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Abstract

Consider a sequential process where agents have individual values at every possible step. A planner is in charge of selecting steps and distributing the accumulated aggregate values among agents. We model this process by a directed network where each edge is associated with a vector of individual values. This model applies to several new and existing problems, e.g., developing a connected public facility and distributing total values received by surrounding districts; selecting a long-term production plan and sharing final profits among partners of a firm; choosing a machine schedule to serve different tasks and distributing total outputs among task owners.

Herein, we provide the first axiomatic study on path selection and value sharing in networks. We consider four sets of axioms from different perspectives, including those related to (1) the sequential consistency of assignments with respect to network decompositions; (2) the monotonicity of assignments with respect to network expansion; (3) the independence of assignments with respect to certain network transformations; and (4) implementation in the case where the planner has no information about the underlying network and individual values. Surprisingly, these four disparate sets of axioms characterize similar classes of solutions — selecting efficient path(s) and assigning to each agent a share of total values which is independent of their individual values. Furthermore, we characterize more general solutions that depend on individual values.

Keywords: *Sequential Values, Sharing, Network, Redistribution*

JEL **classification:** C72, D44, D71, D82.

1 Introduction

The axiomatic division of costs and benefits has been extensively studied over the past 60 years, starting with bargaining (Nash [53]) and cooperative games (Shapley [56]), and followed by applications to problems such as rationing and bankruptcy (O'Neill [55], Aumann and Maschler [3], Thomson [60], Moulin [48, 49]), airport cost-sharing (Littlechild and Owen [41], Thomson [62]), hierarchical ventures (Hougaard et al. [26]), and more general cost-sharing problems (e.g., Sprumont [57, 58], Moulin [47], Friedman and Moulin [22], Moulin and Sprumont [51], Moulin and Shenker [50]). Such studies have characterized a wide variety of sharing rules using axioms motivated by positive and normative perspectives. However, they are largely limited to scenarios with a fixed resource. Little is known regarding scenarios that are more general in two respects: (1) the amount of the resource may not be fixed but can be chosen, and (2) the resource may be generated in a sequence of steps, where the amount in future steps depends on the choices made in previous steps. Such a generalized problem requires resource-generating steps to be determined together with the allocation. This "two-tiered" approach not only expands the range of problems, but also gives rise to a new question on the interdependence of the step selection rule and sharing rule.

To better illustrate our problem, consider a planner in charge of developing a connected public facility (e.g., highways, rail-roads, or irrigation canals). The project might be developed in different steps, each of which might produce different benefits to the agents in a given society. The planner is in charge of choosing the steps and redistributing the benefits of the project among the agents. After proceeding along each step, the planner faces a new problem. This new problem is different from the original one and might depend on the steps preceded (Section 1.1 discusses other applications).

Formally, we model a sequential process as an acyclic-directed-network with a common source and multiple sinks (hereafter referred to as a network), where each edge represents a possible step in the process and each node faces a forward process. There is a finite number of agents. A **problem** consists of a network where each edge is attached with a vector of individual values of agents. The value of a path connecting the source and a sink is the sum of all individual values over all edges in the path. A **solution** selects in each problem one or several paths with the same value, and distributes among the agents the value of the selected path(s).

We provide the first systematic and comprehensive study of this problem by considering axioms appropriate to a wide range of scenarios. Surprisingly, our four sets of axioms from different perspectives characterize similar classes of solutions — selecting the path(s) with the highest value (hereafter referred to as the efficient path(s)) and assigning to each agent a share of the value of the path(s) which is independent of individual values. Moreover, we show the richness of suitable solutions in different scenarios. For instance, we characterize a large class of solutions that depend on individual values in a "rationalizable" way. For an overview of our results, see Section 1.2.

1.1 Applications and Solutions

To see the applicability of our model and rules, consider the following examples:

Sharing the benefits of connected public facilities. For the construction of highways, railroads or irrigation canals, a government usually selects one of several potential routes. Benefits of each edge to agents in surrounding districts depend on the convenience of the access to it. Typically, the government selects the efficient path, and performs no transfer so that each agent receives his individual benefits. This efficient-path-selection and no-transfer solution is denoted by **EFF-NT**.[1] However, the districts located closer to the selected route could benefit substantially more than those farther away, and this could lead to social conflicts. For instance, tens of thousands of people in Linshui county (in China) protested against the government for abandoning the plan of a high speed railway route passing the county in 2015 (see Figure 1). Another solution could have been for the government to select the efficient path and assign to each agent an equal share of the total benefits (denoted by **EFF-ES**). Such redistribution can be achieved via a lump-sum tax and subsidy on the agents.

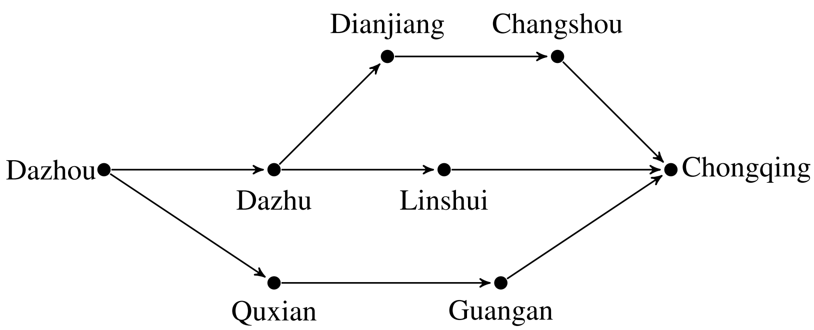

Figure 1: Network outlining the three potential routes to build a high-speed railway connecting Dazhou and Chongqing. People from Linshui protested in 2015 when the government shortlisted the upper and lower routes.

Profit sharing in companies and joint-business ventures. A company chooses a priority over different projects and makes a redistribution of profits among the agents (i.e., employees). Agents generate different profits on different projects. The choice of earlier projects changes the availability and profitability of later projects. Choosing a priority of the projects that maximizes the total profit (efficient-path-selection) is natural for a for-profit company. Agents are often rewarded with bonuses that are tied to the profits they have generated.[2]

[1]When there are multiple efficient paths, a tie-breaking rule is needed. For example, ties are broken in favor of agents with a lexicographic order.

[2]For partnerships between professionals, such as a group of lawyers in a law firm, the redistribution among partners is typically 100% of the total profit (Juarez and Nitta [37]). Our problem can also be applied to the case of other for-profit companies, where the employees are rewarded with a fixed percentage of the total profit, such as Chobani which has committed to redistribute 10% of the profit to its employees.

EFF-NT is a typical solution that rewards agents for the profit they have contributed. Alternatively, EFF-ES rewards each agent an equal share of the total profit. [3]

1.2 Overview of the Results

We study two versions of the problem in relation to the information of a planner. For the first part of the paper, the planner has complete information about the network and individual values of the agents. The planner is interested in systematically selecting a path(s) and share the value of the selected path(s). We provide three axiomatic characterizations in this case.

Our first characterization relates to the independence of the timing of redistribution. Loosely speaking, given any node of a selected path, agents could be paid first based on the "subproblem" from the source to the node and then based on the other subproblem from the node to the original sinks. We require that the two allocations from the two subproblems add up to the allocation chosen for the original problem. Hence, agents are indifferent between receiving a lump-sum payment at the end of the process or receiving installments step by step (**sequential composition**). This rules out renegotiations of agents at intermediate stages.[4] Besides, we impose two basic axioms. First, in each problem with a path of positive value, at least one agent should receive a positive assignment (**non-triviality**). Second, a small change in the individual values should have a small impact on the allocation (**continuity**). These three axioms characterize the class of solutions that selects efficient path(s) and assigns to each agent a proportion of the value of the selected path(s) where the proportion is constant over all problems. For example, in a two-agent case, for each problem, 10% is always assigned to agent 1, and 90% to agent 2.

Our second characterization imposes a monotonicity axiom. It requires that no agent shall get hurt from the technology improvement which brings a new edge and destination to the existing network (**technology monotonicity**). This single axiom characterizes the class of solutions that select efficient path(s) and assigns to each agent a proportion of the value of the selected path(s) where the proportion depends only on this value. For example, in a two-agent case, for each problem, equal sharing between agents 1 and 2 whenever the total value is no less than 100, and for any incremental value above 100, 10% is assigned to agent 1 and 90% to agent 2.

Our third characterization relates to several independence principles with respect to certain network transformations. First, suppose that a step in a process consists of two substeps. Then whether the step is represented by one edge or two consecutive edges should have no impact on the allocation, as long as the value vectors of the two edges add up to that of the single edge (**split invariance**). Second, consider a problem with two subnetworks intersecting only at the source. Each path in one subnetwork is step-wise Pareto dominated by some other path in the other subnetwork. Then removing this subnetwork from the network should

[3]Equal sharing is often used in professional partnership (Encinosa et al. [20], and Farrell and Sctochmer [21]). See Bartling and von Siemens [5], Bose et al. [11], and Kobayashi et al. [40] for justifications under various situations.

[4]For generic problems where no two paths have the same value, this axiom implies a "dynamic consistency" studied in other settings, where once a path selection is made, it does not change at intermediate steps.

not affect the allocation (**irrelevance of dominated paths**). Third, suppose that after solving a problem, an undiscovered disjoint subnetwork connecting to the source is found to be available. To deal with this issue, one procedure is to cancel the initial allocation and select an allocation from the complete problem that augments the original network with the new subnetwork. An alternative procedure is to select an allocation from the simplified problem that augments an edge associated with the initial allocation with the new subnetwork. We require that the two procedures lead to the same allocation to avoid the dispute of agents (**parallel composition**). The three axioms, together with continuity, characterize a general class of "rationalizable" solution. A planner who adopts such a solution selects an efficient path(s) and divides the value in two steps. In each problem, the planner first redistributes the value of each path based on individual values at the path. This gives a set of potential redistributions. Second, the planner selects an optimal allocation based on the set of potential redistributions according to a partial order. To understand the partial order, imagine that the planner have some selection criteria such as Pareto dominance and "fairness". If there is one potential redistribution that dominates every other potential redistribution in the set by some criterion, then the planner selects this potential redistribution (the maximum redistribution). Otherwise, the planner selects one outside allocation that dominates each potential redistribution in the set and has the minimum "departure" from the set (the least upper bound). Thus, the sharing rule is rationalizable by this partial order. This order is incomplete since some redistributions may not be comparable by either criterion. When the partial order is complete, the sharing depends only on the value of the selected path(s). This subclass of rules is characterized by an additional axiom. Loosely speaking, it requires that in each problem with a generic "parallel network", the allocation depends only on the selected path (**irrelevance of parallel outside options**). The solutions obtained in the first two characterizations are special cases of this subclass.

Although the classes of solutions above are characterized from three different perspectives, they all reduce to EFF-ES if we further impose a basic fair requirement.[5] That is, agents having the same individual values should receive the same shares (**equal treatment of equals**).

For the second part of the paper, the planner has no information about the network or individual values of the agents, whereas the agents have full information. After a path is selected, the planner only observes the individual values of the agents in each step along the path. A sharing rule in this case may only depend on what the planner observes. We assume that the planner is interested in the implementation of an efficient path by choosing a sharing rule. We characterize a class of sharing rules that incentivize the agents to collectively select an efficient path.

Our first axiom requires that, for any two paths, the sharing rule assign weakly larger shares to at least k agents at the path with a larger value (***k*-majority**), where k is larger than half the number of agents. This is a stability notion since it guarantees an efficient path to be a Condorcet winner when agents vote for paths. Moreover, when agents sequentially vote

[5]In the third characterization, the subclass of rationalizable solutions satisfying irrelevance of parallel outside options, rather than the general class, reduces to EFF-ES.

at each node for edges to continue, it guarantees an efficient path to be chosen as a subgame perfect equilibrium under the k-majority rule. Second, we require that the identity of agents should not matter (**anonymity**). These two axioms imply that a sharing rule assigns the average value of a path to at least k agents. Furthermore, the equal sharing rule is characterized by adding one of several axioms ranging from sequential composition to other monotonicity axioms.

1.3 Literature review

While the axiomatic study of sharing rules has been widely discussed and applied in many settings, our general two-tiered framework that selects the path along with the sharing rule has not received much attention in the literature. Our model provides an abstract framework for more stylistic two-step problems such as the queuing problem (Chun [15]), the minimal cost spanning tree (Kar [39], Dutta and Kar [18], Bergantiños and Vidal-Puga [7], Hougaard et al. [27], Claus and Kleitman [16]) and other cost-sharing models (Juarez [34], Juarez [35], Juarez and Kumar [36]). In such problems, an ordering of agents (queuing), a network meeting certain conditions (spanning tree) or other decisions (selection of a group or a path) must be made and its benefit/cost divided among agents. In contrast with this literature, we do not assume that the most efficient path (subnetwork, subgraph, ordering or group of agents) is selected, but instead its selection is axiomatized along with a sharing rule.

The division of benefits/cost under exogenous network structures has been recently studied. For instance, allocations in linear river problems are studied by Ambec and Sprumont [2], Ni and Wang [54], and Ambec and Ehlers [1]. More complex river network problems are studied by Brink et al. [10], and Dong et al. [17]. The allocation of benefits in hierarchical ventures is studied by Hougaard et al. [26]. Values of cooperative game under permissible structures are studied by Brink [8], Brink et al. [9] and Gilles et al. [24]. Unlike our paper, this literature does not study the selection of the path, but instead assume that it is given.

The second part of the paper relates to the recent literature on the implementation of the efficient subgraphs in networks. For instance, Juarez and Kumar [36] implement the efficient subgraph in connection networks, Hougaard and Tvede [28, 29] implement the minimal cost spanning tree, and Juarez and Nitta [37] implement the efficient time allocation in production economies. Similar to our paper, the main objective of this literature is to select an "efficient path". The main difference is that we adopt an axiomatic approach that works for a variety of games, including sequential voting for a path.

Our model is the first to jointly address the issue of selecting paths and sharing the total value axiomatically for sequential problems where the information of the individual values of agents is available at every step.

2 The model under complete information

Fix a finite group of agents $N = \{1, ..., n\}$. We refer to a finite directed multigraph[6] with a unique source (possibly multiple sinks) and no cycles as a **network**. A network represents a sequential process in which each edge represents a feasible step to continue and each node faces a number of steps to choose from.[7] Let $\mathcal{G}$ be the set of networks. For each $G \in \mathcal{G}$, a **value function** v associated with G is a function that assigns to each edge e in G a vector $v(e) \in \mathbb{R}^n_+$ where for each $i \in N$, the i-th coordinate $v_i(e)$ represents the value of agent i generated in step e.[8] Let $\mathcal{V}^G$ be the set of all value functions associated with G. A **problem** is a pair (G, v) where $G \in \mathcal{G}$ and $v \in \mathcal{V}^G$. For each $x \in \mathbb{R}^n_+$, we simply use (e, x) to denote a problem where the network contains a single edge e and it is assigned the vector x of individual values. Let $\mathcal{P}$ be the set of all problems.

For each problem $(G, v) \in \mathcal{P}$, and for each edge e and each path L in G,[9] let $v_N(e) := \sum_{i \in N} v_i(e)$ be the **value of *e***, $v_N(L) := \sum_{e \in L} v_N(e)$ the **value of *L***, $v_N(G) := \max_{L \in G} v_N(L)$ the **value of *G***, and L is called **efficient** if $v_N(L) = v_N(G)$.

A **solution** is a pair (φ, μ) of functions on $\mathcal{P}$ such that for each $(G, v) \in \mathcal{P}$, $\varphi(G, v)$ is a nonempty subset of paths in G with the same value, and $\mu(G, v)$ is an element of $\mathbb{R}^n_+$ such that for each $L \in \varphi(G, v)$, $\sum_{i \in N} \mu_i(G, v) = v_N(L)$.[10]

Example 1 (Path selection rules and sharing rules)**.** *First, we discuss two general methods for path selection. Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a utility function over all vectors of individual values (associated with edges).*

i. [Additively separable rules] Consider a planner with a preference over paths that is additively separable across edges. The planner selects a path(s) that maximizes the sum of utilities of its edges. That is, for each $(G, v) \in \mathcal{P}$, an additively separable rule *selects a subset of* $\arg\max_{L \text{ in } G} \sum_{e \text{ in } L} u(v(e))$.[11] *In particular, if for each $x \in \mathbb{R}^n_+$, $u(x) = \sum_{i \in N} x_i$, it selects only efficient path(s). We denote by **EFF** a rule that selects efficient path(s).*

ii. [Myopic rules] Consider a planner myopically selects a path by maximizing the utilities of immediate steps (edges). Only when more than one immediate steps give the same utilities, the planner compares the utilities of additional steps. Formally, for each $(G, v) \in \mathcal{P}$, for each pair of paths L and L′, with consecutive edges $e^1, \dots, e^k$ in L and $\bar{e}^1, \dots, \bar{e}^{k'}$ in L′ respectively, define $L \succeq_{LEX} L'$ if for $z, z' \in \mathbb{R}^{\max\{k,k'\}}_+$ such that

[6] A multigraph is a graph where there can be multiple edges with the same end nodes.

[7] Throughout the paper, we assume that the labels of nodes and edges in each network do not have identity.

[8] For simplicity, we restrict to non-negative values to keep the interpretation of benefits throughout the rest of the paper. This assumption is without loss of generality, as all axioms and results in the paper can be adapted trivially to values in $\mathbb{R}$.

[9] In this paper, for each problem we only consider the paths from the source to a sink.

[10] We interpret $\mu_i(G, v)$ as the share of agent i. We assume that agents only care about their respective share. Note that such shares depend on the graph, the value function and the path(s) selected $\varphi(G, v)$. Furthermore, in case several paths are selected, the allocation does not depend on the which path is actually realized.

[11] In both (i) and (ii), tie-breaking rules are needed when there are multiple maximum paths with different values.

$$z_i = \begin{cases} u(v(e^i)) & \textit{if } i \in \{1, \ldots, k\}, \\ 0 & \textit{if } i \in \{k+1, \ldots, \max\{k, k'\}\}, \end{cases}$$

and

$$z'_i = \begin{cases} u(v(\bar{e}^i)) & \textit{if } i \in \{1, \ldots, k'\}, \\ 0 & \textit{if } i \in \{k'+1, \ldots, \max\{k, k'\}\}, \end{cases}$$

z lexicographically dominates z'. A myopic rule selects the maximum path(s) with respect to $\geq_{LEX}$ in each problem. We denote by MYO a myopic rule when for each $x \in \mathbb{R}^n_+$, $u(x) = \sum_{i \in N} x_i$.

Next, we discuss two traditional sharing rules for a given path selection rule φ.

iii. *[Equal-sharing] The equal sharing rule, denoted by ES, divides the value of the selected paths equally. That is, for each $(G, v) \in \mathcal{P}$, ES assigns each agent $\frac{v_N(L)}{n}$ where $L \in \varphi(G, v)$.*

iv. *[No transfer] The no transfer sharing rule, denoted by NT, assigns agents their average individual values over all selected paths.[12] That is, for each $(G, v) \in \mathcal{P}$, NT selects the allocation $\frac{1}{|\varphi(G,v)|} \sum_{L \in \varphi(G,v)} \sum_{e \text{ in } L} v(e)$. Note that when there is a unique path selected, NT simply assigns to each agent his individual value.*

Each combination of a path selection rule and a sharing rule is a solution. We use EFF-ES to denote a solution that combines an EFF path selection rule and the ES sharing rule, and use EFF-NT, MYO-ES, and MYO-NT to denote analogous solutions.

2.1 Sequential Composition

Consider the following axioms on a solution (φ, μ). We start with two basic axioms. The first one says that each problem with a path that has a positive value should positively benefit at least one agent. This is a basic efficiency property ruling out that all agents get nothing when it is possible to distribute something.

Non-triviality: Given $(G, v) \in \mathcal{P}$, if $v_N(G) > 0$, then $\mu_i(G, v) > 0$ for some $i \in N$.

Non-triviality is equivalent to not selecting paths with zero value when there is a path with a positive value. This axiom is satisfied by EFF-ES, EFF-NT, MYO-ES, and MYO-NT. Furthermore, it is satisfied by a solution with an additively separable path selection rule or a myopic path selection rule, as defined in Example 1, as long as the utility function u is such that for each $x \gneq \mathbf{0}$, $u(x) > u(\mathbf{0})$.

[12] Alternatively, we can interpret this rule as selecting a path with equal probability, and assigning agents their individual values at a path each time the path is selected.

The second basic axiom says that in each network, small changes in individual values should have small impact on the allocation. Such small changes often happen due to measurement errors and the axiom requires that the solution be robust with respect to such errors.

Continuity: Given $(G, v) \in \mathcal{P}$ and a sequence $\{v^t\}_{t=1}^{\infty}$ of elements of $\mathcal{V}^G$, if for each e in G, $\lim_{t\to\infty} v^t(e) = v(e)$, then $\lim_{t\to\infty} \mu(G, v^t) = \mu(G, v)$.

Continuity is a standard topological property in the fair allocation literature. In our model, it has strong implications on both path selection rules and sharing rules. Both MYO-ES and EFF-NT violate *continuity*. To see that, consider the problems in Figure 2. In the left problem, for each $\epsilon > 0$, MYO-ES selects the top path and allocates $(2 + \frac{\epsilon}{2}, 2 + \frac{\epsilon}{2})$; for each $\epsilon < 0$, it selects the bottom path and allocates $(1, 1)$. In the right problem, for each $\epsilon > 0$, EFF-NT selects the top path and allocates $(2 + \epsilon, 0)$; for each $\epsilon < 0$, it selects the lower path and allocates $(0, 2)$.

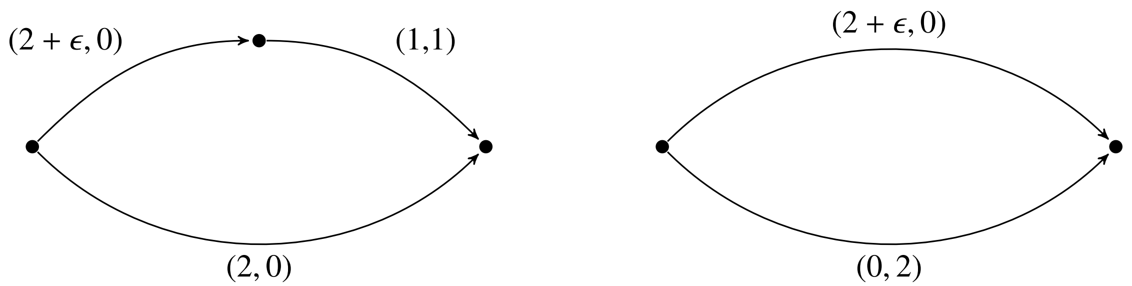

Figure 2: Problems illustrating that MYO-ES and EFF-NT violate *continuity*.

On the other hand, there is a large class of solutions meeting *continuity*. The following example provides an interesting class of such solutions.

Example 2 (Solutions satisfying *continuity*)**.** *Consider a solution that selects paths depending on values of the paths, and divides the value of the selected path(s) continuously depending on individual values of the agents in the network. For example, in a network with T paths, the path with the T^*-th largest value is selected, where $T^* \leq T$ may depend on T. When $T^* = 1$, efficient paths are selected. When $T^* = T$, the least efficient paths are selected. When $T^* = \lfloor \frac{T+1}{2} \rfloor$,[13] the paths with median value are selected whenever T is odd. After the selection of a path(s), each agent receives a share in proportion to the sum of his individual values over all edges in the network.*

Formally, let φ be such that for each $(G, v) \in \mathcal{P}$ with paths $L_1, \ldots, L_T$ in G, $L \in \varphi(G, v)$ if and only if $v_N(L)$ is the T^*-th largest number among $v_N(L_1), \ldots, v_N(L_T)$. Let μ be such that for each $(G, v) \in \mathcal{P}$ and each $i \in N$, $\mu_i(G, v) = \frac{\sum_{e \text{ in } G} v_i(e)}{\sum_{j \in N} \sum_{e \text{ in } G} v_j(e)} v_N(L)$ where $L \in \varphi(G, v)$. The solution (φ, μ) satisfies continuity.

[13] For each $c \in \mathbb{R}$, $\lfloor c \rfloor$ is the largest integer no more than c.

Our main axiom in this section relates to the independence of the timing of redistribution. Consider a process with a selected plan (path) involving at least two steps. The accounting practice may require that interim payment be made in the middle of the plan. Thus, agents receive two installments where the first installment is based on the "subproblem" from the beginning to the middle of the plan, and the second one is based on the other subproblem from the middle to the end of the plan. The axiom requires that agents are indifferent between receiving a lump-sum payment at the end of the process or receiving the two installments. This rules out renegotiation of agents at intermediate stages.

To formally define subproblems, let $(G, v) \in \mathcal{P}$ and let d be a node in G which is neither the source nor a sink. We denote by $G|_d$ the maximum sub-network with d being the sink, i.e., the sub-network which contains all the paths from the original source to node d, and let $v|_d$ be the restriction of v to the edges in $G|_d$. Analogously, we denote by $G|^d$ the maximum sub-network with d being the source, i.e., the sub-network which contains all the paths from node d to original sinks, and let $v|^d$ be the restriction of v to the edges in $G|^d$. In the network depicted in Figure 3, we illustrate $G|_d$ as the dotted sub-network and $G|^d$ the dashed sub-network.

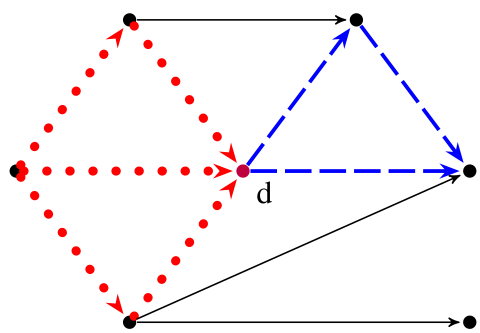

Figure 3: The dotted and dashed subnetworks illustrate $G|_d$ and $G|^d$.

Sequential composition: Given $(G, v) \in \mathcal{P}$ and a path $L \in \varphi(G, v)$. Let d be a node in L which is neither the source nor a sink. Then $\mu(G, v) = \mu(G|_d, v|_d) + \mu(G|^d, v|^d)$.

There is a large class of rules meeting *sequential composition*. All the solutions introduced in Example 1,[14] including EFF-ES, EFF-NT, MYO-ES, and MYO-NT, satisfy this axiom.

We now characterize the solutions that satisfy the three axioms above.

Theorem 1. *A solution (φ, μ) satisfies non-triviality, continuity, and sequential composition if and only if φ only selects efficient path(s), and there is $\alpha \in \mathbb{R}^n_+$ with $\sum_{i \in N} \alpha_i = 1$ such that for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = v_N(G)\alpha$.*

[14]For all general solutions in Example 1, in non-generic cases where there are multiple maximum paths with different values, a tie-breaking rule which selects the same path(s) in subproblems is needed.

One implication, perhaps surprising, is that the efficient path selection is guaranteed with three axioms that are seemingly unrelated to efficiency. Another interesting implication is that the sharing is independent of the network configuration and individual values of the agents.

The characterization is tight. Dropping *sequential composition*, a class of solutions meeting *non-triviality* and *continuity* will be discussed in Theorem 2. Dropping *continuity*, EFF-NT, MYO-ES, and MYO-NT meet *non-triviality* and *sequential composition*. Dropping *non-triviality*, the solution that selects the path(s) with the smallest value and divides the value equally among the agents meets *continuity* and *sequential composition*.

It follows readily from Theorem 1 that if in addition the solution satisfies a basic fairness requirement that agents with the same individual values be assigned the same shares, then μ is the ES rule.

Equal treatment of equals: For each $(G, v) \in \mathcal{P}$ and each pair $i, j \in N$, if for each e in G, $v_i(e) = v_j(e)$, then $\mu_i(G, v) = \mu_j(G, v)$.

Corollary 1. *A solution (φ, μ) satisfies non-triviality, continuity, sequential composition, and equal treatment of equals if and only if φ only selects efficient path(s), and for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_N(G)}{n}\boldsymbol{1}$.*

2.2 Technology Monotonicity

We now propose a monotonicity axiom. It requires that no agent gets worse off from a technology improvement which brings new steps and destinations to the existing network. Formally, given $(G, v) \in \mathcal{P}$, we say that $(G', v') \in \mathcal{P}$ is a **technology improvement** of (G, v) if (i) G' is constructed either by adding a parallel edge connecting two nodes in G or by adding a sink and an edge going out from a node in G to this sink, and (ii) for each e in G, $v'(e) = v(e)$.

Technology monotonicity: For each $(G, v), (G', v') \in \mathcal{P}$, if (G', v') is a technology improvement of (G, v), then $\mu(G', v') \geq \mu(G, v)$.

Theorem 2. *A solution (φ, μ) satisfies technology monotonicity if and only if φ only selects efficient path(s), and there is a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}^n_+$ such that for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = f(v_N(G))$.*

It can be readily shown that the non-decreasing function f in Theorem 2 must be continuous, so *continuity* is implied by *technology monotonicity*. In addition, if we require *equal treatment of equals*, then μ must be the ES rule.

Corollary 2. *A solution (φ, μ) satisfies technology monotonicity and equal treatment of equals if and only if φ only selects efficient path(s), and for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_N(G)}{n}\boldsymbol{1}$.*

2.3 Independence with respect to network transformation

We now consider some independence principles related to several types of network transformation.

Suppose that a step in a process consists of several substeps, and each agent's value generated in the step is the sum of those generated in the substeps. In the network representation of this process, such a step can be represented by a single edge that is assigned the sum of the value vectors. Alternatively, it can be represented by several consecutive edges, each assigned a corresponding substep value vector. We require that the two ways of formulating this problem have no impact on the allocation.

Split invariance: Let $G, G' \in \mathcal{G}$ be such that G' is constructed by splitting an edge e in G into two consecutive edges e_1 and e_2. If $v \in \mathcal{V}^G$ and $v' \in \mathcal{V}^{G'}$ are such that $v(e) = v'(e_1) + v'(e_2)$, and for each e' in G other than e, $v(e') = v'(e')$, then $\mu(G, v) = \mu(G', v')$.

Equivalently, this requirement can be formulated as the merge invariance axiom: In each problem, the allocation should not be affected by merging two consecutive edges, with no ingoing and outgoing edges in between, into one edge and assigning to it the sum of the value vectors of both edges. An immediate consequence of *split invariance* is that in each problem where the network has a single path, the allocation only depends on the sum of the value vectors over all edges at the path.

Our *split invariance* is different from the split and merge proofness axiom in the fair allocation literature (Banker [4], Moulin [44], de Frutos [23], Ju [31], Sprumont [59], and Chun [14]).[15] In these problems, the resource is fixed, and a sharing rule is required to be immune to the split of an agent into several participation units or the merge of several agents into one participation unit.

The next axiom relates to network reduction due to an efficiency concern. Imagine that in a problem, for each path in one "component" of the network, there is an outside path, with the same number edges, in which the value vectors Pareto dominate those in the former path edge by edge. In this case, we require that simplifying the problem by removing the dominated component should not affect the allocation.

Formally, let $(G, v), (G', v') \in \mathcal{P}$ be given. We use $(G, v) \cup (G', v')$ to denote the problem given by combing the sources of G and G' into one source, and assigning the edges in the combined network the same value vectors as in the respective individual problems. An example is given in Figure 4. We say that a path L in (G, v) is **stepwise dominated** by a path L' in (G', v') if L and L' have the same number of edges, say $T \in \mathbb{N}$, and for each pair of the t-th edges e_t in L and e'_t in L', where $t \in \{1, ..., T\}$, $v(e_k) \gtrsim v'(e'_k)$.

Irrelevance of dominated paths: For each pair $(G, v), (G', v') \in \mathcal{P}$, if each path in (G, v) is stepwise dominated by a path in (G', v'), then $\mu((G, v) \cup (G', v')) = \mu(G', v')$.

[15] *Split or merge invariance* is investigated in a unified framework of allocation problems by Ju, Miyagawa, and Sakai [33].

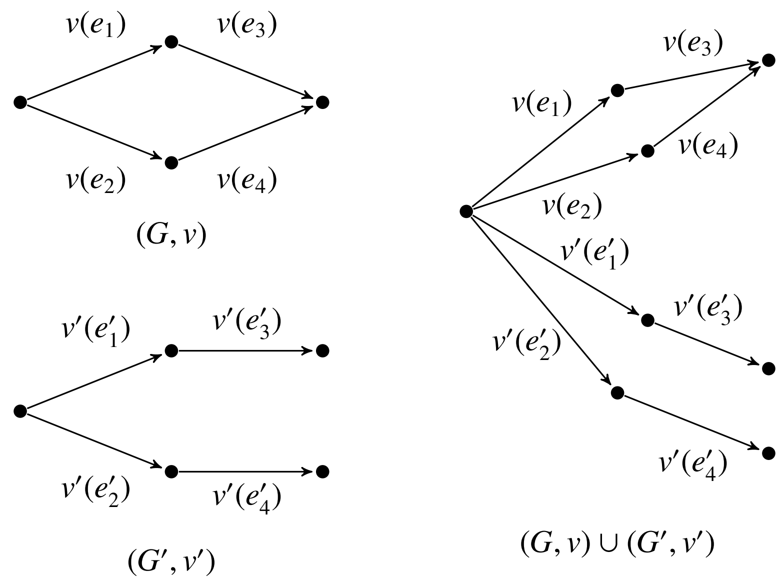

Figure 4: Union of two problems

Suppose that after a problem has been solved, a new component of the network is found to be available. To deal with this issue, one procedure is to cancel the initial allocation, and then find a new allocation based on the complete problem that augments the original network with the new component. An alternative procedure is to save the initial allocation, and then select a new allocation based on the simplified problem that combines the new component with an edge associated with the initial allocation. We require that both ways of dealing with the issue lead to the same allocations so that agents have no dispute on which procedure is better.

Parallel composition: For each pair $(G, v), (G', v') \in \mathcal{P}$, $\mu((G, v) \cup (G', v')) = \mu((e, \mu(G, v)) \cup (G', v'))$.

Parallel composition basically allows that a problem can be solved part by part. This is reminiscent of the "lower composition" axiom in the rationing model (Young [63]) and the "step by step negotiation" property in the axiomatic bargaining model (Kalai [38]).

Recall that the solutions characterized in the previous sections for each problem select efficient path(s) and divide the value of the problem regardless of the network and individual values. Such solutions trivially satisfy the three independence axioms. We now provide an example meeting the axioms and taking into account individual values in a desirable way. For the ease of illustration, we consider the case when $n = 2$ in Example 3. The general case is discussed in Appendix A.2.

Example 3 (Solutions that depend on individual values)**.** *Consider a planner who selects the efficient path(s) in a problem and divides the value in two steps. First, for each path in the problem, the planner obtains a potential allocation that assigns to each agent the sum of his* *individual values over all edges at the path. This gives a set of potential allocations induced by all paths. Second, the planner selects an allocation based on the set of all potential allocations according to the following three criteria of domination.*

The first criterion is Pareto domination. An allocation is dominated by another allocation if it is Pareto dominated by the other allocation. The second criterion relates to egalitarianism. An allocation is dominated by another allocation if the latter has the same total value as the former and is a convex combination of the former and the equal sharing allocation. The third criterion relates to transitivity. If by either of the above two criteria, one allocation is dominated by another allocation which is in turn dominated by a third one, then the first allocation is also dominated by the third one.

The planner selects one potential allocation if every other potential allocation is dominated by this one according to some criterion. Otherwise, the planner selects an outside allocation by which each potential allocation is dominated. There are typically more than one choice for the outside allocation, and the planner selects the one with the minimum "departure" from the set of potential allocations, i.e., the one dominated by all the other choices.[16]

The choice of the planner can be understood as rationalizble by a "preference order" over all allocations which is determined by the three criteria. This preference order is incomplete since some allocations may not be comparable by either criterion. The selected allocation is exactly the least upper bound of the set of potential allocations according to this partial order. Formally, let $\succsim$ *be the binary relation on the set* $\mathbb{R}^2_+$ *of all allocations such that for each pair* $x, y \in \mathbb{R}^2_+$, $x \succsim y$ *if there are* $z \in \mathbb{R}^2_+$ *and* $\lambda \in [0,1]$ *such that*

$$z_1 + z_2 = y_1 + y_2, \text{ and } x \geq z = \lambda y + (1-\lambda)\frac{y_1 + y_2}{n}\boldsymbol{1}. \tag{1}$$

Note that first for each pair $x, y \in \mathbb{R}^2_+$, *if* $x \geq y$, *then* $x \succsim y$ *(the first criterion). Second, if* $x_1 + x_2 = y_1 + y_2$, *then* $x \succsim y$ *if and only if* x *is a convex combination of* y *and* $\frac{x_1+x_2}{2}\boldsymbol{1}$ *(the second criterion). The partial order* $\succsim$ *is the transitive closure (the third criterion) of the binary relation defined by the previous two properties. Moreover, it can be shown that* $(\succsim, \mathbb{R}^2_+)$ *is a join-semilattice, so that the least upper bound of a finite number of allocations always exist. Let* (φ, μ) *be such that for each* $(G, v) \in \mathcal{P}$, φ *selects all efficient path(s) in* (G, v), *and* $\mu(G, v) = \max_{\succsim}\{\sum_{e \text{ in } L} v(e) : L \text{ in } G\}$. *This solution satisfies* split invariance, irrelevance of dominated paths, parallel composition, *and* continuity. *The proofs are given in Appendix A.2.*

For the problem in Figure 5(a), there is only one path, giving rise to one potential allocations: $(0,2)$. *Hence,* $(0,2)$ *is selected. For the problem in Figure 5(b), the two potential allocations are* $(0,2)$ *and* $(1,1)$. *Clearly,* $(1,1)$ *is more egalitarian than* $(0,2)$. *Hence,* $(1,1)$ *is selected. For the problem in Figure 5(c), the two potential allocations are* $(0,2)$ *and* $(2,0)$. *Neither is dominated by the other. Hence, the planner needs to find an outside allocation. It is not difficult to check that* $(1,1)$ *is the least upper bound of* $\{(0,2),(2,0)\}$. *Hence,* $(1,1)$ *is*

[16] The existence of the outside allocation with the minimum departure is guaranteed. See the proof in Appendix A.2.

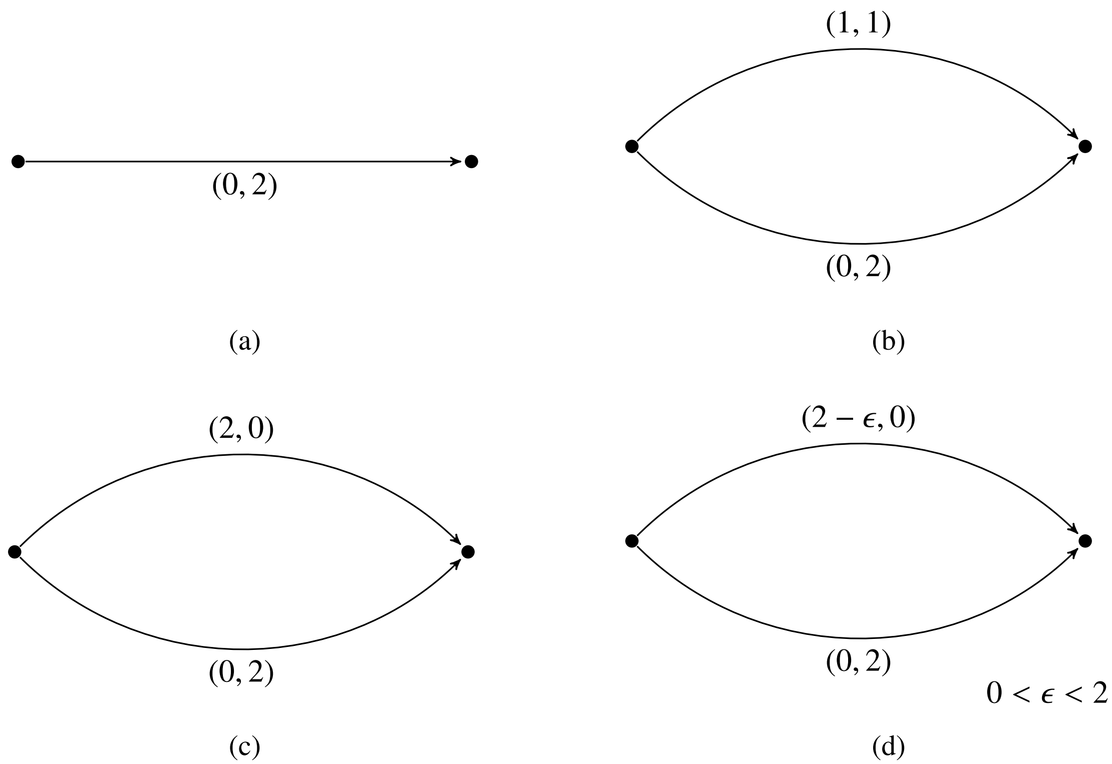

Figure 5: Illustration of the rationalizable solution

selected. For the problem in Figure 5(d), the two potential allocations are $(0, 2)$ *and* $(2-\epsilon, 0)$*. Since* $0 < \epsilon < 2$*, neither allocation is dominated by the other. Both potential allocations are dominated by* $(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2})$*, because it is more egalitarian than* $(0, 2)$*, and it Pareto dominates* $(1-\frac{\epsilon}{2}, 1-\frac{\epsilon}{2})$ *which is more egalitarian than* $(2-\epsilon, 0)$*. Moreover, it is the least upper bound of* $\{(0, 2), (2-\epsilon, 0)\}$*. Figure 6 shows that the intersection of the upper contour sets of* $(0, 2)$ *and* $(2-\epsilon, 0)$ *is the upper contour set of* $(1-\frac{\epsilon}{2}, 1-\frac{\epsilon}{2})$*. Hence, the planner selects* $(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2})$*.*

The problems in Figure 5 clearly illustrates how the sharing can be jointly determined by individual values at all paths (including inefficient ones). In Figure 5(a) where there is only one path, the planner conducts no transfer. In Figure 5(c), an additional path becomes available in Figure 5(c), at which the individual values of the two agents are reversed. Then the planner selects the equal sharing allocation as a compromise between the two potential allocations favoring different agents. In Figure 5(d), even if agent 1's value at the additional path is reduced by ϵ*, so that the path becomes inefficient, the planner still takes the individual values of this path into account and selects* $(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2})$*. When* ϵ *goes to* 0*, the planner treats them in equal and selects* $(1, 1)$ *eventually.*

In fact, the three independence axioms together with *continuity* characterize a class of "rationalizable" solutions generalizing that in Example 3. The generalized solution allows a redistribution to be applied on the set of potential allocations. Each solution in this general class selects only efficient paths, and divides the value of each problem based on a "re-

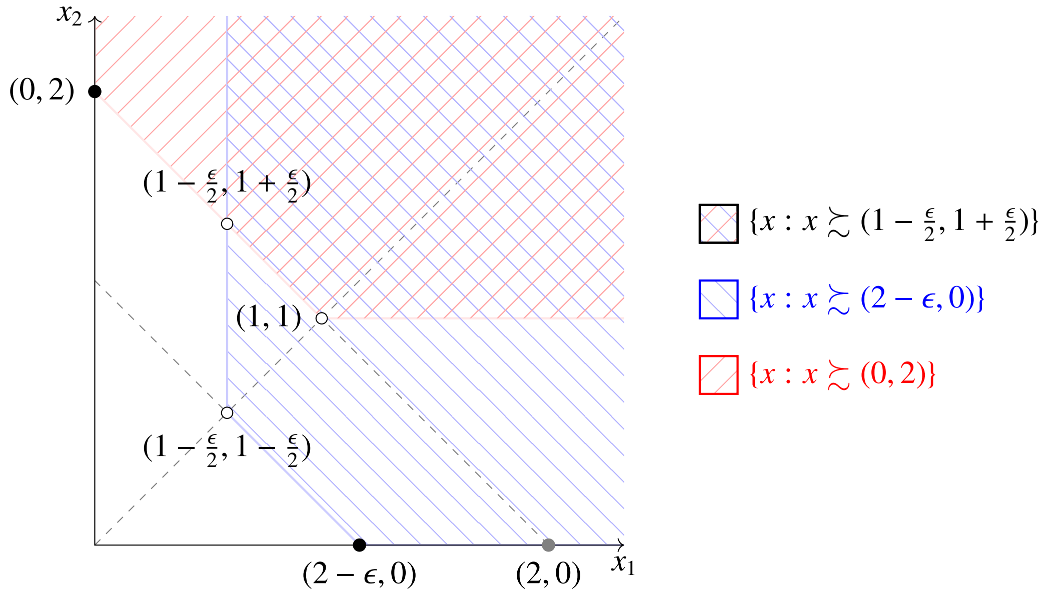

Figure 6: The upper contour sets of $(2-\epsilon, 0)$, $(0, 2)$, and $(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2})$.

distribution function" and a partial order over the set of redistributed allocations. In each problem, for each path, the planner applies the redistribution function to the no transfer allocation given by the path and obtains a potential redistributed allocation. Then the optimal allocation is selected, according to the partial order, based on the set of potential redistributed allocations given by all paths.

Formally, $r : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a **redistribution function** if for each $x \in \mathbb{R}^n_+$, $\sum\limits_{i\in N} r_i(x) = \sum\limits_{i\in N} x_i$, and $r(r(x)) = r(x)$. Given a partial order $\succsim$ on a set $S \subseteq \mathbb{R}^n_+$ and a subset S' of S, we denote by $\max\limits_{\succsim} S'$ the join of S' when it exists. Given a redistribution function r and a partial order $\succsim$ on $r(\mathbb{R}^n_+)$, a solution (φ, μ) is said to be **(r, $\succsim$) - rationalizable** if (1) for each pair $x, y \in \mathbb{R}^n_+$ with $x \geq y$, $r(x) \succsim r(y)$, (2) $(r(\mathbb{R}^n_+), \succsim)$ is a join-semilattice, and (3) for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = x^* := \max\limits_{\succsim}\{r(\sum\limits_{e \text{ in } L} v(e)) : L \text{ in } G\}$, and $\sum\limits_{i\in N} x^*_i = v_N(L)$ where $L \in \varphi(G, v)$. Moreover, (φ, μ) is said to be **continuously (r, $\succsim$) - rationalizable** if (φ, μ) is **(r, $\succsim$) - rationalizable**, r is continuous and $g : r(\mathbb{R}^n_+)^2 \to r(\mathbb{R}^n_+)$, defined by setting for each $(x, y) \in r(\mathbb{R}^n_+)^2$, $g(x, y) = \max\limits_{\succsim}\{x, y\}$, is continuous. Note that if (φ, μ) is continuously $(r, \succsim)$ - rationalizable, then $\succsim$ is continuous.

All the solutions in the previous sections are rationalizable. For example, let $f : \mathbb{R}_+ \to \mathbb{R}^n_+$ be a non-decreasing function and (φ, μ) be a solution in Theorem 2. Define $r : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by setting for each $x \in \mathbb{R}^n_+$, $r(x) = f(\sum\limits_{i\in N} x_i)$. Define $\succsim$ as a linear order on $r(\mathbb{R}^n_+)$ such that for each pair $x, y \in r(\mathbb{R}^n_+)$, $x \succsim y$ if $\sum\limits_{i\in N} x_i \geq \sum\limits_{i\in N} y_i$. Clearly, (φ, μ) is $(r, \succsim)$ - rationalizable. If f is also continuous, then (φ, μ) is continuously $(r, \succsim)$ - rationalizable. This is the case for the solutions in Theorem 1 where f assigns to each agent a fixed share of total values. Moreover, when $n = 2$, if $r : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is the identity function and $\succsim$ is defined by condition (1), then the solution in Example 3 is continuously $(r, \succsim)$ - rationalizable. The proof with general n can be found in the Appendix A.2.

Theorem 3. *A solution (φ, μ) satisfies split invariance, irrelevance of dominated paths, independence, and continuity if and only if φ only selects efficient path(s), and there exist a redistribution function r and a partial order $\succsim$ on $r(\mathbb{R}^n_+)$ such that φ only select efficient path(s) and (φ, μ) is continuously $(r, \succsim)$ - rationalizable.*

When the associated partial order of a solution in Theorem 3 is complete, it disregards individual values in each problem and selects an allocation based only on the value of the problem, like those in the previous sections. Such solutions are singled out from the rationalizable family by the next axiom which essentially implies the completeness of associated partial orders. This axiom says that in each problem with only parallel paths, the allocation depends only on one of the paths.

Irrelevance of parallel outside options: For each $T \in \mathbb{N}$ and each set $\{(G^t, v^t) \in \mathcal{P}$: G^t consists of a single path, $t = 1, ..., T\}$, there is $t' \in \{1, ..., T\}$ such that $\mu(\bigcup_{t=1}^{T}(G^t, v^t)) = \mu(G^{t'}, v^{t'})$.

Generically, if no two paths in the network have the same value, the allocation depends only on the path selected. When this axiom is imposed in addition to those in Theorem 3, *split invariance* and *parallel composition* become redundant.

Theorem 4. *A solution (φ, μ) satisfies irrelevance of dominated paths, irrelevance of parallel outside options, and continuity if and only if φ only selects efficient path(s), and there is a continuous function $f : \mathbb{R}_+ \to \mathbb{R}^n_+$ such that for each $c \in \mathbb{R}_+$, $\sum_{i \in N} f_i(c) = c$, and for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = f(v_N(G))$.*

The characterizations in both Theorem 3 and 4 are tight. Dropping *split invariance*, consider the solution in Appendix A.2 (an n-agent version of Example 3) with a "redistribution function" r that depends on not only the value vector but also on the number of edges in the path that generate the value vector. That is, for each $(G, v) \in \mathcal{P}$ and each L in G, if the number of edges in L is 2, then r divides equally among agents in group S_1 ($S_1 \subseteq N$) the aggregate values generated by agents of the other group S_2 ($S_2 = N \setminus S_1$) along the path, and among agents in S_2 those generated by agents of S_1; if the number of edges in L is other than 2, then r agrees with the redistribution function in Appendix A.2. The solution with modified redistribution function satisfies *irrelevance of dominated paths*, *parallel composition*, and *continuity*. Dropping *irrelevance of dominated paths*, the solution that selects all the paths with the smallest values and divides the value equally among the agents satisfies *split invariance*, *parallel composition*, *irrelevance of parallel outside options*, and *continuity*. Dropping *parallel composition*, consider the solution that selects all the efficient paths and divides the value to each agent in proportion to the maximum aggregate values he can generate over all the paths in a problem. This solution satisfies *split invariance*, *irrelevance of dominated paths*, and *continuity*. Dropping *continuity*, consider a path selection rule that picks in the first round the paths among all the efficient ones that maximizes agent 1's individual aggregate values. Then, it picks among the selected ones in the first round those maximizing agent 2's individual aggregate values, and so on and so forth, to the n-th round. The solution that adopts this path selection rule and assigns to the agents their individual aggregate values along the selected path(s) satisfies *split invariance*, *irrelevance of dominated paths*, *parallel composition*, and *irrelevance of parallel outside options*. Dropping *irrelevance of parallel outside options*, the solution in Example 3 satisfies *irrelevance of dominated paths* and *continuity*.

Lastly, as in the previous sections, *equal treatment of equals* single outs the equal sharing rule.

Corollary 3. *A solution (φ, μ) satisfies irrelevance of dominated paths, irrelevance of parallel outside options, continuity, and equal treatment of equals if and only if φ only selects efficient path(s), and for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_N(G)}{n}\boldsymbol{1}$.*

3 The model under incomplete information

Consider now that the planner has no information about the network and individual values of agents, while the agents have full information. Thus, the planner may *delegate* agents to collectively choose a path.[17] After a path is realized, the planner observes individual values at each edge only for that path and uses a predetermined sharing rule to redistribute the value of the realized path. The agents collectively choose a path taking into account their information and the redistribution scheme. We assume that the planner's objective is to implement an efficient path, which is desirable from both the positive and normative sides (Theorems 1-4). The planner chooses a sharing rule that aligns the objective of the planner with the incentives of the agents.

For example, consider profit sharing in a company. With limited information, a manager (planner) may want the employees (agents) to collectively decide the priority of projects. A compensation scheme (sharing rule) is set beforehand and depend only on the potential profits brought by the agents (individual values). The manager selects a compensation scheme to incentivize the employees to choose a profit-maximizing priority of projects.

Different from the previous setting, the path selection is irrelevant since the planner has delegated the selection of path to the agents. Moreover, the sharing rule only depends on sequences of individual values since the planner only observes the realized path. Formally, a **path** is a finite sequence $\{x^t\}_{t=1}^T$ of elements of $\mathbb{R}^n_+$, where $T \in \mathbb{N}$ is the number of edges in

[17] A more classical way of implementing a path is to elicit the information from agents. Section 3.1 briefly discusses this issue and ways to solve it. We focus here on a more decentralized setting where a complex network may be difficult to elicit. Our analysis also applies to the scenario where the decision has to made by agents regardless of the information of the planner. For example, a planner in charge of building a connected public facility may only choose a sharing rule and have to delegate the agents to collectively choose (e.g. by voting) a path.

the path and x^t is the vector of individual values of the tth edge. We denote a typical path by l, and the set of all paths by $\mathcal{L}$. For each path $l = \{x^t\}_{t=1}^T$ in $\mathcal{L}$, $K \in \mathbb{N}$, let $l_N := \sum_{t=1}^{T} \sum_{i \in N} x_i^t$ be the **value of l**. A **sharing rule** here is a function $\mu : \mathcal{L} \to \mathbb{R}_+^n$ such that for each $l \in \mathcal{L}$, $\sum_{i \in N} \mu_i(l) = l_N$.

Different from the classical implementation literature, we follow an axiomatic approach. Our axioms ensures a variety of games to implement an efficient path.

The first axiom says that for each pair of paths, at least k agents should receive weakly more assignments at the path with a larger value, where k is more than half of the number of the agents. This is a stability requirement which is necessary to guarantee the selection of the efficient path when agents are delegated to make a decision using a Condorcet voting rule and other k-majoritarian rules, as we will see in Applications 1 and 2.

***k*-majority** ($\frac{n}{2} < k \leq n$): For each pair $l, l' \in \mathcal{L}$, if $l_N \geq l'_N$, then there is $N' \subset N$ such that $|N'| \geq k$ and for each $i \in N'$, $\mu_i(l) \geq \mu_i(l')$.

A sharing rule meeting this axiom will be referred to as ***k*-majoritarian**. When $k = \lfloor \frac{n}{2} \rfloor + 1$, a majority of agents always prefer a more efficient path. When $k = n$, all agents prefer a more efficient path.[18]

Application 1 (Path selection using a Condorcet Social Choice Function)**.** *Suppose that a path is selected using a Social Choice Function (SCF) that satisfies the Condorcet property. That is, the SCF elects a Condorcet winner when available.*[19]

For each agent, a sharing rule determines an assignment for each path, which induces an ordinal ranking over paths. Each $k-$majoritarian sharing rule guarantees that an efficient path is a Condorcet winner regardless of the network and individual values. Therefore, each SCF that meets the Condorcet property picks an efficient path. Conversely, if a rule is not $k-$majoritarian for each $k > \frac{n}{2}$, then a SCF that meets the Condocret property may pick an inefficient path.

Application 2 (Sequential voting)**.** *Suppose that agents vote step by step to decide a path. For instance, agents vote sequentially for an route of a railway or other connected public facility. More precisely, consider the dynamic game of complete information where at every node agents vote on the direction to continue. A path is selected using a k-majoritarian voting rule at every node (i.e., if a direction receives at least k-votes, then it is chosen). The payoff of the agents is given by applying the sharing rule to the realized path.*

[18] When $k = n$, we have an *n-majority*. It is reminiscent to the property of Pareto Nash Implementation (PNI) in connection networks by Juarez and Kumar [36], that requires that the efficient Nash equilibrium be preferred by all the agents over any other equilibrium.

[19] Formally, given the set of objects $\mathcal{M}$, let $\mathcal{R}$ be the set of ordinal preferences over $\mathcal{M}$. A social choice function $\Psi : \mathcal{R}^N \to \mathcal{M}$ meets the Condorcet property if for the preference profile $\succeq = (\succeq_1, \dots, \succeq_n) \in \mathcal{R}^N$ there exists $l^* \in M$ such that for any $l \in M$, $|\{i \in N | l^* \succ l\}| > \frac{n}{2}$, then $\Psi(\succeq) = \{l^*\}$. A large class of SCFs that satisfy this property are discussed in Moulin [46].

A k-majoritarian sharing rule will always implement an efficient path as a strong subgame perfect Nash equilibrium. Such an equilibrium is unique when there are no two paths with the same values.

Finally, the implementation of the efficient path(s) is robust to voting only at a subset of nodes. It is also robust to incomplete information of the agents about at which nodes voting will occur.[20]

The following example shows some sharing rules satisfying *k-majority*.

Example 4 (Sharing rules satisfying *k-majority*)**.**

i. *Fix a priority group of at least k agents. Members of the priority group always equally share the value of a path while the others get nothing. Formally, let $S \subseteq N$ be such that $|S| \geq k$. For each path l and each $i \in N$,*

$$\mu_i(l) = \begin{cases} \frac{l_N}{|S|} & \text{if } i \in S, \\ 0 & \text{if } i \in N \setminus S. \end{cases}$$

ii. *Consider a sharing rule that depends on agents' individual values rather than their names. For each realized path, agents are classified into two groups: (1) the top group with the top m agents who have the highest m accumulated individual values along the path, and (2) the bottom group with the remaining $n - m$ agents. Each top group member shares equally the value of the realized path. The bottom group receives nothing. Formally, let $m \in \mathbb{N}$ be such that $k \leq m \leq n$. Given $l = \{x^t\}_{t=1}^T$ and $i \in N$, let $l_i := \sum_t l_i^t$ be the accumulated value of agent i at the path l. For each path l with $l_{i_1} \geq l_{i_2} \geq \cdots \geq l_{i_n}$ (break indifferences arbitrarily) and each $i_j \in N$,*

$$\mu_{i_j}(l) = \begin{cases} \frac{l_N}{m} & \text{if } 1 \leq j \leq m, \\ 0 & \text{if } m+1 \leq j \leq n. \end{cases}$$

The rule μ satisfies k-majority, let l, l' be two paths such that $l_N \geq l'_N$. Then, the m agents in the top group at l is assigned no less than at l'.

iii. *The above rule can be generalized to admit more than two groups. For example, for each realized path, agents are classified into three groups: (1) the top group with the top q agents who have the highest q accumulated individual values, and analogously, (2) the middle group with the subsequent m agents, where $m \geq k$, and (3) the bottom group with the remaining $n - q - m$ agents. Agents in the same group receive the same shares. Each middle group member receives the average value of the realized path.*

[20] This problem often occurs when voters want to re-evaluate a chosen route after it has been partially built, for instance in projects that take several years to construct, like the rail in Honolulu, from Ewa side to Waikiki via Downtown. While the decision to build the rail from Honolulu to Ewa was approved by voters, their construction stopped in the middle to re-evaluate the route chosen and to be confirmed by the voters before further spending on the project occurs.

Each bottom group member receives the average value subtracted by an amount that is based on the "gaps" between top and middle groups as well as middle and bottom groups. Each top group member receives the average value adjusted up by the amount subtracted from the bottom group.

Formally, let q, m be non-negative integers such that $q + m \leq n$ and $m \geq k$. Let $\lambda : \mathbb{R}_+ \to [0, 1]$ be such that $\lambda(0) = 0$ and $\lim_{c\to\infty} \lambda(c) = 1$. For each path l with $l_{i_1} \geq \cdots \geq l_{i_n}$ (break indifferences arbitrarily) and each $i_j \in N$,

$$\mu_{i_j}(l) = \begin{cases} \frac{l_N}{n} + \frac{n-q-m}{q}\lambda((l_{i_q} - l_{i_{q+1}})(l_{i_{q+m}} - l_{i_{q+m+1}}))\frac{l_N}{n} & \text{if } 1 \leq j \leq q, \\ \frac{l_N}{n} & \text{if } q+1 \leq j \leq q+m, \\ \frac{l_N}{n} - \lambda((l_{i_q} - l_{i_{q+1}})(l_{i_{q+m}} - l_{i_{q+m+1}}))\frac{l_N}{n} & \text{if } q+m+1 \leq j \leq n. \end{cases}$$

If $m < 1$, then for each path l and each $i \in N$, $\mu_i(l) = \frac{l_N}{n}$.[21]

We focus on sharing rules that are independent of the agents' names. For instance, redistribution schemes for sharing benefit of public facilities are often anonymous.

For permutation π of N and each path $l = \{x^t\}_{t=1}^T$ in $\mathcal{L}$, where $T \in \mathbb{N}$, let $l^\pi \in \mathcal{L}$ be such that for each $t \in \{1, ..., T\}$ and each $i \in N$, $x^t_i = x^t_{\pi(i)}$. Our second axiom formalizes the requirement that a sharing rule be independent of the names of the agents.

Anonymity: For each permutation π of N, each $l \in \mathcal{L}$, and each $i \in N$, $\mu_i(l) = \mu_{\pi(i)}(l^\pi)$.

Anonymity is a desirable property, as agents have symmetric information. Thus, agents should not be discriminated solely on the basis of names. On the other hand, this does not prevent agents from being discriminated based on their individual values. Example 4 (i) does not satisfy *anonymity* while Examples 4 (ii) and (iii) do.

We now move to the main result of this section. The combination of the above two axioms lead to an important restriction on the sharing rules: at least k agents should be allocated the average of the value of the path.

Proposition 1. *If μ satisfies k-majority and anonymity, then for each $l \in \mathcal{L}$, there is $N' \subseteq N$ such that $|N'| \geq k$ and for each $i \in N'$, $\mu_i(l) = \frac{l_N}{n}$.*

When the population is small, in particular, when $n = 3$ or $n = 4$, Proposition 1 implies that the equal sharing rule is the only rule meeting *k-majority* and *anonymity*. When the population is large, the equal sharing rule can be characterized by adding either one of the following axioms. These axioms include *sequential composition* in Section 2.1; a fairness axiom that requires a more egalitarian sharing at a more egalitarian path; and a basic monotonicity of a sharing rule with respect to transfers between agents.

[21] To see that μ satisfies *k-majority*, let l, l' be two paths such that $l_N \geq l'_N$. If an agent in the middle group at l' belongs to either the top or the middle group at l, then μ assigns no less to him at l than l'. If there are agents in the middle group at l' belonging to the bottom group at l, then there are equal number of agents in the bottom group at l' belonging to either the top or the middle group at l, so that they are assigned no less at l than at l'. Hence, there are at least m agents who receive no less at l than at l'.

Theorem 5. *A rule μ is the equal sharing rule if and only if it satisfies k-majority, anonymity, and either of the following axioms:*

i. Sequential composition: For each $l = \{x^t\}_{t=1}^T$ in $\mathcal{L}$, where $T \in \mathbb{N}$, $\mu(l) = \sum \mu(\{x^t\})$.

ii. Lorenz monotonicity: For each pair $l = \{x^t\}_{t=1}^T$ and $l' = \{x'^t\}_{t=1}^T$ in $\mathcal{L}$, where $T \in \mathbb{N}$, if for each $t \in \{1, ..., T\}$, $x^t \succ_{Lorenz} x'^t$, then $\mu(l') \not\succ_{Lorenz} \mu(l)$.[22]

iii. Transfer monotonicity: For each pair $i, i' \in N$, and each pair $l = \{x^t\}_{t=1}^T$ and $l' = \{x'^t\}_{t=1}^T$ in $\mathcal{L}$, where $T \in \mathbb{N}$, if for each $t \in \{1, ..., T\}$, $x'^t_i - x^t_i = x^t_{i'} - x'^t_{i'} \geq 0$, and for each $j \in N \setminus \{i, i'\}$, $x^t_j = x'^t_j$, then $\mu_i(l') \geq \mu_i(l)$.

3.1 Remarks about Information

So far, we have studied the case of complete information among agents. We can alternatively consider the case where only a few of the agents have complete information about the network and individual values. A natural mechanism to select a path is one that delegates the more informed agents to decide which direction to continue. Consider a rule that incentivizes the delegates to make the efficient decision regardless of the network and individual values. It is easy to show that such a rule should allocate the delegates a share depending only on the value of a path, while non-delegates can be given arbitrary shares. Moreover, we can show that the equal sharing rule is the only anonymous rule in this class.

We can alternatively consider the problem where there is no agent with complete information about the network and individual values. For instance, consider the case where agents only know the network and their own values. In this setting, a more traditional approach from the mechanism design literature would require agents to report their information to the planner, who will use this information to make an estimation of the values in the network and select an efficient path —see Hougaard and Tvede [28, 29] for a related study of implementation of the efficient path in minimal cost spanning trees. A natural issue in this setting is to find the mechanisms and sharing rules that incentivize agents to report their true information. When agents are critical, that is, when they have information about the values at some edges that no one else has, it is easy to prove that every mechanism is manipulable. On the other hand, when agents are not critical, in particular, when for every edge there are at least three agents who have information about all agent's values on that edge, several mechanisms and sharing rules can achieve truth telling as an equilibrium.

4 Conclusion

We have introduced the problem of division of sequential values and provided a comprehensive study of it. In particular, we have addressed the problem from different perspectives,

[22] To define $\succ_{Lorenz}$, for each $x \in \mathbb{R}^n_+$, let $x^* \in \mathbb{R}^n_+$ be such that for each $i \in \{1, ..., n\}$, x^*_i is the i-th smallest number of $x_1, ..., x_n$. For each pair $x, y \in \mathbb{R}^n_+$, we denote by $x \succ_{Lorenz} y$ if for each $m \in \{1, ..., n\}$, $\sum_{i=1}^m x^*_i \geq \sum_{i=1}^m y^*_i$. See Hougaard [25] for recent applications of Lorenz monotonicity to allocation problems.

including the complete and incomplete information case, and used old and new axioms from other strands in the literature to characterize several classes of solutions not uncovered elsewhere. An advantage of covering this problem from a wide variety of angles is that axioms can be chosen according to relevancy of the application in mind. Our analysis highlights the robustness of the EFF-ES solution, in both the complete and incomplete information settings. This rule, however, is by no means the only rule when less stringent axioms are imposed. Indeed, we have also characterized a class of solutions that take into account individual values accumulated along all paths in a rationalizable way.

We see this work as opening new avenues of research in the distribution of sequential costs and benefits, especially from the normative and positive angles. New classes of solutions, like the ones uncovered in Examples 1 and 2, require further study.

A Appendix

A.1 Proofs of results

Proof of Theorem 1. The necessity is readily seen. We only check the sufficiency. Let (φ, μ) satisfy *non-triviality*, *continuity*, and *sequential composition*. Let $(G, v) \in \mathcal{P}$. Suppose that there is a unique efficient path L in (G, v) and for each edge e in L, $v_N(e) > 0$. Let $d_1, e_1, ..., d_m, e_m, d_{m+1}$ be the consecutive nodes and edges in L.

We claim that $\varphi(G, v) = \{L\}$. To see it, let $v^1 \in \mathcal{V}^G$ be such that $v^1(e_1) = v(e_1)$, and for each $e \in G$ with $e \neq e_1$, $v^1(e) = \mathbf{0}$. By *non-triviality*, $\sum\limits_{i \in N} \mu_i(G, v^1) > 0$, and thus for each $L' \in \varphi(G, v^1)$, $e_1 \in L'$. For each $\lambda \in (0, 1]$, let $v^{2\lambda} \in \mathcal{V}^G$ be such that $v^{2\lambda}(e_2) = \lambda v(e_2) + (1 - \lambda)\mathbf{0}$,[23] and for each $e \in G$ with $e \neq e_2$, $v^{2\lambda}(e) = v^1(e)$. Thus, for each $e \in G$, $\lim\limits_{\lambda \to 0} v^{2\lambda}(e) = v^1(e)$. By *continuity*, $\lim\limits_{\lambda \to 0} \sum\limits_{i \in N} \mu_i(G, v^{2\lambda}) = \sum\limits_{i \in N} \mu_i(G, v^1(e))$. Thus, when λ is sufficiently small, for each $L' \in \varphi(G, v^{2\lambda})$, $e_1 \in L'$, and by *sequential composition* and *non-triviality*, $\sum\limits_{i \in N} \mu_i(G, v^{2\lambda}) = \sum\limits_{i \in N} \mu_i(G|_{d_2}, v^{2\lambda}|_{d_2}) + \sum\limits_{i \in N} \mu_i(G|^{d_2}, v^{2\lambda}|^{d_2}) > v_N(e_1)$, so $e_2 \in L'$. Let $\bar{\lambda} := \sup\{\lambda' \in (0, 1] :$ for each $\lambda \in (0, \lambda']$ and each $L' \in \varphi(G, v^{2\lambda})$, $e_1, e_2 \in L'\}$. Then, $\bar{\lambda} > 0$, and by *continuity*, $\sum\limits_{i \in N} \mu_i(G, v^{2\bar{\lambda}}) = v_N(e_1) + \bar{\lambda} v_N(e_2)$. If $\bar{\lambda} < 1$, then there is a sequence $\{\lambda_t\}_{t=1}^{\infty}$ of elements of $(\bar{\lambda}, 1]$ such that $\lim\limits_{t \to \infty} \lambda_t = \bar{\lambda}$, and for each $t \in \mathbb{N}$, there is $L' \in \varphi(G, v^{2\lambda_t})$ such that either $e_1 \notin L'$ or $e_2 \notin L'$. Thus, $\limsup\limits_{t \to \infty} \sum\limits_{i \in N} \mu_i(G, v^{2\lambda_t}) < v_N(e_1) + \bar{\lambda} v_N(e_2) = \sum \mu_i(G, v^{2\bar{\lambda}})$, which is a violation of *continuity*. Hence, $\bar{\lambda} = 1$, and $\sum\limits_{i \in N} \mu_i(G, v^2) = v_N(e_1) + v_N(e_2)$. Thus, for each $L' \in \varphi(G, v^2)$, $e_1, e_2 \in L'$.

For each $\lambda \in [0, 1]$, let $\hat{v}^\lambda \in \mathcal{V}^G$ be such that for each $e \in L$, $\hat{v}^\lambda(e) = v(e)$, and for each $e \notin L$, $\hat{v}^\lambda(e) = \lambda v(e) + (1 - \lambda)\mathbf{0}$. By applying the above argument repeatedly, $L \in \varphi(G, \hat{v}^0)$. By *continuity*, when λ is sufficiently small, $L \in \varphi(G, \hat{v}^\lambda)$. Let $\hat{\lambda} := \{\lambda' \in [0, 1] :$ for each $\lambda \in [0, \lambda'], L \in \varphi(G, \hat{v}^\lambda)\}$. By *continuity*, $L \in \varphi(G, \hat{v}^{\hat{\lambda}})$. If $\hat{\lambda} < 1$, then there is a sequence

[23] We denote by $\mathbf{c}$, $c \in \mathbb{R}$, the n-dimensional vector in which each coordinate equals c.

$\{\lambda_t\}$ of elements of $(\hat{\lambda}, 1]$ such that $\lim_{t\to\infty} \lambda_t = \hat{\lambda}$ and for each $t \in \mathbb{N}$, $L \notin \varphi(G, \hat{v}^{\lambda_t})$. Since $\lim_{t\to\infty} \max_{L'\in G, L'\neq L} \hat{v}_N^{\lambda_t}(L') = \max_{L'\in G, L'\neq L} \hat{v}_N^{\hat{\lambda}}(L') \leq \max_{L'\in G, L'\neq L} \hat{v}_N^{1}(L') < v_N(L)$, then $\limsup_{t\to\infty} \sum_{i\in N} \mu_i(G, \hat{v}^{\lambda_t}) < v_N(L) = \sum \mu_i(G, \hat{v}^{\hat{\lambda}})$, which is a violation of *continuity*. Hence, $\hat{\lambda} = 1$, and $L \in \varphi(G, v)$. Since L is the unique efficient path in (G, v), then $\varphi(G, v) = \{L\}$.

To see that there is $\alpha \in \mathbb{R}_+^n$ with $\sum_{i\in N} \alpha_i = 1$ such that $\mu(G, v) = v_N(G)\alpha$, let $G' \in \mathcal{G}$ be as in Figure 7. That is, G' is obtained by adding a new source d_0, a new parallel path L' with $d_0, e_1', d_1', e_2', d_2'$ being the consecutive nodes and edges in L', and a new edge e_0 going out from d_0 and going into d_1, followed by the orginal network G. For each $\lambda \leq$

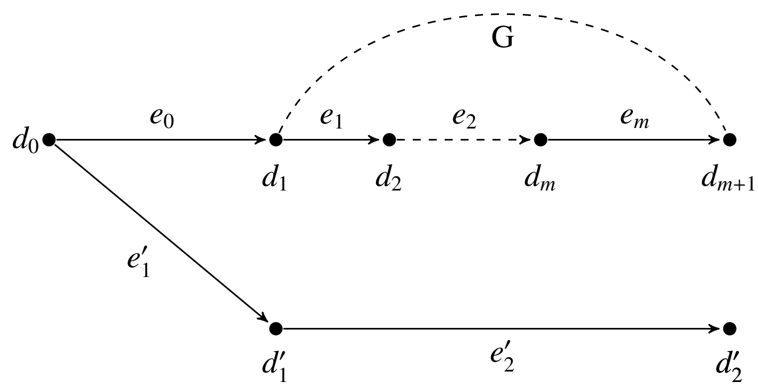

Figure 7: Incremented network G' based on G

$\frac{v_N(G)}{n}$, let $v'^{\lambda} \in \mathcal{V}^{G'}$ be such that for each $e \in G$, $v'^{\lambda}(e) = v(e)$, $v'^{\lambda}(e_0) = v'^{\lambda}(e_1') = \mathbf{0}$, and $v'^{\lambda}(e_2') = (\frac{v_N(G)}{n} - \lambda)\mathbf{1}$. Note that whenever $\lambda > 0$, L incremented by d_0 and e_0 is the unique efficient path in (G', v'^{λ}). By *sequential composition*, $\mu(G', v'^{\lambda}) = \mu(G, v)$. By *continuity*, $\mu(G', v'^{0}) = \mu(G, v)$. Whenever $\lambda < 0$, L' is the unique efficient path in G'. By *continuity* and *sequential composition*, $\mu(G', v'^{0}) = \mu(e, \frac{v_N(G)}{n}\mathbf{1})$. Thus, $\mu(G, v) = \mu(e, \frac{v_N(G)}{n}\mathbf{1})$.

Let $c, c' \in \mathbb{R}_+$, $\bar{G} \in \mathcal{G}$ be as in Figure 8, and $\bar{v} \in \mathcal{V}^{\bar{G}}$ be such that $\bar{v}(e_1) = \frac{c+c'}{n}\mathbf{1}$, $\bar{v}(e_2) = \mathbf{0}$, $\bar{v}(e_1') = \frac{c}{n}\mathbf{1}$, and $\bar{v}(e_2') = \frac{c'}{n}\mathbf{1}$. By *continuity* and *sequential composition*, $\mu(\bar{G}, \bar{v}) = \mu(e, \frac{c+c'}{n}\mathbf{1}) = \mu(e, \frac{c}{n}\mathbf{1}) + \mu(e, \frac{c'}{n}\mathbf{1})$. Thus, for each $i \in N$, $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ defined by setting for each $c \in \mathbb{R}_+$, $f_i(c) = \mu_i(e, \frac{c}{n}\mathbf{1})$ is additive, and by *continuity*, it is continuous. Hence, there is $\alpha_i \in \mathbb{R}$ such that for each $c \in \mathbb{R}_+$, $f_i(c) = \alpha_i c$. Since for each $c > 0$ and $i \in N$, $f_i(c) \geq 0$, and $\sum_{i\in N} f_i(c) = c$, then for each $i \in N$, $\alpha_i \geq 0$, and $\sum_{i\in N} \alpha_i = 1$. Hence, for each $i \in N$, $\mu_i(G, v) = f_i(\frac{v_N(G)}{n}) = \alpha_i v_N(G)$.

Lastly, suppose that there are multiple efficient paths in (G, v) or there are some edges of zero value in an efficient path. Let $\{v^t\}_{t=1}^{\infty}$ be a sequence of elements of $\mathcal{V}^G$ such that for each k, (G, v) has a unique efficient path, each edge of which has a positive value, and for each edge e in G, $\lim_{t\to\infty} v^t(e) = v(e)$. Thus, for each $t \in \mathbb{N}$, $\mu(G, v^t) = v_N^t(G)\alpha$. By *continuity*, $\mu(G, v) = v_N(G)\alpha$. This also shows that each path in $\varphi(G, v)$ is efficient. $\square$

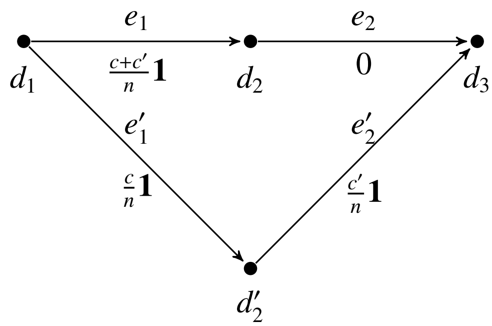

Figure 8: Problem $(\bar{G}, \bar{v})$

Proof of Theorem 2. We shall only check the sufficiency. Suppose that (φ, μ) satisfies *technology improvement*. Define $f : \mathbb{R}_+ \to \mathbb{R}^n_+$ by setting for each $c \in \mathbb{R}_+$, $f(c) = \mu(e, \frac{c}{n}\mathbf{1})$. Let $(G, v) \in \mathcal{P}$. We claim that $\mu(G, v) = f(v_N(G))$.

Let $(G', v') \in \mathcal{P}$ be a technology improvement of (G, v) where G' is constructed by adding an edge e' going out from the source to a sink in G, and $v'(e') = \frac{v_N(G)}{n}\mathbf{1}$. By *technology monotonicity*, $\mu(G', v') \geq \mu(G, v)$. Since $v'_N(G) = v_N(G)$, then $\mu(G', v') = \mu(G, v)$. Note that there is a finite sequence $\{(G^t, v^t)\}_{t=1}^T$, $T \in \mathbb{N}$, of elements of $\mathcal{P}$ such that $(G^1, v^1) = (e, \frac{v_N(G)}{n}\mathbf{1})$, $(G^T, v^T) = (G', v')$, and for each $t = 2, ..., T$, (G^t, v^t) is a technological improvement of (G^{t-1}, v^{t-1}). By applying *technology monotonicity* repeatedly, $\mu(G', v') \geq \mu(e, \frac{v_N(G)}{n}\mathbf{1})$. Since $v'_N(G) = v_N(G)$, then $\mu(G', v') = \mu(e, \frac{v_N(G)}{n}\mathbf{1})$. Hence, $\mu(G, v) = \mu(e, \frac{v_N(G)}{n}\mathbf{1}) = f(v_N(G))$. This also shows that each path in $\varphi(G, v)$ is efficient.

□

Proof of Theorem 3. We shall only prove the sufficiency. Define $r : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by setting for each $x \in \mathbb{R}^n_+$, $r(x) = y$ if $\mu(e, x) = y$. Clearly, $\sum_{i \in N} x_i = \sum_{i \in N} y_i$. By *continuity*, r is continuous. Suppose that $r(x) = y$ for some $x \in \mathbb{R}^n_+$. Then by *parallel composition*, $\mu((e, x) \cup (e', x)) = \mu((e, y) \cup (e', x)) = \mu((e, y) \cup (e', y))$. By *irrelevance of dominated paths* and *continuity*, $\mu((e, x) \cup (e', x)) = \mu(e, x)$ and $\mu((e, y) \cup (e', y)) = \mu(e, y)$. Thus, $\mu(e, y) = \mu(e, x) = y$. Hence, $r(y) = y$.

Define $\succsim$ on $r(\mathbb{R}^n_+)$ by setting for each pair $x, y \in r(\mathbb{R}^n_+)$, $y \succsim x$ if $\mu((e, x) \cup (e', y)) = y$. We claim that $\succsim$ is a partial order. Since for each $y \in r(\mathbb{R}^n_+)$, $\mu((e, y) \cup (e', y)) = y$, then $\succsim$ is reflexive. By definition, $\succsim$ is antisymmetric. To see $\succsim$ is transitive, let $x, y, z \in r(\mathbb{R}^n_+)$ be such that $y \succsim x$ and $z \succsim y$, and let $z' := \mu((e, x) \cup (e', z))$. By *parallel composition*, *irrelevance of dominated paths* and *continuity*,

$$\begin{aligned}
&\mu((e_1, z) \cup (e_2, x) \cup (e_3, y) \cup (e_4, z)) \\
=&\mu((e_1, z) \cup (e_2, x) \cup (e_3, z)) \\
=&\mu((e_1, z) \cup (e_2, x)) = z',
\end{aligned}$$

and

$$\begin{aligned}&\mu((e_1,z)\cup(e_2,x)\cup(e_3,y)\cup(e_4,z))\\=&\mu((e_1,z)\cup(e_2,x)\cup(e_3,y))\\=&\mu((e_1,z)\cup(e_2,y))=z.\end{aligned}$$

Hence, $z'=z$. Thus, $z\succsim x$ as desired. By *parallel composition* and *irrelevance of dominated paths*, for each pair $x,y\in\mathbb{R}^n_+$ such that $x>y$, $\mu((e,r(x))\cup(e',r(y)))=\mu((e,x)\cup(e',y))=\mu(e,x)=r(x)$, so $r(x)\succsim r(y)$. Since $\succsim$ is continuous, for each pair $x,y\in\mathbb{R}^n_+$ such that $x\geq y$, $r(x)\succsim r(y)$.

To see that $(r(\mathbb{R}^n_+),\succsim)$ is a join-semilattice, let $x,y\in r(\mathbb{R}^n_+)$ and $z:=\mu((e,x)\cup(e',y))$. We claim that $z=\max\limits_{\succsim}\{x,y\}$. By *parallel composition*, *irrelevance of dominated paths*, and *continuity*,

$$\begin{aligned}&\mu((e_1,z))=\mu((e_1,z)\cup(e_2,z))\\=&\mu((e_1,x)\cup(e_2,y)\cup(e_3,x)\cup(e_4,y))\\=&\mu((e_1,x)\cup(e_2,y))=z,\end{aligned}$$

and

$$\begin{aligned}&\mu((e_1,x)\cup(e_2,z))\\=&\mu((e_1,x)\cup(e_2,y)\cup(e_3,x))\\=&\mu((e_1,x)\cup(e_2,y))=z.\end{aligned}$$

Hence, $z\in r(R^n_+)$ and $z\succsim x$. Similarly, $z\succsim y$. If there is $z'\in r(\mathbb{R}^n_+)$ such that $z'\succsim x$ and $z'\succsim y$, then by *parallel composition*,

$$\begin{aligned}&\mu((e_1,z')\cup(e_2,z))\\=&\mu((e_1,z')\cup(e_2,x)\cup(e_3,y))\\=&\mu((e_1,z')\cup(e_3,y))=z'.\end{aligned}$$

Hence, $z'\succsim z$. Thus, $z=\max\limits_{\succsim}\{x,y\}\in r(\mathbb{R}^n_+)$ as desired.

Let $(G,v)\in\mathcal{P}$. We claim that φ only selects efficient path(s). If $v_N(G)=0$, then we are done. Suppose that $v_N(G)>0$. Let $\{L_1,...,L_m\}$, $m\in\mathbb{N}$, be the set of paths in (G,v). For each $j\in\{1,...,m\}$, we denote by (L_j,v^j) the problem consisted of a single path L_j such that for each edge e in L, $v^j(e)=v(e)$. By *irrelevance of dominated paths* and *continuity*, $\mu(G,v)=\mu(\bigcup\limits_{j=1}^m(L_j,v^j)\cup(G,v))=\mu(\bigcup\limits_{j=1}^m(L_j,v^j))$.

For each $\lambda\in[0,1]$ and each $j\in\{1,...,m\}$, let $x^{\lambda j}:=\sum\limits_{e\text{ in }L_j}v(e)$ if $v_N(L_j)=v_N(G)$, and $x^{\lambda j}:=\lambda\sum\limits_{e\text{ in }L_j}v(e)+(1-\lambda)\mathbf{0}$ if $v_N(L_j)<v_N(G)$. When $\lambda=1$, we simply write x^j for x^{1j}. By *split invariance*, $\mu(G,v)=\mu(\bigcup\limits_{j=1}^m(e_j,x^j))$. By *irrelevance of dominated paths*,

the $\sum_{i\in N}\mu_i(\bigcup_{j=1}^{m}(e_j, x^{0j})) = v_N(G)$. Let $\bar\lambda := \sup\{\lambda \in [0,1] : \sum_{i\in N}\mu_i(\bigcup_{j=1}^{m}(e_j, x^{\lambda j})) = v_N(G)\}$. By *continuity*, $\sum_{i\in N}\mu_i(\bigcup_{j=1}^{m}(e_j, x^{\bar\lambda j})) = v_N(G)$. Suppose that $\bar\lambda < 1$. Then, there is $\epsilon > 0$ such that for each $\lambda > \bar\lambda$, $\sum_{i\in N}\mu_i(\bigcup_{j=1}^{m}(e_j, x^{\lambda j})) \leq v_N(G) - \epsilon$, which is a violation of *continuity*. Hence, $\bar\lambda = 1$, and $\mu_i(G, v) = \mu_i(\bigcup_{j=1}^{m}(e_j, x^j)) = v_N(G)$.

Next, we claim $\mu(G, v) = \max_{\succsim}\{r(x^j) : j = 1, ..., m\}$. Let $z := \mu(\bigcup_{j=1}^{m}(e_j, x^j))$. It suffices to show that $z = \max_{\succsim}\{r(x^j) : j = 1, ..., m\}$. By *parallel composition*, *irrelevance of dominated paths*, and *continuity*, $\mu(e, z) = z$, and for each $j' \in \{1, ..., m\}$,

$$
\begin{aligned}
&\mu((e, z) \cup (e', r(x^{j'})))\\
=&\mu(\bigcup_{j=1}^{m}(e_j, x^j) \cup (e', x^{j'}))\\
=&\mu(\bigcup_{j=1}^{m}(e_j, x^j)) = z.
\end{aligned}
$$

Hence, $z \in r(\mathbb{R}^n_+)$ and for each $j' \in \{1, ..., m\}$, $z \succsim r(x^{j'})$. If there is $z' \in r(\mathbb{R}^n_+)$ such that for each $j' \in \{1, ..., m\}$, $z' \succsim r(x^{j'})$, then by *parallel composition*,

$$
\begin{aligned}
&\mu((e, z) \cup (e', z')) = \mu(\bigcup_{j=1}^{m}(e_j, x^j) \cup (e', z'))\\
=&\mu(\bigcup_{j=1}^{m}(e_j, r(x^j)) \cup (e', z')) = z',
\end{aligned}
$$

so $z' \succsim z$ as desired. Therefore, (φ, μ) is $(r, \succsim)$ - rationalizable. By *continuity*, (φ, μ) is continuously $(r, \succsim)$ - rationalizable. □

Proof of Theorem 4. We shall only check the sufficiency. Let (φ, μ) satisfy *irrelevance of dominated paths*, *irrelevance of parallel outside options*, and *continuity*. For each $(G, v) \in \mathcal{P}$ and each L in G, we denote by (L, v^L) the problem consisting of the single path L and v^L such that for each edge e in L, $v^L(e) = v(e)$. By *irrelevance of dominated paths* and *continuity*, for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \mu(\bigcup_{L \text{ in } G}(L, v^L))$. Moreover, by a similar argument as in the the proof of Theorem 3, φ selects only efficient path(s). Let $(G, v), (G', v') \in \mathcal{P}$. Let L^* be an efficient path in $(G', v') \cup (G, v)$ and for each $\epsilon > 0$, $v^\epsilon \in \mathcal{V}^{L^*}$ be such that for each e in L^*, $v^\epsilon(e) = v(e) + \epsilon\mathbf{1}$ if L^* is in G, and $v^\epsilon(e) = v'(e) + \epsilon\mathbf{1}$ if L^* is in G'. Suppose that L^* is in G'. By *continuity*, $\mu((G, v) \cup (G', v')) = \lim_{\epsilon\downarrow 0}\mu((\bigcup_{L \text{ in } G}(L, v^L)) \cup (\bigcup_{L \text{ in } G', L\neq L^*}(L, v'^L)) \cup (L^*, v^\epsilon))$. Since φ selects only efficient path(s), then by *irrelevance of parallel outside options*, $\mu((G, v) \cup (G', v')) = \lim_{\epsilon\downarrow 0}\mu(L^*, v^\epsilon) = \mu(L^*, v^{L^*})$. Similarly, $\mu((e, \mu(G, v)) \cup (G', v')) = \mu(L^*, v^{L^*})$, and thus $\mu((G,v)\cup(G',v')) = \mu((e,\mu(G,v))\cup(G',v'))$. Suppose that L^* is in G. Then $\mu((G,v)\cup(G',v')) = \lim_{\epsilon\downarrow 0}\mu((\bigcup_{L \text{ in } G, L\neq L^*}(L,v^L))\cup(\bigcup_{L \text{ in } G'}(L,v'^L))\cup(L^*,v^\epsilon)) = \mu(L^*,v^{L^*})$. Moreover, $\mu((e,\mu(G,v))\cup(G',v')) = \lim_{\epsilon\downarrow 0}\mu((e,\mu(G,v)+\epsilon\mathbf{1})\cup(G',v')) = \lim_{\epsilon\downarrow 0}\mu((e,\mu(G,v)+\epsilon\mathbf{1})\cup(\bigcup_{L \text{ in } G'}(L,v'^L))) = \lim_{\epsilon\downarrow 0}\mu(e,\mu(G,v)+\epsilon\mathbf{1}) = \mu(e,\mu(G,v))$. For each $\epsilon > 0$, $\mu((e,\mu(G,v)+\epsilon\mathbf{1})\cup(L^*,v^{L^*})) = \mu(e,\mu(G,v)+\epsilon\mathbf{1})$ and $\mu((e,\mu(G,v))\cup(L^*,v^\epsilon)) = \mu(L^*,v^\epsilon)$. By *continuity*, $\mu((e,\mu(G,v))\cup(L^*,v^{L^*})) = \mu(e,\mu(G,v)) = \mu(L^*,v^{L^*})$. Hence, $\mu((G,v)\cup(G',v')) = \mu((e,\mu(G,v))\cup(G',v'))$. Thus, (φ,μ) satisfies *parallel composition*.

Moreover, we claim that (φ,μ) satisfies *split invariance*. It suffices to show that for each $(G,v)\in\mathcal{P}$, $\mu(\bigcup_{L \text{ in } G}(L,v^L)) = \mu(\bigcup_{L \text{ in } G}(e^L,\sum_{e \text{ in } L}v(e)))$. Let $(G,v)\in\mathcal{P}$ and L be a path in G. By *parallel composition*, it suffices to show that $\mu(L,v^L) = \mu(e^L,\sum_{e \text{ in } L}v(e))$. For each $\epsilon > 0$, let $v^\epsilon \in \mathcal{V}^L$ be such that for each e in L, $v^\epsilon(e) = v^L(e)+\epsilon\mathbf{1}$. Then, for each $\epsilon > 0$, $\mu((L,v^\epsilon)\cup(e^L,\sum_{e \text{ in } L}v(e))) = \mu(L,v^\epsilon)$ and $\mu((L,v^L)\cup(e^L,\sum_{e \text{ in } L}v(e)+\epsilon\mathbf{1})) = \mu(e^L,\sum_{e \text{ in } L}v(e)+\epsilon\mathbf{1})$. By *continuity*, $\mu((L,v^L)\cup(e^L,\sum_{e \text{ in } L}v(e))) = \mu(L,v^L) = \mu(e^L,\sum_{e \text{ in } L}v(e))$, as desired.

Since (φ,μ) satisfies both *split invariance* and *parallel composition*, then by Theorem 3, there exist a redistribution function r and a partial order $\succsim$ on $r(\mathbb{R}^n_+)$ such that for each pair $x,y\in\mathbb{R}^n_+$ with $x\geq y$, $r(x)\succsim r(y)$, and (φ,μ) is continuously $(r,\succsim)$ - rationalizable. By *irrelevance of parallel outside options*, for each pair $x,y\in r(\mathbb{R}^n_+)$, either $\mu((e,x)\cup(e',y)) = \mu(e,x) = x$ or $\mu((e,x)\cup(e',y)) = \mu(e',y) = y$. Since r is continuous, then $r(\mathbb{R}^n_+)$ is connected. Then by Eilenberg [19],[24] there is a one-to-one and continuous mapping $g : r(\mathbb{R}^n_+)\to\mathbb{R}$.

We claim that for each pair $x,y\in\mathbb{R}^n_+$ such that $\sum_{i\in N}x_i = \sum_{i\in N}y_i$, $r(x) = r(y)$. Suppose to the contrary that $x,y\in\mathbb{R}^n_+$, $\sum_{i\in N}x_i = \sum_{i\in N}y_i$, and $r(x)\neq r(y)$. Thus, $\sum_{i\in N}x_i > 0$. Let $z\in\mathbb{R}^n_+$ be such that $\sum_{i\in N}z_i < \sum_{i\in N}x_i$, so $r(z)\neq r(x)$ and $r(z)\neq r(y)$. For each pair $(x',x'')\in\{(x,y),(y,z),(z,x)\}$, let $D_{(x',x'')} := \{\lambda x' + (1-\lambda)x'' : \lambda\in[0,1]\}$. Since r is continuous and $D_{(x',x'')}$ is convex, then $r(D_{(x',x'')})$ is path-connected. Since $r(D_{(x',x'')})$ is a Hausdorff space, then $r(D_{(x',x'')})$ is arc-connected. Thus, there is a function $h_{(x',x'')} : [0,1]\to r(D_{(x',x'')})$ such that $h_{(x',x'')}(0) = r(x')$, $h_{(x',x'')}(1) = r(x'')$, and $h_{(x',x'')}$ is a homeomorphism between $[0,1]$ and $h_{(x',x'')}([0,1])$. Note that $h_{(x,y)}([0,1])\cap h_{(y,z)}([0,1]) = \{r(y)\}$, $h_{(x,y)}([0,1])\cap h_{(z,x)}([0,1]) = \{r(x)\}$, and $A := h_{(x,y)}([0,1])\cup h_{(y,z)}([0,1])\cup h_{(z,x)}([0,1])$ is connected. Since g is continuous, then $g(A)$ is an interval. Since g is one-to-one, then there is $\bar{c}\in(0,1)$ such that $g(h_{(x,y)}(\bar{c}))$ is an interior point of $g(A)$, and thus $g(A\setminus h_{(x,y)}(\bar{c}))$ is not connected. Since $A\setminus h_{(x,y)}(\bar{c})$ is connected and g is continuous, then $g(A\setminus h_{(x,y)}(\bar{c}))$ is connected, which is a contradiction.

Define $f : \mathbb{R}_+\to\mathbb{R}^n_+$ by setting for each $c\in\mathbb{R}_+$, $f(c) = r(\frac{c}{n}\mathbf{1})$. For each $(G,v)\in\mathcal{P}$, since $\mu(G,v) = \max_{\succsim}\{r(\sum_{e \text{ in } L}v(e)) : L \text{ in } G\}$, then by the result proved in the previous paragraph, $\mu(G,v) = \max_{\succsim}\{f(v_N(L)) : L \text{ in } G\}$. Recall that for each pair $x,y\in\mathbb{R}^n_+$ with $x\geq y$, $r(x)\succsim r(y)$. Thus, $\mu(G,v) = f(v_N(G))$.

□

[24] See his Theorem I and (6.1).

The following lemma is used in the proof of Proposition 1.

Lemma 1. *Let $c > 0$ and $X \subseteq \Delta = \{x \in \mathbb{R}^n_+ : \sum_{i \in N} x_i = c\}$. Suppose that (i) $\frac{c}{n}\mathbf{1} \in X$, (ii) for each pair $x, y \in X$, there is $N' \subseteq N$ such that $|N'| \geq k$ and for each $i \in N'$, $x_i \geq y_i$, and (iii) for each permutation π of N and each $x \in X$, x^π defined by setting for each $i \in N$, $x^\pi_i = x_{\pi(i)}$ belongs to X. Then, for each $x \in X$, there is $N' \subseteq N$ such that $|N'| \geq k$ and for each $i \in N'$, $x_i = \frac{c}{n}$.*

Proof. By assumption (ii), for each pair $x, y \in X$, $|\{i \in N : x_i > y_i\}| \leq n - k$, $|\{i \in N : x_i < y_i\}| \leq n - k$, and thus

$$|\{i \in N : x_i = y_i\}| \geq 2k - n. \tag{2}$$

For each $x \in X$, $N^x_> := \{i \in N : x_i > \frac{c}{n}\}$, $N^x_< := \{i \in N : x_i < \frac{c}{n}\}$, and $N^x_= := \{i \in N : x_i = \frac{c}{n}\}$. Let $x \in X$. By assumption (i), $|N^x_>| \leq n - k$, $|N^x_<| \leq n - k$, and $|N^x_=| \geq 2k - n$. Suppose to the contrary that $|N^x_=| < k$.

If $|N^x_=| \geq \frac{n}{2}$, then let π be a permutation on N such that for each $i \in N^x_> \cup N^x_<$, $\pi(i) \in N^x_=$. By assumption (iii), $x^\pi \in X$. Note that $N^x_> \cup N^x_< \subseteq N^{x^\pi}_=$ and $N^{x^\pi}_> \cup N^{x^\pi}_< \subseteq N^x_=$ (see Figure 9).

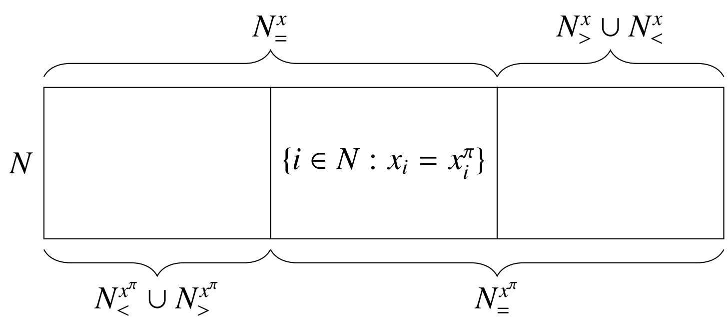

Figure 9: Permutation π of N

Thus, $|\{i \in N : x_i = x^\pi_i\}| = |N^x_=| - |N^x_>| - |N^x_<| = |N^x_=| - (n - |N^x_=|) < 2k - n$, which violates (2).

If $|N^x_=| < \frac{n}{2}$, assume that $|N^x_>| \leq |N^x_<| \leq |N^x_=|$. Let π be a permutation of N such that for each $i \in N^x_=$, $\pi(i) \in N^x_> \cup N^x_<$, and for each $i \in N^x_<$, $\pi(i) \in N^x_=$ (see Figure 10). Since $|N^x_>| \leq |N^x_=| < \frac{n}{2} < |N^x_> \cup N^x_<|$, then for each $i \in N^x_>$, $\pi(i) \in N^x_< \cup N^x_=$. Hence, for each $i \in N$, $x_i \neq x^\pi_i$, which violates (2). A similar contradiction can be obtained when $|N^x_<| \leq |N^x_>| \leq |N^x_=|$, and when $|N^x_=| < |N^x_>|$ or $|N^x_=| < |N^x_<|$, since both $|N^x_>|$ and $|N^x_<|$ are less than $\frac{n}{2}$. □

Proof of Proposition 1. Let $l \in \mathcal{L}$ and $c := l_N$. Let $X = \{\mu(l') : l' \in \mathcal{L}, l'_N = c\}$. By *anonymity*, assumptions (i) and (iii) of Lemma 1 are satisfied. By *k-majority*, assumption (ii) is also satisfied. By Lemma 1, there is $N' \subseteq N$ such that $|N'| \geq k$ and for each $i \in N'$, $\mu_i(l) = \frac{c}{n}$. □

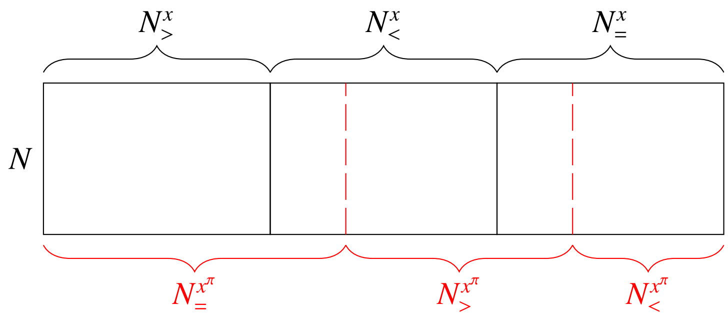

Figure 10: Permutation π of N

Proof of Theorem 5. We only check the sufficiency. Let μ satisfy *k-majority* and *anonymity*.

Suppose that μ satisfies *sequential composition*. For each $l \in \mathcal{L}$, let $N^l_= := \{i \in N : \mu_i(l) = \frac{l_N}{n}\}$ and $N^l_{\neq} := N \setminus N^l_=$. Suppose to the contrary that there is $l' \in \mathcal{L}$ such that $N^{l'}_{\neq} \neq \emptyset$. By Theorem 1, $|N^{l'}_=| \geq k \geq |N^{l'}_{\neq}|$. Let π be a permutation on N such that for each $i \in N^{l'}_{\neq}$, $\pi(i) \in N^{l'}_=$. Let $\bar{l} \in \mathcal{L}$ be a path that connects l' and l'^{π}. By *sequential composition*, $\mu(\bar{l}) = \mu(l') + \mu(l'^{\pi})$. Thus, $N^{\bar{l}}_= = N^{l'}_= \cap N^{l'^{\pi}}_=$, so $N^{\bar{l}}_{\neq} \neq \emptyset$ and $|N^{\bar{l}}_=| = |N^{l'}_=| - |N^{l'}_{\neq}| < |N^{l'}_=|$ (see Figure 11). Repeating

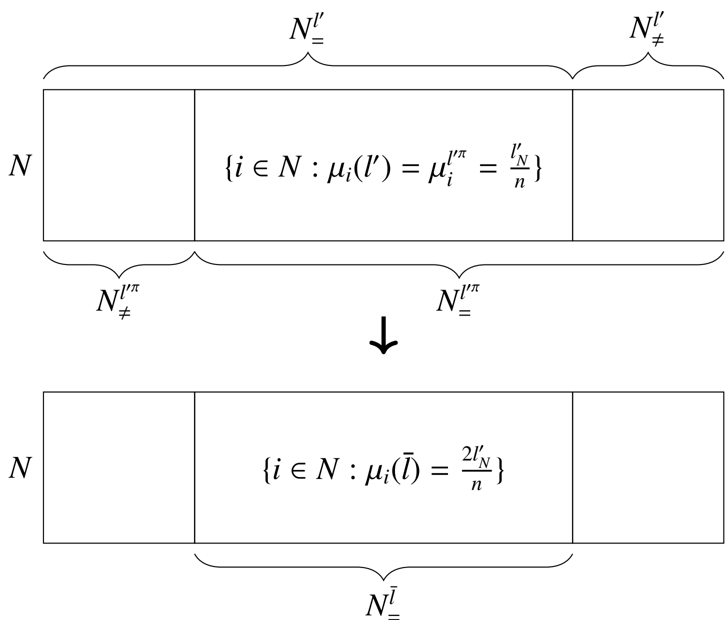

Figure 11: Permutation π of N

the argument, there is $l'' \in \mathcal{L}$ such that $N^{l''}_{\neq} \neq \emptyset$ and $|N^{l''}_=| < |N^{l'}_=|$. Within finitely many steps, we can find $\hat{l} \in \mathcal{L}$ such that $|N^{\hat{l}}_=| < k$, which is a violation of Proposition 1.

Suppose that μ satisfies *Lorenz monotonicity*. Let $l = \{x^t\}_{t=1}^T$, where $T \in \mathbb{N}$. Let $l' = \{x'^t\}_{t=1}^T$ be such that for each $t \in \{1, ..., T\}$ and each $i \in \{1, ..., n-1\}$, $x'^t_i = 0$ and $x'^t_n = \sum_{i \in N} x'^t_i$.

By construction, for each $t \in \{1, ..., T\}$, $x^t \succ_{Lorenz} x'^t$. By *anonymity*, for each pair $i, j \in \{1, ..., n-1\}$, $\mu_i(l') = \mu_j(l')$. By Proposition 1, for each $i \in \{1, ..., n-1\}$, $\mu_i(l') = \frac{l'_N}{n} = \frac{l_N}{n}$, and thus $\mu_n(l') = \frac{l_N}{n}$. By *Lorenz monotonicity*, $\frac{l_N}{n}\mathbf{1} \not\succ_{Lorenz} \mu(l)$. Hence, $\mu(l) = \frac{l_N}{n}\mathbf{1}$.

Suppose that μ satisfies *transfer monotonicity*. Let $l = \{x^t\}_{t=1}^T$, where $T \in \mathbb{N}$. For each $j \in N$, let $l^j = \{x'^t\}_{t=1}^T$ be such that for each $t \in \{1, ..., T\}$ and each $i \in N \setminus \{j\}$, $x_i'^t = 0$ and $x_j'^t = \sum_{i \in N} x_i'^t$. By *anonymity* and Proposition 1, $\mu_j(l^j) = \frac{l_N}{n}$. By *transfer monotonicity*, for each $j \in N$, $\mu_j(l) \leq \mu_j(l^j)$. Thus, $\mu(l) = \frac{l_N}{n}\mathbf{1}$. □

A.2 Local egalitarianism with transfers — an n-agent version of Example 3.

Imagine that n agents are divided into two groups according to their exogenous types. Consider the redistribution function that divides equally within each group the sum of the values of its group members. For instance, a partnership firm runs two businesses and adopts the equal sharing rule respectively for its partners involved in each business (Burrows and Black [12], Baskenille-Morley and Beechey [6]).

Formally, let $S_1, S_2 \subseteq \{1, ..., n\}$ be such that $S_1, S_2 \neq \emptyset$, $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2 = \{1, ..., n\}$. Define $r : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by setting for each $x \in \mathbb{R}^n_+$, each $j \in \{1, 2\}$, and each $i \in S_j$, $r_i(x) = \frac{1}{|S_j|} \sum_{i \in S_j} x_i$. When $n = 2$, r is simply the identity mapping that assigns to both agents their individual values.

Define a binary relation $\succsim$ on $r(\mathbb{R}^n_+)$ by setting for each pair $x, y \in r(\mathbb{R}^n_+)$, $x \succsim y$ if there are $z \in r(\mathbb{R}^n_+)$ and $\lambda \in [0, 1]$ such that

$$\sum_{i \in N} z_i = \sum_{i \in N} y_i, \text{ and } x \geq z = \lambda y + (1 - \lambda)\frac{\sum_{i \in N} y_i}{n}\mathbf{1}.$$

Note that for each pair $x, y \in r(\mathbb{R}^n_+)$, if $x \geq y$, then $x \succsim y$. Moreover, if $\sum_{i \in N} x_i = \sum_{i \in N} y_i$, then $x \succsim y$ if and only if x is a convex combination of y and $\frac{\sum_{i \in N} x_i}{n}\mathbf{1}$.

Let $m := |S_1|$. Then, $|S_2| = n - m$. Assume without loss of generality that $1 \in S_1$ and $2 \in S_2$.

Lemma 2. *For each pair $x, y \in r(\mathbb{R}^n_+)$, $x \succsim y$ if and only if $\sum_{i \in N} x_i \geq \sum_{i \in N} y_i$, $x_1 \geq \min\{y_1, \frac{\sum_{i \in N} y_i}{n}\}$, and $x_2 \geq \min\{y_2, \frac{\sum_{i \in N} y_i}{n}\}$.*

Proof. The "only if" direction is easy to check. To see the "if" direction, let $x, y \in r(\mathbb{R}^n_+)$ be such that $\sum_{i \in N} x_i \geq \sum_{i \in N} y_i$, $x_1 \geq \min\{y_1, \frac{\sum_{i \in N} y_i}{n}\}$, and $x_2 \geq \min\{y_2, \frac{\sum_{i \in N} y_i}{n}\}$. Suppose first that $y_1 \leq \frac{\sum_{i \in N} y_i}{n}$. Then, $y_2 \geq \frac{\sum_{i \in N} y_i}{n}$, $x_1 \geq y_1$, and $x_2 \geq \frac{\sum_{i \in N} y_i}{n}$. If $x_2 \geq y_2$, then $x \geq y$, and thus $x \succsim y$. If $x_2 < y_2$, then $my_1 + (n-m)x_2 \leq my_1 + (n-m)y_2 \leq mx_1 + (n-m)x_2$. Hence, there is $c \in [y_1, x_1]$

such that $mc + (n-m)x_2 = \sum y_i$. Let $z \in r(\mathbb{R}^n_+)$ be such that for each $i \in S_1$, $z_i = c$, and for each $i \in S_2$, $z_i = x_2$. Then, $x \geq z$, $\sum\limits_{i\in N} z_i = \sum\limits_{i\in N} y_i$, and $y_1 \leq z_1$. Since $x_2 \geq \frac{\sum_{i\in N} y_i}{n}$, then $z_1 \leq \frac{\sum_{i\in N} y_i}{n}$. Thus, there is $\lambda \in [0,1]$ such that $z = \lambda y + (1-\lambda)\frac{\sum_{i\in N} y_i}{n}$. Hence, $x \succsim y$. A similar arguments apply to the case of $y_1 > \frac{\sum_{i\in N} y_i}{n}$.

□

Proposition 2. *The relation $\succsim$ is a partial order, and $(r(\mathbb{R}^n_+), \succsim)$ is a join-semilattice.*

Proof. By definition, $\succsim$ is reflexive. To see $\succsim$ is antisymmetric, let $x, y \in r(\mathbb{R}^n_+)$ be such that $x \succsim y$ and $y \succsim x$. By Lemma 2, $\sum\limits_{i\in N} x_i = \sum\limits_{i\in N} y_i$. Suppose that $y_1 \leq \frac{\sum_{i\in N} y_i}{n}$. Then, $y_2 \geq \frac{\sum_{i\in N} y_i}{n}$. Since $x \succsim y$, then by Lemma 2, $x_1 \geq y_1$, $x_2 \geq \frac{\sum_{i\in N} y_i}{n}$, and thus $x_1 \leq \frac{\sum_{i\in N} y_i}{n}$. Since $y \succsim x$, then by Lemma 2, $y_1 \geq x_1$. Hence, $x_1 = y_1$, and thus $x = y$. A similar argument holds when $y_1 > \frac{\sum_{i\in N} y_i}{n}$. To see that $\succsim$ is transitive, let $x, y, z \in r(\mathbb{R}^n_+)$ be such that $x \succsim y$ and $y \succsim z$. By Lemma 2, $\sum\limits_{i\in N} x_i \geq \sum\limits_{i\in N} y_i \geq \sum\limits_{i\in N} z_i$, $x_1 \geq \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\} \geq \min\{\min\{z_1, \frac{\sum_{i\in N} z_i}{n}\}, \frac{\sum y_i}{n}\} = \min\{z_1, \frac{\sum_{i\in N} z_i}{n}\}$, and $x_2 \geq \min\{y_2, \frac{\sum_{i\in N} y_i}{n}\} \geq \min\{\min\{z_2, \frac{\sum_{i\in N} z_i}{n}\}, \frac{\sum_{i\in N} y_i}{n}\} = \min\{z_2, \frac{\sum_{i\in N} z_i}{n}\}$. Again by Lemma 2, $x \succsim z$. Hence, $\succsim$ is a partial order.

To see that $(r(\mathbb{R}^n_+), \succsim)$ is a join-semilattice, let $x, y \in r(\mathbb{R}^n_+)$. Suppose without loss of generality that $\sum\limits_{i\in N} x_i \geq \sum\limits_{i\in N} y_i$. Let *med* denote the operator that takes the median of an odd number of real numbers. Let $z^* \in r(\mathbb{R}^n_+)$ be such that for each $i \in S_1$, $z^*_i = med\{\min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}, x_1, \frac{1}{m}[\sum\limits_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}]\}$, and for each $i \in S_2$, $z^*_i = \frac{\sum_{i\in N} x_i - mz^*_1}{n-m}$. We claim that $z^* = \max\limits_{\succsim}\{x, y\}$. Since $\sum\limits_{i\in N} x_i \geq \sum\limits_{i\in N} y_i$, then $\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\} \leq \frac{\sum_{i\in N} y_i}{n} \leq \frac{\sum_{i\in N} x_i}{n}$, and thus

$$\frac{1}{m}[\sum_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}] \geq \frac{\sum_{i\in N} x_i}{n} \geq \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}. \tag{3}$$

By (3), $\frac{1}{m}[\sum\limits_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}] \geq z^*_1 \geq \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}$, and thus $z^*_2 = \frac{\sum_{i\in N} x_i - mz^*_1}{n-m} \geq \frac{(n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}}{n-m} = \min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}$. Moreover, since $\sum\limits_{i\in N} z^*_i = \sum\limits_{i\in N} x_i \geq \sum\limits_{i\in N} y_i$, then by Lemma 2, $z^* \succsim y$.

Let $z \in r(\mathbb{R}^n_+)$ be such that $z \succsim x$ and $z \succsim y$. Since $z^* \succsim y$, then to show that $z^* = \max\limits_{\succsim}\{x, y\}$, it suffices to check that $z \succsim z^* \succsim x$. If $\frac{1}{m}[\sum\limits_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}] = \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}$, then

$$\frac{1}{m}[\sum_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}] = \frac{\sum_{i\in N} x_i}{n} = \frac{\sum_{i\in N} y_i}{n} = \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}.$$

If $y_1 = \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}$, then $y_1 = \frac{\sum_{i\in N} y_i}{n}$. If $y_1 > \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}$, then $y_2 < \frac{\sum_{i\in N} y_i}{n}$, and thus $y_1 = \frac{1}{m}[\sum_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}] = \frac{\sum_{i\in N} y_i}{n}$. In either case, $y = \frac{\sum_{i\in N} y_i}{n}\mathbf{1} = \frac{\sum_{i\in N} x_i}{n}\mathbf{1} = z^*$. Since $z \succsim y$ and $y = z^*$, then $z \succsim z^*$. Since $z^* = \frac{\sum_{i\in N} x_i}{n}\mathbf{1}$, then $z^* \succsim x$. Suppose from now on that $\frac{1}{m}[\sum_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}] > \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}$. If $z_1^* = x_1$, then $z^* = x$, and we are done. Assume that $z_1^* = \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}$. Then by (3), $\frac{\sum_{i\in N} x_i}{n} \geq \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\} \geq x_1$. Thus, $z_1^* \geq x_1 = \min\{x_1, \frac{\sum_{i\in N} x_i}{n}\}$ and $z_2^* = \frac{\sum_{i\in N} x_i - mz_1^*}{n-m} \geq \frac{\sum_{i\in N} x_i - \frac{m\sum_{i\in N} x_i}{n}}{n-m} = \frac{\sum_{i\in N} x_i}{n} \geq \min\{x_2, \frac{\sum_{i\in N} x_i}{n}\}$. Note that $\sum_{i\in N} z_i^* = \sum_{i\in N} x_i$. Hence, by Lemma 2, $z^* \succsim x$. Moreover, $\min\{z_1^*, \frac{\sum_{i\in N} z_i^*}{n}\} = \min\{z_1^*, \frac{\sum_{i\in N} x_i}{n}\} = \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}$, and $\min\{z_2^*, \frac{\sum_{i\in N} z_i^*}{n}\} = \min\{\frac{\sum_{i\in N} x_i - mz_1^*}{n-m}, \frac{\sum_{i\in N} x_i}{n}\} \leq \min\{\frac{\sum_{i\in N} x_i - mx_1}{n-m}, \frac{\sum_{i\in N} x_i}{n}\} = \min\{x_2, \frac{\sum_{i\in N} x_i}{n}\}$. Since $z \succsim x$ and $z \succsim y$, then by Lemma 2, $\sum_{i\in N} z_i \geq \sum_{i\in N} x_i = \sum_{i\in N} z_i^*$, $z_1 \geq \min\{y_1, \frac{\sum_{i\in N} y_i}{n}\} = \min\{z_1^*, \frac{\sum_{i\in N} z_i^*}{n}\}$, and $z_2 \geq \min\{x_2, \frac{\sum_{i\in N} x_i}{n}\} \geq \min\{z_2^*, \frac{\sum_{i\in N} z_i^*}{n}\}$. By Lemma 2, $z \succsim z^*$. Lastly, assume that $z_1^* = \frac{1}{m}[\sum_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}]$. By (3), $x_1 \geq \frac{1}{m}[\sum_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}] \geq \frac{\sum_{i\in N} x_i}{n}$. Thus, $z_1^* \geq \min\{x_1, \frac{\sum_{i\in N} x_i}{n}\}$ and $z_2^* = \frac{\sum_{i\in N} x_i - mz_1^*}{n-m} \geq \frac{\sum_{i\in N} x_i - mx_1}{n-m} = x_2 \geq \min\{x_2, \frac{\sum_{i\in N} x_i}{n}\}$. By Lemma 2, $z^* \succsim x$. Moreover, $\min\{z_1^*, \frac{\sum_{i\in N} z_i^*}{n}\} = \min\{z_1^*, \frac{\sum_{i\in N} x_i}{n}\} = \frac{\sum_{i\in N} x_i}{n} = \min\{x_1, \frac{\sum_{i\in N} x_i}{n}\}$, and $\min\{z_2^*, \frac{\sum_{i\in N} z_i^*}{n}\} = \min\{\frac{\sum_{i\in N} x_i - mz_1^*}{n-m}, \frac{\sum_{i\in N} x_i}{n}\} = \min\{\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}, \frac{\sum_{i\in N} x_i}{n}\} = \min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}$. By Lemma 2, $\sum_{i\in N} z_i \geq \sum_{i\in N} x_i = \sum_{i\in N} z_i^*$, $z_1 \geq \min\{x_1, \frac{\sum_{i\in N} x_i}{n}\} = \min\{z_1^*, \frac{\sum_{i\in N} z_i^*}{n}\}$, and $z_2 \geq \min\{y_2, \frac{\sum_{i\in N} y_i}{n}\} = \min\{z_2^*, \frac{\sum_{i\in N} z_i^*}{n}\}$. By Lemma 2, $z \succsim z^*$. $\square$

Let (φ, μ) be such that for each $(G, v) \in \mathcal{P}$, φ selects all efficient path(s) in (G, v), and $\mu(G, v) = \max_{\succsim}\{r(\sum_{e \text{ in } L} v(e)) : L \text{ in } G\}$.

Proposition 3. *The solution (φ, μ) is continuously $(r, \succsim)$ - rationalizable.*

Proof. Note that for each pair $x \geq y$, $r(x) \geq r(y)$, and thus $r(x) \succsim r(y)$. Hence, by the definition of (φ, μ) and Proposition 2, (φ, μ) is $(r, \succsim)$ - rationalizable.

To see that the solution is continuously $(r, \succsim)$ - rationalizable, let $g : r(\mathbb{R}^n_+)^2 \to r(\mathbb{R}^n_+)$, $x, y \in r(\mathbb{R}^n_+)$ and $\{x^t\}_{t=1}^{\infty}, \{y^t\}_{t=1}^{\infty}$ be two sequences of elements of $r(\mathbb{R}^n_+)$ such that $\lim_{t\to\infty} x^t = x$ and $\lim_{t\to\infty} y^t = y$. Let $z := \max\{x, y\}$ and for each $t \in \mathbb{N}$, $z^t := \max\{x^t, y^t\}$. Let $m := |S_1|$. Suppose without loss of generality that $\sum_{i\in N} x_i \geq \sum_{i\in N} y_i$. If $\sum_{i\in N} x_i > \sum_{i\in N} y_i$, then for sufficiently large $t \in \mathbb{N}$, $\sum_{i\in N} x_i^t > \sum_{i\in N} y_i^t$. By the proof of Proposition 2, $z_1 = med\{\min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}, x_1, \frac{1}{m}[\sum_{i\in N} x_i - (n - m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}]\}$, $z_2 = \frac{\sum_{i\in N} x_i - mz_1}{n-m}$, and for sufficiently large t, $z_1^t = med\{\min\{y_1^t, \frac{\sum_{i\in N} y_i^t}{n}\}, x_1^t, \frac{1}{m}[\sum_{i\in N} x_i^t -$

$(n-m)\min\{y_2^t, \frac{\sum y_i^t}{n}\}]\}$, and $z_2^t = \frac{\sum_{i\in N} x_i^t - mz_1^t}{n-m}$. Clearly, $\lim_{t\to\infty} z^t = z$. If $\sum_{i\in N} x_i = \sum_{i\in N} y_i$, then $z_1 = med\{\min\{y_1, \frac{\sum_{i\in N} y_i}{n}\}, x_1, \frac{1}{m}[\sum_{i\in N} x_i - (n-m)\min\{y_2, \frac{\sum_{i\in N} y_i}{n}\}]\} = med\{\min\{x_1, \frac{\sum_{i\in N} x_i}{n}\}, y_1, \frac{1}{m}[\sum_{i\in N} y_i - (n-m)\min\{x_2, \frac{\sum_{i\in N} x_i}{n}\}]\}$, and $z_2 = \frac{\sum_{i\in N} x_i - mz_1}{n-m} = \frac{\sum_{i\in N} y_i - mz_1}{n-m}$. For each t, if $\sum_{i\in N} x_i^t \geq \sum_{i\in N} y_i^t$, then $z_1^t = med\{\min\{y_1^t, \frac{\sum_{i\in N} y_i^t}{n}\}, x_1^t, \frac{1}{m}[\sum_{i\in N} x_i^t - (n-m)\min\{y_2^t, \frac{\sum_{i\in N} y_i^t}{n}\}]\}$ and $z_2^t = \frac{\sum_{i\in N} x_i^t - mz_1^t}{n-m}$. If $\sum_{i\in N} x_i^t < \sum_{i\in N} y_i^t$, then $z_1^t = med\{\min\{x_1^t, \frac{\sum_{i\in N} x_i^t}{n}\}, y_1^t, \frac{1}{m}[\sum_{i\in N} y_i^t - (n-m)\min\{x_2^t, \frac{\sum_{i\in N} x_i^t}{n}\}]\}$ and $z_2^t = \frac{\sum_{i\in N} y_i^t - mz_1^t}{n-m}$. It can be readily seen that $\lim_{t\to\infty} z^t = z$. □

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