# On the Behavior of Threshold Models over Finite Networks

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***Abstract*—We study a model for cascade effects over finite networks based on a deterministic binary linear threshold model. Our starting point is a networked coordination game where each agent's payoff is the sum of the payoffs coming from pairwise interaction with each of the neighbors. We first establish that the best response dynamics in this networked game is equivalent to the linear threshold dynamics with heterogeneous thresholds over the agents. While the previous literature has studied such linear threshold models under the assumption that each agent may change actions at most once, a study of best response dynamics in such networked games necessitates an analysis that allows for multiple switches in actions. In this paper, we develop such an analysis. We first establish that agent behavior cycles among different actions in the limit, we characterize the length of such limit cycles, and reveal bounds on the time steps required to reach them. We finally propose a measure of network resilience that captures the nature of the involved dynamics. We prove bounds and investigate the resilience of different network structures under this measure.**

## I. Introduction

Interactions over many different types of networks require coordination between agents and their neighbors. For example, in economic networks, technologies that conform to the standards used by other related firms are more productive, and in social networks, conformity to the behavior of friends is valuable for a variety of reasons. The desire for such coordination can lead to cascading behavior: the adoption decision of some agents can spread to their neighbors and from there to the rest of the network. One of the most commonly used models of such cascading behavior is the linear threshold model originally introduced by Granovetter [1]. This model is used to explain a variety of aggregate level behaviors including diffusion of innovation, voting, propagation of rumors and diseases, spread of riots and strikes, and dynamics of opinions.

Most analyses of this model assume that one of the behaviors adopted by the agents (represented by the nodes of a graph) is irreversible, meaning that agents can only make a single switch into this behavior and can never switch out from it. This assumption may be well justified in some settings (e.g. educational attainment), however it is restrictive for many other applications. For example, many of the behaviors in social networks, corresponding to product choices, opinions and actions, change regularly.

In this paper, we study a model of cascades based on binary linear threshold dynamics. We start from an explicit coordination game set over a finite undirected network. The payoff of each agent is the sum of the payoff in a *two player and two action coordination game* the agent plays pairwise with each of its neighbors (the action is fixed across all interactions). We then study the behavior induced by best response dynamics, whereby each agent changes the played action to that which yields highest payoff given the actions of the neighbors. We first establish that best response dynamics are identical to the dynamics traced by the linear threshold model with heterogeneous thresholds for the agents. However, crucially, actions can change multiple times. Thus the dynamics of interest for the set of problems posed here cannot be studied using existing results and in fact have a different mathematical structure. The main contribution of our paper is to fully characterize the limiting behavior of these dynamics.

We establish that agent behavior cycles among different actions in the limit. Our analysis relies on first embedding the dynamics (over any graph structure) into a bipartite structure while preserving local properties, then transforming the parallel dynamics into sequential dynamics to obtain desired results. Using this technique, we show that the limit cycles consist of at most two action profiles for *any graph structure* and *any threshold distribution* over the agents. Substantively, this means that in the limit, each player either sticks to playing one particular action, or switches actions at every time step. We also establish a uniform upper-bound on the time steps needed to reach this cyclic behavior that is quadratic in the number of agents. We mention that similar results on convergence cycles and quadratic convergence time for linear threshold models (termed differently) have appeared in the literature on Cellular Automata in [12]. We approach the problem from a different perspective and provide different insight. We further improve the convergence time to be uniformly not more than the size of the network whenever the graph in concern is a tree.

Of central importance in the study of cascades over networks is the resilience of networks to invasion by certain types of behavior (e.g., cascades of failures or spread of epidemics). For the new dynamics defined by our problem, we define a measure of resilience of a network to such invasion that captures the the minimal 'cost of recovery' needed when the model is confronted with a perturbation in the agents' action profile. We prove achievable uniform lower-bounds and upper-bounds on the resilience measure, we list the resilience measure of some network structures and provide basic insight on how different network structures affect this measure.

Our paper is related to a large literature on network dynamics and linear threshold models (see e.g., [2]-[7]). A number of papers in this literature investigate the question of whether a behavior initially adopted by a subset of agents (i.e., *the seed set*) will spread to a *large* portion of the network, focusing on the dynamics where agents can make a single switch

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to one of the behaviors. Morris [2], while starting from a multi-switch version of the dynamics, studied without loss of generality the single-switch version to answer whether there exists a finite set of initial adopters (in an infinite network with homogeneous thresholds) such that the behavior diffuses to the entire network. In [5], Watts derives conditions for the behavior to spread to a positive fraction of the network (represented by a random graph with given degree distribution) using a branching process analysis. Similarly, Lelarge [6] provides an explicit characterization of the expected fraction of the agents that adopt the behavior in the limit over such networks. Related work [4] studies how to target a fixed number of agents (and change their behavior) in order to maximize the spread of the behavior in the network in the limit. In the context of network resilience, the recent paper [7] adopts single-switch linear threshold dynamics as a model of failures in a network. This work defines a measure of network resilience that is a function of the graph topology and the distribution over thresholds and studies this measure for different network structures focusing on $d$-regular graphs (hence ignoring the effect of the degree distribution of a graph on cascaded failures). Here we provide a novel resilience measure that highlights the impact of heterogeneity in thresholds and degrees of different agents. Finally, noisy versions of best-response dynamics in networked coordination games were studied in [8] and [9] (see also [10] and [11] in the statistical mechanics literature). The random dynamics in these models can be represented in terms of Markov chains with absorbing states, and therefore do not exhibit the cyclic behavior predicted by the multi-switch linear threshold model studied in this paper.

The rest of the paper proceeds as follows. We begin by a description of the model in section II. We then proceed in section III to describe the general behavioral rules of the dynamics. We branch out to characterize convergence cycles and convergence time in sections IV and V. We finally propose the network resilience measure in section VI.

## II. MODEL

We define a networked coordination game. For a positive integer $n$, we denote by $\mathcal{I}_n$ the set of $n$ players.[1]For technical convenience, we assume that $\mathcal{I}_n \subset \mathcal{I}_m$ for $n < m$.[2] We define $\mathcal{G}_n$ to be the class of all undirected graphs $G(\mathcal{I}_n, E)$ defined over the vertex set $\mathcal{I}_n$, with edge set $E$. To be proper, $E$ is a relation [3]on $\mathcal{I}_n$, but for convenience we will consider the set $E$ to have cardinality exactly equal to the number of undirected edges. We denote an undirected edge in $E$ by $\{i,j\}$, and we abbreviate it to $ij$ when no confusion arises. For $G(\mathcal{I}_n, E)$ in $\mathcal{G}_n$, we use $\mathcal{N}_G(i)$ to denote the neighborhood of player $i$ in $G$, i.e. $\mathcal{N}_G(i) = \{\ j \in \mathcal{I}_n : ij \in E\ \}$. We denote by $d_G(i)$ the degree of player $i$ in $G$, namely the cardinality of $\mathcal{N}_G(i)$. We refer to $\mathcal{N}_G(i)$ and $d_G(i)$ as $\mathcal{N}_i$ and $d_i$ respectively, when the underlying graph is clear from the context. We finally define $\mathcal{Q}_n$ to be the space of type distributions over the agents, namely the set of maps from $\mathcal{I}_n$ into $[0,1]$. For $q$ in $\mathcal{Q}_n$, we refer to $q_i$ as the *type* of player $i$.

Given a graph $G(\mathcal{I}_n, E)$, each player $i$ in $\mathcal{I}_n$ plays one action $a_i$ in $\{\mathbb{B}, \mathbb{W}\}$. For $ij \in E$, we define the payoff received by agent $i$ when playing $a_i$ against agent $j$ playing $a_j$ to be

$$g_{i,j}(a_i, a_j) = \begin{cases} q_i & \text{if } a_i = a_j = \mathbb{W} \\ 1 - q_i & \text{if } a_i = a_j = \mathbb{B} \\ 0 & \text{if } a_i \neq a_j \end{cases} . \tag{1}$$

The utility player $i$ gets is the sum of the payoffs from the pairwise interactions with the players in $\mathcal{N}_i$, namely when player $j$ plays action $a_j$,

$$u_i(a_i, a_{-i}) = \sum_{j \in \mathcal{N}_i} g_{i,j}(a_i, a_j), \tag{2}$$

where $a_{-i}$ denotes the action profile of all players except $i$.

We define $\mathcal{A}_n$ be the space of action profiles[4] played by the agents, namely the set of maps from $\mathcal{I}_n$ into $\{\mathbb{B}, \mathbb{W}\}$. The players are assigned an initial action profile $\underline{a}$, we refer to $\underline{a}$ as the action profile of the players at time step 0. For $T$ in $\mathbb{N}$,[5] every player best responds to the action profile of the players at time step $T-1$, by choosing the action that maximizes its utility. We suppose that players play action $\mathbb{W}$ as a tie breaking rule. Formally we impose a strict order on $\{\mathbb{W}, \mathbb{B}\}$ such that $\min\{\mathbb{W}, \mathbb{B}\} = \mathbb{W}$. Suppose we denote by $a_{i,T}$ the action played by player $i$ at time $T$, then given an initial action configuration $\underline{a}$ in $\mathcal{A}_n$, for every player $i$, we recursively define:

$$\begin{aligned} &a_{i,0} = \underline{a}_i \\ &a_{i,T} = \min \operatorname*{argmax}_{a_i \in \{\mathbb{W}, \mathbb{B}\}} u_i(a_i, a_{-i,T-1}), \quad \text{for } T \in \mathbb{Z}^+, \end{aligned} \tag{3}$$

where the $\min$ operator breaks ties. The rule induced by the recursive definition in (3) is equivalent to the rule provided in the following proposition.

*Proposition 1:* Let $\underline{a}$ be the initial action configuration, namely the action profile of the players at time step 0. For every positive integer $T$, player $i$ plays action $\mathbb{B}$ at time step $T$ if and only if more than $q_i d_i$ neighbors of player $i$ played action $\mathbb{B}$ at time step $T-1$.

*Proof:* We substitute $u_i$ in (3) with the expressions in (1) and (2), and get that player $i$ plays action $\mathbb{B}$ at time $T$ if and only if

$$\sum_{j \in \mathcal{N}_i} (1 - q_i) \mathbf{1}_{\{\mathbb{B}\}}(a_{j,T-1}) > \sum_{j \in \mathcal{N}_i} q_i \mathbf{1}_{\{\mathbb{W}\}}(a_{j,T-1}), \tag{4}$$

[1]We use the words player, agent, node and vertex interchangeably.

[2]We use the letters $i$ and $j$ to denote agents. We reserve the letter $n$ for the number of players in the game. If it is clear from the context to which set $X$ an element $x$ belongs to, we refrain from mentioning the set $X$ explicitly to simplify notation. Moreover, for any function $f$ with domain $\mathcal{I}_n$, we will denote $f(i)$ by $f_i$. In particular, for functions $q$, $k$ and $a$ with domain $\mathcal{I}_n$, $q(i)$, $k(i)$ and $a(i)$ are denoted $q_i$, $k_i$ and $a_i$ respectively.

[3]A (binary) relation $R$ on a set $A$ is a subset of $A \times A$. We use the notation $aRb$ to denote $(a,b) \in R$.

[4]We use the words profile and configuration interchangeably.

[5]We denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{Z}^+$ the set of positive integers.

where $\mathbf{1}_\Gamma(x) = 1$ if and only if $x \in \Gamma$. Equivalently, player $i$ plays action $\mathbb{B}$ at time $T$ if and only if

$$\sum_{j \in \mathcal{N}_i} \mathbf{1}_{\{\mathbb{B}\}}(a_{j,T-1}) > q_i d_i. \tag{5}$$

The left-side term is essentially summing the number of neighbors of player $i$ playing action $\mathbb{B}$. ■

As a technical clarification, every player is capable of switching actions both from $\mathbb{W}$ to $\mathbb{B}$ and $\mathbb{B}$ to $\mathbb{W}$.

## III. Description of the Dynamics

We begin by a coarse description of the involved dynamics. To sum up the model, we consider a *finite* set of players $\mathcal{I}_n$ along with three mathematical objects $\mathcal{G}_n$, $\mathcal{Q}_n$ and $\mathcal{A}_n$. An element $G(\mathcal{I}_n, E)$ of $\mathcal{G}_n$ corresponds to the network structure imposed on the players, an element $q$ of $\mathcal{Q}_n$ refers to the type distribution over the players, and an element $a$ of $\mathcal{A}_n$ consists an action profile played by the players. The triplet $G$, $q$ and $a$ interact as dictated by Proposition 1.

### A. From Types to Thresholds

Proposition 1 implies that playing $\mathbb{B}$ is never a best response for player $i$ if no player in $\mathcal{N}_i$ is playing $\mathbb{B}$. We will generalize our model to provide symmetry between both actions $\mathbb{B}$ and $\mathbb{W}$. We do this for two reasons. The first is to consider the linear threshold model as considered in the literature. The second is a technical reason, mainly to ensure closure of the set $\mathcal{G}_n \times \mathcal{Q}_n$ under certain operations. Nevertheless, any result for the generalized version of the model is inherited by the initial version trivially by inclusion.

We substitute the set $\mathcal{Q}_n$ by a set $\mathcal{K}_n$ and then modify the statement of Proposition 1. We define $\mathcal{K}_n$ to be the space of threshold distributions over the agents, namely the set of maps from $\mathcal{I}_n$ into $\mathbb{N}$. We make a particular distinction between the word *type* attributed to $\mathcal{Q}_n$ and the word *threshold* attributed to $\mathcal{K}_n$. For $k$ in $\mathcal{K}_n$, we refer to $k_i$ as the *threshold* of player $i$. Given a pair $(G, k)$ with $k \in \mathcal{K}_n$, we generalize Proposition 1 as follows:

*Proposition 2:* Let $\underline{a}$ be the initial action configuration, namely the action profile of the players at time step $0$. For every positive integer $T$, player $i$ plays action $\mathbb{B}$ at time step $T$ if and only if at least $k_i$ neighbors of player $i$ played action $\mathbb{B}$ at time step $T-1$.

The rule in Proposition 2 supersets the rule in Proposition 1. Indeed, for every $q$ in $\mathcal{Q}_n$ there exists a $k$ in $\mathcal{K}_n$ such that $q_i d_i$ may be substituted with the integer $k_i$ for all $i$ without changing the behavior of the players. It is also crucial to note that *more than* is replaced by *at least*.

For $G(I_n, E)$ in $\mathcal{G}_n$ and $k$ in $\mathcal{K}_n$, we denote by $G_k$ the map from $\mathcal{A}_n$ into $\mathcal{A}_n$ such that for player $i$, $(G_k a)_i = \mathbb{B}$ if and only if at least $k_i$ players are in $a^{-1}(\mathbb{B}) \cap \mathcal{N}_i$. From this perspective, given an initial configuration $a$ in $\mathcal{A}_n$, the sequence $a, G_k a, G_k^2 a, \cdots$ [6]corresponds to the sequence of action profiles $a, a_1, a_2, \cdots$ where $a_T = G_k^T a$ is the action profile played by the players at time $T$ if they act in accordance with the rule in Proposition 2.

[6]Let $f : A \to B$ and $g : B \to C$ be functions, we denote by $gf$ the function $g \circ f : A \to C$. In particular, if a function $f$ maps a set $A$ to itself, for a non-negative integer $m$, we denote by $f^m$ the function $f \circ f^{m-1}$ where $f^0$ is the identity map on $A$.

### B. The Limiting Behavior

To understand the limiting behavior, we note two fundamental properties: the space $\mathcal{A}_n$ has finite cardinality, and Proposition 2 is deterministic. Since $\mathcal{A}_n$ is finite, if we let $a_0, a_1, a_2, \cdots$ be any infinite sequence of action profiles played by the agents according to Proposition 2, then there exists at least one action profile $\hat{a}$ that will appear infinitely many times along this sequence. This follows from the pigeonhole principle. Since the dynamics are deterministic (and $a_{T+1}$ depends only on $a_T$), the same sequence of action profiles appears between any two consecutive occurrences of $\hat{a}$. This means that after a finite time step, the sequence of action profiles will cycle among action profiles.

Let us consider a different representation of the dynamics. We define a relation $\to$ on $\mathcal{A}_n$ such that for $a$ and $b$ in $\mathcal{A}_n$, $a \to b$ if and only if $b = G_k a$. Consider the graph $H(\mathcal{A}_n, \to)$, it forms a directed graph on the vertex set taken to be the space of action profiles $\mathcal{A}_n$, and an action profile $a$ is connected to an action profile $b$ by a directed edge $(a, b)$ going from $a$ to $b$ if and only if $b = G_k a$. Suppose we pick a vertex $a$, namely an action configuration, and perform a walk on vertices along the edges in $H$ starting from $a$. The walk eventually cycles vertices in the same order. Every initial action profile leads to one cycle, and two action profiles need not lead to the same cycle. We formalize the idea in the following definitions.

*Definition 1:* Given $(G, k)$ in $\mathcal{G}_n \times \mathcal{K}_n$, for two action profiles $a$ and $b$ in $\mathcal{A}_n$, it is said that $a$ can be reached from $b$ with respect to $G_k$ if there exists a non-negative integer $T$ such that $a = G_k^T b$. Formally, we define the relation $\mathcal{R}_{G_k}$ on $\mathcal{A}_n$ such that for $a$ and $b$ in $\mathcal{A}_n$, $a \mathcal{R}_{G_k} b$ if and only if there exists a non-negative integer $T$ such that $a = G_k^T b$.

If we construct a relation $\mathcal{C}_{G_k}$ on $\mathcal{A}_n$ such that for $a$ and $b$ in $\mathcal{A}_n$, $a \mathcal{C}_{G_k} b$ if and only if $a \mathcal{R}_{G_k} b$ or $b \mathcal{R}_{G_k} a$, then $\mathcal{C}_{G_k}$ is an equivalence relation on $\mathcal{A}_n$. Two configurations in $\mathcal{A}_n$ are in the same equivalence class with respect to the relation $C_{G_k}$ if and only if one configuration can be reached from the other by iteratively applying $G_k$. In this case, every equivalence class consists of one cycle, we characterize the set of cycles:

*Definition 2:* Given a pair $(G, k)$ in $\mathcal{G}_n \times \mathcal{K}_n$, we define $CYCLE_n(G, k)$ to be the collection of subsets of $\mathcal{A}_n$, such that for every $C$ in $CYCLE_n(G, k)$, we have $a \mathcal{R}_{G_k} b$ for any $a$ and $b$ in $C$, and for every $c$ in $\mathcal{A}_n \backslash C$, there does not exist an $a$ in $C$ such that $a \mathcal{R}_{G_k} c$.

We refer to the elements of $CYCLE_n(G, k)$ as *convergence cycles*. In this paper, we characterize both the convergence cycles length (the number of action profiles consisting the cycles) and the minimal number of time steps required to reach such cycles from some initial action profile. We refer to the latter as the *convergence time*.

## IV. On Convergence Cycles

We begin by characterizing the length of the cycles in the equivalence classes as a function of the imposed graph structure and the threshold distribution.

*Theorem 1:* For every positive integer $n$, every $(G, k)$ in $\mathcal{G}_n \times \mathcal{K}_n$ and every $C$ in $CYCLE_n(G, k)$, the cardinality of $C$ is less than or equal to 2.

Put differently, given a network structure $G$, a threshold distribution $k$ and an initial action profile $a$, if we iteratively apply $G_k$ on $a$ ad infinitum to get a sequence of best response action profiles, along the sequence of actions considered by player $i$, player $i$ will eventually either settle on playing one action, or switch action on every new application of $G_k$.

To prove the theorem, we begin by a lemma. We define $\mathcal{S}_n$ to be the set of all pairs $(G, k)$ in $\mathcal{G}_n \times \mathcal{K}_n$ such for each player $i$, $d_i$ has odd cardinality and $k_i$ is equal to $(d_i + 1)/2$. We refer to $\mathcal{S}_n$ as the set of *symmetric models*, in the sense that for $(G, k)$ in $\mathcal{S}_n$ the property is such that for any action profile $a$ in $\mathcal{A}_n$, and any player $i$, the action $(G_k(a))_i$ is the action played by the majority in $\mathcal{N}_i$ with respect to the action profile $a$. In this case, the two actions $\mathbb{B}$ and $\mathbb{W}$ are treated as having equal weights by all players in the network.

Let us define $\mathcal{M}$ to be the subset of $\bigcup_n \mathcal{G}_n \times \mathcal{K}_n$ such that for each $(G, k)$ in $\mathcal{M}$ and every $C$ in $CYCLE_n(G, k)$, the cardinality of $C$ is less than or equal to 2.

*Lemma 1:* If $\mathcal{S}_n$ belongs to $\mathcal{M}$ for all $n$, then $\mathcal{G}_n \times \mathcal{K}_n$ belongs to $\mathcal{M}$ for all $n$.

*Proof:* Given a pair $(G, k)$ in $(\mathcal{G}_n \times \mathcal{K}_n) \backslash \mathcal{S}_n$, we construct a pair $(G', k')$ in $\mathcal{G}'_n \times \mathcal{K}'_n$ as follows. We suppose that $G$ is equal to $(\mathcal{I}_n, E)$, and choose a player $i$ in $\mathcal{I}_n$ such that either $d_i$ is even, or $d_i$ is odd and $k_i$ is not equal to $(d_i + 1)/2$. Surely such a node exists since $(G, k)$ does not belong to $\mathcal{S}_n$. We call the node $i$ the **pivot** node in the one-step symmetric-expansion of $(G, k)$ into $(G', k')$. Let $b_i$ be an integer equal to $k_i$, and consider $w_i$ an integer equal to $d_i - b_i + 1$. In this sense, if $a$ is an action configuration in $\mathcal{A}_n$, $b_i$ would be considered to be the least number of $\mathbb{B}$-playing neighbors needed by player $i$ to play $\mathbb{B}$ when $G_k$ acts on $a$, whereas $w_i$ would be the least number of $\mathbb{W}$-playing neighbors needed by player $i$ to play $\mathbb{W}$. We shall construct an instance $(G', k')$ in $\mathcal{G}_{n+3b_i+3w_i} \times \mathcal{K}_{n+3b_i+3w_i}$. We suppose that $G'$ is equal to $(\mathcal{I}_{n+3b_i+3w_i}, E')$ and partition $\mathcal{I}_{n+3b_i+3w_i}$ into $\mathcal{I}_n, P_1^w, \cdots, P_{b_i}^w, P_1^b, \cdots, P_{w_i}^b$ where each partition different than $\mathcal{I}_n$ has cardinality exactly equal 3. We define $E'$ to be the undirected set of edges such that $E'$ contains $E$. Furthermore, for every $m$, suppose $P_m^w = \{j, j', j''\}$, we let $E'$ contain $jj'$, $jj''$ and $ij$. Similarly, for every $l$, suppose $P_l^b = \{j, j', j''\}$ we let $E'$ contain $jj'$, $jj''$ and $ij$. To visualize the obtained graph structure $G'$, we attached $b_i + w_i$ 3-node Y-shaped graphs to node $i$. Finally, we set $k'$ to be equal to $k$ on $\mathcal{I}_n \backslash \{i\}$,[7] equal to $(d_i + b_i + w_i)/2 = d_i + 1$ at $i$, equal to 2 on the remaining nodes having degree 3 and equal to 1 everywhere else.

We define the map $\alpha$ from $\mathcal{A}_n$ into $\mathcal{A}_{n+3b_i+3w_i}$ in such a way that for $a$ in $\mathcal{A}_n$, $\alpha(a)$ is equal to $a$ on $\mathcal{I}_n$, $\mathbb{B}$ on $P_1^b \cup \cdots \cup P_{w_i}^b$ and $\mathbb{W}$ on $P_1^w \cup \cdots \cup P_{b_i}^w$. One could check that for $a$ in $\mathcal{A}_n$:

$$\alpha(G_k a) = G'_{k'} \alpha(a). \tag{6}$$

Let $C = \{a_1, \cdots, a_k\}$ be a cycle in $CYCLE_n(G, k)$, then $C' = \{\alpha(a_1), \cdots, \alpha(a_k)\}$ is a cycle in $CYCLE_n(G, k)$. Since $\alpha$ is an injective map, then $|C'| = |C|$. Therefore, if $(G', k')$ belongs to $\mathcal{M}$, then $(G, k)$ belongs to $\mathcal{M}$. If $(G', k')$ does not belong to $\mathcal{S}_{n'}$, choose a player $j$, where $d_j$ is even or $k_j$ is not equal to $(d_j + 1)/2$. Repeat the procedure described above to obtain a pair $(G'', k'')$. In this case, if $(G'', k'')$ belongs to $\mathcal{M}$, then $(G', k')$ belongs to $\mathcal{M}$. We repeat this procedure until we obtain a pair $(\bar{G}, \bar{k})$ in $\mathcal{S}_{\bar{n}}$, we need only repeat it finitely many times. The result then follows by transitivity. ∎

*Definition 3:* Let $P$ be a subset of $\mathcal{I}_n$, for $(G, k)$ in $\mathcal{G}_n \times \mathcal{K}_n$, we define $G_k|_P$ to be the restriction of $G_k$ to act on the actions of the players in $P$. Formally, for $a$ in $\mathcal{A}_n$,

$$(G_k|_P a)_i = \begin{cases} (G_k a)_i & \text{if } i \in P \\ a_i & \text{if } i \notin P \end{cases} \tag{7}$$

Note that we are not restricting the domain of the function.

*Proof of Theorem 1:* Without loss of generality, let $(G, k)$ be a pair in $\mathcal{S}_n$ (see Lemma 1). We construct a pair $(G', k')$ in $\mathcal{S}_{2n}$ as follows. Suppose $G'$ is equal to $(\mathcal{I}_{2n}, E')$, then partition $\mathcal{I}_{2n}$ into two sets $\mathcal{I}_n$ and $\mathcal{J}$. We will denote $\mathcal{I}_n$ by $\mathcal{I}$ throughout this proof. We define a bijection $\phi$ from $\mathcal{J}$ into $\mathcal{I}$ and we define $E'$ to be the set of edges on $\mathcal{I}_{2n}$ such that for $i, j$ in $\mathcal{I}_{2n}$, $\{i, j\} \in E'$ if and only if $\{i, \phi(j)\} \in E$. Define $\alpha$ to be the map from $\mathcal{A}_n^2$ into $\mathcal{A}_{2n}$ such that $\alpha(a_\mathcal{I}, a_\mathcal{J})$ is equal to $a_\mathcal{I}$ on $\mathcal{I}_n$ and $a_\mathcal{J}$ on $\mathcal{J}_n$. For $(a_\mathcal{I}, a_\mathcal{J})$ in $\mathcal{A}_n^2$, we then get

$$G'_{k'}|_\mathcal{I} \alpha(a_\mathcal{I}, a_\mathcal{J}) = \alpha(G_k a_\mathcal{J}, a_\mathcal{J}), \tag{8}$$

and therefore,

$$G'_{k'}|_\mathcal{J} G'_{k'}|_\mathcal{I} \alpha(a_\mathcal{I}, a_\mathcal{J}) = \alpha(G_k a_\mathcal{J}, G_k^2 a_\mathcal{J}). \tag{9}$$

This said, it can be checked that $CYCLE_n(G, k)$ contains only cycles of cardinality at most two, if and only if for every $a$ in $\mathcal{A}_{2n}$, there exists a point $b$ in $\mathcal{A}_{2n}$, such that

$$b = (G_k|_\mathcal{I} G_k|_\mathcal{J})^T a \tag{10}$$

for some non-negative integer $T$ and

$$(G'_{k'}|_\mathcal{J} G'_{k'}|_\mathcal{I}) b = b. \tag{11}$$

To show the result we seek existence of such a configuration $b$. Let us define the map $\mathcal{E}$ from $\mathcal{A}_{2n}$ into $\mathbb{N}$ such that $\mathcal{E}(a) = |\{\ ij \in E' : a_i \neq a_j\ \}|$ for $a$ in $\mathcal{A}_{2n}$. Then, for every action configuration $a$ in $\mathcal{A}_{2n}$, we have $G'_{k'}|_\mathcal{I} a \neq a$ if and only if $\mathcal{E}(G'_{k'}|_\mathcal{I} a) < \mathcal{E}(a)$. To see that, note that since $(G', k')$ is in $\mathcal{S}_{2n}$, $(G'_{k'} a)_i$ is equal to the majority of the actions in $\mathcal{N}_i$. Then player $i$ switches action if and only if it can decrease the number of players with opposite actions. By symmetry, we get a similar claim for $\mathcal{J}$. It follows that $G'_{k'}|_\mathcal{J} G'_{k'}|_\mathcal{I} a \neq a$ if and only if $\mathcal{E}(G'_{k'}|_\mathcal{J} G'_{k'}|_\mathcal{I} a) < \mathcal{E}(a)$. Since $\mathcal{E}(a)$ is bounded above by $2n^2$ and bounded below by $0$, it follows that such a $b$ exists. ∎

[7] Let $X$ be a set. For $A$ and $B$ subsets of $X$, we denote by $A \backslash B$ the subset of $X$ containing elements in $A$ that are not in $B$.

Given a pair $(G,k)$, we refer to the *symmetric expansion* of $(G,k)$ as the pair $(G',k')$ in $\cup_n \mathcal{S}_n$ generated by the procedure described in Lemma 1. Given a pair $(G,k)$, we refer to the *bipartite expansion* of $(G,k)$ as the pair $(G',k')$ generated by the procedure described in Theorem 1, whereby the graph is first duplicated, then the two copies are cross-connected.

To explain the proof idea, let $(G,k)$ be a pair in $\cup_n \mathcal{S}_n$, let $a_0$ be a point of $\mathcal{A}_n$ and consider the sequence $a_1, a_2 \cdots$ with $a_T = G_k(a_{T-1})$ for every positive integer $T$. By performing a bipartite expansion on $(G,k)$ to get $(G',k')$ and applying $G'_{k'}|_{\mathcal{I}_n} G'_{k'}|_{\mathcal{J}_n}$ iteratively on $\alpha(a,a)$, the players in $\mathcal{I}_n$ will play the action profiles having *even* indices in the sequence, whereas the players in $\mathcal{J}$ will play the action profiles having *odd* indices in the sequence. From this perspective, it is easy to understand that the cycles of $CYCLE_n(G,k)$ consist of at most two configurations if and only if iteratively applying $G_k|_{\mathcal{I}} G_k|_{\mathcal{J}}$, starting from any configuration, always leads to a fixed point. It is better to think of the process as sequential, where players in $\mathcal{I}$ update at even time steps, and players in $\mathcal{J}$ update at odd time steps. The proof idea to follow stems from the fact that two players updating on the same time steps share no edges in common. Let us refer to an edge connecting two players with opposite actions as a *conflict edge*. Since we enforced the symmetric assumption on the model (i.e. $(G,k)$ belongs to $\mathcal{S}_n$), a node switches action (whenever it is allowed to update) if and only if it can decrease the number of conflict edges in the graph. Therefore, every player cannot switch actions infinitely many times since the number of conflict edges cannot keep on decreasing indefinitely.

## V. On Convergence Time

Given a graph structure $G$, a threshold distribution $k$, and an action profile $a$, we characterize the number of times one needs to apply $G_k$ on $a$ to reach a cycle.

*Definition 4:* For every positive integer $n$, and every $(G,k,a)$ in $\mathcal{G}_n \times \mathcal{K}_n \times \mathcal{A}_n$, we define $\delta_n(G,k,a)$ to be equal to the smallest non-negative integer $T$ such that there exists a cycle $C$ in $CYCLE_n(G,k)$ and $b$ in $C$ with $G_k^T a = b$.
The quantity $\delta_n(G,k,a)$ denotes to the minimal number of iterations needed until a given action configuration $a$ reaches a cycle, when iteratively applying $G_k$. We refer to $\delta_n(G,k,a)$ as the convergence time from $a$ under $G_k$.

*Theorem 2:* For all positive integers $n$, and every $(G,k,a)$ in $\mathcal{G}_n \times \mathcal{K}_n \times \mathcal{A}_n$, the convergence time $\delta_n(G,k,a)$ is less than or equal to $mn^2$ for some positive integer $m$.

*Proof:* Let $(G,k)$ be a point in $\mathcal{G}_n \times \mathcal{K}_n$ for some $n$, let $(G',k')$ be the symmetric expansion of $(G,k)$ in $\mathcal{S}_{n'}$, and let $(G'',k'')$ be the bipartite expansion of $(G',k')$ in $\mathcal{S}_{2n'}$. Then:

$$\delta_n(G,k) \leq \delta_{n'}(G',k') \leq \delta_{2n'}(G'',k''). \tag{12}$$

Moreover, we have:

$$\begin{aligned} \delta_{2n'}(G'',k'') &\leq \max_{a \in \mathcal{A}_{n''}} |\{\ ij \in E'' : a_i \neq a_j\ \}| \\ &\leq 2|E'| \leq 2[n^2 + 3 \sum_{i \in \mathcal{I}_n} b_i + w_i]. \end{aligned} \tag{13}$$

Since $\sum_{i \in \mathcal{I}_n} b_i + w_i = \sum_{i \in \mathcal{I}_n} d_i + 1 = 2|E| + n$, the result follows. ∎

The constant $m$ in the theorem statement can be optimized, but it is of no interest. Instead, it would interesting to prove a bound below quadratic. In what follows, we improve the convergence time upper-bound to be linear in the size of the network when the graph structure is a tree.

*Theorem 3:* For all positive integers $n$ and every $(G,k,a)$ in $\mathcal{G}_n \times \mathcal{K}_n \times \mathcal{A}_n$ where $G$ is a tree, the convergence time $\delta_n(G,k,a)$ is less than or equal to $n$.

*Definition 5:* Given a bipartite graph $G(\mathcal{I}_n, E)$ in $\mathcal{G}_n$, a 2-Partition of $\mathcal{I}_n$ with respect to $G$, is a pair $(P_o, P_e)$ of disjoint subsets of $\mathcal{I}_n$ such that $P_o \cup P_e = \mathcal{I}_n$ and there does not exist an $(i,j)$ in $P_o^2 \cup P_e^2$ such that $ij \in E$.

*Definition 6:* Given a triplet $(G,k,a)$ in $\mathcal{G}_n \times \mathcal{K}_n \times \mathcal{A}_n$ and a 2-Partition $(P_o, P_e)$ of $\mathcal{I}_n$ with respect to $G$. We identify $a$ with the pair $(a \restriction P_o, a \restriction P_e)$.[8] It is said that $a \restriction P_o$ is reachable in $(G,k)$ if there exists $a'$ in $\mathcal{A}_n$ such that $a \restriction P_o = (G_k|_{P_o} a') \restriction P_o$. In this case, it is said that $a' \restriction P_e$ induces $a \restriction P_o$. Similarly, $a \restriction P_e$ is reachable in $(G,k)$ if there exists $a'$ in $\mathcal{A}_n$ such that $a \restriction P_e = (G_k|_{P_e} a') \restriction P_e$. And again, it is said that $a' \restriction P_o$ induces $a \restriction P_e$.

*Lemma 2:* For every positive integer $n$, given a triplet $(G,k,a)$ in $\mathcal{G}_n \times \mathcal{K}_n \times \mathcal{A}_n$ where $G$ is a tree and a 2-Partition $(P_o, P_e)$ of $\mathcal{I}_n$ with respect to $G$. If $a \restriction P_o$ is reachable and $a \restriction P_o$ induces $a \restriction P_e$ both in $(G,k)$, then there exists a player $i$ in $P_e$, such that $((G_k|_{P_e} G_k|_{P_o})^T a)_i = a_i$ for all non-negative integers $T$.

Whenever the graph $G$ in concern is bipartite, we *decouple* the process. That is, we consider a 2-Partition $(P_o, P_e)$ of $G$, and (instead of simultaneous update) we let the nodes in $P_o$ update at *odd* time steps and the nodes in $P_e$ update at *even* time steps. The lemma claims that if the stated conditions are met, at least one player in $P_e$ will never switch his action along this sequential process. We refer the reader to [13] for the proof of Lemma 2. We prove Theorem 3.

*Proof of Theorem 3:* Let $G$ be a tree in $\mathcal{G}_n$, and consider a 2-Partition $(P_o, P_e)$ of $\mathcal{I}_n$ with respect to $G$ such that $|P_e| \leq |P_o|$. For any $k$ in $\mathcal{K}_n$ and $a$ in $\mathcal{A}_n$, if we consider $(a^o, a^e) = (G_k a \restriction P_o, G_k^2 a \restriction P_e)$, then $a^o$ is reachable, and $a^o$ induces $a^e$. Then by Lemma 2, there exists at least one node $i$ in $P_e$ such that $((G_k|_{P_e} G_k|_{P_o})^m (a^o, a^e))_i = a_i^e$ for all non-negative integers $m$. We can remove this player from the game, obtain a graph $G'$ and modify the threshold distribution accordingly to obtain a threshold distribution $k'$ (See [13] for details). By successive application of Lemma 2 on the connected components, we exhaust all nodes in $P_e$ in at most $2|P_e|$ time steps. But since $|P_e| \leq |P_o|$, we get $|P_e| \leq n/2$ and the result follows. ∎

We end this section with a conjecture: the convergence time $\delta_n(G,k,a)$ is less than or equal to $n$ whenever $G$ is bipartite. In this case, $\delta_n(G,k,a)$ will necessarily be less than or equal to $2n$ when $G$ is non-bipartite.

[8] Let $f$ be a map from $A$ into $B$, and let $A'$ be a subset of $A$. We denote by $f \restriction A'$ the restriction of the function $f$ to $A'$.

## VI. Resilience of Networks

In this section, we revert back to consider *types* instead of *thresholds*, namely $\mathcal{Q}_n$ instead of $\mathcal{K}_n$. All the needed definitions in this paper including $K_n$ naturally extend to the set $Q_n$. Mainly, for $G(\mathcal{I}_n, E)$ in $\mathcal{G}_n$ and $q$ in $\mathcal{Q}_n$, we denote by $G_q$ the map from $\mathcal{A}_n$ into $\mathcal{A}_n$ such that for player $i$, $(G_k a)_i = \mathbb{B}$ if and only if *more than* $q_i d_i$ players are in $a^{-1}(\mathbb{B}) \cap \mathcal{N}_i$. We further redefine $\mathcal{G}_n$ to be the class of all *connected* undirected graphs defined over the vertex set $\mathcal{I}_n$.

We consider the following resilience problem. We define $||.||_1$ to be the map from $\mathcal{Q}_n$ into $\mathbb{R}$ such that for $q$ in $\mathcal{Q}_n$:

$$||q||_1 = \sum_{i \in \mathcal{I}_n} q_i. \tag{14}$$

We restrict the analysis in the paper to $||.||_1$. Let $K$ be a positive integer, we denote by $\mathcal{A}_n^K$ the subset of $\mathcal{A}_n$ such that, $a$ is in $\mathcal{A}_n^K$ if and only if the cardinality of $a^{-1}(\mathbb{B})$ is at most $K$. We denote respectively by $\mathbb{W}^n$ and $\mathbb{B}^n$ the points $(\mathbb{W}, \cdots, \mathbb{W})$ and $(\mathbb{B}, \cdots, \mathbb{B})$ in $\mathcal{A}_n$, and given a graph $G$ in $\mathcal{G}_n$, define $\mathcal{Q}_n^{G,K}$ to be the subset of $\mathcal{Q}_n$ such that for every $q$ in $\mathcal{Q}_n^{G,K}$ and $a$ in $A_n^K$, $\mathbb{W}^n \mathcal{R}_{G_q} a$. We define the resilience measure of a graph $G$ with respect to $K$ deviations to be:

$$\mu_n^K(G) = \inf\{\ ||q||_1 : q \in \mathcal{Q}_n^{G,K}\ \}. \tag{15}$$

Given a graph structure $G$ and a positive integer $K$, we suppose that at most $K$ players in the network start playing action $\mathbb{B}$. The goal is to allocate a type distribution $q$ to the players, so that the dynamics depicted in Proposition 1 lead the agents to play action $\mathbb{W}$ at the limit. From this perspective, the measure $\mu$ captures the minimal cost of threshold investment required to recover the network $G$ from a perturbation of magnitude $K$. The lower the resilience measure is for a graph $G$, the more robust $G$ is against perturbations, in that we mean the less costly it is to allocate types to have $G$ recover. We state some bounds, we refer the reader to [13] for the proofs.

*Theorem 4:* The resilience measure $\mu_n^K$ is greater than or equal to 1 for every positive integer $K \leq n$.

*Theorem 5:* The resilience measure $\mu_n^K$ is less than or equal to $n/2$ for every positive integer $K \leq n$.

One can show that the lower-bound is achieved by the star network for every positive integer $K$. The upper-bound is achieved by the 2-regular graph for $K > n/2$. As a piece of insight, high degree nodes lower the resilience measure in the graph. One manifestation of this fact lies in the examples that meet the bounds. However, if we consider the complete graph, it has a resilience measure of 1 for $K = 1$ that grows to $n/2$ for $K = n$. This said, although high degree nodes increase the resilience of a network, having a large number of high degree nodes in the network makes the network more fragile against large perturbation, and hence more costly to ensure its recovery.

## VII. Conclusion

In this paper, we focus on characterizing the behavior of a linear threshold model where agents are allowed to switch their actions multiple times. We established that in the limit, the agents in the network cycle among action profiles and proceeded to characterize the lengths of such cycles, and the required number of time steps needed to reach such cycles. In particular, we showed that for *any graph structure* and *any threshold distribution* over the agents, such cycles consist of a most two action profiles. Namely, in the limit, each agent either always plays one specific action or switches action at every single time step. We also showed that over *all graph structure* (of size $n$) and *all threshold distributions* no more than $mn^2$ time steps are required to reach such cycles. Our methods follow a combinatorial approach, and are based on two techniques: transforming the general graph structure into a bipartite structure, and transforming the parallel dynamics on this bipartite structure into sequential dynamics. We further improve the convergence time bound to be not more than $n$ time steps if the graph structure is a tree.

Finally, in the setting of resilience of networks, we defined a measure $\mu$ that captures the minimal cost of threshold investment required to recover the network $G$ from a perturbation of magnitude $K$, whereby we suppose that $K$ agents will initially deviate from action $\mathbb{W}$ and play action $\mathbb{B}$. We show that this measure is lower-bounded by 1, and that this measure is upper-bounded by $n/2$, where $n$ is the size of the network. We finally provide an interpretation of how this measure varies with respect to the network structures. High degree nodes add resilience to the network, however too many high degree nodes can make the network fragile against strong perturbations.

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