $\bar{P}_i$ and any preference of $\hat{\mathbb{D}}$. We next identify preference $\hat{P}_i$: $(0,0)\rightharpoondown(0,1)\rightharpoondown(1,0)\rightharpoondown(2,0)\rightharpoondown(1,1)\rightharpoondown(2,1)$, which belongs to $\hat{\mathbb{D}}$. Note that $r_1(\bar{P}_i)=r_1(\bar{P}_i)=(0,0)$, $\bar{P}_i \sim \hat{P}_i$, and $\Gamma(\bar{P}_i,\hat{P}_i)=\big\{\{(2,1),(1,1)\}\big\}$. Therefore, by the first observation, $\mathbb{D}$ still satisfies the Interior$^+$ property. Next, given $P_i' \in \hat{\mathbb{D}}$ with $r_1(P_i') \neq (0,0)$ and $a,b \in A$ with $a\bar{P}_i b$ and $aP_i'b$, by the second observation, note that $\{a,b\} \neq \{(1,1),(2,1)\}$, and hence $a\hat{P}_i b$. Then, by the first observation, we have a path in $\hat{\mathbb{D}}$ which connects $\hat{P}_i$ and P_i', and ranks a above b in all involved preferences. Last, since $\bar{P}_i \sim \hat{P}_i$, we add $\bar{P}_i$ to the beginning of the path, and hence, construct a path in $\mathbb{D}$ which connects $\bar{P}_i$ and P_i', and ranks a above b in all involved preferences.

E.9. Detailed verification for Example 6

To verify the Interior$^+$ property on domain $\mathbb{D}$, we first identify preference $\bar{P}_i \in \mathbb{D}_{\text{MSP}}$ below:

$$\begin{aligned}\hat{P}_i : &(0,0,0)\rightharpoondown(1,0,0)\rightharpoondown(2,0,0)\rightharpoondown(0,1,0)\rightharpoondown(1,1,0)\rightharpoondown(2,1,0)\rightharpoondown(0,0,1)\rightharpoondown\\ &(1,0,1)\rightharpoondown(2,0,1)\rightharpoondown(0,1,1)\rightharpoondown(1,1,1)\rightharpoondown(2,1,1).\end{aligned}$$

Note that $r_1(\hat{P}_i)=r_1(\bar{P}_i)=(0,0,0)$, $\hat{P}_i \sim \bar{P}_i$ and $\Gamma(\hat{P}_i,\bar{P}_i)=\big\{\{(1,1,1),(2,1,1)\}\big\}$. Then, it is easy to check that domain $\mathbb{D}$ satisfies the Interior$^+$ property. We show by contradiction that no path in $\mathbb{D}$ reconciles the difference between $\bar{P}_i$ and a preference P_i' with peak $(2,1,1)$, and meanwhile keeps $(2,1,1)$ ranked above $(1,1,1)$ in every involved preference. Suppose not, i.e., we have such an admissible path in $\mathbb{D}$. Since the path starts from $\bar{P}_i$ whose peak is $(0,0,0)$ and ends at preference P_i' whose peak is $(2,1,1)$, the path must include a preference with peak $(1,0,0)$ or $(0,1,0)$ or $(1,0,0)$. However, in every preference of $\mathbb{D}$ which has peak $(1,0,0)$ or $(0,1,0)$ or $(1,0,0)$, alternative $(1,1,1)$ always ranks above $(2,1,1)$. Contradiction! Therefore, domain $\mathbb{D}$ violates the Exterior$^+$ property.

We consider a particular mixed multidimensional projection rules $\varphi : \mathbb{D}^2 \to \Delta(A)$: For all $P_i, P_j \in \mathbb{D}$,

$$\varphi(P_i,P_j)=\frac{1}{10}\sum\nolimits_{a\notin\{(1,1,1),(2,1,1)\}} f^a(P_i,P_j).$$

It is evident that φ satisfies unanimity. To verify sd-strategy-proofness of φ, we show that each one of these 10 multidimensional projection rules is sd-strategy-proof. In fact, analogous to the proof of the Theorem of Chatterji et al. (2013), it is easy to show that a multidimensional projection rule $f^a : \mathbb{D}^2 \to A$ is sd-strategy-proof if and only if domain $\mathbb{D}$ is *multidimensional semi-single-peaked* w.r.t. the projector a, i.e., given $P_i \in \mathbb{D}$, say $r_1(P_i)=z$,

1. if $x,y \in \langle z,a\rangle$ and $x \in \langle z,y\rangle$, then $xP_i y$;
2. if $x \notin \langle z,a\rangle$, then $\pi(x,\langle z,a\rangle)P_i x$. Recall that $\pi(x,\langle z,a\rangle)=\big(\pi^s(x^s,\langle z^s,a^s\rangle)\big)_{s\in M}$ denotes the projection of x on $\langle z,a\rangle$.

Since we choose projectors $a \notin \{(1,1,1),(2,1,1)\}$, sd-strategy-proofness of the multidimensional projection rule f^a does not require a preference with peak $(0,0,0)$ rank $(1,1,1)$ over $(2,1,1)$. Therefore, all 10 multidimensional projection rules here are sd-strategy-proof, and hence RSCF φ is sd-strategy-proof. Last, we show that φ satisfies the compromise property. By the proof of Claim 4 in the verification of the sufficiency part of Theorem 1, we here only need to consider the profile $(P_i, P_j) \in \mathbb{D}^2$ such that $\{r_1(P_i), r_1(P_j)\} = \{(0,1,1),(2,1,1)\}$ and $r_2(P_i) = r_2(P_j) = (1,1,1)$. According to Fig. 5, it is true that the projection of $(1,0,0)$ or $(1,0,1)$ or $(1,1,0)$ on $\langle r_1(P_i), r_1(P_j)\rangle$ is $(1,1,1)$. Therefore, at least one of these 10 multidimensional projection rules chooses the compromise alternative $(1,1,1)$ at (P_i, P_j). Hence, φ satisfies the compromise property.

E.10. Details of the discussion in Remark 9

Example 10 *(Indispensability of unanimity (w.r.t. feasibility)).* Let $A = A^1 \times A^2$ and $A^1 = A^2 = \{0,1,2\}$. First, let $\mathbb{D}^1$ and $\mathbb{D}^2$ be two marginal domains such that $\mathbb{D}^1$ is the single-peaked domain over A^1 according to the underlying order $0 < 1 < 2$, and $\mathbb{D}^2$ is the complete marginal domain over A^2. Second, according to $\mathbb{D}^1$ and $\mathbb{D}^2$, construct a separable domain $\mathbb{D} \subset \mathbb{D}_S$ containing all admissible preferences. It is evident that $\mathbb{D}$ is a minimally rich and connected$^+$ domain.

Let $\bar{A} = \{1,2\} \times \{0,1,2\}$. It is evident that $\mathbb{D}$ satisfies Assumptions 1 and 2. However, $\mathbb{D}$ is not multidimensional single-peaked w.r.t. $\bar{A}$. Last, similar to Example 1, we define a two-voter point voting scheme $\phi : [\mathbb{D}_{|\bar{A}}]^2 \to \Delta(\bar{A})$, and extend it to a constrained RSCF $\varphi : \mathbb{D}^2 \to \Delta(\bar{A})$ such that $\varphi(P_1, P_2) = \phi(P_{1|\bar{A}}, P_{2|\bar{A}})$ for all $(P_1, P_2) \in \mathbb{D}^2$. It is easy to show that φ satisfies sd-strategy-proofness and the compromise property (w.r.t. feasibility), but fails unanimity (w.r.t. feasibility). $\square$

Example 11 *(Indispensability of sd-strategy-proofness).* Consider the same domain $\mathbb{D}$ and the same feasible set $\bar{A}$ of Example 10. Then, similar to Example 2 we define a two-voter constrained DSCF $f : \mathbb{D}^2 \to \bar{A}$ such that

$$f(P_i, P_j) = \begin{cases} a & \text{if } r_1(P_i) \neq r_1(P_j) \text{ and } r_2(P_i) = r_2(P_j) \equiv a \in \bar{A}, \\ r_1(P_{i|\bar{A}}) & \text{otherwise.} \end{cases}$$

It is evident that f satisfies unanimity (w.r.t. feasibility) and the compromise property (w.r.t. feasibility), but fails sd-strategy-proofness. $\square$

Example 12 *(Indispensability of the compromise property (w.r.t. feasibility)).* We adopt the same strengthening on recognizing compromise alternatives as that in Example 3. Consider the same domain $\mathbb{D}$ and the same feasible set $\bar{A}$ of Example 10. Note that since $\mathbb{D}_{|\bar{A}}$ remains to be top-separable, we can construct a three-voter generalized random dictatorship $\phi : [\mathbb{D}_{|\bar{A}}]^3 \to \Delta(\bar{A})$ which assigns strictly positive weights to all voters sequences, and is sd-strategy-proof by Proposition 1, and extend it to a constrained RSCF $\varphi : \mathbb{D}^3 \to \Delta(\bar{A})$ such that $\varphi(P_1, P_2, P_3) = \phi(P_{1|\bar{A}}, P_{2|\bar{A}}, P_{3|\bar{A}})$ for all $(P_1, P_2, P_3) \in \mathbb{D}^3$. It is evident that φ is unanimous (w.r.t. feasibility) and sd-strategy-proof. Last, given $(P_1, P_2, P_3) \in \mathbb{D}^3$, assume that peaks are pairwise distinct, the common second best alternative exists, say a, and $a \in \bar{A}$. If $r_1(P_i) \notin \bar{A}$ for some $i \in I$, then we have $\varphi_a(P_1, P_2, P_3) = \phi_a(P_{1|\bar{A}}, P_{2|\bar{A}}, P_{3|\bar{A}}) \geq \gamma(i,i,i) > 0$. If $r_1(P_i) \in \bar{A}$ for all $i \in I$, then a can be assembled by three peaks via some voter sequence $\underline{i}$. We then have

$\varphi_a(P_1, P_2, P_3) = \phi_a\big(P_{1|\bar{A}}, P_{2|\bar{A}}, P_{3|\bar{A}}\big) \geq \gamma(\underline{i}) > 0$. Therefore, φ satisfies the modified compromise property (w.r.t. feasibility). □

Example 13 *(Indispensability of top-separability).* Let $A = A^1 \times A^2$, $A^1 = \{0, 1, 2\}$ and $A^2 = \{0, 1\}$. Recalling the product of two lines in Fig. 4, we generate the multidimensional single-peaked domain $\mathbb{D}_{\text{MSP}}$, and construct a particular preference $\bar{P}_i : (2, 0) \rightharpoondown (1, 0) \rightharpoondown (1, 1) \rightharpoondown (2, 1) \rightharpoondown (0, 0) \rightharpoondown (0, 1)$. Evidently, $\bar{P}_i \notin \mathbb{D}_{\text{MSP}}$ since $(1, 1)\bar{P}_i(2, 1)$, and $\bar{P}_i$ is not top-separable since $r_1(\bar{P}_i) = (2, 0)$ and $(1, 1)\bar{P}_i(2, 1)$. Let $\mathbb{D} = \mathbb{D}_{\text{MSP}} \cup \{\bar{P}_i\}$, and $\bar{A} = \{1, 2\} \times \{0, 1\}$. It is easy to show that $\mathbb{D}$ satisfies Assumptions 1 and 2. However, $\mathbb{D}$ is not multidimensional single-peaked w.r.t. $\bar{A}$.

To show that $\mathbb{D}$ satisfies the Interior$^+$ and Exterior$^+$ properties, we first highlight two preferences $\hat{P}_i, \tilde{P}_i \in \mathbb{D}_{\text{MSP}}$ below which are adjacent to $\bar{P}_i$.

$$\hat{P}_i : (2, 0) \rightharpoondown (1, 0) \rightharpoondown (2, 1) \rightharpoondown (1, 1) \rightharpoondown (0, 0) \rightharpoondown (0, 1), \text{ and}$$
$$\tilde{P}_i : (1, 0) \rightharpoondown (2, 0) \rightharpoondown (1, 1) \rightharpoondown (2, 1) \rightharpoondown (0, 0) \rightharpoondown (0, 1).$$

By Fact 11 of Appendix E.5, given $P_i \in \mathbb{D}_{\text{MSP}} \backslash \{\hat{P}_i\}$ with $r_1(P_i) = (2, 0)$, there exists an adjacency path of $\mathbb{D}_{\text{MSP}}$ connecting P_i and $\hat{P}_i$ such that every involved preference has peak $(2, 0)$. Since $\hat{P}_i \sim \bar{P}_i$, we can generate an adjacency path of $\mathbb{D}$ connecting P_i and $\bar{P}_i$. Therefore, the Interior$^+$ property still holds. Next, given $P_i \in \mathbb{D}_{\text{MSP}}$ with $r_1(P_i) \neq (2, 0)$ and $a, b \in A$ with $a P_i b$ and $a \bar{P}_i b$, we know $\big[\{a, b\} \neq \{(1, 1), (2, 1)\}\big] \Rightarrow [a \hat{P}_i b]$, and $\big[\{a, b\} = \{(1, 1), (2, 1)\}\big] \Rightarrow [a \tilde{P}_i b]$. According to the Exterior$^+$ property satisfied by $\mathbb{D}_{\text{MSP}}$ and Fact 11 of Appendix E.5, we know that there exists a path connecting P_i and $\hat{P}_i$ (respectively, P_i and $\tilde{P}_i$) such that a is ranked above b in every preference of the path. We then extend the path to connect P_i and $\bar{P}_i$ via either $\hat{P}_i$ or $\tilde{P}_i$, and keep a ranked above b along the path. In particular, when $r_1(P_i) \in (A^1, 0)$, since the path connecting P_i and $\hat{P}_i$ (respectively, P_i and $\tilde{P}_i$) satisfies the requirement of the no-detour property, we know that the extended path connecting P_i and $\bar{P}_i$ also satisfies the requirement of the no-detour property. Therefore, domain $\mathbb{D}$ satisfies the Interior$^+$ and Exterior$^+$ properties.

Last, we can construct a constrained random dictatorship on $\mathbb{D}$ where all voters receive strictly positive weights. Similar to Example 7, this constrained random dictatorship is unanimous (w.r.t. feasibility) and sd-strategy-proof, and satisfies the compromise property (w.r.t. feasibility). □

Example 14 *(Indispensability of minimal richness).* We adopt the same domain $\mathbb{D}$ of Example 5. Evidently, $\mathbb{D}$ is not minimally rich. By Appendix E.8, we know that $\mathbb{D}$ is a connected$^+$ domain.

Let $\bar{A} = \{1, 2\} \times \{0, 1\}$. It is easy to show that $\mathbb{D}$ satisfies Assumptions 1 and 2. However, $\mathbb{D}$ is not multidimensional single-peaked w.r.t. $\bar{A}$ due to preference $\bar{P}_i$ of Example 5. Similarly, we refer to an arbitrary constrained random dictatorship which is unanimous (w.r.t. feasibility) and sd-strategy-proof, and satisfies the compromise property (w.r.t. feasibility) vacuously. □

Example 15 *(Indispensability of paths in connectedness$^+$).* We adopt the same domain $\mathbb{D}$ of Example 6. Thus, we know that $\mathbb{D}$ is a minimally rich top-separable domain, satisfies the Interior$^+$ property, but violates the Exterior$^+$ property by Appendix E.9.

Let $\bar{A} = \{1, 2\} \times \{0, 1\} \times \{0, 1\}$. It is easy to show that $\mathbb{D}$ satisfies Assumptions 1 and 2. However, $\mathbb{D}$ is not multidimensional single-peaked w.r.t. $\bar{A}$ due to preference $\bar{P}_i$ of Example 6. Last, since $\bar{A}$ form a box in Fig. 5, we can construct a multidimensional projection rules $\bar{f}^a : \big[\mathbb{D}_{|\bar{A}}\big]^N \rightarrow \bar{A}$ for each $a \in \{(1, 0, 0), (1, 1, 0), (1, 0, 1), (2, 0, 0), (2, 1, 0), (2, 0, 1)\}$, and assemble them as a mixed multidimensional projection rule $\phi : \big[\mathbb{D}_{|\bar{A}}\big]^N \rightarrow \Delta(\bar{A})$. We then extend

ϕ to a constrained RSCF $\varphi : \mathbb{D}^N \to \Delta(\bar{A})$ such that $\varphi(P_1, \ldots, P_N) = \phi\big(P_{1|\bar{A}}, \ldots, P_{N|\bar{A}}\big)$ for all $(P_1, \ldots, P_N) \in \mathbb{D}^N$. By a similar verification in Appendix E.9, we know that φ is unanimous (w.r.t. feasibility) and sd-strategy-proof, and satisfies the compromise property (w.r.t. feasibility). $\square$

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