

Variations on Instant Insanity

Erik D. Demaine¹, Martin L. Demaine¹, Sarah Eisenstat¹,
Thomas D. Morgan², and Ryuhei Uehara³

¹ MIT Computer Science and Artificial Intelligence Laboratory,
32 Vassar St., Cambridge, MA 02139, USA.

email: {[edemaine](mailto:edemaine@mit.edu), [mdemaine](mailto:mdemaine@mit.edu), [seisenst](mailto:seisenst@mit.edu)}@mit.edu

² Harvard University School of Engineering and Applied Sciences,
33 Oxford St., Cambridge, MA 02138, USA.

email: tdmorgan@seas.harvard.edu

³ School of Information Science,
Japan Advanced Institute of Science and Technology (JAIST),
Asahidai 1-1, Nomi, Ishikawa 923-1292, Japan.

email: uehara@jaist.ac.jp

Abstract. In one of the first papers about the complexity of puzzles, Robertson and Munro [12] proved that a generalized form of the then-popular Instant Insanity puzzle is NP-complete. Here we study several variations of this puzzle, exploring how the complexity depends on the piece shapes and the allowable orientations of those shapes.

Prepared in honor of Ian Munro's 66th birthday.

1 Introduction

In the late 1960s, the company Parker Brothers popularized a puzzle known as *Instant Insanity*⁴. Instant Insanity is composed of four cubes, where each face of each cube is colored red, green, white, or blue. The goal is to arrange the cubes in a tower with dimensions $1 \times 1 \times 4$ so that on each of the four long sides of the tower, every color appears (exactly once per side). This puzzle has a rich history — the name *Instant Insanity* dates back to 1967 [11], but there were many earlier variants, released under such names as *Katzenjammer*, *Groceries*, and *The Great Tantalizer* [10].

The mathematics behind the Instant Insanity puzzle have been studied extensively, and the puzzle is used as a sample application of graph theory in some textbooks [1, 3, 14]. In 1978, Robertson and Munro [12] showed that, by generalizing the number of colors and cubes from 4 to n , the puzzle becomes NP-complete to solve; they also proved a two-player variant PSPACE-complete. Their paper was one of the first to study the computational complexity of puzzles and games [4, 6, 9]. More recently, there have been two studies of variants

⁴ The name “Instant Insanity” was originally trademarked by Parker Brothers in 1967. The trademark is currently owned by Winning Moves, Inc.



Fig. 1. A modern instance of *Instant Insanity* (left, distributed by Winning Moves, Inc., 2010), and a much older *The Great Tantalizer* (right, indicated only as British Made, made with wooden blocks).

of Instant Insanity. In 2002, Jebasingh and Simoson [8] studied variants of the problem with Platonic solid pieces. In 2008, Berkove et al. [2] studied a problem in which the goal is to combine cubes to form a larger box with consistent colors on each side.

Our Results. Inspired by Robertson and Munro [12], our goal is to explore how the complexity of the puzzle changes with the shape of the pieces and the set of allowable motions. In particular, we consider puzzles in which all of the pieces are shaped like identical right prisms, and the goal is to stack the pieces to form a taller prism with the same base. In Section 2, we establish a combinatorial definition for the pieces of a puzzle, and give a formal definition of the Instant Insanity problem.

In Section 3.2, we examine the case of pieces where the base of the prism is a regular polygon. When the base is an equilateral triangle, we show that the problem is easy to solve. When the base is a square (but the piece is not a cube), we prove that the problem is NP-complete, even though the number of allowable configurations for each piece is only a third as large as the number of configurations for the original Instant Insanity problem.

In Section 3.3, we consider the case of regular polygon prisms where the motion is restricted, and show that even in the case of equilateral triangle pieces with three possible configurations per piece, the problem remains NP-complete. Finally, in Section 4, we prove results about some irregular prism pieces, using a technique for solving any puzzle in which each piece has two possible configurations.

2 Definitions

2.1 Instant Insanity

Let C be a finite set of colors. Given a polyhedral *piece shape*, let k_1 be the number of potentially visible sides, and let k_2 be the number of sides that are

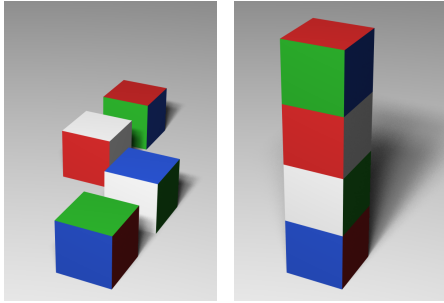


Fig. 2. A rendering of the original Instant Insanity puzzle. Although the back of the tower is not shown, each color occurs exactly once on each of the four long sides of the tower of blocks.

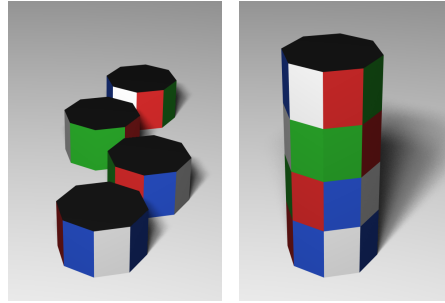


Fig. 3. A generalized version of the puzzle with octagonal prism pieces. The top and bottom of each prism can never lie on any of the eight sides of the tower. The other eight faces are visible at all times.

visible in any given configuration. In this paper, we restrict ourselves to pieces in the shape of a right prism, which often implies that $k_1 = k_2$. For each (potentially visible) side of the piece shape, we assign a unique number from the set $\{1, \dots, k_1\}$. Hence, a single piece can be represented by a tuple in C^{k_1} assigning a color to each of the k_1 sides. When defining a piece, we sometimes use the special symbol $*$, which represents a unique color used once in the entire puzzle.

The set of possible configurations of a single piece is given by the *piece configurations* $P \subseteq \{1, \dots, k_1\}^{k_2}$. Each piece configuration indicates how the colors in the original piece should be assigned to the visible sides. Specifically, a piece configuration is a mapping from each of the k_2 visible sides to the index of one of the sides of the piece shape. A single side of a single piece cannot appear on multiple sides of the puzzle, so each piece configuration has the additional restriction that no element of the tuple is repeated.

For each visible side $1 \leq i \leq k_2$, we define the function $F_i : C^{k_1} \times P \rightarrow C$ to return the color of side i , given a piece and its configuration. Formally, we say:

$$F_i(\langle a_1, \dots, a_{k_1} \rangle, \langle q_1, \dots, q_{k_2} \rangle) = a_{q_i}.$$

As an example, we consider the original Instant Insanity puzzle. Each cube has $k_1 = 6$ sides with colors on them; only $k_2 = 4$ of those sides are visible when the cubes are stacked vertically. Suppose that we number the sides as depicted in Fig. 4. Then there are 24 possible piece configurations: 6 ways to choose the side that faces down, and 4 possible ways to rotate the visible faces. These 24 piece configurations are listed in Fig. 5.

Many piece shapes, including the cube, have a number of symmetries. In particular, many have the following property:

Definition 1. *The set of piece configurations P is rotationally symmetric if P is closed under cyclic shifts.*

Using this combinatorial definition of piece configurations, we can formally define two variants of the Instant Insanity problem:

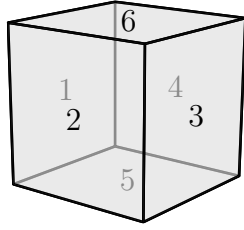


Fig. 4. Labels for each of the faces on a cube.

$\{ \langle 1, 2, 3, 4 \rangle, \langle 2, 3, 4, 1 \rangle, \langle 3, 4, 1, 2 \rangle, \langle 4, 1, 2, 3 \rangle, \langle 1, 4, 3, 2 \rangle, \langle 2, 1, 4, 3 \rangle, \langle 3, 2, 1, 4 \rangle, \langle 4, 3, 2, 1 \rangle, \langle 1, 5, 3, 6 \rangle, \langle 5, 3, 6, 1 \rangle, \langle 3, 6, 1, 5 \rangle, \langle 6, 1, 5, 3 \rangle, \langle 1, 6, 3, 5 \rangle, \langle 5, 1, 6, 3 \rangle, \langle 3, 5, 1, 6 \rangle, \langle 6, 3, 5, 1 \rangle, \langle 2, 5, 4, 6 \rangle, \langle 5, 4, 6, 2 \rangle, \langle 4, 6, 2, 5 \rangle, \langle 6, 2, 5, 4 \rangle, \langle 2, 6, 4, 5 \rangle, \langle 5, 2, 6, 4 \rangle, \langle 4, 5, 2, 6 \rangle, \langle 6, 4, 5, 2 \rangle \}$

Fig. 5. The 24 different piece configurations for the cube piece depicted in Fig. 4.

Definition 2. The COMPLETE-INSANITY(P) problem is defined as follows:

Input: A set of colors C and a sequence of pieces A_1, \dots, A_n , where $n = |C|$.

Output: YES if and only if there is a sequence of configurations $p_1, \dots, p_n \in P$ such that for each side $1 \leq i \leq k_2$, the set of visible colors $\{F_i(A_j, p_j) \mid 1 \leq j \leq n\} = C$.

Definition 3. The PARTIAL-INSANITY(P) problem is defined as follows:

Input: A set of colors C and a sequence of pieces A_1, \dots, A_n , where $n \leq |C|$.

Output: YES if and only if there is a sequence of configurations $p_1, \dots, p_n \in P$ such that for each side $1 \leq i \leq k_2$, all of the visible colors $F_i(A_j, p_j)$ on side i are distinct.

Note that both problems require all visible colors on a single side to be distinct; however, the PARTIAL-INSANITY(P) problem only requires a subset of all of the colors to be visible on a single side, while the COMPLETE-INSANITY(P) problem requires all colors to be visible. Clearly, the PARTIAL-INSANITY(P) problem is at least as hard as the COMPLETE-INSANITY(P) problem, and both are contained in the complexity class NP.

2.2 Positive Not-All-Equal Satisfiability

In this paper, we prove that several variants of the Instant Insanity puzzle are NP-complete to solve by reducing from a known NP-complete problem. The problem we use for these reductions is a variant of the 3-SAT problem known as POSITIVE-NOT-ALL-EQUAL-SAT, or POSITIVE-NAE-SAT.

Definition 4. The POSITIVE-NAE-SAT problem is defined as follows:

Input: A set of n variables x_1, \dots, x_n and m clauses C_1, \dots, C_m . Each clause C_i is composed of three positive literals $c_{i,1}, c_{i,2}, c_{i,3} \in \{x_1, \dots, x_n\}$.

Output: YES if there exists some mapping $\phi : \{x_1, \dots, x_n\} \rightarrow \{T, F\}$ such that each clause C_i has at least one literal c_{i,j_1} such that $\phi(c_{i,j_1}) = T$, and at least one literal c_{i,j_2} such that $\phi(c_{i,j_2}) = F$.

This problem was one of many Boolean satisfiability problems shown to be NP-complete by Schaefer [13].

3 Regular Prism Pieces

3.1 Partial Versus Complete Insanity

In this section, we show that for many types of regular pieces, if the PARTIAL-INSANITY(P) problem is NP-complete, then the COMPLETE-INSANITY(P) problem is NP-complete. We do this by way of a third problem in which the use of each color is restricted. Specifically, we restrict the *number of occurrences of* $c \in C$: the sum over all pieces A_i of the number of sides of A_i with the color c .

Definition 5. *The ONE-OR-ALL-INSANITY(P) problem is defined as follows:*

Input: *A set of colors C and a sequence of pieces A_1, \dots, A_n , where $n \leq |C|$. Furthermore, the number of occurrences of each color $c \in C$ is either 1 or k_2 .*
Output: *YES if and only if there is a sequence of configurations $p_1, \dots, p_n \in P$ such that for each side $1 \leq i \leq k_2$, all of the visible colors $F_i(A_j, p_j)$ on side i are distinct.*

Lemma 1. *Suppose that $k_1 = k_2$ and the set of piece configurations P is rotationally symmetric. Then there exists a polynomial-time reduction from the problem PARTIAL-INSANITY(P) to the problem ONE-OR-ALL-INSANITY(P).*

Proof. Suppose that we are given an instance of the PARTIAL-INSANITY(P) problem consisting of colors C and pieces A_1, \dots, A_n . Using these pieces, we construct an instance of the ONE-OR-ALL-INSANITY(P) problem as follows:

1. The first n pieces are identical to the pieces A_1, \dots, A_n .
2. For each color $c \in C$, let $\#(c)$ be the number of occurrences of c in the set of pieces A_1, \dots, A_n . If $\#(c) = 1$ or $\#(c) = k_2$, then we do not need to do anything. If $\#(c) > k_2$, then the puzzle must be impossible to solve. Otherwise, we generate $k_2 - \#(c)$ new pieces, each defined as follows:

$$\langle c, \underbrace{*, *, \dots, *}_{k_2 - 1 \text{ times}} \rangle.$$

(Recall that each $*$ symbol represents a unique color used exactly once.)

Let D be the set of colors generated by this process, and let B_1, \dots, B_m be the pieces. Our construction ensures that the pieces B_1, \dots, B_m form a valid instance of the ONE-OR-ALL-INSANITY(P) problem. We must additionally show that the puzzle formed by A_1, \dots, A_n can be solved if and only if B_1, \dots, B_m can be solved.

Suppose that we have a solution p_1, \dots, p_n to the problem A_1, \dots, A_n . If we use p_1, \dots, p_n to configure B_1, \dots, B_n , then we are guaranteed that for the first n pieces, no color is used twice on the same side. So our goal is to find a way to configure the remaining $m - n$ pieces. All of the new colors generated in step 2 of the reduction occur exactly once, so they can be used on any side of the puzzle. Because the set of piece configurations P is rotationally symmetric, the

first color in each piece B_i for $n + 1 \leq i \leq m$ can be placed on any side of the puzzle. Each color occurs at most k_2 times, so it is straightforward to arrange the pieces B_{n+1}, \dots, B_m to ensure that no color occurs twice on any side.

Furthermore, if there exists a solution q_1, \dots, q_m to the problem B_1, \dots, B_m , then we can find a solution to A_1, \dots, A_n using the piece configurations q_1, \dots, q_n . This means that our polynomial-time reduction is correct. \square

Lemma 2. *Suppose that $k_1 = k_2$ and the set of piece configurations P is rotationally symmetric. Then there exists a polynomial-time reduction from ONE-OR-ALL-INSANITY(P) to COMPLETE-INSANITY(P).*

Proof. Suppose that we are given an instance of the ONE-OR-ALL-INSANITY(P) problem consisting of colors C and pieces A_1, \dots, A_n . Let C_1 be the set of colors that are used once, and let C_2 be the set of colors that are used k_2 times. For each color $c \in C_2$, let $g(c)$ be a unique index in $\{1, \dots, |C_2|\}$.

For each $1 \leq i \leq k_2$, we construct n pieces $B_{i,1}, \dots, B_{i,n}$ and $|C_2|$ distinct colors $D_i = \{d_{i,1}, \dots, d_{i,|C_2|}\}$. If the piece A_j consists of the sequence of colors $a_{j,1}, \dots, a_{j,k_2}$, then we define the colors $b_{i,j,1}, \dots, b_{i,j,k_2}$ for the piece $B_{i,j}$ as follows:

$$b_{i,j,k} = \begin{cases} d_{i,g(a_{j,k})}, & \text{if } a_{j,k} \in C_2; \\ a_{j,k}, & \text{otherwise.} \end{cases}$$

This ensures that each color $c \in (C_1 \cup D_1 \cup \dots \cup D_{k_2})$ is used exactly k_2 times in the set of pieces $B_{1,1}, \dots, B_{k_2,n}$. Hence the number of colors is $k_2 n$, which is the same as the number of pieces, making this an instance of COMPLETE-INSANITY(P).

Now we wish to show that A_1, \dots, A_n is solvable if and only if $B_{1,1}, \dots, B_{k_2,n}$ is solvable. First, suppose that $B_{1,1}, \dots, B_{k_2,n}$ is solvable. This means that there is a way to configure $B_{1,1}, \dots, B_{1,n}$ such that no color is used twice on a single side. The pieces $B_{1,1}, \dots, B_{1,n}$ are almost identical to the pieces A_1, \dots, A_n , with the exception that each color $c \in C_2$ is replaced with the color $d_{1,g(c)}$. Hence, a configuration of $B_{1,1}, \dots, B_{1,n}$ that does not reuse colors on any side must also be a configuration of A_1, \dots, A_n that does not reuse colors on any side. So if $B_{1,1}, \dots, B_{k_2,n}$ is a solvable instance of COMPLETE-INSANITY(P), then A_1, \dots, A_n is a solvable instance of ONE-OR-ALL-INSANITY(P).

Next, suppose that A_1, \dots, A_n is solvable. This means that there is a sequence of configurations p_1, \dots, p_n such that if we apply p_1, \dots, p_n to the pieces A_1, \dots, A_n , then no color is used twice on a single side. Let σ_i be the circular shift of k_2 elements by i so that $\sigma_i(\langle a_1, \dots, a_{k_2} \rangle) = \langle a_{i+1 \bmod k_2}, \dots, a_{i+k_2 \bmod k_2} \rangle$. Because P is rotationally symmetric, for any $1 \leq i \leq k_2$ and any $1 \leq j \leq n$, $\sigma_i(p_j) \in P$.

Suppose that we use the configuration $\sigma_i(p_j)$ for the piece $B_{i,j}$. Suppose, for the sake of contradiction, that there is some pair of distinct pieces B_{i_1,j_1} and B_{i_2,j_2} that assigns the same color c to the same side of the puzzle. Note that the colors of B_{i_1,j_1} must be a subset of $C_1 \cup D_{i_1}$, while the colors of B_{i_2,j_2} must be a subset of $C_1 \cup D_{i_2}$. We consider three cases:

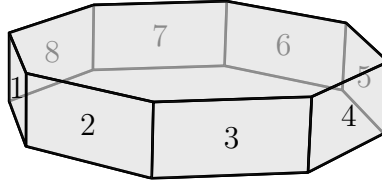


Fig. 6. Labels for each of the faces on an 8-sided regular prism.

Case 1: $i_1 \neq i_2$. Then it must be that $c \in C_1$. Because A_1, \dots, A_n is an instance of ONE-OR-ALL-INSANITY(P), there is exactly one piece A_{j_3} that uses the color c . By construction of the pieces B_{i_1, j_1} and B_{i_2, j_2} , it must be that $j_1 = j_2 = j_3$, and the index of c is the same in both pieces. The configuration of B_{i_1, j_1} is $\sigma_{i_1}(p_{j_1})$, while the configuration of B_{i_2, j_2} is $\sigma_{i_2}(p_{j_2}) = \sigma_{i_2}(p_{j_1})$. Hence the configurations would map the color c to different sides of the puzzle, contradicting our assumption that c occurs twice on the same side.

Case 2: $i_1 = i_2$ and $c \in C_1$. Because $c \in C_1$, it is used exactly once in the original set of pieces A_1, \dots, A_n . Therefore, it is also used exactly once in the set of pieces $B_{i_1, 1}, \dots, B_{i_1, n}$, and so if it occurs both in B_{i_1, j_1} and $B_{i_2, j_2} = B_{i_1, j_2}$, then we must have $j_1 = j_2$. Again, this means that c is not used twice.

Case 3: $i_1 = i_2$ and $c \notin C_1$. Let $i = i_1 = i_2$. By our assumption, we know that there exists some ℓ such that $F_\ell(B_{i, j_1}, \sigma_i(p_{j_1})) = F_\ell(B_{i, j_2}, \sigma_i(p_{j_2})) = c$. Because σ_i is a cyclic shift, this means that there is some ℓ' such that $F_{\ell'}(B_{i, j_1}, p_{j_1}) = F_{\ell'}(B_{i, j_2}, p_{j_2}) = c$. By construction of B_{i, j_1} and B_{i, j_2} , this means that $F_{\ell'}(A_{j_1}, p_{j_1}) = F_{\ell'}(A_{j_2}, p_{j_2})$, which is a contradiction. \square

The combination of Lemmas 1 and 2 yields the following theorem:

Theorem 1. *Suppose that $k_1 = k_2$ and the set of piece configurations P is rotationally symmetric. Then there exists a polynomial-time reduction from the PARTIAL-INSANITY(P) problem to the COMPLETE-INSANITY(P) problem.*

3.2 Regular Prism Pieces

Definition 6. *The k -sided regular prism is a right prism whose base is a regular k -sided polygon. We leave the bases of the prism unlabeled, because they cannot become visible. The other k faces we label in order, as depicted in Fig. 6. We define \mathcal{R}_k to be the set of piece configurations of a k -sided regular prism: in particular, \mathcal{R}_k is the union of all cyclic shifts of $\langle 1, \dots, k \rangle$ and its reverse.*

Note that in the case of $k = 4$, we assume that the height of the prism and the side length of the square base are distinct, so that the pieces do not have the full range of motion of the original Instant Insanity puzzle.

Our first result concerns right prism pieces with an equilateral triangle base:

Theorem 2. *The PARTIAL-INSANITY(\mathcal{R}_3) problem can be solved in polynomial time.*

Proof. Let \mathcal{P}_k be the set of all permutations of $\{1, \dots, k\}$. By definition, $\mathcal{R}_3 = \mathcal{P}_3$. Furthermore, for any k , the set of piece configurations \mathcal{P}_k is rotationally symmetric, so by Theorem 1, if the COMPLETE-INSANITY(\mathcal{P}_k) problem is solvable in polynomial time, then the PARTIAL-INSANITY(\mathcal{P}_k) problem is also solvable in polynomial time. Hence we may examine the COMPLETE-INSANITY(\mathcal{P}_k) instead.

In particular, our goal is to show that COMPLETE-INSANITY(\mathcal{P}_k) can be solved if and only if every color $c \in C$ occurs exactly k times in the pieces A_1, \dots, A_n . It is easy to see that if a particular series of pieces A_1, \dots, A_n can be solved, then the number of occurrences of each color must be k . We may show the converse by induction on k .

First, consider the base case COMPLETE-INSANITY(\mathcal{P}_1). If all colors occur exactly once, then the only possible configuration of the puzzle is in fact a solved state. By induction, we assume that COMPLETE-INSANITY(\mathcal{P}_{k-1}) can be solved if each color is used exactly $k-1$ times. Suppose that we have an instance A_1, \dots, A_n of the COMPLETE-INSANITY(\mathcal{P}_k) puzzle in which each color is used exactly k times. We construct a multigraph G as follows:

1. For each index $1 \leq i \leq n$, construct a node u_i .
2. For each color $c \in C$, construct a node v_c .
3. For each piece $A_i = (a_1, \dots, a_k)$ and each side $1 \leq j \leq k$, construct an edge from u_i to v_{a_j} . (Note that this may result in the creation of parallel edges.)

Because each piece has k colors, and each color is used exactly k times, G is a k -regular bipartite multigraph. By applying Hall's theorem [5], we know that G has a perfect matching, which can be computed in $O(kn\sqrt{n})$ time with the Hopcroft-Karp algorithm [7]. For each edge (u_i, v_c) in the matching, we wish to configure piece A_i so that the color on the first visible side is c . By using the matching to determine the colors on side 1, we ensure that no color will occur twice. Furthermore, because the set of piece configurations \mathcal{P}_k consists of all permutations, we know that for each piece A_i , the remaining $k-1$ sides can be arbitrarily assigned to the remaining $k-1$ visible sides of the puzzle. In particular, we are left with an instance of the COMPLETE-INSANITY(\mathcal{P}_{k-1}) problem in which each color occurs $k-1$ times. Hence, by induction, the puzzle is solvable. In particular, the PARTIAL-INSANITY(\mathcal{R}_3) problem can be solved in $O(n\sqrt{n})$ time. \square

Note that the structure of this proof reveals some interesting properties about the problem COMPLETE-INSANITY(\mathcal{R}_3). In particular, we don't need to know anything about the assignment of colors to pieces to determine whether a given COMPLETE-INSANITY(\mathcal{R}_3) puzzle can be solved — we need only check whether the number of occurrences of each color is 3. Furthermore, careful analysis reveals that the reduction of Theorem 1 will produce an COMPLETE-INSANITY(\mathcal{R}_3) puzzle with 3 copies of each color if and only if the original PARTIAL-INSANITY(\mathcal{R}_3) puzzle has at most 3 copies of each color. Hence, to determine whether a given

instance of PARTIAL-INSANITY(\mathcal{R}_3) can be solved, it is sufficient to check that every color occurs ≤ 3 times.

The PARTIAL-INSANITY(\mathcal{R}_4) problem is not as easy to solve:

Theorem 3. *The PARTIAL-INSANITY(\mathcal{R}_4) problem is NP-complete.*

Proof. We show that PARTIAL-INSANITY(\mathcal{R}_4) is NP-complete by a reduction from the POSITIVE-NAE-SAT problem. Suppose that we are given a POSITIVE-NAE-SAT instance θ with n clauses, each containing three literals. Let m be the number of variables. Then we can represent which literals are associated with which variables by a function $\gamma : \{0, \dots, 3n - 1\} \rightarrow \{0, \dots, m - 1\}$ mapping the index of each literal to the index of the corresponding variable. Let $L(i) = \{x \mid \gamma(x) = i\}$, and let $\ell(i, k)$ be the k th literal index in $L(i)$. Then our reduction proceeds as follows.

For each literal index $0 \leq x \leq 3n - 1$, add three colors: a_x , b_x , and c_x . For each variable $0 \leq i \leq m - 1$ and each index $0 \leq h \leq |L(i)| - 1$, let $x = \ell(i, h)$ and $y = \ell(i, (h + 1) \bmod |L(i)|)$. Use these values to construct two pieces:

$$\begin{aligned} A_{x,0} &= \langle a_x, b_x, c_x, b_x \rangle \\ A_{x,1} &= \langle a_x, c_x, b_y, c_x \rangle \end{aligned}$$

Intuitively, the placement of the color a_x will determine whether or not the corresponding literal is true — in particular, the structure of the pieces $A_{x,0}$ and $A_{x,1}$ ensures that the two copies of the color a_x must occur on opposite sides of the puzzle. Hence the color a_x can either be placed on sides 1 and 3, or on sides 2 and 4. These two possibilities represent the two possible assignments to literal x . The use of the color b_y in $A_{x,1}$ ensures that the assignment to literal x is the same as the assignment to literal y , so that the assignment is consistent.

We must also add pieces to ensure that the assignment is satisfying. For each clause $0 \leq j \leq n - 1$, add a new color d_j and construct the following three pieces:

$$\begin{aligned} B_{3j+0} &= \langle d_j, a_{3j+0}, *, * \rangle \\ B_{3j+1} &= \langle d_j, a_{3j+1}, *, * \rangle \\ B_{3j+2} &= \langle d_j, a_{3j+2}, *, * \rangle \end{aligned}$$

We wish to show that, when taken together, these pieces form a puzzle that can be solved if and only if the original POSITIVE-NAE-SAT problem can be solved.

Suppose that we have a consistent solution to the original POSITIVE-NAE-SAT instance θ . We can represent this solution as a function $\phi : \{1, \dots, 3n\} \rightarrow \{T, F\}$ assigning a value of true or false to each of the literals. This assignment has the property that for any pair of indices j_1, j_2 , if $\gamma(j_1) = \gamma(j_2)$, then $\phi(j_1) = \phi(j_2)$. Using this assignment, we wish to construct configurations $p_{0,0}, p_{0,1}, \dots, p_{3n-1,0}, p_{3n-1,1}$ for the pieces $A_{0,0}, A_{0,1}, \dots, A_{3n-1,0}, A_{3n-1,1}$ and configurations q_0, \dots, q_{3n-1} for the pieces B_0, \dots, B_{3n-1} . We define these configurations as follows:

- For each literal $0 \leq x \leq 3n - 1$, if $\phi(x) = T$, then we set $p_{x,0} = \langle 1, 2, 3, 4 \rangle$ and $p_{x,1} = \langle 3, 4, 1, 2 \rangle$. Otherwise, we set $p_{x,0} = \langle 2, 3, 4, 1 \rangle$ and $p_{x,1} = \langle 4, 1, 2, 3 \rangle$.

- For each clause $0 \leq j \leq n-1$, we assign q_{3j+0} , q_{3j+1} , and q_{3j+2} according to the following table:

$\phi(3j+0)$	$\phi(3j+1)$	$\phi(3j+2)$	q_{3j+0}	q_{3j+1}	q_{3j+2}
T	T	F	$\langle 1, 2, 3, 4 \rangle$	$\langle 3, 4, 1, 2 \rangle$	$\langle 2, 3, 4, 1 \rangle$
T	F	T	$\langle 1, 2, 3, 4 \rangle$	$\langle 2, 3, 4, 1 \rangle$	$\langle 3, 4, 1, 2 \rangle$
T	F	F	$\langle 1, 2, 3, 4 \rangle$	$\langle 2, 3, 4, 1 \rangle$	$\langle 4, 1, 2, 3 \rangle$
F	T	T	$\langle 4, 1, 2, 3 \rangle$	$\langle 3, 4, 1, 2 \rangle$	$\langle 1, 2, 3, 4 \rangle$
F	T	F	$\langle 4, 1, 2, 3 \rangle$	$\langle 3, 4, 1, 2 \rangle$	$\langle 2, 3, 4, 1 \rangle$
F	F	T	$\langle 4, 1, 2, 3 \rangle$	$\langle 2, 3, 4, 1 \rangle$	$\langle 1, 2, 3, 4 \rangle$

It may be verified that by configuring the pieces in this way, we ensure that no color appears twice on the same side of the puzzle. Hence we have shown that the existence of a solution to the POSITIVE-NAE-SAT instance implies the existence of a solution to the constructed PARTIAL-INSANITY(\mathcal{R}_4) puzzle.

Suppose now that the PARTIAL-INSANITY(\mathcal{R}_4) problem can be solved. Let $p_{0,0}, \dots, p_{3n-1,1}$ be the sequence of configurations for the pieces $A_{0,0}, \dots, A_{3n-1,1}$, and let q_0, \dots, q_{3n-1} be the sequence of configurations for B_0, \dots, B_{3n-1} . Then we construct an assignment function $\phi : \{1, \dots, 3n\} \rightarrow \{T, F\}$ as follows: for each literal $0 \leq x \leq 3n-1$, if the configuration $p_{x,0} = (s_1, s_2, s_3, s_4)$, then we define $\phi(x) = T$ if and only if $s_1 \equiv 1 \pmod{2}$. To see that this is a valid assignment, we must show that this assignment is both consistent and satisfying.

Suppose, for the sake of contradiction, that the assignment is not consistent. This means that there exists some variable i such that $\{\phi(j) \mid \gamma(j) = i\} = \{T, F\}$. Then there must be some index h such that $\phi(\ell(i, h)) \neq \phi(\ell(i, (h+1) \bmod |L(i)|))$. Let $x = \ell(i, h)$ and let $y = \ell(i, (h+1) \bmod |L(i)|)$. Without loss of generality, we may assume that $\phi(x) = F$ and $\phi(y) = T$. Hence $F_2(A_{y,0}, p_{y,0}) = F_4(A_{y,0}, p_{y,0}) = b_y$. No color is used twice on the same side, so we know that either $F_1(A_{x,1}, p_{x,1}) = b_y$ or $F_3(A_{x,1}, p_{x,1}) = b_y$. In either case, all possible configurations $p_{x,1}$ have the property that $F_2(A_{x,1}, p_{x,1}) = F_4(A_{x,1}, p_{x,1}) = c_x$. By assumption, $\phi(x) \neq \phi(y)$, so either $F_2(A_{x,0}, p_{x,0}) = c_x$ or $F_4(A_{x,0}, p_{x,0}) = c_x$. In either case, we have a contradiction — colors cannot be repeated on the same side of the puzzle. Hence, the assignment of values to literals must be consistent.

Now suppose that there is some clause j such that $\phi(3j+0) = \phi(3j+1) = \phi(3j+2)$. Without loss of generality, we may assume that all three of these values are T . For each literal index $x \in \{3j+0, 3j+1, 3j+2\}$, pieces $A_{x,0}$ and $A_{x,1}$ must be configured so that they do not place the color c_x on the same side of the puzzle. Furthermore, because $\phi(x) = T$, we know that the configuration of $A_{x,0}$ must place the color a_x on side 1 or side 3. Together, these constraints ensure that the configuration of $A_{x,1}$ must also place the color a_x on side 1 or side 3. Hence between the pieces $A_{x,0}$ and $A_{x,1}$, the color a_x must be placed both on side 1 and on side 3. Then B_x must place a_x on either side 2 or side 4, and must therefore place d_j on side 1 or 3. This holds for all $x \in \{3j+0, 3j+1, 3j+2\}$. As a result, d_j must show up at least twice on either side 1 or side 3. This contradicts our assumption that the puzzle was solved. \square

3.3 Regular Prism Pieces with Restricted Motion

In this section, we consider what happens when the motion of the pieces is limited. In particular, we consider what happens when the pieces of the puzzle are mounted on a central dowel, so that the set of allowable motions is restricted to rotations around the dowel.

Definition 7. We define \mathcal{U}_k to be the unflippable piece configurations of the k -sided regular prism: in particular, \mathcal{U}_k is the set of all cyclic shifts of $\langle 1, \dots, k \rangle$.

Theorem 4. The $\text{PARTIAL-INSANITY}(\mathcal{U}_k)$ problem is NP-complete for all $k \geq 3$.

Proof. Suppose that we are given a POSITIVE-NAE-SAT instance θ with n clauses, each containing three literals. Let m be the number of variables. Again, we represent the relationship between variables and the corresponding literal indices with a function $\gamma : \{0, \dots, 3n - 1\} \rightarrow \{0, \dots, m - 1\}$ mapping the index of each literal to the index of the corresponding variable.

Let $L(i) = \{x \mid \gamma(x) = i\}$, and let $\ell(i, j)$ be the j th literal index in $L(i)$. Then our reduction proceeds as follows. For each literal index $0 \leq x \leq 3n - 1$, construct six colors: $a_x, b_{x,1}, b_{x,2}, c_{x,1}, c_{x,2}$, and d_x . We then use these colors to define three pieces for each literal index x :

$$\begin{aligned} A_x &= \langle \underbrace{a_x, \dots, a_x}_{k-2 \text{ times}}, b_{x,1}, b_{x,2} \rangle \\ B_x &= \langle \underbrace{*, \dots, *}_{k-3 \text{ times}}, a_x, c_{x,1}, c_{x,2} \rangle \\ C_x &= \langle \underbrace{*, \dots, *}_{k-3 \text{ times}}, a_x, d_x, * \rangle \end{aligned}$$

To ensure that all of the pieces A_0, \dots, A_{3n-1} have the same configuration, we use a series of gadgets, each of which consists of $k-1$ identical pieces. Specifically, for every $x \in \{0, \dots, 3n-2\}$, we create $k-1$ copies of the following piece:

$$G_x = \langle \underbrace{*, \dots, *}_{k-2 \text{ times}}, b_{x,1}, b_{x+1,2} \rangle$$

These $k-1$ copies of G_x , combined with the piece A_x , all have the color $b_{x,1}$ on side $k-1$. Hence, each piece must have a different configuration. There are k possible configurations, and k pieces, so each configuration is used exactly once. The configurations are rotationally symmetric, so we may suppose without loss of generality that the configuration of A_x is $\langle 1, \dots, k \rangle$. Every other possible cyclic shift is used to configure one of the pieces G_x . Hence, the color $b_{x+1,2}$ occurs on sides $\{1, 2, 3, \dots, k-1\}$ — everything except side k . So the only possible configuration for piece A_{x+1} is $\langle 1, \dots, k \rangle$. By induction, for any pair x_1, x_2 , the configuration of A_{x_1} must be the same as the configuration of A_{x_2} .

We also wish to ensure that for any x_1, x_2 such that $\gamma(x_1) = \gamma(x_2)$, the configuration of B_{x_1} is the same as the configuration of B_{x_2} . We accomplish

this with a nearly identical construction. For each variable i , and each index $h \in \{0, \dots, |L(i)| - 2\}$, let $x = \ell(i, h)$ and let $y = \ell(i, h + 1)$. Then we create $k - 1$ copies of the following piece:

$$H_x = \langle \underbrace{*, \dots, *}_{k-2 \text{ times}}, c_{x,1}, c_{y,2} \rangle$$

By a similar argument, this enforces the desired constraint.

Finally, for each clause $j \in \{0, \dots, n - 1\}$, we add one more piece:

$$D_j = \langle \underbrace{d_{3j+0}, \dots, d_{3j+0}}_{k-3 \text{ times}}, d_{3j+0}, d_{3j+1}, d_{3j+2} \rangle.$$

This completes the construction. Next, we must show that this construction is in fact a reduction: that there is a solution to this instance of the PARTIAL-INSANITY(\mathcal{U}_k) problem if and only if there is a solution to the original POSITIVE-NAE-SAT problem.

Suppose that we are given an assignment $\phi : \{0, \dots, 3n - 1\} \rightarrow \{T, F\}$ mapping each literal to true or false. Then we can construct a solution to the PARTIAL-INSANITY(\mathcal{U}_k) puzzle as follows. For each A_x , use the configuration $\langle 1, \dots, k \rangle$. If $\phi(x) = T$, then the configuration of B_x is $\langle k, 1, \dots, k - 1 \rangle$ and the configuration of C_x is $\langle k - 1, k, 1, \dots, k - 2 \rangle$. Otherwise, the configuration of B_x is $\langle k - 1, k, 1, \dots, k - 2 \rangle$ and the configuration of C_x is $\langle k, 1, \dots, k - 1 \rangle$. The configuration of D_j is determined according to the following table:

$\phi(3j + 0)$	$\phi(3j + 1)$	$\phi(3j + 2)$	configuration of D_j
T	T	F	$\langle k, 1, \dots, k - 2, k - 1 \rangle$
T	F	T	$\langle k - 1, k, 1, \dots, k - 2 \rangle$
T	F	F	$\langle k - 1, k, 1, \dots, k - 2 \rangle$
F	T	T	$\langle 1, \dots, k - 2, k - 1, k \rangle$
F	T	F	$\langle k, 1, \dots, k - 2, k - 1 \rangle$
F	F	T	$\langle 1, \dots, k - 2, k - 1, k \rangle$

It can be verified that by configuring the pieces in this way, we ensure that each color occurs at most once per side.

Because the configurations of A_{x_1} and A_{x_2} are the same for any $x_1, x_2 \in \{0, \dots, 3n - 1\}$, there is a way to configure the gadgets G_x so that the colors $b_{x,1}$ and $b_{x,2}$ occur exactly once on each visible side of the puzzle. Similarly, because the configurations of A_{x_1} and A_{x_2} are the same for any $x_1, x_2 \in \{0, \dots, 3n - 1\}$ with $\gamma(x_1) = \gamma(x_2)$, there is a way to configure the gadgets H_x .

We have shown that if there exists a satisfying assignment for the original POSITIVE-NAE-SAT problem, then there is a solution to the corresponding PARTIAL-INSANITY(\mathcal{U}_k) puzzle. Now we wish to show the converse. To that end, suppose that we have a solution to the puzzle. For each literal $x \in \{0, \dots, 3n - 1\}$, let p_x be the configuration of A_x , let q_x be the configuration of B_x , and let r_x be the configuration of C_x . Furthermore, for each clause $j \in \{0, \dots, n - 1\}$, let s_j be the configuration of D_j . Without loss of generality, suppose that $p_0 = \langle 1, \dots, k \rangle$.

Then, as we have argued previously, the $k - 1$ copies of the pieces G_0, \dots, G_{3n-2} ensure that $p_x = \langle 1, \dots, k \rangle$ for all literal indices x .

We define our assignment function $\phi : \{0, \dots, 3n - 1\} \rightarrow \{T, F\}$ as follows: $\phi(x) = T$ if and only if $q_x = \langle k, 1, \dots, k - 1 \rangle$. To see that this assignment function is consistent, we recall that the $k - 1$ copies of the pieces H_0, \dots, H_{3n-2} ensure that $q_{x_1} = q_{x_2}$ for all $x_1, x_2 \in \{0, \dots, 3n - 1\}$ with $\gamma(x_1) = \gamma(x_2)$. Hence, if $\gamma(x_1) = \gamma(x_2)$, then $\phi(x_1) = \phi(x_2)$.

It remains to show that our assignment function is satisfying: for all clauses $j \in \{0, \dots, n - 1\}$, $\{\phi(3j + 0), \phi(3j + 1), \phi(3j + 2)\} = \{T, F\}$. Suppose, for the sake of contradiction, that there is some clause with $\phi(3j + 0) = \phi(3j + 1) = \phi(3j + 2)$. We know that $p_x = \langle 1, \dots, k \rangle$ for all $x \in \{3j + 0, 3j + 1, 3j + 2\}$, so the color a_x already shows up on sides $\{1, \dots, k - 2\}$. Hence the configurations q_x and r_x must be in the set $\{\langle k, 1, \dots, k - 1 \rangle, \langle k - 1, k, 1, \dots, k - 2 \rangle\}$. We also know that $q_x \neq r_x$ for all $x \in \{3j + 0, 3j + 1, 3j + 2\}$.

If $\phi(3j + 0) = \phi(3j + 1) = \phi(3j + 2) = T$, then $q_{3j+0} = q_{3j+1} = q_{3j+2} = \langle k, 1, \dots, k - 1 \rangle$, and $r_{3j+0} = r_{3j+1} = r_{3j+2} = \langle k - 1, k, 1, \dots, k - 2 \rangle$. This means that the three pieces C_{3j+0} , C_{3j+1} , and C_{3j+2} map the three colors d_{3j+0} , d_{3j+1} , and d_{3j+2} to side 1. Hence, there is no way to configure D_j to avoid duplicating one of the colors on one of the sides. A similar argument shows that setting $\phi(3j + 0) = \phi(3j + 1) = \phi(3j + 2) = F$ also leads to contradiction. Hence, the assignment is satisfying as well as consistent. \square

4 Irregular Prism Pieces

In this section, we consider more irregular pieces. Because the symmetries of the shape determine the number of possible configurations, puzzles with highly asymmetric shapes are easy to solve. For instance, if all edge lengths are distinct (as in the case of a generic k -gon), then there is exactly one possible configuration for each piece, making the puzzle trivial to solve. Nonetheless, there are a few shapes with interesting symmetries, some of which we discuss here.

The results in this section derive from the following theorem:

Theorem 5. *For any set of piece configurations P , if $|P| = 2$, then PARTIAL-INSANITY(P) can be solved in polynomial time.*

Proof. We may show this by a reduction to 2-SAT. Let p_1, p_2 be the two piece configurations in P . Suppose that we are given a set of pieces A_1, \dots, A_n . Then we construct two variables for each piece A_i : a variable $x_{i,1}$ that is true if and only if A_i is in configuration p_1 , and a variable $x_{i,2}$ that is true if and only if A_i is in configuration p_2 . We additionally construct two clauses for each A_i : $(x_{i,1} \vee x_{i,2})$ and $(\neg x_{i,1} \vee \neg x_{i,2})$. This ensures that A_i must be placed in exactly one of the two configurations.

Next, we construct constraints to ensure that no color is used twice. For each pair of piece indices $1 \leq i_1 \neq i_2 \leq n$ and for each pair of (not necessarily distinct) configurations $j_1, j_2 \in \{1, 2\}$, we examine the k_2 visible sides. If there exists some $\ell \in \{1, \dots, k_2\}$ such that $F_\ell(A_{i_1}, p_{j_1}) = F_\ell(A_{i_2}, p_{j_2})$, then we know

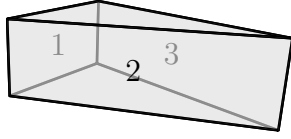


Fig. 7. Labels for the faces on a right prism with an isosceles triangle base.

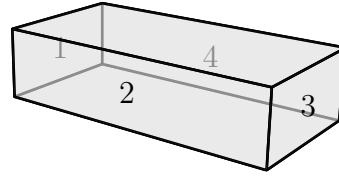


Fig. 8. Labels for each of the potentially visible faces on a box.

that the pair of assigned configurations is invalid, and so we add a constraint $(\neg x_{i_1, j_1} \vee \neg x_{i_2, j_2})$ to prevent the reuse of colors on a single side.

Because these constraints completely encapsulate the requirements of the $\text{PARTIAL-INSANITY}(P)$ problem, we may use a standard 2-SAT algorithm to find a satisfying assignment, then use the satisfying assignment to determine how we configure the pieces. \square

For example, we can make use of Theorem 5 to help analyze the complexity of solving a puzzle with the following shape:

Definition 8. *Suppose that the piece shape is a right prism with an isosceles triangle base. If we number the sides of the prism in order around the base, as depicted in Fig. 7, then the set of configurations is $\mathcal{T} = \{\langle 1, 2, 3 \rangle, \langle 1, 3, 2 \rangle\}$.*

Because $|\mathcal{T}| = 2$, we can show the following:

Theorem 6. *The problem $\text{PARTIAL-INSANITY}(\mathcal{T})$ can be solved in polynomial time.*

We can also use Theorem 5 to help us analyze more complex shapes:

Definition 9. *Suppose that the piece shape is a box with dimensions $w \times h \times d$, where the goal is to stack pieces into a tower with base $w \times h$. If we number the sides of the prism in order around the base, as depicted in Fig. 8, then the set of configurations is $\mathcal{B} = \{\langle 1, 2, 3, 4 \rangle, \langle 3, 4, 1, 2 \rangle, \langle 1, 4, 3, 2 \rangle, \langle 3, 2, 1, 4 \rangle\}$.*

Theorem 7. *The problem $\text{PARTIAL-INSANITY}(\mathcal{B})$ can be solved in polynomial time.*

Proof. The set of configurations \mathcal{B} is generated by allowing side 1 to swap freely with side 3, and allowing side 2 to swap freely with side 4. Hence, we can decompose the $\text{PARTIAL-INSANITY}(\mathcal{B})$ problem into two instances of the $\text{PARTIAL-INSANITY}(\mathcal{P}_2)$ problem: one for sides 1 and 3, and one for sides 2 and 4. If we solve both halves of the problem with the technique of Theorem 5, it is straightforward to combine the solutions to compute the configuration of each piece. \square

5 Conclusion

In this paper, we have examined several variants of Instant Insanity, exploring how the complexity changes as the geometry of the pieces changes. We have also explored how restricting the motion of the pieces can change the complexity. In particular, we have analyzed several types of triangular prism puzzles and rectangular prism puzzles, discovering which variants are NP-complete and which can be solved in polynomial time.

Our results leave open a few problems. In particular, the complexity of $\text{PARTIAL-INSANITY}(\mathcal{R}_k)$ for $k \geq 5$ is still unknown. Because the $\text{PARTIAL-INSANITY}(\mathcal{R}_4)$ problem is NP-complete, it is likely (but not yet proven) that the $\text{PARTIAL-INSANITY}(\mathcal{R}_k)$ problem is NP-complete for $k \geq 5$ as well.

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