4-2018

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DOI: https://doi.org/10.1145/3200692

Published in ACM Transactions on Intelligent Systems
and Technology,
Vol 9, Issue 5, April 2018,
DOI 10.1145/3200692

Combination Forecasting Reversion Strategy for Online Portfolio Selection

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Machine learning and artificial intelligence techniques have been applied to construct online portfolio selection strategies recently. A popular and state-of-the-art family of strategies is to explore the reversion phenomenon through online learning algorithms and statistical prediction models. Despite gaining promising results on some benchmark datasets, these strategies often adopt a single model based on a selection criterion (e.g., breakdown point) for predicting future price. However, such model selection is often unstable and may cause unnecessarily high variability in the final estimation, leading to poor prediction performance in real datasets and thus non-optimal portfolios. To overcome the drawbacks, in this article, we propose to exploit the reversion phenomenon by using combination forecasting estimators and design a novel online portfolio selection strategy, named *Combination Forecasting Reversion* (CFR), which outputs optimal portfolios based on the improved reversion estimator. We further present two efficient CFR implementations based on online Newton step (ONS) and online gradient descent (OGD) algorithms, respectively, and theoretically analyze their regret bounds, which guarantee that the online CFR model performs as well as the best CFR model in hindsight. We evaluate the proposed algorithms on various real markets with extensive experiments. Empirical results show that CFR can effectively overcome the drawbacks of existing reversion strategies and achieve the state-of-the-art performance.

CCS Concepts: • **Applied computing** → *Law, social and behavioral sciences*; *Economics*; • **Computing methodologies** → *Artificial intelligence*; *Machine learning*; *Online learning settings*;

Additional Key Words and Phrases: Portfolio selection, online learning, mean reversion, combination forecasting reversion, combination forecasting estimators

This work was partially supported by the National Natural Science Foundation of China (11501204, 71401128, U1711262), the Natural Science Foundation of Shanghai (15ZR1408300), the Program of Science and Technology Innovation Action of Science and Technology Commission of Shanghai Municipality (STCSM) (17511105204), Academic Team Building Plan for Young Scholars from Wuhan University (WHU2016012), and Singapore Ministry of Education Academic Research Fund Tier 1 Grant (14-C220-SMU-016).
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1 INTRODUCTION

Online portfolio selection (OLPS) aims to determine an investment allocation for a set of assets and dynamically change it on the fly. Recent years have witnessed much research effort from artificial intelligence and machine-learning researchers [11, 48, 49] to design OLPS strategy [1, 7, 14, 15, 24, 33] through online learning algorithms and statistical prediction models. Among them, one class of representative and state-of-the-art studies is the reversion strategies [7, 29, 34, 38, 41], which capture and utilize the mean or median reversion phenomena in the financial markets to maximize the cumulative return on investment.

Though these reversion algorithms achieve promising results, they perform poorly on certain datasets, e.g., DJIA [38, 41]. A key reason is that these strategies often select a single model and ignore the non-stationary nature of financial time series [4], such as mean or median model, whose prediction is based on a selection criterion (e.g., breakdown point) or general data characteristics (e.g., noisy data and outliers) [29]. However, the issue of model selection is highly non-trivial in time series analysis and forecasting. One major drawback of model selection is its instability, which may cause an unnecessarily high variability in final estimation/prediction [60] and thus likely lead to sub-optimal portfolios. Furthermore, the assumption of single-period prediction [33, 41] also leads to estimation error, thus making the performance extremely poor [34].

To address the above drawbacks, we propose a novel multi-period OLPS strategy named *Combination Forecasting Reversion* (CFR), which explicitly estimates the next price relative by combining forecasting estimator and is more accurate than traditional simple mean/median estimators. To handle the non-stationary characteristics of price time series, we first use the ARIMA model to preestimate the next price [42]. Then, we apply online learning theory to find the optimal portfolios by exploiting the reversion property with the combination forecasting estimator. The proposed combination forecasting scheme in this work specifically aggregates four types of different predictive models, including mean estimator [34], median estimator [29], ARMA [2], and ARIMA models [42]. In particular, ARIMA can deal with the non-stationary characteristics of price time series [42], while the median estimator is robust to noisy data and outliers [29]. Thus, we can apply online learning algorithm with CFR for modeling non-stationarity of time series data with ARIMA for OLPS and for dynamically improving the robustness of OLPS in complex scenarios. We also develop two efficient implementations of the combination forecasting estimator based on online Newton step (ONS) and online gradient descent (OGD) algorithms, and we analyze their regret bounds theoretically, which guarantee that the online CFR strategy is provably as well as the best CFR strategy in hindsight.

To the best of our knowledge, CFR is the first OLPS algorithm that exploits the reversion phenomenon by applying the combination forecasting estimator of aggregating multiple time series predictive models with online learning theory. Though simple in nature, CFR is more robust than existing algorithms and empirically achieves significantly better performance in terms of cumulative wealth. Besides, CFR is robust to different parameter settings and can withstand a reasonable transaction cost.

As a summary, the main contributions of this article include the following:

(1) We propose a novel multi-period OLPS strategy named CFR, which explicitly estimates the next price relative via combination forecasting estimator and is more accurate than simple mean/median estimators.
(2) We exploit two types of combination forecasting estimation based on ONS and OGD algorithms to obtain optimal approximal solution.
(3) We obtain the theoretical regret bound of the CFR algorithms, which guarantee that the online CFR strategy is as good as the best CFR strategy in hindsight.
(4) We conduct extensive experiments to empirically evaluate the proposed CFR algorithms by comparing with various state-of-the-art algorithms, in particular mean reversion algorithms.

The rest of the article is organized as follows. Section 2 formulates the online portfolio selection problem and reviews some related work. Section 3 presents the proposed algorithm and gives some theoretical results. Section 4 empirically evaluates CFR's efficacy on real markets. Section 5 summarizes the article.

2 PRELIMINARIES AND RELATED WORK

In this section, we first describe the problem setting and then introduce and analyze related work.

2.1 Problem Setting

Consider an investment task over a financial market with d assets for n trading periods. On the tth period, the asset prices are represented by a *close price vector* $\mathbf{p}_t \in \mathbb{R}_+^d$, and each element p_t^i represents the close price of asset i. The changes of asset prices are represented by a *price relative vector* $\mathbf{x}_t = (x_t^1, \dots, x_t^d) \in \mathbb{R}_+^d$, where x_t^i indicates the ratio of close price to last close price of asset i at the tth period, i.e., $x_t^i = p_t^i / p_{t-1}^i$. Let us denote $\mathbf{x}_1^n = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ as the sequence of price relative vectors for n periods.

At the beginning of the tth period, we diversify the capital among the d assets according to a *portfolio vector* $\mathbf{b}_t = (b_t^1, \dots, b_t^d) \in \mathbb{R}_+^d$, where b_t^i represents the proportion of wealth invested in the asset i. Typically, we assume the portfolio is self-financed and no margin/short is allowed, which means $\mathbf{b}_t \in \Delta_d$, where $\Delta_d = \{\mathbf{b}_t : \mathbf{b}_t \in \mathbb{R}_+^d, \sum_{i=1}^d b_t^i = 1\}$. The investment procedure is represented by a *portfolio strategy*, that is, $\mathbf{b}_1 = \frac{1}{d}\mathbf{1}$ and following sequence of mappings $\mathbf{b}_t : (\mathbb{R}_+^d)^{t-1} \to \Delta_d$, $t = 2, 3, \dots$, where $\mathbf{b}_t = \mathbf{b}_t(\mathbf{x}_1^{t-1})$ is the portfolio used on the tth trading period given past market sequence $\mathbf{x}_1^{t-1} = (\mathbf{x}_1, \dots, \mathbf{x}_{t-1})$. We then denote by $\mathbf{b}_1^n = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ the portfolio strategy for n periods.

On the tth trading period, a portfolio $\mathbf{b}_t$ achieves a *portfolio period return* s_t, that is, the wealth increases by a factor of $s_t = \mathbf{b}_t^T \mathbf{x}_t = \sum_{i=1}^d b_t^i x_t^i$. Since we reinvest and adopt price relative, the portfolio wealth would multiplicatively grow. Thus, after n trading periods, a portfolio strategy $\mathbf{b}_1^n$ produces a *portfolio cumulative wealth* S_n, which increases the initial wealth by a factor of $\prod_{t=1}^n \mathbf{b}_t^T \mathbf{x}_t$, that is, $S_n(\mathbf{b}_1^n, \mathbf{x}_1^n) = S_0 \prod_{t=1}^n (\mathbf{b}_t^T \mathbf{x}_t)$, where S_0 is the initial wealth, which is set to 1 in this article.

The OLPS problem can be formulated as a sequential decision task. The portfolio manager aims to design a strategy $\mathbf{b}_1^n$ to maximize the portfolio cumulative wealth S_n. The portfolios are selected in a sequential fashion. In each period t, given the historical information, the manager learns to select a new portfolio vector $\mathbf{b}_t$ for the next price relative vector $\mathbf{x}_t$, where the decision criterion varies among different managers. The resulting portfolio $\mathbf{b}_t$ is scored based on the portfolio period return of s_t. Such procedure repeats until the end of trading periods and the portfolio strategy is finally scored by the cumulative wealth S_n.

Table 1. Description of Notations

Notation
d: number of assets
n: trading periods to be invested
p_t^i: close price of asset i
$\mathbf{p}_t$: close price vector
$x_t^i = p_t^i / p_{t-1}^i$: ratio of close price to last close price of asset i
$\mathbf{x}_t = (x_t^1, \ldots, x_t^d) \in \mathbb{R}_+^d$: price relative vector
$\mathbf{x}_1^n = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$: sequence of price relative vectors for n periods
$\tilde{\mathbf{x}}_i^t$: predictive value of ith estimator on tth period
$\Delta_d = \{\mathbf{b}_t \in \mathbb{R}_+^d : \sum_{i=1}^d b_t^i = 1\}$: portfolio is self-financed and no margin/short is allowed
b_t^i: proportion of wealth invested in the asset i
$\mathbf{b}_t = (b_t^1, \ldots, b_t^d) \in \mathbb{R}_+^d$: portfolio vector
$\mathbf{b}_1^n = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$: sequence of portfolio vectors for n periods
s_t: portfolio period return
S_n: portfolio cumulative wealth
A_0, B_0, C_0: exact price sequences
A_1, B_1, C_1: sequences contaminated by an outlier of 10
"?": price to be estimated
Acc: accurate target price
k: number of estimators
w_i: weight of ith estimator
$\mathbf{w}$: weight vector
ℓ_t: loss function

As with most of the existing works [34], in the above model, we make several general assumptions:

(1) *Transaction cost*: We assume no transaction cost or taxes in this OLPS model;
(2) *Market liquidity*: We assume that one can buy and sell required quantities at last closing price of any given trading period;
(3) *Impact cost*: We assume that market behavior is not affected by a OLPS strategy.

These assumptions are not trivial, which has been explained in all existing work (refer to Section 2.2 for detail). We will empirically analyze the effects of transaction costs in Section 4.

We also summarize all the frequently used symbols that appear in the above model and the rest of the article in Table 1, to make it easy to follow.

2.2 Related Work

OLPS is an important topic in online finance [32, 56] and in the AI community [1, 24, 29, 34, 36, 38, 54]. It has been extensively explored following the principle of Kelly investment [31]. *Constantly rebalanced portfolios* (CRP) keeps fixed weight for each asset on all periods. *Best CRP* (BCRP) [14], the best CRP strategy over a whole market sequence in hindsight, is an optimal strategy if the market is i.i.d. [16]. *Successive CRP* (SCRP) [22] and *online Newton step* (ONS) [1] implicitly estimate the next price relative via all historical price relatives with a uniform probability.

Besides estimation via all historical price relatives, some strategies predict the next price relatives by selecting a set of similar price relatives. *Nonparametric kernel based moving window* (B^K) [24] measures the similarity by kernel method. Following the same framework, *nonparametric nearest neighbor* (B^{NN}) [25] locates the set of price relatives via nearest-neighbor method. Li et al. [36] proposed *correlation-driven nonparametric learning* (CORN), which measures the similarity via correlation.

Moreover, another type of estimation is to predict the next price relative via a single-value prediction. *Exponential gradient* (EG) [28] estimates next price relative as last price relative. *Passive-aggressive mean reversion* (PAMR) [41] and *confidence-weighted mean reversion* (CWMR) [38] estimate next price as the inverse of last price relative, which is in essence the mean reversion principle. Recently, Li and Hoi [34, 37] proposed *online moving average reversion* (OLMAR), which predicts the next price relative using moving averages and explores the multi-period mean reversion. Huang et al. [29] proposed a *robust median reversion* (RMR) algorithm by using L_1-median estimator. Some other algorithms do not focus on estimation. *Universal portfolios* (UP) [14, 15] is the historical performance-weighted average of all CRPs. *Anti-correlation* (Anticor) [7] adopts the consistency of positive lagged cross-correlation and negative autocorrelation to adjust the portfolio.

2.3 Analysis of Existing Work

We analyze the mean reversion strategies PAMR, OLMAR, and RMR, which belong to "follow the loser" category [35] and are closely related to this article. Let us consider the estimation methods of existing work. In practice, a Kelly portfolio manager [31, 52] first predicts $\tilde{\mathbf{x}}_{t+1}$ in terms of k possible values $\tilde{\mathbf{x}}^1_{t+1}, \ldots, \tilde{\mathbf{x}}^k_{t+1}$ and their corresponding probabilities $p_1, \ldots, p_k$, where $\tilde{\mathbf{x}}^i_{t+1}$ denotes one possible combination vector of individual price relative predictions. Then he/she can figure out a portfolio by maximizing the expected log return on these possible combinations, i.e.,

$$\mathbf{b}_{t+1} = \arg\max_{\mathbf{b}\in\Delta_d} \sum_{i=1}^{k} p_i \log\left(\mathbf{b} \cdot \tilde{\mathbf{x}}^i_{t+1}\right).$$

As different estimation methods result in different $\tilde{\mathbf{x}}^i_{t+1}$ and p_i and leading to different portfolios, an accurate estimation method is crucial to the success of a strategy.

PAMR implicitly assumes $\tilde{\mathbf{x}}^1_{t+1} = \frac{1}{\mathbf{x}_t}$ with $p_1 = 1$, i.e., it estimates next price relative as the inverse of last price relative. From the price perspective [34], it implicitly assumes that next price $\tilde{\mathbf{p}}_{t+1}$ will revert to last price $\mathbf{p}_{t-1}$,

$$\tilde{\mathbf{x}}_{t+1} = \frac{1}{\mathbf{x}_t} \Rightarrow \frac{\tilde{\mathbf{p}}_{t+1}}{\mathbf{p}_t} = \frac{\mathbf{p}_{t-1}}{\mathbf{p}_t} \Rightarrow \tilde{\mathbf{p}}_{t+1} = \mathbf{p}_{t-1},$$

where $\mathbf{x}$ and $\mathbf{p}$ are all vectors and the above operations are element-wise. Rather than $\tilde{\mathbf{p}}_{t+1} = \mathbf{p}_{t-1}$, OLMAR estimates the next price as a moving average at the end of the tth period, i.e., $\tilde{\mathbf{p}}_{t+1} = MA_t(w) = \frac{1}{w}\sum_{i=t-w+1}^{t} \mathbf{p}_i$, where $MA_t(w)$ denotes the moving average with a w-window. RMR estimates the next price as a median at the end of the tth period, i.e., $\tilde{\mathbf{p}}_{t+1} = L_1 med_{t+1}(w) = \arg\min_{\boldsymbol{\mu}} \sum_{i=0}^{w-1} \|\mathbf{p}_{t-i} - \boldsymbol{\mu}\|^2$, where $\|\cdot\|$ is L_2 norm.

These estimators pursue simple mean value evaluation or robustness to noise and outlier. However, they do not consider the statistically intrinsic properties of the data and prediction model. There are two potential problems with the estimation process. First, stock price sequences are always non-stationary, and hence the estimated values of the existing mean reversion are not very accurate. Second, almost all the mean or median reversion algorithms use single prediction model,

Table 2. Illustration of Different Price Estimation Methods on Toy Data

Price: $t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \ldots$	Acc	OLMAR	RMR	CFR
A_0 : 1, 2, 4, 8, ?	16	3.75	3	16
A_1 : 1, 2, (10), 8, ?	16	5.25	5	15.16
B_0 : 1, 2, 4, 2, 4, 8, ?	4	3.5	3	4
B_1 : 1, 2, 4, 2, (10), 8, ?	4	4.5	3	3.98
C_0 : 1, 0.5, 0.25, 0.5, 0.25, 0.125, ?	0.25	0.44	0.25	0.25
C_1 : 1, 0.5, 0.25, 0.5, (10), 0.125, ?	0.25	2.06	0.25	0.25

Note: A_0 and A_1, B_0 and B_1, and C_0 and C_1 represent price sequences. A_0, B_0, and C_0 are the exact price sequences, and A_1, B_1, and C_1 are price sequences contaminated by an outlier of 10. "Acc" is the accurate target price for forecasting.

which usually has unstable estimated values and thus makes the non-optimal portfolios. These two drawbacks motivate the proposed methods in this article.

3 COMBINATION FORECASTING REVERSION STRATEGY

In this section, before we present the proposed CFR strategy, we first consider a toy example to show the drawbacks of non-stationary price sequences and the unstability of single-model prediction.

3.1 Motivating Example

The toy market consists of one volatile stock, and t_i ($i \geq 0$) denotes the period that requires estimation. Several sequences of price time series are listed in Table 2. Note that the price sequences are non-stationary due to the long-term trend. A_0 and A_1 are exponential-type price sequences and their prices change by exponential factor 2. B_0 and B_1 are growth-oriented price sequences and their prices change by sequent factor of $2, 2, \frac{1}{2}, 2, 2, \frac{1}{2}, \ldots$. For example, let P_{t_i} be the price of the ith period, then $P_{t_1} = P_{t_0} \times 2 = 1 \times 2 = 2, P_{t_2} = P_{t_1} \times 2 = 2 \times 2 = 4, P_{t_3} = P_{t_2} \times \frac{1}{2} = 4 \times \frac{1}{2} = 2, \ldots$. C_0 and C_1 are attenuation-type price sequences, and the price changes by sequent factor of $\frac{1}{2}, \frac{1}{2}, 2, \frac{1}{2}, \frac{1}{2}, 2, \ldots$. Moreover, A_0, B_0, and C_0 are exact price sequences, while A_1, B_1, and C_1 are the sequences contaminated by an outlier of 10. "?" denotes the price to be estimated and *Acc* is the accurate target price. The estimated prices clearly show that the next prices estimated by either OLMAR or RMR (except C_0, C_1) are far away from the accurate values, which thus leads to inaccurate price and sub-optimal portfolios.

In contract to OLMAR and RMR, the estimated next prices of different scenarios by the proposed CFR approach in Table 2 (the detailed calculation is omitted) are clearly more accurate. In particular, for all types of price sequences A_0, A_1, B_0, B_1, and C_0, C_1, our new estimator is much closer to the targets than OLMAR and RMR, indicating that the new method can better deal with non-stationary multiple-period price sequences. For the contaminated price sequences A_1, B_1, and C_1, CFR can handle noise and outliers as well as RMR, and it is also closer to the target prices than other estimators, validating its robustness to noise and/or outliers. Although the toy example has only a single asset, it can be easily extended to the scenario of multiple assets. The key motivation behind the proposed CFR strategy is that the financial market is often highly complex and very difficult to make accurate forecasting by any single forecasting model. Therefore, instead of using a single model for forecasting, we are prepared to explore the combination forecasting [60], which is often considered as a successful alternative for time series forecasting in References [5, 55].

3.2 Formulation

To better exploit the reversion property meanwhile avoid suffering from the limitations of existing strategies, we propose a new strategy for online portfolio selection, named *Combination Forecasting Reversion* (CFR), which is based on the idea of obtaining the next price relative $\tilde{\mathbf{x}}_{t+1}$ by using *Combination Forecasting* (CF) estimator, and then maximize the expected return $\mathbf{b} \cdot \tilde{\mathbf{x}}_{t+1}$ while keeping last portfolio information via regularization.

Specifically, rather than $\tilde{\mathbf{p}}_{t+1} = MA_t(w)$ or $\tilde{\mathbf{p}}_{t+1} = L_1 med_{t+1}(w)$ obtained by single estimator, it estimates next price by combining multiple estimators at the end of the tth period, i.e., $\tilde{\mathbf{p}}_{t+1} = CF_{t+1}(k) = \sum_{i=1}^{k} w_i \tilde{\mathbf{p}}_i^{t+1}$. Then, the expected price relative with the CF estimator is

$$\tilde{\mathbf{x}}_{t+1}(k) = \frac{CF_{t+1}(k)}{\mathbf{p}_t} = \sum_{i=1}^{k} w_i \tilde{\mathbf{x}}_i^{t+1}, \tag{1}$$

where k is the number of estimators, w_i is the weight of the ith estimator, and $\tilde{\mathbf{x}}_i^{t+1}$ is the predictive value of the ith estimator on $(t+1)$th period.

The combination forecasting community has been extensively explored [13, 30, 57]. In general, there are two modes for combination forecasting, that is, weight synthesis and regional synthesis, where the former is widely used in research community. In this regard, there are many ways to determine the combined weight, such as arithmetic mean, mean square reciprocal and variance reciprocal [3, 53, 60]. In this article, to estimate the next price relative $\tilde{\mathbf{x}}_{t+1}$ of Equation (1), we will calculate the CF estimator of historical prices based on online learning theory.

Online learning is a method of machine learning in which data becomes available in a sequential order and is used to update our best predictor for future data at each step, as opposed to batch learning techniques that generate the best predictor by learning on the entire training data set at once. Online learning is generally defined in a game-theoretic framework, where the data, rather than being chosen stochastically, are chosen arbitrarily, possibly by a powerful adversary with full knowledge of our learning algorithm. In this article, the procedure of online setting is as follows.

At period t, we need to make a prediction $\tilde{\mathbf{p}}_t$ for the price, after which the real price $\mathbf{p}_t$ is revealed, and we suffer a loss denoted by $\ell_t(\mathbf{p}_t, \tilde{\mathbf{p}}_t)$. Our goal is to minimize the sum of losses over a predefined number of iterations T. A reasonable benchmark is to try to be not much worse than the best case. More precisely, let

$$\ell_t(\mathbf{w}) = \ell_t(\mathbf{p}_t, \tilde{\mathbf{p}}_t(\mathbf{w})) = \ell_t\left(\mathbf{p}_t, \sum_{i=1}^{k} w_i \tilde{\mathbf{p}}_i^t\right) \tag{2}$$

denote the loss of the prediction given by CF estimator with weight $\mathbf{w}$. We define the regret as

$$\text{Regret}_T(\text{Alg}) = \sum_{t=1}^{T} \ell_t(\mathbf{w}^t) - \min_{\mathbf{w}} \sum_{t=1}^{T} \ell_t(\mathbf{w}). \tag{3}$$

We wish to obtain an efficient algorithm, whose regret grows sublinearly in T, corresponding to an average per-round regret going to zero as T increases. Below, we present two combination forecasting estimators by using two popular online convex optimization methods: the first is so called "Combination Forecasting based on Online Newton Step" [26, 27, 59], we name it as *CF-ONS*; the second is so called "Combination Forecasting based on Online Gradient Descent" [26, 27, 59], we name it as *CF-OGD*. These two estimators are illustrated in Section 3.3.

To this end, we can calculate the expected price relative following the idea of so called combination forecasting reversion. Based on the two combination forecasting estimators, we can infer two types of CFR by Equation (4):

Combination Forecasting Estimators: CF-ONS and CF-OGD

$$\tilde{\mathbf{x}}_{t+1}(k) = \begin{cases} \dfrac{\text{CF-ONS}(k)}{\mathbf{p}_t} = \displaystyle\sum_{i=1}^{k} w_i^{ONS} \tilde{\mathbf{x}}_i^{t+1} \\ \dfrac{\text{CF-OGD}(k)}{\mathbf{p}_t} = \displaystyle\sum_{i=1}^{k} w_i^{OGD} \tilde{\mathbf{x}}_i^{t+1} \end{cases}, \tag{4}$$

where k is the number of estimators, w_i^{ONS} and w_i^{OGD} are the weights of the ith estimator obtained by CF-ONS and CF-OGD, respectively.

Based on the obtained price relative $\tilde{\mathbf{x}}_{t+1}$ in Equation (4), we further adopt the idea of an effective online learning algorithm, that is, passive-aggressive (PA) learning [17] to exploit combination forecasting reversion. Generally proposed for classification, PA passively keeps the previous solutions if the classification is correct, while aggressively approaches a new solution if the classification is incorrect. Thus, following similar idea to PAMR/OLMAR [34, 41], we can formulate the optimization as follows.

Optimization Problem: CFR for OLPS

$$\mathbf{b}_{t+1} = \arg\min_{\mathbf{b}\in\Delta_d} \frac{1}{2}\|\mathbf{b} - \mathbf{b}_t\|^2 \;\; s.t. \;\; \mathbf{b}\cdot\tilde{\mathbf{x}}_{t+1} \geq \epsilon \tag{5}$$

The above optimization problem attempts to find an optimal portfolio by minimizing the deviation from last portfolio $\mathbf{b}_t$ subject to $\mathbf{b}\cdot\tilde{\mathbf{x}}_{t+1} \geq \epsilon$. On the one hand, if the constraint is satisfied, that is, the expected return is higher than a threshold, then the resulting portfolio equals to the previous portfolio. On the other hand, if the constraint is not satisfied, then the formulation will figure out a new portfolio such that the expected return is higher than the threshold, while the new portfolio is not far from the last one. This explicitly reflects the idea of exploiting reversion principle in our OLPS method. In fact, $\tilde{\mathbf{x}}_{t+1}$ is the price relative estimated via these estimators, while the constraint $\mathbf{b}\cdot\tilde{\mathbf{x}}_{t+1} \geq \epsilon$ means that next price will revert to the estimators and guarantees a certain return on the portfolio.

3.3 Algorithms

Before presenting the CF-ONS and CF-OGD estimators for the combination forecasting of historical prices, we need to define the following parameters. The decision set K is the set of available candidates (k-dimensional vector) at each iteration, which is defined as $K = \{\mathbf{w} \in \mathbb{R}^k : \sum_{j=1}^{k} w_j = 1, w_j \in [0, 1]\}$. We denote by D the diameter of K, and bound as, $D = \sup_{\mathbf{w}_1,\mathbf{w}_2\in K} \| \mathbf{w}_1 - \mathbf{w}_2 \|_2 = \sqrt{k}$. Next, we denote by G the upper-bound of $\| \nabla\ell_t(\mathbf{w}) \|$ for all t and $\mathbf{w} \in K$. This parameter depends on the loss function, and its computation is done accordingly. For example, for squared loss, we get that $G = D$. Finally, we denote by α the exp-concavity parameter of the loss functions $\{\ell_t\}_{t=1}^{T}$, i.e., it holds that $exp(-\alpha\ell_t(\mathbf{w}))$ is concave for all t. This parameter is relevant only for exp-concave loss functions, and its computation is also done according to the loss function considered. It can be shown that $\alpha = \frac{1}{2k}$ when the squared loss is considered. Using these parameters, the CF-ONS and CF-OGD estimators are illustrated by Algorithm 1 and 2, respectively.

Algorithm 1 uses ONS to provide CF estimation. It shows how to choose $\mathbf{w}^t$ in each iteration, when the loss functions $\{\ell_t\}_{t=1}^{T}$ are assumed to be α-exp-concave in $\mathbf{w}$. There are two key updates as follows:

$$A_j = A_{j-1} + \nabla_j\nabla_j^T, \;\; \mathbf{w}^{j+1} = \prod_{\mathcal{K}}^{A_j}\left(\mathbf{w}^j - \frac{1}{\eta}A_j^{-1}\nabla_j\right).$$

ALGORITHM 1: CF-ONS(k, η)

Input: Parameter $k \geq 2$, learning rate η.
Output: Estimated $\tilde{\mathbf{x}}_{t+1}$.
Procedure:
Initialization: $\mathbf{w}^0 \in K$, $A_0 = \varepsilon I$;
for $j = 1$ ***to*** $t+1$ **do**
Predict next price vector:

$$\tilde{\mathbf{p}}_j(\mathbf{w}^j) = \sum_{i=1}^{k} w_i^j \tilde{\mathbf{p}}_i^j;$$

Receive $\mathbf{p}_j$ and incur loss $\ell_j(\mathbf{w}^j)$;
Let $\nabla_j = \nabla \ell_j(\mathbf{w}^j)$, update the matrix:

$$A_j = A_{j-1} + \nabla_j \nabla_j^T;$$

Update the parameter vector:

$$\mathbf{w}^{j+1} = \prod_K^{A_j}(\mathbf{w}^j - \frac{1}{\eta} A_j^{-1} \nabla_j);$$

end
$\tilde{\mathbf{x}}_{t+1} = \tilde{\mathbf{p}}_{t+1}(\mathbf{w}^{t+1})/\mathbf{p}_t$.

Here, $\prod_K^{A_j}$ is the projection in the norm induced by A_j, i.e., $\prod_K^{A_j}(y) = \arg\min_{x \in K}(x-y)^T A_j (x-y)$. In case the dimension k of A_t is large, we note that its inverse can be efficiently re-computed after each update using the Sherman-Morrison formula, i.e.,

$$A_t^{-1} = \left(A_{t-1} + \nabla_t \nabla_t^T\right)^{-1} = A_{t-1}^{-1} - \frac{A_{t-1}^{-1} \nabla_t \nabla_t^T A_{t-1}^{-1}}{1 + \nabla_t^T A_{t-1}^{-1} \nabla_t}.$$

Algorithm 2 for choosing $\mathbf{w}^t$ at each time point is based on OGD. Specifically, it updates the weight $\mathbf{w}^t$ by

$$\mathbf{w}^{j+1} = \prod_K \left(\mathbf{w}^j - \frac{1}{\eta} \nabla_j\right).$$

Here, $\prod_K$ refers to the Euclidean projection onto K, i.e., $\prod_K(y) = \arg\min_{x \in K} \parallel x - y \parallel_2$. This algorithm is applicable to general convex loss functions, as well as to exp-concave ones. It is computationally simpler but has a somewhat worse theoretical result compared to the previous one, when considering an exp-concave loss function.

After obtaining the next price relative, we can obtain the final portfolio selection formula by solving the Optimization Problem. Note that Equation (5) is a convex optimization problem with constrained, and thus, it can be solved directly via the generalised Lagrange multiplier method [8]. Finally, parameter λ_{t+1} and portfolio $\mathbf{b}$ are updated as follows. Please refer to Appendix A.1 for detailed derivation.

Proposition 3.1. *The solution of optimization problem without considering the non-negativity constraint is*

$$\mathbf{b}_{t+1} = \mathbf{b}_t - \lambda_{t+1}(\tilde{\mathbf{x}}_{t+1} - \overline{x}_{t+1} \cdot \mathbf{1}),$$

where $\overline{x}_{t+1} = \frac{1}{d}(\mathbf{1} \cdot \tilde{\mathbf{x}}_{t+1})$ *denotes the average predicted price relative and* λ_{t+1} *is the Lagrangian multiplier calculated as*

$$\lambda_{t+1} = \min\left\{0, \frac{\tilde{\mathbf{x}}_{t+1}\mathbf{b}_t - \epsilon}{\|\tilde{\mathbf{x}}_{t+1} - \overline{x}_{t+1} \cdot \mathbf{1}\|^2}\right\}.$$

ALGORITHM 2: CF-OGD(k, η)

Input: Parameter $k \geq 2$, learning rate η.
Output: Estimated $\tilde{\mathbf{x}}_{t+1}$.
Procedure:
Initialization: $\mathbf{w}^0 \in K$;
for $j = 1$ ***to*** $t + 1$ **do**
Predict next price vector:

$$\tilde{\mathbf{p}}_j(\mathbf{w}^j) = \sum_{i=1}^{k} w_i^j \tilde{\mathbf{p}}_i^j;$$

Receive $\mathbf{p}_j$ and incur loss $\ell_j(\mathbf{w}^j)$;
Let $\nabla_j = \nabla \ell_j(\mathbf{w}^j)$, update the parameter vector:

$$\mathbf{w}^{j+1} = \prod_K (\mathbf{w}^j - \frac{1}{\eta}\nabla_j);$$

end
$\tilde{\mathbf{x}}_{t+1} = \tilde{\mathbf{p}}_{t+1}(\mathbf{w}^{t+1})/\mathbf{p}_t$.

ALGORITHM 3: CFR($\epsilon, \tilde{\mathbf{x}}_{t+1}, \mathbf{b}_t$)

Input: Reversion threshold $\epsilon > 1$, predicted next price relative vector $\tilde{\mathbf{x}}_{t+1}$, current portfolio $\mathbf{b}_t$.
Output: Next portfolio $\mathbf{b}_{t+1}$.
Procedure:
Calculate the following variable:

$$\lambda_{t+1} = \min \left\{ 0, \frac{\tilde{\mathbf{x}}_{t+1}\mathbf{b}_t - \epsilon}{\|\tilde{\mathbf{x}}_{t+1} - \overline{x}_{t+1} \cdot \mathbf{1}\|^2} \right\};$$

Update the portfolio:

$$\mathbf{b}_{t+1} = \mathbf{b}_t - \lambda_{t+1}(\tilde{\mathbf{x}}_{t+1} - \overline{x}_{t+1} \cdot \mathbf{1});$$

Normalization:

$$\mathbf{b}_{t+1} = \arg\min_{\mathbf{b} \in \Delta_d} \|\mathbf{b} - \mathbf{b}_{t+1}\|^2.$$

Note that, to ensure that the portfolio is non-negative, we finally project the above portfolio to the simplex domain [20], as shown in the last step of Algorithm 3.

To sum up, Algorithms 1 and 2 illustrate the CF estimation, where the price relatives $\tilde{\mathbf{x}}_{t+1}$ are based on ONS and OGD, respectively. Note that in Algorithms 1 and 2, in practice, we first use four basic estimators, MA [34], L_1-median [29], online ARMA [2], and online ARIMA [42] to calculate their next price $\tilde{\mathbf{p}}_{t+1}$, respectively, and then use the results to obtain the CF estimation. Algorithm 3 shows the proposed CFR procedure, which exploits the online PA learning. Finally, Algorithm 4 presents the overall CFR strategy (including CFR-ONS and CFR-OGD, which are based on CF-ONS and CF-OGD, respectively) for OLPS under the generic framework. Below, we give the theoretical results of regret for the main Algorithm 4.

3.4 Theoretical Results

Before giving the main theorems, we first put some necessary assumptions:

(1) The loss function $\{\ell_t\}_{t=1}^T$ are α-exp-concave in CFR-ONS. While the loss functions $\{\ell_t\}_{t=1}^T$ are strongly convex in CFR-OGD.

ALGORITHM 4: CFR Strategy for Online PS

Input: Reversion threshold $\epsilon>1$, window size $w\geq 2$, market sequence $\mathbf{x}_1^n$.
Output: Cumulative wealth after n periods S_n.
Procedure:
Initialization: $b_1 = \frac{1}{d}\mathbf{1}, S_0 = 1, \mathbf{p}_0 = \mathbf{1}$;
for $i = 1$ ***to*** n **do**
  $\mathbf{p}_i = \mathbf{x}_i \cdot \mathbf{p}_{i-1}$;
end
for $t = 1, 2, \ldots, n$ **do**
  Receive stock price: $\mathbf{x}_t$;
  Update cumulative return: $S_t = S_{t-1} \times (\mathbf{b}_t \cdot \mathbf{x}_t)$;
  Predict next price relative vector:

$$\tilde{\mathbf{x}}_{t+1} = \text{CF-ONS}(k, \eta) \text{ or CF-OGD}(k, \eta);$$

  Update the portfolio:

$$\mathbf{b}_{t+1} = \text{CFR}(\epsilon, \tilde{\mathbf{x}}_{t+1}, \mathbf{b}_t).$$

end

(2) The weight vector $\mathbf{w}$ satisfy $\mathbf{w} \in K$. It is also a standard assumption, and the decision set is generally set to be bounded.
(3) The price is bounded by a constant that is independent of T. Without loss of generality, we assume that $\mid p_t^i \mid < 1$ for all t.

For Algorithm 4, we can prove the following theorems, and thus bound the regrets of CFRs. Please refer to Appendices A.2 and A.3 for detailed proofs.

Theorem 3.2. *Let $k \geq 1$, and set $A_0 = \varepsilon I_k$, $\varepsilon = \frac{1}{\eta^2 D^2}$, $\eta = \frac{1}{2}\min\{4GD, \alpha\}$. Then, for any data sequence $\{\mathbf{p}_t\}_{t=1}^T$ that satisfies the assumption 3, Algorithm 1 generates an online sequence $\{\mathbf{w}^t\}_{t=1}^T$, for which the following holds:*

$$\text{Regret}_T(\text{CFR-ONS}) \leq O(\log(T)). \tag{6}$$

Theorem 3.3. *Let $k \geq 1$, and set $\eta = \frac{1}{H_t}$. Then, for any data sequence $\{\mathbf{p}_t\}_{t=1}^T$ that satisfies the assumption 3, Algorithm 2 generates an online sequence $\{\mathbf{w}^t\}_{t=1}^T$, for which the following holds:*

$$\text{Regret}_T(\text{CFR-OGD}) \leq O(\log T). \tag{7}$$

Remark 1. It should be noted that the combination method used in this article like ensemble-learning method in machine learning. Specifically, from the combination of learning algorithms [18], the CFR is equivalent to the Stacking [10] in ensemble learning. Ensemble learning [21, 45, 51, 58] is an important method in machine learning, it virtually shares the same theme as portfolio selection [43, 49]: namely, diversification. In Section 4, we will compare the CFR with two classical ensemble-learning algorithms (Bagging and AdaBoost) for online portfolio selection.

4 EXPERIMENTS

In this section, we use the *cumulative wealth*, *statistical test*, and other *performance criteria* to measure the performance of the proposed CFR algorithms and evaluate their effectiveness, by comparing with 14 existing strategies and 2 classical ensemble-learning methods [45], including the family of state-of-the-art strategies by exploiting reversion properties. We also conduct experiments on online ARIMA algorithm as a comparison, which is called *OLAR*.

Table 3. Summary of Real-World Benchmark Datasets

Dataset	Region	Time Frame	#Days	#Assets
NYSE(O)	U.S.	07/03/1962–12/31/1984	5,651	36
NYSE(N)	U.S.	01/01/1985–06/30/2010	6,431	23
DJIA	U.S.	01/01/2001–01/14/2003	507	30
MSCI	Global	04/01/2006–03/31/2010	1,043	24

4.1 Datasets

In our experiments, we adopt the historical daily prices in stock markets, which are easily obtained from Yahoo Finance.[1] Data from other types of markets, such as high-frequency intra-day quotes, currency, and commodity markets, are either expensive or hard to obtain and process, which can reduce the experimental reproducibility. Here, we test the portfolio strategies on four public datasets from real markets,[2] which are summarized in Table 3. For more detailed data statistics about the datasets, refer to Appendix A.4.

The first dataset "NYSE(O)" is the well-known NYSE dataset, one "standard" dataset pioneered by Cover [14] and followed by most subsequent researchers on the field of on-line portfolio selection in References [1, 24, 28]. This dataset contains 5,651 daily price relatives of 36 stocks in New York Stock Exchange (NYSE) for a 22-year period from July 3, 1962 to December 31, 1984.[3] The principle of the data collection is to first rank the NYSE stocks by market cap and then select the first 36 stocks, such as Coca-Cola Bottling Co. Consolidated (COKE), 3M Company (MMM), International Business Machines Corporation (IBM), and so on, as components.

The second dataset "NYSE(N)" is the extended version of NYSE(O) and is collected by Gábor Gelencsér and Li Bin [36].[4] For consistency, this dataset is from January 1, 1985 to June 30, 2010, which consists of 6,431 trading days and covers the global financial crisis in 2008. It is worth noting that this new dataset consists of 23 stocks rather than the previous 36 stocks owing to amalgamations and bankruptcies, such as Sears and Kmart. All self-collected price relatives are adjusted for splits and dividends, which is consistent with the previous "NYSE(O)" dataset.

The third dataset "DJIA" is collected by Borodin et al. [7], which consists of 30 stocks from Dow Jones Industrial Average containing price relatives of 507 trading days, ranging from Jan. 1st 2001 to January 14, 2003. The fourth dataset is "MSCI," a collection of global equity indices that are the constituents of MSCI World Index. It contains 24 indices that represent the equity markets of 24 countries around the world and consists of a total of 1,043 trading days, ranging from April 1, 2006 to March 31, 2010.[5]

As we can see, the above testbed covers much long trading periods from 1962 to 2010 and diversified markets, which enables us to examine how the proposed CFR strategy performs under different events and crises. For example, it covers several well-known events in the stock markets, such as dotcom bubble from 1995 to 2000 and subprime mortgage crisis from 2007 to 2009. The first three datasets are chosen to test strategy's capability on stocks, while the MSCI dataset aims to test the proposed strategy on global indices, which may be potentially applicable to "Fund on

[1] http://finance.yahoo.com.

[2] All datasets and their compositions can be downloaded from http://olps.stevenhoi.org/.

[3] According to El-Yaniv's homepage http://www.cs.technion.ac.il/rani/portfolios/ and Helmbold et al. [28], the dataset was originally collected by Hal Stern.

[4] The dataset before 2007 was collected by Gábor Gelencsér, the remaining data from 2007 to 2010 was collected by Li Bin via Yahoo Finance.

[5] The constituents of MSCI World Index are available on MSCI Barra (http://www.mscibarra.com).

Fund" (FOF). As a remark, although we numerically test the proposed algorithm on stock markets, we note that the proposed strategy could be generally applied to any type of financial markets.

4.2 Experimental Setup and Metrics

In our experiments, we implement the proposed CFR-ONS and CFR-OGD algorithms. The proposed CFR strategy has two parameters w and ε, since it combined OLMAR and RMR. To compare different methods fairly and consistently, we follow the OLPS toolbox implementation[6] [39], and empirically set the parameters w=5, ϵ=10, and p+m=7 on all the settings. For ARMA, we set the initial parameter $\gamma^0 = [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7]/10^4$ on the tth period, $\eta = 10^3$ and $\varepsilon = 10^{-5.5}$. For ARIMA, we set the initial parameter $\gamma^0 = [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7]/t^2$ on the tth period, $\eta = 1.75$ and $\varepsilon = 10^{-0.5}$.

We adopt the most common metric, cumulative wealth, to primarily compare different trading strategies. In addition to the cumulative wealth, we also adopt annualized Sharpe Ratio (SR) to compare the performance of different trading algorithms. In general, higher values of the cumulative wealth and annualized Sharpe Ratio indicate better performance. Besides, we also adopt Maximum Drawdown (MDD) and Calmar Ratio (CR) to analyze a strategy's downside risk. The lower the MDD, the less the strategy's (downside) risk is. The higher the CR, the better the strategy's (downside) risk-adjusted return is.

4.2.1 Performance Criteria. One of the standard criteria to evaluate the performance of a strategy is portfolio cumulative wealth achieved by the strategy until the end of the whole trading period. In our experiments, we simply set the initial wealth $S_0 = 1$, and thus S_n denotes portfolio cumulative wealth at the end of the nth trading day, which is the ratio of the portfolio cumulative wealth divided by the initial wealth. Another equivalent criterion is *Annualized Percentage Yield* (APY), which takes the compounding effect into account, that is, $\text{APY} = \sqrt[y]{S_n}$, where y is the number of years corresponding to n trading days. APY measures the average wealth increment that one strategy could achieve compounded in a year. *Winning Ratio* (WR) denotes the percentage of cases when the proposed strategy beats the baselines. Typically, the higher the value of portfolio cumulative wealth or annualized percentage yield and WR, the more performance preferable the trading strategy is.

For some process-dependent investors [44], it is important to evaluate *risk* and *risk-adjusted* return of portfolios [46, 47]. One common way to achieve this is to use *annualized standard deviation* of daily returns to measure the volatility risk and *annualized Sharpe Ratio* (SR) to evaluate the risk-adjusted return. For portfolio risk, we calculate the standard deviation of daily returns and multiply by $\sqrt{252}$ (here, 252 is the average number of annual trading days) to obtain annualized standard deviation. For risk-adjusted return, we calculate *annualized Sharpe Ratio* according to $\text{SR} = \frac{\text{APY}-R_f}{\sigma_p}$, where R_f is the risk-free return, and σ_p is the annualized standard deviation of daily returns. Basically, higher annualized SR indicate better performance of a trading strategy concerning the volatility risk.

We also adopt *Calmar Ratio* (CR) to measure the return relative of the drawdown risk of a portfolio, calculated as $\text{CR} = \frac{\text{APY}}{\text{MDD}}$, where MDD is the *Maximum DrawDown* and measures the downside risk of different strategies. Generally speaking, higher Calmar Ratios indicate better performance of a trading strategy concerning the drawdown risk.

To test whether simple luck can generate the return of the proposed strategy, we can also conduct a statistical test to measure the probability of this situation, as is popularly done in the fund management industry [23]. First, we separate the portfolio daily returns into two components:

[6] Our source code will be released and made available in the Github project: https://github.com/OLPS.

one benchmark-related and the other non-benchmark-related by regressing the portfolio excess returns against the benchmark excess returns. Formally, $s_t - s_t(\mathrm{F}) = \alpha + \beta(s_t(\mathrm{B}) - s_t(\mathrm{F}) + \epsilon(t))$, where s_t stands for the portfolio daily returns, $s_t(\mathrm{B})$ denotes the daily returns of the benchmark (market index) and $s_t(\mathrm{F})$ is the daily returns of the risk-free assets (here, we simply choose Treasury bill and set it to 1.000156, or equivalently, annual interest of 4%). This regression estimates the portfolio's alpha(α), which indicates the performance of the investment after accounting for the involved risk. Then, we conduct a statistical t-test to evaluate whether alpha is significantly different from zero, by using the t statistic $\frac{\alpha}{\mathrm{SE}(\alpha)}$, where SE($\alpha$) is the standard error for the estimated alpha. Thus, by assuming the α is normally distributed, we can obtain the probability that the returns of the proposed strategy are generated by simple luck. Generally speaking, the smaller the probability, the higher confidence the trading strategy.

4.2.2 Transaction Cost. In reality, an important and unavoidable issue is *transaction cost* [40]. While our model in Section 2.1 is concise and not complicated to understand, it omits the transaction costs. We shall now relax some constraints to address these issues. Generally, there are two ways to deal with this problem. The first is that the portfolio selection process does not consider the transaction costs while the following re-balancing incurs transaction costs and this method has been commonly adopted by learning to select portfolio strategies. The second way is that the transaction costs are directly involved in the portfolio selection process. In our experiments, we take the first way and adopt *proportional transaction cost* model, which is proposed by References [6, 7]. Specifically, rebalancing the portfolio incurs a transaction cost on every buy and sell operation with regarding to a transaction cost rate $\gamma \in (0, 1)$. At the beginning of the tth trading day, the portfolio manager rebalances the portfolio from the previous closing price adjusted portfolio $\hat{\mathbf{b}}_{t-1}$ to a new portfolio $\hat{\mathbf{b}}_t$, incurring a transaction cost of $\frac{\gamma}{2} \times \sum_i \mid b_{(t,i)} - \hat{b}_{(t-1,i)} \mid$, where the initial portfolio is set to $(0, \dots, 0)$. Thus, with transaction cost rate γ, the cumulative wealth achieved by the end of the nth trading day can be expressed as

$$S_n = S_0 \prod_{t=1}^{n} \left[(\mathbf{b}_t \cdot \mathbf{x}_t) \times \left(1 - \frac{\gamma}{2} \times \sum_i \mid b_{(t,i)} - \hat{b}_{(t-1,i)} \mid \right) \right].$$

To the best of our knowledge, this model cannot work for high-frequency data, since even a small rate will cause all methods to approach to zero.

4.3 Comparison Approaches

In our experiments, we compare the proposed algorithms with a number of benchmarks and existing strategies. We implement the proposed CFR strategy. Below, we summarize a list of compared algorithms, all of which provide extensive empirical evaluations in their respective studies. All parameters are set according to their original studies:

(1) *Market*: Market strategy that is uniform buy-and-hold (BAH) strategy
(2) *Best-Stock*: Best stock in the market, which is a strategy in hindsight
(3) *BCRP*: Best constant rebalanced portfolios strategy in hindsight
(4) *UP*: Universal portfolios strategy with parameters $\delta_0 = 0.004$, $\delta = 0.005$, $m = 100$, and $S = 500$ [50]
(5) *EG*: Exponential gradient algorithm with the best learning rate $\eta = 0.05$ [46]
(6) *ONS*: Online Newton step with the parameters $\eta = 0$, $\beta = 1$, and $\gamma = \frac{1}{8}$ [1]
(7) *Anticor*: $\mathbf{BAH}_{30}$(Anticor) as a variant of Anticor to smooth the performance, which achieves the best performance among the three solutions [12]

Table 4. Cumulative Wealth of Various Strategies on the Four Datasets

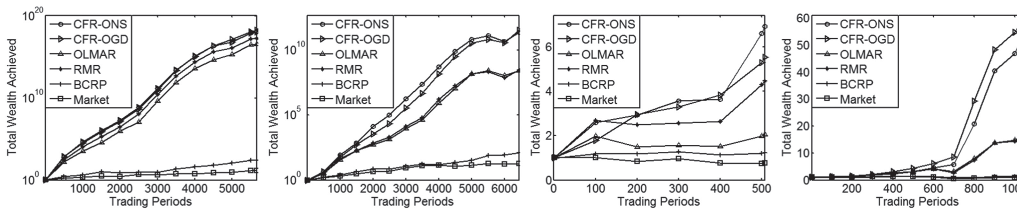
Methods	NYSE(O)	NYSE(N)	DJIA	MSCI
Market	14.50	18.06	0.76	0.91
Best-stock	54.14	83.51	1.19	1.50
BCRP	250.60	120.32	1.24	1.51
UP	26.68	31.49	0.81	0.92
EG	27.09	31.00	0.81	0.93
ONS	109.91	21.59	1.53	0.86
B^k	1.08E+09	4.64E+03	0.68	2.64
B^{NN}	3.35E+11	6.80E+04	0.88	13.47
CORN	1.48E+13	5.37E+05	0.84	26.19
Anticor	2.41E+08	6.21E+06	2.29	3.22
PAMR	5.14E+15	1.25E+06	0.68	15.23
CWMR	6.49E+15	1.41E+06	0.68	17.28
OLMAR	3.68E+16	2.54E+08	2.12	16.39
RMR	2.07E+17	2.70E+08	2.58	16.36
AdaBoost.R2	2.41E+17	5.10E+08	2.76	17.50
Bagging	4.02E+17	3.36E+09	3.00	18.04
OLAR	9.07E+17	3.99E+08	2.61	22.45
CFR-ONS	1.21E+18	2.28E+11	**6.93**	54.55
CFR-OGD	**1.82E + 18**	**3.33E + 11**	5.54	**60.11**

Note: The best results on each dataset are highlighted in bold.

(8) B^K: Nonparametric kernel-based moving window strategy with $W = 5$, $L = 10$, and $c = 1.0$ for daily datasets that has the best empirical performance according to Reference [24]
(9) B^{NN}: Nonparametric nearest-neighbor-based strategy with parameters $W = 5$, $L = 10$, and $p_{\varphi\ell} = 0.02 + 0.5\frac{\ell-1}{L-1}$ [25]
(10) *CORN*: Correlation-driven nonparametric learning approach with parameters $W = 5$ and $\rho = 0.1$ [36]
(11) *PAMR*: Passive-aggressive mean reversion strategy with parameter $\epsilon = 0.5$ [41]
(12) *CWMR*: Confidence-weighted mean reversion strategy with parameters $\phi = 2$ and $\epsilon = 0.5$ [38]
(13) *OLMAR*: Online moving average reversion strategy with parameters $\epsilon = 10$ and $w = 5$ [34]
(14) *RMR*: Robust median reversion with parameters $\epsilon = 10$ and $w = 5$ [29]
(15) *AdaBoost.R2*: Adaptive boosting regressor with linear loss function [19]
(16) *Bagging*: Bootstrap aggregating with simple averaging [9].

4.4 Experimental Results

4.4.1 Cumulative Wealth. Table 4 summarizes the cumulative wealth achieved by various methods without considering transaction costs. As we can see, the proposed CFR strategy outperforms all the existing methods on all the datasets in our experiments. Especially, CFR-ONS and CFR-OGD achieve better results than OLMAR and RMR among the multiple-period mean reversion strategies, which further shows the robustness of the proposed algorithms. Besides the preceding final cumulative wealth, we are also interested in examining how the total wealth achieved by various strategies changes over different trading periods. In Figure 1, we plot the wealth achieved



(a) NYSE(O) (b) NYSE(N) (c) DJIA (d) MSCI

Fig. 1. Trend of cumulative wealth achieved by six strategies during the entire period on the four datasets.

Table 5. Statistical Test of CFR-ONS

Stat.	NYSE(O)	NYSE(N)	DJIA	MSCI
Size	5651	6431	507	1043
MER (CFR-ONS)	0.0080	0.0047	0.0043	0.0041
MER (Market)	0.0005	0.0005	−0.0004	0.0000
WR	0.5733	0.5548	0.5444	0.6098
α	0.0074	0.0041	0.0048	0.0041
β	1.2829	1.1651	1.1925	1.1287
t-statistics	16.4504	9.9704	4.2565	7.7861
p-value	0.0000	0.0000	0.0000	0.0000

Table 6. Statistical Test of CFR-OGD

Stat.	NYSE(O)	NYSE(N)	DJIA	MSCI
Size	5651	6431	507	1043
MER (CFR-OGD)	0.0081	0.0047	0.0038	0.0042
MER (Market)	0.0005	0.0005	−0.0004	0.0000
WR	0.5746	0.5556	0.5661	0.6107
α	0.0075	0.0042	0.0043	0.0042
β	1.2449	1.1163	1.0661	1.1090
t-statistics	16.7756	10.2011	3.9446	7.7321
p-value	0.0000	0.0000	0.0000	0.0000

by the proposed CFR algorithms (CFR-ONS and CFR-OGD), state-of-the-art algorithms (OLMAR, RMR), plus two benchmarks (Market and BCRP). From the results, we can see that the proposed CFR strategy consistently surpassed the benchmarks and the competing strategies over the entire trading period on all datasets, which again validates the efficacy of the proposed technique. In Figure 1(b), there is a temporary drop in cumulative wealth at 6,000 trading periods, which corresponds roughly to October 14, 2008. During 2007 and 2009, due to the financial crisis, the cumulative wealth of CFR-OGD has dropped from 1.02E+11 to 2.20E+10, decreasing around 78.43%. While, the CFR-ONS's cumulative wealth has dropped from 7.36E+10 to 2.58E+10, decreasing around 64.95%. However, the curves of CFR-ONS and CFR-OGD are still above other algorithms, i.e., the CFR strategy achieved the biggest cumulative wealth. Finally, to measure whether the results are generated by simple luck, we conduct widely accepted statistical test. Tables 5 and 6 further show some statistics [23] of CFR-ONS and CFR-OGD, from which we can see that they have small p-values, which means that their excellent performance is not due to luck but owing to their strategy principles. To be specific, the probabilities for achieving the excess returns by luck are almost 0.

Table 7. The Comparison of APY, Volatility, Sharpe Ratio, MDD, and CR Among OLMAR, RMR, OLAR, AdaBoost.R2, Bagging, and CFR (CFR-ONS, CFR-OGD) Strategies

Criteria	Strategy	NYSE(O)	NYSE(N)	DJIA	MSCI
APY	OLMAR	4.6626	1.1051	0.4346	1.0121
	RMR	5.1251	1.1100	0.6058	1.0112
	OLAR	5.5501	1.1420	0.6168	1.1766
	AdaBoost.R2	5.1676	1.1623	0.6661	1.0453
	Bagging	5.3124	1.3247	0.7309	1.0610
	CFR-ONS	5.6369	1.7341	**1.6316**	1.7177
	CFR-OGD	**5.7607**	**1.7744**	1.3544	**1.7845**
Volatility	OLMAR	0.5657	0.5684	0.5216	0.3913
	RMR	0.5699	0.5667	0.5164	0.3929
	OLAR	0.5756	0.5687	0.4988	0.4091
	AdaBoost.R2	0.5671	0.5694	0.5270	0.3957
	Bagging	0.5682	0.5691	0.5278	0.4031
	CFR-ONS	0.5695	0.5659	0.4988	**0.3886**
	CFR-OGD	**0.5626**	**0.5587**	**0.4654**	0.3910
Sharpe Ratio	OLMAR	8.1708	1.8739	0.7565	2.4846
	RMR	8.9235	1.8880	1.0957	2.4721
	OLAR	9.5725	1.9378	1.1563	2.7781
	AdaBoost.R2	9.0419	1.9712	1.1880	2.5407
	Bagging	9.2793	2.2575	1.3091	2.5329
	CFR-ONS	9.8280	2.9935	**3.1907**	4.3171
	CFR-OGD	**10.1687**	**3.1043**	2.8244	**4.4620**
MDD	OLMAR	0.4362	0.9334	0.4395	0.4537
	RMR	0.4248	0.9052	0.3705	0.5085
	OLAR	0.4525	0.9296	0.3639	0.4632
	AdaBoost.R2	0.4629	0.9198	0.3794	0.4806
	Bagging	0.4323	0.8895	0.3775	0.4883
	CFR-ONS	0.4255	0.8847	0.2784	0.2384
	CFR-OGD	**0.4051**	**0.8146**	**0.2049**	**0.2333**
CR	OLMAR	10.6898	1.1840	0.9889	2.2306
	RMR	12.0633	1.2262	1.6349	1.9887
	OLAR	12.2665	1.2286	1.6952	2.5402
	AdaBoost.R2	11.1629	1.2637	1.7558	2.1750
	Bagging	12.2895	1.4894	1.9364	2.1730
	CFR-ONS	13.2488	1.9601	5.8614	7.2063
	CFR-OGD	**14.2216**	**2.1782**	**6.6105**	**7.6489**

Note: The best results on each dataset are highlighted in bold.

The results show that the CFR strategy is a promising and reliable portfolio selection technique to achieve high return with high confidence. Besides, we can find that the winning ratio (WR) of CFR-ONS against market strategy is bigger than 54% (55% for CFR-OGD) on all datasets, which further shows the superiority of the proposed strategy.

4.4.2 APY, Volatility, Sharpe Ratio, MDD, CR. Table 7 summarizes the APY, volatility, annualized Sharpe Ratio, MDD, and CR. From the results, we observe that on the four datasets, the

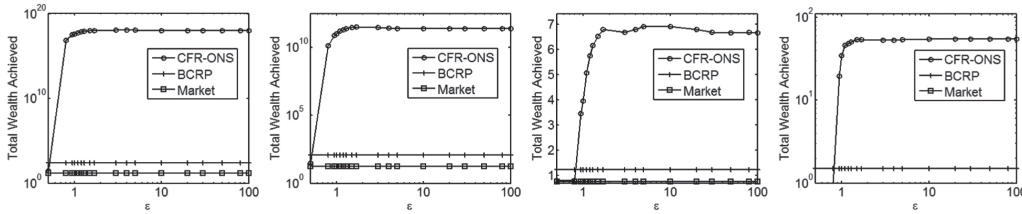
Table 8. The Comparison of Turnover Among OLMAR, RMR, AdaBoost.R2, Bagging, and CFR (CFR-ONS, CFR-OGD) Strategies

Strategy	NYSE(O)	NYSE(N)	DJIA	MSCI
OLMAR	1.478456	1.386448	1.434590	1.482887
RMR	1.401743	1.295016	1.318602	1.401225
AdaBoost.R2	1.366466	1.246057	1.296507	1.364286
Bagging	1.391969	1.287180	1.301647	1.362730
CFR-ONS	1.401689	1.233792	1.172301	1.395220
CFR-OGD	1.369171	1.211833	1.206060	1.401072

CFR strategy achieves the best performance in all criteria. Specifically, CFR-OGD achieves the highest APY, Sharpe Ratio, CR, and lowest volatility, MDD among seven algorithms on NYSE(O) and NYSE(N) datasets. On DJIA dataset, CFR-OGD achieves the lowest volatility, MDD, and highest CR, while CFR-ONS achieves the highest APY and Sharpe Ratio. On MSCI dataset, CFR-OGD achieves the best performance in all criteria, except volatility, while CFR-ONS achieves the lowest volatility. However, CFR-ONS's volatility and MDD on NYSE(O) are two inconsistent expected performances. CFR-ONS's volatility is higher than that of the three algorithms (OLMAR, AdaBoost.R2 and Bagging), and its MDD is higher than that of RMR, but they are almost the same as these four algorithms (OLMAR, RMR, AdaBoost.R2, and Bagging). These encouraging results show that CFR strategy reaches a good trade-off between return and risk, even though the risk is not explicitly taken into account in our problem formulation.

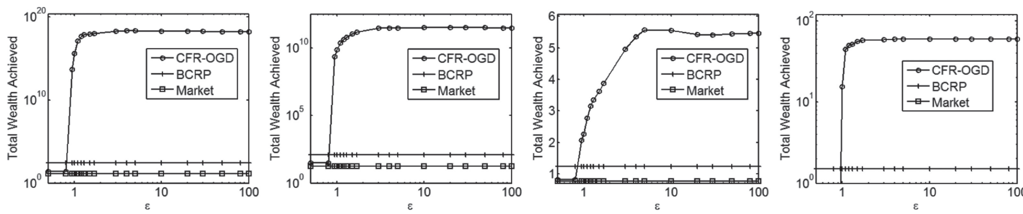
4.4.3 Turnover. Roughly speaking, the turnover is the average percentage of wealth traded in each period. In our experiment, we compare the turnover of CFR with that of the state-of-the-art strategies (OLMAR, RMR) and two classical algorithms of ensemble learning (AdaBoost.R2, Bagging). They are all designed as multiple-period reversion strategies, and therefore the comparison among them is more significant. Table 8 presents the explicit turnover value of some strategies. As we observed, the turnover of CFR-ONS and CFR-OGD are smaller than that of OLMAR and RMR on all datasets. Compared with AdaBoost.R2 and Bagging, the proposed strategies obtain smaller turnover on NYSE(N) and DJIA, and almost the same turnover on NYSE(O) and MSCI. First, the smaller turnover means that the portfolio is more stable, which can be attributed to the resistance to the noise or outliers. Hence, the small turnover empirically shows the robustness of the proposed strategy. Second, the smaller turnover usually results in less transaction costs. From Table 4, we can see that the proposed algorithms achieve highest cumulative wealth without considering transaction costs. At the same time, they get smaller turnover. Thus, they will perform well when transaction costs are taken into account.

4.4.4 Parameter Sensitivity. Now, we experimentally evaluate how different choices of parameters affect the cumulative wealth. CFR algorithms (CFR-ONS, CFR-OGD) contain two parameters, the sensitivity parameter ϵ and window size w, since they combine the MA and L_1-median estimators. First, we examine the performance of the proposed algorithms by varying sensitivity parameter ϵ from 0 to 100 with fixed $w = 5$. Figures 2 and 3 show the effect of sensitivity parameter ϵ on cumulative wealth of CFR-ONS and CFR-OGD, respectively. The cumulative wealth sharply grows as ϵ increases and then tends to stable when ϵ exceeds 10. Second, we evaluate the parameter window size by varying w from 3 to 100. With fixed $\epsilon = 10$, Figures 4 and 5 show the effect of window size w of CFR-ONS and CFR-OGD, respectively. On three of the four datasets (except DJIA), we can see a clear trend of the cumulative wealth with w: first sharply up and then



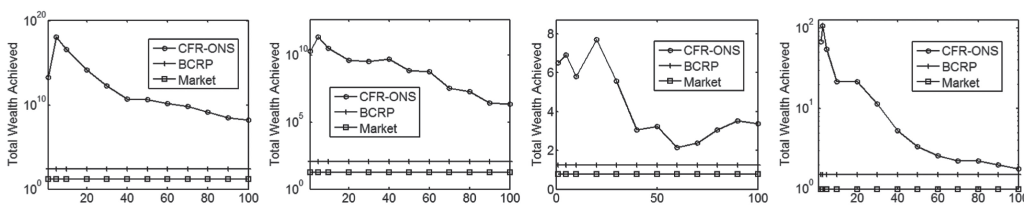
(a) NYSE(O) (b) NYSE(N) (c) DJIA (d) MSCI

Fig. 2. Parameter sensitivity of CFR-ONS w.r.t. ϵ with fixed w=5.



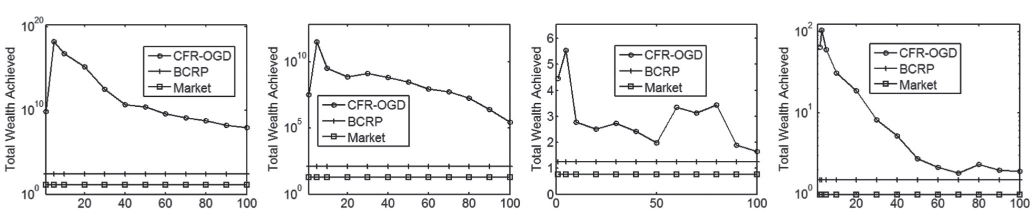
(a) NYSE(O) (b) NYSE(N) (c) DJIA (d) MSCI

Fig. 3. Parameter sensitivity of CFR-OGD w.r.t. ϵ with fixed w=5.



(a) NYSE(O) (b) NYSE(N) (c) DJIA (d) MSCI

Fig. 4. Parameter sensitivity of CFR-ONS w.r.t. w with fixed ϵ=10.

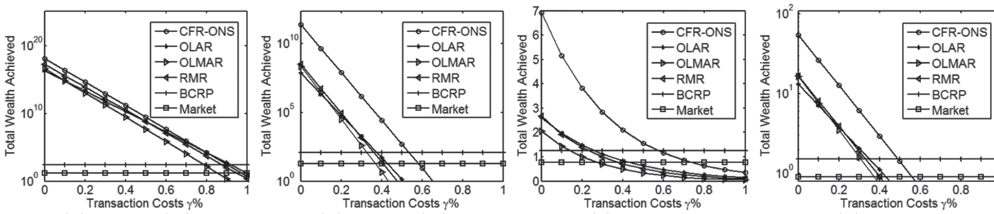


(a) NYSE(O) (b) NYSE(N) (c) DJIA (d) MSCI

Fig. 5. Parameter sensitivity of CFR-OGD w.r.t. w with fixed ϵ=10.

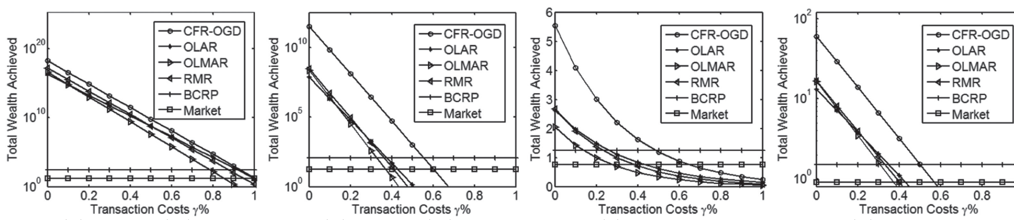
slowly down when w exceeds 5. Hence, $w = 5$ is the best choice when $\epsilon = 10$ (except DJIA). In summary, for most values of ϵ and w, our methods outperform the existing ones.

4.4.5 Transaction Costs. In practice, transaction cost is an important and practical issue for OLPS. In our experiment, we adopt proportional transaction costs model and test the effect of *proportional* transaction costs with the transaction cost rate γ varies from 0 to 1%. Our method is also compared with Market, BCRP, OLMAR, and RMR. Figures 6 and 7 present the results of CFR-ONS and CFR-OGD on four daily datasets, respectively. As we can see, the performance with transaction



(a) NYSE(O) (b) NYSE(N) (c) DJIA (d) MSCI

Fig. 6. Scalability of the total wealth achieved by CFR-ONS with respect to transaction cost rate $\gamma\%$.



(a) NYSE(O) (b) NYSE(N) (c) DJIA (d) MSCI

Fig. 7. Scalability of the total wealth achieved by CFR-OGD with respect to transaction cost rate $\gamma\%$.

costs is market dependent. When the transaction costs increase, the total wealth achieved by CFR strategy drops considerably, which is consistent with other strategies. At the same time, the CFR strategy always achieves the maximum cumulative wealth when the transaction cost rate is fixed. We also observe that the proposed strategy can withstand reasonable transaction cost rates, and can beat the existing methods with the transaction cost rate above 0.62% on the first three datasets (around 0.6% on MSCI). Compared with the benchmarks, the results clearly demonstrate that, on all datasets, the performance of the two algorithms (CFR-ONS, CFR-OGD) are considerably robust with respect to the transaction costs. In summary, the proposed strategy can withstand moderate transaction costs, even though we do not explicitly tackle it during the portfolio selection process, and hence it is practical applicable.

5 CONCLUSION

In this article, we propose a novel multiple-period OLPS strategy named CFR, which exploits the reversion phenomenon in financial markets by combination forecasting estimator and online learning techniques. In general, different prediction models can provide different useful information; if we simply drop some models that have the higher forecast deviations, then it will lose some useful information. Combination forecasting, however, can make use of various prediction models to solve the problem more comprehensively, rather than a single predictive model, and can effectively reduce the influence of some random factors in a single prediction model, to improve the prediction accuracy. At the same time, CFR can overcome the drawbacks of the existing OLPS techniques due to the instability of single prediction model, noisy data and outliers, and the non-stationarity of the price time series. For CFR strategy, we achieve some good theoretical results. Extensive experiments on real markets also show that the CFR strategy beats all the state-of-the-art strategies in literature and achieves promising performance on the benchmark datasets. In the future, we will study other robust estimation methods and performance measures. Furthermore, in a similar way to Cover's study about the side information for universal portfolio, we can also consider incorporating other additional information (including the impact historical price data, market liquidity, and trading volume) into the proposed CFR strategy, which is somewhat more reasonable way to do back test in practice.

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Received September 2017; revised February 2018; accepted March 2018