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Deformations of Legendrian Curves

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Doutoramento em Matemática

Especialidade de Geometria e Topologia

Marco Silva Mendes

Tese orientada por:

Prof. Doutor Orlando Manuel Bartolomeu Neto

Documento especialmente elaborado para a obtenção do grau de doutor

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Abstract

In chapters 1 and 2 we study deformations of Legendrian curves in $\mathbb{P}^*\mathbb{C}^2$.

In chapter 1 we construct versal and semiuniversal objects in the category of deformations of the parametrization of a germ of a Legendrian curve as well as in the subcategory of equimultiple deformations. We show that these objects are given by the conormal or fake conormal of an hypersurface in $\mathbb{C}^2 \times \mathbb{C}^r$.

In chapter 2 we prove the existence of equisingular versal and semiuniversal deformations of a Legendrian curve, on this instance making use of deformations of the equation. By equisingular we mean that the plane projection of the fibres have fixed topological type. We prove in particular that the base space of such an equisingular versal deformation is smooth and construct it explicitly when the special fibre has semiquasihomogeneous or Newton non-degenerate plane projection.

Chapter 3 concerns the construction of a moduli space for Legendrian curves singularities which are contactomorphic-equivalent and equisingular through a contact analogue of the Kodaira-Spencer map for curve singularities. We focus on the specific case of Legendrian curves which are the conormal of a plane curve with one Puiseux pair. To do so, it is fundamental to understand how deformations of such singularities behave, which was done in the previous chapter. The equisingular semiuniversal microlocal deformations constructed in chapter 2 already contain in their base space all the relevant fibres in the construction of such a moduli space. This is so because all deformations are isomorphic through a contact transformation to the pull-back of a semiuniversal deformation.

Key-words: Algebraic Geometry; Relative Contact Geometry; Deformations of Legendrian Curves; Deformation Theory; Legendrian Curves; Moduli Spaces; Plane Curves; Singularity theory.

Resumo

Seja X uma variedade complexa de dimensão 3 e $\mathcal{O}_X$ o feixe das funções holomorfas sobre X. Seja Ω_X^1 o $\mathcal{O}_X$-módulo das formas diferenciais de grau 1 sobre X. Uma forma diferencial ω em Ω_X^1 diz-se uma *forma de contacto* se $\omega \wedge d\omega$ não se anula em nenhum ponto de X. Pelo Teorema de Darboux para formas de contacto existe localmente um sistema de coordenadas (x, y, p) tal que $\omega = dy - pdx$. Um sub-feixe localmente livre $\mathcal{L}$ de Ω_X^1 diz-se uma *estrutura de contacto* sobre X se cada ponto de X possui uma vizinhança aberta tal que sobre essa vizinhança $\mathcal{L}$ é gerado enquanto $\mathcal{O}_X$-módulo por uma forma de contacto. Se $\mathcal{L}$ é uma estrutura de contacto, o par $(X, \mathcal{L})$ diz-se uma *variedade de contacto*. Uma aplicação holomorfa χ entre duas variedades de contacto $(X_1, \mathcal{L}_1)$, $(X_2, \mathcal{L}_2)$ diz-se uma *transformação de contacto* se $\chi^*\omega$ é um gerador local de $\mathcal{L}_1$ sempre que ω seja um gerador local de $\mathcal{L}_2$. Seja L um subconjunto analítico de $(X, \mathcal{L})$ de dimensão 1. Diz-se que L é uma *curva Legendriana* se qualquer secção de $\mathcal{L}$ se anula sobre a parte regular de L.

Consideremos sobre $\mathbb{C}^2$ com coordenadas (x, y) o fibrado cotangente $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$ munido da forma diferencial canónica de grau 1, $\theta = \xi dx + \eta dy$, onde (ξ, η) são coordenadas do espaço dual de $\mathbb{C}^2$. Seja $\pi : \mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ o *fibrado cotangente projectivo* de $\mathbb{C}^2$ tal que $\pi(x, y; \xi : \eta) = (x, y)$. Os abertos $U\,[V]$ definidos por $\eta \neq 0\,[\xi \neq 0]$ definem uma estrutura de variedade complexa sobre $\mathbb{P}^*\mathbb{C}^2$. Munido das formas diferenciais $\theta/\eta = dy - pdx\,[\theta/\xi = dx - qdy]$, onde $p = -\xi/\eta\,[q = -\eta/\xi]$, $\mathbb{P}^*\mathbb{C}^2$ tem estrutura de variedade de contacto.

Dada uma curva plana Y de $\mathbb{C}^2$ definimos o *conormal* de Y como sendo a "menor" curva Legendriana de $\mathbb{P}^*\mathbb{C}^2$ que se projecta sobre Y. Consideremos uma parametrização

$$\varphi(t) = (x(t), y(t))$$

de um germe na origem de uma curva plana irredutível Y com cone tangente definido por $ax + by = 0$, com $(a, b) \neq (0, 0)$. O germe de curva no ponto (a, b) de $\mathbb{P}^*\mathbb{C}^2$ parametrizada por

$$\psi(t) = (x(t), y(t); -y'(t) : x'(t))$$

é o conormal de Y. Se Y é um germe de curva plana com cone tangente irredutível, a união dos conormais das componentes irredutíveis de Y define um germe de curva Legendriana, o conormal de Y.

Os capítulos 1 e 2 estudam propriedades de deformações de curvas Legendrianas em $\mathbb{P}^*\mathbb{C}^2$.

Uma *deformação* de um germe de espaço complexo (X, x) sobre um *espaço base* (S, s) é definida por um morfismo flat $\phi : (\mathscr{X}, x) \to (S, s)$ tal que (X, x) é isomorfo à fibra $(\phi^{-1}(s), x)$. Se (X, x) puder ser imerso em $(\mathbb{C}^n, 0)$ e $(\mathscr{X}, x)$ puder ser imerso em $(\mathbb{C}^n, 0) \times (S, s)$ de tal forma que o morfismo ϕ respeite essas imersões, a deformação diz-se *imersa*. Uma deformação ϕ de (X, x) diz-se *versal* se, para além de uma condição técnica, exigirmos que toda uma outra deformação de (X, x) possa ser obtida a partir de ϕ a menos de isomorfismo. Um germe diz-se *rígido* se uma sua deformação trivial for versal.

No capítulo 1 adoptamos o ponto de vista de deformações da parametrização de germes de curvas Legendrianas, obtendo como resultados principais expressões para deformações cujos conormais definem deformações versais na categoria das deformações de um germe de curva Legendriana e na subcategoria das deformações que preservem a multiplicidade da curva.

No capítulo 2 estudamos deformações de um germe definido por equações no espaço cotangente projectivo. Este ponto de vista tem a vantagem de poder ser estendido a dimensões superiores. Estamos interessados em particular em deformações que mantenham fixo o tipo topológico da sua projecção, ditas equisingulares. No entanto, a definição óbvia de deformação neste caso tem alguns problemas: nem toda a deformação de uma curva legendriana teria como fibras curvas legendrianas, além de que todas as fibras de uma deformação flat seriam rígidas. Adoptamos portanto também aqui a definição introduzida em [4], em que as deformações de uma curva em $\mathbb{P}^*\mathbb{C}^2$ são conormais de deformações em $\mathbb{C}^2$ da sua projecção plana. Temos como resultados principais deste capítulo:

- Existência de uma deformação versal equisingular de uma curva Legendriana. Em particular provamos que o espaço base de uma tal deformação é suave.
- Construção de uma deformação versal equisingular de uma curva Legendriana que tenha como projecção uma curva semi-quasi-homogénea ou Newton-não-degenerada, estendendo os resultados de [4].

No capítulo 3 abordamos a questão da não universalidade das deformações semiuniversais obtidas no capitulo 2 para curvas com planas com um par de Puiseux. Pretendemos, dentro do espaço base das deformações semi-universais microlocais, identificar exactamente que fibras é que são microlocalmente equivalentes, isto é, cujos conormais são isomorfos por transformações de contacto. Um espaço com "boa estrutura" em que cada ponto corresponde a uma classe de uma certa relação de equivalência é dito um *espaço de moduli* para essa relação de equivalência. De uma forma geral, o espaço base das deformações semi-universais microlocais não é um espaço de moduli para a relação de equivalência microlocal. Existe no entanto uma estratificação desse espaço de tal forma a que, em cada estrato, o quociente pela relação de equivalência tenha de facto essa "boa estrutura" e seja portanto um espaço de moduli. As técnicas aqui usadas são inspiradas no trabalho desenvolvido por Gert-Martin Greuel e Gerhard Pfister sobre quocientes geométricos por acções de grupos unipotentes (ver [7] e [10]).

Palavras chave: Curvas Planas; Curvas Legendrianas; Espaços de Moduli; Deformações de Curvas Legendrianas; Geometria Algébrica; Teoria das Deformações; Teoria das Singularidades.

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Marco Silva Mendes
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Contents

Chapter 1

Deformations of Legendrian Curves

1.1 Introduction

Legendrian varieties are analytic subsets of the projective cotangent bundle of a smooth manifold or, more generally, of a contact manifold. They are projectivizations of conic Lagrangian varieties. These are specifically important in $\mathcal{D}$-modules theory and microlocal analysis (see [15], [16], [17]). Its deformation theory is still an almost virgin territory (see [24]).

In sections 1.2, 1.3 and 1.4 we introduce the languages of contact geometry and deformation theory. In sections 1.5 and 1.6 we construct the semiuniversal and equimultiple semiuniversal deformations of the parametrization of a germ of a Legendrian curve, extending to Legendrian curves previous results on deformations of germs of plane curves (see [9]).

These results will be useful to the study of equisingular deformations of Legendrian curves and its moduli spaces in chapters 2 and 3.

1.2 Contact Geometry

Let $(X, \mathcal{O}_X)$ be a complex manifold of dimension 3. A differential form ω of degree 1 is said to be a *contact form* if $\omega \wedge d\omega$ never vanishes. Let ω be a contact form. By Darboux's theorem for contact forms there is locally a system of coordinates (x, y, p) such that $\omega = dy - pdx$. If ω is a contact form and f is a holomorphic function that never vanishes, $f\omega$ is also a contact form. We say that a locally free subsheaf $\mathcal{L}$ of Ω^1_X is a *contact structure* on X if $\mathcal{L}$ is locally generated by a contact form. If $\mathcal{L}$ is a contact structure on X the pair $(X, \mathcal{L})$ is said to be a *contact manifold*. Let $(X_1, \mathcal{L}_1)$ and $(X_2, \mathcal{L}_2)$ be contact manifolds. Let $\chi : X_1 \to X_2$ be a holomorphic map. We say that χ is a *contact transformation* if $\chi^*\omega$ is a local generator of $\mathcal{L}_1$ whenever ω is a local generator of $\mathcal{L}_2$.

Let $\theta = \xi dx + \eta dy$ denote the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. Let $\pi : \mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ be the *projective cotangent bundle* of $\mathbb{C}^2$, where $\pi(x, y; \xi : \eta) = (x, y)$. Let $U\,[V]$ be the open subset of $\mathbb{P}^*\mathbb{C}^2$ defined by $\eta \neq 0\,[\xi \neq 0]$. Then $\theta/\eta\,[\theta/\xi]$ defines a contact form $dy - pdx\,[dx - qdy]$ on $U\,[V]$, where $p = -\xi/\eta\,[q = -\eta/\xi]$. Moreover, $dy - pdx$ and $dx - qdy$ define a structure of contact manifold on $\mathbb{P}^*\mathbb{C}^2$.

If $\Phi(x, y) = (a(x, y), b(x, y))$ with $a, b \in \mathbb{C}\{x, y\}$ is an automorphism of $(\mathbb{C}^2, (0, 0))$, we associate to Φ the germ of contact transformation

$$\chi : (\mathbb{P}^*\mathbb{C}^2, (0, 0; 0 : 1)) \to \big(\mathbb{P}^*\mathbb{C}^2, (0, 0; -\partial_x b(0, 0) : \partial_x a(0, 0)\big)$$

defined by

$$\chi(x, y; \xi : \eta) = (a(x, y), b(x, y); \partial_y b\xi - \partial_x b\eta : -\partial_y a\xi + \partial_x a\eta)\,. \tag{1.2.1}$$

If $D\Phi_{(0,0)}$ leaves invariant $\{y = 0\}$, then $\partial_x b(0, 0) = 0$, $\partial_x a(0, 0) \neq 0$ and $\chi(0, 0; 0 : 1) = (0, 0; 0 : 1)$. Moreover,

$$\chi(x, y, p) = (a(x, y), b(x, y), (\partial_y bp + \partial_x b)/(\partial_y ap + \partial_x a))\,.$$

Let $(X, \mathcal{L})$ be a contact manifold. A curve L in X is said to be *Legendrian* if $\imath^*\omega = 0$ for each section ω of $\mathcal{L}$, where $\imath : L \hookrightarrow X$.

Let Z be the germ at $(0, 0)$ of an irreducible plane curve parametrized by

$$\varphi(t) = (x(t), y(t)). \tag{1.2.2}$$

We define the *conormal* of Z as the curve parametrized by

$$\psi(t) = (x(t), y(t); -y'(t) : x'(t)). \tag{1.2.3}$$

The conormal of Z is the germ of a Legendrian curve of $\mathbb{P}^*\mathbb{C}^2$.

We will denote the conormal of Z by $\mathbb{P}^*_Z\mathbb{C}^2$ and the parametrization (1.2.3) by $\mathcal{C}on\,\varphi$.

Assume that the tangent cone $C(Z)$ is defined by the equation $ax + by = 0$, with $(a, b) \neq (0, 0)$. Then $\mathbb{P}^*_Z\mathbb{C}^2$ is a germ of a Legendrian curve at $(0, 0; a : b)$.

Let $f \in \mathbb{C}\{t\}$. We say the f has order k and write $ord\, f = k$ or $ord_t\, f = k$ if f/t^k is a unit of $\mathbb{C}\{t\}$.

Remark 1.2.1. Let Z be the plane curve parametrized by (1.2.2). Let $L = \mathbb{P}^*_Z\mathbb{C}^2$. Then:

(i) $C(Z) = \{y = 0\}$ if and only if $ord\, y > ord\, x$. If $C(Z) = \{y = 0\}$, L admits the parametrization
$$\psi(t) = (x(t), y(t), y'(t)/x'(t))$$
on the chart (x, y, p).

(ii) $C(Z) = \{y = 0\}$ and $C(L) = \{x = y = 0\}$ if and only if $ord\, x < ord\, y < 2ord\, x$.

(iii) $C(Z) = \{y = 0\}$ and $\{x = y = 0\} \not\subseteq C(L) \subset \{y = 0\}$ if and only if $ord\, y \geq 2ord\, x$.

(iv) $C(L) = \{y = p = 0\}$ if and only if $ord\, y > 2ord\, x$.

(v) $mult\, L \leq mult\, Z$. Moreover, $mult\, L = mult\, Z$ if and only if $ord\, y \geq 2ord\, x$.

If L is the germ of a Legendrian curve at $(0, 0; a : b)$, $\pi(L)$ is a germ of a plane curve of $(\mathbb{C}^2, (0, 0))$. Notice that all branches of $\pi(L)$ have the same tangent cone.

If Z is the germ of a plane curve with irreducible tangent cone, the union L of the conormal of the branches of Z is a germ of a Legendrian curve. We say that L is the *conormal* of Z.

If $C(Z)$ has several components, the union of the conormals of the branches of Z is a union of several germs of Legendrian curves.

If L is a germ of Legendrian curve, L is the conormal of $\pi(L)$.

Consider in the vector space $\mathbb{C}^2$, with coordinates x, p, the symplectic form $dp \wedge dx$. We associate to each symplectic linear automorphism

$$(p, x) \mapsto (\alpha p + \beta x, \gamma p + \delta x)$$

of $\mathbb{C}^2$ the contact transformation

$$(x, y, p) = (\gamma p + \delta x, y + \frac{1}{2}\alpha\gamma p^2 + \beta\gamma x p + \frac{1}{2}\beta\delta x^2, \alpha p + \beta x). \tag{1.2.4}$$

We say that (1.2.4) a *paraboloidal contact transformation.*

In the case $\alpha = \delta = 0$ and $\gamma = -\beta = 1$ we get the so called *Legendre* transformation

$$\Psi(x, y, p) = (p, y - px, -x).$$

We say that a germ of a Legendrian curve L of $(\mathbb{P}^*\mathbb{C}^2, (0, 0; a : b))$ is in *generic position* if $C(L) \not\supset \pi^{-1}(0, 0)$.

Remark 1.2.2. Let L be the germ of a Legendrian curve on a contact manifold $(X, \mathcal{L})$ at a point o. By the Darboux's theorem for contact forms there is a germ of a contact transformation $\chi : (X, o) \to (U, (0, 0, 0))$, where $U = \{\eta \neq 0\}$ is the open subset of $\mathbb{P}^*\mathbb{C}^2$ considered above. Hence $C(\pi(\chi(L))) = \{y = 0\}$. Applying a paraboloidal transformation to $\chi(L)$ we can assume that $\chi(L)$ is in generic position. If $C(L)$ is irreducible, we can assume $C(\chi(L)) = \{y = p = 0\}$.

Following the above remark, from now on we will always assume that every Legendrian curve germ is embedded in $(\mathbb{C}^3_{(x,y,p)}, \omega)$, where $\omega = dy - pdx$.

Example 1.2.3. The plane curve $Z = \{y^2 - x^3 = 0\}$ admits a parametrization $\varphi(t) = (t^2, t^3)$. The conormal L of Z admits the parametrization $\psi(t) = (t^2, t^3, \frac{3}{2}t)$. Hence $C(L) = \pi^{-1}(0, 0)$ and L is not in generic position. If χ is the Legendre transformation, $C(\chi(L)) = \{y = p = 0\}$ and L is in generic position. Moreover, $\pi(\chi(L))$ is a smooth curve.

Example 1.2.4. The plane curve $Z = \{(y^2 - x^3)(y^2 - x^5) = 0\}$ admits a parametrization given by

$$\varphi_1(t_1) = ({t_1}^2, {t_1}^3), \quad \varphi_2(t_2) = ({t_2}^2, {t_2}^5).$$

The conormal L of Z admits the parametrization given by

$$\psi_1(t_1) = ({t_1}^2, {t_1}^3, \frac{3}{2}t_1), \quad \psi_2(t_2) = ({t_2}^2, {t_2}^5, \frac{5}{2}{t_2}^3).$$

Hence $C(L_1) = \pi^{-1}(0,0)$ and L is not in generic position. If χ is the paraboloidal contact transformation

$$\chi : (x, y, p) \mapsto (x + p, y + \frac{1}{2}p^2, p),$$

then $\chi(L)$ has branches with parametrization given by

$$\chi(\psi_1)(t_1) = ({t_1}^2 + \frac{3}{2}t_1, {t_1}^3 + \frac{9}{8}{t_1}^2, \frac{3}{2}t_1),$$
$$\chi(\psi_2)(t_2) = ({t_2}^2 + \frac{5}{2}{t_2}^3, {t_2}^5 + \frac{25}{8}{t_2}^6, \frac{5}{2}{t_2}^3).$$

Then

$$C(\chi(L_1)) = \{y = p - x = 0\}, \quad C(\chi(L_2)) = \{y = p = 0\}$$

and L is in generic position.

1.3 Relative Contact Geometry

Set $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{z} = (z_1, \ldots, z_m)$. Let I be an ideal of the ring $\mathbb{C}\{\mathbf{z}\}$. Let $\widetilde{I}$ be the ideal of $\mathbb{C}\{\mathbf{x}, \mathbf{z}\}$ generated by I. Let $f \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}$. We will denote by $\int f dx_i$ the solution of the Cauchy problem

$$\partial_{x_i} g = f, \qquad g \in (x_i)\mathbb{C}\{\mathbf{x}, \mathbf{z}\}.$$

Lemma 1.3.1. *(a) Let $f \in C\{\mathbf{x}, \mathbf{z}\}$, $f = \sum_\alpha a_\alpha \mathbf{x}^\alpha$ with $a_\alpha \in \mathbb{C}\{\mathbf{z}\}$. Then $f \in \widetilde{I}$ if and only if $a_\alpha \in I$ for each α.*

(b) If $f \in \widetilde{I}$, then $\partial_{x_i} f, \int f dx_i \in \widetilde{I}$ for $1 \leq i \leq n$.

(c) Let $a_1, \ldots, a_{n-1} \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}$. Let $b, \beta_0 \in \widetilde{I}$. Assume that $\partial_{x_n} \beta_0 = 0$. If β is the solution of the Cauchy problem

$$\partial_{x_n} \beta - \sum_{i=1}^{n-1} a_i \partial_{x_i} \beta = b, \qquad \beta - \beta_0 \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\} x_n, \tag{1.3.1}$$

then $\beta \in \widetilde{I}$.

Proof. There are $g_1, \ldots, g_\ell \in \mathbb{C}\{\mathbf{z}\}$ such that $I = (g_1, \ldots, g_\ell)$. If $a_\alpha \in I$ for each α, there are $h_{i,\alpha} \in \mathbb{C}\{\mathbf{z}\}$ such that $a_\alpha = \sum_{i=1}^{\ell} h_{i,\alpha} g_i$. Hence $f = \sum_{i=1}^{\ell} (\sum_\alpha h_{i,\alpha} \mathbf{x}^\alpha) g_i \in \widetilde{I}$.

If $f \in \widetilde{I}$, there are $H_i \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\}$ such that $f = \sum_{i=1}^{\ell} H_i g_i$. There are $b_{i,\alpha} \in \mathbb{C}\{\mathbf{z}\}$ such that $H_i = \sum_\alpha b_{i,\alpha} \mathbf{x}^\alpha$. Therefore $a_\alpha = \sum_{i=1}^{\ell} b_{i,\alpha} g_i \in I$ and (a) follows.

In order to prove (b), note that $\partial_{x_i} f = \sum_\alpha a_\alpha \partial_{x_i} \mathbf{x}^\alpha = \sum_{\alpha'} b_{\alpha'} \mathbf{x}^{\alpha'}$ where, if $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha' = (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n)$ and $b_{\alpha'} = \alpha_i a_\alpha$. From (a), we get that $\partial_{x_i} f \in \widetilde{I}$. In the same manner, $\int f dx_i \in \widetilde{I}$.

We can perform a change of variables that rectifies the vector field $\partial_{x_n} - \sum_{i=1}^{n-1} a_i \partial_{x_i}$ (see for example [2], pp 227-229), reducing the Cauchy problem (1.3.1) to the Cauchy problem

$$\partial_{x_n} \beta = b, \qquad \beta - \beta_0 \in \mathbb{C}\{\mathbf{x}, \mathbf{z}\} x_n.$$

Hence, as $\beta = \int \partial_{x_n} \beta dx_n$ statement (c) follows from (b). □

Let J be an ideal of $\mathbb{C}\{\mathbf{z}\}$ contained in I. Let X, S and T be analytic spaces with local rings $\mathbb{C}\{\mathbf{x}\}, \mathbb{C}\{\mathbf{z}\}/I$ and $\mathbb{C}\{\mathbf{z}\}/J$. Hence $X \times S$ and $X \times T$ have local rings $\mathcal{O} := \mathbb{C}\{\mathbf{x}, \mathbf{z}\}/\widetilde{I}$ and $\widetilde{\mathcal{O}} := \mathbb{C}\{\mathbf{x}, \mathbf{z}\}/\widetilde{J}$. Let $\mathbf{a_1}, \ldots, \mathbf{a_{n-1}}, \mathbf{b} \in \mathcal{O}$ and $\mathbf{g} \in \mathcal{O}/x_n\mathcal{O}$. Let $a_i, b \in \widetilde{\mathcal{O}}$ and $g \in \widetilde{\mathcal{O}}/x_n\widetilde{\mathcal{O}}$ be representatives of $\mathbf{a_i}, \mathbf{b}$ and $\mathbf{g}$. Consider the Cauchy problems

$$\partial_{x_n} f + \sum_{i=1}^{n-1} a_i \partial_{x_i} f = b, \qquad f + x_n \widetilde{\mathcal{O}} = g \tag{1.3.2}$$

and

$$\partial_{x_n} \mathbf{f} + \sum_{i=1}^{n-1} \mathbf{a_i} \partial_{x_i} \mathbf{f} = \mathbf{b}, \qquad \mathbf{f} + x_n \mathcal{O} = \mathbf{g}. \tag{1.3.3}$$

Theorem 1.3.2. *(a) There is one and only one solution of the Cauchy problem (1.3.2).*

(b) If f is a solution of (1.3.2), $\mathbf{f} = f + \widetilde{I}$ is a solution of (1.3.3).

(c) If $\mathbf{f}$ is a solution of (1.3.3) there is a representative f of $\mathbf{f}$ that is a solution of (1.3.2).

Proof. By Lemma 1.3.1 (b), $\partial_{x_i} \widetilde{I}$, the ideal generated by the partial derivatives in order to x_i of elements of $\widetilde{I}$, is equal to $\widetilde{I}$. Hence (b) holds.

Assume $J = (0)$. The existence and uniqueness of the solution of (1.3.2) is a special case of the classical Cauchy-Kowalevski Theorem. There is one and only one formal solution of (1.3.2). Its convergence follows from the majorant method.

The existence of a solution of (1.3.3) follows from (b).

Let $\mathbf{f_1}, \mathbf{f_2}$ be two solutions of (1.3.3). Let f_j be a representative of $\mathbf{f_j}$ for $j = 1, 2$. Then $\partial_{x_n}(f_2 - f_1) + \sum_{i=1}^{n-1} a_i \partial_{x_i}(f_2 - f_1) \in \widetilde{I}$ and $f_2 - f_1 + x_n\widetilde{\mathcal{O}} \in \widetilde{I} + x_n\widetilde{\mathcal{O}}$. By Lemma 1.3.1, $f_2 - f_1 \in \widetilde{I}$. Therefore $\mathbf{f_1} = \mathbf{f_2}$. This ends the proof of statement (a).

If $\mathbf{f}$ is a solution of (1.3.3), it follows from (a) that there is a unique f that is a solution of (1.3.2). It remains to see that f is a representative of $\mathbf{f}$. This follows from (b) and from the uniqueness of the solution of the Cauchy problem. Hence (c) holds.

□

Set $\Omega^1_{X|S} = \bigoplus_{i=1}^n \mathcal{O}dx_i$. We call the elements of $\Omega^1_{X|S}$ *germs of relative differential forms* on $X \times S$. The map $d : \mathcal{O} \to \Omega^1_{X|S}$ given by $df = \sum_{i=1}^n \partial x_i f dx_i$ is called the *relative differential* of f.

Assume that $dim\, X = 3$ and let $\mathcal{L}$ be a contact structure on X. Let $\rho : X \times S \to X$ be the first projection. Let ω be a generator of $\mathcal{L}$. We will denote by $\mathcal{L}_S$ the sub $\mathcal{O}$-module of $\Omega^1_{X|S}$ generated by $\rho^*\omega$. We call $\mathcal{L}_S$ a *relative contact structure* of $X \times S$. We call $(X \times S, \mathcal{L}_S)$ a relative contact manifold. We say that an isomorphism of analytic spaces

$$\chi : X \times S \to X \times S \tag{1.3.4}$$

is a *relative contact transformation* if $\chi(\mathbf{0}, s) = (\mathbf{0}, s)$, $\chi^*\omega \in \mathcal{L}_S$ for each $\omega \in \mathcal{L}_S$ and the diagram

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, hook] & X \arrow[d, hook] \\ X \times S \arrow[r, "\chi"] \arrow[d] & X \times S \arrow[d] \\ S \arrow[r, "id_S"] & S \end{tikzcd} \tag{1.3.5}$$

commutes.

The demand of the commutativeness of diagram (1.3.5) is a very restrictive condition but these are the only relative contact transformations we will need. We can and will assume that the local ring of X equals $\mathbb{C}\{x, y, p\}$ and that $\mathcal{L}$ is generated by $dy - pdx$.

Set $\mathcal{O} = \mathbb{C}\{x, y, p, \mathbf{z}\}/\widetilde{I}$ and $\widetilde{\mathcal{O}} = \mathbb{C}\{x, y, p, \mathbf{z}\}/\widetilde{J}$. Let $\mathfrak{m}_X$ be the maximal ideal of $\mathbb{C}\{x, y, p\}$. Let $\mathfrak{m}\,[\widetilde{\mathfrak{m}}]$ be the maximal ideal of $\mathbb{C}\{\mathbf{z}\}/I\,[\mathbb{C}\{\mathbf{z}\}/J]$. Let $\mathfrak{n}\,[\widetilde{\mathfrak{n}}]$ be the ideal of $\mathcal{O}\,[\widetilde{\mathcal{O}}]$ generated by $\mathfrak{m}_X\mathfrak{m}\,[\mathfrak{m}_X\widetilde{\mathfrak{m}}]$.

Remark 1.3.3. If (1.3.4) is a relative contact transformation, there are $\alpha, \beta, \gamma \in \mathfrak{n}$ such that $\partial_x\beta \in \mathfrak{n}$ and

$$\chi(x, y, p, \mathbf{z}) = (x + \alpha, y + \beta, p + \gamma, \mathbf{z}). \tag{1.3.6}$$

Theorem 1.3.4. *(a) Let $\chi : X \times S \to X \times S$ be a relative contact transformation. There is $\beta_0 \in \mathfrak{n}$ such that $\partial_p\beta_0 = 0$, $\partial_x\beta_0 \in \mathfrak{n}$, β is the solution of the Cauchy problem*

$$\left(1 + \frac{\partial\alpha}{\partial x} + p\frac{\partial\alpha}{\partial y}\right)\frac{\partial\beta}{\partial p} - p\frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial x} = p\frac{\partial\alpha}{\partial p}, \qquad \beta - \beta_0 \in p\mathcal{O} \tag{1.3.7}$$

and

$$\gamma = \left(1 + \frac{\partial\alpha}{\partial x} + p\frac{\partial\alpha}{\partial y}\right)^{-1}\left(\frac{\partial\beta}{\partial x} + p\left(\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial x} - p\frac{\partial\alpha}{\partial y}\right)\right). \tag{1.3.8}$$

(b) Given $\alpha, \beta_0 \in \mathfrak{n}$ such that $\partial_p\beta_0 = 0$ and $\partial_x\beta_0 \in \mathfrak{n}$, there is a unique contact transformation χ verifying the conditions of statement (a). *We will denote χ by χ_{α,β_0}.*

(c) Given a relative contact transformation $\widetilde{\chi} : X \times T \to X \times T$ there is one and only one contact transformation $\chi : X \times S \to X \times S$ such that the diagram

$$\begin{tikzcd} X \times S \arrow[r, "\chi"] \arrow[d, hook] & X \times S \arrow[d, hook] \\ X \times T \arrow[r, "\widetilde{\chi}"] & X \times T \end{tikzcd} \tag{1.3.9}$$

commutes.

(d) Given $\alpha, \beta_0 \in \mathfrak{n}$ and $\widetilde{\alpha}, \widetilde{\beta}_0 \in \widetilde{\mathfrak{n}}$ such that $\partial_p \beta_0 = 0$, $\partial_p \widetilde{\beta}_0 = 0$, $\partial_x \beta_0 \in \mathfrak{n}$, $\partial_x \widetilde{\beta}_0 \in \widetilde{\mathfrak{n}}$ and $\widetilde{\alpha}, \widetilde{\beta}_0$ are representatives of α, β_0, set $\chi = \chi_{\alpha,\beta_0}$, $\widetilde{\chi} = \chi_{\widetilde{\alpha},\widetilde{\beta}_0}$. Then diagram (1.3.9) commutes.

Proof. Statements (a) and (b) are a relative version of Theorem 3.2 of [1]. In [1] we assume $S = \{0\}$. The proof works as long S is smooth. The proof in the singular case is a consequence of the singular variant of the Cauchy-Kowalevski Theorem introduced in 1.3.2. Statement (c) follows from statement (b) of Theorem 1.3.2. To see that (d) holds, note that from (c) of Theorem 1.3.2 it follows that If $\widetilde{\beta}_0$ is a representative β_0, then $\widetilde{\beta}$ (unique by (b)) is a representative of β.

□

Remark 1.3.5. (i) The inclusion $S \hookrightarrow T$ is said to be a *small extension* if the surjective map between local rings $\varphi : \mathcal{O}_T \twoheadrightarrow \mathcal{O}_S$ has one dimensional kernel as vector space over $\mathbb{C}$. If the kernel of φ is generated by ε, we have that, as complex vector spaces, $\mathcal{O}_T = \mathcal{O}_S \oplus \varepsilon\mathbb{C}$. Every extension of Artinian local rings factors through small extensions.

(ii) $\varepsilon \mathfrak{m}_T = 0$: let $\mathfrak{m}_S$ $[\mathfrak{m}_T]$ denote the maximal of $\mathcal{O}_S$ $[\mathcal{O}_T]$. If $a \in \mathfrak{m}_T$, as $a\varepsilon \in Ker\, \varphi$, one has $(\lambda - a)\varepsilon = 0$ for some $\lambda \in \mathbb{C}$ which we suppose non-zero. Now, as $\lambda - a \notin \mathfrak{m}_T$ and $\mathcal{O}_T$ is local, $\lambda - a$ is a unit meaning that $\varepsilon = 0$ which is absurd. We conclude that $\lambda = 0$ and so $\varepsilon \mathfrak{m}_T = 0$.

(iii) $\varepsilon \in \mathfrak{m}_T$: suppose ε is a unit. There is $a \in \mathcal{O}_T$ such that $a\varepsilon = 1$ which implies $\varphi(a)\varphi(\varepsilon) = 1$ which is absurd. We conclude that ε is a non-unit and as $\mathcal{O}_T$ is local $\varepsilon \in \mathfrak{m}_T$.

Theorem 1.3.6. *Let $S \hookrightarrow T$ be a small extension such that $\mathcal{O}_S \cong \mathbb{C}\{\mathbf{z}\}$ and*

$$\mathcal{O}_T \cong \mathbb{C}\{\mathbf{z}, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \dots \varepsilon z_m) = \mathbb{C}\{\mathbf{z}\} \oplus \mathbb{C}\varepsilon.$$

Assume $\chi : X \times S \to X \times S$ is a relative contact transformation given at the ring level by

$$(x, y, p) \mapsto (H_1, H_2, H_3),$$

$\alpha, \beta_0 \in \mathfrak{m}_X$, such that $\partial_p \beta_0 = 0$ and $\beta_0 \in (x^2, y)$. Then, there are uniquely determined $\beta, \gamma \in \mathfrak{m}_X$ such that $\beta - \beta_0 \in p\mathcal{O}_X$ and $\widetilde{\chi} :\ X \times T \to X \times T$, given by

$$\widetilde{\chi}(x, y, p, \mathbf{z}, \varepsilon) = (H_1 + \varepsilon\alpha, H_2 + \varepsilon\beta, H_3 + \varepsilon\gamma, \mathbf{z}, \varepsilon),$$

is a relative contact transformation extending χ (see diagram (1.3.9)). Moreover, the Cauchy problem (1.3.7) for $\widetilde{\chi}$ takes the simplified form

$$\frac{\partial \beta}{\partial p} = p\frac{\partial \alpha}{\partial p}, \qquad \beta - \beta_0 \in \mathbb{C}\{x, y, p\}p \tag{1.3.10}$$

and

$$\gamma = \frac{\partial \beta}{\partial x} + p(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x}) - p^2\frac{\partial \alpha}{\partial y}. \tag{1.3.11}$$

Proof. We have that $\widetilde{\chi}$ is a relative contact transformation if and only if there is $f := f' + \varepsilon f'' \in \mathcal{O}_T\{x, y, p\}$ with $f \notin (x, y, p)\mathcal{O}_T\{x, y, p\}$, $f' \in \mathcal{O}_S\{x, y, p\}$, $f'' \in \mathbb{C}\{x, y, p\} = \mathcal{O}_X$ such that

$$d(H_2 + \varepsilon\beta) - (H_3 + \varepsilon\gamma)d(H_1 + \varepsilon\alpha) = f(dy - pdx). \tag{1.3.12}$$

Since χ is a relative contact transformation we can suppose that

$$dH_2 - H_3dH_1 = f'(dy - pdx).$$

Using the fact that $\varepsilon\mathfrak{m}_{\mathcal{O}_T} = 0$ (see Remark 1.3.5 (ii)) we see that (1.3.12) is equivalent to

$$\begin{aligned} \frac{\partial \beta}{\partial p} &= p\frac{\partial \alpha}{\partial p}, \\ \gamma &= \frac{\partial \beta}{\partial x} + p(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x}) - p^2\frac{\partial \alpha}{\partial y}, \\ f'' &= \frac{\partial \beta}{\partial y} - p\frac{\partial \alpha}{\partial y}. \end{aligned}$$

As $\beta - \beta_0 \in (p)\mathbb{C}\{x, y, p\}$ we have that β, and consequently γ, are completely determined by α and β_0. $\square$

Remark 1.3.7. Set $\alpha = \sum_k \alpha_k p^k$, $\beta = \sum_k \beta_k p^k$, $\gamma = \sum_k \gamma_k p^k$, where $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}\{x, y\}$ for each $k \geq 0$ and $\beta_0 \in (x^2, y)$. Under the assumptions of Theorem 1.3.6,

(i) $\beta_k = \frac{k-1}{k}\alpha_{k-1}, \qquad k \geq 1$.

(ii) Moreover,

$$\gamma_0 = \frac{\partial \beta_0}{\partial x}, \; \gamma_1 = \frac{\partial \beta_0}{\partial y} - \frac{\partial \alpha_0}{\partial x}, \; \gamma_k = -\frac{1}{k}\frac{\partial \alpha_{k-1}}{\partial x} - \frac{1}{k-1}\frac{\partial \alpha_{k-2}}{\partial y}, \qquad k \geq 2.$$

Since,

$$\frac{\partial}{\partial y}\gamma_0 = \frac{\partial}{\partial x}(\frac{\partial \alpha_0}{\partial x} + \gamma_1),$$

β_0 is the solution of the Cauchy problem

$$\frac{\partial \beta_0}{\partial x} = \gamma_0, \quad \frac{\partial \beta_0}{\partial y} = \frac{\partial \alpha_0}{\partial x} + \gamma_1, \qquad \beta_0 \in (x^2, y).$$

1.4 Categories of Deformations

A category $\mathfrak{C}$ is called a *groupoid* if all morphisms of $\mathfrak{C}$ are isomorphisms.

Let $p : \mathfrak{F} \to \mathfrak{C}$ be a functor. Let S be an object of $\mathfrak{C}$. We will denote by $\mathfrak{F}(S)$ the subcategory of $\mathfrak{F}$ given by the following conditions:

- Ψ is an object of $\mathfrak{F}(S)$ if $p(\Psi) = S$.
- χ is a morphism of $\mathfrak{F}(S)$ if $p(\chi) = id_S$.

Let $\chi\,[\Psi]$ be a morphism [an object] of $\mathfrak{F}$. Let $f\,[S]$ be a morphism [an object] of $\mathfrak{C}$. We say that $\chi\,[\Psi]$ is a morphism [an object] of $\mathfrak{F}$ over $f\,[S]$ if $p(\chi) = f\,[p(\Psi) = S]$.

A morphism $\chi' : \Psi' \to \Psi$ of $\mathfrak{F}$ over $f : S' \to S$ is called *cartesian* if for each morphism $\chi'' : \Psi'' \to \Psi$ of $\mathfrak{F}$ over f there is exactly one morphism $\chi : \Psi'' \to \Psi'$ over $id_{S'}$ such that $\chi' \circ \chi = \chi''$.

If the morphism $\chi' : \Psi' \to \Psi$ over f is cartesian, Ψ' is well defined up to a unique isomorphism. We will denote Ψ' by $f^*\Psi$ or $\Psi \times_S S'$.

We say that $\mathfrak{F}$ is a *fibered category* over $\mathfrak{C}$ if

1. For each morphism $f : S' \to S$ in $\mathfrak{C}$ and each object Ψ of $\mathfrak{F}$ over S there is a morphism $\chi' : \Psi' \to \Psi$ over f that is cartesian.
2. The composition of cartesian morphisms is cartesian.

A *fibered groupoid* is a fibered category such that $\mathfrak{F}(S)$ is a groupoid for each $S \in \mathfrak{C}$. The functor $p : \mathfrak{F} \to \mathfrak{C}$ is said to be a *cofibered groupoid* if the dual functor $p^\circ : \mathfrak{F}^\circ \to \mathfrak{C}^\circ$ is a fibered groupoid. Let us denote the element $(f^\circ)^*\Psi$ by $f_*\Psi$ or $\Psi \otimes_A A'$ where $A := \mathcal{O}_S$ $[A' := \mathcal{O}_{S'}]$.

Lemma 1.4.1. *If $p : \mathfrak{F} \to \mathfrak{C}$ is a fibered category each map in $\mathfrak{F}$ is cartesian. In particular, if $p : \mathfrak{F} \to \mathfrak{C}$ satisfies condition* 1. *above and $\mathfrak{F}(S)$ is a groupoid for each object S of $\mathfrak{C}$, then $\mathfrak{F}$ is a fibered groupoid over $\mathfrak{C}$.*

Proof. Let $\chi : \Phi \to \Psi$ be an arbitrary morphism of $\mathfrak{F}$. It is enough to show that χ is cartesian. Set $f = p(\chi)$. Let $\chi' : \Phi' \to \Psi$ be another morphism over f. Let $f^*\Psi \to \Psi$ be a cartesian morphism over f. There are morphisms $\alpha : \Phi' \to f^*\Psi$, $\beta : \Phi \to f^*\Psi$ such that the solid diagram

$$\begin{tikzcd}
f^*\Psi \arrow[dr] & \Phi \arrow[l, "\beta"'] \arrow[d, "\chi"] & \Phi' \arrow[l, dotted] \arrow[ll, bend right, "\alpha"'] \arrow[dl, "\chi'"] \\
& \Psi &
\end{tikzcd} \tag{1.4.1}$$

commutes. Hence $\beta^{-1} \circ \alpha$ is the only morphism over f such that diagram (1.4.1) commutes.

□

Lemma 1.4.2. *If $p : \mathfrak{F} \to \mathfrak{C}$ is a fibered groupoid χ is an isomorphism of $\mathfrak{F}$ if and only if $p(\chi)$ is an isomorphism of $\mathfrak{C}$.*

Proof. The only if part is just a consequence of the functorial proprties of p. Suppose $\chi : \Phi \to \Psi$ is a morphism in $\mathfrak{F}$ such that $p(\chi) : S \to T$ is an isomorphism. There is $g : T \to S$ with $p(\chi) \circ g = id_T$ and $g \circ p(\chi) = id_S$. From (1) of the definition of fibered category we conclude the existence of $\chi' : \Psi \to \Phi$ cartesian over g. As $\chi' \circ \chi \in \mathfrak{F}(S)$, $\chi \circ \chi' \in \mathfrak{F}(T)$ and $\mathfrak{F}(S), \mathfrak{F}(T)$ are groupoids $\chi' \circ \chi, \chi \circ \chi'$ and consequently χ(as well as χ') are isomorphisms. $\square$

Let $\mathfrak{A}n$ be the category of analytic complex space germs. Let 0 denote the complex vector space of dimension 0. Let $p : \mathfrak{F} \to \mathfrak{A}n$ be a fibered category.

Definition 1.4.3. Let T be an analytic complex space germ. Let $\psi\,[\Psi]$ be an object of $\mathfrak{F}(0)\,[\mathfrak{F}(T)]$. We say that Ψ is a *versal deformation* of ψ if given

- a closed embedding $f : T'' \hookrightarrow T'$,
- a morphism of complex analytic space germs $g : T'' \to T$,
- an object Ψ' of $\mathfrak{F}(T')$ such that $f^*\Psi' \cong g^*\Psi$,

there is a morphism of complex analytic space germs $h : T' \to T$ such that

$$h \circ f = g \qquad \text{and} \qquad h^*\Psi \cong \Psi'.$$

If Ψ is versal and for each Ψ' the tangent map $T(h) : T_{T'} \to T_T$ is determined by Ψ', Ψ is called a *semiuniversal deformation* of ψ.

Let T be a germ of a complex analytic space. Let A be the local ring of T and let $\mathfrak{m}$ be the maximal ideal of A. Let T_n be the complex analytic space with local ring $A/\mathfrak{m}^n$ for each positive integer n. The canonical morphisms

$$A \to A/\mathfrak{m}^n \qquad \text{and} \qquad A/\mathfrak{m}^n \to A/\mathfrak{m}^{n+1}$$

induce morphisms $\alpha_n : T_n \to T$ and $\beta_n : T_{n+1} \to T_n$.

A morphism $f : T'' \to T'$ induces morphisms $f_n : T''_n \to T'_n$ such that the diagram

$$\begin{tikzcd}
T'' \arrow[r, "f"] & T' \\
T''_n \arrow[u, hook, "\alpha''_n"] \arrow[r, "f_n"] & T'_n \arrow[u, hook, "\alpha'_n"'] \\
T''_{n+1} \arrow[u, hook, "\beta''_n"] \arrow[r, "f_{n+1}"] & T'_{n+1} \arrow[u, hook, "\beta'_n"']
\end{tikzcd}$$

commutes.

Definition 1.4.4. We will follow the terminology of Definition 1.4.3. Let $g_n = g \circ \alpha''_n$. We say that Ψ is a *formally versal deformation* of ψ if there are morphisms $h_n : T'_n \to T$ such that

$$h_n \circ f_n = g_n, \quad f_n^*\Psi \cong g_n^*\Psi, \quad h_n \circ \beta'_n = h_{n+1} \quad \text{and} \quad h_n^*\Psi \cong {\alpha'_n}^*\Psi'.$$

If Ψ is formally versal and for each Ψ' the tangent maps $T(h_n) : T_{T'_n} \to T_T$ are determined by ${\alpha'_n}^*\Psi'$, Ψ is called a *formally semiuniversal deformation* of ψ.

If $p : \mathfrak{F} \to \mathfrak{C}$ is a cofibered groupoid $\Psi \in \mathfrak{F}$ is said to be a versal [formally versal] object if Ψ is versal as object of $\mathfrak{F}^\circ$ in the fibered groupoid $p^\circ : \mathfrak{F}^\circ \to \mathfrak{C}^\circ$.

Remark 1.4.5. Actually, the usual definition of formal versality comes from Definition 1.4.3 demanding that $\mathcal{O}_{T''}$ and $\mathcal{O}_T$ are Artinian. Definition 1.4.4 is inspired by the following:

Let $p : \mathfrak{F} \to \mathfrak{C}$ be a cofibered groupoid over the category of analytic local $\mathbb{C}$-algebras. Through completion, this functor naturally extends to a functor $\hat{p} : \hat{\mathfrak{F}} \to \hat{\mathfrak{C}}$ over the category of complete local $\mathbb{C}$-algebras. Then a formally versal object in $\hat{\mathfrak{F}}$ is just a projective system $(\alpha_n^*\Psi \in \mathfrak{F}(T_n))_{n\geq 0}$.

By Schlessinger's Theorem (see [26], Theorem 1.11) each $\psi \in \mathfrak{F}(\mathbb{C})$ has a formal semiuniversal deformation in $\hat{\mathfrak{F}}$. Restricting to formal versality, that is, restricting to T'' and T' Artinian in Definition 1.4.3, has the advantage of letting us assume that $T'' \hookrightarrow T'$ is a small extension (see Remark 1.3.5).

The next result will be useful in the proofs of Lemma 1.4.7 and Theorem 1.4.8.

Lemma 1.4.6. *Let $\mathfrak{F} \to \mathfrak{C}$ be a cofibered groupoid. A deformation $b \in \mathfrak{F}(B)$ is versal [formally versal] if and only if $b \otimes_B B\{\mathbf{x}\}$ $[\,b \otimes_B B[[\mathbf{x}]]\,]$ is versal [formally versal], where $B\{\mathbf{x}\} := B\{x_1, \dots, x_n\}$.*

Proof. Let us prove the convergent case. We prove only that $b \in \mathfrak{F}(B)$ is complete if and only if $b \otimes_B B\{\mathbf{x}\}$ is complete, as the proof of the equivalence of versalities is basically the same but with more complicated notation.

Let $i : B \to B\{\mathbf{x}\}$ denote the natural inclusion such that $i_*b = b \otimes_B B\{\mathbf{x}\}$ and let $c \in \mathfrak{F}(C)$. Suppose b is complete. There is $f : B \to C$ such that $c \cong f_*b$ inducing a natural morphism $f' : B\{\mathbf{x}\} \to C$ such that $f = f' \circ i$. Then $c \cong f_*b = f'_*(i_*b)$ and $b \otimes_B B\{\mathbf{x}\}$ is complete.

Conversely, suppose $b \otimes_B B\{\mathbf{x}\}$ is complete. There is $f' : B\{\mathbf{x}\} \to C$ such that $c \cong f'_*(i_*b) = (f' \circ i)_*b$, hence b is complete.

□

Lemma 1.4.7 and Theorem 1.4.8 are proven in [6] which is originally written in German. As, to our knowledge, there exists no English translation of [6], we present the proofs of these results for convenience of the reader.

Lemma 1.4.7 ([6], Lemma 5.3)**.** *Suppose $p : \mathfrak{F} \to \mathfrak{C}$ is cofibered groupoid. Let $\bar{b} \in \hat{\mathfrak{F}}(\bar{B})$ be formally semiuniversal, $\bar{a} \in \hat{\mathfrak{F}}(\bar{A})$ formally versal and $\bar{b} \to \bar{a}$ a morphism. Then $\bar{A}$ is a (formal) power series ring over $\bar{B}$.*

Proof. Since $\bar{a}$ is formally versal there is a morphism $\bar{a} \to \bar{b}$. The composition $\bar{b} \to \bar{a} \to \bar{b}$ is an endomorphism of $\bar{b}$, for which the associated map

$$\mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2 \to \mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2$$

is necessarily the identity because of the formal semiuniversality. In particular, the map

$$\alpha : \mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2 \to \mathfrak{m}_{\bar{A}}/\mathfrak{m}_{\bar{A}}^2$$

is injective. Let $n = dim_{\mathbb{C}} Coker(\alpha)$. If $\bar{C} := \bar{B}[[\mathbf{x}]]_n := \bar{B}[[x_1, \ldots, x_n]]$ we get an isomorphism

$$\beta : \mathfrak{m}_{\bar{A}}/\mathfrak{m}_{\bar{A}}^2 \to \mathfrak{m}_{\bar{C}}/\mathfrak{m}_{\bar{C}}^2$$

inducing an isomorphism $\bar{C}_1 \cong \bar{A}_1$, where $\bar{C}_1 := \bar{C}/({\mathfrak{m}_{\bar{C}}}^2)$ and $\bar{A}_1 := \bar{A}/({\mathfrak{m}_{\bar{A}}}^2)$. As $\bar{a}$ is formally versal, we're able to complete (indicated by a dashed arrow) the following diagram with solid arrows

$$\begin{tikzcd}
\bar{a} \arrow[r, dashed] \arrow[d] & \bar{b} \otimes_{\bar{B}} \bar{C} \arrow[d] \\
\bar{a} \otimes_{\bar{A}} \bar{A}_1 \arrow[r, "\sim"] & \bar{b} \otimes_{\bar{B}} \bar{C}_1.
\end{tikzcd}$$

The morphism $\bar{a} \to \bar{b} \otimes_{\bar{B}} \bar{C}$ induces a $\mathbb{C}$-homomorphism $\varphi : \bar{A} \to \bar{C}$ such that $\dot{\varphi} = \beta$. As β is an isomorphism, by the inverse function theorem (see Theorem I.1.21 of [9]) φ is an isomorphism. $\square$

Theorem 1.4.8 ([6], Theorem 5.2)**.** *Let $\mathfrak{F} \to \mathfrak{C}$ be a cofibered groupoid. Let $a \in \mathfrak{F}(A)$ be a versal deformation of $\psi \in \mathfrak{F}(\mathbb{C})$. Then:*

(a) There is a semiuniversal deformation of ψ in $\mathfrak{F}$.

(b) Every formally versal [semiuniversal]deformation of ψ is versal [semiuniversal].

Proof. (a) : Let $\bar{b}$ be a formally semiuniversal object and $\bar{b} \to \hat{a}$ a morphism. By Lemma 1.4.7 $\hat{A}$ is a (formal) power series ring over $\bar{B}$ such that $\hat{A} = \bar{B}[[\mathbf{x}]]$. Let $a_1, \ldots, a_n$ be elements in A whose images in $\hat{A}$ are mapped through β (see proof of Lemma 1.4.7) to $x_1, \ldots, x_n$. We claim that if $B := A/(a_1, \ldots, a_n)$, $b := a \otimes_A B$ is semiuniversal. Firstly, we notice that according to the contruction $\bar{b} \cong \hat{b}$. Let $C = B\{\mathbf{x}\}$. As in the proof of Lemma 1.4.7 we find a B-isomorphism $A_1 \cong C_1$, and again it's possible to lift the induced morphism $a \to a \otimes_A A_1 \cong b \otimes_B C_1$ to a morphism $a \to b \otimes_B C$, which in turn induces an isomorphism $\varphi : A \to C$. In particular, by Lemma 1.4.2 $a \cong b \otimes_B C$ and so by Lemma 1.4.6 b is also versal.

(b): Let c in $\mathfrak{F}$ be a formally versal object, b a semiuniversal deformation of $c \otimes_C C/\mathfrak{m}_C$ and $b \to c$ a morphism. According to Lemma 1.4.7 $\hat{C}$ is a formal power series ring over

$\hat{B}$ and so C is a convergent power series ring over B. Consequently, as for b, c is also versal (as in the proof of (a)).

□

Let Z be a curve of $\mathbb{C}^n$ with irreducible components $Z_1, \dots, Z_r$. Set $\bar{\mathbb{C}} = \bigsqcup_{i=1}^r \bar{C}_i$ where each $\bar{C}_i$ is a copy of $\mathbb{C}$. Let φ_i be a parametrization of Z_i, $1 \leq i \leq r$. Let $\varphi : \bar{\mathbb{C}} \to \mathbb{C}^n$ be the map such that $\varphi|_{\bar{C}_i} = \varphi_i$, $1 \leq i \leq r$. We call φ the *parametrization* of Z.

Let T be an analytic space. A morphism of analytic spaces $\Phi : \bar{\mathbb{C}} \times T \to \mathbb{C}^n \times T$ is called a *deformation of* φ *over* T if the diagram

$$\begin{tikzcd}
\bar{\mathbb{C}} \arrow[r, "\varphi"] \arrow[d, hook] & \mathbb{C}^n \arrow[d, hook] \\
\bar{\mathbb{C}} \times T \arrow[r, "\Phi"] \arrow[d] & \mathbb{C}^n \times T \arrow[d] \\
T \arrow[r, "id_T"] & T
\end{tikzcd}$$

commutes. The analytic space T is called de *base space* of the deformation.

We will denote by Φ_i the composition

$$\bar{C}_i \times T \hookrightarrow \bar{\mathbb{C}} \times T \xrightarrow{\Phi} \mathbb{C}^n \times T \to \mathbb{C}^n, \qquad 1 \leq i \leq r.$$

The maps Φ_i, $1 \leq i \leq r$, determine Φ.

Let Φ be a deformation of φ over T. Let $f : T' \to T$ be a morphism of analytic spaces. We will denote by $f^*\Phi$ the deformation of φ over T' given by

$$(f^*\Phi)_i = \Phi_i \circ (id_{\bar{C}_i} \times f).$$

We call $f^*\Phi$ the *pullback* of Φ by f.

Let $\Phi' : \bar{\mathbb{C}} \times T \to \mathbb{C}^n \times T$ be another deformation of φ over T. A morphism from Φ' into Φ is a pair (χ, ξ) where $\chi : \mathbb{C}^n \times T \to \mathbb{C}^n \times T$ and $\xi : \bar{\mathbb{C}} \times T \to \bar{\mathbb{C}} \times T$ are isomorphisms of analytic spaces such that the diagram

$$\begin{tikzcd}
T & \bar{\mathbb{C}} \times T \arrow[l] \arrow[r, "\Phi"] & \mathbb{C}^n \times T \arrow[r] & T \\
 & \bar{\mathbb{C}} \arrow[u, hook] \arrow[d, hook] \arrow[r, "\varphi"] & \mathbb{C}^n \times \{0\} \arrow[u, hook] \arrow[d, hook] & \\
T \arrow[uu, "id_T"] & \bar{\mathbb{C}} \times T \arrow[l] \arrow[r, "\Phi'"] \arrow[uu, bend left=60, "\xi"] & \mathbb{C}^n \times T \arrow[r] \arrow[uu, bend right=60, "\chi"'] & T \arrow[uu, "id_T"']
\end{tikzcd}$$

commutes.

Let Φ' be a deformation of φ over S and $f : S \to T$ a morphism of analytic spaces. A *morphism of* Φ' *into* Φ *over* f is a morphism from Φ' into $f^*\Phi$. There is a functor p

that associates T to a deformation Ψ over T and f to a morphism of deformations over f.

Given $t \in T$ let Z_t be the curve parametrized by the composition

$$\bar{\mathbb{C}} \times \{t\} \hookrightarrow \bar{\mathbb{C}} \times T \xrightarrow{\Phi} \mathbb{C}^n \times T \to \mathbb{C}^n.$$

We say that Z_t is the *fiber of the deformation* Φ *at the point* t.

All analytic spaces considered are identified with a germ taken at some point of a representative space. Let $\mathbf{0}$ $[\bar{0}]$ denote the point of representatives of $T, \mathbb{C}^n, \mathbb{C}^n \times T$ $[\bar{\mathbb{C}}]$ where the germ is taken.

The deformation Φ is said be a a *deformation with section* if there are morphisms

$$\sigma : T \to \mathbb{C}^n \times T$$

and

$$\bar{\sigma} : T \to \bar{C} \times T,\ s \mapsto \coprod_{i=1}^{r} (\bar{\sigma}_i(s), s),\ \bar{\sigma}_i(\mathbf{0}) = \bar{0}_i$$

such that

$$\sigma = \Phi_i \circ \bar{\sigma}_i$$

for each $i = 1, \dots, r$. A section is said to be *trivial* if $\sigma(s) = (\mathbf{0}, s)$ such that

$$(\Phi_{i,1}(\bar{\sigma}_i(s)), \dots, \Phi_{i,n}(\bar{\sigma}_i(s))) = (0, \dots, 0) \in \mathbb{C}^n,$$

for each $i = 1, \dots, r$.

Deformations with section equipped with isomorphisms compatible with the section define a subcategory of the deformations of φ. Suppose Φ is a deformation with section of φ. Let

$$\mathbf{ord}\,\varphi := (ord\,\varphi_1, \dots, ord\,\varphi_r),$$

where $ord\,\varphi_i = max\{m : \varphi_i^*(\mathfrak{m}_{\mathbb{C}^n}) \subset \mathfrak{m}_{\bar{C}_i}^m\}$. Let

$$\begin{aligned} I_{\bar{\sigma}_i} &:= Ker(\bar{\sigma}_i^* : \mathcal{O}_{\bar{C}_i \times T} \to \mathcal{O}_T) \\ I_\sigma &:= Ker(\sigma^* : \mathcal{O}_{\mathbb{C}^2 \times T} \to \mathcal{O}_T). \end{aligned}$$

Set

$$\mathbf{ord}\,\Phi := (ord\,\Phi_1, \dots, ord\,\Phi_r),$$

where

$$ord\,\Phi_i := max\{m : \Phi_i^*(I_\sigma) \subset I_{\bar{\sigma}_i}^m\}.$$

Then Φ is said to be *equimultiple* if $\mathbf{ord}\,\varphi = \mathbf{ord}\,\Phi$. Note that if the section is trivial then

$$\begin{aligned} I_\sigma &= (x_1, \dots, x_n), \\ I_{\bar{\sigma}_i} &= (t_i), \end{aligned}$$

where $x_1, \ldots, x_n$ $[t_i]$ are coordinates for $\mathbb{C}^n$ $[\bar{\mathbb{C}}_i]$.

All deformations with section are isomorphic to a deformation with trivial section (see Proposition I.2.2 of [9]). Assume Φ has trivial section, $\Phi_i(t_i, \mathbf{s}) = (X_{1,i}(t_i, \mathbf{s}), \ldots, X_{n,i}(t_i, \mathbf{s}))$ with $1 \leq i \leq r$ and that Z_i has multiplicity m_i. Then, Φ_i is equimultiple if and only if $X_{j,i} \in (t^{m_i})$ for each $1 \leq i \leq r$, $1 \leq j \leq n$ and Φ is *equimultiple* if and only if each Φ_i is equimultiple

Assume Z is a plane curve. Set

$$\Phi_i(t_i, \mathbf{s}) = (X_i(t_i, \mathbf{s}), Y_i(t_i, \mathbf{s})), \qquad 1 \leq i \leq r. \tag{1.4.2}$$

We will denote by $\mathcal{D}ef_\varphi$ $[\mathcal{D}ef_\varphi^{em}]$ the *category of deformations* $[$*equimultiple deformations* $]$*of* φ. We say that Φ is an object of $\overrightarrow{\mathcal{D}ef}_\varphi[\overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi]$ if Φ is equimultiple and $Y_i \in (t_i x_i)$ $[Y_i \in (x_i^2)]$, $1 \leq i \leq r$.

If T is reduced, $\Phi \in \mathcal{D}ef_\varphi^{em}[\overrightarrow{\mathcal{D}ef}_\varphi, \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi]$ if and only if all fibres of Φ are equimultiple [have tangent cone $\{y = 0\}$, and are in generic position].

Consider in $\mathbb{C}^3$ the contact structure given by the differential form $\omega = dy - pdx$. Assume Z is a Legendrian curve parametrized by $\psi : \bar{\mathbb{C}} \to \mathbb{C}^3$. Let Ψ be a deformation of ψ given by

$$\Psi_i(t_i, \mathbf{s}) = (X_i(t_i, \mathbf{s}), Y_i(t_i, \mathbf{s}), P_i(t_i, \mathbf{s})). \tag{1.4.3}$$

for $1 \leq i \leq r$. We say that Ψ is a *Legendrian deformation of* ψ if $\Psi_i^*(\rho^*\omega) = 0$, for $1 \leq i \leq r$. We say that (χ, ξ) is an isomorphism of Legendrian deformations if χ is a relative contact transformation. We will denote by $\widehat{\mathcal{D}ef}_\psi$ $[\widehat{\mathcal{D}ef}_\psi^{em}]$ the category of Legendrian [equimultiple Legendrian] deformations of ψ. All deformations are assumed to have trivial section.

Assume that $\psi = \mathcal{C}on\, \varphi$ parametrizes a germ of a Legendrian curve L, in generic position. If (1.4.2) defines an object of $\overrightarrow{\mathcal{D}ef}_\varphi$, setting

$$P_i(t_i, \mathbf{s}) := \partial_{t_i} Y_i(t_i, \mathbf{s}) / \partial_{t_i} X_i(t_i, \mathbf{s}), \qquad 1 \leq i \leq r,$$

the deformation Ψ given by (1.4.3) is a Legendrian deformation of ψ. We say that Ψ is the *conormal of* Φ and denote Ψ by $\mathcal{C}on\, \Phi$. If $\Psi \in \widehat{\mathcal{D}ef}_\psi$ is given by (1.4.3), the deformation Φ of φ given by (1.4.2) is said to be the *plane projection of* Ψ. We will denote Φ by Ψ^π.

We define in this way the functors

$$\mathcal{C}on : \overrightarrow{\mathcal{D}ef}_\varphi \to \widehat{\mathcal{D}ef}_\psi, \qquad \pi : \widehat{\mathcal{D}ef}_\psi \to \mathcal{D}ef_\varphi.$$

Notice that the conormal of the plane projection of a Legendrian deformation always exists and we have that $\mathcal{C}on\,(\Psi^\pi) = \Psi$ for each $\Psi \in \widehat{\mathcal{D}ef}_\psi$ and $(\mathcal{C}on\, \Phi)^\pi = \Phi$ where $\Phi \in \overrightarrow{\mathcal{D}ef}_\varphi$.

Example 1.4.9. Set $\varphi(t) = (t, 0)$, $\psi = \mathcal{C}on\, \varphi$ and $X(t, s) = t$, $Y(t, s) = st$. Then we get $P(t, s) = s$ and although X, Y define an object of $\mathcal{D}ef_\varphi^{em}$, its conormal Ψ is not an element of $\widehat{\mathcal{D}ef}_\psi$, because Ψ is a deformation with section $s \mapsto (0, 0, s, s)$.

Example 1.4.10. Set $\varphi(t) = (t^2, t^5)$, $X(t,s) = t^2$, $Y(t,s) = t^5 + st^3$. Then we get $2P(t,s) = 5t^3 + 3st$. Altough X, Y defines an object of $\overrightarrow{\mathcal{D}ef}_\varphi$, its conormal is not equimultiple.

Remark 1.4.11. Under the assumptions above,

$$\mathcal{C}on\,(\overrightarrow{\twoheadrightarrow}\!\!\mathcal{D}ef_\varphi) \subset \widehat{\mathcal{D}ef}_\psi^{em} \quad \text{and} \quad (\widehat{\mathcal{D}ef}_\psi^{em})^\pi \subset \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi.$$

Lemma 1.4.12. *If $\mathfrak{C}$ is one of the categories $\widehat{\mathcal{D}ef}_\psi$, $\widehat{\mathcal{D}ef}_\psi^{em}$, $p : \mathfrak{C} \to \mathfrak{A}n$ is a fibered groupoid.*

Proof. Let $f : S \to T$ be a morphism of $\mathfrak{A}n$. Let Ψ be a deformation over T. Then, $(\widetilde{\chi}, \widetilde{\xi}) : f^*\Psi \to \Psi$ is cartesian, with

$$\widetilde{\xi}(t_i, \mathbf{s}) = (t_i, \mathbf{s}), \qquad \widetilde{\chi}(x, y, p, \mathbf{s}) = (x, y, p, \mathbf{s}).$$

This is because if $(\chi, \xi) : \Psi' \to \Psi$ is a morphism over f, then by definition of morphism of deformations over different base spaces, (χ, ξ) is a morphism from Ψ' into $f^*\Psi$ over id_S. □

1.5 Equimultiple Versal Deformations

For Sophus Lie a contact transformation was a transformation that takes curves into curves, instead of points into points. We can recover the initial point of view. Given a plane curve Z at the origin, with tangent cone $\{y = 0\}$, and a contact transformation χ from a neighbourhood of $(0; dy)$ into itself, χ acts on Z in the following way: $\chi \cdot Z$ is the plane projection of the image by χ of the conormal of Z. We can define in a similar way the action of a relative contact transformation on a deformation of a plane curve Z, obtainning another deformation of Z.

We say that $\Phi \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi(T)$ is *trivial* (relative to the action of the group of relative contact transformations over T) if there is χ such that $\chi \cdot \Phi := \pi \circ \chi \circ \mathcal{C}on\,\Phi$ is the constant deformation of ϕ over T, given by

$$(t_i, \mathbf{s}) \mapsto \varphi_i(t_i), \qquad i = 1, \ldots, r.$$

Let Z be the germ of a plane curve parametrized by $\varphi : \bar{\mathbb{C}} \to \mathbb{C}^2$. In the following we will identify each ideal of $\mathcal{O}_Z$ with its image by $\varphi^* : \mathcal{O}_Z \to \mathcal{O}_{\bar{\mathbb{C}}}$. Hence

$$\mathcal{O}_Z = \mathbb{C}\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix} \right\} \subset \bigoplus_{i=1}^r \mathbb{C}\{t_i\} = \mathcal{O}_{\bar{\mathbb{C}}}.$$

Set $\dot{\mathbf{x}} = [\dot{x}_1, \ldots, \dot{x}_r]^t$, where $\dot{x}_i$ is the derivative of x_i in order to t_i, $1 \leq i \leq r$. Let

$$\dot{\varphi} := \dot{\mathbf{x}}\frac{\partial}{\partial x} + \dot{\mathbf{y}}\frac{\partial}{\partial y}$$

be an element of the free $\mathcal{O}_{\bar{\mathbb{C}}}$-module

$$\mathcal{O}_{\bar{\mathbb{C}}}\frac{\partial}{\partial x} \oplus \mathcal{O}_{\bar{\mathbb{C}}}\frac{\partial}{\partial y}. \tag{1.5.1}$$

Notice that (1.5.1) has a structure of $\mathcal{O}_Z$-module induced by φ^*.

Let m_i be the multiplicity of Z_i, $1 \leq i \leq r$. Consider the $\mathcal{O}_{\bar{\mathbb{C}}}$-module

$$\left(\bigoplus_{i=1}^{r} t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^{r} t_i^{2m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right). \tag{1.5.2}$$

Let $\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi}$ be the sub $\mathcal{O}_{\bar{\mathbb{C}}}$-module of (1.5.2) generated by

$$(a_1, \ldots, a_r)\left(\dot{\mathbf{x}}\frac{\partial}{\partial x} + \dot{\mathbf{y}}\frac{\partial}{\partial y}\right),$$

where $a_i \in t_i\mathbb{C}\{t_i\}$, $1 \leq i \leq r$. For $i = 1, \ldots, r$ set $p_i = \dot{y}_i/\dot{x}_i$. For each $k \geq 0$ set

$$\mathbf{p}^k = \left[p_1^k, \ldots, p_r^k\right]^t.$$

Let $\widehat{I}$ be the sub $\mathcal{O}_Z$-module of (1.5.2) generated by

$$\mathbf{p}^k\frac{\partial}{\partial x} + \frac{k}{k+1}\mathbf{p}^{k+1}\frac{\partial}{\partial y}, \qquad k \geq 1.$$

Set

$$\widehat{M}_\varphi = \frac{\left(\bigoplus_{i=1}^{r} t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^{r} t_i^{2m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right)}{\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x,y)\frac{\partial}{\partial x} \oplus (x^2,y)\frac{\partial}{\partial y} + \widehat{I}}.$$

Given a category $\mathfrak{C}$ we will denote by $\underline{\mathfrak{C}}$ the set of isomorphism classes of elements of $\mathfrak{C}$.

Theorem 1.5.1. *Let ψ be the parametrization of a germ of a Legendrian curve L of a contact manifold X. Let $\chi : X \to \mathbb{C}^3$ be a contact transformation such that $\chi(L)$ is in generic position. Let φ be the plane projection of $\chi \circ \psi$. Then there is a canonical isomorphism*

$$\underline{\widehat{\mathcal{D}ef}}_\psi^{em}(T_\varepsilon) \xrightarrow{\sim} \widehat{M}_\varphi.$$

Proof. Let $\Psi \in \widehat{\mathcal{D}ef}_\psi^{em}(T_\varepsilon)$. Then, Ψ is the conormal of its projection $\Phi \in \overrightarrow{\mathcal{D}ef}_\varphi(T_\varepsilon)$ (see Remark 1.4.11). Moreover, Ψ is given by

$$\Psi_i(t_i, \varepsilon) = (x_i + \varepsilon a_i, y_i + \varepsilon b_i, p_i + \varepsilon c_i),$$

where $a_i, b_i, c_i \in \mathbb{C}\{t_i\}$, $ord\, a_i \geq m_i$, $ord\, b_i \geq 2m_i$, $i = 1, \ldots, r$. The deformation Ψ is trivial if and only if Φ is trivial for the action of the relative contact transformations. Φ is trivial if and only if there are

$$\xi_i(t_i) = \widetilde{t_i} = t_i + \varepsilon h_i,$$
$$\chi(x, y, p, \varepsilon) = (x + \varepsilon\alpha, y + \varepsilon\beta, p + \varepsilon\gamma, \varepsilon),$$

such that χ is a relative contact transformation, ξ_i is an isomorphism,

$$\alpha, \beta, \gamma \in (x, y, p)\mathbb{C}\{x, y, p\},\ h_i \in t_i\mathbb{C}\{t_i\},\ 1 \le i \le r$$

and

$$\begin{aligned} x_i(t_i) + \varepsilon a_i(t_i) &= x_i(\widetilde{t_i}) + \varepsilon\alpha(x_i(\widetilde{t_i}), y_i(\widetilde{t_i}), p_i(\widetilde{t_i})), \\ y_i(t_i) + \varepsilon b_i(t_i) &= y_i(\widetilde{t_i}) + \varepsilon\beta(x_i(\widetilde{t_i}), y_i(\widetilde{t_i}), p_i(\widetilde{t_i})), \end{aligned}$$

for $i = 1, \ldots, r$. By Taylor's formula $x_i(\widetilde{t_i}) = x_i(t_i) + \varepsilon\dot{x}_i(t_i)h_i(t_i)$, $y_i(\widetilde{t_i}) = y_i(t_i) + \varepsilon\dot{y}_i(t_i)h_i(t_i)$ and

$$\begin{aligned} \varepsilon\alpha(x_i(\widetilde{t_i}), y_i(\widetilde{t_i}), p_i(\widetilde{t_i})) &= \varepsilon\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \\ \varepsilon\beta(x_i(\widetilde{t_i}), y_i(\widetilde{t_i}), p_i(\widetilde{t_i})) &= \varepsilon\beta(x_i(t_i), y_i(t_i), p_i(t_i)), \end{aligned}$$

for $i = 1, \ldots, r$. Hence Φ is trivialized by χ if and only if

$$a_i(t_i) = \dot{x}_i(t_i)h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \tag{1.5.3}$$
$$b_i(t_i) = \dot{y}_i(t_i)h_i(t_i) + \beta(x_i(t_i), y_i(t_i), p_i(t_i)), \tag{1.5.4}$$

for $i = 1, \ldots, r$. By Remark 1.3.7 (i), (1.5.3) and (1.5.4) are equivalent to the condition

$$\mathbf{a}\frac{\partial}{\partial x} + \mathbf{b}\frac{\partial}{\partial y} \in \mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x, y)\frac{\partial}{\partial x} \oplus (x^2, y)\frac{\partial}{\partial y} + \widehat{I}.$$

□

Set

$$\begin{aligned} M_\varphi &= \frac{\left(\bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right)}{\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x, y)\frac{\partial}{\partial x} \oplus (x, y)\frac{\partial}{\partial y}}, \\ \overset{\twoheadrightarrow}{M}_\varphi &= \frac{\left(\bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^r t_i^{2m_i}\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\right)}{\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x, y)\frac{\partial}{\partial x} \oplus (x^2, y)\frac{\partial}{\partial y}}. \end{aligned}$$

By Proposition $II.2.27$ of [9],

$$\underline{\mathcal{D}ef}_\varphi^{\,em}(T_\varepsilon) \cong M_\varphi.$$

A similar argument shows that

$$\overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi(T_\varepsilon) \cong \overset{\twoheadrightarrow}{M}_\varphi.$$

We have linear maps

$$M_\varphi \overset{\imath}{\hookleftarrow} \overset{\twoheadrightarrow}{M}_\varphi \twoheadrightarrow \widehat{M}_\varphi. \tag{1.5.5}$$

Theorem 1.5.2 ([9], II Theorem 2.38 (3)). *Set $k = dim\, M_\varphi$. Let $\mathbf{a}^j, \mathbf{b}^j \in \bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}$, $1 \leq j \leq k$. If*

$$\mathbf{a}^j \frac{\partial}{\partial x} + \mathbf{b}^j \frac{\partial}{\partial y} = \begin{bmatrix} a_1^j \\ \vdots \\ a_r^j \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} b_1^j \\ \vdots \\ b_r^j \end{bmatrix} \frac{\partial}{\partial y}, \tag{1.5.6}$$

$1 \leq j \leq k$, represents a basis of M_φ, the deformation $\Phi : \bar{\mathbb{C}} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k$ given by

$$X_i(t_i, \mathbf{s}) = x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i)s_j,\ Y_i(t_i, \mathbf{s}) = y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i)s_j, \tag{1.5.7}$$

$i = 1, \dots, r$, is a semiuniversal deformation of φ in $\mathcal{D}ef_\varphi^{em}$.

Lemma 1.5.3. *Set $\overset{\twoheadrightarrow}{k} = dim\, \overset{\twoheadrightarrow}{M}_\varphi$. Let $\mathbf{a}^j \in \bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}$, $\mathbf{b}^j \in \bigoplus_{i=1}^r t_i^{2m_i}\mathbb{C}\{t_i\}$, $1 \leq j \leq \overset{\twoheadrightarrow}{k}$. If (1.5.6) represents a basis of $\overset{\twoheadrightarrow}{M}_\varphi$, the deformation $\overset{\twoheadrightarrow}{\Phi}$ given by (1.5.7), $1 \leq i \leq r$, is a semiuniversal deformation of φ in $\overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi$. Moreover, $\mathcal{C}on\, \overset{\twoheadrightarrow}{\Phi}$ is a versal deformation of ψ in $\widehat{\mathcal{D}ef}_\psi^{em}$.*

Proof. We will only show the completeness of $\overset{\twoheadrightarrow}{\Phi}$ and $\mathcal{C}on\, \overset{\twoheadrightarrow}{\Phi}$. Since the linear inclusion map $\imath$ referred in (1.5.5) is injective, the deformation $\overset{\twoheadrightarrow}{\Phi}$ is the restriction to $\overset{\twoheadrightarrow}{M}_\varphi$ of the deformation Φ introduced in Theorem 1.5.2. Let $\Phi_0 \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi(T)$. Since $\Phi_0 \in \mathcal{D}ef_\varphi^{em}(T)$, there is a morphism of analytic spaces $f : T \to M_\varphi$ such that $\Phi_0 \cong f^*\Phi$. Since $\Phi_0 \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi(T)$, $f(T) \subset \overset{\twoheadrightarrow}{M}_\varphi$. Hence $f^*\overset{\twoheadrightarrow}{\Phi} = f^*\Phi$.

If $\Psi \in \widehat{\mathcal{D}ef}_\psi^{em}(T)$, $\Psi^\pi \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_\varphi(T)$. Hence there is $f : T \to \overset{\twoheadrightarrow}{M}_\varphi$ such that $\Psi^\pi \cong f^*\overset{\twoheadrightarrow}{\Phi}$. Therefore $\Psi = \mathcal{C}on\, \Psi^\pi \cong \mathcal{C}on\, f^*\overset{\twoheadrightarrow}{\Phi} = f^*\mathcal{C}on\, \overset{\twoheadrightarrow}{\Phi}$. □

Theorem 1.5.4. *Let $\mathbf{a}^j \in \bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}$, $\mathbf{b}^j \in \bigoplus_{i=1}^r t_i^{2m_i}\mathbb{C}\{t_i\}$, $1 \leq j \leq \ell$. Assume that (1.5.6) represents a basis [a system of generators]of $\widehat{M}_\varphi$. Let Φ be the deformation given by (1.5.7), $1 \leq i \leq r$. Then $\mathcal{C}on\, \Phi$ is a semiuniversal [versal]deformation of ψ in $\widehat{\mathcal{D}ef}_\psi^{em}$.*

Proof. By Theorem 1.4.8 and Lemma 1.5.3 it is enough to show that $\mathcal{C}on\, \Phi$ is formally semiuniversal [versal].

Let $\imath : T' \hookrightarrow T$ be a small extension. Let $\Psi \in \widehat{\mathcal{D}ef}_\psi^{em}(T)$. Set $\Psi' = \imath^*\Psi$. Let $\eta' : T' \to \mathbb{C}^\ell$ be a morphism of complex analytic spaces. Assume that (χ', ξ') define an isomorphism

$$\eta'^*\mathcal{C}on\, \Phi \cong \Psi'.$$

We need to find $\eta : T \to \mathbb{C}^\ell$ and χ, ξ such that $\eta' = \eta \circ \imath$ and χ, ξ define an isomorphism

$$\eta^*\mathcal{C}on\, \Phi \cong \Psi$$

that extends (χ', ξ'). Let $A\,[A']$ be the local ring of $T\,[T']$. Let δ be the generator of $Ker(A \twoheadrightarrow A')$. We can assume $A' \cong \mathbb{C}\{\mathbf{z}\}/I$, where $\mathbf{z} = (z_1, \ldots, z_m)$. Set

$$\widetilde{A}' = \mathbb{C}\{\mathbf{z}\} \quad \text{and} \quad \widetilde{A} = \mathbb{C}\{\mathbf{z}, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \ldots, \varepsilon z_m).$$

Let $\mathfrak{m}_A$ be the maximal ideal of A. Since $\mathfrak{m}_A \delta = 0$ and $\delta \in \mathfrak{m}_A$, there is a morphism of local analytic algebras from $\widetilde{A}$ onto A that takes ε into δ such that the diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & \widetilde{A}' \arrow[d] \\ A \arrow[r] & A' \end{tikzcd} \tag{1.5.8}$$

commutes. Assume $\widetilde{T}\,[\widetilde{T}']$ has local ring $\widetilde{A}\,[\widetilde{A}']$. We also denote by $\imath$ the morphism $\widetilde{T}' \hookrightarrow \widetilde{T}$. We denote by κ the morphisms $T \hookrightarrow \widetilde{T}$ and $T' \hookrightarrow \widetilde{T}'$. Let $\widetilde{\Psi} \in \widehat{\mathcal{D}ef}_{\psi}^{em}(\widetilde{T})$ be a lifting of Ψ.

We fix a linear map $\sigma : A' \hookrightarrow \widetilde{A}'$ such that $\kappa^* \sigma = id_{A'}$. Set $\widetilde{\chi}' = \chi_{\sigma(\alpha), \sigma(\beta_0)}$, where $\chi' = \chi_{\alpha, \beta_0}$. Define $\widetilde{\eta}'$ by $\widetilde{\eta}'^* s_i = \sigma(\eta'^* s_i)$, $i = 1, \ldots, \ell$. Let $\widetilde{\xi}'$ be the lifting of ξ' determined by σ. Then

$$\widetilde{\Psi}' := \widetilde{\chi}'^{-1} \circ \widetilde{\eta}'^* \mathcal{C}on\, \Phi \circ \widetilde{\xi}'^{-1}$$

is a lifting of Ψ' and

$$\widetilde{\chi}' \circ \widetilde{\Psi}' \circ \widetilde{\xi}' = \widetilde{\eta}'^* \mathcal{C}on\, \Phi. \tag{1.5.9}$$

By Theorem 1.3.4 it is enough to find liftings $\widetilde{\chi}, \widetilde{\xi}, \widetilde{\eta}$ of $\widetilde{\chi}', \widetilde{\xi}', \widetilde{\eta}'$ such that

$$\widetilde{\chi} \cdot \widetilde{\Psi}^{\pi} \circ \widetilde{\xi} = \widetilde{\eta}^* \Phi$$

in order to prove the theorem.

Consider the following commutative diagram

$$\begin{tikzcd} \bar{\mathbb{C}} \times \widetilde{T}' \arrow[r, hook] \arrow[d, "\widetilde{\Psi}'"] & \bar{\mathbb{C}} \times \widetilde{T} \arrow[r, dotted] \arrow[d, "\widetilde{\Psi}"] & \bar{\mathbb{C}} \times \mathbb{C}^\ell \arrow[d, "\mathcal{C}on\, \Phi"] \\ \mathbb{C}^3 \times \widetilde{T}' \arrow[r, hook] \arrow[d, "pr"] & \mathbb{C}^3 \times \widetilde{T} \arrow[r, dotted] \arrow[d, "pr"] & \mathbb{C}^3 \times \mathbb{C}^\ell \arrow[d] \\ \widetilde{T}' \arrow[r, hook] \arrow[rr, bend right, "\widetilde{\eta}'"'] & \widetilde{T} \arrow[r, dotted, "\widetilde{\eta}"] & \mathbb{C}^\ell. \end{tikzcd}$$

If $\mathcal{C}on\, \Phi$ is given by

$$X_i(t_i, \mathbf{s}),\ Y_i(t_i, \mathbf{s}),\ P_i(t_i, \mathbf{s})\ \in \mathbb{C}\{\mathbf{s}, t_i\},$$

then $\widetilde{\eta}'^*\, \mathcal{C}on\, \Phi$ is given by

$$X_i(t_i, \widetilde{\eta}'(\mathbf{z})),\ Y_i(t_i, \widetilde{\eta}'(\mathbf{z})),\ P_i(t_i, \widetilde{\eta}'(\mathbf{z}))\ \in \widetilde{A}'\{t_i\} = \mathbb{C}\{\mathbf{z}, t_i\}$$

for $i = 1, \ldots, r$. Suppose that $\widetilde{\Psi}'$ is given by

$$U_i'(t_i, \mathbf{z}),\ V_i'(t_i, \mathbf{z}),\ W_i'(t_i, \mathbf{z})\ \in \mathbb{C}\{\mathbf{z}, t_i\}.$$

Then, $\widetilde{\Psi}$ must be given by

$$U_i = U_i' + \varepsilon u_i,\ V_i = V_i' + \varepsilon v_i,\ W_i = W_i' + \varepsilon w_i\ \in \widetilde{A}\{t_i\} = \mathbb{C}\{\mathbf{z}, t_i\} \oplus \varepsilon\mathbb{C}\{t_i\}$$

with $u_i, v_i, w_i \in \mathbb{C}\{t_i\}$ and $i = 1, \ldots, r$. By definition of deformation we have that, for each i,

$$(U_i, V_i, W_i) = (x_i(t_i), y_i(t_i), p_i(t_i))\ mod\ \mathfrak{m}_{\widetilde{A}}.$$

Suppose $\widetilde{\eta}' : \widetilde{T}' \to \mathbb{C}^\ell$ is given by $(\widetilde{\eta}_1', \ldots, \widetilde{\eta}_\ell')$, with $\widetilde{\eta}_i' \in \mathbb{C}\{\mathbf{z}\}$. Then $\widetilde{\eta}$ must be given by $\widetilde{\eta} = \widetilde{\eta}' + \varepsilon\widetilde{\eta}^0$ for some $\widetilde{\eta}^0 = (\widetilde{\eta}_1^0, \ldots, \widetilde{\eta}_\ell^0) \in \mathbb{C}^\ell$. Suppose that $\tilde{\chi}' : \mathbb{C}^3 \times \widetilde{T}' \to \mathbb{C}^3 \times \widetilde{T}'$ is given at the ring level by

$$(x, y, p) \mapsto (H_1', H_2', H_3'),$$

such that $H' = id\ mod\ \mathfrak{m}_{\widetilde{A}'}$ with $H_i' \in (x, y, p)A'\{x, y, p\}$. Let the automorphism $\widetilde{\xi}' : \bar{\mathbb{C}} \times \widetilde{T}' \to \bar{\mathbb{C}} \times \widetilde{T}'$ be given at the ring level by

$$t_i \mapsto h_i'$$

such that $h' = id\ mod\ \mathfrak{m}_{\widetilde{A}'}$ with $h_i' \in (t_i)\mathbb{C}\{\mathbf{z}, t_i\}$.

Then, from 1.5.9 follows that

$$\begin{aligned} X_i(t_i, \widetilde{\eta}') &= H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')), \\ Y_i(t_i, \widetilde{\eta}') &= H_2'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')), \\ P_i(t_i, \widetilde{\eta}') &= H_3'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')). \end{aligned} \tag{1.5.10}$$

Now, $\widetilde{\eta}'$ must be extended to $\widetilde{\eta}$ such that the first two previous equations extend as well. That is, we must have

$$\begin{aligned} X_i(t_i, \widetilde{\eta}) &= (H_1' + \varepsilon\alpha)(U_i(h_i' + \varepsilon h_i^0), V_i(h_i' + \varepsilon h_i^0), W_i(h_i' + \varepsilon h_i^0), \\ Y_i(t_i, \widetilde{\eta}) &= (H_2' + \varepsilon\beta)(U_i(h_i' + \varepsilon h_i^0), V_i(h_i' + \varepsilon h_i^0), W_i(h_i' + \varepsilon h_i^0). \end{aligned} \tag{1.5.11}$$

with $\alpha, \beta \in (x, y, p)\mathbb{C}\{x, y, p\}$, $h_i^0 \in (t_i)\mathbb{C}\{t_i\}$ such that

$$(x, y, p) \mapsto (H_1' + \varepsilon\alpha, H_2' + \varepsilon\beta, H_3' + \varepsilon\gamma)$$

gives a relative contact transformation over $\widetilde{T}$ for some $\gamma \in (x, y, p)\mathbb{C}\{x, y, p\}$. The existence of this extended relative contact transformation is guaranteed by Theorem 1.3.6. Moreover, again by Theorem 1.3.6 this extension depends only on the choices of α and β_0. So, we need only to find α, β_0, $\widetilde{\eta}^0$ and h_i^0 such that (1.5.11) holds. Using

Taylor's formula and $\varepsilon^2 = 0$ we see that

$$\begin{aligned} X_i(t_i, \widetilde{\eta}' + \varepsilon \widetilde{\eta}^0) &= X_i(t_i, \widetilde{\eta}') + \varepsilon \sum_{j=1}^{\ell} \frac{\partial X_i}{\partial s_j}(t_i, \widetilde{\eta}') \widetilde{\eta}_j^0 \\ (\varepsilon \mathfrak{m}_{\widetilde{A}} = 0) \qquad &= X_i(t_i, \widetilde{\eta}') + \varepsilon \sum_{j=1}^{\ell} \frac{\partial X_i}{\partial s_j}(t_i, 0) \widetilde{\eta}_j^0, \\ Y_i(t_i, \widetilde{\eta}' + \varepsilon \widetilde{\eta}^0) &= Y_i(t_i, \widetilde{\eta}') + \varepsilon \sum_{j=1}^{\ell} \frac{\partial Y_i}{\partial s_j}(t_i, 0) \widetilde{\eta}_j^0. \end{aligned} \tag{1.5.12}$$

Again by Taylor's formula and noticing that $\varepsilon \mathfrak{m}_{\widetilde{A}} = 0$, $\varepsilon \mathfrak{m}_{\widetilde{A}'} = 0$ in $\widetilde{A}$, $h' = id \ mod \ \mathfrak{m}_{\widetilde{A}'}$ and $(U_i, V_i) = (x_i(t_i), y_i(t_i)) \ mod \ \mathfrak{m}_{\widetilde{A}}$ we see that

$$\begin{aligned} U_i(h_i' + \varepsilon h_i^0) &= U_i(h_i') + \varepsilon \dot{U}_i(h_i') h_i^0 \\ &= U_i'(h_i') + \varepsilon(\dot{x}_i h_i^0 + u_i), \\ V_i(h_i' + \varepsilon h_i^0) &= V_i'(h_i') + \varepsilon(\dot{y}_i h_i^0 + v_i). \end{aligned} \tag{1.5.13}$$

Now, $H' = id \ mod \ \mathfrak{m}_{\widetilde{A}'}$, so

$$\frac{\partial H_1'}{\partial x} = 1 \ mod \ \mathfrak{m}_{\widetilde{A}'}, \qquad \frac{\partial H_1'}{\partial y}, \frac{\partial H_1'}{\partial p} \in \mathfrak{m}_{\widetilde{A}'} \widetilde{A}'\{x, y, p\}.$$

In particular,

$$\varepsilon \frac{\partial H_1'}{\partial y} = \varepsilon \frac{\partial H_1'}{\partial p} = 0.$$

By this and arguing as in (1.5.12) and (1.5.13) we see that

$$\begin{aligned} &(H_1' + \varepsilon\alpha)(U_i'(h_i') + \varepsilon(\dot{x}_i h_i^0 + u_i), V_i'(h_i') + \varepsilon(\dot{y}_i h_i^0 + v_i), W_i'(h_i') + \varepsilon(\dot{p}_i h_i^0 + w_i)) \\ &= H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + \varepsilon(\alpha(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + 1(\dot{x}_i h_i^0 + u_i)) \\ &= H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + \varepsilon(\alpha(x_i, y_i, p_i) + \dot{x}_i h_i^0 + u_i), \\ &(H_2' + \varepsilon\beta)(U_i'(h_i') + \varepsilon(\dot{x}_i h_i^0 + u_i), V_i'(h_i') + \varepsilon(\dot{y}_i h_i^0 + v_i), W_i'(h_i') + \varepsilon(\dot{p}_i h_i^0 + w_i)) \\ &= H_2'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + \varepsilon(\beta(x_i, y_i, p_i) + \dot{y}_i h_i^0 + v_i). \end{aligned}$$

Substituting this in (1.5.11) and using (1.5.10) and (1.5.12) we see that we have to find $\widetilde{\eta}^0 = (\widetilde{\eta}_1^0, \ldots, \widetilde{\eta}_\ell^0) \in \mathbb{C}^\ell$, h_i^0 such that

$$\begin{aligned} (u_i(t_i), v_i(t_i)) = \sum_{j=1}^{\ell} \widetilde{\eta}_j^0 \left(\frac{\partial X_i}{\partial s_j}(t_i, 0), \frac{\partial Y_i}{\partial s_j}(t_i, 0) \right) - \\ -h_i^0(t_i)((\dot{x}_i(t_i), \dot{y}_i(t_i)) - (\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))). \end{aligned} \tag{1.5.14}$$

Note that, because of Remark 1.3.7 (i), $(\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))) \in \widehat{I}$ for each i. Also note that $\widetilde{\Psi} \in \widehat{\mathcal{D}ef}_{\psi}^{em}(\widetilde{T})$ means that $u_i \in t_i^{m_i}\mathbb{C}\{t_i\}, v_i \in t_i^{2m_i}\mathbb{C}\{t_i\}$. Then, if the vectors

$$\begin{aligned}&\left(\frac{\partial X_1}{\partial s_j}(t_1, 0), \ldots, \frac{\partial X_r}{\partial s_j}(t_r, 0)\right)\frac{\partial}{\partial x} + \left(\frac{\partial Y_1}{\partial s_j}(t_1, 0), \ldots, \frac{\partial Y_r}{\partial s_j}(t_r, 0)\right)\frac{\partial}{\partial y}\\ &= (a_1^j(t_1), \ldots, a_r^j(t_r))\frac{\partial}{\partial x} + (b_1^j(t_1), \ldots, b_r^j(t_r))\frac{\partial}{\partial y}, \qquad j = 1, \ldots, \ell\end{aligned}$$

form a basis of [generate] $\widehat{M_\varphi}$, we can solve (1.5.14) with unique $\widetilde{\eta}_1^0, \ldots, \widetilde{\eta}_\ell^0$ [respectively, solve] for all $i = 1, \ldots, r$. This implies that the conormal of Φ is a formally semiuniversal [respectively, versal] equimultiple deformation of ψ over $\mathbb{C}^\ell$. $\square$

1.6 Versal Deformations

Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$. According to the notation introduced in section 1.3, we will denote by $\int f dx_i$ the solution of the Cauchy problem

$$\frac{\partial g}{\partial x_i} = f, \qquad g \in (x_i).$$

Let ψ be a Legendrian curve with parametrization given by

$$t_i \mapsto (x_i(t_i), y_i(t_i), p_i(t_i)) \qquad i = 1, \ldots, r. \tag{1.6.1}$$

We will say that the *fake plane projection* of (1.6.1) is the plane curve σ with parametrization given by

$$t_i \mapsto (x_i(t_i), p_i(t_i)) \qquad i = 1, \ldots, r. \tag{1.6.2}$$

We will denote σ by ψ^{π_f}.

Given a plane curve σ with parametrization (1.6.2), we will say that the *fake conormal* of σ is the Legendrian curve ψ with parametrization (1.6.1), where

$$y_i(t_i) = \int p_i(t_i)\dot{x}_i(t_i)dt_i.$$

We will denote ψ by $\mathcal{C}on_f\,\sigma$. Applying the construction above to each fibre of a deformation we obtain functors

$$\pi_f : \widehat{\mathcal{D}ef}_\psi \to \mathcal{D}ef_\sigma, \qquad \mathcal{C}on_f : \mathcal{D}ef_\sigma \to \widehat{\mathcal{D}ef}_\psi.$$

Notice that

$$\mathcal{C}on_f\,(\Psi^{\pi_f}) = \Psi, \qquad (\mathcal{C}on_f\,(\Sigma))^{\pi_f} = \Sigma \tag{1.6.3}$$

for each $\Psi \in \widehat{\mathcal{D}ef}_\psi$ and each $\Sigma \in \mathcal{D}ef_\sigma$.

Let ψ be the parametrization of a Legendrian curve given by (1.6.1). Let σ be the fake plane projection of ψ. Set $\dot{\sigma} := \dot{\mathbf{x}}\frac{\partial}{\partial x} + \dot{\mathbf{p}}\frac{\partial}{\partial p}$. Let I^f be the linear subspace of

$$\mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial p} = \left(\bigoplus_{i=1}^{r} t_i\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\right) \oplus \left(\bigoplus_{i=1}^{r} t_i\mathbb{C}\{t_i\}\frac{\partial}{\partial p}\right)$$

generated by

$$\alpha_0\frac{\partial}{\partial x} - \left(\frac{\partial\alpha_0}{\partial x} + \frac{\partial\alpha_0}{\partial y}\mathbf{p}\right)\mathbf{p}\frac{\partial}{\partial p}, \qquad \left(\frac{\partial\beta_0}{\partial x} + \frac{\partial\beta_0}{\partial y}\mathbf{p}\right)\frac{\partial}{\partial p},$$

and

$$\alpha_k\mathbf{p}^k\frac{\partial}{\partial x} - \frac{1}{k+1}\left(\frac{\partial\alpha_k}{\partial x}\mathbf{p}^{k+1} + \frac{\partial\alpha_k}{\partial y}\mathbf{p}^{k+2}\right)\frac{\partial}{\partial p}, \quad k \geq 1,$$

where $\alpha_k \in (x, y), \beta_0 \in (x^2, y)$ for each $k \geq 0$. Set

$$M^f_\sigma = \frac{\mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{\bar{\mathbb{C}}}\frac{\partial}{\partial p}}{\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\sigma} + I^f}.$$

Theorem 1.6.1. *Assuming the notations above,* $\underline{\widehat{\mathcal{D}ef}}_\psi(T_\varepsilon) \cong M^f_\sigma$.

Proof. Let $\Psi \in \widehat{\mathcal{D}ef}_\psi(T_\varepsilon)$ be given by

$$\Psi_i(t_i, \varepsilon) = (X_i, Y_i, P_i) = (x_i + \varepsilon a_i, y_i + \varepsilon b_i, p_i + \varepsilon c_i),$$

where $a_i, b_i, c_i \in \mathbb{C}\{t_i\}t_i$ and $Y_i = \int P_i\partial_{t_i}X_i dt_i$, $i = 1, \ldots, r$. Hence

$$b_i = \int(\dot{x}_i c_i + \dot{a}_i p_i)dt_i, \qquad i = 1, \ldots, r.$$

By (1.6.3) Ψ is trivial if and only if there an isomorphism $\xi : \bar{\mathbb{C}} \times T_\varepsilon \to \bar{\mathbb{C}} \times T_\varepsilon$ given by

$$t_i \to \widetilde{t_i} = t_i + \varepsilon h_i, \qquad h_i \in \mathbb{C}\{t_i\}t_i,\ i = 1, \ldots, r,$$

and a relative contact transformation $\chi : \mathbb{C}^3 \times T_\varepsilon \to \mathbb{C}^3 \times T_\varepsilon$ given by

$$(x, y, p, \varepsilon) \mapsto (x + \varepsilon\alpha, y + \varepsilon\beta, p + \varepsilon\gamma, \varepsilon)$$

such that

$$X_i = x_i(\widetilde{t_i}) + \varepsilon\alpha(x_i(\widetilde{t_i}), y_i(\widetilde{t_i}), p_i(\widetilde{t_i})),$$
$$P_i = p_i(\widetilde{t_i}) + \varepsilon\gamma(x_i(\widetilde{t_i}), y_i(\widetilde{t_i}), p_i(\widetilde{t_i})),$$

$i = 1, \ldots, r$. Following the argument of the proof of Theorem 1.5.1, Ψ^{π_f} is trivial if and only if

$$a_i(t_i) = \dot{x}_i(t_i)h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)),$$
$$c_i(t_i) = \dot{p}_i(t_i)h_i(t_i) + \gamma(x_i(t_i), y_i(t_i), p_i(t_i)),$$

$i = 1, \ldots, r$. The result follows from Remark 1.3.7 (ii). $\square$

Lemma 1.6.2. *Let ψ be the parametrization of a Legendrian curve. Let Φ be the semiuniversal deformation in $\mathcal{D}ef_\sigma$ of the fake plane projection σ of ψ. Then $\mathcal{C}on_f\,\Phi$ is a versal deformation of ψ in $\widehat{\mathcal{D}ef}_\psi$.*

Proof. It follows the argument of Lemma 1.5.3. □

Theorem 1.6.3. *Let $\mathbf{a}^j, \mathbf{c}^j \in \mathfrak{m}_{\bar{\mathbb{C}}}$ such that*

$$\mathbf{a}^j \frac{\partial}{\partial x} + \mathbf{c}^j \frac{\partial}{\partial p} = \begin{bmatrix} a_1^j \\ \vdots \\ a_r^j \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} c_1^j \\ \vdots \\ c_r^j \end{bmatrix} \frac{\partial}{\partial p}, \tag{1.6.4}$$

$1 \leq j \leq \ell$, represents a basis [a system of generators]of M_σ^f. Let $\Phi \in \mathcal{D}ef_\sigma$ be given by

$$X_i(t_i, \mathbf{s}) = x_i(t_i) + \sum_{j=1}^{\ell} a_i^j(t_i) s_j, \; P_i(t_i, \mathbf{s}) = p_i(t_i) + \sum_{j=1}^{\ell} c_i^j(t_i) s_j, \tag{1.6.5}$$

$i = 1, \dots, r$. Then $\mathcal{C}on_f\,\Phi$ is a semiuniversal [versal]deformation of ψ in $\widehat{\mathcal{D}ef}_\psi$.

Proof. It follows the argument of Theorem 1.5.4, using Remark 1.3.7 (ii). □

1.7 Examples

Example 1.7.1. Let $\varphi(t) = (t^3, t^{10})$, $\psi(t) = (t^3, t^{10}, \frac{10}{3}t^7)$, $\sigma(t) = (t^3, \frac{10}{3}t^7)$. The deformations given by

- $X(t, \mathbf{s}) = t^3$, $\quad Y(t, \mathbf{s}) = s_1 t^4 + s_2 t^5 + s_3 t^7 + s_4 t^8 + t^{10} + s_5 t^{11} + s_6 t^{14}$;
- $X(t, \mathbf{s}) = s_1 t + s_2 t^2 + t^3$, $\quad Y(t, \mathbf{s}) = s_3 t + s_4 t^2 + s_5 t^4 + s_6 t^5 + s_7 t^7 + s_8 t^8 + t^{10} + s_9 t^{11} + s_{10} t^{14}$;

are respectively

- an equimultiple semiuniversal deformation;
- a semiuniversal deformation

of φ. The conormal of the deformation given by

$$X(t, \mathbf{s}) = t^3, \qquad Y(t, \mathbf{s}) = s_1 t^7 + s_2 t^8 + t^{10} + s_3 t^{11};$$

is an equimultiple semiuniversal deformation of ψ. The fake conormal of the deformation given by

$$X(t, \mathbf{s}) = s_1 t + s_2 t^2 + t^3, \qquad P(t, \mathbf{s}) = s_3 t + s_4 t^2 + s_5 t^4 + s_6 t^5 + \frac{10}{3} t^7 + s_7 t^8;$$

is a semiuniversal deformation of the fake conormal of σ. The conormal of the deformation given by

$$X(t,\mathbf{s}) = s_1t + s_2t^2 + t^3, \qquad Y(t,\mathbf{s}) = \alpha_2t^2 + \alpha_3t^3 + \alpha_4t^4 + \alpha_5t^5 + \alpha_6t^6 + \\ + \alpha_7t^7 + \alpha_8t^8 + \alpha_9t^9 + \alpha_{10}t^{10} + \alpha_{11}t^{11};$$

with

$$\begin{array}{lll} \alpha_2 = \dfrac{s_1s_3}{2}, & \alpha_3 = \dfrac{s_1s_4 + 2s_2s_3}{3}, & \alpha_4 = \dfrac{3s_3 + 2s_2s_4}{4}, \\ \alpha_5 = \dfrac{3s_4 + s_1s_5}{5}, & \alpha_6 = \dfrac{2s_2s_5 + s_1s_6}{6}, & \alpha_7 = \dfrac{3s_5 + 2s_2s_6}{7}, \\ \alpha_8 = \dfrac{10s_1 + 9s_6}{24}, & \alpha_9 = \dfrac{3s_1s_7 + 20s_2}{27}, & \alpha_{10} = 1 + \dfrac{s_2s_7}{5}, \\ \alpha_{11} = \dfrac{3s_7}{11}, & & \end{array}$$

is a semiuniversal deformation of ψ.

Example 1.7.2. Let $Z = \{(x,y) \in \mathbb{C}^2 : (y^2 - x^5)(y^2 - x^7) = 0\}$. Consider the parametrization φ of Z given by

$$x_1(t_1) = t_1^2,\ y_1(t_1) = t_1^5 \qquad x_2(t_2) = t_2^2,\ y_2(t_2) = t_2^7.$$

Let σ be the fake projection of the conormal of φ given by

$$x_1(t_1) = t_1^2,\ p_1(t_1) = \frac{5}{2}t_1^3 \qquad x_2(t_2) = t_2^2,\ p_2(t_2) = \frac{7}{2}t_2^5.$$

The deformations given by

- $X_1(t_1,\mathbf{s}) = t_1^2, \qquad Y_1(t_1,\mathbf{s}) = s_1t_1^3 + t_1^5,$
  $X_2(t_2,\mathbf{s}) = t_2^2, \qquad Y_2(t_2,\mathbf{s}) = s_2t_2^2 + s_3t_2^3 + s_4t_2^4 + s_5t_2^5 + s_6t_2^6 + t_2^7 + s_7t_2^8;$
- $X_1(t_1,\mathbf{s}) = s_1t_1 + t_1^2, \qquad Y_1(t_1,\mathbf{s}) = s_3t_1 + s_4t_1^3 + t_1^5,$
  $X_2(t_2,\mathbf{s}) = s_2t_2 + t_2^2, \qquad Y_2(t_2,\mathbf{s}) = s_5t_2 + s_6t_2^2 + s_7t_2^3 + s_8t_2^4 + s_9t_2^5 + s_{10}t_2^6 + t_2^7 + s_{11}t_2^8;$

are respectively

- an equimultiple semiuniversal deformation;
- a semiuniversal deformation

of φ. The conormal of the deformation given by

$$X_1(t_1,\mathbf{s}) = t_1^2, \qquad Y_1(t_1,\mathbf{s}) = t_1^5,$$
$$X_2(t_2,\mathbf{s}) = t_2^2, \qquad Y_2(t_2,\mathbf{s}) = s_1t_2^4 + s_2t_2^5 + s_3t_2^6 + t_2^7 + s_4t_2^8;$$

is an equimultiple semiuniversal deformation of the conormal of φ.

The fake conormal of the deformation given by

$$X_1(t_1,\mathbf{s}) = s_1t_1 + t_1^2, \qquad P_1(t_1,\mathbf{s}) = s_3t_1 + \frac{5}{2}t_1^3,$$
$$X_2(t_2,\mathbf{s}) = s_2t_2 + t_2^2, \qquad P_2(t_2,\mathbf{s}) = s_4t_2 + s_5t_2^2 + s_6t_2^3 + s_7t_2^4 + \frac{7}{2}t_2^5 + s_8t_2^6;$$

is a semiuniversal deformation of the fake conormal of σ. The conormal of the deformation given by

$$\begin{aligned} X_1(t_1,\mathbf{s}) &= s_1t_1 + t_1^2, & Y_1(t_1,\mathbf{s}) &= \alpha_2t_1^2 + \alpha_3t_1^3 + \alpha_4t_1^4 + t_1^5,\\ X_2(t_2,\mathbf{s}) &= s_2t_2 + t_2^2, & Y_2(t_2,\mathbf{s}) &= \beta_2t_2^2 + \beta_3t_2^3 + \beta_4t_2^4 + \beta_5t_2^5 + \beta_6t_2^6 +\\ & & &+ \beta_7t_2^7 + \beta_8t_2^8; \end{aligned}$$

with

$$\begin{aligned} \alpha_2 &= \frac{s_1s_3}{2}, & \alpha_3 &= \frac{2s_3}{3}, & \alpha_4 &= \frac{5s_1}{8},\\ \beta_2 &= \frac{s_2s_4}{2}, & \beta_3 &= \frac{2s_4 + s_2s_5}{3}, & \beta_4 &= \frac{2s_5 + s_2s_6}{4},\\ \beta_5 &= \frac{2s_6 + s_2s_7}{5}, & \beta_6 &= \frac{4s_7 + 7s_2}{12}, & \beta_7 &= 1 + \frac{s_2s_8}{7},\\ \beta_8 &= \frac{2s_8}{8}, \end{aligned}$$

is a semiuniversal deformation of the conormal of φ.

Example 1.7.3. Let

$$Z = \{(x,y) \in \mathbb{C}^2 : y(y^4 - 2x^5y^2 + x^{10} - 4x^8y - x^{11}) = 0\}.$$

Consider the parametrization φ of Z given by

$$x_1(t_1) = t_1^4,\ y_1(t_1) = t_1^{10} + t_1^{11} \qquad x_2(t_2) = t_2,\ y_2(t_2) = 0.$$

The deformation given by

$$\begin{aligned} X_1(t_1,\mathbf{s}) &= t_1^4, & Y_1(t_1,\mathbf{s}) &= t_1^{10} + t_1^{11} + s_{11}t_1^5 + s_{12}t_1^6 + s_{13}t_1^7 + s_{14}t_1^9 + s_{15}t_1^{10} + s_{16}t_1^{13}\\ & & &+ s_{17}t_1^{14} + s_{18}t_1^{17} + s_{19}t_1^{18} + s_{20}t_1^{21} + s_{21}t_1^{25} + s_{22}t_1^{29},\\ X_2(t_2,\mathbf{s}) &= t_2, & Y_2(t_2,\mathbf{s}) &= t_2 + s_1t_2 + s_2t_2^2 + s_3t_2^3 + s_4t_2^4 + s_5t_2^5 + s_6t_2^6\\ & & &+ s_7t_2^7 + s_8t_2^8 + s_9t_2^9 + s_{10}t_2^{10}; \end{aligned}$$

is an equimultiple semiuniversal deformation of φ.

The conormal of the deformation given by

$$\begin{aligned} X_1(t_1,\mathbf{s}) &= t_1^4, & Y_1(t_1,\mathbf{s}) &= t_1^{10} + t_1^{11} + s_2t_1^9 + s_3t_1^{10} + s_4t_1^{13} + s_5t_1^{14} + s_6t_1^{17} + s_7t_1^{18},\\ X_2(t_2,\mathbf{s}) &= t_2, & Y_2(t_2,\mathbf{s}) &= t_2 + s_1t_2^2. \end{aligned}$$

is an equimultiple semiuniversal deformation of the conormal of φ.

Let us exemplify how to get rid of

$$(0,0)\frac{\partial}{\partial x}+(0,t_2^4)\frac{\partial}{\partial y} \tag{1.7.1}$$

when we go from the plane to the Legendrian case. All the other parameters that don't figure in the Legendrian case but do so in the plane case can be eliminated proceeding in a similar way. Let J denote the sub $\mathcal{O}_Z$-module of (1.5.2) given by

$$\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi}+(x,y)\frac{\partial}{\partial x}\oplus(x^2,y)\frac{\partial}{\partial y}+\widehat{I}.$$

We have that

$$\mathbf{xp}\frac{\partial}{\partial x}+\frac{1}{2}\mathbf{xp}^2\frac{\partial}{\partial y}\in\widehat{I},$$

which is equal to

$$\left(\frac{10}{4}t_1^{10}+\frac{11}{4}t_1^{11},0\right)\frac{\partial}{\partial x}+\frac{1}{2}\left(\left(\frac{10}{4}\right)^2t_1^{16}+2\frac{10}{4}\frac{11}{4}t_1^{17}+\left(\frac{11}{4}\right)^2t_1^{18},0\right)\frac{\partial}{\partial y}.$$

We have that

$$\dot{\varphi}=(4t_1^3,1)\frac{\partial}{\partial x}+(10t_1^9+11t_1^{10},0)\frac{\partial}{\partial y},$$

and

$$\frac{1}{4}(t_1^7,0)\dot{\varphi}=(t_1^{10},0)\frac{\partial}{\partial x}+\left(\frac{10}{4}t_1^{16}+\frac{11}{4}t_1^{17},0\right)\frac{\partial}{\partial y},$$

which means that

$$(t_1^{10},0)\frac{\partial}{\partial x}+(0,0)\frac{\partial}{\partial y}=-(0,0)\frac{\partial}{\partial x}+\left(\frac{10}{4}t_1^{16}+\frac{11}{4}t_1^{17},0\right)\frac{\partial}{\partial y}\ mod\, J.$$

Similarly

$$(t_1^{11},0)\frac{\partial}{\partial x}+(0,0)\frac{\partial}{\partial y}=-(0,0)\frac{\partial}{\partial x}+\left(\frac{10}{4}t_1^{17}+\frac{11}{4}t_1^{18},0\right)\frac{\partial}{\partial y}\ mod\, J.$$

As $\mathbf{x}^4=(t_1^{16},t_2^4)$ we have that

$$(0,0)\frac{\partial}{\partial x}+\left(t_1^{16},0\right)\frac{\partial}{\partial y}=-(0,0)\frac{\partial}{\partial x}+\left(0,t_2^4\right)\frac{\partial}{\partial y}\ mod\, J.$$

This means that, $mod\, J$, we can write (1.7.1) as a linear combination of $(0,0)\frac{\partial}{\partial x}+\left(t_1^{17},0\right)\frac{\partial}{\partial y}$ and $(0,0)\frac{\partial}{\partial x}+\left(t_1^{18},0\right)\frac{\partial}{\partial y}$.

Chapter 2

Equisingular Deformations of Legendrian Curves

2.1 Introduction

To consider deformations of the parametrization of a Legendrian curve is a good first approach in order to understand Legendrian curves. Unfortunately, this approach cannot be generalized to higher dimensions. On the other hand the obvious definition of deformation has its own problems. First, not all deformations of a Legendrian curve are Legendrian. Second, flat deformations of the conormal of $y^k - x^n = 0$ are all rigid, as we recall in example 2.5.3, hence there would be too many rigid Legendrian curves.

We pursue here the approach initiated in [4], following Sophus Lie original approach to contact transformations: to look at [relative] contact transformations as maps that take [deformations of] plane curves into [deformations of] plane curves. We study the category of equisingular deformations of the conormal of a plane curve Y replacing it by an equivalent category $\mathcal{D}ef_Y^{es,\mu}$, a category of equisingular deformations of Y where the isomorphisms do not come only from diffeomorphisms of the plane but also from contact transformations. Here μ stands for "microlocal", which means "locally" in the cotangent bundle (cf. [15], [16]).

Example 2.4.4 presents contact transformations that transform a germ of a plane curve Y into the germ of a plane curve Y^χ such that Y and Y^χ are not topologically equivalent or are topologically equivalent but not analytically equivalent.

We call equisingular deformation of a Legendrian curve to a deformation with equisingular plane projection. The flatness of the plane projection is a constraint strong enough to avoid the problems related with the use of a naive definition of deformation and loose enough so that we have enough deformations.

In section 2.6 we use the results of section 2.5 on equisingular deformations of the parametrization of a Legendrian curve to show that there are semiuniversal equisingular deformations of a Legendrian curve. In particular, we show that the base space of the semiuniversal equisingular deformation is smooth. This argument does not produce a constructive proof of the existence of the semiuniversal deformation in its standard form.

In section 2.7 we construct a semiuniversal equisingular deformation of a Legendrian curve L when L is the conormal of a Newton non-degenerate plane curve, generalizing the results of [4]. This type of assumption was already necessary when dealing with plane curves (see [9]). This construction is used in [18] (chapter 3) to extend the results of [4] and [10], constructing moduli spaces for Legendrian curves that are the conormal of a semiquasihomogeneous plane curve with a fixed equisingularity class.

In section 2.2 we recall some basic results on deformations of curves. In sections 2.3 and 2.4 we introduce relative contact geometry (see [1], [22] and [19]).

2.2 Deformations

We will only consider germs of complex spaces, maps and ideals, although sometimes we chose convenient representatives. We will follow the definitions and notations of [9].

Let S be the germ of an complex space at a point o. Let $\mathfrak{m}_S$ be the maximal ideal of the local ring $\mathcal{O}_{S,o}$ Let T_oS be the dual of the vector space $\mathfrak{m}_S/\mathfrak{m}_S^2$. Let X be a smooth manifold and $x \in X$. We denote by $\imath$ or $\imath_S$ $[\imath_X]$ the immersions $(S,o) \hookrightarrow (T_oS,0)$ $[(X \times S,(x,o)) \hookrightarrow (X \times T_oS,(x,0))]$.

Let $\widetilde{\mathfrak{M}}$ be an $\mathcal{O}_{T_oS,0}$-module $[\widetilde{\alpha}$ be a section of $\widetilde{\mathfrak{M}}$, $\widetilde{Y}$ be an analytic set of $(T_oS,0)]$. Let $\mathfrak{M}$ be an $\mathcal{O}_{S,o}$-module $[\alpha$ be a section of $\mathfrak{M}$, Y be an analytic set of $(S,o)]$. We say that $\widetilde{\mathfrak{M}}$ $[\widetilde{\alpha}, \widetilde{Y}]$ is a lifting of $\mathfrak{M}$ $[\alpha, Y]$ if $\imath^*\widetilde{\mathfrak{M}} = \mathfrak{M}$ $[\imath^*\widetilde{\alpha} = \alpha, \imath^* I_{\widetilde{Y}} = I_Y]$.

Let Y be a reduced analytic set of $(\mathbb{C}^n,0)$. In order to define a deformation of Y over S we need to choose a section σ of the projection $q : \mathbb{C}^n \times S \to S$. We say that a section $\widetilde{\sigma} : T_oS \to \mathbb{C}^n \times T_oS$ is a lifting of σ if $\widetilde{\sigma} \circ i = i \circ \sigma$. Unless we say otherwise we assume σ to be trivial. If S is reduced, σ is trivial if and only if $\sigma(S) = \{0\} \times S$. In general, σ is trivial if and only if it admits a trivial lifting to T_oS.

Let $\mathcal{Y}$ be an analytic subset of $\mathbb{C}^n \times S$. For each $s \in S$, let $\mathcal{Y}_s$ be the fiber of

$$\mathcal{Y} \hookrightarrow \mathbb{C}^n \times S \to S. \tag{2.2.1}$$

Let $i : Y \hookrightarrow \mathcal{Y}$ be a morphism of complex spaces that defines an isomorphism of Y into $\mathcal{Y}_o$. We say that $Y \hookrightarrow \mathcal{Y}$ defines the *deformation* (2.2.1) of Y over S if (2.2.1) is flat.

Every deformation is isomorphic to a deformation with trivial section.

Assume that Y is an hypersurface of $\mathbb{C}^n$ and f is a generator of the defining ideal of Y. Let j be the immersion $\mathbb{C}^n \to \mathbb{C}^n \times T$ and let r be the projection $\mathbb{C}^n \times T \to \mathbb{C}^n$. There is a generator F of the defining ideal of $\mathcal{Y}$ such that $j^*F = f$. We say that F defines a *deformation of the equation* of Y.

Let $Y \hookrightarrow \mathcal{Y}_i \hookrightarrow \mathbb{C}^n \times T \to T$ be two deformations of a reduced analytic set Y over T. We say that an isomorphism $\chi : \mathbb{C}^n \times T \to \mathbb{C}^n \times T$ is an isomorphism of deformations if $q \circ \chi = q$, $r \circ \chi \circ j = id_{\mathbb{C}^n}$ and χ induces an isomorphism from $\mathcal{Y}_1$ onto $\mathcal{Y}_2$.

Given a morphism of complex spaces $f : S \to T$ and a deformation $\mathcal{Y}$ of Y over T, $f^*\mathcal{Y} = S \times_T \mathcal{Y}$ defines a deformation of Y over S.

We say that a deformation $\mathcal{Y}$ of Y over T is a *versal deformation* of Y if given

- a closed embedding of complex space germs $f : T'' \hookrightarrow T'$,

- a morphism $g : T'' \to T$,
- a deformation $\mathcal{Y}'$ of Y over T' such that $f^*\mathcal{Y}' \cong g^*\mathcal{Y}$,

there is a morphism of complex analytic space germs $h : T' \to T$ such that

$$h \circ f = g \qquad \text{and} \qquad h^*\mathcal{Y} \cong \mathcal{Y}'.$$

If $\mathcal{Y}$ is versal and for each $\mathcal{Y}'$ the tangent map $T(h) : T_{T'} \to T_T$ is determined by $\mathcal{Y}'$, $\mathcal{Y}$ is called a *semiuniversal deformation* of Y.

We will now introduce deformations of a parametrization.

Assume the curve Y has irreducible components $Y_1, \ldots, Y_r$. Set $\bar{\mathbb{C}} = \bigsqcup_{i=1}^r \bar{C_i}$ where each $\bar{C_i}$ is a copy of $\mathbb{C}$. Let φ_i be a parametrization of Y_i, $1 \leq i \leq r$. The map $\varphi : \bar{\mathbb{C}} \to \mathbb{C}^n$ such that $\varphi|_{\bar{C_i}} = \varphi_i$, $1 \leq i \leq r$ is called a *parametrization* of Y.

Let $\imath, \imath_n$ denote the inclusions $\bar{\mathbb{C}} \hookrightarrow \bar{\mathbb{C}} \times T$, $\mathbb{C}^n \hookrightarrow \mathbb{C}^n \times T$. Let $\bar{q}$ denote the projection $\bar{\mathbb{C}} \times T \to T$. We say that a morphism of complex spaces $\Phi : \bar{\mathbb{C}} \times T \to \mathbb{C}^n \times T$ is a *deformation of* φ *over* T if $\imath_n \circ \varphi = \Phi \circ \imath$ and $q_n \circ \Phi = \bar{q}$.

We denote by Φ_i the composition $\bar{C_i} \times T \hookrightarrow \bar{\mathbb{C}} \times T \to \mathbb{C}^n \times T \to \mathbb{C}^n$, $1 \leq i \leq r$. The maps Φ_i, $1 \leq i \leq r$, determine Φ. Let Φ be a deformation of φ over T. Let $f : S \to T$ be a morphism of complex spaces. We denote by $f^*\Phi$ the deformation of φ over S given by

$$(f^*\Phi)_i = \Phi_i \circ (id_{\bar{C_i}} \times f).$$

Let $\Phi' : \bar{\mathbb{C}} \times T \to \mathbb{C}^n \times T$ be another deformation of φ over T. A morphism from Φ' into Φ is a pair (χ, ξ) where $\chi : \mathbb{C}^n \times T \to \mathbb{C}^n \times T$ and $\xi : \bar{\mathbb{C}} \times T \to \bar{\mathbb{C}} \times T$ are isomorphisms of complex spaces such that the diagram

$$\begin{tikzcd}
T & \bar{\mathbb{C}} \times T \arrow[l] \arrow[r, "\Phi"] & \mathbb{C}^n \times T \arrow[r] & T \\
 & \bar{\mathbb{C}} \arrow[u, hook] \arrow[d, hook] \arrow[r, "\varphi"] & \mathbb{C}^n \times \{0\} \arrow[u, hook] \arrow[d, hook] & \\
T \arrow[uu, "id_T"] & \bar{\mathbb{C}} \times T \arrow[l] \arrow[r, "\Phi'"] \arrow[uu, bend left, "\xi"] & \mathbb{C}^n \times T \arrow[r] \arrow[uu, bend right, "\chi"'] & T \arrow[uu, "id_T"']
\end{tikzcd} \tag{2.2.2}$$

commutes.

Let Φ' be a deformation of φ over S and $f : S \to T$ a morphism of complex spaces. A *morphism of* Φ' *into* Φ *over* f is a morphism from Φ' into $f^*\Phi$. There is a functor p that associates T to a deformation Ψ over T and f to a morphism of deformations over f.

Given a parametrization φ of a plane curve Y and a deformation Φ of φ, Φ is the parametrization of a hypersurface $\mathcal{Y}$ of $\mathbb{C}^2 \times T$ that defines a deformation of (the equation of) Y.

Let Y, Z be two germs of plane curves of $(\mathbf{C}^2, 0)$.

Definition 2.2.1. Two plane curves Y, Z are *equisingular* if there are neighborhoods V, W of 0 and an homeomorphism $\varphi : V \to W$ such that $\varphi(Y \cap V) = Z \cap W$.

Theorem 2.2.2. *Let $(Y_i)_{i\in I}$ $[(Z_j)_{j\in J}]$ be the set of branches Y $[Z]$. The curves Y, Z are equisingular if and only if there is a bijection $\varphi : I \to J$ such that Y_i and $Z_{\varphi(i)}$ have the same Puiseux exponents for each $i \in I$ and the contact orders $o(Y_i, Y_j)$, $o(Z_{\varphi(i)}, Z_{\varphi(j)})$ are equal, for each $i, j \in I$, $i \neq j$.*

The definition of *equisingular deformation* of the parametrization [equation] of a plane curve over an complex space is very long and technical. We will omit it. See definitions $II.2.36$ and $II.2.6$ of [9]. We will present now the main properties of equisingular deformations, which characterize them completely.

Theorem 2.2.3. (II Theorem 2.64 of [9]) *Let Y be a reduced plane curve. Let φ be a parametrization of Y. Let f be an equation of Y. Every equisingular deformation of φ induces a unique equisingular deformation of f. Every equisingular deformation of f comes from a deformation of φ.*

Theorem 2.2.4. (II Corollary 2.68 of [9]) *A deformation of the equation of a reduced plane curve Y over a reduced complex space is equisingular if and only if the topology of the fibers does not change.*

Theorem 2.2.5. *Let $S \hookrightarrow (C^k, 0)$ be an immersion of complex spaces. Let φ be a parametrization of a reduced plane curve. A deformation of φ over S is equisingular if and only it admits a lifting to an equisingular deformation of φ over $(\mathbb{C}^k, 0)$.*

Proof. It follows from Theorem $II.2.38$ of [9]. □

Proposition 2.2.6. (II Proposition 2.11 of [9]) *Assume $f_1, ..., f_\ell$ define germs of reduced irreducible curves of $(\mathbb{C}^2, 0)$ and F defines an equisingular deformation over a germ of complex space S of the curve defined by $f_1 \cdots f_\ell$. Then $F = F_1 \cdots F_\ell$, where each F_i defines an equisingular deformation of f_i over S.*

2.3 Relative contact geometry

We usually identify a subset of $\mathbb{P}^{n-1}$ with a conic subset of $\mathbb{C}^n$. Given a manifold M we will also identify a subset of the projective cotangent bundle $\mathbb{P}^*M$ with a conic subset of the cotangent bundle T^*M (for the canonical $\mathbb{C}^*$-action of T^*M).

Let $q : X \to S$ be a morphism of complex spaces. Let p_i, $i = 1, 2$ be the canonical projections from $X \times_S X$ to X. Let Δ denote the diagonal of $X \to X \times_S X$ and the diagonal immersion $X \hookrightarrow X \times_S X$. Let I_Δ be the defining ideal of the diagonal of $X \times_S X$. We say that the coherent $\mathcal{O}_X$-module $\Omega^1_{X/S} = \Delta^*(I_\Delta/I_\Delta^2)$ is the sheaf of *relative differential forms of $X \to S$* (see [11]).

Given a local section f of $\mathcal{O}_X$ set $f_i = f \circ p_i$, $i = 1, 2$. Consider the morphism $d : \mathcal{O}_X \to \Omega^1_{X/S}$ given by

$$f \mapsto f_1 - f_2 \qquad \mod I_\Delta^2.$$

Notice that, given an open set U of X and $f, g \in \mathcal{O}_X(U)$, $\varphi \in q^{-1}\mathcal{O}_S$,

$$d(fg) = f dg + g df, \qquad \text{and} \qquad d(\varphi f) = \varphi df \tag{2.3.1}$$

If $x_1, ..., x_n \in \mathcal{O}_X(U)$ are such that $\Omega^1_{X/S}|_U \xrightarrow{\sim} \oplus_{i=1}^n \mathcal{O}_U dx_i$, we say that $(x_1, ..., x_n)$ is a *partial system of local coordinates on U of $X \to S$.*

Notice that $(x_1, ..., x_n)$ is a partial system of local coordinates of $X \to S$ on U if and only if $\Omega^n_{X/S}|_U = \mathcal{O}_U dx_1 \wedge \cdots \wedge dx_n$.

If $(x^1, ..., x^n)$ is a partial system of local coordinates on U of $X \to S$, $x_1^i - x_2^i$, $i = 1, ..., n$, generate $I_\Delta|_U$. Given $f \in \mathcal{O}_X(U)$, there are $a_i \in \mathcal{O}_X(U)$ such that $df = \sum_{i=1}^n a_i dx^i$. We set

$$\frac{\partial f}{\partial x_i} = a_i, \qquad i = 1, ..., n.$$

When M, S are manifolds, $X = M \times S$ and q is the projection $M \times S \to S$ this definition of partial derivative coincides with the usual one because of (2.3.1). When S is a point, $\Omega^1_{X/S}$ equals the sheaf of differential forms Ω^1_X.

If $\Omega^1_{X/S}$ is a locally free $\mathcal{O}_X$-module, we denote by $\pi = \pi_{X/S} : T^*(X/S) \to X$ the vector bundle with sheaf of sections $\Omega^1_{X/S}$. Whenever it is reasonable we will write π instead of $\pi_{X/S}$. We denote by $\tau_{X/S} : T(X/S) \to X$ the dual vector bundle of $T^*(X/S)$. We say that $T(X/S)$ $[T^*(X/S)]$ is the *relative tangent bundle* [*cotangent bundle*] of $X \to S$.

Let $\varphi : X_1 \to X_2$, $q_i : X_i \to S$ be morphisms of complex spaces such that $q_2\varphi = q_1$. Let $\Delta_i : X_i \to X_i \times_S X_i$ be the diagonal map, $i = 1, 2$. If we denote by φ_S the canonical map from $X_1 \times_S X_1$ to $X_2 \times_S X_2$, $\varphi_S^* : I_{\Delta_2} \to I_{\Delta_1}$ induces a morphism $\varphi^* : \Omega^1_{X_2/S} \to \Omega^1_{X_1/S}$ that generalizes the pullback of differential forms. Moreover, φ^* induces a morphism of $\mathcal{O}_{X_1}$-modules

$$\widehat{\rho}_\varphi : \varphi^*\Omega^1_{X_2/S} = \mathcal{O}_{X_1} \otimes_{\varphi^{-1}\mathcal{O}_{X_2}} \varphi^{-1}\Omega^1_{X_2/S} \to \Omega^1_{X_1/S}. \tag{2.3.2}$$

If $\Omega^1_{X_i/S}$, $i = 1, 2$, and the kernel and cokernel of (2.3.2) are locally free, we have a morphism of vector bundles

$$\rho_\varphi : X_1 \times_{X_2} T^*(X_2/S) \to T^*(X_1/S). \tag{2.3.3}$$

If φ is an inclusion map, we say that the kernel of (2.3.3), and its projectivization, are the *conormal bundle of X_1 relative to S.* We will denote by $T^*_{X_1}(X_2/S)$ or $\mathbb{P}^*_{X_1}(X_2/S)$ the conormal bundle of X_1 relative to S. We denote by

$$\varpi_\varphi : T(X_1/S) \to X_1 \times_{X_2} T(X_2/S)$$

the dual morphism of ρ_φ. We say that ϖ_φ is the *relative tangent morphism of φ over S.* These are straightforward generalizations of the constructions of [15].

If $(x_1, ..., x_n)$ is a partial system of local coordinates of $X \to S$ and $(y_1, ..., y_m)$ is a system of local coordinates of a manifold Y, $(x_1, ..., x_n, y_1, ..., y_m)$ is a partial system of local coordinates of $X \times Y \to X \to S$. Hence $\Omega^1_{X/S}$ locally free implies $\Omega^1_{X \times Y/S}$ locally free. Moreover, if $\Omega^1_{X/S}$ is locally free and $E \to X$ is a vector bundle, $\Omega^1_{E/S}$ is locally free.

Let $(x_1, ..., x_n)$ be a partial system of local coordinates of $X \to S$ on an open set U of X. Set $V = \pi_{X/S}^{-1}(U)$. There are $\xi_1, ..., \xi_n \in \mathcal{O}_{T^*(X/S)}(V)$ such that, for each $\sigma \in V$,

$$\sigma = \textstyle\sum_{i=1}^n \xi_i(\sigma) dx_i.$$

Notice that $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ is a partial system of local coordinates of $T^*(X/S) \to S$. Let $o \in X$, $u \in T_\sigma T^*(X/S)$. Let

$$\varpi_\pi(\sigma) : T_\sigma(T^*(X/S)/S) \to T_o(X/S)$$

be the relative tangent morphism of π over S at σ. There is one and only one $\theta \in \Omega^1_{T^*(X/S)/S}$ such that,

$$\theta(\sigma)(u) = \sigma(\varpi_\pi(\sigma)(u)),$$

for each $o \in X$, each $\sigma \in T_o^*(X/S)$ and each $u \in T_\sigma(T^*(X/S)/S)$. Given a partial system of local coordinates $(x_1, ..., x_n)$ of $X \to S$ on an open set U,

$$\theta|_{\pi^{-1}(U)} = \textstyle\sum_{i=1}^n \xi_i dx_i.$$

We say that $\theta_{X/S} = \theta$ is the *canonical* 1-form of $T^*(X/S)$.

Notice that $(d\theta)(\sigma)$ is a symplectic form of $T_\sigma(T^*(X/S)/S)$, for each $\sigma \in T^*(X/S)$. We say that $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ is a *partial system of symplectic coordinates of* $T^*(X/S)$ (associated to $(x_1, ..., x_n)$).

Assume M is a manifold. When q is the projection $M \times S \to S$ we will replace "$M \times S/S$" by "$M|S$". Let r be the projection $M \times S \to M$. Notice that $\Omega^1_{M|S} \xrightarrow{\sim} \mathcal{O}_{M\times S} \otimes_{r^{-1}\mathcal{O}_M} r^{-1}\Omega^1_M$ is a locally free $\mathcal{O}_{M\times S}$-module. Moreover, $T^*(M|S) = T^*M \times_M (M \times S)$. If $\imath$ is the inclusion $T^*(M|S) \hookrightarrow T^*(M \times S)$, $\imath^*\theta_{M\times S} = \theta_{M|S}$. A system of local coordinates of M is a partial system of local coordinates of $M \times S \to S$.

We say that $\Omega^1_{M|S}$ is the *sheaf of relative differential forms of M over S*. We say that $T^*(M|S)$ is the *relative cotangent bundle of M over S*.

Let N be a complex manifold of dimension $2n - 1$. Let S be a complex space. We say that a section ω of $\Omega^1_{N|S}$ is a *relative contact form of N* over S if $\omega \wedge d\omega^{n-1}$ is a local generator of $\Omega^{2n-1}_{N|S}$. Let $\mathfrak{C}$ be a locally free subsheaf of $\Omega^1_{N|S}$. We say that $\mathfrak{C}$ is a *structure of relative contact manifold on N* over S if $\mathfrak{C}$ is locally generated by a relative contact form of N over S. We say that $(N \times S, \mathfrak{C})$ is a *relative contact manifold over S*. When S is a point we obtain the usual notion of contact manifold.

Let $(N_1 \times S, \mathfrak{C}_1)$, $(N_2 \times S, \mathfrak{C}_2)$ be relative contact manifolds over S. Let χ be a morphism from $N_1 \times S$ into $N_2 \times S$ such that $q_{N_2} \circ \chi = q_{N_1}$. We say that χ is a *relative contact transformation* of $(N_1 \times S, \mathfrak{C}_1)$ into $(N_2 \times S, \mathfrak{C}_2)$ if the pull-back by χ of each local generator of $\mathfrak{C}_2$ is a local generator of $\mathfrak{C}_1$.

We say that the projectivization $\pi_{X/S} : \mathbb{P}^*(X/S) \to X$ of the vector bundle $T^*(X/S)$ is the *projective cotangent bundle* of $X \to S$.

Let $(x_1, ..., x_n)$ be a partial system of local coordinates on an open set U of X. Let $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ be the associated partial system of symplectic coordinates of $T^*(X/S)$ on $V = \pi^{-1}(U)$. Set $p_{i,j} = \xi_i\xi_j^{-1}$, $i \neq j$,

$$V_i = \{(x, \xi) \in V : \xi_i \neq 0\}, \qquad \omega_i = \xi_i^{-1}\theta, \qquad i = 1, ..., n.$$

each ω_i defines a relative contact form $dx_j - \sum_{i\neq j} p_{i,j} dx_i$ on $\mathbb{P}^*(X/S)$, endowing $\mathbb{P}^*(X/S)$ with a structure of relative contact manifold over S.

Let ω be a germ at (x,o) of a relative contact form of $\mathfrak{C}$. A lifting $\widetilde{\omega}$ of ω defines a germ $\widetilde{\mathfrak{C}}$ of a relative contact structure of $N \times T_oS \to T_oS$. Moreover, $\widetilde{\mathfrak{C}}$ is a lifting of the germ at o of $\mathfrak{C}$.

Let $(N \times S, \mathfrak{C})$ be a relative contact manifold over a complex manifold S. Assume N has dimension $2n-1$ and S has dimension ℓ. Let $\mathcal{L}$ be a reduced analytic set of $N \times S$ of pure dimension $n+\ell-1$. We say that $\mathcal{L}$ is a *relative Legendrian variety* of $N \times S$ over S if for each section ω of $\mathfrak{C}$, ω vanishes on the regular part of $\mathcal{L}$. When S is a point, we say that $\mathcal{L}$ is a *Legendrian variety* of N.

Let $\mathcal{L}$ be an analytic set of $N \times S$. Let $(x,o) \in \mathcal{L}$. Assume S is an irreducible germ of a complex space at o. We say that $\mathcal{L}$ is a *relative Legendrian variety of N over S at* (x,o) if there is a relative Legendrian variety $\widetilde{\mathcal{L}}$ of (N,x) over $(T_oS, 0)$ that is a lifting of the germ of $\mathcal{L}$ at (x,o). Assume S is a germ of a complex space at o with irreducible components $S_i, i \in I$. We say that $\mathcal{L}$ is a *relative Legendrian variety of N over S at* (x,o) if $S_i \times_S \mathcal{L}$ is a relative Legendrian variety of $S_i \times_S N$ over S_i at (x,o), for each $i \in I$.

We say that $\mathcal{L}$ is a *relative Legendrian variety of* $N \times S$ if $\mathcal{L}$ is a relative Legendrian variety of $N \times S$ at (x,o) for each $(x,o) \in \mathcal{L}$.

The main problem of defining relative Legendrian variety over a complex space S comes from the fact that S does not have to be pure dimensional, hence we cannot a assign a pure dimension to the Legendrian variety.

Lemma 2.3.1. *Let χ be a relative contact transformation from $(N_1 \times S, \mathfrak{C}_1)$ into $(N_2 \times S, \mathfrak{C}_2)$. Let $\mathcal{L}_1$ be a relative Legendrian curve of $(N_1 \times S, \mathfrak{C}_1)$. If $\mathcal{L}_2$ is the analytic subset of $N_2 \times S$ defined by the pull back by χ^{-1} of the defining ideal of $\mathcal{L}_1$, $\mathcal{L}_2$ is a relative Legendrian variety of $(N_2 \times S, \mathfrak{C}_2)$.*

Proof. Let $\chi : (N_1 \times S, \mathfrak{C}_1) \to (N_2 \times S, \mathfrak{C}_2)$ be a relative contact transformation over S. Let (x_1, o) be a point of $N_1 \times S$. Set $(x_2, o) = \chi(x_1, o)$. There is a morphism of germs of complex spaces

$$\widetilde{\chi} : (N_1 \times T_oS, (x_1, o)) \to (N_2 \times T_oS, (x_2, o))$$

such that $\widetilde{\chi} \circ \imath_{N_1} = \imath_{N_2} \circ \chi$. We say that such a morphism is a *lifting* of χ. Let $\widetilde{\mathfrak{C}}_2$ be a lifting of $\mathfrak{C}_2$ at (x_2, o). Then $\widetilde{\mathfrak{C}}_1 = \widetilde{\chi}^* \widetilde{\mathfrak{C}}_2$ is a lifting of $\mathfrak{C}_1$ at (x_1, o). Moreover, $\widetilde{\chi}$ is a germ of a relative contact transformation.

Let $\mathcal{L}_1$ be a germ of a relative Legendrian variety at (x_1, o). There is a lifting $\widetilde{\mathcal{L}}_1$ of $\mathcal{L}_1$ that is a germ of relative Legendrian variety of $N_1 \times T_oS$. Hence $\widetilde{\chi}(\widetilde{\mathcal{L}}_1)$ is a germ of a relative Legendrian variety of $N_2 \times T_oS$ and $\widetilde{\chi}(\widetilde{\mathcal{L}}_1)$ is a lifting of $\mathcal{L}_2$ at (x_2, o). □

Let Y be a reduced analytic set of M. Let $\mathcal{Y}$ be a flat deformation of Y over S. Set $X = M \times S \setminus \mathcal{Y}_{\mathrm{sing}}$. We say that the Zariski closure of $\mathbb{P}^*_{\mathcal{Y}_{\mathrm{reg}}}(X/S)$ in $\mathbb{P}^*(M|S)$ is the *conormal* $\mathbb{P}^*_{\mathcal{Y}}(M|S)$ *of* $\mathcal{Y}$ *over* S.

Theorem 2.3.2. *The conormal of $\mathcal{Y}$ over S is a relative Legendrian variety of $\mathbb{P}^*(M|S)$. If $\mathcal{Y}$ has irreducible components $\mathcal{Y}_1, ..., \mathcal{Y}_r$,*

$$\mathbb{P}^*_{\mathcal{Y}}(M|S) = \cup_{i=1}^r \mathbb{P}^*_{\mathcal{Y}_i}(M|S). \tag{2.3.4}$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{Y}_{\mathrm{reg}} \arrow[r, hook, "i"] \arrow[d, "\Delta_{\mathcal{Y}_{\mathrm{reg}}}"] & X \arrow[d, "\Delta_X"] \\
\mathcal{Y}_{\mathrm{reg}} \times_S \mathcal{Y}_{\mathrm{reg}} \arrow[r, hook, "j"] & X \times_S X
\end{tikzcd}$$

Since $I_{\Delta_{\mathcal{Y}_{\mathrm{reg}}}} = j^*\left((I_{\Delta_X} + I_{\mathcal{Y}_{\mathrm{reg}} \times_S \mathcal{Y}_{\mathrm{reg}}})/I_{\mathcal{Y}_{\mathrm{reg}} \times_S \mathcal{Y}_{\mathrm{reg}}}\right)$,

$$\Delta^*_{\mathcal{Y}_{\mathrm{reg}}}(I_{\Delta_{\mathcal{Y}_{\mathrm{reg}}}}/I^2_{\Delta_{\mathcal{Y}_{\mathrm{reg}}}}) \xrightarrow{\sim} i^*\Delta^*_X((I_{\Delta_X} + I_{\mathcal{Y}_{\mathrm{reg}} \times_S \mathcal{Y}_{\mathrm{reg}}})/(I^2_{\Delta_X} + I_{\mathcal{Y}_{\mathrm{reg}} \times_S \mathcal{Y}_{\mathrm{reg}}})). \tag{2.3.5}$$

Let $(x, o) \in Y_{\mathrm{reg}}$. Let $\widetilde{\mathfrak{m}}$ be the ideal of $\mathcal{O}_{M\times S,(x,o)}$ generated by $\mathfrak{m}_o$. Let $(y_1, ..., y_n)$ be a system of local coordinates of (M, x) such that $I_{Y,x} = (y_{k+1}, ..., y_n)$. There are $F_j \in \mathcal{O}_{M\times S,(x,o)}$, $j = k + 1, ..., n$ such that $I_{\mathcal{Y},(x,o)} = (F_{k+1}, ..., F_n)$ and $F_j - y_j \in \widetilde{\mathfrak{m}}$, $j = k + 1, ..., n$. Set

$$x^i = y_i, \qquad i = 1, ..., k, \qquad x^i = F_i, \qquad i = k + 1, ..., n.$$

Notice that $(x^1, ..., x^n)$ is a partial system of local coordinates of $X \to S$. Since near (x, o)

$$I_{\Delta_X} = (x^1_1 - x^1_2, ..., x^n_1 - x^n_2) \text{ and } I_{\mathcal{Y}\times_s\mathcal{Y}} = (x^{k+1}_1, ..., x^n_1, x^{k+1}_2, ..., x^n_2),$$

it follows from (2.3.5) that $dx^1, ..., dx^k$ is a local basis of $\Omega^1_{\mathcal{Y}/S}$, $dx^1, ..., dx^n$ is a local basis of $\Omega^1_{M|S}$,

$$\widehat{\rho}_i(dx^j) = dx^j, \quad j = 1, ..., k, \qquad \text{and} \qquad \widehat{\rho}_i(dx^j) = 0, \quad j = k + 1, ..., n.$$

Hence the kernel of $\widehat{\rho}_i$ at (x, o) equals $\oplus_{j=k+1}^n \mathbb{C}\{x^1, ..., x^k\}dx^j$. Given the partial system of symplectic coordinates $(x^1, ..., x^n, \xi^1, ..., \xi^n)$, the ideal of the kernel of

$$\rho_i : \mathcal{Y}_{\mathrm{reg}} \times_X T^*(X/S) \to T^*(\mathcal{Y}_{\mathrm{reg}}/S)$$

is generated by $x^{k+1}, ..., x^n, \xi^1, ..., \xi^k$.

It is enough to prove the second statement when S is smooth. Its proof relies on the fact that each connected component of $\mathcal{Y}$ is dense in one of the irreducible components of $\mathcal{Y}$. $\square$

Let $q : X \to S$ be a morphism of complex spaces. Let $y \in Y \subset X$. We say that Y is a *submanifold of* $X \to S$ *at* y if there is a partial system of local coordinates $(x_1, ..., x_n)$ of $X \to S$ near y and $1 \le k \le n$ such that $Y = \{x_1 = \cdots = x_k = 0\}$ near y. We say that Y is a *submanifold* of $X \to S$ if Y is a submanifold of $X \to S$ at y for each $y \in Y$.

Notice that a submanifold of $X \to S$ is not necessarily a manifold, although it behaves like one in several ways.

Let $Y \subset X$. Let $\gamma : \Delta_\varepsilon = \{t \in \mathbb{C} : |t| < \varepsilon\} \to Y$ be an holomorphic curve such that $\gamma(0) = y$. We associate to γ a tangent vector u of Y at y setting $u \cdot f = (f \circ \gamma)'(0))$, for each $f \in \mathcal{O}_{X,y}$. We associate to γ an element u of $T_y(X/S)$ setting

$$u \cdot f = df(y)(\gamma'(0)), \qquad f \in \mathcal{O}_{X,y}. \tag{2.3.6}$$

If Y is a submanifold of $X \to S$ the set of relative vector fields (2.3.6) define a linear subspace $T_y(Y/S)$ of $T_y(X/S)$.

Let us fix a point o of S. Consider the canonical maps

$$T^*M \xrightarrow{i} T^*(M|S) = (T^*M) \times S \xrightarrow{r} T^*M.$$

Since $T_\sigma(T^*(M|S)/S) = T_{r(\sigma)}T^*M$ and

$$(d\theta_{M|S})(\sigma) = (i^* d\theta_M)(r(\sigma)),$$

$(d\theta_{M|S})(\sigma)$ is a symplectic form of $T_\sigma(T^*(M|S)/S)$.

The Poisson bracket of (T^*M) induces a Poisson bracket in $T^*(M|S)$. Let $f \in \mathcal{O}_{T^*(M|S)}$. Setting $f_s(x, \xi) = f(x, \xi, s)$

$$\{f, g\}_{T^*(M|S)}(x, \xi, s) = \{f_s, g_s\}_{T^*M}(x, \xi).$$

Let W be a submanifold of $T^*(M|S)$. Setting $W_s = \{(x, \xi) \in T^*M : (x, \xi, s) \in W\}$, W is an involutive submanifold of $T^*(M|S)$ if and only if W_s is an involutive submanifold of T^*M for each $s \in S$. It is well known that W_s is an involutive submanifold of T^*M if and only if $T_\sigma W_s$ is an involutive linear subspace of $T_\sigma T^*M$ for each $\sigma \in W_s$

Lemma 2.3.3. *Let $\mathcal{L}$ be a conic submanifold of $T^*(M|S)$. The manifold $\mathcal{L}$ is a Legendrian submanifold of $\mathbb{P}^*(M|S)$ if and only if $T_\sigma(\mathcal{L}/S)$ is a linear Lagrangian subspace of $T_\sigma(T^*(M|S)/S)$ for each $\sigma \in \mathcal{L}$.*

Proof. The submanifold W considered above is an involutive submanifold of $T^*(M|S)$ if and only if $T_\sigma(W/S)$ is a linear involutive subspace of $T_\sigma(T^*(M|S)/S)$ for each $\sigma \in W$. The result follows from an argument of dimension. □

Theorem 2.3.4. *Let $\mathcal{L}$ be an irreducible germ of a relative Legendrian analytic set of $\mathbb{P}^*(M|S)$. If the analytic set $\pi(\mathcal{L})$ is a flat deformation over S of an analytic set of M, $\mathcal{L} = \mathbb{P}^*_{\pi(\mathcal{L})}(M|S)$.*

Proof. There is $s \in S$ such that $Y \times \{s\} \subset \mathcal{Y}$. Let o be a smooth point of Y. There is an open set U of Y and a system of local coordinates $(y_1, \ldots, y_n)$ on U such that $Y \cap U = \{y_1 = \cdots = y_k = 0\}$. Since Y is flat, there is a neighborhood V of s and a system of partial symplectic coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ on $\pi^{-1}(U \times V)$ such that

$$\pi(\mathcal{L}) \cap U \times V = \{x_1 = \cdots = x_k = 0\}.$$

Repeating the argument of Lemma 2.3.3,

$$\mathcal{L} \cap \pi^{-1}(\pi(\mathcal{L})_{reg}) = \mathbb{P}^*_{\mathcal{Y}_{reg}}(M \times S \setminus \mathcal{Y}_{sing}/S).$$

Since $\mathcal{L}$ is closed $\mathbb{P}^*_{\mathcal{Y}}(M|S) \subseteq \mathcal{L}$. Since $\mathcal{L}$ is irreducible and both spaces have the same dimension, the inclusion is an equality. □

We present now an alternative construction of the conormal of a flat deformation of an hypersurface. This construction is more suitable to compute the conormal using computer algebra methods. For this purpose it is enough to consider the case where S is smooth.

Let F be a generator of the defining ideal of $\mathcal{Y}$. Let $\mathfrak{I}_{F,(x_i)}$ be the ideal of $\mathbb{C}\{c, x, \xi, s\}$ generated by

$$F, \ \xi_i - cF_{x_i}, \qquad i = 1, ..., n. \tag{2.3.7}$$

The ideal

$$\mathfrak{K}_{F,(x_i)} = \mathfrak{I}_{F,(x_i)} \cap \mathbb{C}\{x, \xi, s\}.$$

defines a conic analytic subset of $T^*M \times S$, hence it also defines an analytic subset $\mathcal{C}on_S\mathcal{Y}$ of $\mathbb{P}^*(M|S)$.

Lemma 2.3.5. *The ideal $\mathfrak{K}_{F,(x_i)}$ does not depend on the choice of F or (x_i).*

Proof. Given another system of local coordinates (y_i) there are function η_i such that $\sum_i \eta_i dy_i = \sum_i \xi_i dx_i$. Since

$$\textstyle\sum_i \eta_i dy_i = \sum_i \eta_i \Sigma_j \frac{\partial y_i}{\partial x_j} dx_j = \Sigma_j \sum_i \frac{\partial y_i}{\partial x_j} \eta_i dx_j,$$

$$\textstyle\xi_j - cF_{x_j} = \sum_i \frac{\partial y_i}{\partial x_j} \eta_i - c \sum_i F_{y_i} \frac{\partial y_i}{\partial x_j} = \sum_i \frac{\partial y_i}{\partial x_j} (\eta_i - cF_{y_i}).$$

Since the Jacobian matrix of the coordinate change is invertible, $\mathfrak{I}_{F,(x_i)}$ does not depend on (x_i).

Assume that φ does not vanish. Since $\xi_i - c(\varphi F)_{x_i} = \xi_i - c\varphi F_{x_i} - cF\varphi_{x_i}$, $\mathfrak{I}_{\varphi F}$ is generated by

$$F, \ \xi_i' - cF_{x_i}, \qquad i = 1, ..., n, \tag{2.3.8}$$

where $\xi_i' = \varphi^{-1}\xi_i$, $i = 1, ..., n$.

Consider the actions of $\mathbb{C}^*$ into $T^*M \times S \times \mathbb{C}$ and $T^*M \times S$ given by

$$t \cdot ((x_i), (\xi_i), (s_j), c) = ((x_i), (t\xi_i), (s_j), tc),$$

$$t \cdot ((x_i), (\xi_i), (s_j)) = ((x_i), (t\xi_i), (s_j)).$$

By (2.3.7), the ideals $\mathfrak{I}_F$ $[\mathfrak{K}_F]$ are generated by homogeneous polynomials on $\xi_1, ..., \xi_n, c$ $[\xi_1, ..., \xi_n]$. Assume that $\mathfrak{K}_F$ is generated by the homogeneous polynomials

$$P_k(\xi_1, ..., \xi_n), k = 1, ..., m.$$

It follows from (2.3.7) and (2.3.8) that $\mathfrak{K}_{\varphi F}$ is generated by $P_k(\xi_1', ..., \xi_n'), k = 1, ..., m$. If P_k is homogeneous of degree d_k, $P_k(\xi_1', ..., \xi_n') = \varphi^{-d_k} P_k(\xi_1, ..., \xi_n)$. Hence $\mathfrak{K}_F = \mathfrak{K}_{\varphi F}$. □

Theorem 2.3.6. *If $\mathcal{Y}$ is a flat deformation over S of an hypersurface of M, $\mathbb{P}^*_{\mathcal{Y}}(M|S) = \mathcal{C}on_S\mathcal{Y}$.*

Proof. If $\mathcal{Y}$ is non singular at a point o, there is a partial system of symplectic coordinates $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ such that $F = x_1$ in a neighborhood U of o. Hence $\mathfrak{I}_{F,(x_i)}$ is generated by

$$\xi_1 - c, \xi_2, ..., \xi_n, x_1. \tag{2.3.9}$$

Therefore $\mathfrak{K}_{F,(x_i)}$ is generated by $x_1, \xi_2, ..., \xi_n$. Hence $\mathbb{P}^*_{\mathcal{Y}}(M|S) = \mathcal{C}on_S\mathcal{Y}$ in $\pi^{-1}(U)$. Therefore $\mathcal{C}on_S\mathcal{Y}$ contains $\mathbb{P}^*_{\mathcal{Y}}(M|S)$. Assume that there is an irreducible component Γ of $\mathcal{C}on_S\mathcal{Y}$ that is not contained in $\mathbb{P}^*_{\mathcal{Y}}(M|S)$. Then Γ is contained in $\mathcal{Y}_{\text{sing}} \times_{M\times S} \mathbb{P}^*(M\times S|S)$. Hence the set of zeros of $\mathfrak{I}_{f,(x_i)}$ contains points of

$$\mathcal{Y}_{\text{sing}} \times_{M\times S} T^*M \times S \times \mathbb{C} \setminus M \times S \times \mathbb{C}.$$

But it follows from (2.3.7) that the intersection of the set of zeros of $\mathfrak{I}_{F,(x_i)}$ with $\mathcal{Y}_{\text{sing}} \times_{M\times S} T^*M \times S \times \mathbb{C}$ is contained in $M \times S \times \mathbb{C}$. □

The following Singular routine (see [5]) computes the relative conormal of the hypersurface defined by $z^2 + y^3 + sx^4$ when we assume $\theta = udx + vdy + wdz$ and we look at s has a deformation parameter.

```
ring r=0,(c,u,v,w,x,y,z,s),dp;
poly F=z2+y3+sx4;
ideal I=F,u-c*diff(F,x),v-c*diff(F,y),w-c*diff(F,z);
ideal J=eliminate(I,c);
J;
```

If we consider the suitable contact coordinates, and choose a different ordering we can reduce substantially the number of equations we obtain.

Let T_ε be the complex space with local ring $\mathbb{C}\{\varepsilon\}/(\varepsilon^2)$. Let I, J be ideals of the ring $\mathbb{C}\{s_1, ..., s_m\}$. Assume $J \subset I$. Let X, S, T be the germs of complex spaces with local rings $\mathbb{C}\{x, y, p\}$, $\mathbb{C}\{s\}/I, \mathbb{C}\{s\}/J$. Consider the maps $i : X \hookrightarrow X \times S$, $j : X \times S \hookrightarrow X \times T$ and $q : X \times S \to S$.

Let $\mathfrak{m}_X, \mathfrak{m}_S$ be the maximal ideals of $\mathbb{C}\{x, y, p\}$, $\mathbb{C}\{s\}/I$. Let $\mathfrak{n}_S$ be the ideal of $\mathcal{O}_{X\times S}$ generated by $\mathfrak{m}_X\mathfrak{m}_S$.

Let $\chi : X \times S \to X \times S$ be a relative contact transformation. If χ verifies

$$\chi \circ i = i,\; q \circ \chi = q \;\text{ and }\; \chi(0,s) = (0,s) \;\text{ for each } s. \tag{2.3.10}$$

there are $\alpha, \beta, \gamma \in \mathfrak{n}_S$ such that

$$\chi(x,y,p,s) = (x+\alpha, y+\beta, p+\gamma, s). \tag{2.3.11}$$

Theorem 2.3.7. (*a*) *Let $\chi : X \times S \to X \times S$ be a relative contact transformation that verifies* (2.3.10). *Then γ is determined by α and β. Moreover, there is $\beta_0 \in \mathfrak{n}_S + p\mathcal{O}_{X\times S}$ such that β is the solution of the Cauchy problem*

$$\left(1 + \frac{\partial \alpha}{\partial x} + p\frac{\partial \alpha}{\partial y}\right)\frac{\partial \beta}{\partial p} - p\frac{\partial \alpha}{\partial p}\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial p}\frac{\partial \beta}{\partial x} = p\frac{\partial \alpha}{\partial p}, \tag{2.3.12}$$

$\beta + p\mathcal{O}_{X\times S} = \beta_0$.

(*b*) *Given $\alpha \in \mathfrak{n}_S$, $\beta_0 \in \mathfrak{n}_S + p\mathcal{O}_{X\times S}$, there is a unique relative contact transformation χ that verifies* (2.3.10) *and the conditions of statement* (*a*). *We denote χ by χ_{α,β_0}.*

(*c*) *If $S = T_\varepsilon$ the Cauchy problem* (2.3.12) *simplifies into*

$$\frac{\partial \beta}{\partial p} = p\frac{\partial \alpha}{\partial p}, \qquad \beta + p\mathcal{O}_{X\times T_\varepsilon} = \beta_0. \tag{2.3.13}$$

(*d*) Let $\chi = \chi_{\alpha,\beta_0} : X \times T \to X \times T$ be a relative contact transformation. Then, χ is a lifting to T of $j^*\chi = \chi_{j^*\alpha, j^*\beta_0} : X \times S \to X \times S$. *If χ equals* (2.3.11),

$$j^*\chi(x,y,p,s) = (x + j^*\alpha, y + j^*\beta, p + j^*\gamma, s).$$

(*e*) *Assume $\mathcal{O}_T = \mathbb{C}\{s\}$, $\mathcal{O}_{T_0} = \mathbb{C}\{s,\varepsilon\}/(\varepsilon^2, \varepsilon s_1, \ldots \varepsilon s_m)$. Given a relative contact transformation*

$$\chi(x,y,p,s) = (x+A, y+B, p+C, s) \tag{2.3.14}$$

over T and $\alpha, \beta, \gamma \in \mathfrak{m}_X$,

$$\chi_0(x,y,p,s,\varepsilon) = (x + A + \varepsilon\alpha, y + B + \varepsilon\beta, p + C + \varepsilon\gamma, s, \varepsilon) \tag{2.3.15}$$

is a relative contact transformation over T_0 if and only if

$$(x,y,p,\varepsilon) \mapsto (x + \varepsilon\alpha, y + \varepsilon\beta, p + \varepsilon\gamma) \tag{2.3.16}$$

is a relative contact transformation over T_ε. Moreover, all liftings of χ to T_0 are of the type (2.3.15).

Proof. See Theorems 2.4 and 2.6 of [19]. □

2.4 Relative Legendrian Curves

Let $\theta = \xi dx + \eta dy$ be the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. Hence $\pi = \pi_{\mathbb{C}^2} : \mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ is given by $\pi(x, y; \xi : \eta) = (x, y)$. Let $U\,[V]$ be the open subset of $\mathbb{P}^*\mathbb{C}^2$ defined by $\eta \neq 0\,[\xi \neq 0]$. Then $\theta/\eta\,[\theta/\xi]$ defines a contact form $dy - pdx\,[dx - qdy]$ on $U\,[V]$, where $p = -\xi/\eta\,[q = -\eta/\xi]$. Moreover, $dy - pdx$ and $dx - qdy$ define the structure of contact manifold on $\mathbb{P}^*\mathbb{C}^2$.

If L is a germ of a Legendrian curve of $\mathbb{P}^*M$ and L is not a fiber of π_M, $\pi_M(L)$ is a germ of plane curve with irreducible tangent cone and $L = \mathbb{P}^*_{\pi_M(L)}M$.

Let Y be the germ of a plane curve with irreducible tangent cone at a point o of a surface M. Let L be the conormal of Y. Let σ be the only point of L such that $\pi_M(\sigma) = o$. Let k be the multiplicity of Y. Let f be a defining function of Y. In this situation we will always choose a system of local coordinates (x, y) of M such that the tangent cone $C(Y)$ of Y equals $\{y = 0\}$.

Lemma 2.4.1. *The following statements are equivalent*:

(*a*) *mult*$_\sigma(L) =$*mult*$_o(Y)$;

(*b*) $C_\sigma(L) \not\supset (D\pi(\sigma))^{-1}(0, 0)$;

(*c*) $f \in (x^2, y)^k$;

(*d*) *if* $t \mapsto (x(t), y(t))$ *parametrizes a branch of* Y, x^2 *divides* y.

Proof. The equivalence of statements holds if and only if it holds for each branch. Assume Y irreducible. Assume $x(t) = t^k$ and $y(t) = t^n\varphi(t) = \widetilde{\varphi}(t)$, where φ is a unit of $\mathbb{C}\{t\}$. Since $C(Y) = \{y = 0\}$, $n > k$. There is an unit ψ of of $\mathbb{C}\{t\}$ such that $p(t) = t^{n-k}\psi(t)$. Statements (a) and (b) hold if and only if $n - k \geq k$. Statement (d) holds if and only if $n \geq 2k$. Remark that

$$f = y^k + \textstyle\sum_{i=1}^k a_i y^{k-i} = \prod_{i=1}^k (y - \widetilde{\varphi}(\theta^i t))$$

where $\theta = \exp(2\pi i/k)$. Since a_i is an homogeneous polynomial of degree i on the $\widetilde{\varphi}(\theta^j t)$, $j = 1, .., k$, $a_i \in (x^{[in/k]})$ and a_k generates (x^n). Therefore (c) is verified if and only if $n/k \geq 2$. □

We say that a plane curve Y is *generic* [a Legendrian curve L is in *generic position*] if it is verifies the conditions of Lemma 2.4.1.

Given a germ of a Legendrian curve L of U at σ there is a germ of a contact transformation $\chi : (U, \sigma) \to (U, \sigma)$ such that $\chi(L)$ is in generic position (see [16] Corollary 1.6.4.).

Lemma 2.4.2. *Let* σ *denote the origin of* U. *Assume* L, L_1, L_2 *are germs of Legendrian branches in generic position.*

(*a*) $C_\sigma(L) = \{y = p = 0\}$ *if and only if given a parametrization* $t \mapsto (x(t), y(t))$ *of a branch of* Y, $x^2 \notin (y)$.

(*b*) $C_\sigma(L_1) \neq C_\sigma(L_2)$ *if and only if* $\pi(L_1)$ *and* $\pi(L_2)$ *have contact of order* 2.

Proof. Under the notations of Lemma 2.4.1, $C_\sigma(L) = \{y = p = 0\}$ if $n \geq 2k+1$ and $C_\sigma(L) = \{y = p - \psi(0)x = 0\}$ if $n = 2k$. $\square$

Remark that if Y is a germ of a plane curve of $\mathbb{C}^2$ at the origin and $C(Y) = \{y = 0\}$, its conormal is a Legendrian variety contained in U. By Darboux's Theorem each germ of a contact manifold of dimension 3 is isomorphic to the germ of U at σ, endowed with the contact structure of U defined by $dy - pdx$.

Definition 2.4.3. Let S be a reduced complex space. Let Y be a reduced plane curve. Let $\mathcal{Y}$ be a deformation of Y over S. We say that $\mathcal{Y}$ is *generic* if its fibers are generic. If S is a non reduced complex space we say that $\mathcal{Y}$ is *generic* if $\mathcal{Y}$ admits a generic lifting.

Given a flat deformation $\mathcal{Y}$ of a plane curve Y over a complex space S we will denote $\mathbb{P}^*_{\mathcal{Y}}(\mathbb{C}^2|S)$ by $\mathcal{C}on(\mathcal{Y})$.

Consider the contact transformations from $\mathbb{C}^3$ to $\mathbb{C}^3$ given by

$$\Phi(x,y,p) = (\lambda x, \lambda\mu y, \mu p),\ \lambda, \mu \in \mathbb{C}^*, \tag{2.4.1}$$

$$\Phi(x,y,p) = (ax+bp, y + \frac{ac}{2}x^2 + \frac{bd}{2}p^2 + bcxp, cx+dp),\ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1, \tag{2.4.2}$$

$$\rho_\lambda(x,y,p) = (x, y - \lambda x^2/2, p - \lambda x),\ \lambda \in \mathbb{C}. \tag{2.4.3}$$

The contact transformations (2.4.2) are called *paraboloidal contact transformations.*

Example 2.4.4. (*a*) Let k, n be integers such that $(k,n) = 1$ and $0 < k < n$. Let $Y = \{y^k - x^n = 0\}$. Consider the contact transformation $\chi(x,y,p) = (p, y - xp, -x)$. The conormal L of Y is parametrized by

$$x = t^k,\ y = t^n,\ p = \frac{n}{k}t^{n-k}.$$

Therefore, $Y^\chi = \pi(\chi(L))$ admits the equation $(xy/(k-n))^k = x^{n-k}$. We say that Y^χ is the action of the contact transformation χ on the plane curve Y.

(*b*) Setting $Y = \{y^3 - x^7 = 0\}$, $\chi(x,y,p) = (x+p, y+p^2/2, p)$, Y^χ admits a parametrization

$$x = t^3 + (7/3)t^4,\ y = t^7 + (49/18)t^8.$$

Changing parameters we get

$$x = s^3,\ y = s^7 + \lambda s^8 + h.o.t.,$$

with $\lambda \neq 0$. Following [30], Y^χ and Y have the same topological type but are not analytically equivalent.

Theorem 2.4.5. (See [1] or [22].) *Let $\Phi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ the the germ of a contact transformation. Then $\Phi = \Phi_1\Phi_2\Phi_3$, where Φ_1 is of type* (2.4.1)*, Φ_2 is of type* (2.4.2) *and Φ_3 is of type* (2.3.11)*, with $\alpha, \beta, \gamma \in \mathbb{C}\{x, y, p\}$. Moreover, there is $\beta_0 \in \mathbb{C}\{x, y\}$ such that β verifies the Cauchy problem* (2.3.12)*, $\beta - \beta_0 \in (p)$ and*

$$\alpha, \beta, \gamma, \beta_0, \frac{\partial \alpha}{\partial x}, \frac{\partial \beta_0}{\partial x}, \frac{\partial \beta}{\partial p}, \frac{\partial^2 \beta}{\partial x \partial p} \in (x, y, p). \tag{2.4.4}$$

If $D\Phi(0)(\{y = p = 0\}) = \{y = p = 0\}$, $\Phi_2 = id_{\mathbf{C}^3}$.

Let Σ be an additive submonoid of the set of non negative integers. Let Σ_0 be a minimal set of generators of Σ. Let $\mathcal{O}_\Sigma$ be the set of power series $\sum_i a_i t^i$ such that $a_i = 0$ if $i \notin \Sigma$. Let $\mathcal{O}^*_\Sigma$ be the set of power series $\sum_i a_i t^i \in \mathcal{O}_\Sigma$ such that $a_i \neq 0$ if $i \in \Sigma_0$.

Lemma 2.4.6. (Lemma 3.5.4 of [28]) *Let $\alpha, \beta, \gamma \in \mathbb{C}\{t\}$. Assume $\alpha(0) \neq 0$.*
(a) If $(t\alpha)^k = t^k\gamma$, $\alpha \in \mathcal{O}_\Sigma$ if and only if $\gamma \in \mathcal{O}_\Sigma$ and $\alpha \in \mathcal{O}^*_\Sigma$ if and only if $\gamma \in \mathcal{O}^*_\Sigma$.
(b) If $t = s\beta(s)$ solves $s = t\alpha(t)$, $\alpha \in \mathcal{O}_\Sigma$ if and only if $\beta \in \mathcal{O}_\Sigma$ and $\alpha \in \mathcal{O}^*_\Sigma$ if and only if $\beta \in \mathcal{O}^*_\Sigma$.

Theorem 2.4.7 (Theorem 1.3, [4])**.** *Let $\chi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a germ of a contact transformation. Let L be a germ of a Legendrian curve of $\mathbb{C}^3$ at the origin. If L and $\chi(L)$ are in generic position, $\pi(L)$ and $\pi(\chi(L))$ are equisingular.*

Proof. Assume $C_\sigma(L)$ is irreducible. Since when $\chi = \rho_\lambda$ or χ is of type (2.4.1) $\pi(L)$ and $\pi(\chi(L))$ are equisingular, we can assume that

$$C_\sigma(L) = C_\sigma(\chi(L)) = \{y = p = 0\}$$

and χ is of type (2.3.11). Let L_1, L_2 be branches of L. Let $S[k]$ be the semigroup [multiplicity] of $\pi(L_1)$. Let S' be the semigroup generated by $(S_0 - k) \cap \mathbb{N}$. There are parametrizations

$$t \mapsto (x_i(t), y_i(t), p_i(t)) \tag{2.4.5}$$

of L_i, $i = 1, 2$ such that $x_1(t) = t^k$, $y_1 \in \mathcal{O}^*_S$ and $p_1 \in \mathcal{O}_{S'}$. By (2.4.4) $\chi(L_1)$ admits a parametrizaton (2.4.5) with $x_1(t) = t^k$·unit, $x_1 \in \mathcal{O}_{S'}$, $y_1 \in \mathcal{O}^*_S$. By Lemma 2.4.6 we can assume that, after a reparametrization, $x_1(t) = t^k$ and $y_1 \in \mathcal{O}^*_S$. Hence $\pi(L_1)$ and $\pi(\chi(L_1))$ are equisingular.

Assume $\pi(L_i)$ has multiplicity k_i, $i = 1, 2$ and k is the least common multiple of k_1, k_2. Assume $\pi(L_1)$ and $\pi(L_2)$ have contact of order ν. Then we can assume that $x_i(t) = t^{k_i k/k_j}$, $\{i, j\} = \{1, 2\}$,

$$y_2 \equiv y_1 \mod \mathcal{O}_S \text{ and } y_2 \not\equiv y_1 \mod \mathcal{O}_{S_+}, \tag{2.4.6}$$

where $S_\ell = \{0\} \cup \ell + \mathbb{N}$, $S = S_{\nu k}$, $S_+ = S_{\nu k+1}$ and $S' = S_{\nu k - k}$. Therefore $p_2 \equiv p_1 \mod \mathcal{O}_{S'}$. Composing χ with (2.4.5) we obtain a parametrization (2.4.5) of $\chi(L_i)$ such that

$$x_i = t^k \cdot \text{unit},\ x_2 \equiv x_1 \mod \mathcal{O}_{S'} \text{ and } y_2 \equiv y_1 \mod \mathcal{O}_S,\ i = 1, 2.$$

By Lemma 2.4.6, after reparametrization, (2.4.6) holds. The theorem is proved when $C_\sigma(L)$ is irreducible.

Assume there is λ_i such that $\pi(L_i) = \{y = \lambda_i x^2\}$, $i = 1, 2$ and $\lambda_1 \neq \lambda_2$. If χ is paraboloidal, there are μ_i such that $\pi(\chi(L_i)) = \{y = \mu_i x^2\}$, $i = 1, 2$ and $\mu_1 \neq \mu_2$. By Lemma 2.4.2 if $C_\sigma(L_1) \neq C_\sigma(L_2)$, the contact order of $\pi(L_1)$ and $\pi(L_2)$ equals 2. Hence the truncation of the Puiseux expansion of $\pi(L_i)$ equals $\lambda_i x^2$, $i = 1, 2$. Therefore the contact order of $\pi(\chi(L_1))$ and $\pi(\chi(L_2))$ equals 2. $\square$

Definition 2.4.8. Two Legendrian curves are *equisingular* if their generic plane projections are equisingular.

Lemma 2.4.9. *Assume Y is a generic plane curve and $Y \hookrightarrow \mathcal{Y}$ defines an equisingular deformation of Y with trivial normal cone along its trivial section. Then $\mathcal{Y}$ is generic.*

Proof. By Proposition 2.2.6 we can assume that Y is irreducible. Moreover, we can assume that $\mathcal{Y}$ is a deformation over a vector space and $C_{\{x=y=0\}}(\mathcal{Y}) = \{y = 0\}$. Let $x = t^k$, $y = t^n + \sum_{i \geq n+1} a_i t^i$, $n \geq 2k$ be a parametrization of Y. After reparametrization, we can assume that $\mathcal{Y}$ admits a parametrization of the type

$$x = t^k, \qquad y = \textstyle\sum_i \alpha_i t^i, \tag{2.4.7}$$

with $\alpha_i \in \mathcal{O}_S$, $\alpha_i = 0$ if $i < n$ and k does note divide i. Since the normal cone of $\mathcal{Y}$ along its section is trivial, $\alpha_k = 0$. Since (2.4.7) and

$$p = \textstyle\sum_i i\alpha_i t^{i-k}$$

define a parametrization of $\mathcal{C}on(\mathcal{Y})$,

$$C_{\{x=y=0\}}(\mathcal{C}on(\mathcal{Y})) = \{y = p - 2k\alpha_{2k} x = 0\}.$$

$\square$

Definition 2.4.10. Let L be (a germ of) a Legendrian curve of $\mathbb{C}^3$ in generic position. Let $\mathcal{L}$ be a relative Legendrian curve over (a germ of) a complex space S at o. We say that an imersion $i : L \hookrightarrow \mathcal{L}$ defines a *deformation*

$$\mathcal{L} \hookrightarrow \mathbb{C}^3 \times S \to S \tag{2.4.8}$$

of the Legendrian curve L over S if i induces an isomorphism of L onto $\mathcal{L}_o$ and there is a generic deformation $\mathcal{Y}$ of a plane curve Y over S such that $\chi(\mathcal{L})$ is isomorphic to $\mathcal{C}on\mathcal{Y}$ by a relative contact transformation verifying (2.3.10).

We say that the deformation (2.4.8) is *equisingular* if $\mathcal{Y}$ is equisingular. We denote by $\widehat{\mathcal{D}ef}_L^{es}$ the category of equisingular deformations of L.

Remark 2.4.11. We do not demand the flatness of the morphism (2.4.8).

Lemma 2.4.12. *Using the notations of definition* 2.4.10*, given a section $\sigma : S \to \mathcal{L}$ of $\mathbb{C}^3 \times S \to S$, there is a relative contact transformation χ such that $\chi \circ \sigma$ is trivial. Hence $\mathcal{L}$ is isomorphic to a deformation with trivial section.*

Proof. We can assume that S is the germ at the origin of a vector space. Set $\sigma(s) = (\overline{x}(s), \overline{y}(s), \overline{p}(s), s)$. Setting $\chi(x, y, p, s) = (x - \overline{x}(s), y - \overline{y}(s), p, s)$, we can assume that $\overline{x}, \overline{y}$ vanish. Now $\chi(x, y, p, s) = (x, y - \overline{p}(s)x, p - \overline{p}(s), s)$ trivializes σ. $\square$

Theorem 2.4.13. *Assume $\mathcal{Y}$ defines an equisingular deformation of a generic plane curve Y with trivial normal cone along its trivial section. Let χ be a relative contact transformation verifying* (2.3.10). *Then $\mathcal{Y}^\chi = \pi(\chi(\mathcal{C}on\mathcal{Y}))$ is a generic equisingular deformation of Y.*

Proof. We can assume that S is the germ of a vector space. We only have to prove that (i) $(\mathcal{Y}^\chi)_s$ is generic and (ii) $(\mathcal{Y}^\chi)_s$ are equisingular, for small enough s. Let $(\mathcal{Y}^\chi)_{s,i}$ be one branch of $(\mathcal{Y}^\chi)_s$. Since $(\mathcal{Y}^\chi)_{s,i}$ is generic its conormal admits a parametrization

$$\psi(t) = (t^k, t^n + h.o.t., (n/k)t^{n-k} + h.o.t.),$$

with $n \geq 2k$ (see Lemma 2.4.1). By Theorem 2.4.5, $\chi_s = \Phi_1\Phi_2\Phi_3$. Since Φ_1 preserves genericity, we can assume $\Phi_1 = id$. Notice that $(\mathcal{Y}_{s,i})^{\Phi_2}$ is parametrized by

$$t \mapsto (x(t), y(t)), \tag{2.4.9}$$

where $x(t) = at^k + b(n/k)t^{n-k} + h.o.t.$ and $y \in (t^{2k})$. If s is small enough we can assume a close to 1 and b close to 0. Hence $(x) = (t^k)$. Therefore we can assume $\Phi_2 = id$. Finally $(\mathcal{Y}_{s,i})^{\Phi_3}$ is parametrized by (2.4.9), with

$$x(t) = t^k + \psi^*(\alpha),\ y(t) = t^n + \psi^*(\beta).$$

By (2.4.4) $(x) = (t^k)$ and $y \in (t^{2k})$ for small s. Now (ii) follows from Theorem 2.4.7, for s small enough. $\square$

2.5 Deformations of the parametrization

Let $\psi : \bar{\mathbb{C}} \to \mathbb{C}^3$ be the parametrization of a Legendrian curve L. We say that a deformation Ψ of ψ is a *Legendrian deformation of* ψ if the analytic set parametrized by Ψ is a relative Legendrian curve. We say that (χ, ξ) is an isomorphism of Legendrian deformations if $\chi : \mathbb{C}^3 \times T \to \mathbb{C}^3 \times T$ is a relative contact transformation (see (2.2.2)).

Definition 2.5.1. Let $\varphi : \bar{\mathbb{C}} \to \mathbb{C}^2$ be the parametrization of a generic plane curve Y with tangent cone $\{y = 0\}$. Let $\mathcal{D}ef_\varphi^{es}$ be the category of equisingular deformations of φ.

Let $\mathcal{Y}$ be an object of $\mathcal{D}ef_\varphi^{es}$. We say that $\mathcal{Y}$ is an object of the full subcategory $\overrightarrow{\mathcal{D}ef}_\varphi^{es}$ of $\mathcal{D}ef_\varphi^{es}$ if $\mathcal{Y}$ is generic and the normal cone of $\mathcal{Y}$ along $\{x = y = 0\}$ equals $\{y = 0\}$.

Let $\psi : \bar{\mathbb{C}} \to \mathbb{C}^3$ be the parametrization of a curve L in generic position. We will denote by $\widehat{\mathcal{D}ef}_\psi^{es}$ the category of equisingular Legendrian deformations of ψ.

Theorem 2.5.2. *Let $\varphi : \bar{\mathbb{C}} \to \mathbb{C}^2$ be the parametrization of a generic plane curve Y with tangent cone $\{y = 0\}$. Then the semiuniversal deformation of φ in $\mathcal{D}ef_\varphi^{es}$ is also a semiuniversal deformation in $\overrightarrow{\mathcal{D}ef}_\varphi^{es}$.*

Proof. Assume $\varphi_i(t_i) = (x_i(t_i), y_i(t_i))$, $i = 1, \ldots, r$. Let I_φ^{es} be the vector space of the $a\partial_x + b\partial_y$ such that $a = [a_1, \ldots, a_r]^t$, $b = [b_1, \ldots, b_r]^t$, where $a_i, b_i \in \mathbb{C}\{t_i\}t_i$ and

$$t_i \mapsto (x_i(t_i) + \varepsilon a_i(t_i), y_i(t_i) + \varepsilon b_i(t_i)),$$

$i = 1, \ldots, r$, is an equisingular deformation of φ along the trivial section over T_ε. Let $T_\varphi^{1,es}$ be the quotient of I_φ^{es} by the linear subspace of its elements that define trivial deformations. Let

$$a^j\partial_x + b^j\partial_y, \qquad j = 1, \ldots, \ell,$$

be a family of representatives of a basis of $T_\varphi^{1,es}$. Set

$$X_i = x_i + \textstyle\sum_{j=1}^{\ell} a_i^j s_j,\ Y_i = y_i + \sum_{j=1}^{\ell} b_i^j s_j$$

$i = 1, \ldots, r$. By Theorem II.2.38 of [9],

$$\Phi_i(t_i) = (X_i(t_i), Y_i(t_i), \qquad i = 1, \ldots, r,$$

defines a semiuniversal deformation of φ in $\mathcal{D}ef_\varphi^{es}$. It is enough to show that Φ_i, $i = 1, \ldots, r$ is an element of $\overset{\twoheadrightarrow}{\mathcal{D}ef}{}_\varphi^{es}$. Let m_i be the multiplicity of Φ_i. Then $(x_i) = (t_i^{m_i})$. Since Φ_i is equimultiple $X_i, Y_i \in (t_i^{m_i})$. Since $y_i \in (t_i^{2m_i})$ and Φ_i is equisingular

$$t_i \mapsto (X_i(t_i), Y_i(t_i)/X_i(t_i))$$

is equimultiple (see II of [9]). Therefore $Y_i \in (t_i^{2m_i})$. $\square$

Assume ψ is a parametrization of the conormal of the curve parametrized by φ. Let $\Phi[\Psi]$ be the deformation [Legendrian deformation] of $\varphi[\psi]$ given by

$$\Phi_i(t_i, s) = (X_i(t_i, s), Y_i(t_i, s)), \qquad [\Psi_i(t_i, s) = (X_i(t_i, s), Y_i(t_i, s), P_i(t_i, s))].$$

There are functors $\mathcal{C}on : \overset{\twoheadrightarrow}{\mathcal{D}ef}{}_\varphi^{es} \to \widehat{\mathcal{D}ef}{}_\psi^{es}$, $\pi : \widehat{\mathcal{D}ef}{}_\psi^{es} \to \overset{\twoheadrightarrow}{\mathcal{D}ef}{}_\varphi^{es}$ given by

$$(\mathcal{C}on\Phi)_i = \left(X_i, Y_i, \frac{\partial Y_i}{\partial t} \left(\frac{\partial X_i}{\partial t} \right)^{-1} \right), \qquad (\Psi^\pi)_i = (X_i, Y_i).$$

Example 2.5.3. Let Φ be the deformation $x = t^3$, $y = t^{10} + st^{11}$ of the plane curve Y given by the equation $y^3 - x^{10}$ and parametrized by $x = t^3$, $y = t^{10}$. The deformation Φ induces the flat deformation given by

$$y^3 - x^{10} - 3sx^7y - s^3x^{11}.$$

The conormal Ψ of Φ is given by $x = t^3$, $y = t^{10} + st^{11}$, $3p = 10t^7 + 11st^8$.

The semigroup of the conormal curve of $\{y^3 - x^{10} = 0\}$ equals $\{3, 6, 7, 9, 10\} \cup \mathbb{N} + 12$. The semigroup of the conormal of the deformed curve also contains the number 11. Hence the deformation is not flat (see [3]).

It is shown in [4] that each flat deformation of the conormal of $y^k - x^n = 0$ is rigid. This result shows that the obvious choice of a definition of deformation of a Legendrian variety is not a very good one. This is the reason to introduce Definitions 2.4.10 and 2.5.1.

Definition 2.5.4. Let $\mathcal{D}ef_\varphi^{es,\mu}$ be the category given in the following way: the objects of $\mathcal{D}ef_\varphi^{es,\mu}$ are the objects of $\overrightarrow{\mathcal{D}ef}_\varphi^{es}$; the morphisms of $\mathcal{D}ef_\varphi^{es,\mu}$ are the pairs (χ, ξ) where $\chi : \mathbb{C}^3 \times T \to \mathbb{C}^3 \times T$ is a relative contact transformation that acts on a deformation Φ by

$$(\chi \cdot \Phi)_i = (\chi \circ \mathcal{C}on\Phi_i)^\pi,$$

and leaves invariant the normal cone along $\{x = y = 0\}$ of the image of Φ.

Notice that, by Theorem 2.4.13 $\chi \cdot \Phi$ defined above is in fact an object of $\mathcal{D}ef_\varphi^{es,\mu}$.

Let $\mathfrak{C}_\varphi$ be a category of deformations of a curve parametrized by φ. Let S be a complex space. We will denote by $\mathfrak{C}_\varphi(S)$ the category of deformations of $\mathfrak{C}_\varphi$ over S. We will denote by $\underline{\mathfrak{C}}_\varphi(S)$ the set of isomorphism classes of objects of $\mathfrak{C}_\varphi(S)$.

The functors $\mathcal{C}on : \mathcal{D}ef_\varphi^{es,\mu} \to \widehat{\mathcal{D}ef}_\psi^{es}$, $\pi : \widehat{\mathcal{D}ef}_\psi^{es} \to \mathcal{D}ef_\varphi^{es,\mu}$ are surjective and define natural equivalences between the functors

$$T \mapsto \underline{\mathcal{D}ef}_\varphi^{es,\mu}(T) \qquad \text{and} \qquad T \mapsto \underline{\widehat{\mathcal{D}ef}}_\psi^{es}(T).$$

Let $\varphi : \overline{\mathbb{C}} \to \mathbb{C}^2$ be a parametrization of a generic plane curve Y with irreducible components $Y_1, ..., Y_r$. Assume $\varphi_i(t) = (x_i(t_i), y_i(t_i))$, $i = 1, ..., r$.

We will identify each ideal of $\mathcal{O}_Y$ with its image by $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_{\bar{\mathbb{C}}}$:

$$\mathcal{O}_Y = \mathbb{C}\left\{[x_1 \dots x_r]^t, [y_1 \dots y_r]^t\right\} \subset \oplus_{i=1}^r \mathbb{C}\{t_i\} = \mathcal{O}_{\bar{\mathbb{C}}}.$$

Set $\dot{\mathbf{x}} = [\dot{x}_1, \dots, \dot{x}_r]^t$, where $\dot{x}_i$ is the derivative of x_i in order to t_i, $1 \le i \le r$. Let $\dot{\varphi} := \dot{\mathbf{x}}\partial_x + \dot{\mathbf{y}}\partial_y$ be an element of the free $\mathcal{O}_{\bar{\mathbb{C}}}$-module $\mathcal{O}_{\bar{\mathbb{C}}}\partial_x \oplus \mathcal{O}_{\bar{\mathbb{C}}}\partial_y$, which has a structure of $\mathcal{O}_Y$-module induced by φ^*.

Let $u_1, ..., u_r, v_1, ..., v_r \in \mathbb{C}\{t_i\}$. We say that

$$(u_1, ..., u_r)\partial_x \oplus (v_1, ..., v_r)\partial_y$$

belongs to the *equisingularity module* Σ_φ^{es} (see II of [9]) of φ if the deformation Φ given by $\Phi_i(t_i, \varepsilon) = (x_i(t_i) + \varepsilon u_i(t_i), y_i(t_i) + \varepsilon v_i(t_i))$ is equisingular and has trivial normal cone along its trivial section.

Let $\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi}$ be the sub $\mathcal{O}_{\bar{\mathbb{C}}}$-module of Σ_φ^{es} generated by

$$(a_1, \dots, a_r)\left(\dot{\mathbf{x}}\partial_x + \dot{\mathbf{y}}\partial_y\right), \qquad a_i \in t_i\mathbb{C}\{t_i\}, \quad 1 \le i \le r.$$

For $i = 1, \dots, r$ set $p_i = \dot{y}_i/\dot{x}_i$. For each $k \ge 0$ set $\mathbf{p}^k = \left[p_1^k, \dots, p_r^k\right]^t$. Let $\widehat{I}$ be the sub $\mathcal{O}_Y$-module of $\mathcal{O}_{\bar{\mathbb{C}}}\partial_x \oplus \mathcal{O}_{\bar{\mathbb{C}}}\partial_y$ generated by $(k+1)\mathbf{p}^k\partial_x + k\mathbf{p}^{k+1}\partial_y$, $k \ge 1$.

Theorem 2.5.5. *The module $\widehat{I}$ is contained in Σ_φ^{es} and*

$$\underline{\mathcal{D}ef}_\varphi^{es,\mu}(T_\varepsilon) \simeq \Sigma_\varphi^{es}/(\mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x, y)\partial_x \oplus (x^2, y)\partial_y + \widehat{I}).$$

Proof. Let $(u_1, ..., u_r)\partial_x + (v_1, ..., v_r)\partial_y \in \widehat{I}$ and Φ be the deformation given by

$$\Phi_i(t_i, \varepsilon) = (x_i(t_i) + \varepsilon u_i(t_i), y_i(t_i) + \varepsilon v_i(t_i)). \tag{2.5.1}$$

We can suppose that for each $i = 1, \ldots, r$

$$u_i = p_i^\ell,\ v_i = \frac{\ell}{\ell+1} p_i^{\ell+1}$$

for some $\ell \geq 1$. Because Y is generic we have that $ord_{t_i} p_i > ord_{t_i} x_i$, $2ord_{t_i} p_i > ord_{t_i} y_i$ and, by Lemma 2.4.1, Φ has generic fibres. The deformation Φ is the result of the action over the trivial deformation of Y of the relative contact transformation

$$\chi(x, y, p, \varepsilon) = (x + \varepsilon p^\ell, y + \varepsilon \frac{\ell}{\ell+1} p^{\ell+1}, p, \varepsilon).$$

As the trivial deformation is equisingular, Φ is equisingular.

Let $\Phi \in \mathcal{D}ef_\varphi^{es,\mu}$ be given as in (2.5.1), where $u_i, v_i \in \mathbb{C}\{t_i\}$, $ord_{t_i} u_i \geq m_i$, $ord_{t_i} v_i \geq 2m_i$, $i = 1, \ldots, r$, where m_i is the multiplicity of Y_i. We have that Φ is trivial if and only if there are

$$\begin{aligned} \xi_i(t_i) &= \widetilde{t}_i = t_i + \varepsilon h_i, \\ \chi(x, y, p, \varepsilon) &= (x + \varepsilon\alpha, y + \varepsilon\beta, p + \varepsilon\gamma, \varepsilon), \end{aligned}$$

such that χ is a relative contact transformation, ξ_i is an isomorphism,

$$\alpha, \beta, \gamma \in (x, y, p)\mathbb{C}\{x, y, p\},\ h_i \in t_i\mathbb{C}\{t_i\},\ 1 \leq i \leq r$$

and

$$\begin{aligned} x_i(t_i) + \varepsilon u_i(t_i) &= x_i(\widetilde{t}_i) + \varepsilon\alpha(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i), p_i(\widetilde{t}_i)), \\ y_i(t_i) + \varepsilon v_i(t_i) &= y_i(\widetilde{t}_i) + \varepsilon\beta(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i), p_i(\widetilde{t}_i)), \end{aligned}$$

for $i = 1, \ldots, r$. By Taylor's formula $x_i(\widetilde{t}_i) = x_i(t_i) + \varepsilon\dot{x}_i(t_i)h_i(t_i)$, $y_i(\widetilde{t}_i) = y_i(t_i) + \varepsilon\dot{y}_i(t_i)h_i(t_i)$ and

$$\begin{aligned} \varepsilon\alpha(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i), p_i(\widetilde{t}_i)) &= \varepsilon\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \\ \varepsilon\beta(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i), p_i(\widetilde{t}_i)) &= \varepsilon\beta(x_i(t_i), y_i(t_i), p_i(t_i)), \end{aligned}$$

for $i = 1, \ldots, r$. Hence Φ is trivialized by χ if and only if

$$u_i(t_i) = \dot{x}_i(t_i)h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \tag{2.5.2}$$
$$v_i(t_i) = \dot{y}(t_i)h_i(t_i) + \beta(x_i(t_i), y_i(t_i), p_i(t_i)), \tag{2.5.3}$$

for $i = 1, \ldots, r$. By Theorem 2.3.7 (c), (2.5.2) and (2.5.3) are equivalent to the condition

$$\mathbf{u}\partial_x + \mathbf{v}\partial_y \in \mathfrak{m}_{\bar{\mathbb{C}}}\dot{\varphi} + (x, y)\partial_x \oplus (x^2, y)\partial_y + \widehat{I}.$$

$\square$

Theorem 2.5.6. *Set $\ell = dim\, \underline{\mathcal{D}ef}_{\varphi}^{es,\mu}(T_\varepsilon)$. Assume that*

$$\mathbf{a}^j \frac{\partial}{\partial x} + \mathbf{b}^j \frac{\partial}{\partial y} = \begin{bmatrix} a_1^j \\ \vdots \\ a_r^j \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} b_1^j \\ \vdots \\ b_r^j \end{bmatrix} \frac{\partial}{\partial y}, \tag{2.5.4}$$

$1 \leq j \leq \ell$, represents a basis of $\underline{\mathcal{D}ef}_{\varphi}^{es,\mu}(T_\varepsilon)$. Let $\Phi : \bar{\mathbb{C}} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k$ be the deformation of φ given by

$$X_i(t_i, \mathbf{s}) = x_i(t_i) + \sum_{j=1}^{\ell} a_i^j(t_i) s_j,\ Y_i(t_i, \mathbf{s}) = y_i(t_i) + \sum_{j=1}^{\ell} b_i^j(t_i) s_j, \tag{2.5.5}$$

$i = 1, \ldots, r$. Then $\mathcal{C}on\, \Phi$ is a semiuniversal deformation of ψ in $\widehat{\mathcal{D}ef}_{\psi}^{es}$.

This Theorem is the equivalent for Legendrian curves of Theorem II.2.38 of [9] for plane curves.

Remark 2.5.7. Set

$$\overset{\twoheadrightarrow}{M}_{\varphi} = \Sigma_{\varphi}^{es} / \left(\mathfrak{m}_{\bar{\mathbb{C}}} \dot{\varphi} + (x, y)\partial_x \oplus (x^2, y)\partial_y \right).$$

Then

$$\underline{\overset{\twoheadrightarrow}{\mathcal{D}ef}}_{\varphi}^{es}(T_\varepsilon) \cong \overset{\twoheadrightarrow}{M}_{\varphi}.$$

Let $k = dim\, \overset{\twoheadrightarrow}{M}_{\varphi}$ and assume that (2.5.4), $1 \leq j \leq k$, represents a basis of $\overset{\twoheadrightarrow}{M}_{\varphi}$. Let $\Phi : \bar{\mathbb{C}} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k$ be the deformation of φ given by

$$X_i(t_i, \mathbf{s}) = x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i) s_j,\ Y_i(t_i, \mathbf{s}) = y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i) s_j.$$

Then Φ is semiuniversal in $\overset{\twoheadrightarrow}{\mathcal{D}ef}_{\varphi}^{es}$ (see [9] II Theorem 2.38). If $\Psi \in \widehat{\mathcal{D}ef}_{\psi}^{es}(T)$, then $\Psi^\pi \in \overset{\twoheadrightarrow}{\mathcal{D}ef}_{\varphi}^{es}(T)$. Hence there is $f : T \to \overset{\twoheadrightarrow}{M}_{\varphi}$ such that $\Psi^\pi \cong f^*\Phi$. Therefore $\Psi = \mathcal{C}on\, \Psi^\pi \cong \mathcal{C}on\, f^*\Phi = f^*\mathcal{C}on\, \Phi$. This shows that $\mathcal{C}on\, \Phi$ is complete in $\widehat{\mathcal{D}ef}_{\psi}^{es}$. It is actually versal and the proof is only technically more complicated.

Proof. (of Theorem 2.5.6) It is enough to show that $\mathcal{C}on\, \Phi$ is formally semiuniversal (see remark 2.5.7 and [6] Satz 5.2).

Let $\imath : T' \hookrightarrow T$ be a small extension. Let $\Psi \in \widehat{\mathcal{D}ef}_{\psi}^{es}(T)$. Set $\Psi' = \imath^*\Psi$. Let $\eta' : T' \to \mathbb{C}^\ell$ be a morphism of complex analytic spaces. Assume that (χ', ξ') define an isomorphism

$$\eta'^*\mathcal{C}on\, \Phi \cong \Psi'.$$

We need to find $\eta : T \to \mathbb{C}^\ell$ and χ, ξ such that $\eta' = \eta \circ \imath$ and χ, ξ define an isomorphism

$$\eta^* \mathcal{C}on\, \Phi \cong \Psi$$

that extends (χ', ξ').

Let $A\,[A']$ be the local ring of $T\,[T']$. Let δ be the generator of $Ker(A \twoheadrightarrow A')$. We can assume $A' \cong \mathbb{C}\{\mathbf{z}\}/I$, where $\mathbf{z} = (z_1, \ldots, z_m)$. Set

$$\widetilde{A}' = \mathbb{C}\{\mathbf{z}\} \quad \text{and} \quad \widetilde{A} = \mathbb{C}\{\mathbf{z}, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \ldots, \varepsilon z_m).$$

Let $\mathfrak{m}_A$ be the maximal ideal of A. Since $\mathfrak{m}_A \delta = 0$ and $\delta \in \mathfrak{m}_A$, there is a morphism of local analytic algebras from $\widetilde{A}$ onto A that takes ε into δ such that the diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & \widetilde{A}' \arrow[d] \\ A \arrow[r] & A' \end{tikzcd} \tag{2.5.6}$$

commutes. Assume $\widetilde{T}\,[\widetilde{T}']$ has local ring $\widetilde{A}\,[\widetilde{A}']$. We also denote by $\imath$ the morphism $\widetilde{T}' \hookrightarrow \widetilde{T}$. We denote by κ the morphisms $T \hookrightarrow \widetilde{T}$ and $T' \hookrightarrow \widetilde{T}'$. Let $\widetilde{\Psi} \in \widehat{\mathcal{D}ef}^{\,em}_{\psi}(\widetilde{T})$ be a lifting of Ψ.

We fix a linear map $\sigma : A' \hookrightarrow \widetilde{A}'$ such that $\kappa^* \sigma = id_{A'}$. Set $\widetilde{\chi}' = \chi_{\sigma(\alpha), \sigma(\beta_0)}$, where $\chi' = \chi_{\alpha, \beta_0}$. Define $\widetilde{\eta}'$ by $\widetilde{\eta}'^* s_i = \sigma(\eta'^* s_i)$, $i = 1, \ldots, l$. Let $\widetilde{\xi}'$ be the lifting of ξ' determined by σ. Then

$$\widetilde{\Psi}' := \widetilde{\chi}'^{-1} \circ \widetilde{\eta}'^* \mathcal{C}on\, \Phi \circ \widetilde{\xi}'^{-1}$$

is a lifting of Ψ' and

$$\widetilde{\chi}' \circ \widetilde{\Psi}' \circ \widetilde{\xi}' = \widetilde{\eta}'^* \mathcal{C}on\, \Phi. \tag{2.5.7}$$

By Theorem 2.3.7 it is enough to find liftings $\widetilde{\chi}, \widetilde{\xi}, \widetilde{\eta}$ of $\widetilde{\chi}', \widetilde{\xi}', \widetilde{\eta}'$ such that

$$\widetilde{\chi} \cdot \widetilde{\Psi}^{\pi} \circ \widetilde{\xi} = \widetilde{\eta}^* \Phi$$

in order to prove the theorem.

Consider the following commutative diagram

$$\begin{tikzcd} \bar{\mathbb{C}} \times \widetilde{T}' \arrow[r, hook] \arrow[d, "\widetilde{\Psi}'"] & \bar{\mathbb{C}} \times \widetilde{T} \arrow[r, dotted] \arrow[d, "\widetilde{\Psi}"] & \bar{\mathbb{C}} \times \mathbb{C}^\ell \arrow[d, "\mathcal{C}on\, \Phi"] \\ \mathbb{C}^3 \times \widetilde{T}' \arrow[r, hook] \arrow[d, "pr"] & \mathbb{C}^3 \times \widetilde{T} \arrow[r, dotted] \arrow[d, "pr"] & \mathbb{C}^3 \times \mathbb{C}^\ell \arrow[d] \\ \widetilde{T}' \arrow[r, hook] \arrow[rr, bend right, "\widetilde{\eta}'"'] & \widetilde{T} \arrow[r, dotted, "\widetilde{\eta}"] & \mathbb{C}^\ell. \end{tikzcd}$$

If $\mathcal{C}on\,\Phi$ is given by
$$X_i(t_i,\mathbf{s}),\ Y_i(t_i,\mathbf{s}),\ P_i(t_i,\mathbf{s})\ \in \mathbb{C}\{\mathbf{s},t_i\},$$
then $\widetilde{\eta}'^*\,\mathcal{C}on\,\Phi$ is given by
$$X_i(t_i,\widetilde{\eta}'(\mathbf{z})),\ Y_i(t_i,\widetilde{\eta}'(\mathbf{z})),\ P_i(t_i,\widetilde{\eta}'(\mathbf{z}))\ \in \widetilde{A}'\{t_i\}=\mathbb{C}\{\mathbf{z},t_i\}$$
for $i=1,\dots,r$. Suppose that $\widetilde{\Psi}'$ is given by
$$U_i'(t_i,\mathbf{z}),\ V_i'(t_i,\mathbf{z}),\ W_i'(t_i,\mathbf{z})\ \in \mathbb{C}\{\mathbf{z},t_i\}.$$
Then, $\widetilde{\Psi}$ must be given by
$$U_i=U_i'+\varepsilon u_i,\ V_i=V_i'+\varepsilon v_i,\ W_i=W_i'+\varepsilon w_i\ \in \widetilde{A}\{t_i\}=\mathbb{C}\{\mathbf{z},t_i\}\oplus\varepsilon\mathbb{C}\{t_i\}$$
with $u_i,v_i,w_i\in\mathbb{C}\{t_i\}$ and $i=1,\dots,r$. By definition of deformation we have that, for each i,
$$(U_i,V_i,W_i)=(x_i(t_i),y_i(t_i),p_i(t_i))\ mod\ \mathfrak{m}_{\widetilde{A}}.$$
Suppose $\widetilde{\eta}':\widetilde{T}'\to\mathbb{C}^\ell$ is given by $(\widetilde{\eta}_1',\dots,\widetilde{\eta}_\ell')$, with $\widetilde{\eta}_i'\in\mathbb{C}\{\mathbf{z}\}$. Then $\widetilde{\eta}$ must be given by $\widetilde{\eta}=\widetilde{\eta}'+\varepsilon\widetilde{\eta}^0$ for some $\widetilde{\eta}^0=(\widetilde{\eta}_1^0,\dots,\widetilde{\eta}_\ell^0)\in\mathbb{C}^\ell$. Suppose that $\tilde{\chi}':\mathbb{C}^3\times\widetilde{T}'\to\mathbb{C}^3\times\widetilde{T}'$ is given at the ring level by
$$(x,y,p)\mapsto(H_1',H_2',H_3'),$$
such that $H'=id\ mod\ \mathfrak{m}_{\widetilde{A}'}$ with $H_i'\in(x,y,p)A'\{x,y,p\}$. Let the automorphism $\widetilde{\xi}':\bar{\mathbb{C}}\times\widetilde{T}'\to\bar{\mathbb{C}}\times\widetilde{T}'$ be given at the ring level by
$$t_i\mapsto h_i'$$
such that $h'=id\,mod\,\mathfrak{m}_{\widetilde{A}'}$ with $h_i'\in(t_i)\mathbb{C}\{\mathbf{z},t_i\}$.

Then, from (2.5.7) it follows that
$$\begin{aligned}X_i(t_i,\widetilde{\eta}')&=H_1'(U_i'(h_i'),V_i'(h_i'),W_i'(h_i')),\\ Y_i(t_i,\widetilde{\eta}')&=H_2'(U_i'(h_i'),V_i'(h_i'),W_i'(h_i')),\\ P_i(t_i,\widetilde{\eta}')&=H_3'(U_i'(h_i'),V_i'(h_i'),W_i'(h_i')).\end{aligned}\tag{2.5.8}$$
Now, $\widetilde{\eta}'$ must be extended to $\widetilde{\eta}$ such that the first two previous equations extend as well. That is, we must have
$$\begin{aligned}X_i(t_i,\widetilde{\eta})&=(H_1'+\varepsilon\alpha)(U_i(h_i'+\varepsilon h_i^0),V_i(h_i'+\varepsilon h_i^0),W_i(h_i'+\varepsilon h_i^0),\qquad(2.5.9)\\ Y_i(t_i,\widetilde{\eta})&=(H_2'+\varepsilon\beta)(U_i(h_i'+\varepsilon h_i^0),V_i(h_i'+\varepsilon h_i^0),W_i(h_i'+\varepsilon h_i^0).\end{aligned}$$
with $\alpha,\beta\in(x,y,p)\mathbb{C}\{x,y,p\}$, $h_i^0\in(t_i)\mathbb{C}\{t_i\}$ such that
$$(x,y,p)\mapsto(H_1'+\varepsilon\alpha,H_2'+\varepsilon\beta,H_3'+\varepsilon\gamma)$$
gives a relative contact transformation over $\widetilde{T}$ for some $\gamma\in(x,y,p)\mathbb{C}\{x,y,p\}$. The existence of this extended relative contact transformation is guaranteed by Theorem

2.3.7 (e). Moreover, this extension depends only on the choices of α and β_0. So, we need only to find α, β_0, $\widetilde{\eta}^0$ and h_i^0 such that (2.5.9) holds. Using Taylor's formula and $\varepsilon^2 = 0$ we see that

$$\begin{aligned}
X_i(t_i, \widetilde{\eta}' + \varepsilon\widetilde{\eta}^0) &= X_i(t_i, \widetilde{\eta}') + \varepsilon \sum_{j=1}^{\ell} \partial_{s_j} X_i(t_i, \widetilde{\eta}')\widetilde{\eta}_j^0 \\
(\varepsilon\mathfrak{m}_{\widetilde{A}} = 0) \qquad &= X_i(t_i, \widetilde{\eta}') + \varepsilon \sum_{j=1}^{\ell} \partial_{s_j} X_i(t_i, 0)\widetilde{\eta}_j^0, \qquad (2.5.10) \\
Y_i(t_i, \widetilde{\eta}' + \varepsilon\widetilde{\eta}^0) &= Y_i(t_i, \widetilde{\eta}') + \varepsilon \sum_{j=1}^{\ell} \partial_{s_j} Y_i(t_i, 0)\widetilde{\eta}_j^0.
\end{aligned}$$

Again by Taylor's formula and noticing that $\varepsilon\mathfrak{m}_{\widetilde{A}} = 0$, $\varepsilon\mathfrak{m}_{\widetilde{A}'} = 0$ in $\widetilde{A}$, $h' = id \ mod \ \mathfrak{m}_{\widetilde{A}'}$ and $(U_i, V_i) = (x_i(t_i), y_i(t_i)) \ mod \ \mathfrak{m}_{\widetilde{A}}$ we see that

$$\begin{aligned}
U_i(h_i' + \varepsilon h_i^0) &= U_i(h_i') + \varepsilon\dot{U}_i(h_i')h_i^0 \\
&= U_i'(h_i') + \varepsilon(\dot{x}_i h_i^0 + u_i), \qquad (2.5.11) \\
V_i(h_i' + \varepsilon h_i^0) &= V_i'(h_i') + \varepsilon(\dot{y}_i h_i^0 + v_i).
\end{aligned}$$

Now, $H' = id \ mod \ \mathfrak{m}_{\widetilde{A}'}$, so

$$\partial_x H_1' = 1 \ mod \ \mathfrak{m}_{\widetilde{A}'}, \qquad \partial_y H_1', \partial_p H_1' \in \mathfrak{m}_{\widetilde{A}'}\widetilde{A}'\{x, y, p\}.$$

In particular,

$$\varepsilon\partial_y H_1' = \varepsilon\partial_p H_1' = 0.$$

By this and arguing as in (2.5.10) and (2.5.11) we see that

$$\begin{aligned}
&(H_1' + \varepsilon\alpha)(U_i'(h_i') + \varepsilon(\dot{x}_i h_i^0 + u_i), V_i'(h_i') + \varepsilon(\dot{y}_i h_i^0 + v_i), W_i'(h_i') + \varepsilon(\dot{p}_i h_i^0 + w_i)) \\
&= H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + \varepsilon(\alpha(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + 1(\dot{x}_i h_i^0 + u_i)) \\
&= H_1'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + \varepsilon(\alpha(x_i, y_i, p_i) + \dot{x}_i h_i^0 + u_i), \\
&(H_2' + \varepsilon\beta)(U_i'(h_i') + \varepsilon(\dot{x}_i h_i^0 + u_i), V_i'(h_i') + \varepsilon(\dot{y}_i h_i^0 + v_i), W_i'(h_i') + \varepsilon(\dot{p}_i h_i^0 + w_i)) \\
&= H_2'(U_i'(h_i'), V_i'(h_i'), W_i'(h_i')) + \varepsilon(\beta(x_i, y_i, p_i) + \dot{y}_i h_i^0 + v_i).
\end{aligned}$$

Substituting this in (2.5.9) and using (2.5.8) and (2.5.10) we see that we have to find $\widetilde{\eta}^0 = (\widetilde{\eta}_1^0, \ldots, \widetilde{\eta}_\ell^0) \in \mathbb{C}^\ell$, h_i^0 such that

$$\begin{aligned}
(u_i(t_i), v_i(t_i)) = \sum_{j=1}^{\ell} \widetilde{\eta}_j^0 \left(\partial_{s_j} X_i(t_i, 0), \partial_{s_j} Y_i(t_i, 0)\right) - \qquad (2.5.12) \\
-h_i^0(t_i)((\dot{x}_i(t_i), \dot{y}_i(t_i)) - (\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))).
\end{aligned}$$

Note that, because of Theorem 2.3.7 (c),

$$(\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))) \in \widehat{I}$$

for each i. Also note that $\widetilde{\Psi} \in \widehat{\mathcal{D}ef}{}^{es}_{\psi}(\widetilde{T})$ means that $(u_i, v_i) \in \Sigma^{es}_{\varphi}$. Then, if the vectors

$$\begin{aligned}&\left(\partial_{s_j} X_1(t_1,0), \ldots, \partial_{s_j} X_r(t_r,0)\right)\partial_x + \left(\partial_{s_j} Y_1(t_1,0), \ldots, \partial_{s_j} Y_r(t_r,0)\right)\partial_y \\ &= (a_1^j(t_1), \ldots, a_r^j(t_r))\partial_x + (b_1^j(t_1), \ldots, b_r^j(t_r))\partial_y, \qquad j = 1, \ldots, \ell\end{aligned}$$

form a basis of $\underline{\mathcal{D}ef}{}^{es,\mu}_{\varphi}(T_\varepsilon)$, we can solve (2.5.12) with unique $\widetilde{\eta}^0_1, \ldots, \widetilde{\eta}^0_\ell$ for all $i = 1, \ldots, r$. This implies that the conormal of Φ is a formally semiuniversal equisingular deformation of ψ over $\mathbb{C}^\ell$. □

2.6 Deformations of the equation I

Let Y be a generic curve with parametrization φ and equation f. Let L be the conormal of Y.

Definition 2.6.1. We will denote by $\overrightarrow{\mathcal{D}ef}{}^{es}_{f}$ (or $\overrightarrow{\mathcal{D}ef}{}^{es}_{Y}$) the full subcategory of generic *equisingular deformations* of (the equation f of) the plane curve Y such that its normal cone along $\{x = y = 0\}$ equals $\{y = 0\}$.

Let T be a complex space. We associate to a deformation Φ of φ the deformation $\mathcal{Y}$ defined by the kernel of $\Phi^* : \mathcal{O}_{\mathbb{C}^2\times T} \to \mathcal{O}_{\overline{C}\times T}$. We obtain in this way a functor

$$\vartheta : \overrightarrow{\mathcal{D}ef}{}^{es}_{\varphi} \to \overrightarrow{\mathcal{D}ef}{}^{es}_{f}.$$

Theorem 2.6.2. *The functor ϑ is surjective and induces a natural equivalence between the functors $T \mapsto \underline{\overrightarrow{\mathcal{D}ef}}{}^{es}_{\varphi}(T)$ and $T \mapsto \underline{\overrightarrow{\mathcal{D}ef}}{}^{es}_{f}(T)$.*

Given a morphism of complex spaces $\sigma : T \to S$ and $\Phi \in \overrightarrow{\mathcal{D}ef}{}^{es}_{\varphi}(S)$,

$$\sigma^*\vartheta(\Phi) = \vartheta(\sigma^*\Phi).$$

Proof. See Theorem II.2.64 of [9]. □

Let $\mathcal{Y}$ be an object of $\overrightarrow{\mathcal{D}ef}{}^{es}_{\varphi}$. Since the normal cone of $\mathcal{Y}$ along $\{x = y = 0\}$ equals $\{y = 0\}$, $\mathcal{C}on(\mathcal{Y}) \subset U \times T$.

Let ψ be the parametrization of the conormal of φ. Let $\Phi \in \overrightarrow{\mathcal{D}ef}{}^{es}_{\varphi}(T)$. Let Ψ be the conormal of Φ. Let $\widehat{\vartheta}(\Psi)$ denote the image of Ψ. By Theorem 2.3.4

$$\widehat{\vartheta}(\Psi) = \mathcal{C}on(\vartheta(\Psi^\pi)). \tag{2.6.1}$$

Lemma 2.6.3. *The functor $\widehat{\vartheta}$ is surjective and induces a natural equivalence between the functors $T \mapsto \underline{\widehat{\mathcal{D}ef}}^{es}_{\psi}(T)$ and $T \mapsto \underline{\widehat{\mathcal{D}ef}}^{es}_{L}(T)$.*

Given a morphism of complex spaces $\sigma : T \to S$ and $\Psi \in \widehat{\mathcal{D}ef}^{es}_{\psi}(S)$,

$$\sigma^* \widehat{\vartheta}(\Psi) = \widehat{\vartheta}(\sigma^* \Psi). \tag{2.6.2}$$

Proof. If $\mathcal{L}$ is in $\widehat{\mathcal{D}ef}^{es}_{L}(T)$, $\mathcal{L}^{\pi}$ is in $\overrightarrow{\mathcal{D}ef}^{es}_{f}(T)$. Therefore $\mathcal{L}^{\pi} = \vartheta(\Phi)$, for some $\Phi \in \overrightarrow{\mathcal{D}ef}^{es}_{\varphi}(T)$. Setting $\Psi = \mathcal{C}on(\Phi)$, $\widehat{\vartheta}(\Psi) = \mathcal{L}$.

By Theorem 2.6.2 and (2.6.1), $\widehat{\vartheta}$ induces a natural equivalence and (2.6.2) holds. □

Theorem 2.6.4. *For each Legendrian curve L there is a semiuniversal deformation $\mathcal{L}$ of L in the category $\widehat{\mathcal{D}ef}^{es}_{L}$. Moreover, $\mathcal{L}$ is defined over a smooth analytic manifold.*

Proof. Let Ψ be the semiuniversal deformation of the parametrization ψ of L in the category $\widehat{\mathcal{D}ef}^{es}_{\psi}$. By Lemma 2.6.3, we can take $\mathcal{L} = \widehat{\vartheta}(\Psi)$. □

2.7 Deformations of the equation II

Definition 2.7.1. Let $\mathcal{D}ef^{es,\mu}_{f}$ (or $\mathcal{D}ef^{es,\mu}_{Y}$) be the category given in the following way: the objects of $\mathcal{D}ef^{es,\mu}_{f}$ are the objects of $\overrightarrow{\mathcal{D}ef}^{es}_{f}$; two objects $\mathcal{Y}, \mathcal{Z}$ of $\mathcal{D}ef^{es,\mu}_{f}(T)$ are *isomorphic* if there is a relative contact transformation χ over T such that $\mathcal{Z} = \mathcal{Y}^{\chi}$.

Lemma 2.7.2. *Assume $f \in \mathbb{C}\{x, y\}$ is the defining function of a generic plane curve Y. Let L be the conormal of Y. For each $\ell \geq 1$ there is $h_\ell \in \mathbb{C}\{x, y\}$ such that*

$$(\ell + 1)p^{\ell} f_x + \ell p^{\ell+1} f_y \equiv h_\ell \ mod\ I_L.$$

Moreover, h_ℓ is unique modulo I_Y.

Proof. Let Δ be the germ of $\mathbb{C}$ at the origin. Let k_τ $[c_\tau]$ be the multiplicity [the conductor] of the branch Y_τ of Y, $\tau = 1, ..., n$. Let $\sigma_\tau : \Delta \to L_\tau$ be the normalization of the conormal L_τ of Y_τ, $\tau = 1, ..., n$. Let v_τ be the valuation of $\mathbb{C}\{x, y, p\}$ associated to σ_τ, $\tau = 1, ..., n$. The restriction of v_τ to $\mathbb{C}\{x, y\}$ defines the valuation of $\mathbb{C}\{x, y\}$ associated to the normalization of Y_τ, $\tau = 1, ..., n$. By [30], Section I.2

$$v_\tau(f_{\tau,y}) = c_\tau + k_\tau - 1, \qquad \text{and} \qquad v_\tau(x f_{\tau,x}) = v_\tau(y f_{\tau,y}), \tag{2.7.1}$$

for $\tau = 1, ..., n$. By (2.7.1) and [30] there is $a_{\tau,\ell} \in \mathbb{C}\{x, y\}$ such that $v_\tau(\ell p^{\ell+1} f_{\tau,y} - a_{\tau,\ell}) = +\infty$, $\tau = 1, ..., n$, for each $\ell \geq 1$. Setting $a_\ell = \sum_{\tau=1}^{n} a_{\tau,\ell} \prod_{j \neq \tau} f_j$,

$$v_\tau(\ell p^{\ell+1} f_y - a_\ell) = +\infty, \qquad \text{for } \ell \geq 1, \quad \tau = 1, ..., n.$$

A similar reasoning shows that there are $b_\ell \in \mathbb{C}\{x, y\}$ such that

$$v_\tau((\ell + 1)p^{\ell} f_x - b_\ell) = +\infty, \qquad \text{for } \ell \geq 1, \quad \tau = 1, ..., n.$$

□

Remark 2.7.3. Assume Y is irreducible with multiplicity ν. Suppose $\mathcal{Y} \in \overrightarrow{\mathcal{D}ef}^{es}_Y(T)$, where T is a reduced complex space and let $\mathcal{L}$ be the relative conormal of $\mathcal{Y}$. Let Φ be the deformation of the parametrization of Y such that $\vartheta(\Phi) = \mathcal{Y}$. Let Ψ be the conormal of Φ. There $A_i \in \mathcal{O}_T$ such that

$$\Psi^* x = t^\nu, \quad \Psi^* y = t^n + \sum_{i \geq n+1} A_i t^i \quad \text{and} \quad \Psi^* p = \frac{n}{\nu} t^{n-\nu} + \sum_{i \geq n+1} \frac{i}{\nu} A_i t^{i-\nu}.$$

Given $f \in \mathcal{O}_T\{x, y, p\}$, $f \in I_{\mathcal{L}}$ if and only if $\Psi^* f = 0$.

Theorem 2.7.4. *Let Y be a generic curve. Let T be a complex space. Let $\imath_0 : T \hookrightarrow T_0$ be a small extension and χ_0 be a relative contact transformation over T_0. Let $\mathcal{Y}_0 \in \overrightarrow{\mathcal{D}ef}_f(T_0)$, $\mathcal{Y} = \imath_0^* \mathcal{Y}_0$ and $\chi = \imath_0^* \chi_0$. Assume χ_0 equals* (2.3.15) *and $\mathcal{Y}$ $[\mathcal{Y}_0, \mathcal{Y}^\chi, \mathcal{Y}_0^{\chi_0}]$ are defined by F $[F_0, F^\chi, F_0^{\chi_0}]$, where $F_0 = F + \varepsilon g$, $g \in \mathbb{C}\{x, y\}$, and F^χ is a lifting of f. Then, if $F_0^{\chi_0}$ is a lifting of F^χ,*

$$F_0^{\chi_0} = F^\chi + \varepsilon g + \varepsilon \alpha_0 f_x + \varepsilon \beta_0 f_y + \varepsilon {\textstyle\sum_{k \geq 1}} \frac{\alpha_k}{k+1} h_k. \tag{2.7.2}$$

Proof. Remark that if χ equals (2.3.14) and $I_{\mathcal{Y}}$ is generated by F, $I_{\mathcal{Y}^\chi}$ is generated by $F^\chi \in \mathcal{O}_{\mathbb{C}^2 \times S}$ such that

$$F^\chi(x, y, s) \equiv F(x + A, y + B, s) \bmod I_{\chi(\mathcal{L})}.$$

Let L denote the conormal of Y. Let $\mathcal{L}[\mathcal{L}_0]$ denote the relative conormal of $\mathcal{Y}[\mathcal{Y}_0]$. We can assume $s = (s_1, ..., s_m)$,

$$\mathcal{O}_T = \mathbb{C}\{s\}, \; \mathcal{O}_{T_0} = \mathbb{C}\{s, \varepsilon\}/\mathfrak{n}_\varepsilon, \; \mathfrak{n}_\varepsilon = (s_1\varepsilon, ..., s_m\varepsilon, \varepsilon^2).$$

Since $I_{\chi_0(\mathcal{L}_0)} = I_{\chi(\mathcal{L})} + \varepsilon \mathcal{O}_{\mathbb{C}^3 \times T_0} \cap I_{\chi_0(\mathcal{L}_0)} = I_{\chi(\mathcal{L})} + \varepsilon I_L$ we have the following congruences modulo $I_{\chi_0(\mathcal{L}_0)}$:

$$\begin{aligned} F_0^{\chi_0} &\equiv F_0(x + A + \varepsilon\alpha, y + B + \varepsilon\beta, s, \varepsilon) \\ &\equiv F(x + A + \varepsilon\alpha, y + B + \varepsilon\beta, s) + \varepsilon g \\ &\equiv F(x + A, y + B, s) + \varepsilon g + \varepsilon\alpha\partial_x F + \varepsilon\beta\partial_y F \\ &\equiv F^\chi + \varepsilon g + \varepsilon\alpha_0 f_x + \varepsilon\beta_0 f_y + \varepsilon {\textstyle\sum_{k\geq 1}} \frac{\alpha_k}{k+1} h_k. \end{aligned}$$

□

Corollary 2.7.5. *Let $F = f + \varepsilon g$ be a defining function of a deformation $\mathcal{Y} \in \overrightarrow{\mathcal{D}ef}^{es}_f(T_\varepsilon)$. Let χ_{α,β_0} be a contact transformation over T_ε. Then*

$$f + \varepsilon g + \varepsilon\alpha_0 f_x + \varepsilon\beta_0 f_y + \varepsilon {\textstyle\sum_{k \geq 1}} \frac{\alpha_k}{k+1} h_k \tag{2.7.3}$$

defines the action of χ_{α,β_0} on $\mathcal{Y}$.

Definition 2.7.6. Let f be a generic plane curve with tangent cone $\{y = 0\}$. We will denote by I_f the ideal of $\mathbb{C}\{x, y\}$ generated by the functions g such that $f + \varepsilon g$ is equisingular over T_ε and has trivial normal cone along its trivial section. We call I_f the *equisingularity ideal of* f.

We will denote by I_f^μ the ideal of $\mathbb{C}\{x, y\}$ generated by $f, (x, y)f_x$, $(x^2, y)f_y$ and h_ℓ, $\ell \geq 1$.

Let $f = \sum_{k,\ell} a_{k,\ell}$ be a convergent power series. Let u, v, d be positive integers. Assume u, v coprime. If $a_{k,\ell} \neq 0$ implies $uk + v\ell \geq d$ and there are k_1, ℓ_1, k_2, ℓ_2 such that $(k_1, \ell_1) \neq (k_2, \ell_2)$ and $a_{k_i,\ell_i} \neq 0$, $i = 1, 2$, we call

$$f_{u,v,d}(x, y) = \textstyle\sum_{uk+v\ell=d} a_{k,\ell} x^k y^\ell$$

a *face* of f. We say that f is *semiquasihomogeneous* (*SQH*) of type $(u, v; d)$ if $f_{u,v,d}$ is a face of f and $f_{u,v,d}$ has isolated singularities. We say that f is *Newton non-degenerate* (*NND*) if x, y do not divide f and the singular locus of each face of f is contained in $\{xy = 0\}$.

Lemma 2.7.7. *If f is generic, $I_f^\mu \subset I_f$.*

Proof. Let $\alpha \in (x, y)$, $\beta \in (x^2, y)$. Set $\chi = \chi_{\alpha,0}$ $[\chi = \chi_{0,\beta}, \chi = \chi_{p^\ell,0}]$. By Lemma 2.7.4, f^χ equals

$$f + \varepsilon\alpha f_x, \qquad [f + \varepsilon\beta f_y, \quad f + \varepsilon h_\ell/(\ell + 1)].$$

By Lemma 2.4.13, f^χ is equisingular. Since the derivative of χ leaves invariant $\{y = 0\}$, then $(x, y)f_x$, $(x^2, y)f_y \subset I_f$ and $h_\ell \in I_f$, for each $\ell \geq 1$. $\square$

Theorem 2.7.8. *If f is generic,*

$$\underline{\mathcal{D}ef}_f^{es,\mu}(T_\varepsilon) \simeq I_f/I_f^\mu.$$

Proof. Let $G \in \mathcal{D}ef_f^{es,\mu}(T_\varepsilon)$. There is $g \in I_f$ such that $G = f + \varepsilon g$. The deformation $f + \varepsilon g$ is trivial in $\mathcal{D}ef_f^{es,\mu}(T_\varepsilon)$ if and only if there are $h \in \mathbb{C}\{x, y\}$ and a contact transformation (2.3.11) such that

$$G(x + \alpha, y + \beta, \varepsilon) = (1 + \varepsilon h)f \qquad \mathrm{mod}\ \varepsilon I_L. \tag{2.7.4}$$

By Corollary 2.7.5, (2.7.4) holds if and only if

$$g + \alpha_0 f_x + \beta_0 f_y + \sum_\ell \frac{\alpha_\ell}{\ell + 1} h_\ell = hf \ \mathrm{mod}\ (f).$$

Hence G is trivial if and only if $g \in I_f^\mu$. $\square$

Remark 2.7.9. Each equisingular deformation F of a SQH or NND plane curve f is isomorphic to a deformation $\widetilde{F}$, such that $\widetilde{F}$ is equisingular via trivial sections (see [29] and [9]). This means that, in the SQH or NND case, if $A \twoheadrightarrow A'$ is a small extension with kernel ε such that $\mathcal{Y}' \in \mathcal{D}ef_f^{es,\mu}(A'), \mathcal{Y} \in \mathcal{D}ef_f^{es,\mu}(A)$ defined by F', respectively $F = F' + \varepsilon a(x, y)$, then $f + \varepsilon a(x, y)$ defines a deformation in $\mathcal{D}ef_f^{es,\mu}(T_\varepsilon)$(see Theorem 8.2 of [29]).

Theorem 2.7.10. *Assume Y is a generic plane curve with conormal L, defined by a power series f. Assume f is SQH or f is NND. If $g_1, ..., g_n \in I_f$ represent a basis of I_f/I_f^μ with Newton order ≥ 1, the deformation $\mathcal{G}$ defined by*

$$G(x, y, s_1, ..., s_n) = f(x, y) + \sum_{i=1}^{n} s_i g_i \tag{2.7.5}$$

is a semiuniversal deformation of f in $\mathcal{D}ef_f^{es,\mu}$.

Proof. The choice of $g_1, ..., g_n$ identifies I_f/I_f^μ with $\mathbb{C}^n$. It is enough to show that (3.2.4) is a formally versal deformation of f in $\mathcal{D}ef_f^{es,\mu}$ and there is a versal deformation of f in $\mathcal{D}ef_f^{es,\mu}$ (see [6] Satz 5.2). The second requirement follows from Theorem 2.6.4. Let us prove that the first requirement is fulfilled. We will follow the terminology of the proof of Theorem 2.7.4. Let $\eta : T \to \mathbb{C}^n$ be a morphism of complex spaces and let χ be a relative contact transformation over T such that $\eta^*\mathcal{G} = \mathcal{Y}^\chi$. It is enough to show that there are a unique morphism $\eta_0 : T_0 \to \mathbb{C}^n$ and a relative contact transformation χ_0 over T_0 such that

$$\eta_0 \circ \imath_0 = \eta \qquad \text{and} \qquad \eta_0^*\mathcal{G} = \mathcal{Y}_0^{\chi_0}. \tag{2.7.6}$$

Because $\eta^*\mathcal{G} = \mathcal{Y}^\chi$ there is $h \in (s)\mathcal{O}_{\mathbb{C}^2 \times T}$ such that

$$(1 + h)\eta^* G = F^\chi.$$

In order for 2.7.6 to hold, we need to find $a \in \mathbb{C}^n$, $\sigma \in \mathcal{O}_{\mathbb{C}^2}$ and χ_0 such that

$$\eta^0 = \eta + \varepsilon a, \qquad \text{and} \qquad (1 + h + \varepsilon\sigma)\eta_0^* G = F_0^{\chi_0}.$$

By Theorem 2.3.7 there are A, B_0 such that

$$\chi = \chi_{A,B_0}$$

and χ_0 exists if and only if there are α, β_0 such that

$$\chi_0 = \chi_{A+\varepsilon\alpha, B_0+\varepsilon\beta_0}.$$

By Theorem 2.7.4, $F_0^{\chi_0}$ equals (2.7.2). Moreover,

$$\begin{aligned}(1 + h + \varepsilon\sigma)\eta_0^* G &= (1 + h)\eta^* G + \varepsilon\sigma\eta^* G + \varepsilon(1 + h)\textstyle\sum_{i=1}^n a_i g_i \\ &= F^\chi + \varepsilon\sigma f + \varepsilon(1 + h)\textstyle\sum_{i=1}^n a_i g_i.\end{aligned} \tag{2.7.7}$$

Hence we need to solve the equation

$$g(1 + h)^{-1} = \textstyle\sum_{i=1}^n a_i g_i - (1 + h)^{-1}(\varepsilon\sigma f + \alpha_0 f_x + \beta_0 f_y + \sum_\ell \frac{\alpha_\ell}{\ell+1} h_\ell). \tag{2.7.8}$$

Since, as noted in Remark 2.7.9, $g(1 + h)^{-1} \in I_f$ there are unique $a_1, ..., a_n$ such that

$$g(1 + h)^{-1} - \textstyle\sum_{i=1}^n a_i g_i \in I_f^\mu.$$

Hence there are $\alpha_\ell, \beta_0, \sigma$ such that (2.7.8) holds. □

Corollary 2.7.11. *The relative conormal of $\mathcal{G}$ is a semiuniversal deformation of the conormal L of Y on $\widehat{\mathcal{D}ef}_L^{es}$.*

Proof. Suppose $\imath : T' \hookrightarrow T$ is an embedding of complex spaces, $\mathcal{L} \in \widehat{\mathcal{D}ef}_L^{es}(T)$, $\mathcal{L}' = \imath^*\mathcal{L} \in \widehat{\mathcal{D}ef}_L^{es}(T')$. Let $\eta' : T' \to \mathbb{C}^n$ be a morphism of complex spaces and χ' a relative contact transformation such that

$$\chi'(\mathcal{L}') = \eta'^* \mathcal{C}on(\mathcal{G}). \tag{2.7.9}$$

Let $\mathcal{Y}' = \pi(\mathcal{L}')$ and $\mathcal{Y} = \pi(\mathcal{L})$. Equation (2.7.9) implies that $\mathcal{Y}'^{\chi'} = \eta'^*\mathcal{G} \in \mathcal{D}ef_f^{es,\mu}(T')$. Because $\mathcal{G}$ is semiuniversal, there is $\eta : T \to \mathbb{C}^n$ with $\eta' = \eta \circ \imath$ and χ relative contact transformation extending χ' such that $\mathcal{Y}^{\chi} = \eta^*\mathcal{G}$. This means that $\eta^*\mathcal{C}on(\mathcal{G}) = \chi(\mathcal{L})$, hence $\mathcal{C}on(\mathcal{G})$ is semiuniversal. $\square$

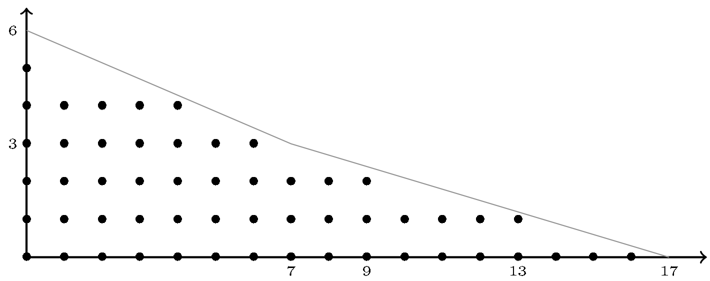

Figure 2.1: Monomial base for $\frac{\mathbb{C}\{x,y\}}{I_{fix}^{es}(f)}$.

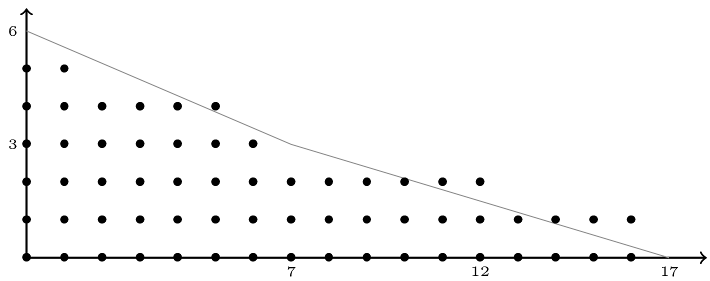

Figure 2.2: Monomial base for $\frac{\mathbb{C}\{x,y\}}{(f,(x,y)f_x,(x^2,y)f_y)}$.

Example 2.7.12. If $f(x,y) = (y^3 + x^7)(y^3 + x^{10})$, f is NND and I_f is generated by the polynomials $x^2 f_y, y f_x$ and $x^i y^j$ such that $3i + 7j \geq 42$ and $3i + 10j \geq 51$ (see Proposition II.2.17 of [9]).

Let us first find a basis for $\underline{\mathcal{D}ef}_f^{\twoheadrightarrow\, es}(T_\varepsilon)$. Then using the h_ℓ's we see which terms can be eliminated in order to get a basis for $\underline{\mathcal{D}ef}_f^{es,\mu}(T_\varepsilon) \cong I_f/I_f^\mu$.

Consider the SINGULAR session (see [5]):

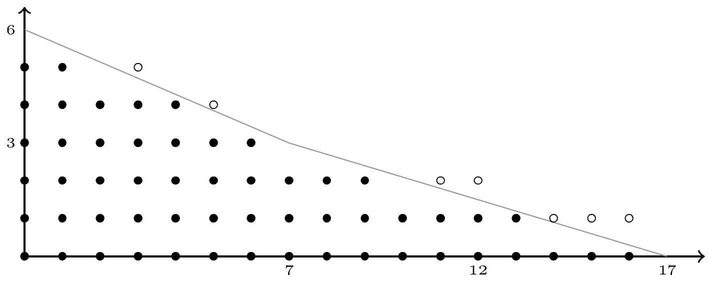

Figure 2.3: Monomial base for $\frac{\mathbb{C}\{x,y\}}{(f,(x,y)f_x,(x^2,y)f_y)}$.

```
> LIB "equising.lib";
> ring r=0,(x,y),ls;
> poly f=(y3+x7)*(y3+x10);
> ideal yJ=f,x*diff(f,x),y*diff(f,x),x2*diff(f,y),y*diff(f,y);
> list Ies=esIdeal(f,1);
polynomial is Newton non-degenerate

//
// equisingularity ideal is computed!
> vdim(std(Ies[2]));        // Ies[2] is the equisingularity ideal // with
fixed trivial section
54
> vdim(std(yJ));
62
```

Let $I^{es}_{fix}(f)$ denote the equisingularity ideal with fixed trivial section. The command `kbase(std(Ies[2]));` provides us with a monomial base (54 monomials) for the vector space (see fig. 2.1)

$$\frac{\mathbb{C}\{x,y\}}{I^{es}_{fix}(f)}.$$

The command `kbase(std(yJ));` provides us with a monomial base (62 monomials) for the vector space (see fig. 2.2).

$$\frac{\mathbb{C}\{x,y\}}{(f,(x,y)f_x,(x^2,y)f_y)}.$$

Note that the monomial xy^5 does not belong to I_f as it changes the tangent cone of f. The monomial $x^{10}y^2$ is under the Newton diagram of f, therefore it can't be a part of an equisingular deformation of f. As x^3y^5 and $x^{10}y^2$ are congruent in $\underline{\mathcal{D}ef}^{\,\overset{\twoheadrightarrow}{es}}_f(T_\varepsilon)$ a monomial base for $\underline{\mathcal{D}ef}^{\,\overset{\twoheadrightarrow}{es}}_f(T_\varepsilon)$ is given by the monomials marked with circles in fig. 2.3. This already tells us that $dim_{\mathbb{C}}\,\underline{\mathcal{D}ef}^{\,es,\mu}_f(T_\varepsilon) \leq 7$, which makes sense, since $dim_{\mathbb{C}}\,\underline{\mathcal{D}ef}^{\,es}_f(T_\varepsilon) = 7$ as can be seen continuing the previous SINGULAR session:

```
> tjurina(f)-tau_es(f);       // tau_es(f) is equal to the
// codimension of the stratum with fixed Milnor number
7
```

A semiuniversal object in $\overrightarrow{\mathcal{D}ef}_f^{es}$ (see Proposition II.2.69, Corollary II.2.71 of [9] and Remark 2.7.13) is given by:

$$f(x,y) + s_1x^3y^5 + s_2x^5y^4 + s_3x^{11}y^2 + s_4x^{12}y^2 + s_5x^{14}y + s_6x^{15}y + s_7x^{16}y.$$

The following SINGULAR session confirms that this deformation is in fact equisingular:

```
> LIB "all.lib";
> ring R = 0,(s1,s2,s3,s4,s5,s6,s7,x,y),ds;
> poly f = (y3+x7)*(y3+x10);
> poly F = f+s1*x3y5+s2*x11y2+s3*x12y2+s4*x14y+s5*x15y+s6*x16y+s7*x5y4;
> list L = esStratum(F);
> L;
[1]:
   [1]:
      [1]=0
   [1]:
      [2]=0
[2]:
   0
```

Using the expressions (for both branches as the h_ℓ's are only unique $mod\, I_Y$) for h_1 we see that $x^{15}y, x^{16}y$ can be eliminated from the basis of $\underline{\overrightarrow{\mathcal{D}ef}}_f^{es}(T_\varepsilon)$. Using the expressions of h_2 we can eliminate $x^{11}y^2, x^{12}y^2$. So, a basis for I_f/I_f^μ is given by the monomials $x^3y^5, x^5y^4, x^{14}y$. According to Theorem 2.7.10, the deformation defined by

$$f(x,y) + s_1x^3y^5 + s_2x^5y^4 + s_3x^{14}y$$

is a semiuniversal deformation of f in $\mathcal{D}ef_f^{es,\mu}$.

Remark 2.7.13. Proposition II.2.69 (b) of [9] should say that *there is* a basis of $I^{es}(f)/(f, j(f))$ such that φ^{es} is a semiuniversal equisingular deformation of $(C,0)$, as for an arbitrary basis the result doesn't always hold. In the example above for instance, the monomial $x^{10}y^2$ belongs to $I^{es}(f)$ but $f + s1x^{10}y^2$ is not equisingular. Thus, in Corollary II.2.71 (a) of [9] the authors should further require that the g_i have Newton order ≥ 1.

Chapter 3

Moduli Spaces of germs of Semiquasihomogeneous Legendrian Curves

3.1 Introduction

Greuel, Laudal, Pfister et all (see [10], [13]) constructed moduli spaces of germs of plane curves equisingular to a plane curve $\{y^k + x^n = 0\}$, $(k, n) = 1$. Their main tools are the Kodaira Spencer map of the equisingular semiuniversal deformation of the curve and the results of [7]. We extend their results to Legendrian curves.

Let Y be the germ of a plane curve that is a generic plane projection of a Legendrian curve L. The equisingularity type of Y does not depend on the projection (see [25]). Two Legendrian curves are equisingular if their generic plane projections are equisingular. We say that an irreducible Legendrian curve L is semiquasihomogeneous if its generic plane projection is equisingular to a quasihomogeneous plane curve $\{y^k+n^n = 0\}$, for some k, n such that $(k, n) = 1$. Hence the generic plane projection of L is a semiquasihomogeneous plane curve.

In section 3.2 we recall the main results of relative contact geometry. In section 3.3 we construct the microlocal Kodaira Spencer map and study its kernel $\mathcal{L}_B$, a Lie algebra of vector fields over the base space $\mathbb{C}^B$ of the semiuniversal equisingular deformation of the plane curve $\{y^k + n^n = 0\}$. We use $\mathcal{L}_B$ in order to construct a Lie algebra of vector fields $\mathcal{L}_C$ over the base space $\mathbb{C}^C$ of the microlocal semiuniversal equisingular deformation of $\{y^k + n^n = 0\}$. In section 3.4 we recall some results of [7]. In section 3.5 we study the stratification of $\mathbb{C}^C$ induced by $\mathcal{L}_C$ and show that the conormals of two fibers $\mathcal{F}_b$, $\mathcal{F}_c$ of the microlocal semiuniversal equisingular deformation of $\{y^k + n^n = 0\}$ are isomorphic if and only if b and c are in the same integral manifold of $\mathcal{L}_C$. Moreover, we construct the moduli spaces. The final section in dedicated to presenting an example.

3.2 Relative contact geometry

Let $q : X \to S$ be a morphism of complex spaces. We can associate to q a coherent $\mathcal{O}_X$-module $\Omega^1_{X/S}$, the *sheaf of relative differential forms of* $X \to S$, and a differential morphism $d : \mathcal{O}_X \to \Omega^1_{X/S}$ (see [11] or [20]).

If $\Omega^1_{X/S}$ is a locally free $\mathcal{O}_X$-module, we denote by $\pi = \pi_{X/S} : T^*(X/S) \to X$ the vector bundle with sheaf of sections $\Omega^1_{X/S}$. We say that $T(X/S)$ $[T^*(X/S)]$ is the *relative tangent bundle* [*cotangent bundle*] of $X \to S$.

Let $\varphi : X_1 \to X_2$, $q_i : X_i \to S$ be morphisms of complex spaces such that $q_2\varphi = q_1$. There is a morphism of $\mathcal{O}_{X_1}$-modules

$$\widehat{\rho}_\varphi : \varphi^*\Omega^1_{X_2/S} = \mathcal{O}_{X_1} \otimes_{\varphi^{-1}\mathcal{O}_{X_2}} \varphi^{-1}\Omega^1_{X_2/S} \to \Omega^1_{X_1/S}. \tag{3.2.1}$$

If $\Omega^1_{X_i/S}$, $i = 1, 2$, and the kernel and cokernel of (3.2.1) are locally free, we have a morphism of vector bundles

$$\rho_\varphi : X_1 \times_{X_2} T^*(X_2/S) \to T^*(X_1/S). \tag{3.2.2}$$

If φ is an inclusion map, we say that the kernel of (3.2.2), and its projectivization, are the *conormal bundle of* X_1 *relative to* S. We will denote by $T^*_{X_1}(X_2/S)$ or $\mathbb{P}^*_{X_1}(X_2/S)$ the conormal bundle of X_1 relative to S.

Assume M is a manifold. When q is the projection $M \times S \to S$ we will replace "$M \times S/S$" by "$M|S$". Let r be the projection $M \times S \to M$. Notice that $\Omega^1_{M|S} \xrightarrow{\sim} \mathcal{O}_{M\times S} \otimes_{r^{-1}\mathcal{O}_M} r^{-1}\Omega^1_M$ is a locally free $\mathcal{O}_{M\times S}$-module. Moreover, $T^*(M|S) = T^*M \times S$.

We say that $\Omega^1_{M|S}$ is the *sheaf of relative differential forms of* M *over* S. We say that $T^*(M|S)$ is the *relative cotangent bundle of* M *over* S.

Let N be a complex manifold of dimension $2n - 1$. Let S be a complex space. We say that a section ω of $\Omega^1_{N|S}$ is a *relative contact form of* N over S if $\omega \wedge d\omega^{n-1}$ is a local generator of $\Omega^{2n-1}_{N|S}$. Let $\mathfrak{C}$ be a locally free subsheaf of $\Omega^1_{N|S}$. We say that $\mathfrak{C}$ is a *structure of relative contact manifold on* N over S if $\mathfrak{C}$ is locally generated by a relative contact form of N over S. We say that $(N \times S, \mathfrak{C})$ is a *relative contact manifold over* S. When S is a point we obtain the usual notion of contact manifold.

Let $(N_1 \times S, \mathfrak{C}_1)$, $(N_2 \times S, \mathfrak{C}_2)$ be relative contact manifolds over S. Let χ be a morphism from $N_1 \times S$ into $N_2 \times S$ such that $q_{N_2} \circ \chi = q_{N_1}$. We say that χ is a *relative contact transformation* of $(N_1 \times S, \mathfrak{C}_1)$ into $(N_2 \times S, \mathfrak{C}_2)$ if the pull-back by χ of each local generator of $\mathfrak{C}_2$ is a local generator of $\mathfrak{C}_1$.

We say that the projectivization $\pi_{X/S} : \mathbb{P}^*(X/S) \to X$ of the vector bundle $T^*(X/S)$ is the *projective cotangent bundle* of $X \to S$.

Let $(x_1, ..., x_n)$ be a partial system of local coordinates on an open set U of X. Let $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ be the associated partial system of symplectic coordinates of $T^*(X/S)$ on $V = \pi^{-1}(U)$. Set $p_{i,j} = \xi_i\xi_j^{-1}$, $i \neq j$,

$$V_i = \{(x, \xi) \in V : \xi_i \neq 0\}, \qquad \omega_i = \xi_i^{-1}\theta, \qquad i = 1, ..., n.$$

each ω_i defines a relative contact form $dx_j - \sum_{i\neq j} p_{i,j} dx_i$ on $\mathbb{P}^*(X/S)$, endowing $\mathbb{P}^*(X/S)$ with a structure of relative contact manifold over S.

Let ω be a germ at (x,o) of a relative contact form of $\mathfrak{C}$. A lifting $\widetilde{\omega}$ of ω defines a germ $\widetilde{\mathfrak{C}}$ of a relative contact structure of $N \times T_oS \to T_oS$. Moreover, $\widetilde{\mathfrak{C}}$ is a lifting of the germ at o of $\mathfrak{C}$.

Let $(N \times S, \mathfrak{C})$ be a relative contact manifold over a complex manifold S. Assume N has dimension $2n-1$ and S has dimension ℓ. Let $\mathcal{L}$ be a reduced analytic set of $N \times S$ of pure dimension $n+\ell-1$. We say that $\mathcal{L}$ is a *relative Legendrian variety* of $N \times S$ over S if for each section ω of $\mathfrak{C}$, ω vanishes on the regular part of $\mathcal{L}$. When S is a point, we say that $\mathcal{L}$ is a *Legendrian variety* of N.

Let $\mathcal{L}$ be an analytic set of $N \times S$. Let $(x,o) \in \mathcal{L}$. Assume S is an irreducible germ of a complex space at o. We say that $\mathcal{L}$ is a *relative Legendrian variety of N over S at (x,o)* if there is a relative Legendrian variety $\widetilde{\mathcal{L}}$ of (N,x) over $(T_oS, 0)$ that is a lifting of the germ of $\mathcal{L}$ at (x,o). Assume S is a germ of a complex space at o with irreducible components $S_i, i \in I$. We say that $\mathcal{L}$ is a *relative Legendrian variety of N over S at (x,o)* if $S_i \times_S \mathcal{L}$ is a relative Legendrian variety of $S_i \times_S N$ over S_i at (x,o), for each $i \in I$.

We say that $\mathcal{L}$ is a *relative Legendrian variety of* $N \times S$ if $\mathcal{L}$ is a relative Legendrian variety of $N \times S$ at (x,o) for each $(x,o) \in \mathcal{L}$.

Let Y be a reduced analytic set of M. Let $\mathcal{Y}$ be a flat deformation of Y over S. Set $X = M \times S \setminus \mathcal{Y}_{\text{sing}}$. We say that the Zariski closure of $\mathbb{P}^*_{\mathcal{Y}_{\text{reg}}}(X/S)$ in $\mathbb{P}^*(M|S)$ is the *conormal* $\mathbb{P}^*_{\mathcal{Y}}(M|S)$ *of* $\mathcal{Y}$ *over* S.

Theorem 3.2.1. *The conormal of $\mathcal{Y}$ over S is a relative Legendrian variety of $\mathbb{P}^*(M|S)$. If $\mathcal{Y}$ has irreducible components $\mathcal{Y}_1, ..., \mathcal{Y}_r$,*

$$\mathbb{P}^*_{\mathcal{Y}}(M|S) = \cup_{i=1}^r \mathbb{P}^*_{\mathcal{Y}_i}(M|S).$$

Theorem 3.2.2. *Let $\mathcal{L}$ be an irreducible germ of a relative Legendrian analytic set of $\mathbb{P}^*(M|S)$. If the analytic set $\pi(\mathcal{L})$ is a flat deformation over S of an analytic set of M, $\mathcal{L} = \mathbb{P}^*_{\pi(\mathcal{L})}(M|S)$.*

Let $\theta = \xi dx + \eta dy$ be the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. Hence $\pi = \pi_{\mathbb{C}^2}$: $\mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ is given by $\pi(x,y;\xi:\eta) = (x,y)$. Let $U\,[V]$ be the open subset of $\mathbb{P}^*\mathbb{C}^2$ defined by $\eta \neq 0\,[\xi \neq 0]$. Then $\theta/\eta\,[\theta/\xi]$ defines a contact form $dy - pdx\,[dx - qdy]$ on $U\,[V]$, where $p = -\xi/\eta\,[q = -\eta/\xi]$. Moreover, $dy - pdx$ and $dx - qdy$ define the structure of contact manifold on $\mathbb{P}^*\mathbb{C}^2$.

If L is a germ of a Legendrian curve of $\mathbb{P}^*M$ and L is not a fiber of π_M, $\pi_M(L)$ is a germ of plane curve with irreducible tangent cone and $L = \mathbb{P}^*_{\pi_M(L)}M$.

Let Y be the germ of a plane curve with irreducible tangent cone at a point o of a surface M. Let L be the conormal of Y. Let σ be the only point of L such that $\pi_M(\sigma) = o$. Let k be the multiplicity of Y. Let f be a defining function of Y. In this situation we will always choose a system of local coordinates (x,y) of M such that the tangent cone $C(Y)$ of Y equals $\{y=0\}$.

Lemma 3.2.3. *The following statements are equivalent:*

1. $mult_\sigma(L) = mult_o(Y)$;
2. $C_\sigma(L) \not\supset (D\pi(\sigma))^{-1}(0,0)$;
3. $f \in (x^2, y)^k$;
4. *if* $t \mapsto (x(t), y(t))$ *parametrizes a branch of* Y, x^2 *divides* y.

Definition 3.2.4. Let S be a reduced complex space. Let Y be a reduced plane curve. Let $\mathcal{Y}$ be a deformation of Y over S. We say that $\mathcal{Y}$ is *generic* if its fibers are generic. If S is a non reduced complex space we say that $\mathcal{Y}$ is *generic* if $\mathcal{Y}$ admits a generic lifting.

Given a flat deformation $\mathcal{Y}$ of a plane curve Y over a complex space S we will denote $\mathbb{P}^*_{\mathcal{Y}}(\mathbb{C}^2|S)$ by $\mathcal{C}on(\mathcal{Y})$.

Theorem 3.2.5 (Theorem 1.3, [4])**.** *Let* $\chi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ *be a germ of a contact transformation. Let* L *be a germ of a Legendrian curve of* $\mathbb{C}^3$ *at the origin. If* L *and* $\chi(L)$ *are in generic position,* $\pi(L)$ *and* $\pi(\chi(L))$ *are equisingular.*

Definition 3.2.6. Two Legendrian curves are *equisingular* if their generic plane projections are equisingular.

Lemma 3.2.7. *Assume* Y *is a generic plane curve and* $Y \hookrightarrow \mathcal{Y}$ *defines an equisingular deformation of* Y *with trivial normal cone along its trivial section. Then* $\mathcal{Y}$ *is generic.*

Definition 3.2.8. Let L be (a germ of) a Legendrian curve of $\mathbb{C}^3$ in generic position. Let $\mathcal{L}$ be a relative Legendrian curve over (a germ of) a complex space S at o. We say that an immersion $i : L \hookrightarrow \mathcal{L}$ defines a *deformation*

$$\mathcal{L} \hookrightarrow \mathbb{C}^3 \times S \to S \tag{3.2.3}$$

of the Legendrian curve L *over* S if i induces an isomorphism of L onto $\mathcal{L}_o$ and there is a generic deformation $\mathcal{Y}$ of a plane curve Y over S such that $\chi(\mathcal{L})$ is isomorphic to $\mathcal{C}on\mathcal{Y}$ by a relative contact transformation verifying (3.2.6).

We say that the deformation (3.2.3) is *equisingular* if $\mathcal{Y}$ is equisingular. We denote by $\widehat{\mathcal{D}ef}^{\,es}_L$ the category of equisingular deformations of L.

Remark 3.2.9. We do not demand the flatness of the morphism (3.2.3).

Lemma 3.2.10. *Using the notations of definition* 3.2.8*, given a section* $\sigma : S \to \mathcal{L}$ *of* $\mathbb{C}^3 \times S \to S$*, there is a relative contact transformation* χ *such that* $\chi \circ \sigma$ *is trivial. Hence* $\mathcal{L}$ *is isomorphic to a deformation with trivial section.*

Consider the maps $i : X \hookrightarrow X \times S$ and $q : X \times S \to S$.

Theorem 3.2.11. *Assume $\mathcal{Y}$ defines an equisingular deformation of a generic plane curve Y with trivial normal cone along its trivial section. Let $\chi : X \times S \to X \times S$ be a relative contact transformation verifying*

$$\chi \circ i = i, \ q \circ \chi = q \ \text{ and } \ \chi(0,s) = (0,s) \ \text{ for each } s.$$

Then $\mathcal{Y}^\chi = \pi\left(\chi(\mathcal{C}on\mathcal{Y})\right)$ is a generic equisingular deformation of Y.

Definition 3.2.12. Let $\mathcal{D}ef_f^{es,\mu}$ (or $\mathcal{D}ef_Y^{es,\mu}$) be the category given in the following way: the objects of $\mathcal{D}ef_f^{es,\mu}$ are the objects of $\overrightarrow{\mathcal{D}ef}_f^{es}$; two objects $\mathcal{Y}, \mathcal{Z}$ of $\mathcal{D}ef_f^{es,\mu}(T)$ are *isomorphic* if there is a relative contact transformation χ over T such that $\mathcal{Z} = \mathcal{Y}^\chi$.

Lemma 3.2.13. *Assume $f \in \mathbb{C}\{x,y\}$ is the defining function of a generic plane curve Y. Let L be the conormal of Y. For each $\ell \geq 1$ there is $h_\ell \in \mathbb{C}\{x,y\}$ such that*

$$(\ell+1)p^\ell f_x + \ell p^{\ell+1} f_y \equiv h_\ell \ mod\ I_L.$$

Moreover, h_ℓ is unique modulo I_Y.

Definition 3.2.14. Let f be a generic plane curve with tangent cone $\{y=0\}$. We will denote by I_f the ideal of $\mathbb{C}\{x,y\}$ generated by the functions g such that $f + \varepsilon g$ is equisingular over T_ε and has trivial normal cone along its trivial section. We call I_f the *equisingularity ideal of f*.

We will denote by I_f^μ the ideal of $\mathbb{C}\{x,y\}$ generated by $f, (x,y)f_x$, $(x^2,y)f_y$ and h_ℓ, $\ell \geq 1$.

Theorem 3.2.15. *Assume Y is a generic plane curve with conormal L, defined by a power series f. Assume f is SQH or f is NND. If $g_1,...,g_n \in I_f$ represent a basis of I_f/I_f^μ with Newton order ≥ 1, the deformation $\mathcal{G}$ defined by*

$$G(x,y,s_1,...,s_n) = f(x,y) + \sum_{i=1}^n s_i g_i \tag{3.2.4}$$

is a semiuniversal deformation of f in $\mathcal{D}ef_f^{es,\mu}$.

Lemma 3.2.16. *Let S be the germ of a complex space. Assume F defines an object $\mathcal{F}$ in $\overrightarrow{\mathcal{D}ef}_f^{es}(S)$. Given $\gamma \geq 1$ there are $H^\gamma \in \mathcal{O}_S\{x,y\}$ such that*

$$H^\gamma \equiv p^\gamma \partial_x F \qquad mod\ I_{Con(\mathcal{F})} + \Delta_F.$$

If f has multiplicity k, $H^\gamma \equiv 0$ for $\gamma \geq k-1$.

Proof. Let us first show that

$$H^\gamma \equiv (\gamma+1)p^\gamma \partial_x F + \gamma p^{\gamma+1}\partial_y F \qquad mod\ I_{Con(\mathcal{F})}.$$

This is a relative version of Lemma 7.2 of [20]. Since $\mathcal{F}$ is equisingular, the multiplicity and the conductor are constant. Moreover, there are parametrizations of each component of $\mathcal{F}$. Therefore, we can generalize the argument in the proof of the quoted Lemma.

Now it is enough to show that

$$\partial_x F + p\partial_y F \equiv 0 \qquad mod\ I_{Con(\mathcal{F})}. \tag{3.2.5}$$

Assume $\mathcal{F}$ is irreducible. Let $(t,s) \mapsto (X,Y,P)$ be a parametrization of $Con(\mathcal{F})$. Since $F(X,Y) = 0$ we conclude that

$$\partial_x F \partial_t X + \partial_y F \partial_t Y = 0.$$

Since $P = \partial_t Y/\partial_t X$, (3.2.5) holds. □

Let T_ε be the complex space with local ring $\mathbb{C}\{\varepsilon\}/(\varepsilon^2)$. Let I, J be ideals of the ring $\mathbb{C}\{s_1, ..., s_m\}$. Assume $J \subset I$. Let X, S, T be the germs of complex spaces with local rings $\mathbb{C}\{x,y,p\}$, $\mathbb{C}\{s\}/I, \mathbb{C}\{s\}/J$. Consider the maps $i : X \hookrightarrow X \times S$, $j : X \times S \hookrightarrow X \times T$ and $q : X \times S \to S$.

Let $\mathfrak{m}_X, \mathfrak{m}_S$ be the maximal ideals of $\mathbb{C}\{x,y,p\}$, $\mathbb{C}\{s\}/I$. Let $\mathfrak{n}_S$ be the ideal of $\mathcal{O}_{X\times S}$ generated by $\mathfrak{m}_X \mathfrak{m}_S$.

Let $\chi : X \times S \to X \times S$ be a relative contact transformation. If χ verifies

$$\chi \circ i = i,\ \ q \circ \chi = q \ \text{ and } \ \chi(0,s) = (0,s) \ \text{ for each } s. \tag{3.2.6}$$

there are $\alpha, \beta, \gamma \in \mathfrak{n}_S$ such that

$$\chi(x,y,p,s) = (x+\alpha, y+\beta, p+\gamma, s). \tag{3.2.7}$$

Theorem 3.2.17. (1) *Let $\chi : X \times S \to X \times S$ be a relative contact transformation that verifies* (3.2.6). *Then γ is determined by α and β. Moreover, there is $\beta_0 \in \mathfrak{n}_S + p\mathcal{O}_{X\times S}$ such that β is the solution of the Cauchy problem*

$$\left(1 + \frac{\partial\alpha}{\partial x} + p\frac{\partial\alpha}{\partial y}\right)\frac{\partial\beta}{\partial p} - p\frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial p}\frac{\partial\beta}{\partial x} = p\frac{\partial\alpha}{\partial p}, \tag{3.2.8}$$

$\beta + p\mathcal{O}_{X\times S} = \beta_0$.
(2) *Given $\alpha \in \mathfrak{n}_S$, $\beta_0 \in \mathfrak{n}_S + p\mathcal{O}_{X\times S}$, there is a unique relative contact transformation χ that verifies* (3.2.6) *and the conditions of statement* (*a*). *We denote χ by χ_{α,β_0}.*
(3) *If $S = T_\varepsilon$ the Cauchy problem* (3.2.8) *simplifies into*

$$\frac{\partial\beta}{\partial p} = p\frac{\partial\alpha}{\partial p}, \qquad \beta + p\mathcal{O}_{X\times T_\varepsilon} = \beta_0. \tag{3.2.9}$$

Consider the contact transformations from $\mathbb{C}^3$ to $\mathbb{C}^3$ given by

$$\Phi(x,y,p) = (\lambda x, \lambda\mu y, \mu p),\ \ \lambda, \mu \in \mathbb{C}^*, \tag{3.2.10}$$

$$\Phi(x,y,p) = (ax+bp, y + \frac{ac}{2}x^2 + \frac{bd}{2}p^2 + bcxp, cx+dp),\ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1, \tag{3.2.11}$$

Theorem 3.2.18. (See [1] or [22].) *Let $\Phi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ the the germ of a contact transformation. Then $\Phi = \Phi_1\Phi_2\Phi_3$, where Φ_1 is of type* (3.2.10)*, Φ_2 is of type* (3.2.11) *and Φ_3 is of type* (3.2.7)*, with $\alpha, \beta, \gamma \in \mathbb{C}\{x, y, p\}$. Moreover, there is $\beta_0 \in \mathbb{C}\{x, y\}$ such that β verifies the Cauchy problem* (3.2.8)*, $\beta - \beta_0 \in (p)$ and*

$$\alpha, \beta, \gamma, \beta_0, \frac{\partial \alpha}{\partial x}, \frac{\partial \beta_0}{\partial x}, \frac{\partial \beta}{\partial p}, \frac{\partial^2 \beta}{\partial x \partial p} \in (x, y, p). \tag{3.2.12}$$

If $D\Phi(0)(\{y = p = 0\}) = \{y = p = 0\}$, $\Phi_2 = id_{\mathbf{C}^3}$.

Proposition 3.2.19. *Let f and g be two microlocally equivalent SQH or NND generic plane curves. Then, f and g have equisingular semiuniversal microlocal deformations with isomorphic base spaces.*

Proof. Let X, Y denote the germs of analytic subsets at the origin of $\mathbb{C}^3$ defined by $Con\, f$ and $Con\, g$ respectively. Let $\chi : \mathbb{C}^3 \to \mathbb{C}^3$ be a contact transformation such that $\chi(Y) = X$ and $\mathcal{X} := (i, \Phi) : X \hookrightarrow \mathbb{C}^3 \times \mathbb{C}^\ell \to \mathbb{C}^\ell$ be a semiuniversal equisingular deformation of X (to see that such an object exists see Theorem 3.2.15 and Chapter 2). Let us show that $(i \circ \chi, \Phi)$ is a semiuniversal equisingular deformation of Y:

Let $\mathcal{Y} := (j, \Psi) : Y \hookrightarrow \mathbb{C}^3 \times \mathbb{C}^k \to \mathbb{C}^k$ be an equisingular deformation of Y. Because $\mathcal{X}$ is versal there is $\varphi : \mathbb{C}^k \to \mathbb{C}^\ell$ such that $\varphi^*\mathcal{X} \cong (j \circ \chi^{-1}, \Psi)$.

$$\begin{tikzcd} Y \arrow[d, hook, "j"] & & X \arrow[ll, "\chi^{-1}"'] \arrow[d, hook, "\varphi^* i"] \\ \mathbb{C}^3 \times \mathbb{C}^k \arrow[d, "\Psi"] & \cong & \mathbb{C}^3 \times \mathbb{C}^k \arrow[d, "\varphi^*\Phi"] \\ \mathbb{C}^k & \cong & \mathbb{C}^k \end{tikzcd} \tag{3.2.13}$$

Then, $(\varphi^* i \circ \chi, \varphi^*\Phi) \cong (j, \Psi)$ which means that $\varphi^*(i \circ \chi, \Phi) \cong \mathcal{Y}$. The result follows from the fact that a semiuniversal deformation is unique up to isomorphism (see Lemma II.1.12 of [9]). □

Recall that, for a SQH or NND generic plane curve f, there is a semiuniversal microlocal equisingular deformation with base space $\mathbb{C}^k$, where k is the the dimension as vector space over $\mathbb{C}$ of I_f / I_f^μ. So, because of Proposition 3.2.19 and Proposition II.2.17 of [9], the following defines an invariant between microlocally equivalent fibers of F.

Definition 3.2.20. Let f be a SQH or NND generic plane curve. Then

$$\widehat{\tau}(f) := dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{I_f^\mu}$$

is the *microlocal Tjurina* number of f.

3.3 The microlocal Kodaira-Spencer map

Assume k, n are coprime integers, $0 < 2k < n$. Set $f = y^k - x^n$, $\mu = (n-2)(k-2)$. Consider in $\mathbb{C}[x, y]$ the grading given by $o(x^i y^j) = ki + nj$, $(i, j) \in \mathbb{N}^2$. Set $\omega = o(x^{n-2}y^{k-2}) - kn$, $\varpi = o(x^{n-k}y^{k-2}) - kn$, $e(x^i y^j) = (i, j) \in \mathbb{N}^2$,

$$\begin{aligned} B &= \{(i,j) \in \mathbb{N}^2 : i \leq n-2,\ j \leq k-2\}, \\ C &= \{(i,j) \in B : i + j \leq n-2\}, \\ D &= \{(i,j) \in B : o(x^i y^j) - kn \leq \varpi\}, \\ A_0 &= \{(i,j) \in A : ki + nj > kn\}, \text{for each } A \subseteq B. \end{aligned}$$

Let $m_1, \ldots, m_\mu$ be the family $x^i y^j$, $(i, j) \in B$, ordered by degree. Set $b = \# B_0$. If $\mu - b + 1 \leq \ell \leq \mu$, set $o(\ell) = o(m_\ell) - kn$ and $o(s_{o(\ell)}) = -o(\ell)$.

Let $A \subseteq B$. Set $I_A = \{\ell : e(m_\ell) \in A_0\}$, $s_A = \left(s_{o(\ell)}\right)_{\ell \in I_A}$. Set $\mathbb{C}^A = \mathbb{C}^{\# A_0}$ with coordinates s_A. Notice that $I_B = \{\mu - b + 1, \ldots, \mu\}$. Moreover,

$$F_A = f + \sum_{\ell \in I_A} s_{o(\ell)} m_\ell$$

is homogeneous of degree kn.

Let Y be the plane curve defined by f. Let Γ be the conormal of Y. Let $\mathcal{F}_A$ be the deformation defined by F_A. Notice that

- $\mathcal{F}_B$ is a semiuniversal equisingular deformation of Y,
- $\mathcal{F}_C$ is a semiuniversal equisingular microlocal deformation of Y,
- if $C \subseteq A \subseteq B$, $\mathcal{F}_A$ is a complete equisingular microlocal deformation of Y.

Let Δ_{F_A} be the ideal of $\mathbb{C}[s_A]$ generated by $\partial_x F_A$ and $\partial_y F_A$. Assume $o(p) = n - k$ in order to guarantee that the contact form $dy - pdx$ is homogeneous.

Lemma 3.3.1. *Assume $C \subseteq A \subseteq B$ and $\gamma \geq 1$. There is $H_A^\gamma \in \mathbb{C}[s_A]\{x, y\}$ such that $H_A^\gamma \equiv p^\gamma \partial_x F_A \bmod I_{Con(\mathcal{F}_A)} + \Delta_{F_A}$ where H_A^γ is homogeneous of degree $\gamma(n-k) + kn - k$. If $\gamma \geq k-1$, $H_A^\gamma \in \Delta_{F_A}$. If $C \subseteq A' \subseteq A \subseteq B$, $H_{A'}^\gamma = H_A^\gamma|_{\mathbb{C}^{A'}}$.*

Proof. Set $\psi_0 = \theta$, where $\theta^k = -1$. There are $\psi_i \in (s_A)\mathbb{C}[s_A]$, $i \geq 1$, such that

$$X(t, s_A) = t^k, \qquad Y(t, s_A) = \sum_{i \geq 0} \psi_i t^{n+i}$$

defines a parametrization Φ of $\mathcal{F}_A$. Setting $P(t, s_A) = \sum_{i \geq 0} \frac{n+i}{k} \psi_i t^{n-k+i}$, X, Y, P defines a parametrization Ψ of $Con(\mathcal{F}_A)$. Since x is homogeneous of degree k and $x = t^k$, we assume t homogeneous of degree 1. Let us show that Y is homogeneous of degree n. The $\mathbb{C}^*$-action acts on Φ by

$$a \cdot \Phi(t, s_A) = \left(a^k t^k, a^n \left(\theta t^n + \sum_{i \geq 1} (a \cdot \psi_i) a^i t^{n+i} \right) \right).$$

Since F_A is homogeneous, for each s_A,

$$t \mapsto \Phi_a(t, s_A) = \left(t^k, \theta t^n + \sum_{i \geq 1} (a \cdot \psi_i) a^i t^{n+i} \right)$$

is another parametrization of the curve defined by $(x, y) \mapsto F_A(x, y, s_A)$. Since the first term of both parametrizations coincide, $\Phi_a = \Phi$, $a \cdot \psi_i = a^{-i}\psi_i$ and Φ is homogeneous. Therefore, Ψ is homogeneous.

There is an integer c such that $\Phi^*(\Delta_{F_A}) \supset t^c\mathbb{C}[s_A]\{t\}$. Remark that $p^\gamma \partial_x F_A$ is homogeneous of degree $\gamma(n-k) + kn - k$. We construct H_A^γ in the following manner. There is a monomial ax^iy^j, $a \in \mathbb{C}[s_A]$ such that the monomials of lowest t-order $\Phi^*(ax^iy^j)$ and $\Psi^*(p^\gamma \partial_x F_A)$ coincide. Replace $p^\gamma \partial_x F_A$ by $p^\gamma \partial_x F_A - ax^iy^j$ and iterate the procedure. After a finite number of steps we construct H_A^γ such that

$$\Psi^*(p^\gamma \partial_x F_A - H_A^\gamma) \in t^c\mathbb{C}[s_A]\{t\}.$$

Therefore,

$$p^\gamma \partial_x F_A - H_A^\gamma \in I_{Con(\mathcal{F}_A)} + \Delta_{F_A}.$$

Remark that the monomial ax^iy^j is homogeneous of degree $\gamma(n-k) + kn - k$. □

Set $\Theta_B = Der_{\mathbb{C}}\mathbb{C}[s_B]$, $\partial_{o(\ell)} = \partial_{s_{o(\ell)}}$ and $o(\partial_{o(\ell)}) = o(\ell)$ for each $\ell \in I_B$. Assume $C \subseteq A' \subseteq A \subseteq B$. Let $\Theta_{A,A'}$ be the $\mathbb{C}[s_A]$-submodule of Θ_B generated by $\partial_{o(\ell)}$, $\ell \in I_{A'}$. Set $\Theta_A = \Theta_{A,A}$. There are maps

$$\Theta_A \hookleftarrow \Theta_{A,A'} \xrightarrow{r_{A,A'}} \Theta_{A'},$$

where $r_{A,A'}$ is the restriction to $\mathbb{C}^{A'}$.

Definition 3.3.2. Let I_F^μ be the ideal of $\mathbb{C}[s_B][[x, y]]$ generated by F_B, ΔF_B and H_B^γ, $\gamma = 1, \ldots, k-2$. We say that the map

$$\rho : \Theta_B \to \mathbb{C}[s_B][[x, y]]/I_F^\mu,$$

given by $\rho(\delta) = \delta F_B + I_F^\mu$ is the *microlocal Kodaira-Spencer map* of f. We will denote the kernel of ρ by $\mathcal{L}_B$.

Assume we have defined $\mathcal{L}_A$. We set

$$\mathcal{L}_{A,A'} = \mathcal{L}_A \cap \Theta_{A,A'} \text{ and } \mathcal{L}_{A'} = r_{A,A'}(\mathcal{L}_{A,A'}).$$

Let L be a Lie subalgebra of Θ_A. Consider in $\mathbb{C}^A$ the binary relation $\sim$ given by $p \sim q$ if there is a vector field δ of L and an integral curve γ of δ such that p and q are in the trajectory of γ. We denote by L the equivalence relation generated by $\sim$. We say that a subset M of $\mathbb{C}^A$ is an integral manifold of L if M is an equivalence class of L.

Assume $C \subseteq A \subseteq B$. The family m_ℓ, $1 \leq \ell \leq \mu$, defines a basis of the $\mathbb{C}[s_A]$-module

$$R_A = \mathbb{C}[s_A][[x,y]]/\Delta F_A.$$

Set $H_A^0 = F_A$. The relations

$$m_\ell H_A^\gamma \equiv \sum_{\upsilon=1}^{\mu} c_{\ell,\upsilon}^\gamma m_\upsilon \qquad mod\ \Delta F_A \tag{3.3.1}$$

define $c_{\ell,\upsilon}^\gamma \in \mathbb{C}[s_A]$ for each $0 \leq \gamma \leq k-2$, $1 \leq \ell, \upsilon \leq \mu$. Assume $A = B$ and set

$$\delta_\ell^\gamma = \sum_{\upsilon=\mu-b+1}^{\mu} c_{\ell,\upsilon}^\gamma \partial_{s_{o(\upsilon)}}, \qquad \ell = 1, \ldots, \mu,\ \gamma = 0, \ldots, k-2. \tag{3.3.2}$$

If $m_\ell = x^i y^j$ we will also denote δ_ℓ^γ by $\delta_{i,j}^\gamma$. For $1 \leq \gamma \leq k-2$, set

$$\begin{array}{ll} \alpha_\ell^0 = o(m_\ell), \quad \alpha_\ell^\gamma = \alpha_\ell^0 + \gamma(n-k) - k, & \ell = 1, \ldots, \mu, \\ \alpha_{i,j}^0 = o(x^i y^j), \quad \alpha_{i,j}^\gamma = \alpha_{i,j}^0 + \gamma(n-k) - k, & (i,j) \in B. \end{array}$$

Lemma 3.3.3. *With the previous notations, we have that:*

1. *The vector fields δ_ℓ^γ ($\delta_{i,j}^\gamma$) are homogeneous of degree α_ℓ^γ ($\alpha_{i,j}^\gamma$), $0 \leq \gamma \leq k-2$, $1 \leq \ell \leq \mu$ ($(i,j) \in B$).*
2. *$\delta_{i,j}^\gamma(0) \neq 0$ if and only if $\gamma \geq 1$, $i \leq \gamma - 1$, $\gamma + j \leq k-2$.*
3. *$\delta_{i,j}^\gamma = 0$ if $\alpha_{i,j}^\gamma > \omega$.*
4. *The Lie algebra $\mathcal{L}_B$ is generated as $\mathbb{C}[s_B]$-module by $\{\delta_\ell^\gamma : 0 \leq \gamma \leq k-2,\ \alpha_\ell^\gamma \leq \omega\}$.*
5. *If $\sigma > \varpi$, $\partial_{s_\sigma} \in \mathcal{L}_B$.*
6. *If $(u,v) \in B \setminus C$ there is $\delta \in \mathcal{L}_B$ such that $\delta = \partial_{s_\sigma} + \varepsilon$ is homogeneous of degree $\sigma = ku + nv - kn$, where ε is a linear combination of $\partial_{s_{o(i)}}$, $i \in I_B$, $i > \sigma$, with coefficients in $\mathbb{C}[s_B]$.*

Proof. (3): Just notice that if $\alpha_{i,j}^0 > \omega = o(m_\mu) - kn$ then $o(m_{i,j}F_B) > o(m_\mu)$. Now, because $n > 2k$, $o(H_B^\gamma) > kn = o(F_B)$ for any $\gamma = 1, \ldots, k-2$, the result holds for $\gamma > 0$.

(4): For $\gamma = 0$ ($1 \leq \gamma \leq k-2$) and each $\ell = 1, \ldots, \mu$ such that $o(m_\ell) \leq \omega$, we have that $\rho(\delta_\ell^\gamma) = \delta_\ell^\gamma F_B + I_F^\mu = m_\ell F_B + I_F^\mu$ ($m_\ell H_B^\gamma + I_F^\mu$) $= 0 + I_F^\mu$. So, $\{\delta_\ell^\gamma : 0 \leq \gamma \leq k-2,\ \alpha_\ell^\gamma \leq \omega\} \subset \mathcal{L}_B$.

Now, let

$$\delta = \sum_{\upsilon=\mu-b+1}^{\mu} w_\upsilon \partial_{s_{o(\upsilon)}} \in \Theta_B$$

such that $\rho(\delta) = 0$. Then

$$\delta F_B = \sum_{\upsilon=\mu-b+1}^{\mu} w_\upsilon m_\upsilon = M_0 F_B + M_1 H_B^1 + \ldots + M_{k-2} H_B^{k-2} \; mod \; \Delta F_B,$$

with $M_0, \ldots, M_{k-2} \in \mathbb{C}[s_B][[x,y]]$. Suppose

$$\begin{aligned} M_0 =& \sum_{\ell=1}^{\mu} M_{0,\ell} m_\ell \; mod \; \Delta F_B, \\ & \ldots \\ M_{k-2} =& \sum_{\ell=1}^{\mu} M_{k-2,\ell} m_\ell \; mod \; \Delta F_B, \end{aligned}$$

where the $M_{\gamma,\ell} \in \mathbb{C}[s_B]$ for each $\ell = 1, \ldots, \mu$, $\gamma = 0, \ldots, k-2$. Then

$$\begin{aligned} M_0 F_B =& M_{0,1} m_1 F_B + \ldots + M_{0,\mu} m_\mu F_B \; mod \; \Delta F_B \\ =& M_{0,1} m_1 F_B + \ldots + M_{0,b} m_b F_B \; mod \; \Delta F_B \\ =& M_{0,1} \delta_1^0 F_B + \ldots + M_{0,b} \delta_b^0 F_B \; mod \; \Delta F_B. \end{aligned}$$

Similarly, for any $\gamma = 1, \ldots, k-2$

$$\begin{aligned} M_\gamma H_B^\gamma =& M_{\gamma,1} m_1 H_B^\gamma + \ldots + M_{\gamma,b} m_b H_B^\gamma \; mod \; \Delta F_B \\ =& M_{\gamma,1} \delta_1^\gamma F_B + \ldots + M_{\gamma,b} \delta_b^\gamma F_B \; mod \; \Delta F_B. \end{aligned}$$

So,

$$\delta F_B = \sum_{\gamma=0}^{k-2} \sum_{\ell=1}^{b} M_{\gamma,\ell} \delta_\ell^\gamma F_B \; mod \; \Delta F_B,$$

which means that

$$\delta = \sum_{\gamma=0}^{k-2} \sum_{\ell=1}^{b} M_{\gamma,\ell} \delta_\ell^\gamma.$$

□

Let L_B be the Lie algebra generated by δ_ℓ^γ, $\gamma = 0, \ldots, k-2$, $\ell = 1, \ldots, b$. Remark that $\mathbb{C}^B/\mathcal{L}_B \cong \mathbb{C}^B/L_B$. Consider a matrix with lines given by the coefficients of the vector fields δ_ℓ^γ, $\gamma = 0, \ldots, k-2$, $\ell = 1, \ldots, b$. After performing Gaussian diagonalization we can assume that:

- For each $\sigma \in I_B \setminus I_C$ there is a line corresponding to a vector field $\partial_{s_{o(\sigma)}} + \varepsilon$, where $\varepsilon \in \Theta_{B,C}$.
- The remaining lines correspond to vector fields δ'_ℓ, $\ell \in J$, of $\Theta_{B,C}$.

The vector fields δ'_ℓ, $\ell \in J$, generate $\mathcal{L}_{B,C}$ as a $\mathbb{C}[s_B]$-module. Let δ_ℓ be the restriction of δ'_ℓ to $\mathbb{C}^C$ for each $\ell \in J$. The vector fields δ_ℓ, $\ell \in J$, generate $\mathcal{L}_C$ as $\mathbb{C}[s_C]$-module. Note that $\{\delta_\ell,\ \ell \in J\}$ is in general not uniquely determined but the $\mathbb{C}[s_C]$-module generated by them is. Let L_C be the Lie algebra generated by $\{\delta_\ell,\ \ell \in J\}$. Since $L_C \subseteq L_B$ the inclusion map $\mathbb{C}^C \hookrightarrow \mathbb{C}^B$ defines a map $\mathbb{C}^C/L_C \to \mathbb{C}^B/L_B$. By statement (6) of Lemma 3.3.3, this map is surjective.

Assume there is a vector field δ_ℓ, $\ell \in J$, of order α. Let $\{\delta^{\alpha,i} : i \in I_\alpha\}$ be the set of vector fields δ_ℓ, $\ell \in J$, of order α, with $I_\alpha = \{1, \ldots, \# I_\alpha\}$. If there is ℓ_0 such that $\delta^{\alpha,j}(s_\ell) = 0$ for $\ell \leq \ell_0$ and $\delta^{\alpha,i}(s_{\ell_0}) \neq 0$, we assume that $i < j$. If $I_\alpha = \{1\}$, set $\delta^\alpha = \delta^{\alpha,1}$.

Remark 3.3.4. If $k = 7, n = 15$, we have that a semiuniversal equisingular microlocal deformation of f given by

$$\begin{aligned} F_C = {} & y^7 + x^{15} + s_2 x^{11} y^2 + s_3 x^9 y^3 + s_4 x^7 y^4 + s_5 x^5 y^5 + s_{10} x^{10} y^3 + s_{11} x^8 y^4 \\ & + s_{12} x^6 y^5 + s_{18} x^9 y^4 + s_{19} x^7 y^5 + s_{26} x^8 y^5. \end{aligned}$$

Notice that the vector fields $\delta^0_{0,1}$ and $\delta^1_{2,0}$ give origin to the linearly independent vector fields

$$\delta^{15,1} = 3 s_3 \partial_{s_{18}} + 4 s_4 \partial_{s_{19}} + \cdots .$$

and

$$\delta^{15,2} = \left(\frac{7^2}{15} \left(\frac{4}{7} \psi_2^2 - 3 \left(\frac{15}{7} \right)^2 s_4 \right) - 4 s_4 \right) \partial_{s_{19}} + \cdots .$$

Theorem 3.3.5. *The map $\mathbb{C}^C/L_C \to \mathbb{C}^B/L_B$ is bijective.*

Proof. Let I_p be the subset of I_B that contains I_C and the p smallest elements of $I_B \setminus I_C$. Set $C_p = \{(i,j) \in B : ki + nj - kn \in I_p\}$. The Lie algebra $L_{C_p} = \mathcal{L}_{C_p} \cup L_B$ generates $\mathcal{L}_{C_p}$ as $\mathbb{C}[s_{C_p}]$-module. There is p such that $C_p = D$. By statement (5) of Lemma 3.3.3 the integral manifolds of L_B are of the type $M \times \mathbb{C}^{B \setminus D}$, where M is an integral manifold of L_{C_p}. Therefore, $\mathbb{C}^D/L_D \cong \mathbb{C}^B/L_B$. Assume $\mathbb{C}^{C_{p+1}}/L_{C_{p+1}} \cong \mathbb{C}^B/L_B$ and $I_{C_{p+1}} \setminus C_p = \{\sigma\}$. The Lie algebra $\mathcal{L}_{C_{p+1}}$ is generated by $\mathcal{L}_{C_p}$ and a vector field $\partial_{s_{o(\sigma)}} + \varepsilon$, where $\varepsilon \in \mathcal{L}_{C_{p+1},C}$. Consider the flow of $\partial_{s_{o(\sigma)}} + \varepsilon$ with initial condition at a point of $\mathbb{C}^{C_p}$. We can use this flow to construct an homogeneous affine isomorphism of $\mathbb{C}^{C_{p+1}}$ into itself that equals the identity on $\mathbb{C}^{C_p}$ and rectifies $\partial_{s_{o(\sigma)}} + \varepsilon$, leaving invariant $\mathcal{L}_{C_p}$. Hence, $\mathbb{C}^{C_p}/L_{C_p} \cong \mathbb{C}^{C_{p+1}}/L_{C_{p+1}}$.

□

Remark 3.3.6. Let us denote by $P(s_C)$ the restriction of $P \in \mathbb{C}[s_B][[x,y]]$ to $\mathbb{C}^C$. Then, $F_B(s_C) = F_C$, $\Delta F_B(s_C) = \Delta F_C$ and $H_B^\gamma(s_C) = H_C^\gamma$ for each $\gamma = 1, \ldots, k-2$. Let $\{\delta_{\ell,\mu},\ \ell \in J\} \subset Der_{\mathbb{C}}\, \mathbb{C}[s_C]$ be the set of vector fields obtained if we proceed as in the definition of $\{\delta'_\ell,\ \ell \in J\}$, now with C in the place of B. Then $< \{\delta_{\ell,\mu}\} > = < \{\delta_\ell\} >$ as

$\mathbb{C}[s_C]$-modules. To see this just notice that, if

$$m_i F_B = \sum_{j=1}^{\mu} c_{i,j}^0 m_j \; mod \, \Delta F_B$$
$$m_i H_B^\gamma = \sum_{j=1}^{\mu} c_{i,j}^\gamma m_j \; mod \, \Delta F_B$$

then

$$m_i F_B(s_C) = \sum_{j=1}^{\mu} c_{i,j}^0(s_C) m_j \; mod \, \Delta F(s_C)$$
$$m_i H_B^\gamma(s_C) = \sum_{j=1}^{\mu} c_{i,j}^\gamma(s_C) m_j \; mod \, \Delta F(s_C).$$

3.4 Geometric Quotients of Unipotent Group Actions

An affine algebraic group is said to be *unipotent* if it is isomorphic to a group of upper triangular matrices of the form $Id + \varepsilon$, where ε is nilpotent. If $\mathcal{G}$ is unipotent its Lie algebra L is nilpotent and the map $exp : L \to \mathcal{G}$ is algebraic. Given a nilpotent Lie algebra L, there is a unipotent group $\mathcal{G} = exp\ L$ such that L is the Lie algebra of $\mathcal{G}$.

Let A be a Noetherian $\mathbb{C}$-algebra. A linear map $D : A \to A$ is a derivation of A if $D(fg) = fD(g) + gD(f)$. A derivation D of A is nilpotent if for each $f \in A$ there is n such that $D^n(f) = 0$. Let $Der^{nil}(A)$ denote the Lie algebra of nilpotent derivations of A. Here, we set $A = \mathbb{C}[s_C]$.

Let $\mathcal{G}$ be an algebraic group acting algebraically on an algebraic variety X. If Y is an algebraic variety and $\pi : X \to Y$ a morphism then π is called a *geometric quotient*, if

1. π is surjective and open,
2. $(\pi_* \mathcal{O}_X)^{\mathcal{G}} = \mathcal{O}_Y$,
3. π is a orbit map, i.e. the fibres of π are orbits of $\mathcal{G}$.

If a geometric quotient exists it is uniquely determined and we just say that $X/\mathcal{G}$ exists. Here, $\mathcal{G}$ will act on each strata of $\mathbb{C}^c = Spec\, A$ through the action of $\mathcal{G}$ on each fiber of G. On Theorem 3.5.3 we prove that $\mathbb{C}^c/_c$ is a classifying space for germs of Legendrian curves with generic plane projection $\{y^k + x^n = 0\}$. The integral manifolds of L_C are the orbits of the action of $\mathcal{G}_0 := exp\, L_C$. Set $L := [L_C, L_C]$ and $\mathcal{G} = exp\, L$. Note that L is nilpotent ($\mathcal{G}$ unipotent) and $L_C/L \cong \mathbb{C}\delta_0$, where δ_0 is the Euler field.

Definition 3.4.1. Let $\mathcal{G}$ be a unipotent algebraic group, $Z = Spec\, A$ an affine $\mathcal{G}$-variety and $X \subseteq Z$ open and $\mathcal{G}$-stable. Let $\pi : X \to Y := Spec\, A^{\mathcal{G}}$ be the canonical map. A point $x \in X$ is called *stable* under the action of $\mathcal{G}$ *with respect to* A (*or with respect to*

Z) if the following holds:
There exists an $f \in A^{\mathcal{G}}$ such that $x \in X_f = \{y \in X, f(y) \neq 0\}$ and $\pi : X_f \to Y_f := Spec\, A_f^{\mathcal{G}}$ is open and an orbit map. If $X = Z = Spec\, A$ we call a point stable with respect to A just *stable*.

Let $X^s(A)$ denote the set of stable points of X (under $\mathcal{G}$ with respect to A).

Proposition 3.4.2 ([7]). *With the previous notations, we have that:*

1. *$X^s(A)$ is open and $\mathcal{G}$-stable.*
2. *$X^s(A)/\mathcal{G}$ exists and is a quasiaffine algebraic variety.*
3. *If $V \subset Spec\, A^{\mathcal{G}}$ is open, $U = \pi^{-1}(V)$ and $\pi : U \to V$ is a geometric quotient then $U \subset X^s(A)$.*
4. *If X is reduced then $X^s(A)$ is dense in X.*

Definition 3.4.3. A geometric quotient $\pi : X \to Y$ is *locally trivial* if an open covering $\{V_i\}_{i \in I}$ of Y and $n_i \geq 0$ exist, such that $\pi^{-1}(V_i) \cong V_i \times A_{\mathbb{C}}^{n_i}$ over V_i.

We use the following notations:
Let $L \subseteq Der^{nil}(A)$ be a nilpotent Lie-algebra and $d : A \to Hom_{\mathbb{C}}(L, A)$ the differential defined by $da(\delta) = \delta(a)$. If $B \subset A$ is a subalgebra then $\int B := \{a \in A : \delta(a) \in B \text{ for all } \delta \in L\}$. If $\mathfrak{a} \subset A$ is an ideal, $V(\mathfrak{a})$ denotes the closed subscheme $Spec\, A/\mathfrak{a}$ of $Spec\, A$ and $D(\mathfrak{a})$ the open subscheme $Spec\, A - V(\mathfrak{a})$.

Let A be a noetherian $\mathbb{C}$-algebra and $L \subseteq Der^{nil}(A)$ a finite dimensional nilpotent Lie-algebra. Suppose that $A = \cup_{i \in \mathbb{Z}} F^i(A)$ has a filtration

$$F^\bullet : 0 = F^{-1}(A) \subset F^0(A) \subset F^1(A) \subset \dots$$

by sub-vector spaces $F^i(A)$ such that

$$(F) \qquad \delta F^i(A) \subseteq F^{i-1}(A) \text{ for all } i \in \mathbb{Z} \text{ and all } \delta \in L.$$

Assume, furthermore, that

$$Z_\bullet : L = Z_0(L) \supseteq Z_1(L) \supseteq \dots \supseteq Z_\ell(L) \supseteq Z_{\ell+1}(L) = 0$$

is filtered by sub-Lie-algebras $Z_j(L)$ such that

$$(Z) \qquad [L, Z_j(L)] \supseteq Z_{j+1}(L) \text{ for all } j \in \mathbb{Z}.$$

The filtration $Z_\bullet$ of L induces projections

$$\pi_j : Hom_{\mathbb{C}}(L, A) \to Hom_{\mathbb{C}}(Z_j(L), A).$$

For a point $t \in Spec\, A$ with residue field $\kappa(t)$ let

$$r_i(t) := dim_{\kappa(t)} AdF^i(A) \otimes_A \kappa(t) \qquad i = 1, \dots, \rho,$$

with ρ minimal such that $AdF^{\rho}(A) = AdA$,

$$s_i(t) := dim_{\kappa(t)} \pi_j(AdA) \otimes_A \kappa(t) \qquad j = 1, \dots, \ell,$$

such that $s_j(t)$ is the orbit dimension of $Z_j(L)$ at t.

Let $Spec\, A = \cup U_\alpha$ be the flattening stratification of the modules

$$Hom_{\mathbb{C}}(L, A)/AdF^i(A), \qquad i = 1, \dots, \rho$$

and

$$Hom_{\mathbb{C}}(Z_j(L), A)/\pi_j(AdA), \qquad j = 1, \dots, \ell.$$

Theorem 3.4.4 ([7]). *Each stratum U_α is invariant by L and admits a locally trivial geometric quotient with respect to the action of L. The functions $r_i(t)$ and $s_i(t)$ are constant along U_α. Let $x_1, \dots, x_p \in A, \delta_1, \dots, \delta_q \in L$ satisfying the following properties:*

- *there are $\nu_1, \dots, \nu_\rho, 0 \leq \nu_1 < \dots < \nu_\rho = p$, such that $dx_1, \dots, dx_{\nu_i}$ generate the A-module $AdF^i(A)$;*
- *there are $\mu_0, \dots, \mu_\ell, 1 = \mu_0 < \mu_1 < \dots < \mu_\ell$ such that $\delta_{\mu_j}, \dots, \delta_m \in Z_j(L)$ and $Z_j(L) \subseteq \sum_{i \geq \mu_j} A\delta_i$.*

Then

$$rank(\delta_\alpha(x_\beta)(t))_{\beta \leq \nu_i} = r_i(t) \qquad i = 1, \dots, \rho, \tag{3.4.1}$$
$$rank(\delta_\alpha(x_\beta)(t))_{\alpha \geq \mu_j} = s_j(t) \qquad j = 1, \dots, \ell. \tag{3.4.2}$$

The strata U_α are defined set theoretically by fixing (3.4.1) and (3.4.2).

3.5 Filtrations and Strata

Set $L = [L_C, L_C]$. Fix a integer a such that $k \geq a \geq 0$. For each $i \in \mathbb{Z}$ let F_a^i be the $\mathbb{C}$-vector space generated by monomials in $\mathbb{C}[s_C]$ of degree $\geq -(a + ik)$. Since $o(\delta) \geq k$ for each homogeneous vector field of L, $LF_a^j \subseteq F_a^{j-1}$ for each j. For each $m \in \mathbb{Z}$ let I_a^m be the ideal of $\mathbb{C}[[x, y]]$ generated by the monomials of degree $\geq a + mk$. Let ρ be the smallest i such that dF_a^i generates $\mathbb{C}[s_C]d\mathbb{C}[s_C]$ as a $\mathbb{C}[s_C]$-module.

Given $\alpha \in \mathbb{Z}$, set $\alpha^\vee := nk - k^2 - 2n - \alpha$. For each integer j set $S_j = \{\alpha : s_{\alpha^\vee} \in F_a^{\rho-j},\ \alpha \neq 0\}$ and let Z_j^a be the sub-Lie algebra of L generated by the homogeneous vector fields $\delta \in L$ such that $o(\delta) \in S_j$. Remark that

$$Z_1^a = L,\ Z_{\rho+1}^a = 0 \qquad \text{and} \qquad [L, Z_j^a] \subseteq Z_{j+1}^a.$$

For each $t \in \mathbb{C}^C$ let I_t^μ be the ideal of $\mathbb{C}[[x, y]]$ generated by $F_t, \Delta F_t$ and $H_t^1, \dots, H_t^{k-2}$. Set

$$\widehat{\tau}_{a,1}^m(t) = dim_{\mathbb{C}}\, \mathbb{C}[[x, y]]/(I_t^\mu, I_a^m),$$
$$\widehat{\tau}_{a,2}^m(t) = dim_{\mathbb{C}}\, \mathbb{C}[[x, y]]/(\Delta F_t, (F_t, H_t^1, \dots, H_t^{k-2}) \cap I_a^{\rho-1+2n-m}),$$

for $m = n, \ldots, n+\rho$ and

$$\widehat{\tau}_a^{\bullet}(t) = (\widehat{\tau}_{a,1}^{n}(t), \ldots, \widehat{\tau}_{a,1}^{n+\rho}(t); \widehat{\tau}_{a,2}^{n}(t), \ldots, \widehat{\tau}_{a,2}^{n+\rho}(t)).$$

We say that $\widehat{\tau}_a^{\bullet}(t)$ is the *microlocal Hilbert function* of X_t. Set

$$\begin{aligned}
\widehat{\mu} &= \#C = \mu - (k-2)(k-1)/2, \\
\widehat{\mu}_1^k &= \widehat{\mu} - \#\{m_\ell \in I_a^k : \ell \in I_C\}, \\
\widehat{\mu}_2^k &= \mu - \#\{m_\ell \in I_a^{\rho-1+2n-k} : \ell \in I_{B\setminus C}\}.
\end{aligned}$$

We only define $\widehat{\tau}_a^{\bullet}(t)$ for $m = n, \ldots, n+\rho$ because

$$\widehat{\tau}_{a,1}^{m}(t) = \widehat{\tau}_{a,2}^{m}(t) = \widehat{\tau}(X_t)$$

(the microlocal Tjurina number of X_t) if m is big and

$$\widehat{\tau}_{a,1}^{m}(t) = dim_{\mathbb{C}}\, \mathbb{C}[[x,y]]/I_a^m, \qquad \widehat{\tau}_{a,2}^{m}(t) = \widehat{\mu}_2^m$$

(hence independent of t) if m is small.

Let $\{U_\alpha^a\}$ be the flattening stratification of $\mathbb{C}^C$ corresponding to $F_a^{\bullet}$ and $Z_{\bullet}^a$. It follows from Theorem 3.4.4 that $U_\alpha^a \to U_\alpha^a/L$ is a geometric quotient. Moreover, $L_C/L \cong \mathbb{C}^*$ acts on U_α^a/L and $U_\alpha^a/\mathcal{L}_C = U_\alpha^a/L_C$ is a geometric quotient of U_α^a by L_C. For $t \in \mathbb{C}^C$ let us define

$$\underline{e}^a(t) = (u_0^a(t), \ldots, u_\rho^a(t); v_0^a(t), \ldots, v_\rho^a(t)) \in \mathbb{N}^{2\rho+2},$$

where

$$u_j^a(t) = rank(\delta({s'}_\beta)(t))_{o(r(\beta)) \le a+jk}, \qquad j = 0, \ldots, \rho,$$

and

$$v_j^a(t) = rank(\delta({s'}_\beta)(t))_{o(\delta) \in S_j}, \qquad j = 0, \ldots, \rho.$$

Lemma 3.5.1. *The function $t \mapsto \underline{e}^a(t)$ is constant on U_α^a and takes different values for different α. The analytic structure of U_α^a is defined by the corresponding subminors of $(\delta(s_C)(t))$. Moreover, $u_j^a(t) = \widehat{\mu}_1^{n+j} - \widehat{\tau}_{a,1}^{n+j}(t)$ and $v_j^a(t) = \widehat{\mu}_2^{n+j} - \widehat{\tau}_{a,2}^{n+j}(t)$. In particular, $u_\rho^a(t) = v_\rho^a(t) = \widehat{\mu} - \widehat{\tau}(X_t)$ where $\widehat{\tau}(X_t)$ is the microlocal Tjurina number of the curve singularity X_t.*

Proof. That $\underline{e}^a(t)$ is constant on U_α^a and takes different values for different α is a consequence of Theorem, 3.4.4, as is the claim about the analytic structure of each strata.

Let $t \in U_\alpha^a$ and consider for each $m \in \{n, \ldots, n+\rho\}$ the induced $\mathbb{C}$-base $\{m_{\ell \in J_m}(t)\} = \{m_{\ell \in J_m}\}$ of $\mathbb{C}\{x,y\}/(\Delta F_t, I_a^m)$. Then, for each $\ell \in J_m$

$$m_\ell F_t = \sum_{j=1}^{b} \delta_\ell^0(s_{o(j)})(t) m_{\mu-b+j} \; mod\, (\Delta F_t, I_a^m)$$

and

$$m_\ell H_t^\gamma = \sum_{j=1}^{b} \delta_\ell^\gamma(s_{o(j)})(t)m_{\mu-b+j}\ mod\,(\Delta F_t, I_a^m)$$

for $\gamma = 1,\dots,k-2$. Then, by definition of $\widehat{\tau}_a^\bullet(t)$ and from the definition of $\{\delta_\mu\}$, $u_j^a(t) = \widehat{\mu}_1^{n+j} - \widehat{\tau}_{a,1}^{n+j}(t)$.

The proof of the claim about the $v_j^a(t)$ is similar with the difference that we're now interested in the relations $mod\ \Delta F_t$ between the $m_\ell F_t, m_\ell H_t^\gamma$ that belong to $I_a^{\rho-1+2n-m}$ for each $m \in \{n,\dots,n+\rho\}$. Note that $m_\ell F_t, m_\ell H_t^\gamma \in I_a^{\rho-1+2n-m}$ if and only if $\alpha_\ell^0, \alpha_\ell^\gamma \in S_{m-n}$.

□

Lemma 3.5.2. *If $a,b \in \mathbb{C}^B$ are such that $\mathcal{C}on(\mathcal{F}_a) \cong \mathcal{C}on(\mathcal{F}_b)$, there is $\psi : \mathbb{C} \to \mathbb{C}^B$ microlocally trivial such that $\psi(0) = a$ and $\psi(1) = b$.*

Proof. Let χ_0 be a contact transformation given by α, β_0 such that $F_b = uF_a^{\chi_0}$ for some unit $u \in \mathbb{C}\{x,y\}$. We can assume $deg\ \chi_0 > 0$. There is a relative contact transformation $\chi(t)$ over $\mathbb{C}$ such that $\chi(0) = id_{\mathbb{C}^3}$ and $\chi(1) = \chi_0$. Then

$$G(t) = u(tx,ty)F_B^\chi(x,y,a)$$

is an unfolding of F_a such that $G(1) = F_b$. By versality of F_B and because F_a is semiquasihomogeneous ($j(F_a) = (F_a, j(F_a))$) there is a relative coordinate transformation

$$\begin{aligned}\Phi :&\mathbb{C}\times\mathbb{C}^2 \to \mathbb{C}\times\mathbb{C}^2\\ &(t,x,y) \mapsto (t,\Phi_1,\Phi_2)\end{aligned}$$

and $\psi : \mathbb{C} \to \mathbb{C}^B$ such that

$$\Phi\,(G(t)) = F_{\psi(t)}.$$

(see Remark 1.1 and Corollary 3.3 of [8]). Now, because F_B is semiuniversal (hence does not contain trivial subfamilies with respect to right equivalence) $\Phi(1)\,(G(1)) = \Phi(1)\,(F_b) = F_{\psi(1)}$ implies that $\psi(1) = b$.

□

Theorem 3.5.3. *Given $a,b \in \mathbb{C}^C$, $\mathcal{C}on(\mathcal{F}_a) \cong \mathcal{C}on(\mathcal{F}_b)$ if and only if a and b are in the same integral manifold of $\mathcal{L}_C$.*

Proof. By Theorem 3.3.5 we can replace C by B.

Let us first prove sufficiency. Let $C \subset A \subset B$ and S be a complex space. We say that a holomorphic map $\psi : S \to \mathbb{C}^A$ is *trivial* if for each $o \in S$, $\psi^*\mathcal{F}_A$ is a trivial deformation of $\mathcal{D}ef_f^{es,\mu}(S,o)$. Assume $\psi : (\mathbb{C},0) \to \mathbb{C}^B$ is the germ of an integral curve of a vector field δ in $\mathcal{L}_B$. Set $q = \psi(0)$. Let $\psi_\varepsilon : T_\varepsilon \to \mathbb{C}^B$ be the morphism induced by ψ. There are $a_0, a_1, \dots, a_l, \alpha_0, \beta_0 \in \mathbb{C}\{s_B\}[[x,y]]$ such that

$$\delta F_B = a_0F_B + \sum_{j=1}^{\ell} a_jH_B^j + \alpha_0\partial_xF_B + \beta_0\partial_yF_B.$$

Set $u = 1 + \varepsilon a_0(q)$, $\alpha = \alpha(q) + \sum_{j=1}^{\ell} a_j(q)p^j$ and $\beta = \beta(q) + \sum_{j=1}^{\ell} \frac{j}{j+1} a_j(q)p^{j+1}$. By Theorem 3.2.17 there is $\gamma \in \mathbb{C}\{x, y, p\}$ such that

$$(x, y, p, \varepsilon) \mapsto (x + \alpha\varepsilon, y + \beta\varepsilon, p + \gamma\varepsilon, \varepsilon)$$

defines a relative contact transformation χ^ε over T_ε. Let $G \in \mathbb{C}\{x, y, p, \varepsilon\}$ be defined by $G(x, y, p, \varepsilon) = F_B(x + \alpha\varepsilon, y + \beta\varepsilon, q)$. Since $\psi^*\mathcal{F}_B \equiv uG \; mod \; (\varepsilon)$ and

$$\partial_\varepsilon \psi^* \mathcal{F}_B \equiv \partial_\varepsilon uG \; mod \; I_{\mathcal{C}on(\mathcal{F}_q)} + (\varepsilon),$$

we have that

$$\psi^*\mathcal{F}_B \equiv uG \; mod \; I_{\chi^\varepsilon(\mathcal{C}on(\mathcal{F}_q))} + (\varepsilon^2).$$

Therefore, $\psi_\varepsilon^* F_B$ is a trivial deformation of $\mathcal{D}ef_f^{es,\mu}(T_\varepsilon)$. Then $\Psi^* F_B$ is a trivial deformation of $\mathcal{D}ef_f^{es,\mu}(\mathbb{C}, 0)$ (see the proof of Theorem 3.2.15 in chapter 2).

Conversely, assume that there is a germ of contact transformation χ_1 such that $(F_a^{\chi_1}) = (F_b)$. We can assume $deg \; \chi_1 > 0$. If χ_1 is of type (3.2.10), by Lemma 3.5.2 there is a trivial curve $\psi : \mathbb{C} \to \mathbb{C}^B$ such that $\psi(0) = a$ and $\psi(1) = b$. Moreover, ψ is an integral curve of the Euler vector field. Since the derivative of χ_1 leaves $\{y = p = 0\}$ invariant, we can assume by Theorem 3.2.18 that χ_1 is of type (3.2.7). Set $\chi = \chi_{t\alpha, t\beta_0}$. There is a curve with polynomial coefficients $\psi : \mathbb{C} \to \mathbb{C}^B$ such that $F_a^\chi = \psi^* F_B$, $\psi(0) = a$ and $\psi(1) = b$.

Let Ω be an open set of $\mathbb{C}$. Let $\psi : \Omega \to \mathbb{C}^B$ be a trivial curve. Let us show that ψ is contained in an integral manifold of $\mathcal{L}_B$. Let U be the union of the strata U_α such that, for each $c \in U$ the microlocal Tjurina number of F_c equals the microlocal Tjurina number of F_a. Remark that the trajectory of ψ is contained in U. By Theorem 3.4.4 $\mathcal{L}_{B|U}$ verifies the Frobenius Theorem. Hence, it is enough to show that, for each $t_0 \in \Omega$, there is $\delta \in \mathcal{L}_B$ such that $\psi'(t_0) = \delta(\psi(t_0))$. We can assume $t_0 = 0$. Since ψ is trivial, there are a relative contact transformation χ and $u \in \mathbb{C}\{x, y, t\}$ such that $u(x, y, 0) = 1$ and

$$F(x, y, \psi(t)) \equiv uF^\chi(x, y, q) \; mod \; I_{\chi(\mathcal{C}on(\mathcal{F}_q))}.$$

If χ is of type 3.2.10, we can assume δ is the Euler field. Hence we can assume that χ is of type (3.2.7). Therefore there are $\ell \geq 1$ and $a, b, a_i \in \mathbb{C}\{x, y\}$, $1 \leq i \leq \ell$, such that

$$F(x, y, \psi(t)) = uF(x, y, q) + \sum_{\ell=1}^{k-2} a_\ell t H_q^\ell + at\partial_x F_q + bt\partial_y F_q \; mod \; (t^2).$$

Deriving in order to t and evaluating at 0, there is $a_0 \in \mathbb{C}\{x, y\}$ such that

$$\textstyle\sum_{(i,j)\in C_0} \psi'_{i,j}(0)x^i y^j = a_0 F_q + \sum_{\ell=1}^{k-2} a_\ell H_q^\ell + a\partial_x F_q + b\partial_y F_q.$$

There are $\delta \in L_B$ and $\varepsilon \in \Delta_{F_B}$ such that

$$\delta F_B = a_0 F_B + \textstyle\sum_{\ell=1}^{k-2} a_\ell H_B^\ell + \varepsilon.$$

Hence

$$\textstyle\sum_{(i,j)\in B_0} \psi'_{i,j}(0)x^i y^j - \delta(q)F_B = \varepsilon(q) + a\partial_x F_q + b\partial_y F_q.$$

If $\delta = \sum_{(i,j)\in B_0} a_{i,j}\partial_{s_{i,j}}$, $a_{i,j}(\psi(0)) = \psi'_{i,j}(0)$ for each $(i, j) \in B_0$. $\square$

Theorem 3.5.4. (1) *Let $\underline{e} = (e_1, \ldots, e_\rho) \in \mathbb{N}^{\rho+1}$ and let $U^a_{\underline{e}}$ denote the unique stratum (assumed to be not empty) such that $\underline{e}^a(t) = \underline{e}$ for each $t \in U^a_{\underline{e}}$. The geometric quotient $U^a_{\underline{e}}/\mathcal{L}$ is quasiaffine and of finite type over $\mathbb{C}$. It is a coarse moduli space for the functor which associates to any complex space S the set of isomorphism classes of flat families (with section) over S of plane curve singularities with fixed semigroup $\langle k, n\rangle$ and fixed microlocal Hilbert function $\widehat{\tau}^\bullet_a$.*
(2) *Let $T_{\widehat{\tau}_{min}}$ be the open dense set defined by singularities with minimal microlocal Tjurina number $\widehat{\tau}_{min}$. Then the geometric quotient $T_{\widehat{\tau}_{min}}/\mathcal{L}_C$ exists and is a coarse moduli space for curves with semigroup $\langle k, n\rangle$ and microlocal Tjurina number $\widehat{\tau}_{min}$. Moreover, $T_{\widehat{\tau}_{min}}/\mathcal{L}_C$ is locally isomorphic to an open subset of a weighted projective space.*

Proof. It follows from Lemma 3.5.1 and Theorems 3.4.4 and 3.5.3. □

3.6 Example

The function

$$F_C = y^6 + x^{13} + s_2x^9y^2 + s_3x^7y^3 + s_4x^5y^4 + s_9x^8y^3 + s_{10}x^6y^4 + s_{16}x^7y^4.$$

is a semiuniversal equisingular microlocal deformation of $f = y^6 + x^{13}$.

The Lie algebra L_C is generated by the vector fields

$$\begin{aligned}
\delta^0 &= 2s_2\partial_{s_2} + 3s_3\partial_{s_3} + 4s_4\partial_{s_4} + 9s_9\partial_{s_9} + 10s_{10}\partial_{s_{10}} + 16s_{16}\partial_{s_{16}} \\
\delta^6 &= 3s_3\partial_{s_9} + (4s_4 - \frac{58}{39}s_2^2)\partial_{s_{10}} + 10s_{10}\partial_{s_{16}} \\
\delta^7 &= 2s_2\partial_{s_9} + 3s_3\partial_{s_{10}} \\
\delta^{12} &= 4s_4\partial_{s_{16}} \\
\delta^{13} &= 3s_3\partial_{s_{16}} \\
\delta^{14} &= 2s_2\partial_{s_{16}}.
\end{aligned}$$

Choosing $a = 6$ we get $F^0_a = \langle s_2, s_3, s_4\rangle$, $F^1_a = \langle s_2, s_3, s_4, s_9, s_{10}\rangle$, $F^2_a = \langle s_2, s_3, s_4, s_9, s_{10}, s_{16}\rangle$.
So, $\rho = 2$ and the stratification $\{U^a_\alpha\}$ given by fixing $\underline{e}^a(t) = (u^a_0(t), u^a_1(t), u^a_2(t); v^a_0(t), v^a_1(t), v^a_2(t))$

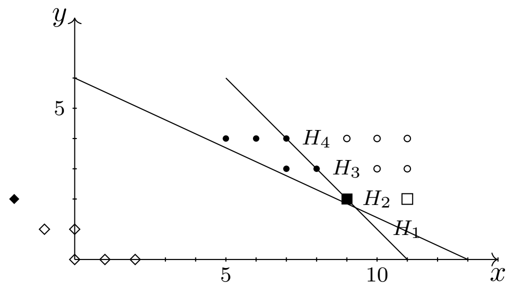

Figure 3.1: This figure concerns the case $k = 6$ and $n = 13$. The diamonds represent the set of orders of the vector fields generating L_C. The black circles and black squares represent C_0. The leading monomials of $H_1, \dots, H_4$ are represented as well. The white square represents the leading monomial of xH_2 and yH_1 which produce the vector field δ^{14} with order represented by a black diamond. The order of the vector fields δ_0^0, δ_1^0, δ_1^1, δ_2^0 and δ_3^0 are represented by white diamonds.

is given by

$$\begin{aligned}
U_1 =& \{t = (t_2, t_3, t_4, t_9, t_{10}, t_{16}) \in Spec\,\mathbb{C}[s_C] : \underline{e}^a(t) = (1,3,4;1,3,4)\}\\
=& \{t : 9t_3^2 - 8t_2t_4 + \frac{116}{39}t_2^3 \neq 0\}.\\
U_2 =& \{t \in Spec\,\mathbb{C}[s_C] : \underline{e}^a(t) = (1,2,3;1,2,3)\}\\
=& \{t : 9t_3^2 - 8t_2t_4 + \frac{116}{39}t_2^3 = 0 \text{ and } t_2 \neq 0 \text{ or } t_3 \neq 0 \text{ or } t_4 \neq 0\}.\\
U_3 =& \{t \in Spec\,\mathbb{C}[s_C] : \underline{e}^a(t) = (0,1,2;0,1,2)\}\\
=& \{t : t_2 = t_3 = t_4 = 0 \text{ and } t_{10} \neq 0\}.\\
U_4 =& \{t \in Spec\,\mathbb{C}[s_C] : \underline{e}^a(t) = (0,1,1;0,0,1)\}\\
=& \{t : t_2 = t_3 = t_4 = t_{10} = 0 \text{ and } t_9 \neq 0\}.\\
U_5 =& \{t \in Spec\,\mathbb{C}[s_C] : \underline{e}^a(t) = (0,0,1;0,0,1)\}\\
=& \{t : t_2 = \cdots = t_{10} = 0 \text{ and } t_{16} \neq 0\}.\\
U_6 =& \{t \in Spec\,\mathbb{C}[s_C] : \underline{e}^a(t) = (0,0,0;0,0,0)\}\\
=& \{t : t_2 = \cdots = t_{16} = 0\}.
\end{aligned}$$

U_1 is the stratum with minimal microlocal Tjurina number.

Let us present detailed calculations concerning the generators of L_C in the previous example. Let Y denote the germ at the origin of $\{F_C = 0\}$. The relative conormal $\mathcal{L}$ of

Y can be parametrized by

$$
\begin{aligned}
&x = -t^6, \\
&y = t^{13} + \psi_2 t^{15} + \psi_3 t^{16} + \psi_4 t^{17} + \psi_5 t^{18} + \psi_6 t^{19} + \psi_7 t^{20} + \psi_8 t^{21} + \psi_9 t^{22} + \cdots, \\
&p = -\frac{13}{6} t^7 - \frac{5}{2}\psi_2 t^9 - \frac{8}{3}\psi_3 t^{10} - \frac{17}{6}\psi_4 t^{11} - 3\psi_5 t^{12} - \frac{19}{6}\psi_6 t^{13} - \frac{10}{3}\psi_7 t^{14} - \frac{7}{2}\psi_8 t^{15} \\
&\quad - \frac{11}{3}\psi_9 t^{16} + \cdots,
\end{aligned}
$$

where $\psi_i \in (s_C)\mathbb{C}[s_C]$ are homogeneous of degree $-i$. These are the a_i such that the polynomial in $\mathbb{C}[t]$ given by the following SINGULAR session is zero:

```
> ring r=(0,a2,a3,a4,a5,a6,a7,a8,a9,s2,s3,s4,s9,s10,s16),(x,y,t),dp;
> poly F=y6+x13+s2*x9y2+s3*x7y3+s4*x5y4+s9*x8y3+s10*x6y4+s16*x7y4;
> subst(F,x,-t6);
-t^78 +(-s2)*y^2*t^54+(s9)*y^3*t^48+(-s16)*y^4*t^42+(-s3)*y^3*t^42
+(s10)*y^4*t^36+(-s4)*y^4*t^30+y^6
> subst(-t^78 +(-s2)*y^2*t^54+(s9)*y^3*t^48+(-s16)*y^4*t^42+(-s3)*y^3*t^42
+(s10)*y^4*t^36+(-s4)*y^4*t^30+y^6,y,t^13+a2*t^15+a3*t^16+a4*t^17+a5*t^18
+a6*t^19+a7*t^20+a8*t^21+a9*t^22)
```

As we'll see, the only ψ_i we actually need to find the generators of L_C is

$$\psi_2 = s_2/6.$$

Let us calculate the vector fields generating L_C. Here, all equalities are $mod\,\Delta F_C$ and in the vector fields we identify, by abuse of language, the monomials and the corresponding ∂'s :

- δ^{14}:

$$xH_2 = xp^2\partial_x F_C = 13p^2x^{13} + 9s_2p^2x^9y^2 + \cdots$$

  Notice that, as a consequence of Lemma 3.3.3, the monomials occurring with order bigger than $deg(x^7y^4)$ can be ignored in this calculation. From now on, whenever we use the symbol $\cdots$ we mean that bigger order monomials can be ignored. Now, continuing the previous SINGULAR session:

```
> poly p=(-13t7-15*a2*t9-16*a3*t10-17*a4*t11-18*a5*t12-19*a6*t13-20*a7*t14
-21*a8*t15-22*a9*t16)/6;
> poly X=-t6;
> poly Y=t13+a2*t15+a3*t16+a4*t17+a5*t18+a6*t19+a7*t20+a8*t21+a9*t22;
> p^2*X^13-(13/6)^2*X^11*Y^2;
```

```
(-35*a9^2)/4*t^110+(-293*a8*a9)/18*t^109+(-271*a7*a9-136*a8^2)/18*t^108+(-
249*a6*a9-251*a7*a8)/18*t^107+(-454*a5*a9-460*a6*a8-231*a7^2)/36*t^106+(-
205*a4*a9-209*a5*a8-211*a6*a7)/18*t^105+(-183*a3*a9-188*a4*a8-191*a5*a7-
96*a6^2)/18*t^104+(-161*a2*a9-167*a3*a8-171*a4*a7-173*a5*a6)/18*t^103+(-
292*a2*a8-302*a3*a7-308*a4*a6-155*a5^2)/36*t^102+(-131*a2*a7-135*a3*a6-
137*a4*a5-117*a9)/18*t^101+(-116*a2*a6-119*a3*a5-60*a4^2-104*a8)/18*t^100
+(-101*a2*a5-103*a3*a4-91*a7)/18*t^99+(-172*a2*a4-87*a3^2-156*a6)/36*t^98
+(-71*a2*a3-65*a5)/18*t^97+(-14*a2^2-26*a4)/9*t^96+(-13*a3)/6*t^95+(-
13*a2)/9*t^94
```

we see that

$$p^2x^{13} = \left(\frac{13}{6}\right)^2 x^{11}y^2 + \frac{13\psi_2}{9}x^7y^4 + \cdots$$

Now, δ_3^1, given by yH_1, which has the same order as xH_2 can be used to, through elementary operations, eliminate from δ_1^2 the monomial $x^{11}y^2$. Thus,

$$\delta^{14} = s_2x^7y^4.$$

- δ^{13}:

$$-6.13yF_C = 2s_2x^9y^3 + 3s_3x^7y^4 + \cdots$$

But, as

$$H_3 = \left(\frac{13}{6}\right)^3 .13x^9y^3 + \cdots$$

we see that, through elementary operations involving δ_0^3, we can eliminate from δ_3^0 the monomial x^9y^3. Thus,

$$\delta^{13} = 3s_3x^7y^4.$$

- δ^{12}:

$$-6.13x^2F_C = 2s_2x^{11}y^2 + 3s_3x^9y^3 + 4s_4x^7y^4 + \cdots$$

through elementary operations involving δ_3^1 and δ_0^3 we can eliminate the monomials $x^{11}y^2$ and x^9y^3 from δ_2^0 and get:

$$\delta_2^0 = (4s_4 + *s_2^2)x^7y^4, \qquad * \in \mathbb{C}.$$

Finally, using δ^{14} to eliminate $*s_2^2x^7y^4$, we have that

$$\delta^{12} = 4s_4x^7y^4.$$

- δ^7:

$$xH_1 = xp\partial_xF_C = 13px^{13} + 9s_2px^9y^2 + 7s_3px^7y^3 + 5s_4px^5y^4 + 8s_9px^8y^3 + \cdots$$

and

$$px^{13} = \frac{13}{6}x^{12}y + \frac{s_2}{18}x^8y^3 + \frac{s_3}{12}x^6y^4 + \cdots$$

Remark 3.6.1. The reason why we can ignore in px^{13} the monomials that occur after x^6y^4 is that

1. All monomials after x^6y^4, except for x^7y^4, can be eliminated because of Lemma 3.3.3 and through elementary operations involving δ_3^1 and δ_0^3.
2. Even x^7y^4 can be ignored, observing that px^{13} is homogeneous of degree 7 and as such, the only variables involved in the coefficient (in $\mathbb{C}[s_C]$) of x^7y^4 may be s_2, s_3 or s_4. Now, using δ^{14}, δ^{13} and δ^{12} we can eliminate, through elementary operations, the monomial x^7y^4 from δ^7.

From

$$y\partial_x F_C = 13x^{12}y + 9s_2x^8y^3 + 7s_3x^6y^4 + 5s_4x^4y^5 + 8s_9x^7y^4 + \cdots$$

we get that

$$\frac{13}{6}x^{12}y = -\frac{3}{2}s_2x^8y^3 - \frac{7}{6}s_3x^6y^4 - \frac{5}{6}s_4x^4y^5 - \frac{8}{6}s_9x^7y^4 + \cdots$$

Reasoning as in remark 3.6.1 we see that $s_4x^4y^5$ can be ignored. Thus,

$$13px^{13} = 13\left(\left(-\frac{3}{2}s_2 + \frac{s_2}{18}\right)x^8y^3 + \left(-\frac{7}{6}s_3 + \frac{s_3}{12}\right)x^6y^4 - \frac{8}{6}s_9x^7y^4 + ..\right)$$

Now,

$$\begin{aligned} px^9y^2 &= \frac{13}{6}x^8y^3 + \cdots \\ px^7y^3 &= \frac{13}{6}x^6y^4 + ... \\ px^8y^3 &= \frac{13}{6}x^7y^4 + ... \end{aligned}$$

Once again, the monomials ignored can be eliminated, reasoning as in Remark 3.6.1. So,

$$\begin{aligned} xH_1 = &\left(13\left(-\frac{3}{2}s_2 + \frac{s_2}{18}\right) + 9\frac{13}{6}s_2\right)x^8y^3 + \left(13\left(-\frac{7}{6}s_3 + \frac{s_3}{12}\right) + 7\frac{13}{6}s_3\right)x^6y^4 + \\ &+ \left(-13\frac{8}{6}s_9 + 8\frac{13}{6}s_9\right)x^7y^4 \\ =&\frac{13}{18}s_2x^8y^3 + \frac{13}{12}s_3x^6y^4. \end{aligned}$$

We get that

$$\delta^7 = \frac{s_2}{3}x^8y^3 + \frac{s_3}{2}x^6y^4.$$

- δ^6:

$$-6.13xF_C = 2s_2x^{10}y^2 + 3s_3x^8y^3 + 4s_4x^6y^4 + 9s_9x^9y^3 + 10s_{10}x^7y^4 + \cdots$$

Because (monomials ignored as in Remark 3.6.1)

$$H_2 = p^2\partial_x F_C = 13p^2x^{12} + 9s_2p^2x^8y^2 + \cdots ,$$

$$p^2x^{12} = \left(\frac{13}{6}\right)^2 x^{10}y^2 + \frac{13}{6.9}s_2x^6y^4 + \cdots ,$$

and

$$p^2x^8y^2 = \left(\frac{13}{6}\right)^2 x^6y^4 + \cdots$$

wet get

$$\begin{aligned}
&-6.13xF_C - 2s_2\left(\frac{6}{13}\right)^2\frac{H_2}{13} = \\
=&3s_3x^8y^3 + \left(4s_4 - 2s_2\left(\frac{6}{13}\right)^2\frac{13}{6.9}s_2 - 2s_2\left(\frac{6}{13}\right)^2\frac{9}{13}\left(\frac{13}{6}\right)^2 s_2\right)x^6y^4 + 10s_{10}x^7y^4 = \\
=&3s_3x^8y^3 + \left(4s_4 - \frac{58}{39}s_2^2\right)x^6y^4 + 10s_{10}x^7y^4.
\end{aligned}$$

So,

$$\delta^6 = 3s_3x^8y^3 + \left(4s_4 - \frac{58}{39}s_2^2\right)x^6y^4 + 10s_{10}x^7y^4.$$

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