HIGHER ORDER MAASS FORMS

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Abstract. The linear structure of the space of Maass forms of even weight and of arbitrary order is determined.

1. Introduction

In this work, the structure of the space of Maass forms of general order and integral weight as a linear vector space is determined. It is proved that, under suitable conditions, this space is as large as one would expect it to be.

There are mainly two objects and associated problems that suggest the study of specifically this type of higher-order form. The first is Eisenstein series modified with modular symbols defined by

$$E^*(z,s) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \langle f,\gamma\rangle \,\mathrm{Im}(\gamma z)^s\,, \tag{1.1}$$

where Γ_∞ is the subgroup of translations of the congruence group $\Gamma_0(N)$, f a weight 2 newform and $\langle f,\gamma\rangle := -2\pi i\int_\infty^{\gamma\infty} f(w)dw$. The study of this function has led to important results, such as the proof that the suitably normalised modular symbols follow the normal distribution ([19]). The function $E^*(-,s)$ is not automorphic but transforms as a second-order automorphic form.

We recall that, for a group Γ of motions on the upper half-plane $\mathfrak{H}$, a function is said to be Γ-invariant of *order* $q\in\mathbb{N}$ and weight 0, if it satisfies

$$f|(\gamma_1-1)(\gamma_2-1)\cdots(\gamma_q-1) \;=\; 0 \qquad \text{for all } \gamma_1,\gamma_2,\ldots,\gamma_n\in\Gamma\,. \tag{1.2}$$

Here, the action $|$ of Γ on functions on $\mathfrak{H}$ is given by

$$f|\gamma(z) \;:=\; f(\gamma z)\,.$$

and it is extended linearly to an action of the group ring $\mathbb{C}[\Gamma]$.

Clearly, several types of conditions on holomorphicity, growth etc. can be imposed on functions of general order. The function $E^*(-,s)$ in particular, is an eigenfunction of the Laplacian and therefore we view it as a Maass form of order 2.

The second object leading to functions that are Γ-invariant of second-order arises from considerations related to values of derivatives of L-functions of cusp forms: In [11] and [8] certain "period integrals" are associated to derivatives of L-functions of weight 2 cusp forms in a way analogous to the link between values of L-functions and modular integrals ([17]). Specifically, let f be a newform of weight 2 for $\Gamma_0(N)$ and let $L_f(s)$ be its L-function. If $L_f(1)=0$, then, for each prime p, $(p,N)=1$, $L'_f(1)$ can be written as a linear combination of integrals of the form

$$\int_0^{\gamma(0)} f(z)\,u(z)\,dz\,, \qquad \gamma\in\Gamma_0(N) \tag{1.3}$$

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plus some "lower order terms". Here $u(z) := \log\eta(z) + \log\eta(Nz)$, where η is the Dedekind η-function. The differential $f(z)u(z)\,dz$ is not $\Gamma_0(N)$-invariant. It does satisfy a transformation law which is reminiscent of (1.2), but is not quite $\Gamma_0(N)$-invariant of order 2 in the narrow sense. If it were, the value of the derivative at 1 would be expressed as the value of the actual L-function of second-order $\Gamma_0(N)$ at 1. That could be advantageous for the study of $L'_f(1)$ in terms of the outstanding conjectures, especially since there is now evidence that a motivic structure underlies higher order forms (see [10] and [22]).

Here we show that it is indeed possible to obtain a second-order $\Gamma_0(N)$-invariant function from $u(z)$ provided we move to a different domain. This domain is the universal covering group which we will be defining in detail in §5.1.

As will become apparent in the sequel, it is natural, in higher orders, to unify the study of Maass forms and that of forms on universal covering groups. The full definition of the *higher-order Maass forms with generalised weight on the universal covering group* is discussed in §6. Theorem 6.4 then allows us to translate results on the universal covering group to the analogous results on the upper-half plane.

A fundamental question is how "large" this space is. In the case of *holomorphic* higher-order cusp forms, the corresponding spaces are finite-dimensional and the answer can be given by computing the dimensions ([7] and [9]). In the present case, where the relevant space is not finite dimensional, a different characterisation of "size" is required. Such a characterisation is proposed in §3.

Although our results imply that there are "many" higher order Maass forms, the proofs are highly inductive and do not easily lead to explicit examples. In §4.3 and §6.4 we address this problem, by illustrating various methods that lead to explicit examples of higher order Maass. Surprisingly, these examples are derived very naturally from the theory which was developed in a completely different context in [2, 3].

Finally, a particular aspect of the proof that deserves to be singled out because of its independent interest is the definition of genuinely higher-order Fourier expansions. Higher order automorphic forms need not be invariant under the group fixing a cusp, so there is no obvious Fourier expansion. To date, to address this problem one had to partially revert to the classical setting by imposing the somewhat unnatural extra condition of invariance under the parabolic elements of the group. In §7, appropriate higher-order Fourier terms are constructed, thus avoiding additional invariance conditions.

2. Structure of the paper

In §3 we first discuss higher-order invariants for general groups and modules. This allows a precise definition of the concept of "as large as possible" (*maximally perturbable*). A first maximal perturbability result for a general space of maps is also proved.

In §4, Maass forms on $\mathfrak{H}$ (both general and holomorphic) are defined and the first two main theorems of the paper (4.2 and 4.3) are stated. The section includes an extended discussion of concrete examples of low-order forms on $\mathfrak{H}$.

In the next section the universal covering group $\tilde{G}$ is introduced and the basic facts about $\tilde{G}$ are given.

Maass forms on the universal covering group are defined in §6 and the counterparts of Theorems 4.2 and 4.3 for forms on the universal covering group are stated. The section concludes with concrete examples of low-order forms on $\tilde{G}$.

Section 7 is of independent interest. A theory of Fourier expansions for higher-order forms is developed.

The proof of Theorems 4.2 and 4.3 is the content of §8. The proof involves the construction of two spaces with support conditions. To deduce their maximal perturbability we employ spectral techniques.

3. Higher order invariants

In this section, we discuss higher order invariants in general and then specialise their study to discrete cofinite subgroups $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$. We introduce the concept of a "maximally perturbable" Γ-module to make precise the statement that there are as many higher order invariants of a given type as one can expect. A first maximal perturbability result in a general context is proved.

3.1. **Higher order invariants on general groups.** The concept of higher order invariant functions on the upper half plane is a special case of the concept "higher order invariants" for any group Γ and any Γ-module V. We work with *right* Γ-modules, an write the action as $v \mapsto v|\gamma$. It should be clear from the context when we refer to this general meaning of $|$ and when to the more narrow meaning given in the Introduction. We define the *higher order invariants* inductively:

$$\begin{aligned} V^{\Gamma,1} &= V^{\Gamma} = \{v \in V \,:\, \forall \gamma \in \Gamma,\, v|\gamma = v\}, \\ V^{\Gamma,q+1} &= \{v \in V \,:\, \forall \gamma \in \Gamma,\, v|(\gamma - 1) \in V^{\Gamma,q}\}. \end{aligned} \tag{3.1}$$

We set $V^{\Gamma,0} = \{0\}$.

Let now Γ be finitely generated and let I be the augmentation ideal in the group ring $\mathbb{C}[\Gamma]$, generated by $\gamma - 1$ with $\gamma \in \Gamma$. A fundamental role in the paper will be played by the map

$$\mathsf{m}_q : V^{\Gamma,q+1} \to \hom_{\mathbf{C}[\Gamma]}(I^{q+1}\backslash I^q, V^{\Gamma}).$$

To define it we first quote from [5] (before Proposition 1.2):

$$V^{\Gamma,q} \;\cong\; \hom_{\mathbb{C}[\Gamma]}(I^q\backslash\mathbb{C}[\Gamma], V)\,. \tag{3.2}$$

Next, we note that $I^{q+1}\backslash I^q$ is generated by

$$I^{q+1} + (\gamma_1 - 1)\cdots(\gamma_q - 1)\,,$$

with $\gamma_i \in \Gamma$. To each $v \in V^{\Gamma,q+1}$ we associate the map on $I^{q+1}\backslash I^q$ sending this element to $v|(\gamma_1-1)\cdots(\gamma_q-1)$. This map is well-defined because $v|(\gamma_1 - 1)\cdots(\gamma_{q+1} - 1) = 0$. In this way, we obtain a map m_q from $V^{\Gamma,q+1}$ to

$$\hom_{\mathbb{C}[\Gamma]}(I^{q+1}\backslash I^q, V) \cong \hom_{\mathbb{C}[\Gamma]}(I^{q+1}\backslash I^q, V^{\Gamma})$$

(since the action induced on $I^{q+1}\backslash I^q$ by the operation of Γ is trivial). It is easy to see that the kernel of m_q is $V^{\Gamma,q}$ and thus we obtain the exact sequence

$$0 \longrightarrow V^{\Gamma,q} \longrightarrow V^{\Gamma,q+1} \xrightarrow{\mathsf{m}_q} \hom_{\mathbb{C}[\Gamma]}(I^{q+1}\backslash I^q, V) \tag{3.3}$$

The map m_q may or may not be surjective and we will interpret the phrase "as large as possible" as surjectivity of m_q for all $q \in \mathbb{N}$.

Definition 3.1. Let Γ be a finitely generated group. We will call a Γ-module V *maximally perturbable* if the linear map $\mathsf{m}_q : V^{\Gamma,q+1} \to \hom_{\mathbb{C}[\Gamma]}(I^{q+1}\backslash I^q, V^{\Gamma})$ is surjective for all $q \geq 1$.

A reformulation of this definition which is occasionally easier to use, uses the finite dimension

$$n(\Gamma, q) := \dim_{\mathbb{C}}(I^{q+1}\backslash I^q). \tag{3.4}$$

V is maximally perturbable if and only if $V^{\Gamma,q+1}/V^{\Gamma,q} \cong (V^{\Gamma})^{n(\Gamma,q)}$ for all $q \in \mathbb{N}$.

In [9] higher order cusps forms of weight k for a discrete group Γ are considered in the space of holomorphic functions on $\mathfrak{H}$ with exponential decay at the cusps that moreover are invariant under the parabolic

transformations. The dimensions of these spaces are computed and generally turn out to be strictly smaller than $n(\Gamma, q)$. So the corresponding Γ-module is not maximally perturbable.

A useful definition is based on the isomorphism $\hom_{\mathbb{C}[\Gamma]}(I^{q+1}\backslash I^q, V^\Gamma) \cong \mathrm{Mult}^q(\Gamma, V^\Gamma)$, the space of maps $\Gamma^q \to V^\Gamma$ inducing group homomorphisms $\Gamma \to \mathbb{C}$ on each of their coordinates. For a finitely generated group Γ, $\mathrm{Mult}^q(\Gamma, V^\Gamma) \cong \mathrm{Mult}^q(\Gamma, \mathbb{C}) \otimes_{\mathbb{C}} V^\Gamma$ where $\mathrm{Mult}^q(\Gamma, \mathbb{C})$ is the q-th tensor power of the abelianised group $\Gamma^{\mathrm{ab}} = \Gamma/[\Gamma, \Gamma]$. With this notation we define

Definition 3.2. Let $q \in \mathbb{N}$. For any group Γ and any Γ-module V we call $f \in V^{\Gamma^q}$ a *perturbation* of $\varphi \in V^\Gamma$ if there exists $\mu_f \in \mathrm{Mult}^q(\Gamma, \mathbb{C})$ such that for all $\gamma_1, \cdots, \gamma_q \in \Gamma$:

$$f|(\gamma_1 - 1)\cdots(\gamma_q - 1) \;=\; \mu_f(\gamma_1, \ldots, \gamma_q)\,\varphi\,. \tag{3.5}$$

We call a perturbation *commutative* if μ_f is invariant under all permutations of its arguments. If not, we call it non-commutative.

3.2. **Higher order invariants on subgroups of** $\mathrm{PSL}_2(\mathbb{R})$**.**

3.2.1. *Canonical generators.* In this section we recall the "canonical generators" of cofinite discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$, and use them to show that certain modules are maximally perturbable.

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a cofinite discrete group of motions in the upper half-plane $\mathfrak{H}$. A system of *canonical generators* for Γ consists of

- Parabolic generators $P_1, \ldots, P_{n_{\mathrm{par}}}$, each conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to $\pm\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. We shall assume that Γ has cusps: $n_{\mathrm{par}} \geq 1$.
- Elliptic generators $E_1, \ldots, E_{n_{\mathrm{ell}}}$, with $n_{\mathrm{ell}} \geq 0$. Each E_j is conjugate to $\pm\left(\begin{smallmatrix}\cos(\pi/v_j) & \sin(\pi/v_j)\\ -\sin(\pi/v_j) & \cos(\pi/v_j)\end{smallmatrix}\right)$ in $\mathrm{PSL}_2(\mathbb{R})$ for some $v_j \geq 2$.
- Hyperbolic generators $H_1, \ldots, H_{2g}$, with $g \geq 0$, each conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to the image $\pm\left(\begin{smallmatrix}t & 0\\ 0 & t^{-1}\end{smallmatrix}\right)$, $t > 1$, $t \neq 1$, of a diagonal matrix

See, *e.g.*, [15], Chap. VII.4, p. 241, or [18], §3. The relations are given by the condition that each $E_j^{v_j} = I$ for $j = 1, \ldots, n_{\mathrm{ell}}$, and one large relation

$$P_1 \cdots P_{n_{\mathrm{par}}} E_1 \cdots E_{n_{\mathrm{ell}}} [H_1, H_2] \cdots [H_{2g-1} H_{2g}] \;=\; \mathrm{Id}\,. \tag{3.6}$$

The choice of canonical generators is not unique, but the numbers n_{par}, n_{ell} and g, and the elliptic orders $v_1, \ldots, v_{n_{\mathrm{ell}}}$ are uniquely determined by Γ.

Each group homomorphism $\Gamma \to \mathbb{C}$ vanishes on the E_j, and is determined by its values on $H_1, \ldots, H_{2g}$, $P_1, \ldots, P_{n_{\mathrm{par}}-1}$, hence

$$\dim \hom(\Gamma, \mathbb{C}) \;=\; n_{\mathrm{par}} - 1 + 2g\,. \tag{3.7}$$

We put $t(\Gamma) = n_{\mathrm{par}} + 2g$, and denote $A_1 = P_1, \ldots, A_{n_{\mathrm{par}}-1} = P_{n_{\mathrm{par}}-1}, A_{n_{\mathrm{par}}} = H_1, A_{n_{\mathrm{par}}+1} = H_2, \ldots, A_{t(\Gamma)-1} = H_{2g}$. The group Γ is generated by $E_1, \ldots, E_{n_{\mathrm{ell}}}$ and $A_1, \ldots, A_{t(\Gamma)-1}$.

For the modular group we have $n_{\mathrm{par}} = 1$, $P_1 = \pm\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, $n_{\mathrm{ell}} = 2$, $E_1 = \pm\left(\begin{smallmatrix}1 & 1\\ -1 & 0\end{smallmatrix}\right)$, $E_2 = \pm S := \pm\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, $g = 0$, and hence $\hom(\Gamma_{\mathrm{mod}}, \mathbb{C}) = \{0\}$ and $t(\Gamma_{\mathrm{mod}}) = 1$.

In the sequel, we will need a basis for $I^{q+1}\backslash I^q$. Arguing as in Lemma 2.1 in [5] we can deduce that the elements

$$\mathbf{b}(\mathbf{i}) = (A_{\mathbf{i}(1)} - 1)\cdots(A_{\mathbf{i}(q)} - 1)\,, \tag{3.8}$$

where **i** runs over all $(t(\Gamma)-1)$-tuples of elements of $\{1,\dots,t(\Gamma)-1\}$ form a basis of $I^{q+1}\backslash I^q$. We do not give a proof here, since it follows from the more general result Proposition 5.3.

3.2.2. *A first maximal perturbability result.* We view the space $\mathrm{Map}(\Gamma,\mathbb{C})$ of all maps from Γ to $\mathbb{C}$ as a right Γ-module for the action $|$ by left translation.

Proposition 3.3. *If Γ is a discrete cofinite subgroup of $\mathrm{PSL}_2(\mathbb{R})$ with cusps, then $\mathrm{Map}(\Gamma,\mathbb{C})$ is maximally perturbable.*

Proof. We construct functions $\mathbf{g}_\mathbf{i} \in \mathrm{Map}(\Gamma,\mathbb{C})$ for n-tuples **i** from $\{1,\dots,t(\Gamma)-1\}$. Firstly, let Γ_0 be the free subgroup of Γ which is generated by the elements A_j, $1 \le j \le t(\Gamma)-1$. It is clear that there is a unique system of functions $\{\mathbf{g}_\mathbf{i}\}$ on Γ_0 such that

$$\begin{aligned} \mathbf{g}_() &= 1\,, \\ \mathbf{g}_{(j,\mathbf{i})}|(A_j-1) &= \mathbf{g}_\mathbf{i}\,, \\ \mathbf{g}_\mathbf{i}|(A_j-1) &= 0 \quad \text{if } \mathbf{i}(1)\neq j, \\ \mathbf{g}_\mathbf{i}(1) &= 0 \quad \text{if } |\mathbf{i}|\ge 1\,. \end{aligned} \tag{3.9}$$

By $|\mathbf{i}|$ we denote the length of the tuple **i**.

We next set $\mathbf{g}_\mathbf{i}(\gamma) = \mathbf{g}_\mathbf{i}(\varphi_0(\gamma))$ for $\gamma\in\Gamma$, where φ_0 is the homomorphism defined by $\varphi_0(E_j)=I$, $\varphi_0(A_j)=A_j$. With the map m_q in (3.3) and for for $|\mathbf{i}|=|\mathbf{j}|$ we have on the basis elements in (3.8),

$$\begin{aligned} (\mathsf{m}_q\mathbf{g}_\mathbf{i})(\mathbf{b}(\mathbf{j})) &= \mathbf{g}_\mathbf{i}|(A_{\mathbf{j}(1)}-1)\cdots(A_{\mathbf{j}(q)}-1) \\ &= \delta_{\mathbf{i}(1),\mathbf{j}(1)}\mathbf{g}_{\mathbf{i}'}|(A_{\mathbf{j}(2)}-1)\cdots(A_{\mathbf{j}(q)}-1)\,, \end{aligned} \tag{3.10}$$

where $\mathbf{i}'$ is the tuple $(\mathbf{i}(2),\dots,\mathbf{i}(q))$. Inductively we obtain

$$(\mathsf{m}_q\mathbf{g}_\mathbf{i})(\mathbf{b}(\mathbf{j})) = \delta_{\mathbf{i},\mathbf{j}} := \prod_{l=1}^{q}\delta_{\mathbf{i}(l),\mathbf{j}(l)}\,. \tag{3.11}$$

Hence the $\mathbf{g}_\mathbf{i}$ with $|\mathbf{i}|=q$ form a dual system for the generators $\mathbf{b}(\mathbf{i})$. This implies that the image $\mathsf{m}_qV^{\Gamma,q+1}$ has maximal dimension $n(\Gamma,q)$. □

4. Maass forms

We turn to spaces of functions on the upper half-plane that contain the classical holomorphic automorphic forms and the more general Maass forms. The first main results of this paper are stated in Theorems 4.2 and 4.3. In §4.3 we give some explicit examples of higher order Maass forms.

4.1. **General Maass forms.** Let Γ be a cofinite discrete subgroup Γ of the group $G=\mathrm{PSL}_2(\mathbb{R})$. For each cusp κ, we choose $g_\kappa\in\mathrm{PSL}_2(\mathbb{R})$ such that

$$\kappa = g_\kappa\infty \qquad \text{and } g_\kappa^{-1}\Gamma_\kappa g_\kappa = \left\{\pm\begin{pmatrix}1&n\\0&1\end{pmatrix} : n\in\mathbb{Z}\right\} \tag{4.1}$$

Here, Γ_κ is the set of elements of Γ fixing κ. The elements g_κ are determined up to right multiplication by elements $\pm\left(\begin{smallmatrix}a&b\\0&a^{-1}\end{smallmatrix}\right)\in G$. We choose the g_κ for cusps in the same Γ-orbit so that $g_{\gamma\kappa}\in\gamma g_\kappa\Gamma_\infty$.

We further consider a generalisation of the action $|$ considered in the last section. For a fixed k and for a $f:\mathfrak{H}\to\mathbb{C}$ we set

$$f\,\Big|_k\begin{pmatrix}a&b\\c&d\end{pmatrix}(z) = (cz+d)^{-k}\,f((az+b)/(cz+d)). \tag{4.2}$$

We finally set

$$L_k \;=\; -y^2\partial_x^2 - y^2\partial_y^2 + iky\partial_x - ky\partial_y + \frac{k}{2}(1-\frac{k}{2})\,. \tag{4.3}$$

With this notation we have

Definition 4.1. Let $k \in 2\mathbb{Z}$ and $\lambda \in \mathbb{C}$.

i) $\mathcal{M}_k(\Gamma,\lambda)$ denotes the space of smooth functions $f : \mathfrak{H} \to \mathbb{C}$ such that $L_k f = \lambda f$ and for which there is some $a \in \mathbb{R}$ such that

$$f(g_\kappa(x+iy)) \;=\; \mathrm{O}(y^a) \quad (y\to\infty) \qquad \text{uniformly for } x \text{ in compact sets in } \mathbb{R}, \text{ for } \textit{all} \text{ cusps } \kappa \text{ of } \Gamma\ . \tag{4.4}$$

ii) $\mathcal{E}_k(\Gamma,\lambda)$ denotes the space of smooth functions f such that $L_k f = \lambda f$ and for which there is some $a \in \mathbb{R}$ such that

$$f(g_\kappa(x+iy)) \;=\; \mathrm{O}(e^{ay}) \quad (y\to\infty) \qquad \text{uniformly for } x \text{ in compact sets in } \mathbb{R}, \text{ for } \textit{all} \text{ cusps } \kappa \text{ of } \Gamma. \tag{4.5}$$

iii) We denote the invariants in these spaces by

$$E_k(\Gamma,\lambda) \;:=\; \mathcal{E}_k(\Gamma,\lambda)^\Gamma \quad \text{and } M_k(\Gamma,\lambda) \;:=\; \mathcal{M}_k(\Gamma,\lambda)^\Gamma\,. \tag{4.6}$$

We call the elements of $E_k(\Gamma,\lambda)$ (resp. $M_k(\Gamma,\lambda)$) *Maass forms of polynomial (resp. exponential) growth of weight k and eigenvalue* $\lambda \in \mathbb{C}$ *for* Γ.

Remarks.

i) Since L_k is elliptic, all its eigenfunctions are automatically real-analytic. (See, *e.g.*, [14], §5 of App. A4, and the references therein.) If f is holomorphic, then it is an eigenfunction of L_k with eigenvalue $\frac{k}{2}(1-\frac{k}{2})$.

ii) The space $M_k(\Gamma,\lambda)$ is known to have finite dimension. The space $E_k(\Gamma,\lambda)$ has, for groups Γ with cusps, infinite dimension. The subspace of $E_k(\Gamma,\lambda)$ corresponding to a fixed value of a in the bound $\mathrm{O}(e^{ay})$ has finite dimension.

iii) In an alternative definition, suitable for functions not necessarily holomorphic, one replaces the Maass forms f as defined above by $h(z) = y^{k/2} f(z)$. Then invariance under (4.2) becomes invariance under the action

$$f\,\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) \;=\; e^{-ik\arg(cz+d)}\, f((az+b)/(cz+d)) \tag{4.7}$$

and the eigenproperty in the terms of the Laplacian

$$(-y^2\partial_x^2 - y^2\partial_y^2 + iky\partial_x)h \;=\; \lambda h. \tag{4.8}$$

The formulation of the growth conditions remains unchanged. Now antiholomorphic automorphic forms $a(z)$ of weight k give Maass forms $h(z) = y^{k/2}a(z)$ of weight $-k$.

Our main result for general Maass forms on $\mathfrak{H}$ is

Theorem 4.2. *Let* Γ *be a cofinite discrete group of motions in* $\mathfrak{H}$ *with cusps. Then the* Γ*-module* $\mathcal{E}_k(\Gamma,\lambda)$ *is maximally perturbable for each* $k \in 2\mathbb{Z}$ *and each* $\lambda \in \mathbb{C}$.

In the course of the proof in §8 we will see that even if we start with Maass forms with polynomial growth the construction of higher order invariants will lead us to functions that have exponential growth.

4.2. **Holomorphic automorphic forms.** For even k the space $\mathcal{E}_k(\Gamma,\lambda_k)$, with $\lambda_k = \frac{k}{2}(1-\frac{k}{2})$ contains the subspace $\mathcal{E}_k^{\rm hol}(\Gamma,\lambda_k)$ where the condition $L_k f = \lambda_k f$ is replaced by the stronger condition that f is holomorphic. In the alternative definition, condition (4.8) is replaced by the condition that $z \mapsto y^{-k} f(z)$ is holomorphic. The space $\mathcal{E}_k^{\rm hol}(\Gamma,\lambda_k)$ is a Γ-submodule of $\mathcal{E}_k(\Gamma,\lambda_k)$. We also have the Γ-submodule $\mathcal{M}_k^{\rm hol}(\Gamma,\lambda_k) = \mathcal{M}_k(\Gamma,\lambda_k) \cap \mathcal{E}_k^{\rm hol}(\Gamma,\lambda_k)$ of $\mathcal{M}_k(\Gamma,\lambda_k)$.

The space $\mathcal{M}_k^{\rm hol}(\Gamma,\lambda_k)^\Gamma$ is the usual space of entire weight k automorphic forms for Γ, and $\mathcal{E}_k^{\rm hol}(\Gamma,\lambda_k)^\Gamma$ is the space of meromorphic automorphic forms with singularities only at cusps. Sometimes, e.g. in [1], the elements of $\mathcal{E}_k^{\rm hol}(\Gamma,\lambda_k)^\Gamma$ are called *weakly holomorphic*. There the elements of $\mathcal{E}_k(\Gamma,\lambda_k)^\Gamma$ are called *harmonic weak Maass forms*. We prefer to use the term *harmonic* for Maass forms in $\mathcal{E}_k(\Gamma,0)^\Gamma$. (Note that $\lambda_k \neq 0$ for $k \neq 0, 2$.)

Our main result for holomorphic automorphic forms on $\mathfrak{H}$ is:

Theorem 4.3. *Let Γ be a cofinite discrete group of motions in $\mathfrak{H}$ with cusps. Then $\mathcal{E}_k^{\rm hol}(\Gamma, k/2 - k^2/4)$ is maximally perturbable for each $k \in 2\mathbb{Z}$.*

4.3. **Examples of harmonic and holomorphic forms of order two and three.** According to Theorems 4.2 and 4.3 there are plenty of examples of higher order Maass forms for cofinite groups with cusps for which $\dim_{\mathbb{C}} \hom(\Gamma,\mathbb{C}) \geq 1$. It is, however, not very easy to exhibit explicit examples.

For the modular group $\Gamma_{\rm mod} = {\rm PSL}_2(\mathbb{Z})$ the space $\hom(\Gamma_{\rm mod},\mathbb{C})$ is zero. Hence it does not accept higher order invariants. For the commutator subgroup $\Gamma_{\rm com} = [\Gamma_{\rm mod},\Gamma_{\rm mod}]$ we will employ three different approaches to exhibit full sets of perturbations of 1 (as defined in Definition 3.2) of orders two and three. A reader only interested in the existence of higher order forms may prefer to skip this subsection.

4.3.1. *Holomorphic perturbation of* 1. In [15], Chap. XI, §3E, p. 362, one finds various facts concerning $\Gamma_{\rm com}$. It is freely generated by $D = \pm\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $C = \pm\left(\begin{smallmatrix} 2 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. It has no elliptic elements, and one cuspidal orbit $\Gamma_{\rm com}\,\infty = \mathbb{P}^1_{\mathbb{Q}}$. The group $(\Gamma_{\rm com})_\infty$ fixing ∞ is generated by $\pm\left(\begin{smallmatrix} 1 & 6 \\ 0 & 1 \end{smallmatrix}\right)$. We have $t(\Gamma_{\rm com}) = 3$.

The space of holomorphic cusp forms of weight 2 has dimension $g = 1$. We use the basis element η^4 (power of the Dedekind eta-function). The map

$$H(z) \;=\; -2\pi i \int_\infty^z \eta(\tau)^4\, d\tau \;=\; -6e^{\pi i z/3} + {\rm O}(e^{7\pi i z/3}) \tag{4.9}$$

induces an embedding of $\Gamma_{\rm com}\backslash\mathfrak{H}$ into an elliptic curve, which can be described as $\mathbb{C}/\Lambda$, with

$$\Lambda \;=\; \varpi\,\mathbb{Z}[\rho]\,, \qquad \varpi \;=\; \pi^{1/2}\Gamma(1/6)/(6\sqrt{3}\,\Gamma(2/3))\,, \quad \rho \;=\; e^{\pi i/3}\,. \tag{4.10}$$

(See computations in §15.2–3 in [3].) The map H maps $\mathfrak{H}$ onto $\mathbb{C} \setminus \Lambda$, and satisfies for $\gamma \in \Gamma_{\rm com}$

$$H(\gamma z) \;=\; H(z) + \lambda(\gamma)\,, \qquad \lambda(\gamma) \;=\; -2\pi i \int_\infty^{\gamma\infty} \eta(\tau)^4\, d\tau\,, \tag{4.11}$$

where $\lambda(C) = \rho\varpi$ and $\lambda(D) = \bar\rho\varpi$. So the lattice Λ is the image of $\lambda : \Gamma_{\rm com} \to \mathbb{C}$, and $\hom(\Gamma_{\rm com},\mathbb{C}) = {\rm Mult}^1(\Gamma_{\rm com},\mathbb{C})$ has λ, $\bar\lambda$ as a basis. We note that the kernel $\ker(\lambda)$ is a subgroup with infinite index in $\Gamma_{\rm com}$; it is in fact the commutator subgroup of $\Gamma_{\rm com}$. The element $\pm\left(\begin{smallmatrix} 1 & 6 \\ 0 & 1 \end{smallmatrix}\right)$ generating the subgroup of $\Gamma_{\rm com}$ fixing ∞ is in $\ker(\lambda)$. Since $\ker(\lambda)$ has no elliptic elements, composition with H gives a bijection from the holomorphic functions on $\mathbb{C} \setminus \Lambda$ to the holomorphic $\ker(\lambda)$-invariant functions on $\mathfrak{H}$.

Clearly, H is a holomorphic second order perturbation of 1 with linear form λ. It is also a *harmonic perturbation* of 1, i.e., a perturbation which is harmonic as a function. By conjugation we obtain the antiholomorphic harmonic perturbation of 1 with linear form $\bar\lambda$.

According to Theorem 4.3 there should also be a holomorphic second order perturbation of 1 with a linear form that is linearly independent of λ. Here we can use the Weierstrass zeta-function

$$\zeta(u;\Lambda) \;=\; \frac{1}{u} + \sum_{\omega\in\Lambda}{}'\Big(\frac{1}{u-\omega}+\frac{1}{\omega}+\frac{u}{\omega^2}\Big). \tag{4.12}$$

See, *e.g.*, [13], Chap. I, §6. It is holomorphic on $\mathbb{C}\smallsetminus\Lambda$ and satisfies $\zeta(u+\omega;\Lambda)=\zeta(u;\Lambda)+\mathsf{h}(\omega)$ for all $\omega\in\Lambda$, where $\mathsf{h}\in\hom(\Lambda,\mathbb{C})$ is linearly independent of $\omega\mapsto\omega$. (The classical notation for h is η. We write h to avoid confusion with the Dedekind eta function.) Pulling back this zeta-function to $\mathfrak{H}$ we get a second order holomorphic perturbation of 1

$$W(z) \;=\; \zeta(H(z);\Lambda) \tag{4.13}$$

with the linear form $\gamma\mapsto\mathsf{h}(\lambda(\gamma))$. The Laurent expansion of the Weierstrass zeta-function at 0 starts with $\zeta(u;\Lambda)=u^{-1}+\mathrm{O}(u^3)$. Hence W has a Fourier expansion at ∞ starting with

$$W(z) \;=\; \frac{-1}{6}\,e^{-\pi i z/3}+\mathrm{O}(e^{\pi i z}). \tag{4.14}$$

This shows that W has exponential growth at the cusps.

We may carry this out also for holomorphic forms of order three, to obtain the following commutative perturbations of 1 of order 3:

f	$H(z)^2$	$H(z)\,W(z)$	$W(z)^2$
μ_f	$2\,\lambda\otimes\lambda$	$\lambda\otimes(\mathsf{h}\circ\lambda)+(\mathsf{h}\circ\lambda)\otimes\lambda$	$2(\mathsf{h}\circ\lambda)\otimes(\mathsf{h}\circ\lambda)$

(4.15)

We know that there also exist non-commutative holomorphic perturbations of order 3. To find an explicit example, we have to work on $\mathfrak{H}$, since the group Λ acting on $\mathbb{C}$ is abelian.

The closed holomorphic 1-forms

$$\omega=-2\pi i\,\eta(\tau)^4\,d\tau \quad\text{ and }\quad \omega_1=-2\pi i\,W(\tau)\,\eta(\tau)^4\,d\tau$$

on $\mathfrak{H}$ transform as follows under Γ_{com}:

$$\omega|\gamma \;=\; \omega\,,\qquad \omega_1|\gamma \;=\; \omega_1+\mathsf{h}(\lambda(\gamma))\,\omega\,. \tag{4.16}$$

For an arbitrary base point $z_0\in\mathfrak{H}$ we put

$$K(z) \;=\; \int_{z_0}^{z}\omega_1\,. \tag{4.17}$$

This defines a holomorphic function on $\mathfrak{H}$ that satisfies for $\gamma\in\Gamma_{\mathrm{com}}$:

$$K|(\gamma-1)(z) \;=\; \int_z^{\gamma z}\omega_1\,,$$

and hence for $\gamma,\delta\in\Gamma_{\mathrm{com}}$:

$$\begin{aligned} K|(\gamma-1)(\delta-1)(z) \;&=\; \Big(\int_{\gamma z}^{\gamma\delta z}-\int_z^{\delta z}\Big)\omega_1 \;=\; \int_z^{\delta z}\omega_1|\gamma-\int_z^{\delta z}\omega_1\\ &=\; \mathsf{h}(\lambda(\gamma))\int_z^{\delta z}\omega \;=\; \mathsf{h}(\lambda(\gamma))\,\lambda(\delta)\,. \end{aligned}$$

Thus, we have a holomorphic third order non-commutative perturbation K of 1 with non-symmetric multilinear form $(\mathsf{h}\circ\lambda)\otimes\lambda$. Since holomorphic forms are harmonic in weight zero these perturbations are also harmonic perturbations of 1.

4.3.2. *Iterated integrals.* The construction of the third order form K in (4.17) is closely related to the iterated integrals used in [10] to prove maximal perturbability of spaces of smooth functions.

The idea is that we have two closed Γ_{com}-invariant differential forms on $\mathfrak{H}$, $dH(z) = \omega = -2\pi i\,\eta(z)^4\,dz$, and

$$\omega_0 \;=\; dW(z) \;=\; -\wp(H(z))\,d(H(z))\,,$$

where $\wp(u;\Lambda) = -\frac{d}{du}\zeta(u;\Lambda)$ is the Weierstrass $\wp$-function. If $t \mapsto z(t)$, $0 \le t \le 1$ is a path in $\mathfrak{H}$ from z_0 to z_1, then

$$\begin{aligned}\int_{t_2=0}^{1}\int_{t_1=0}^{t_2}\omega_0(z(t_1))\,\omega(z(t_2)) \;&=\; \int_{t_2=0}^{1}(W(z(t_2))-W(z_0))\,dH(z(t_2))\\ &=\; -2\pi i\int_{t=0}^{1}W(z(t))\,\eta(z(t))^4\,z'(t)\,dt - W(z_0)(H(z_1)-H(z_0))\\ &=\; K(z_1)-W(z_0)(H(z_1)-H(z_0))\end{aligned}$$

depends only on z_0 and z_1, not on the actual path. For a fixed base point z_0 the holomorphic function $z_1 \mapsto W(z_0)(H(z_1)-H(z_0))$ is invariant of order two. So up to lower order terms the invariant K is given by an iterated integral, as in (3) of [10]; see also [4].

4.3.3. *Differentiation of families.* We start by considering a general finitely generated group Γ acting on a space X. We will use the notation $f|\gamma(x) = f(\gamma x)$ for the action induced on functions defined on X. We consider a family of characters of Γ of the form $\chi_r(\gamma) = e^{ir\cdot\alpha(\gamma)}$, where $r\cdot\alpha(\gamma) = r_1\alpha_1(\gamma)+\cdots+r_n\alpha_n(\gamma)$, $\alpha_1,\dots,\alpha_n \in \hom(\Gamma,\mathbb{R})$, r varying over an open set U in $\mathbb{R}^n$. In this way χ_r is a family of unitary characters.

We consider a C^∞ family $r \mapsto f_r$ on a neighborhood $U \subset \mathbb{R}^n$ of 0 of functions $X \to \mathbb{C}$ that satisfy

$$f_r(\gamma x) \;=\; \chi_r(\gamma)\,f_r(x) \qquad (\gamma\in\Gamma)\,. \tag{4.18}$$

We assume that χ_0 is the trivial character and that f_0 is a Γ-invariant function f.

We now set $h(x) = \partial_{r_j} f_r(x)\big|_{r=0}$, for one of the coordinates of r. The transformation behaviour gives $h(\gamma x) \;=\; i\alpha_j(\gamma)\,f(x) + h(x)$, or, rewritten,

$$h|\gamma - h \;=\; i\alpha_j(\gamma)\,f\,.$$

The function h is a second order perturbation of f, with $i\alpha_j$ as the corresponding element of $\hom(\Gamma,\mathbb{C})$. This can be generalised:

Proposition 4.4. *For all multi-indices $a \in \mathbb{N}^n$ the derivative*

$$f^{(a)}(x) := \partial_r^a f_r(x)\big|_{r=0}$$

is a commutative perturbation of f with order $1+|a|$.

We use the notations $\partial_r^a = \partial_{r_1}^{a_1}\cdots\partial_{r_n}^{a_n}$ and $|a| = a_1+a_2+\cdots+a_n$.

Proof. We use induction on the length $|a|$ of the multi-index. The case $|a| = 1$ has already been handled above. For $|a| > 1$ we have

$$f^{(a)}(\gamma x) \;=\; \sum_{0\le b\le a}(i\alpha(\gamma))^{a-b}\binom{a}{b}f^{(b)}(x)\,,$$

where b runs over the multi-indices with $0 \le b_j \le a_j$, where $\binom{b}{a} = \prod_j \binom{a_j}{b_j}$, and where $\alpha(\gamma)^c = \prod_j \alpha_j(\gamma)^{c_j}$. Hence

$$f^{(a)}|(\gamma - 1) \;=\; \sum_{0\le b<a} (i\alpha(\gamma))^{a-b} \binom{a}{b} f^{(b)} \tag{4.19}$$

is a linear combination of higher order forms $f^{(b)}$ of orders $1, \dots, |a|$. So $f^{(a)}$ is an invariant of order at most $1 + |a|$. Furthermore

$$f^{(a)}|(\gamma_1 - 1)\cdots(\gamma_{|a|} - 1) \;=\; \sum_{0\le b<a} (i\alpha(\gamma_1))^{a-b} \binom{a}{b} f^{(b)}|(\gamma_2 - 1)\cdots(\gamma_{|a|} - 1)\,. \tag{4.20}$$

By induction assumption, the $f^{(b)}|(\gamma_2 - 1)\cdots(\gamma_{|a|} - 1)$ are multiples of f (zero if $|b| < |a| - 1$). So $f^{(a)}$ is a perturbation of f.

For the commutativity of the perturbation we note by induction that, for all $g_1, \dots, g_s \in \Gamma$

$$(g_1 - 1)(g_2 - 1)\cdots(g_s - 1) = \sum_{l=0}^{s} (-1)^{s-l} \sum_{i_1<i_2<\cdots<i_l} (g_{i_1} g_{i_2} \cdots g_{i_l} - 1)\,,$$

where the i_j run through the set $\{1, \dots, s\}$. Application of (4.19) leads to

$$f^{(a)}\big|(\gamma_1 - 1)\cdots(\gamma_{|a|} - 1) = \sum_{l=0}^{|a|} (-1)^{|a|-l} \sum_{i_1<i_2<\cdots<i_l} \sum_{0\le b<|a|} (i\alpha(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_l}))^{a-b} \binom{a}{b} f^{(b)}\,.$$

Since α is a homomorphism, the factor $\alpha(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_l})$ does not depend on the order of the γ_{i_j}. Hence we may rewrite the expression as follows.

$$f^{(a)}\big|(\gamma_1 - 1)\cdots(\gamma_{|a|} - 1) = \sum_{l=0}^{|a|} \frac{(-1)^{|a|-l}}{l!} \sum_{\mathbf{i}} \sum_{0\le b<|a|} (i\alpha(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_l}))^{a-b} \binom{a}{b} f^{(b)}\,,$$

where $\mathbf{i}$ in the sum $\sum_{\mathbf{i}}$ runs over the subsets of $\{1, \dots, |a|\}$ with l elements. This is an expression that is invariant under permutations of the γ_j, which shows that $f^{(a)}$ is a commutative perturbation. □

Remark. Proposition 4.4 shows that commutative perturbations can arise as infinitesimal perturbations of a family of automorphic forms. That is our motivation to use the word *perturbation* in Definition 3.2.

Application to harmonic perturbations of 1. We use the method of differentiation of families to produce explicit harmonic higher order forms for Γ_{com} of order 3. We employ families studied in [3].

Since Γ_{com} is free on the generators $C = \pm\left(\begin{smallmatrix} 2 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ and $D = \pm\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, the character group of Γ_{com} is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. We can parametrise the characters by

$$\chi_{v,w}(\gamma) \;=\; e^{iv\lambda(\gamma) + iw\overline{\lambda(\gamma)}}\,, \tag{4.21}$$

where (v, w) runs through $\mathbb{C}^2$, and where $\lambda \in \hom(\Gamma_{\mathrm{com}}, \mathbb{C})$ is as defined in (4.11). We are interested only in (v, w) in a neighborhood of $0 \in \mathbb{C}^2$.

In [3], §15.5 it is shown that there is a meromorphic Eisenstein family $E(v, w, s)$ of automorphic forms for Γ_{com}, with the character $\chi_{v,w}$ and eigenvalue $\frac{1}{4} - s^2$ for $\omega_0 = -y^2\left(\partial_x^2 + \partial_y^2\right)$. (In [3] the discussion of the family E is made in the context of families of automorphic forms of varying weight which are thus defined on the covering group $\tilde{\Gamma}_{\mathrm{com}}$. However, in §15.5 the weight is zero, and the automorphic forms are, in effect, on the discrete group Γ_{com}.) The restriction to $s = \frac{1}{2}$ exists ([3], §15.6) and forms a meromorphic

family $(v,w)\mapsto f(v,w;z)$ on $\mathbb{C}^2$ such that $f(v,w;\gamma z)=\chi_{v,w}(\gamma)\,f(v,w;z)$, and $L_0 f(v,w;z)=0$ for the dense set of (v,w) at which f is holomorphic. There is a meromorphic family $(v,w)\mapsto h(v,w;\cdot)$ on $\mathbb{C}$, such that $f(v,w;z)=h(v,w;H(z))$, satisfying $h(v,w;u+\lambda)=e^{iv\lambda+iw\lambda}\,h(v,w;u)$ ([3], §15.1–6). Chapter 15 of [3] gives a complicated but explicit construction (obtained with the help of D.Zagier) of such a family h with Jacobi theta-functions.

Specifically, in §15.6.11 the function h is expressed as a sum

$$h(v,w;u) \;=\; G_{(v+w)\varpi/2\pi}(u,w)+G_{-(v+w)\varpi/2\pi}(-\bar u,-v)\,, \tag{4.22}$$

where the function $G_\mu(u,w)$, for $\mu\notin\mathbb{Z}$ and $0<\operatorname{Im}u<\frac12\varpi\sqrt3$ is given by

$$G_\mu(u,w) \;=\; \sum_{m=-\infty}^{\infty}\frac{1}{\mu+m}\,\frac{\xi^{\mu+m}}{\eta\,q^m-1}\,, \tag{4.23}$$

with $q=-e^{-\pi\sqrt3}$, $\xi=e^{2\pi iu/\varpi}$, and $\eta=e^{-w\varpi\sqrt3}$ We consider this for u, w, and μ near zero, but not equal to zero. Hence $\eta\approx1$ but $\eta\neq1$, and $|q|<|\xi|<1$. The latter inequalities imply absolute convergence of the series. We shall derive the Taylor expansion of $\tilde h(v,w;u):=vw\,h(v,w;u)$ in terms of (v,w) near zero up to order two, from which we can obtain higher order forms by Proposition 4.4.

The term of $G_\mu(u,w)$ with $m=0$

$$\frac{1}{\mu}\,\frac{\xi^\mu}{\eta-1}\,, \tag{4.24}$$

has singularities at $\mu=0$, and, due to $\frac{1}{\eta-1}$, also at $w=0$. This term has the following contribution to $h(v,w;u)$ in (4.22).

$$\frac{2\pi}{\varpi(v+w)}\,\frac{e^{iu\,(v+w)}}{e^{-w\varpi\sqrt3}-1}-\frac{2\pi}{\varpi(v+w)}\,\frac{e^{i\bar u\,(v+w)}}{e^{v\varpi\sqrt3}-1} \tag{4.25}$$

We write the corresponding contribution to $\tilde h(v,w;u)=vw\,h(v,w;u)$ as follows.

$$\begin{aligned}\frac{2\pi}{\varpi}\,\frac{v\,w}{(e^{-w\varpi\sqrt3}-1)(e^{v\varpi\sqrt3}-1)}\Big(&\frac{e^{-w\varpi\sqrt3}(e^{(v+w)\varpi\sqrt3}-1)}{v+w}\\&+\frac{(e^{iu(v+w)}-1)(e^{v\varpi\sqrt3}-1)}{v+w}-\frac{(e^{i\bar u(v+w)}-1)(e^{-w\varpi\sqrt3}-1)}{v+w}\Big).\end{aligned}$$

The last three quotients are holomorphic as a function of $v+w$ in a neighborhood of 0. We replace them by their Taylor expansion up to the term $(v+w)^2$ and after that the Taylor expansion in both v and w up to order 2 is computed. This gives

$$\begin{aligned}\frac{-2\pi}{\varpi^2\sqrt3}\Big(&1+iu\,v+i\bar u\,w-\frac12u^2\,v^2-\frac12\bar u^2w^2\\&-\frac{\sqrt3}{2\pi}\Big(-\frac{\pi\varpi^2}{2\sqrt3}-\pi i\varpi u+\pi i\varpi\bar u+\frac{\pi}{\sqrt3}(u^2+\bar u^2)\Big)\,vw\Big)+\cdots\,.\end{aligned} \tag{4.26}$$

In the terms with $m\neq0$ in (4.23) we write $\xi=e^{2\pi iu/\varpi}$, $\tilde\xi=e^{-2\pi i\bar u/\varpi}$, $\eta_1=e^{-w\varpi\sqrt3}$, $\eta_2=e^{v\varpi\sqrt3}$, $q=-e^{-\pi\sqrt3}$, and $\mu=(v+w)\varpi/2\pi$. We find the following contribution to $\tilde h(v,w;u)$:

$$\sum_{m=1}^{\infty}\Big(\frac{vw}{m+\mu}\,\frac{\xi^{m+\mu}}{\eta_1q^m-1}+\frac{vw}{\mu-m}\,\frac{\xi^{\mu-m}}{\eta_1q^{-m}-1}+\frac{vw}{m-\mu}\,\frac{\tilde\xi^{m-\mu}}{\eta_2q^m-1}+\frac{vw}{-m-\mu}\,\frac{\tilde\xi^{-m-\mu}}{\eta_2q^{-m}-1}\Big).$$

This contribution is holomorphic near $v = w = 0$. Its expansion starts with the term vw. So for third order forms we need only the contribution to $h(0,0;u)$:

$$\sum_{m=1}^{\infty} \frac{1}{m}\Big(\frac{\xi^m}{q^m-1} + \frac{(q/\xi)^m}{q^m-1} + \frac{\tilde{\xi}^m}{q^m-1} + \frac{(q/\tilde{\xi})^m}{q^m-1}\Big). \tag{4.27}$$

Each of these terms gives a convergent series on the region $0 < \operatorname{Im} u < \frac{1}{2}\varpi\sqrt{3}$.

Commutative perturbations. In this expansion we find various higher order harmonic forms that we have seen above. Denoting $f = \frac{-2\pi}{\varpi^2\sqrt{3}}$ we find:

(4.28)

term of	on $\mathbb{C}$	on $\mathfrak{H}$
1	f	f (constant function)
v	$if\,u$	$if\,H(z)$
w	$if\,\overline{u}$	$if\,\overline{H(z)}$
v^2	$\frac{-f}{2}\,u^2$	$\frac{-f}{2}\,H(z)^2$
w^2	$\frac{-f}{2}\,\overline{u}^2$	$\frac{-f}{2}\,\overline{H(z)}^2$

The coefficient of vw gives a third order form

$$\begin{aligned} b_{1,1}(u) \;:=\; & \frac{\pi}{\sqrt{3}}\Big((\frac{u}{\varpi} - \frac{i\sqrt{3}}{2})^2 + (\frac{\overline{u}}{\varpi} + \frac{i\sqrt{3}}{2})^2 + 1\Big) \\ & + S(u) + S(\varpi\rho - u) + S(-\overline{u}) + S(\varpi\rho + \overline{u}), \end{aligned} \tag{4.29}$$

$$\text{with}\quad S(u) \;:=\; \sum_{m=1}^{\infty} \frac{e^{2\pi i m u/\varpi}}{m(q^m-1)}, \qquad \rho \;=\; \frac{1}{2} + \frac{i}{2}\sqrt{3}.$$

By $B_{1,1}(z) = b_{1,1}(H(z))$ we denote the corresponding harmonic third order perturbation of 1 on $\mathfrak{H}$. The way $B_{1,1}$ has been derived, together with the proof of Proposition 4.4, ensures that it is a perturbation of 1 with a multilinear form that is a multiple of $\lambda\otimes\overline{\lambda} + \overline{\lambda}\otimes\lambda$.

However, $b_{1,1}(u)$ is represented by (4.29) only on the region $0 < \operatorname{Im} u < \frac{1}{2}\varpi\sqrt{3}$. In [3], §15.3.5, the image under H of the fundamental domain

$$\bigcup_{n=-2}^{3} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mathfrak{F}_{\mathrm{mod}}$$

(where $\mathfrak{F}_{\mathrm{mod}}$ is the standard fundamental domain of the modular group) is shown to be the regular hexagon with centre 0 and one corner at $-\frac{1}{3}(e^{\pi i/3}+1)\varpi$. Only the upper half of this hexagon is in the region where we have an expression for $b_{1,1}$. We shall continue this function to the entire $\mathbb{C}$.

We first note that the series in (4.29) defining $S(u)$ converges absolutely for $\operatorname{Im} u > 0$ yielding a holomorphic function in that region. To extend $S(u)$ to other values we use the following identity, valid for $\operatorname{Im} u > \frac{1}{2}\varpi\sqrt{3}$:

$$S(u) \;=\; \sum_{m=1}^{\infty} \frac{e^{2\pi i m(u/\varpi+\rho)}}{m(q^m-1)} - \sum_{m=1}^{\infty} \frac{e^{2\pi i m u/\varpi}}{m} \;=\; S(u+\varpi\rho) + \log(1 - e^{2\pi i u/\varpi}). \tag{4.30}$$

Via this identity, we can define $S(u)$ in the region $\operatorname{Im} u > -\frac{1}{2}\varpi\sqrt{3}$. This extension of S is multivalued, since it depends on the way in which we extend the function $u \mapsto \log(1 - e^{2\pi i u/\varpi})$, which is given by the

second series in (4.30) only for $\operatorname{Im} u > 0$. However, the sum

$$S(u) + S(-\bar{u}) \;=\; S(u+\varpi\rho) + S(-\bar{u}+\varpi\rho) + 2\log|1-e^{2\pi i u/\varpi}| \tag{4.31}$$

is single-valued on $\operatorname{Im} u > -\frac{1}{2}\varpi\sqrt{3}$, with logarithmic singularities at $u = \varpi n$, $n \in \mathbb{Z}$. Applying (4.31) repeatedly, we can extend $S(u) + S(-\bar{u})$ to all $\mathbb{C}$ to obtain a harmonic function with singularities at the points in $\Lambda = \varpi\,\mathbb{Z}[\rho]$ which have non-positive imaginary part.

Via (4.29), we then obtain the continuation of the function $b_{1,1}$. It is harmonic on $\mathbb{C}\setminus\Lambda$, with logarithmic singularities at all points of Λ.

Let us explicitly check the transformation behaviour: Since S is periodic with period ϖ (and, equivalently, $S(u+\varpi\rho) \;=\; S(u-\varpi\bar{\rho})$),

$$\begin{aligned}
b_{1,1}(u+\varpi) - b_{1,1}(u) \;&=\; \frac{\pi}{\sqrt{3}}\Big((\frac{u}{\varpi}+1-\frac{i\sqrt{3}}{2})^2 - (\frac{u}{\varpi}-\frac{i\sqrt{3}}{2})^2 \\
&\qquad + (\frac{\bar{u}}{\varpi}+1+\frac{i\sqrt{3}}{2})^2 - (\frac{\bar{u}}{\varpi}+\frac{i\sqrt{3}}{2})^2\Big) \\
&= \frac{\pi}{\sqrt{3}}\Big(\frac{2u}{\varpi}+1-i\sqrt{3}+\frac{2\bar{u}}{\varpi}+1+i\sqrt{3}\Big) \;=\; \frac{2\pi}{\varpi\sqrt{3}}(u+\bar{u}+\varpi)\,; \\
b_{1,1}(u+\varpi\rho) - b_{1,1}(u) \;&=\; \frac{\pi}{\sqrt{3}}\Big((\frac{u}{\varpi}+\frac{1}{2})^2 - (\frac{u}{\varpi}-\frac{i\sqrt{3}}{2})^2 \\
&\qquad + (\frac{\bar{u}}{\varpi}+\frac{1}{2})^2 - (\frac{\bar{u}}{\varpi}+\frac{i\sqrt{3}}{2})^2\Big) \\
&\qquad - 2\log\big|1-e^{2\pi i u/\varpi}\big| + S(-u) + S(\bar{u}+\varpi) - S(\varpi\rho-u) - S(\varpi\rho+\bar{u}) \\
&= \frac{\pi}{\sqrt{3}}\Big(\rho(\frac{2u}{\varpi}+\rho^{-1}) + \rho^{-1}(\frac{2\bar{u}}{\varpi}+\rho)\Big) \\
&\qquad - 2\log\big|1-e^{2\pi i u/\varpi}\big| + 2\log\big|1-e^{-2\pi i u/\varpi}\big| \\
&= \frac{2\pi}{\sqrt{3}}(1+\frac{\rho u+\overline{\rho u}}{\varpi}) - 2\pi i(u-\bar{u})/\varpi \;=\; \frac{2\pi}{\sqrt{3}}(1+\rho^{-1}\,\frac{u}{\varpi}+\rho\,\frac{\bar{u}}{\varpi})\,.
\end{aligned}$$

Let us denote by T_ω the translation by $\omega \in \Lambda$, and use the notations $b_{1,0}(u) = u$, $b_{0,1}(u) = \bar{u}$. With the notations $f = \frac{-2\pi}{\varpi^2\sqrt{3}}$ and $a = \frac{2\pi}{\varpi\sqrt{3}} = -f\varpi$ we have

$$\begin{aligned}
b_{1,0}|(T_\varpi - 1) \;&=\; \omega\,, \qquad b_{0,1}|(T_\omega - 1) \;=\; \bar{\omega}\,, \\
b_{1,1}|(T_\varpi - 1) \;&=\; a\,(b_{1,0} + b_{0,1} + \varpi)\,, \\
b_{1,1}|(T_{\rho\varpi} - 1) \;&=\; a\,(\bar{\rho}b_{1,0} + \rho b_{0,1} + \varpi)\,, \\
b_{1,1}|(T_\varpi - 1)^2 \;&=\; 2a\varpi \;=\; -2f\varpi^2\,, \\
b_{1,1}|(T_\varpi - 1)(T_{\rho\varpi} - 1) \;&=\; a\,(\rho\varpi + \bar{\rho}\varpi) \;=\; -f\,(\bar{\varpi}\cdot\rho\varpi + \varpi\cdot\overline{\rho\varpi})\,, \\
b_{1,1}|(T_{\rho\varpi} - 1)^2 \;&=\; 2a\varpi \;=\; -2f(\rho\varpi)(\overline{\rho\varpi})\,.
\end{aligned} \tag{4.32}$$

Since Λ is commutative we need not consider $b_{1,1}|(T_{\rho\varpi} - 1)(T_\varpi - 1)$. We conclude that the pull-back $-f^{-1}B_{1,1} = -f^{-1}b_{1,1}\circ H$ is a harmonic commutative perturbation of 1 for the multilinear form μ determined

by the following values at the generators C and CD of Γ_{com}:

$$\mu(g,h) \;=\; \begin{cases} 2\varpi^2 & \text{if } g=h=C \text{ or } CD\,,\\ \varpi^2 & \text{if } g=C,\ h=CD\,,\ \text{ or if } g=CD,\ h=C\,. \end{cases} \tag{4.33}$$

We have used the values of λ given below (4.11).) With these values at the generators, μ coincides with $\lambda\otimes\overline{\lambda}+\overline{\lambda}\otimes\lambda$ as predicted above by the way $B_{1,1}$ was constructed.

Non-commutative perturbation. Proposition 4.4 shows that differentiation of families produces only commutative perturbations. However, by Theorem 4.2, there are non-commutative third order harmonic perturbations of 1. We can obtain such perturbations from $B_{1,1}$ upon decomposing it as $B_{1,1}=A+B$ for a holomorphic function A and an anti-holomorphic function B.

Specifically, in view of (4.29), for those $z\in\mathfrak{H}$ for which $H(z)$ is in the upper half of the fundamental hexagon for $\mathbb{C}/\Lambda$, we can set

$$\begin{aligned} A(z) \;&=\; \frac{\pi}{2\sqrt{3}}+\frac{\pi}{\sqrt{3}}\Big(\frac{H(z)}{\varpi}-\frac{i\sqrt{3}}{2}\Big)^2+S\big(H(z)\big)+S\big(\varpi\rho-H(z)\big)\,,\\ B(z) \;&=\; \frac{\pi}{2\sqrt{3}}+\frac{\pi}{\sqrt{3}}\Big(\frac{\overline{H(z)}}{\varpi}+\frac{i\sqrt{3}}{2}\Big)^2+S\big(-\overline{H(z)}\big)+S\big(\varpi\rho+\overline{H(z)}\big)\,. \end{aligned} \tag{4.34}$$

As shown above, $B_{1,1}|(\gamma-1)(\delta-1)=-f\,\lambda\otimes\overline{\lambda}-f\,\overline{\lambda}\otimes\lambda$. Hence,

$$A|(\gamma-1)(\delta-1) \;=\; -B|(\gamma-1)(\delta-1)-f\,\lambda\otimes\overline{\lambda}-f\,\overline{\lambda}\otimes\lambda.$$

gives an equality between a holomorphic and an antiholomorphic function, and therefore, there is $\nu:\Gamma^2\to\mathbb{C}$ such that

$$A|(\gamma-1)(\delta-1) \;=\; \nu(\gamma,\delta)\,,\ B|(\gamma-1)(\delta-1) \;=\; -f\,\lambda\otimes\overline{\lambda}-f\,\overline{\lambda}\otimes\lambda-\nu(\gamma,\delta)$$

for all $\gamma,\delta\in\Gamma$. This implies that A and B are third order invariants, and that $\nu\in\mathrm{Mult}^2(\Gamma,\mathbb{C})$.

To determine the bilinear form ν, we recall that $\lambda(C)=\rho\varpi$ and $\lambda(D)=\overline{\rho}\varpi=(1-\rho)\varpi$. We consider the following four functions:

$$\begin{aligned} &A|(C-1)+f(\overline{\rho}\varpi\,H+\tfrac{\varpi^2}{2})\,, && B|(C-1)+f(\rho\varpi\,\overline{H}+\tfrac{\varpi^2}{2})\,,\\ &A|(D-1)+f(\rho\varpi\,H+\tfrac{\varpi^2}{2})\,, && B|(D-1)+f(\overline{\rho}\varpi\,\overline{H}+\tfrac{\varpi^2}{2})\,, \end{aligned} \tag{4.35}$$

The functions on the left are holomorphic, and those on the right are antiholomorphic. We consider the sum of the two functions on the first row, and denote $u=H(z)$. With (4.32):

$$\begin{aligned} B_{1,1}|(C-1)(z)+f\overline{\rho}\varpi H(z)+f\rho\varpi\overline{H(z)}+f\varpi^2&\\ =\; b_{1,1}|(T_{\rho\varpi}-1)(u)+f\varpi(\overline{\rho}u+\rho\overline{u})+f\varpi^2&\\ =\; -f\varpi(\overline{\rho}u+\rho\overline{u}+\varpi)+f\varpi(\overline{\rho}u+\rho\overline{u})+f\varpi^2 \;=\; 0\,.& \end{aligned}$$

Similarly the sum of the two functions on the second row gives

$$\begin{aligned}
&B_{1,1}|(D-1)(z) + f(\rho\varpi H(z) + \bar\rho\varpi\overline{H(z)} + \varpi^2)\\
&\quad = b_{1,1}|(T_{\bar\rho\varpi}-1)(u) + f\varpi(\rho u + \overline{\rho u} + \varpi)\\
&\quad = b_{1,1}|(T_{\varpi}-1)T_{\rho\varpi}^{-1} - b_{1,1}|(T_{\rho\varpi}-1)T_{\rho\varpi}^{-1} + f\varpi(\rho u + \overline{\rho u} + \varpi)\\
&\quad = (-f\varpi(u+\bar u+\varpi) + f\varpi(\bar\rho u + \rho\bar u + \varpi))|T_{\rho\varpi}^{-1} + f\varpi(\rho u + \overline{\rho u} + \varpi)\\
&\quad = f\varpi(-u + \rho\varpi - \bar u + \bar\rho\varpi - \varpi + \bar\rho(u-\rho\varpi) + \rho(\bar u - \bar\rho\varpi) + \varpi\\
&\qquad + \rho u + \overline{\rho u} + \varpi) \;=\; 0\,.
\end{aligned}$$

The sums of the rows in (4.35) are zero, so the individual functions are constant. We do not try to determine these constants.

For A we have

$$\begin{aligned}
A|(C-1)(C-1) &= -f\,(\bar\rho\varpi H|(C-1)+0) \;=\; -f\bar\rho\varpi\lambda(C) \;=\; -f\overline{\lambda(C)}\,\lambda(C)\,,\\
A|(C-1)(D-1) &= -f(\bar\rho\varpi\lambda(D)) \;=\; -f\overline{\lambda(C)}\,\lambda(D)\,,\\
A|(D-1)(C-1) &= -f\overline{\lambda(D)}\,\lambda(C)\,,\\
A|(D-1)(D-1) &= -f\overline{\lambda(D)}\,\lambda(C)\,.
\end{aligned}$$

We conclude that $-f^{-1}A$ is a non-commutative holomorphic third order holomorphic perturbation of 1 with multilinear form $\bar\lambda\otimes\lambda$. Then the multilinear form of the anticommutative third order perturbation of 1 given by $-f^{-1}B = -f^{-1}(B_{1,1}-A)$ is $(\lambda\otimes\bar\lambda + \bar\lambda\otimes\lambda) - \bar\lambda\otimes\lambda = \lambda\otimes\bar\lambda$.

5. Universal covering group

5.1. **Universal covering group of** $\mathrm{SL}_2(\mathbb{R})$**.** To define the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, which is also the universal covering group of $G = \mathrm{PSL}_2(\mathbb{R})$, we first note that, as an analytic variety, $\mathrm{SL}_2(\mathbb{R})$ is isomorphic to $\mathfrak{H}\times(\mathbb{R}/2\pi\mathbb{Z})$, by the Iwasawa decomposition expressing each element of $\mathrm{SL}_2(\mathbb{R})$ uniquely as a product

$$\begin{pmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{pmatrix}\begin{pmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{pmatrix},$$

with $x+iy\in\mathfrak{H}$ and $\vartheta\in\mathbb{R}/2\pi\mathbb{Z}$. Left multiplication by $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{R})$ amounts to

$$(z,\vartheta+2\pi\mathbb{Z}) \mapsto \Big(\frac{az+b}{cz+d}, \vartheta - \arg\big(j\big(\begin{pmatrix}a & b\\ c & d\end{pmatrix},z\big)\big) + 2\pi\mathbb{Z}\Big)\,. \tag{5.1}$$

Here, $j(\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right),z) := cz+d$. This describes an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}\times(\mathbb{R}/2\pi\mathbb{Z})$.

We define for each $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{R})$ the operator

$$\widetilde{\begin{pmatrix}a & b\\ c & d\end{pmatrix}} : (z,\vartheta) \mapsto \Big(\frac{az+b}{cz+d}, \vartheta - \arg\big(j\big(\begin{pmatrix}a & b\\ c & d\end{pmatrix},z\big)\big)\Big) \tag{5.2}$$

from $\mathfrak{H}\times\mathbb{R}$ to itself, where we choose the argument such that $-\pi < \arg(cz+d)\le\pi$. We note that the map $g\mapsto\tilde g$ is injective.

Definition 5.1. The universal covering group $\tilde G$ of G is the group of operators $\mathfrak{H}\times\mathbb{R}\to\mathfrak{H}\times\mathbb{R}$ generated by the operators $\tilde g$ in (5.2) for all $g\in\mathrm{SL}_2(\mathbb{R})$.

A lengthy but routine calculation shows

Lemma 5.2. *If the vertical maps in the diagram*

$$\begin{array}{ccccc} \mathfrak{H}\times\mathbb{R} & \longrightarrow & \mathfrak{H}\times\mathbb{R}/\mathbb{Z} & = & \mathrm{SL}_2(\mathbb{R}) \\ \downarrow{\scriptstyle \tilde{g}} & & \downarrow & & \downarrow{\scriptstyle g} \\ \mathfrak{H}\times\mathbb{R} & \longrightarrow & \mathfrak{H}\times\mathbb{R}/\mathbb{Z} & = & \mathrm{SL}_2(\mathbb{R}) \end{array}$$

are given by (5.2), (5.1) *and by left multiplication by g respectively, then the diagram is commutative. (The last horizontal maps are defined by the Iwasawa decomposition.)*

Suppose now that $\tilde{g}_1\tilde{g}_2\cdots\tilde{g}_n$ is the identity as an operator on $\mathfrak{H}\times\mathbb{R}$. Then $z \mapsto g_1g_2\cdots g_nz$ is the identity operator on $\mathfrak{H}$. So $g_1g_2\cdots g_n \in \{I, -I\} \subset \mathrm{SL}_2(\mathbb{R})$. By Lemma 5.2, it is impossible that $g_1g_2\cdots g_n = -I$ while $\tilde{g}_1\tilde{g}_2\cdots\tilde{g}_n$ is the identity operator. So $g_1g_2\cdots g_n = I$. This implies that the map $\tilde{g} \mapsto g$ on the generators extends to a group homomorphism

$$\mathrm{pr}_2 : \tilde{G} \longrightarrow \mathrm{SL}_2(\mathbb{R})\,.$$

The composition of pr_2 with the natural projection $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ gives a map

$$\mathrm{pr} : \tilde{G} \longrightarrow \mathrm{PSL}_2(\mathbb{R})\,.$$

We single out the following following families of elements of $\tilde{G}$.

a) For $x \in \mathbb{R}$ we put $n(x) = \widetilde{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)}$ in $\tilde{G}$. This induces an injective group homomorphism $n : \mathbb{R} \to \tilde{G}$.
b) For $y \in \mathbb{R}_+^*$ we set $a(y) = \widetilde{\left(\begin{smallmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{smallmatrix}\right)}$. This induces an injective group homomorphism $a : \mathbb{R}_+^* \to \tilde{G}$.
c) For $\vartheta \in \mathbb{R}$, we set

$$k(\vartheta)(z, \vartheta_1) = \Big(\frac{z\cos\vartheta + \sin\vartheta}{-z\sin\vartheta + \cos\vartheta}, \vartheta_1 + \vartheta - \arg(e^{i\vartheta}(-z\sin\vartheta + \cos\vartheta))\Big). \tag{5.3}$$

This defines $k(\vartheta) \in \tilde{G}$ satisfying $\mathrm{pr}_2 k(\vartheta) = \left(\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right)$. For fixed $(z, \vartheta_1) \in \mathfrak{H}\times\mathbb{R}$, the quantity $k(\vartheta)(z, \vartheta_1)$ is real-analytic in ϑ. If both ϑ and ϑ' have values near zero then $k(\vartheta + \vartheta') = k(\vartheta)k(\vartheta')$, since pr_2 is locally an isomorphism. By analyticity this relation extends to all $\vartheta, \vartheta' \in \mathbb{R}$. So we have a group homomorphism $k : \mathbb{R} \to \tilde{G}$. The kernel of the composition $\mathrm{pr}_2 \circ k$ is $2\pi\mathbb{Z}$. For each $n \in \mathbb{Z}$ the element $k(n\pi)$ acts as $(z, \vartheta_1) \mapsto (z, \vartheta_1 + \pi n)$. This implies that k is an injective group homomorphism. Although it satisfies $\mathrm{pr}_2 k(\vartheta) = \left(\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right)$ for all $\vartheta \in \mathbb{R}$, the relation $\widetilde{\left(\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right)} = k(\vartheta)$ holds only for $\vartheta \in [-\pi, \pi)$.

With these definitions and notations we deduce some basic facts about $\tilde{G}$.

Centre of $\tilde{G}$: The elements $k(\pi n)$ with $n \in \mathbb{Z}$ form the centre $\tilde{Z}$ of $\tilde{G}$.

Transitivity of action of $\tilde{G}$ on $\mathfrak{H}\times\mathbb{R}$: This is implied by $n(x)a(y)k(\vartheta)\,(i, 0) = (x + iy, \vartheta)$ for all $x + iy \in \mathfrak{H}$ and $\vartheta \in \mathbb{R}$.

Generators of $\tilde{G}$: The elements $n(x)$, $a(y)$ and $k(\vartheta)$ generate $\tilde{G}$, and each element of $\tilde{G}$ can be written uniquely as $n(x)a(y)k(\vartheta)$. This is a consequence of the relations

$$a(y)n(x) = n(y^2x)a(y) \qquad \text{and} \tag{5.4}$$

$$k(\vartheta)n(x)a(y) = n(x_\vartheta)a(y_\vartheta)k(\vartheta - \arg(e^{i\vartheta}(-z\sin\vartheta + \cos\vartheta)))\,) \tag{5.5}$$

with $z = x + iy$ and $x_\vartheta + iy_\vartheta = \frac{z\cos\vartheta + \sin\vartheta}{-z\sin\vartheta + \cos\vartheta}$.

$\tilde{G} \cong \mathfrak{H} \times \mathbb{R}$. Because of the last two facts, we can identify $\tilde{G}$ with $\mathfrak{H} \times \mathbb{R}$ as analytic varieties. Furthermore, the group operations are analytic with respect to the structure of $\mathfrak{H} \times \mathbb{R}$ as an analytic variety. So $\tilde{G}$ is a Lie group. The maps pr and pr_2 are covering maps. One can show that any covering of $\mathrm{SL}_2(\mathbb{R})$ factors through $\tilde{G}$, hence $\tilde{G}$ is the universal covering group of $\mathrm{SL}_2(\mathbb{R})$.

Section $g \to \tilde{g}$: This is a homeomorphism for g near the unit element of $\mathrm{SL}_2(\mathbb{R})$, but it is discontinuous at $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ with $c = 0$ and $d < 0$. This section is not a group homomorphism but instead there is a $\mathbb{Z}$-valued 2-cocycle w on $\mathrm{SL}_2(\mathbb{R})$ such that $\tilde{g}\tilde{g}_1 = \widetilde{gg_1} k(2\pi w(g, g_1))$ for all $g, g_1 \in \mathrm{SL}_2(\mathbb{R})$. See Theorem 16 on p. 115 of [16] for an explicit description of this cocycle. Each element of $\tilde{G}$ has a unique decomposition as $\tilde{g}\, k(2\pi n)$ with $g \in \mathrm{SL}_2(\mathbb{R})$ and $n \in \mathbb{Z}$. In this paper we will not use this description of the group structure of $\tilde{G}$. We work with the interpretation as a group of operators in $\mathfrak{H} \times \mathbb{R}$, and occasionally use the "one-parameter subgroups" n, a and k.

The action of $\tilde{G}$ on $\mathfrak{H}^* := \mathfrak{H} \cup \{\text{cusps}\}$ is given by $\gamma z := \mathrm{pr}(\gamma)\, z$.

5.2. **The Lie algebra of the universal covering group.** The direction of the three one-parameter subgroups n, a and k at the origin determines elements of the (real) Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $\tilde{G}$. The groups $\tilde{G}$, $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R})$ have the same Lie algebra, since they are locally isomorphic. The Lie algebra elements corresponding to n, a and k are, respectively,

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \frac{1}{2}\mathbf{H} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \text{and} \quad \mathbf{W} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.6}$$

The Lie algebra acts on the functions on $\tilde{G}$ by differentiation on the right: $\mathbf{Y}F(g) = \partial_t F(g \exp(t\mathbf{Y}))|_{t=0}$ for $\mathbf{Y} \in \mathfrak{g}_{\mathbb{R}}$. This action can be extended to the complexified Lie algebra $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$, and to the universal enveloping algebra of $\mathfrak{g}$. All the resulting differential operators commute with the action of $\tilde{G}$ by left translation. With the identification of $\tilde{G}$ as $\mathfrak{H} \times \mathbb{R}$ we have in the coordinates given by $(x + iy, \vartheta)$:

$$\begin{aligned} \mathbf{X} &= \partial_x\,, \qquad \mathbf{H} = 2y\partial_y\,, \qquad \mathbf{W} = \partial_\vartheta\,, \\ \mathbf{E}^+ &= \mathbf{H} + i(2\mathbf{X} - \mathbf{W}) = e^{2i\vartheta}(2iy\partial_x + 2y\partial_y - i\partial_\vartheta)\,, \\ \mathbf{E}^- &= \mathbf{H} - i(2\mathbf{X} - \mathbf{W}) = e^{-2i\vartheta}(-2iy\partial_x + 2y\partial_y + i\partial_\vartheta)\,, \\ \omega &= -\frac{1}{4}\mathbf{E}^{\pm}\mathbf{E}^{\mp} + \frac{1}{4}\mathbf{W}^2 \mp \frac{i}{2}\mathbf{W} = -y^2\partial_y^2 - y^2\partial_x^2 + y\partial_x\partial_\vartheta\,. \end{aligned} \tag{5.7}$$

The *Casimir operator* ω generates the centre of the enveloping algebra of $\mathfrak{g}$. The corresponding differential operator commutes with left and right translations in $\tilde{G}$.

5.3. **Cofinite discrete subgroups.** To a cofinite discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ we associate its full original $\tilde{\Gamma} := \mathrm{pr}^{-1}\Gamma$ in $\tilde{G}$. This gives a bijective correspondence between cofinite discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ and cofinite discrete subgroups of $\tilde{G}$ that contain the centre $\tilde{Z} = \langle\zeta\rangle$, where $\zeta := k(\pi)$. The projection pr induces an isomorphism $\Gamma \cong \tilde{\Gamma}/\tilde{Z}$.

As an example we consider the *modular group* $\Gamma_{\mathrm{mod}} = \mathrm{PSL}_2(\mathbb{Z})$, with corresponding group $\tilde{\Gamma}_{\mathrm{mod}} \subset \tilde{G}$. It is known that $\mathrm{PSL}_2(\mathbb{Z})$ is presented by the generators $S = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and relations $S^2 = (TS)^2 = I$.

Set $s := k(-\pi/2) = \widetilde{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}$ and $t := n(1) = \widetilde{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}$ with $\mathrm{pr}(s) = S$ and $\mathrm{pr}(t) = T$. Then $s^2 = k(-\pi) = \zeta^{-1} \in \tilde{Z}$, so s and t generate $\tilde{\Gamma}_{\mathrm{mod}}$. The relation $S^2 = I$ is replaced by the centrality of s^2. We have

$$ts(i,0) \;=\; t\widetilde{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}(i,0) \;=\; \widetilde{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}(i, -\arg i) \;=\; (\frac{i-1}{i}, -\pi/2 - \arg 1) = \widetilde{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}}(i,0) \;=\; (i+1, -\pi/2)\,.$$

So $ts = \widetilde{\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)}$, and it corresponds to $(i+1, -\pi/2)$ in $\mathfrak{H} \times \mathbb{R} \cong \tilde{G}$. Hence

$$\begin{aligned} (ts)^3 \;&=\; ts\widetilde{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}}(i+1, -\pi/2) \;=\; ts\,(\frac{i}{i+1}, -\frac{\pi}{2} - \arg(i+1)) \\ &=\; \widetilde{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}}(\frac{i+1}{2}, -\frac{3\pi}{4}) \;=\; (\frac{i-1}{i+1}, -\frac{3\pi}{4} - \arg(i-1)) \\ &=\; (i, -\pi) \;=\; \zeta^{-1} \;=\; s^2\,. \end{aligned}$$

The conclusion is that $\tilde{\Gamma}_{\mathrm{mod}}$ has the presentation with generators s and t and relations $s^2t = ts^2$ and $tstst = s$. This implies that the linear space $\hom(\tilde{\Gamma}_{\mathrm{mod}}, \mathbb{C})$ has dimension 1, and is generated by $\alpha : t \mapsto \frac{\pi}{6}, \alpha : s \mapsto \frac{-\pi}{2}$. For reasons that will become clear later, we take this basis element, and not an integral-valued one.

5.4. **Canonical generators.** The canonical generators of Γ induce canonical generators of $\tilde{\Gamma}$:

- Elements $\pi_1, \ldots, \pi_{n_{\mathrm{par}}}$ of the form $\pi_j = \tilde{g}_{\kappa_j} n(1) \tilde{g}_{\kappa_j}^{-1}$ fixing a system of representatives $\kappa_1, \ldots, \kappa_{n_{\mathrm{par}}}$ of the $\tilde{\Gamma}$-orbits of cusps.
- Elements $\varepsilon_1, \ldots, \varepsilon_{n_{\mathrm{ell}}}$ conjugate in $\tilde{G}$ to $k(\pi/v_j)$ with $v_j \geq 2$.
- Elements $\eta_1, \ldots, \eta_{2g}$ conjugate in $\tilde{G}$ to elements $a(t_j)$ with $t_j > 1$.
- The generator $\zeta = k(\pi)$ of the centre $\tilde{Z}$ of $\tilde{\Gamma}$.

The relations are:

$$\begin{aligned} &\zeta \text{ is central}\,, \\ &\varepsilon_j^{v_j} = \zeta \text{ for } 1 \leq j \leq n_{\mathrm{ell}}\,, \\ &\pi_1 \cdots \pi_{n_{\mathrm{par}}} \varepsilon_1 \cdots \varepsilon_{n_{\mathrm{ell}}} [\eta_1, \eta_2] \cdots [\eta_{2g-1}, \eta_{2g}] \;=\; \zeta^{2g-2+n_{\mathrm{par}}+n_{\mathrm{ell}}}\,. \end{aligned} \tag{5.8}$$

The integer $2g - 2 + n_{\mathrm{par}} + n_{\mathrm{ell}}$ is always positive. For these facts see [3], §3.3.

If $n_{\mathrm{ell}} > 0$ or if $2g - 2 + n_{\mathrm{par}} = 1$ and $n_{\mathrm{ell}} = 0$, we do not need ζ as a generator. If $n_{\mathrm{ell}} = 0$ the group $\tilde{\Gamma}$ is free on $\pi_1, \ldots, \pi_{n_{\mathrm{par}}-1}, \eta_1, \ldots, \eta_{2g}, \zeta$.

Among the canonical generators we single out the following elements: $\alpha_1 = \pi_1, \ldots, \alpha_{n_{\mathrm{par}}-1} = \pi_{n_{\mathrm{par}}-1}$, $\alpha_{n_{\mathrm{par}}} = \eta_1, \ldots, \alpha_{t(\Gamma)-1} = \eta_{2g}, \alpha_{t(\Gamma)} = \zeta$. (We recall that $t(\Gamma) = n_{\mathrm{par}} + 2g$.) The α_j together with the ε_j generate $\tilde{\Gamma}$, with $\varepsilon_j^{v_j} = \zeta$ and the centrality of ζ as the sole relations.

For the *modular group* $\tilde{\Gamma}_{\mathrm{mod}}$ we have $n_{\mathrm{par}} = 1$, $n_{\mathrm{ell}} = 2$, $g = 0$, and $t(\Gamma_{\mathrm{mod}}) = 1$. We may take $\pi_1 = t = n(1)$, $\varepsilon_1 = t^{-1}s^{-1}$, and $\varepsilon_2 = s^{-1} = k(\pi/2) = p^{-1}k(\pi/3)p$, with $p = n(-1/2)a(\sqrt{3}/2)$.

By I we now denote the augmentation ideal of the group ring $\mathbb{C}[\tilde{\Gamma}]$. In $\mathbb{C}[\tilde{\Gamma}]$ we have the elements

$$\mathbf{b}(\mathbf{i}) \;=\; (\alpha_{\mathbf{i}(1)} - 1) \cdots (\alpha_{\mathbf{i}(q)} - 1) \qquad \mathbf{i} \in \{1, \ldots, t(\Gamma)\}^q\,. \tag{5.9}$$

We allow ourselves to use the same notation as in (3.8), since from now on we will use $\tilde{\Gamma}$. The centrality of ζ allows us to move $(\zeta - 1)$ through the product. So it suffices to consider only q-tuples $\mathbf{i}$ for which all $\mathbf{i}(l) = t(\Gamma)$ occur at the end. Such q-tuples we will call $\tilde{\Gamma}$-*q-tuples*.

Proposition 5.3. *A $\mathbb{C}$-basis of $I^{q+1}\backslash I^q$ is induced by the elements*

$$\mathbf{b(i)} \;=\; (\alpha_{\mathbf{i}(1)}-1)\cdots(\alpha_{\mathbf{i}(q)}-1)\,, \tag{5.10}$$

where $\mathbf{i}$ *runs over the* $\tilde\Gamma$*-q-tuples.*

Proof. The ideal I^q is generated by the products of the form $(\gamma_1-1)\cdots(\gamma_q-1)$ with $\gamma_1,\dots,\gamma_q\in\tilde\Gamma$. (Lemma 1.1 in [5].) With the relation

$$(\gamma\delta-1)=(\gamma-1)(\delta-1)+(\gamma-1)+(\delta-1)\,,$$

we can take the γ_j in a system of generators, for instance $\alpha_1,\dots,\alpha_{t(\Gamma)},\varepsilon_1,\dots,\varepsilon_{n_{\rm ell}}$. For the elliptic elements ε_j we use $\zeta-1=\sum_{k=0}^{v_j-1}\varepsilon_j^k(\varepsilon_j-1)\equiv v_j(\varepsilon_j-1)$ mod I^2 to see that the α_j suffice. (Note that v_j is invertible in $\mathbb{C}$.) Since $\alpha_{t(\Gamma)}=\zeta$ is central, we can move all occurrences of $\zeta-1$ to the right to see that the $\mathbf{b(i)}$ in the proposition generate $I^{q+1}\backslash I^q$.

To see that the $\mathbf{b(i)}$ are linearly independent over $\mathbb{C}$ we proceed in rewriting terms $\xi(\alpha_{\mathbf{i}(1)}-1)\cdots(\alpha_{\mathbf{i}(q)}-1)$ by replacing $\xi\in R:=\mathbb{C}[\tilde\Gamma]$ by $n+\eta$ with $n\in\mathbb{C}$ and $\eta\in I$. In this way, we express each element of I^q as a $\mathbb{C}$-linear combination of products of q factors α_j-1 plus a term in I^N, with $N>q$. To eliminate I^N we consider the I-adic completion $\hat R$ of $\mathbb{C}[\tilde\Gamma]$, with closure $\hat I^q$ of I^q. Each element of $\hat I\supset I$ is a countable sum of products of a complex number and finitely many factors α_j-1. Since $\hat I^{q+1}\backslash\hat I^q$ and $I^{q+1}\backslash I^q$ are isomorphic, it suffices to prove that the $\mathbf{b(i)}$ are linearly independent as elements of $\hat I^{q+1}\backslash\hat I^q$.

We suppose that there are $x_{\mathbf{i}}\in\mathbb{C}$ for all q-tuples $\mathbf{i}$ such that

$$\sum_{\mathbf{i}} x_{\mathbf{i}}(\alpha_{\mathbf{i}(1)}-1)\cdots(\alpha_{\mathbf{i}(q)}-1)\;\in\;\hat I^{q+1}\,. \tag{5.11}$$

We can write this element of $\hat I^{q+1}$ as $\sum_{\mathbf{j}}c_{\mathbf{j}}\,\xi_{\mathbf{j}}$ with $c_{\mathbf{j}}\in\mathbb{C}$, and $\xi_{\mathbf{j}}$ running over the countably many products $(\alpha_{\mathbf{j}(1)}-1)\cdots(\alpha_{\mathbf{j}(m)}-1)$ with m-tuples from $\{1,\dots,t(\Gamma)\}$ for all $m>q$.

We form the ring $N=\mathbb{C}\langle\Xi_1,\dots,\Xi_{t(\Gamma)}\rangle$ of power series in the non-commuting, algebraically independent (over $\mathbb{C}$) variables $\Xi_1,\dots,\Xi_t$, and the two-sided ideal Z in N generated by the commutators

$$\Xi_j\,\Xi_{t(\Gamma)}-\Xi_{t(\Gamma)}\,\Xi_j\qquad\text{for } 1\le j\le t(\Gamma)\,.$$

The quotient ring $M:=N/Z$ is non-commutative if $t(\Gamma)\ge 3$. The relations between the generators imply that there is a group homomorphism $\varphi:\tilde\Gamma\to M^*$ given by $\varphi(\alpha_j)=1+\Xi_j$ for $1\le j\le t(\Gamma)$, and

$$\varphi(\varepsilon_j)\;=\;(1+\Xi_{t(\Gamma)})^{1/v_j}\;=\;\sum_{l\ge0}\binom{1/v_j}{l}\Xi_{t(\Gamma)}^l\,.$$

This group homomorphism induces a ring homomorphism $\hat\varphi:\hat R\longrightarrow M$, for which

$$\hat\varphi(\xi_{\mathbf{i}})=\hat\varphi(\alpha_{\mathbf{i}(1)}-1)\,\hat\varphi(\alpha_{\mathbf{i}(2)}-1)\cdots\hat\varphi(\alpha_{\mathbf{i}(|\mathbf{i}|)}-1)\;=\;\Xi^{\mathbf{i}}:=\Xi_{\mathbf{i}(1)}\,\Xi_{\mathbf{i}(2)}\cdots\Xi_{\mathbf{i}(|\mathbf{i}|)}\,.$$

Now we have

$$\sum_{\mathbf{i}}x_{\mathbf{i}}\Xi^{\mathbf{i}}\;=\;\hat\varphi\Big(\sum_{\mathbf{i}}x_{\mathbf{i}}\,\xi_{\mathbf{i}}\Big)\;=\;\hat\varphi\Big(\sum_{\mathbf{j}}c_{\mathbf{j}}\xi_{\mathbf{j}}\Big)\;=\;\sum_{\mathbf{j}}c_{\mathbf{j}}\Xi^{\mathbf{j}}\,,$$

where $\mathbf{i}$ runs over q-tuples, and $\mathbf{j}$ runs over countably many tuples with length strictly larger than q. Hence all $x_{\mathbf{i}}$ (and $c_{\mathbf{j}}$) vanish. □

So for $\tilde\Gamma$ with cusps the trivial $\tilde\Gamma$-module $I^{q+1}\backslash I^q$ is always non-trivial. The dimension is equal to the number of all $\tilde\Gamma$-q-tuples. Thus we have

$$\dim_{\mathbb{C}}(I^{q+1}\backslash I^q) \;=\; n(\tilde\Gamma,q) \;=\; \sum_{m=0}^{q}(t(\Gamma)-1)^m \;=\; \begin{cases} 1 & \text{if } t(\Gamma)=1\,,\\ q+1 & \text{if } t(\Gamma)=2\,,\\ \frac{(t(\Gamma)-1)^{q+1}-1}{t(\Gamma)-2} & \text{if } t(\Gamma)\geq 3\,.\end{cases} \tag{5.12}$$

We obtain for each $\tilde\Gamma$-module V an exact sequence

$$0 \longrightarrow V^{\tilde\Gamma,q} \longrightarrow V^{\tilde\Gamma,q+1} \xrightarrow{\mathsf{m}_q} (V^{\tilde\Gamma})^{n(\tilde\Gamma,q)}$$

with

$$(\mathsf{m}_q f)_{\mathbf{i}} \;=\; f|(\alpha_{\mathbf{i}(1)}-1)\cdots(\alpha_{\mathbf{i}(q)}-1)\,. \tag{5.13}$$

For the modular group, we have $n_{\mathrm{par}}=1$, $n_{\mathrm{ell}}=2$ and $g=0$, hence $t(\Gamma_{\mathrm{mod}})=1$, and $n(\tilde\Gamma_{\mathrm{mod}},q)=1$ for all q. So in contrast to Γ_{mod}, for $\tilde\Gamma_{\mathrm{mod}}$ we may hope for non-trivial higher order automorphic forms.

6. Maass forms with generalised weight on the universal covering group

6.1. **The logarithm of the Dedekind eta function.** In the introduction we mentioned that one of the motivating objects for the study of higher order forms on the universal covering group is the logarithm of the Dedekind eta function. Its branch is fixed by the second of the following expressions:

$$\log\eta(z) \;=\; \frac{\pi i z}{12}+\sum_{n=1}^{\infty}\log(1-e^{2\pi i n z}) \;=\; \frac{\pi i z}{12}-\sum_{n=1}^{\infty}\sigma_{-1}(n)\,e^{2\pi i n z}\,. \tag{6.1}$$

where $\sigma_u(n)=\sum_{d\mid n} d^u$. One can show that its behaviour under Γ_{mod} is given by

$$\log\eta(z+1) \;=\; \log\eta(z)+\frac{\pi i}{12}\,,\quad \log\eta(-1/z) \;=\; \log\eta(z)+\frac{1}{2}\log z-\frac{\pi i}{4}\,. \tag{6.2}$$

Except for the term $\frac{1}{2}\log z$ this looks like a second order holomorphic modular form of weight zero. In the next few sections we make this precise by generalizing the concept "weight" of Maass forms, and replacing the group Γ_{mod} by the discrete subgroup $\tilde\Gamma_{\mathrm{mod}}$ of the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, using the notation we introduced in the last section.

We first define the following function on $\mathfrak{H}\times\mathbb{R}$:

$$L(z,\vartheta) \;=\; \frac{1}{2}\log y+2\log\eta(z)+i\vartheta\,. \tag{6.3}$$

With (6.2) we check easily that $L(\gamma(z,\vartheta)) \;=\; L(z,\vartheta)+i\alpha(\gamma)$ for $\gamma=t$ and $\gamma=s$, where $\alpha:\tilde\Gamma_{\mathrm{mod}}\to\frac{\pi}{6}\mathbb{Z}$ is the group homomorphism at the end of §5.3. Thus L has the transformation behaviour of a second order invariant in the functions on $\tilde G$ for the action by left translation.

Routine computations show that L satisfies $\mathbf{E}^-L=0$, $\mathbf{W}L=i$ and $\omega L=\frac{1}{2}$.

6.2. **General Maass forms on the universal covering group.** The considerations on the function L on $\tilde{G}$ induced by the logarithm of the eta functions lead us to the definition of Maass forms on $\tilde{G}$.

We first establish appropriate notions of weight and holomorphicity. We say that a function f on $\tilde{G}$ has *(strict) weight* $r \in \mathbb{C}$ if $f(z,\vartheta) = e^{ir\vartheta} f(z,0)$. Such a function is completely determined by the function $f_r(z) = f(z,0)$ on $\mathfrak{H}$ and satisfies $\mathbf{W} f = ir f$.

The left translation of f by $\tilde{g}$, with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, induces an action $|$ of $\tilde{G}$ on the space of functions of strict weight on $\tilde{G}$. On the other hand, $\tilde{G}$ acts on the space of corresponding functions f_r on $\mathfrak{H}$ via

$$f_r|\tilde{g}(z) \;=\; e^{-ir\arg(cz+d)}\, f_r\Big(\frac{az+b}{cz+d}\Big),$$

The latter action corresponds to (4.7) when $r \in \mathbb{Z}$. In general, this is an action of $\tilde{G}$, not of $\mathrm{SL}_2(\mathbb{R})$. The map $f \to f_r$ defined above on the space of functions of strict weight is then equivariant in terms of these actions.

Many important functions on $\tilde{G}$, such as L, are not eigenfunctions of the operator $\mathbf{W}$, but they are annihilated by a power of $\mathbf{W}$. This suggests the following definition.

Definition 6.1. An $f \in C^\infty(\tilde{G})$ has *generalised weight* $r \in \mathbb{C}$ if $(\mathbf{W} - ir)^n f = 0$ for some $n \in \mathbb{N}$.

Thus, L and all its powers have generalised weight 0.

Next, holomorphy of $F_r = y^{-r/2} f_r$ corresponds to the property $\mathbf{E}^- f = 0$.

Definition 6.2. We call any differentiable function f on $\tilde{G}$ *holomorphic* (resp. *antiholomorphic*) if $\mathbf{E}^- f = 0$, (resp. $\mathbf{E}^+ f = 0$). We call any twice differentiable function f on $\tilde{G}$ *harmonic* if it satisfies $\omega f = 0$.

Note that, for functions of non-zero weight, this definition of harmonicity does not correspond to the use of the word harmonic in "harmonic weak Maass forms" in, *e.g.*, [1].

With these definitions we set

Definition 6.3. Let $k, \lambda \in \mathbb{C}$. Let $\tilde{\Gamma}$ be a discrete cofinite subgroup of $\tilde{G}$.
i. The space $\tilde{\mathcal{E}}_k(\tilde{\Gamma}, \lambda)$ consists of the smooth functions $f : \mathfrak{H} \times \mathbb{R} \to \mathbb{C}$ that satisfy:

a) *(Eigenfunction Casimir operator)* $\omega f = \lambda f$.
b) *(Generalised weight)* $(\mathbf{W} - ik)^n f = 0$ for some $n \in \mathbb{N}$.
c) *(Exponential growth)* There exists $a \in \mathbb{R}$ such that for all compact sets X and $\Theta \subset \mathbb{R}$ and for all cusps κ of $\tilde{\Gamma}$ we have

$$f(\tilde{g}_\kappa(x+iy,\vartheta)) = \mathrm{O}(e^{ay}) \tag{6.4}$$

as $y \to \infty$ uniformly in $x \in X$ and $\vartheta \in \Theta$.

ii.

$$\tilde{E}_k(\tilde{\Gamma}, \lambda) := \tilde{\mathcal{E}}_k(\tilde{\Gamma}, \lambda)^{\tilde{\Gamma}}$$

(where $\tilde{\Gamma}$ acts by left translation). The elements of $\tilde{E}_k(\tilde{\Gamma}, \lambda)$ are called *Maass forms on* $\tilde{G}$ *of generalised weight* k *and eigenvalue* λ *for* $\tilde{\Gamma}$.

The space $\tilde{E}_r(\tilde{\Gamma}, \lambda)$ is infinite dimensional. Further, since ω and $\mathbf{W}$ commute with left translations in $\tilde{G}$, the space $\tilde{\mathcal{E}}_k(\tilde{\Gamma}, \lambda)$ is invariant under left translation by elements of $\tilde{\Gamma}$.

When $k \in 2\mathbb{Z}$, the space $E_k(\Gamma, \lambda)$ can be identified with $\tilde{E}_k(\tilde{\Gamma}, \lambda)$. We prove the following slightly stronger statement.

Theorem 6.4. *Let $\tilde\Gamma$ be a cofinite discrete subgroup of $\tilde G$, and let $k, \lambda \in \mathbb{C}$. If $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde Z}$ contains a non-zero element f, then $k \in 2\mathbb{Z}$ and $\partial_\vartheta f(z,\vartheta) = ik f(z,\vartheta)$.*

If $k \in 2\mathbb{Z}$, then the elements $f \in \tilde E_k(\tilde\Gamma,\lambda)$ correspond bijectively to the Maass forms $F \in E_k(\Gamma,\lambda)$ by

$$f(z,\vartheta) = y^{k/2} F(z)\, e^{ik\vartheta}.$$

So the condition of $\tilde Z$-invariance implies that the weight k is even, and that the weight is *strict*, i.e., condition b) holds with $n = 1$.

Proof of Theorem 6.4. Any smooth function $f \in C^\infty(\mathfrak{H}\times\mathbb{R})$ satisfying b) in Definition 6.3 can be written in the form $f(z,\vartheta) = \sum_{j=0}^{n-1} \varphi_j(z)\, e^{ik\vartheta}\, \vartheta^j$, with $\varphi_j \in C^\infty(\mathfrak{H})$.

If such a function is left-invariant under $\tilde Z$, then the action of $k(\pi m) \in \tilde Z \subset \tilde\Gamma$, implies for each $m \in \mathbb{Z}$:

$$e^{\pi i k m} \sum_j \varphi_j(z) e^{ik\vartheta} (\vartheta + \pi m)^j = \sum_j \varphi_j(z) e^{ik\vartheta}\vartheta^j \qquad \text{for all } m \in \mathbb{Z}\,.$$

With induction this gives $k \in 2\mathbb{Z}$ and $\varphi_j = 0$ for $j \geq 1$, hence $f(z,\vartheta) = \varphi_0(z) e^{ik\vartheta}$. Moreover, the stronger condition $f \in \tilde E_k(\tilde\Gamma,\lambda) = \tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde\Gamma}$ can be checked to be equivalent to $F_k \in E_k(\Gamma,\lambda)$ for $F_k(z) = y^{-k/2} f(z,0)$. □

We have the following generalisation of Theorem 4.2.

Theorem 6.5. *Let $\tilde\Gamma$ be a cofinite discrete subgroup of $\tilde G$ with cusps. Then the $\tilde\Gamma$-module $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)$ is maximally perturbable for each $k \in 2\mathbb{Z}$ and each $\lambda \in \mathbb{C}$.*

In Section 8 we will prove this theorem. In this section we will show that it implies the corresponding result for $E_k(\Gamma,\lambda)$. We first give some facts that are of more general interest.

The map identifying $E_k(\Gamma,\lambda)$ and $\tilde E_k(\tilde\Gamma,\lambda)$ can be extended to an isomorphism

$$\mu : \mathcal{E}_k(\Gamma,\lambda) \longrightarrow \tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde Z}.$$

Since the centre $\tilde Z$ of $\tilde\Gamma$ acts trivially on $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde Z}$, it can be considered as a Γ-module. With this interpretation we obtain an identification of the Γ-modules $\mathcal{E}_k(\Gamma,\lambda)$ and $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde Z}$. Specifically, for $F \in \mathcal{E}_k(\Gamma,\lambda)$, $g \in \tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde Z}$ we have

$$\begin{aligned}
(\mu f)(z,\vartheta) &= y^{k/2} F(z)\, e^{ik\vartheta}\,,\\
(\mu^{-1} g)(z) &= y^{-k/2}\, g(z,0)\,,\\
\mu(F|_k\gamma) &= \mu(F)|\nu(\gamma) \qquad (\gamma \in \Gamma)\,,\\
\mu^{-1}(g|\tilde Z\delta) &= \mu^{-1}(g)|_k \nu^{-1}(\tilde Z\delta) \quad (\delta \in \tilde\Gamma)\,,
\end{aligned} \tag{6.5}$$

where ν denotes the isomorphism identifying Γ with $\tilde Z\backslash\tilde\Gamma$.

Proposition 6.6. *Let Γ be a cofinite discrete subgroup of G with cusps, and let $\tilde\Gamma = \mathrm{pr}^{-1}\Gamma$. If the $\tilde\Gamma$-module V is maximally perturbable, then the subspace $V^{\tilde Z}$, considered as a Γ-module, is maximally perturbable.*

Proof. The projection $\mathrm{pr} : \tilde\Gamma \to \Gamma$ induces linear maps $\mathrm{pr} : \mathbb{C}[\tilde\Gamma] \to \mathbb{C}[\Gamma]$ between the group rings, $\mathrm{pr} : I_{\tilde\Gamma} \to I_\Gamma$ between the augmentation ideals, and $\mathrm{pr} : I_{\tilde\Gamma}^{q+1}\backslash I_{\tilde\Gamma}^q \to I_\Gamma^{q+1}\backslash I_\Gamma^q$ for all $q \in \mathbb{N}$. Since, $\mathrm{pr}(A_{\mathbf{i}}) = \alpha_{\mathbf{i}}$, on the basis elements $\mathbf{b}_{\tilde\Gamma}(\mathbf{i})$ in Proposition 5.3 and $\mathbf{b}_\Gamma(\mathbf{i})$ in (3.8) we have for $\tilde\Gamma$-q-tuples:

$$\mathrm{pr}\,\mathbf{b}_{\tilde\Gamma}(\mathbf{i}) = \begin{cases} \mathbf{b}_\Gamma(\mathbf{i}) & \text{if } \mathbf{i}(l) < t(\Gamma) \text{ for } l = 1,\dots,q\,,\\ 0 & \text{if } \mathbf{i}(q) = t(\Gamma)\,. \end{cases} \tag{6.6}$$

This means that we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^{\tilde\Gamma,q} & \longrightarrow & V^{\tilde\Gamma,q+1} & \xrightarrow{\mathsf{m}_q} & \hom(I_{\tilde\Gamma}^{q+1}\backslash I_{\tilde\Gamma}^{q}, V^{\tilde\Gamma}) & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ 0 & \longrightarrow & (V^{\tilde Z})^{\Gamma,q} & \longrightarrow & (V^{\tilde Z})^{\Gamma,q+1} & \xrightarrow{\mathsf{m}_q} & \hom(I_{\Gamma}^{q+1}\backslash I_{\Gamma}^{q}, (V^{\tilde Z})^{\Gamma}) & & \end{array}$$

where the vertical arrow sends $f : I_\Gamma^{q+1}\backslash I_\Gamma^q \to (V^{\tilde Z})^\Gamma = V^{\tilde\Gamma}$ to $\tilde f : I_{\tilde\Gamma}^{q+1}\backslash I_{\tilde\Gamma}^q \to V^{\tilde\Gamma}$ such that $\tilde f(\mathbf{b}_{\tilde\Gamma}(\mathbf{i})) = f(\mathbf{b}_\Gamma(\mathbf{i}))$ if $\mathbf{i} \in \{1,\dots,t(\Gamma)-1\}^q$, and $\tilde f(\mathbf{b}_{\tilde\Gamma}(\mathbf{i})) = 0$ otherwise.

We want to write a given $f : I_\Gamma^{q+1}\backslash I_\Gamma^q \to (V^{\tilde Z})^\Gamma$ as $\mathsf{m}_q v_0$ with $v_0 \in (V^{\tilde Z})^{\Gamma,q+1}$. By assumption, there is an element $v \in V^{\tilde\Gamma,q+1}$ such that $\mathsf{m}_q v = \tilde f$. If $v|(\zeta-1) = 0$, then $v \in V^{\tilde\Gamma,q+1} \cap V^{\tilde Z} = (V^{\tilde Z})^{\Gamma,q+1}$, and we are done.

Suppose that $w = v|(\zeta-1) \neq 0$. Take $r \in [1,q]$ minimal such that $w \in V^{\tilde\Gamma,r}$. We will show that we can replace v by another element $v_1 \in v + V^{\tilde\Gamma,q}$ with $v_1|(\zeta-1) \in V^{\tilde\Gamma,r_1}$ and $r_1 < r$. Repeating this process brings us eventually to $v_j|(\zeta-1) = 0$. For this v_j we will have $\mathsf{m}_q v_j = \tilde f$ and $v_j|(\zeta-1) = 0$ which, according to the remark of the last paragraph suffices to prove the proposition.

From $w|(\gamma_1-1)\cdots(\gamma_{q-1}-1) = v|(\gamma_1-1)\cdots(\gamma_{q-1}-1)(\zeta-1) = \tilde f(\gamma_1,\cdots,\gamma_{q-1},\zeta) = 0$ we conclude that $r \le q-1$. Define $\tilde g \in \hom(I_{\tilde\Gamma}^{r+1}\backslash I_{\tilde\Gamma}^r, V^{\tilde\Gamma})$ by $\tilde g(\mathbf{b}_{\tilde\Gamma}(\mathbf{j})) = w|(\alpha_{\mathbf{j}(1)}-1)\cdots(\alpha_{\mathbf{j}(r-1)}-1)$ if the $\tilde\Gamma$-r-tuple $\mathbf{j}$ satisfies $\mathbf{j}(r) = t(\Gamma)$ and $\tilde g(\mathbf{b}_{\tilde\Gamma}(\mathbf{j})) = 0$ otherwise. There is $u \in V^{\tilde\Gamma,r+1} \subset V^{\tilde\Gamma,q}$ with $\mathsf{m}_r u = \tilde g$. We take $v_1 = v - u \in v + V^{\tilde\Gamma,q}$. We check that for all $\tilde\Gamma$-$(r-1)$-tuples $\mathbf{j}$

$$\begin{aligned} & v_1|(\zeta-1)(\alpha_{\mathbf{j}(1)}-1)\cdots(\alpha_{\mathbf{j}(r-1)}-1) \\ &\quad = w|(\alpha_{\mathbf{j}(1)}-1)\cdots(\alpha_{\mathbf{j}(r-1)}-1) - u|(\alpha_{\mathbf{j}(1)}-1)\cdots(\alpha_{\mathbf{j}(r-1)}-1)(\zeta-1) \\ &\quad = 0\,. \end{aligned}$$

This shows that $v_1|(\zeta-1)$ has order less than r. □

Proof of Theorem 4.2. From Theorem 6.5, $V = \tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda_k)$ is maximally perturbable. Therefore, by Proposition 6.6, the space $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda_k)^{\tilde Z} \cong \mathcal{E}_k(\Gamma,\lambda_k)$ is maximally perturbable too. □

This proof illustrates the fact that, for groups with cusps, there are really more higher order forms with generalised weight than with strict weight: The basis in Proposition 5.3 is for all such discrete groups larger than the corresponding basis in §3.2.1.

6.3. **Holomorphic forms on the universal covering group.**

Definition 6.7. For $k \in 2\mathbb{Z}$ we define $\mathcal{H}_k(\tilde\Gamma)$ as the space of elements of $C^\infty(\mathfrak{H}\times\mathbb{R})$ that satisfy

(1) *(Holomorphy)* $\mathbf{E}^- f = 0$.
(2) *(Generalised weight)* $(\mathbf{W} - ik)^n f = 0$ for some $n \in \mathbb{N}$.
(3) *(Exponential growth)* as described in condition c) in Definition 6.3.

This is a $\tilde\Gamma$-module for the action by left translation. We denote by $\mathcal{H}_k^p(\tilde\Gamma)$ (resp. $\mathcal{H}_k^c(\tilde\Gamma)$) the space of $f \in \mathcal{H}_k(\tilde\Gamma)$ satisfying $f(\tilde g_\kappa(x+iy,\vartheta)) = \mathrm{O}(y^C)$ for some $C \in \mathbb{R}$ (resp. $f(\tilde g_\kappa(x+iy,\vartheta)) = \mathrm{O}(e^{ay})$ for some $a < 0$) instead of (6.4).

We will prove:

Theorem 6.8. *Let $\tilde\Gamma$ be a cofinite discrete subgroup of $\tilde G$ with cusps. Then the $\tilde\Gamma$-module $\mathcal{H}_k(\tilde\Gamma)$ is maximally perturbable for each $k \in 2\mathbb{Z}$.*

Proof of Theorem 4.3. As in the case of general Maass forms, we can show that, for $k \in 2\mathbb{Z}$, $\mathcal{E}_k^{\mathrm{hol}}(\Gamma, \lambda_k) \cong \mathcal{H}_k(\tilde{\Gamma})^{\tilde{Z}}$. Then, Proposition 6.6 implies Theorem 4.3. □

Second order forms and derivatives of L-functions. With this definition, L is a second order invariant belonging to $\mathcal{H}_0(\tilde{\Gamma}_{\mathrm{mod}})^{\tilde{\Gamma}_{\mathrm{mod}},2}$. (Incidentally, this example shows that, for generalised weight k, the space $\mathcal{H}_k(\tilde{\Gamma})$ need not be contained in $\tilde{\mathcal{E}}_k(\tilde{\Gamma}, \lambda_k)$.)

Based on L we can construct a second-order form which is related to derivatives of classical modular forms. Specifically, for positive integer N, denote by G_N the group generated by $\tilde{g}$, $g \in < \Gamma_0(N), W_N >$ where $W_N := \begin{pmatrix} 0 & -\sqrt{N}^{-1} \\ \sqrt{N} & 0 \end{pmatrix}$. Set

$$L_1(z, \vartheta) = L(z, \vartheta) + L(Nz, \vartheta).$$

Using the transformation law for L and the identity $\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & Nb \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)$, a routine calculation implies that, for some $\beta \in \mathrm{Hom}(G_N, \mathbb{C})$,

$$L_1(\gamma(z, \vartheta)) = L_1(z, \vartheta) + i\beta(\gamma), \qquad \text{for all } \gamma \in G_N.$$

Let now f be a newform in the space S_2 of cusp forms of weight 2 for $\Gamma_0(N)$ such that its L-function $L_f(s)$ vanishes at 1. Then, $f(W_N w)d(W_N w) = f(w)dw$ and, for all $\vartheta \in \mathbb{R}$,

$$\begin{aligned} \int_0^\infty f(iy)L_1(iy, \vartheta)diy = -\int_{W_N 0}^{W_N \infty} f(iy)L_1(iy, \vartheta)diy &= -\int_0^\infty f(W_N iy)L_1(W_N iy, \vartheta)d(W_N iy) \\ &= -\int_0^\infty f(iy)L_1(W_N iy, \vartheta)diy. \end{aligned} \tag{6.7}$$

Since $L_1(z, \vartheta + x) = L_1(z, \vartheta) + 2ix$ and $L_f(1) = 2\pi \int_0^\infty f(iy)dy = 0$, our integral is independent of ϑ. It further equals

$$-\int_0^\infty f(iy)L_1(\tilde{W_N}(iy, 0))diy = -\int_0^\infty f(iy)(L_1(iy, 0) + i\beta(\tilde{W_N}))diy = -\int_0^\infty f(iy)L_1(iy, 0)diy \tag{6.8}$$

Therefore, $\int_0^\infty f(iy)L_1(iy, 0)dy = -\int_0^\infty f(iy)L_1(iy, 0)dy$, i.e.

$$\int_0^\infty f(iy)L_1(iy, 0)dy = 0 \quad \text{and hence}$$

$$\int_0^\infty f(iy)\log y\, dy + 2\int_0^\infty f(iy)u(iy)\, dy = 0$$

where $u(z) := \log(\eta(z)) + \log(\eta(Nz))$. From this we see that, since, $L_f'(s) = 2\pi \int_0^\infty f(iy)\log(y)dy$, we can retrieve, from a alternative perspective, the formula

$$L_f'(1) = -4\pi \int_0^\infty f(iy)u(iy)dy$$

first derived in [11].

Thus, Goldfeld's expression of $L_f'(1)$ is equivalent to the orthogonality of $L_1 \in \mathcal{H}_0^p(G_N)^{G_N,2}$ to $S_2 \hookrightarrow \mathcal{H}_2^c(G_N)^{G_N}$ in terms of the pairing

$$\langle \cdot, \cdot \rangle : \mathcal{H}_2^c(G_N)^{G_N} \times \mathcal{H}_0^p(G_N)^{G_N,2} \to \mathbb{C}$$

defined by

$$\langle g, h \rangle = \int_0^\infty g(iy, 0)h(iy, 0)\frac{dy}{y}.$$

6.4. **Examples of higher order forms for the full modular group.** Theorems 6.5 and 6.8 show that there are perturbations of 1 for the full original $\tilde\Gamma_{\mathrm{mod}}$ of $\mathrm{SL}_2(\mathbb{Z})$ in the universal covering group. Since $t(\Gamma_{\mathrm{mod}}) = 1$ all these perturbations are commutative (see (5.12)).

1. The function L can lead to second order *harmonic perturbations of* 1. Specifically, although $L \notin \tilde{\mathcal{E}}_0(0)^{\tilde\Gamma,2}$ (because $\omega L = \frac{1}{2}$), the imaginary part $\operatorname{Im} L : (z,\vartheta) \mapsto 2\operatorname{Im}\log\eta(z) + \vartheta$ is harmonic, has second order, and corresponds to the linear form $\alpha \in \mathrm{Mult}^1(\tilde\Gamma_{\mathrm{mod}}, \mathbb{C})$. It has generalised weight 0, and it is not holomorphic.

2. Set $\chi_r = e^{ir\alpha}$, $r \in \mathbb{C}$, where $\alpha \in \hom(\tilde\Gamma_{\mathrm{mod}}, \mathbb{C})$ is given by $\alpha(n(1)) = \frac{\pi}{6}$ and $\alpha(k(\pi/2)) = \frac{\pi}{2}$. The family

$$r \mapsto e^{rL(z,\vartheta)} \;=\; y^{r/2}\,\eta(z)^{2r}\,e^{ir\vartheta} \tag{6.9}$$

consists of elements of $\mathcal{H}_r(\tilde\Gamma)$ that are $\tilde\Gamma_{\mathrm{mod}}$-invariant under the action given by

$$(f|\gamma)(z) = f(\gamma z)\overline{\chi_r(\gamma)}.$$

By Proposition 4.4, for $k \geq 1$ the derivative

$$\partial_r^k e^{rL(z,\vartheta)}\big|_{r=0} = L(z,\vartheta)^k$$

is a holomorphic perturbation of 1 of order $k+1$. The corresponding element of $\mathrm{Mult}^k(\tilde\Gamma_{\mathrm{mod}}, \mathbb{C})$ is $i^k\,k!\,\alpha^{\otimes k}$.

3. It is possible to obtain a more or less explicit description of a harmonic perturbation of 1 of order 3. We sketch how this can be done with the meromorphic continuation of the Eisenstein in weight and spectral parameter jointly. This family is studied in [2]. In that work, automorphic forms are described as functions on $\mathfrak{H}$ transforming according to a multiplier system of Γ_{mod}. These correspond to functions on $\tilde G$ that transform according to a character of $\tilde\Gamma_{\mathrm{mod}}$. Carrying out the reformulation, we can rephrase §2.18 in [2] as stating that there is a meromorphic family of Maass forms on $U \times \mathbb{C}$, where U is some neighborhood of $(-12, 12)$ in $\mathbb{C}$. We retrieve the exact family studied in [2] by considering $z \mapsto E(r, s; z, 0)$. For each $(r, s) \in U \times \mathbb{C}$ at which E is not singular it is an automorphic form of weight r for the character $\chi_r = e^{ir\alpha}$ of $\tilde\Gamma_{\mathrm{mod}}$ with eigenvalue $\lambda_s = \frac{1}{4} - s^2$. It is a meromorphic family of automorphic forms on $\tilde\Gamma_{\mathrm{mod}}$ with character χ_r with a Fourier expansion of the form

$$E(r,s) \;=\; \mu_r(r/12, s) + C_0(r,s)\,\mu_r(r/12, -s) + \sum_{n\neq 0} C_n(r,s)\,\omega_r(n + r/12, s)\,, \tag{6.10}$$

where the $C_n(r, s)$ are meromorphic functions, and where we use the following notations.

$$\begin{aligned} \omega_r(\nu, s; z, \vartheta) &= e^{2\pi i\nu x}\,W_{r\,\mathrm{Sign}(\mathrm{Re}\,\nu)/2,s}(4\pi\nu\,\mathrm{Sign}(\mathrm{Re}\,\nu)y)\,e^{ir\vartheta}\,, \\ \mu_r(\nu, s; z, \vartheta) &= e^{2\pi i\nu z}\,y^{\frac{1}{2}+s}\,{}_1F_1(\tfrac{1}{2} + s - \tfrac{r}{2}; 1 + 2s; 4\pi\nu y)\,e^{ir\vartheta}\,. \end{aligned} \tag{6.11}$$

This family and its Fourier coefficient C_0 satisfy the following functional equations.

$$\begin{aligned} E(r,-s) &= C_0(r,-s)E(r,s)\,, \\ E(r,s;-x+iy,-\vartheta) &= E(-r,s;x+iy,\vartheta)\,. \end{aligned} \tag{6.12}$$

Further, the restriction of this family to the (complex) line $r = 0$ exists, and gives a meromorphic family of automorphic forms depending on one parameter s. This is a family of weight zero, so it does not depend on the parameter ϑ on $\tilde G$. The resulting family on $\mathfrak{H}$ is the meromorphic continuation of the Eisenstein

series for Γ_{mod} in weight 0, with Fourier expansion

$$\begin{aligned} E(0,s) &= \mu_0(0,s) + \frac{\sqrt{\pi}\,\Gamma(s)\,\zeta(2s)}{\Gamma(s+\frac{1}{2})\,\zeta(2s+1)}\,\mu_0(0,-s) \\ &\quad + \frac{\pi^{s+\frac{1}{2}}}{\Gamma(s+\frac{1}{2})\,\zeta(2s+1)} \sum_{n\neq 0} \frac{\sigma_{2s}(|n|)}{|n|^{s+\frac{1}{2}}}\,\omega_0(n,s)\,. \end{aligned} \tag{6.13}$$

where

$$\begin{aligned} \mu_0(0,s;z,\vartheta) &= y^{\frac{1}{2}+s}\,, \\ \omega_0(n,s;z,\vartheta) &= e^{2\pi i n x}\,W_{0,s}(4\pi|n|y) = e^{2\pi i n x}\,2|n|^{1/2}\,K_s(2\pi|n|y)\,. \end{aligned}$$

At $(0,-\frac{1}{2})$ the family E is holomorphic in both variables r and s, with a constant as its value at $(0,-\frac{1}{2})$. (This is a consequence of Proposition 6.5 ii) in [2].) So in principle, we obtain higher order harmonic perturbations of 1 by differentiating $r \mapsto E(r,-\frac{1}{2})$. Here we encounter the problem that we have an explicit Fourier expansion (6.13) only for $E(0,s)$ and thus we cannot describe the derivatives in the direction of r directly. To overcome this problem we use the fact that for r near 0 we have

$$\begin{aligned} E(r,-\frac{1-r}{2};z,\vartheta) &= H_r(z,\vartheta) = e^{rL(z,\vartheta)}\,, \\ E(r,-\frac{1+r}{2};z,\vartheta) &= H_{-r}(-\bar{z},-\vartheta) = e^{-r\overline{L(z,\vartheta)}}\,. \end{aligned} \tag{6.14}$$

The proof of the first equality is contained in 6.10 in [2]. The second one follows from the second functional equation in (6.12). Now we use the Taylor expansion of E of degree 2 at $(r,s)=(0,-\frac{1}{2})$:

$$\begin{aligned} E(r,s) &= 1 + r\,A_{1,0} + (s+\frac{1}{2})\,A_{0,1} \\ &\quad + \frac{1}{2}r^2\,A_{2,0} + r\,(s+\frac{1}{2})\,A_{1,1} + \frac{1}{2}(s+\frac{1}{2})^2\,A_{0,2} + \cdots \end{aligned} \tag{6.15}$$

By Proposition 4.4, the coefficients $A_{1,0}$ and $A_{2,0}$ are harmonic perturbations of 1 of order 2 and 3, respectively. From (6.14) we obtain the following results:

$$\begin{aligned} A_{1,0} &= i\,\mathrm{Im}\,L\,, \qquad A_{0,1} = 2\mathrm{Re}\,L\,, \\ A_{2,0} + \frac{1}{4}A_{0,2} &= \mathrm{Re}\,L^2\,, \qquad A_{1,1} = i\,\mathrm{Im}\,L^2\,. \end{aligned} \tag{6.16}$$

This confirms that $\mathrm{Im}\,L$ is a second order harmonic perturbation of 1. Differentiation in the direction of s preserves $\tilde{\Gamma}_{\mathrm{mod}}$-invariance. So $A_{0,1} = 2\mathrm{Re}\,L$ and $A_{0,2}$ are $\tilde{\Gamma}_{\mathrm{mod}}$-invariant. However these functions are not in the kernel of ω.

Thanks to the identity $A_{2,0} + \frac{1}{4}A_{0,2} = \mathrm{Re}\,L^2$, to determine the third order harmonic perturbation $A_{2,0}$ it suffices to explicitly compute $A_{0,2}$ because $\mathrm{Re}\,L^2$ is known in a fairly explicit way. The function $A_{0,2}$ can be obtained as the coefficient of $\frac{1}{2}(s+\frac{1}{2})^2$ in the Taylor expansion of $E(0,s)$ at $s=-\frac{1}{2}$. As a by-product of this computation we will also obtain the $\tilde{\Gamma}_{\mathrm{mod}}$-invariant function $A_{0,1}$ as the coefficient of $s+\frac{1}{2}$ in the same expansion. We shall examine each term of the expansion separately.

Set $\xi := s+\frac{1}{2}$. The first term of our expansion is

$$\mu_0(0,s;z,0) = y^{\frac{1}{2}+s} = 1 + \xi\log y + \xi^2\frac{1}{2}(\log y)^2 + \cdots \tag{6.17}$$

For the next term $\frac{\Lambda(2s)}{\Lambda(2s+1)}\,\mu(0,-s;z,0) = \frac{\Lambda(2-2\xi)}{\Lambda(1-2\xi)}\,\mu(0,-s;z,0)$ with $\Lambda(u) = \pi^{-u/2}\,\Gamma(\frac{u}{2})\,\zeta(u) = \Lambda(1-u)$, we define a_0 and b_1 by

$$\Lambda(1+h) \;=\; h^{-1} + a_0 + \cdots\,, \qquad \Lambda(2+h) \;=\; \frac{\pi}{6} + b_1 h + \cdots\,. \tag{6.18}$$

We get

$$\frac{\Lambda(2s)}{\Lambda(2s+1)}\mu_0(0,-s;z,0) \;=\; -\frac{\pi}{3}\,y\,\xi + (4b_1 - \frac{2\pi a_0}{3} + \frac{\pi}{3}\,\log y)\,y\,\xi^2 + \cdots\,. \tag{6.19}$$

For the other terms we use

$$\begin{aligned}
W_{0,-s}(t) \;&=\; W_{0,s} \;=\; \frac{e^{-\frac{t}{2}}}{\Gamma(\frac{1}{2}+s)}\int_0^\infty e^{-x}(x(1+\frac{x}{t}))^{s-\frac{1}{2}}\,dx\,,\\
W_{0,1/2}(t) \;&=\; \frac{e^{-\frac{t}{2}}}{1}\cdot 1 \;=\; e^{-\frac{t}{2}}\,,\\
-\partial_s W_{0,s}(t)\big|_{s=-\frac{1}{2}} \;&=\; \partial_s W_{0,s}(t)\big|_{s=\frac{1}{2}} \;=\; -\frac{e^{-\frac{t}{2}}}{1^2}\Gamma'(1)\cdot 1 + e^{-\frac{t}{2}}\int_0^\infty e^{-x}\,\log(x(1+\frac{x}{t}))\,dx\\
&=\; e^{-\frac{t}{2}}\Big(-\Gamma'(1) + \Gamma'(1) + \int_0^\infty e^{-x}\,\log(1+\frac{x}{t})\,dx\Big)\\
\text{(part. int.)} \qquad &=\; e^{-\frac{t}{2}}\int_0^\infty e^{-x}\,\frac{dx}{x+t} \;=\; e^{\frac{t}{2}}\int_t^\infty e^{-x}\,\frac{dx}{x} = e^{\frac{t}{2}}\,\Gamma(0,t)\,,
\end{aligned}$$

with the incomplete gamma-function $\Gamma(a,t) = \int_t^\infty e^{-x}\,x^{a-1}\,dx$. With these ingredients:

$$\begin{aligned}
\frac{\sigma_{2s}(|n|)}{\Lambda(2s+1)\,|n|^{s+\frac{1}{2}}}\omega_0(n,s;z,0) \;&=\; \sum_{d\Big||n|}\frac{1}{d}\Big(-2e^{-2\pi|n|y}\,\xi\\
&+ (2e^{2\pi|n|y}\Gamma(0,4\pi|n|y) - 2e^{-2\pi|n|y}\log\frac{d^2}{|n|} - 4a_0\,e^{-2\pi|n|y})\,\xi^2 + \ldots\Big)\,e^{2\pi i n x}\,.
\end{aligned} \tag{6.20}$$

The results in (6.17), (6.19) and (6.20) confirm that the constant term equals 1, and that

$$A_{0,1}(z,0) = \log y - \frac{\pi}{3}y - 2\sum_{n\geq 1}\sum_{d|n}\frac{1}{d}(q^n+\overline{q}^n) = 2\mathrm{Re}\,(\frac{1}{2}\log y + \frac{\pi i}{6}z - \sum_{n=1}^\infty \sigma_{-1}(n)\,q^n) \;=\; 2\mathrm{Re}\,L(z,0)\,,$$

with the notation $q = e^{2\pi i z}$. The term of order 2 leads to:

$$\begin{aligned}
A_{0,2}(z,0) \;&=\; (\log y)^2 + (8b_1 - \frac{4\pi\,a_0}{3} + \frac{2\pi}{3}\log y)\,y\\
&+ \sum_{n=1}^\infty\Big(-4a_0\,\sigma_{-1}(n)(q^n+\overline{q}^n) + 2\sigma_{-1}(n)\,(q^{-n}+\overline{q}^{-n})\,\Gamma(0,4\pi n y) - 2(q^n+\overline{q}^n)\sum_{d|n}\frac{\log(d^2/n)}{d}\Big),
\end{aligned} \tag{6.21}$$

which is a complicated, but explicit expression.

A remarkable aspect of this computation that we have used an explicit computation of the derivatives of the Eisenstein series in weight zero to compute the second derivative in the r-direction of the more complicated Eisenstein family in two variables. The basic observation is (6.14), which shows that the Eisenstein family has easy derivatives in two directions. The Taylor expansion of E at $(0,-\frac{1}{2})$ has three monomials in order 2. So it suffices to compute a second order derivative in one more direction to get

hold of all terms. Higher order terms in the Taylor expansion have too many monomials for this method to work. We do not know how to compute all harmonic perturbations of 1 of higher order.

7. Higher order Fourier expansions

This section is needed for the constructions on which the proofs of Theorems 6.5 and 6.8 are based, but it is also of independent interest. It provides a higher-order analogue of the classical Fourier expansions.

7.1. **Fourier expansion of Maass forms.** If f is in $\tilde{E}_r(\tilde{\Gamma},\lambda)$, then for each cusp κ of Γ there is a Fourier expansion

$$f(\tilde{g}_\kappa g) \;=\; \sum_\nu F_{\kappa,\nu}\, f(g)\,, \qquad F_{\kappa,\nu} f(g) \;=\; \int_0^1 e^{-2\pi i\nu x}\, f(\tilde{g}_\kappa n(x) g)\, dx\,, \tag{7.1}$$

where ν runs through a class in $\mathbb{C} \bmod \mathbb{Z}$ determined by χ and the cusp κ. The function $F_\nu f$ satisfies $F_{\kappa,\nu}\, f(z,\vartheta) = e^{2\pi i\nu x} F_{\kappa,\nu}\, f(iy,0)\, e^{ir\vartheta}$ and $\omega F_{\kappa,\nu}\, f = \lambda\, F_{\kappa,\nu}\, f$.

For each given ν, r and s set

$$\mathcal{W}_r(\nu,s) \;:=\; \{f : \tilde{G} \to \mathbb{C}\;;\; \omega f = (\frac{1}{4} - s^2) f,\;\; f(z,\theta) = e^{2\pi i\nu x + ir\theta} f(iy,0)\}. \tag{7.2}$$

Because of the second relation in the definition, $f \in \mathcal{W}_r(\nu,s)$ can be thought of as a function of y. Therefore the space $\mathcal{W}_r(\nu,s)$ is isomorphic to the space of $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$-y^2 h''(y) + (4\pi^2\nu^2 y^2 - 2\pi\nu r\, y - \tfrac{1}{4} + s^2)\, h(y) \;=\; 0. \tag{7.3}$$

It is convenient to write $\lambda = \lambda_s = \frac{1}{4} - s^2$ with $s \in \mathbb{C}$. We can choose a fixed s with $\operatorname{Re} s \geq 0$ corresponding to the eigenvalue $\lambda = \lambda_s$ under consideration. The spaces $\mathcal{W}_r(\nu,s)$ are two-dimensional. We will use the basis elements in §4.2 of [3].

- For $\operatorname{Re}\nu \neq 0$ a basis of $\mathcal{W}_r(\nu,s)$ is formed by

$$\begin{aligned} \omega_r(\nu,s;z,\vartheta) \;&=\; e^{2\pi i\nu x}\, W_{r\,\mathrm{Sign}(\operatorname{Re}\nu)/2,s}(4\pi\nu\,\mathrm{Sign}(\operatorname{Re}\nu) y)\, e^{ir\vartheta}\,, \\ \hat{\omega}_r(\nu,s;z,\vartheta) \;&=\; e^{2\pi i\nu x}\, W_{-r\,\mathrm{Sign}(\operatorname{Re}\nu)/2,s}(-4\pi\nu\,\mathrm{Sign}(\operatorname{Re}\nu) y)\, e^{ir\vartheta}\,. \end{aligned} \tag{7.4}$$

Here $W_{\mu,s}(t)$ is the Whittaker function that decreases exponentially as $t \to \infty$. We use the branch of $W_{\kappa,s}(z)$ that is holomorphic for $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$. The asymptotic behaviour as $y \to \infty$, by §4.2.1 in [21] is:

$$\omega_r(\nu,s;z,\vartheta) \;\sim\; (4\pi\nu\varepsilon y)^{r\varepsilon/2}\, e^{2\pi\nu(ix-\varepsilon y)+ir\vartheta}, \tag{7.5}$$

$$\hat{\omega}_r(\nu,s;z,\vartheta) \;\sim\; e^{-\pi i r\varepsilon/2}\, (4\pi\varepsilon\nu y)^{-r\varepsilon/2}\, e^{2\pi\nu(ix+\varepsilon y)+ir\vartheta}\,, \tag{7.6}$$

where ε denotes $\mathrm{Sign}(\operatorname{Re}\nu)$. The subspace of $\mathcal{W}_r(\nu,s)$ generated by $\omega_r(\nu,s)$ is denoted by $\mathcal{W}_r^0(\nu,s)$.

- For $\nu = 0$, a basis is given by $\{y^{\frac{1}{2}+s}e^{ir\vartheta},\, y^{\frac{1}{2}-s}e^{ir\vartheta}\}$ if $s \neq 0$ and $\{y^{\frac{1}{2}}e^{ir\vartheta},\, y^{\frac{1}{2}}\log y e^{ir\vartheta}\}$ if $s = 0$.

The following proposition characterises functions with exponential growth in terms of Fourier series.

Proposition 7.1. *Let $k \in 2\mathbb{Z}$, $\operatorname{Re} s \geq 0$. Suppose that the function $f \in C^\infty(\tilde{\Gamma}\backslash\tilde{G})$ satisfies $\omega f = \lambda_s f$ and $\mathbf{W} f = ik f$. Then it has at each cusp κ an absolutely converging Fourier expansion*

$$f(\tilde{g}_\kappa g) \;=\; \sum_{n\in\mathbb{Z}} F_{\kappa,n}\, f(g) \tag{7.7}$$

with $F_{\kappa,n}\, f \in \mathcal{W}_k(n,s)$. Moreover, $f \in \tilde{E}_k(\tilde{\Gamma},\lambda_s)$ if and only if there exists $N > 0$ such that all Fourier terms $F_{\kappa,n}\, f$ with $|n| \geq N$ are in $\mathcal{W}_k^0(n,s)$ for all cusps κ.

Proof. The existence of such a Fourier expansion is a standard result. A detailed proof in a more general setting can be found in [3], §4.1–3.

Fourier terms of automorphic forms inherit the growth behaviour of the automorphic form. So if f is in $\mathcal{E}_k(\tilde{\Gamma}, \lambda_s)$, all Fourier terms satisfy $F_{\kappa,\nu} f(z,\vartheta) = \mathrm{O}(e^{ay})$ as $y \to \infty$ for some a depending on f. Each Fourier term of non-zero order is a linear combination of $\omega_k(n,s)$ and $\hat{\omega}_k(n,s)$. From (7.6) we conclude that $F_{\kappa,n} f$ is a multiple of $\omega_k(n,s)$ for all but finitely many n.

Conversely, suppose that for the cusp κ we have $F_{\kappa,n} f = c_n\, \omega_k(n,s)$ for all n with $|n| \geq N$. Then (7.5) and the convergence of the Fourier expansion at $(z,\vartheta) = (iy_0, 0)$ with $y_0 > 0$ implies that $c_n = \mathrm{O}(y_0^{-k\,\mathrm{Sign}(n)/2} e^{2\pi|n|y_0})$. This in turn shows that the sum over $|n| \geq N$ gives a bounded contribution in (7.7) for all y large enough. The terms with $|n| < N$ cannot give a growth at the cusp κ larger than $\mathrm{O}(y^a\, e^{2\pi(N-1)y})$ for some $a > 0$. □

Remark 7.2. The $\tilde{\Gamma}$-invariance in Proposition 7.1 is not necessary. Invariance under only the parabolic elements of $\tilde{\Gamma}$ suffices. If we work with functions f on $\{(z,\vartheta) \;:\; y \geq y_0\}$ for some $y_0 > 0$ that satisfy $\omega f = \lambda_s f$, $\mathbf{W} f = ikf$ and are left-invariant under $\{n(l) \;:\; l \in \mathbb{Z}\}$, then there is an expansion like in (7.7) on the set $y \geq y_0$, and exponential growth of such a function is equivalent to the statement that all Fourier terms of sufficiently large order are in $\mathcal{W}_k^0(n,s)$.

7.2. **Higher order Fourier terms.** The higher order invariants of $\mathcal{V}_k(n,s)$ that we will define now are the higher-order analogues of the classical Fourier terms.

Definition 7.3. Let $k \in 2\mathbb{Z}$, $n \in \mathbb{Z}$, and $s \in \mathbb{C}$. By $\mathcal{V}_k(n,s)$ we denote the space of functions f on $\tilde{G}$ that satisfy $\omega f = \lambda_s f$, have generalised weight k, and satisfy $(\partial_x - 2\pi i n)^m f = 0$ for some $m \in \mathbb{N}$ (which may depend on f).

For $n \neq 0$ we denote by $\mathcal{V}_k^0(n,s)$ the subspace of $f \in \mathcal{V}_k(n,s)$ that satisfy $f(z,\vartheta) = \mathrm{O}(y^a\, e^{-2\pi|n|y})$ as $y \to \infty$ for some $a \in \mathbb{R}$.

The free commutative group $\tilde{\Delta}$ generated by $\tau = n(1)$ and $\zeta = k(\pi)$ acts on these spaces by left translation.

Proposition 7.4. *Let k, n, s be as above. The $\tilde{\Delta}$-modules $\mathcal{V}_k(n,s)$ and $\mathcal{V}_k^0(n,s)$ are maximally perturbable.*

For each $q \in \mathbb{N}$ the elements $f \in \mathcal{V}_k(n,s)^{\tilde{\Delta},q}$ satisfy, for each $\delta > 0$,

$$f(z,\vartheta) \ll_\delta e^{(2\pi|n|+\delta)y} \qquad (y \to \infty) \tag{7.8}$$

uniformly for x and ϑ in compact sets. If $n \neq 0$ then for each $q \in \mathbb{N}$ the elements $f \in \mathcal{V}_k^0(n,s)^{\tilde{\Delta},q}$ satisfy, for each $\delta > 0$,

$$f(z,\vartheta) \ll_\delta e^{(\delta-2\pi|n|)y} \qquad (y \to \infty) \tag{7.9}$$

uniformly for x and ϑ in compact sets.

Proof. To prove that $\mathcal{V}_k(n,s)$ is maximally perturbable, we start with a characterisation of the space $\mathcal{V}_k(n,s)^{\tilde{\Delta}}$. We first note that $\mathcal{W}_k(n,s) \subset \mathcal{V}_k(n,s)^{\tilde{\Delta}}$. Conversely, if $f \in \mathcal{V}_k(n,s)^{\tilde{\Delta}}$, then the reasoning in the proof of Theorem 6.4 shows that the weight of f is strict, and also that $\partial_x f = 2\pi i n f$, hence $f(z,\vartheta) = e^{2\pi i n x} f(iy,\vartheta)$. So $f \in \mathcal{W}_k(n,s)$. If, for $n \neq 0$, the function f is also exponentially decreasing it has to be a multiple of $\omega_k(n,s)$. Therefore, $\mathcal{V}_k^0(n,s)^{\tilde{\Delta}} = \mathcal{W}_k^0(n,s)$.

Let f be an arbitrary element of $\mathcal{W}_k(n,s)$. Since each of the basis elements of $\mathcal{W}_k(n,s)$ is a specialisation of a holomorphic family of elements of $\mathcal{W}_r(\nu,s)$, there is a holomorphic family of $h(r,\nu) \in \mathcal{W}_r(\nu,s)$ such that $h(k,n) = f$. We have $h(r,\nu; n(\xi)k(\ell\pi)(z,\vartheta)) = e^{2\pi i\nu\xi + \pi i r\ell}\, h(r,\nu; z,\vartheta)$ for $\xi \in \mathbb{R}$ and $\ell \in \mathbb{Z}$.

Next consider the polynomials $Q_q \in \mathbb{Q}[X]$ of degree q defined by

$$Q_0 \;=\; 1\,, \quad Q_{q+1}(X+1) - Q_{q+1}(X) \;=\; Q_q(X) \text{ and } Q_q(0) \;=\; 0 \text{ for } q \geq 1\,. \tag{7.10}$$

Then for each $\mathbf{m} = (m_1, m_2)$, $m_j \geq 0$ set

$$h_k^{\mathbf{m}}(n,s) \;=\; Q_{m_1}(\tfrac{1}{\pi i}\partial_r)\, Q_{m_2}(\tfrac{1}{2\pi i}\partial_\nu)\, h(r,\nu)\big|_{\nu=n,\, r=k}\,. \tag{7.11}$$

Upon applying the differential operator $\frac{1}{2\pi i}\partial_\nu^a$ on $h(r,\nu)|(\tau-1) \;=\; (e^{2\pi i\nu}-1)h(r,\nu)$ we obtain

$$(2\pi i)^{-a}\frac{\partial^a h(r,\nu)}{\partial \nu^a}\;\Big|\;(\tau-1) \;=\; \sum_{b=0}^{a-1}\binom{a}{b}(2\pi i)^{-b}\frac{\partial^b h(r,\nu)}{\partial \nu^b} \;=\; \Big((\tfrac{1}{2\pi i}\partial_\nu+1)^a - (\tfrac{1}{2\pi i}\partial_\nu)^a\Big)h(r,\nu)\,. \tag{7.12}$$

Therefore,

$$Q_{m_2}\Big(\tfrac{1}{2\pi i}\partial_\nu\Big)h(r,\nu)|(\tau-1) \;=\; \Big(Q_{m_2}(\tfrac{1}{2\pi i}\partial_\nu+1) - Q_{m_2}(\tfrac{1}{2\pi i}\partial_\nu)\Big)h(r,\nu) \;=\; Q_{m_2-1}(\tfrac{1}{2\pi i}\partial_\nu)h(r,\nu)\,. \tag{7.13}$$

Since τ, ζ commute, this implies $h_k^{\mathbf{m}}(n,s)|(\tau-1) = h_k^{(m_1,m_2-1)}(n,s)$. Likewise, we obtain the transformation law $h_k^{\mathbf{m}}(n,s)|(\zeta-1) = h_k^{(m_1-1,m_2)}(n,s)$. Therefore, for $l_1+l_2 = m_1+m_2$ ($l_1, l_2 \geq 0$),

$$h_k^{(m_1,m_2)}(n,s)|(\zeta-1)^{l_1}(\tau-1)^{l_2} = \delta_{m_1,l_1}\delta_{m_2,l_2}\, f\,, \tag{7.14}$$

thus obtaining the maximal perturbability of $\mathcal{V}_k(n,s)$. For convenience, we shall call perturbations satisfying the transformation law (7.14) *perturbations of type* $\mathbf{m}$.

Based on $\mathcal{V}_k^0(n,s)^{\tilde{\Delta}} = \mathcal{W}_k^0(n,s)$, we deduce in an analogous way the maximal perturbability of $\mathcal{V}_k^0(n,s)$.

To prove (7.8) and (7.9), we first note that the maximal perturbability we have just shown implies that the functions $h^{\mathbf{m}}$ constructed from f's ranging over a basis of $\mathcal{W}_k(n,s)$ (resp. $\mathcal{W}_k^0(n,s)$) induce a basis of the quotients $\mathcal{V}^{\tilde{\Delta},q+1}/\mathcal{V}^{\tilde{\Delta},q}$. Therefore, it suffices to show (7.8) and (7.9) for $h^{\mathbf{m}}$ only. In the case $n \neq 0$, the family h may be taken to be $\omega_r(\nu,s)$ or $\hat{\omega}_r(\nu,s)$ in (7.4). For these functions the question reduces to the asymptotic behaviour of $\partial_t^j\partial_\kappa^l W_{\kappa,s}(t)$, since the factors $e^{2\pi i\nu x}$ and $e^{ir\vartheta}$ produce polynomials in x and ϑ, which yield constants when they vary through compact sets. The differentiation of $4\pi\,\mathrm{Sign}(\mathrm{Re}\,\nu)\,\nu\,y$ yields only a power of y, which can be absorbed by the factor $e^{\delta y}$.

Differentiation of $W_{\kappa,s}(t)$ with respect to t does not change the exponential part of the asymptotic behaviour, since derivatives of $W_{\kappa,s}(t)$ are linear combinations of $W_{\kappa,s}(t)$ and $W_{\kappa+1,s}(t)$ with powers of t in the factors. See (2.4.24) in [21]. So we have to look only at differentiation with respect to κ.

For $t \in \mathbb{R}$ with $t > 0$, $\kappa - \frac{1}{2} - s \neq -1, -2, \ldots$, we shall use the integral representation (3.5.18) in [21]:

$$W_{\kappa,s}(t) \;=\; \frac{-1}{2\pi i}\Gamma(\kappa+\frac{1}{2}-s)e^{-t/2}t^\kappa\int_{(0+)}^{\infty} e^{-x}(-x)^{s-\kappa-\frac{1}{2}}(1+\frac{x}{t})^{s+\kappa-\frac{1}{2}}\,dx \tag{7.15}$$

where the contour comes from ∞ along a line slightly above the positive real axis, encircles 0 with radius $\delta < 1$ and then goes back to ∞ on a line slightly below the positive real axis. By a routine computation we see that the part of the integral over the circular part is $O(e^{\delta|t|})$. The integral over the remaining part of the contour is $O(|t|^A)$ ($A \in \mathbb{R}$). In all cases, the implied constants does not depend on t. Differentiation in terms of κ on $W_{\kappa,s}(t)$ leads to the appearance of additional factors $\log(-x)$ and $\log(1+x/t)$ in the integrand. The arguments used in the last paragraph imply the same estimate. Thus we get the desired exponential decay of the perturbations of $\omega_k(n,s)$.

The representation (7.15) is valid as long as $-t = e^{-\pi i}t$ is outside the path of integration. If we tilt the path of integration anti-clockwise by an angle ϕ we get a representation of $W_{\kappa,s}(t)$ for $e^{-\pi i}t$ outside the new path of integration, provided we keep $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ to have convergence. For $0 < \varphi < \frac{\pi}{2}$ this gives

a representation that can be used for $\arg(e^{-\pi i}t) = 0$ with $|t| > \delta$, which leads to the desired growth of perturbations of $\hat\omega_k(n,s)$.

If $\kappa - \frac12 - s = -1, -2, \dots$ we take $0 < \varphi < \frac{\pi}{2}$ and transform the integral representation (7.15) into

$$W_{\kappa,s}(t) \;=\; \frac{e^{-\frac12 t}\, t^\kappa\, e^{i\varphi(s-\kappa+1/2)}}{\Gamma(s+\frac12-\kappa)} \int_0^\infty e^{-e^{i\varphi}u}\, u^{s-\kappa-\frac12}\,(1+e^{i\varphi}u/t)^{s+\kappa-\frac12}\,du\,, \tag{7.16}$$

Proceeding as before we obtained the same estimates.

All these estimates taken together prove (7.8), (7.9) (when $n \neq 0$). They further show that the derivatives of a family with exponential decay have exponential decay and thus $\mathcal{V}_k^0(n,s)$ is also maximally perturbable.

In the case $n = 0$ we might use the same method. However, many families of special functions have to be considered to cover all cases. Instead we argue directly that we can find functions $h_k^{\mathbf{m}}(0,s)$ in $\mathcal{V}_k(0,s)$ of the form $p_{\mathbf{m}}(x,y,\vartheta)\,y^{\frac12\pm2}\,e^{ik\vartheta}$ where $p_{\mathbf{m}}$ is a polynomial in three variables with degree m_1 in ϑ and degree m_2 in x. If the coefficient of $\vartheta^{m_1}x^{m_2}$ in this polynomial does not depend on y, this leads to a perturbation of $y^{\frac12\pm s}e^{ik\vartheta}$ of type $\mathbf{m}$. Such functions satisfy the required estimates, with a polynomial factor y^A instead of $e^{\delta y}$. The remaining task is to check that they can be chosen to satisfy $(\omega - \frac14 + s^2)\,h_k^{\mathbf{m}}(0,s) = 0$. We do this by induction in the degrees in ϑ and x. We check that

$$(\omega - \frac14 + s^2)x^{m_2}y^{\frac12\pm s+a}\vartheta^{m_1}e^{ik\vartheta} \;=\; -a(a\pm 2s)x^{m_2}y^{\frac12\pm s+a}\vartheta^{m_1}e^{ik\vartheta} + \text{terms of lower degree in } x \text{ or } \vartheta\,.$$

With $a = 0$ this gives the top coefficient of $p_{\mathbf{m}}$. Moreover, the terms of lower degree all are multiples of $x^{\tilde m_2}y^{\frac12\pm s+a}\vartheta^{\tilde m_1}e^{ik\vartheta}$ with $\tilde m_j \le m_j$, $\tilde m_1 < m_1$ or $\tilde m_2 < m_2$, and $a \in \mathbb{Z}_{\ge0}$. Successively we can determine the lower degree terms, and arrange that $h_k^{\mathbf{m}}(0,s)$ is an eigenfunction of the Casimir operator ω with eigenvalue $\frac14 - s^2$.

This takes care of the case $n = 0$, except if $s = 0$. It that case we also have to perform a computation involving $y^{\frac12+a}\log y$, which we leave to the reader. □

Holomorphic Fourier terms on $\tilde G$ are multiples of

$$\eta_r(\nu;z,\vartheta) \;=\; y^{r/2}\,e^{2\pi i\nu z}\,e^{ir\vartheta}\,. \tag{7.17}$$

Thus we have the spectral parameter $s = \pm\frac{r-1}{2}$. For real values of ν and r we have

$$\eta_r(\nu) \;=\; \begin{cases} (4\pi\nu)^{-r/2}\,\omega_r(\nu,\pm\frac{r-1}{2}) & \text{if } \nu > 0\,,\\ \mu_r(0,\frac{r-1}{2}) & \text{if } \nu = 0\,,\\ e^{-\pi ir}\,(4\pi|\nu|)^{-r/2}\,\hat\omega_r(\nu,\pm\frac{r-1}{2}) & \text{if } \nu < 0\,,\end{cases} \tag{7.18}$$

with notations as in (7.4) and (6.11). The functions

$$\eta_k^{\mathbf{m}}(n;z,\vartheta) \;=\; Q_{m_1}(\tfrac{2i\vartheta+\log y}{2\pi i})\,Q_{m_2}(z)\,\eta_k(n;z,\vartheta) \tag{7.19}$$

satisfy

$$\mathsf{m}_{m_1+m_2}\eta_k^{\mathbf{m}} : (\zeta-1)^{l_1}\,(\tau-1)^{l_2} \mapsto \delta_{m_1,l_1}\delta_{m_2,l_2}\,\eta_k(n) \tag{7.20}$$

for $l_1 + l_2 = m_1 + m_2$, and as $y \to \infty$ their growth is of order $\mathrm{O}(e^{(\delta-2\pi n)y})$. For the commutative group $\tilde\Delta$ and for a fixed $\mathbf{m}$ they yield a basis of the space of forms of order $m_1 + m_2 + 1$ modulo lower order forms.

As an example we note that the Fourier expansion (6.1) can be written in the following way:

$$L(z,\vartheta) \;=\; \pi i\,\eta_0^{(1,0)}(0;z,\vartheta) + \frac{\pi i}{6}\,\eta_0^{(0,1)}(0;z,\vartheta) - 2\sum_{n\geq 1}\sigma_{-1}(n)\,\eta_0^{(0,0)}(n;z,\vartheta)\,. \tag{7.21}$$

8. Proofs of Theorems 6.5 and 6.8

The method of the proof is highly inductive. At each step we use the maximal perturbability of other spaces which has been proved in a previous step. The starting point for this process is the space $\mathrm{Map}(\tilde\Gamma,\mathbb{C})$ whose maximal perturbability is proved based on general algebraic principles in Proposition 8.1. This implies directly the maximal perturbability of the $\tilde\Gamma$-module $\mathrm{Map}(\mathfrak{H}\times\mathbb{R},\mathbb{C})$. We proceed by imposing increasingly stringent regularity conditions on the functions $\mathfrak{H}\times\mathbb{R}\to\mathbb{C}$. We consider $C^\infty(\mathfrak{H}\times\mathbb{R}) = C^\infty(\tilde G)$, the subspace $C_k^\infty(\tilde G)$ of functions in $C^\infty(\tilde G)$ with generalised weight k and the subspace $\mathcal{C}_k$ of $C_k^\infty(\tilde G)$ of functions that have compact support modulo $\tilde\Gamma$. In §7 we have considered higher order invariant functions for the group $\tilde\Delta$ generated by $n(1)$ and $k(\pi)$. These functions are related to the Fourier expansions of Maass forms. After proving that some more auxiliary subspaces of $C_k^\infty(\mathfrak{H}\times\mathbb{R})$ are maximally perturbable, we finally prove in §8.5 the maximal perturbability of $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)$ and $\mathcal{H}_k(\tilde\Gamma)$.

8.1. Higher order invariants in maps on $\tilde\Gamma$.

A generalisation of Proposition 3.3 is the following:

Proposition 8.1. *If $\tilde\Gamma$ is a discrete cofinite subgroup of $\tilde G$ with cusps, then the $\tilde\Gamma$-module* $\mathrm{Map}(\tilde\Gamma,\mathbb{C})$ *(with the action by left translation) is maximally perturbable.*

Proof. We first define (similarly to Proposition 3.3) $\mathbf{g_i}$ on the free subgroup $\tilde\Gamma_0$ of $\tilde\Gamma$ generated by $\alpha_1,\dots,\alpha_{t(\Gamma)-1}$ for $\mathbf{i}\in\{1,\dots,t(\Gamma)-1\}^q$ by the relations in (3.9), with A_j replaced by α_j.

Let $\varphi_0:\tilde\Gamma\to\tilde\Gamma_0$ be the surjective group homomorphism given by $\varphi_0(\alpha_j)=\alpha_j$ for $1\leq j\leq t(\Gamma)-1$, $\varphi_0(\zeta)=1$ and $\varphi_0(\varepsilon_j)=1$ for $1\leq j\leq n_{\mathrm{ell}}$. For $1\leq j\leq t(\Gamma)$ we define $\psi_j\in\hom(\tilde\Gamma,\mathbb{C})$ such that $\psi_j(\alpha_{j'}) = \delta_{j,j'}$. This determines ψ_j completely, since values on elliptic generators are given by $\psi_j(\varepsilon_j)=\frac{1}{v_j}\psi_j(\zeta)$. For $\mathbf{i}=(\mathbf{i}',t(\Gamma),\dots,t(\Gamma))$ with m coordinates $t(\Gamma)$ at the end and $\mathbf{i}'\in\{1,\dots,t(\Gamma)-1\}^{q-m}$, we put

$$\mathbf{f_i}(\gamma) \;=\; \mathbf{g}_{\mathbf{i}'}(\varphi_0(\gamma))\,Q_m(\psi_{t(\Gamma)}(\gamma)) \tag{8.1}$$

where Q_n are the polynomials defined in (7.10). Now we can check the following properties of $\mathbf{f_i}$:

$$\mathbf{f}_{()} \;=\; 1\,,\qquad (\text{empty tuple, } q=0)\,; \tag{8.2}$$

$$\mathbf{f_i}(1) \;=\; 0\qquad \text{if } |\mathbf{i}|\geq 1\;; \tag{8.3}$$

$$\mathbf{f_i}|(\zeta-1) \;=\; \begin{cases}\mathbf{f}_{\mathbf{i}'} & \text{if } \mathbf{i} \;=\; (\mathbf{i}',t(\Gamma))\,,\\ 0 & \text{if } \mathbf{i} \text{ does not end with a } t(\Gamma)\,;\end{cases} \tag{8.4}$$

$$\mathbf{f_i}|(\alpha_j-1) \;=\; \begin{cases}\mathbf{f}_{\mathbf{i}'} & \text{if } \mathbf{i}=(j,\mathbf{i}') \text{ with } j<t(\Gamma)\,,\\ 0 & \text{if } j<t(\Gamma),\ j\neq\mathbf{i}(1)\,.\end{cases} \tag{8.5}$$

Using this we can see that

$$(\mathsf{m}_q\mathbf{f_i})(\mathbf{b(j)}) \;=\; \delta_{\mathbf{i},\mathbf{j}}\,. \tag{8.6}$$

Now, the choice of the basis $\mathbf{b(i)}$ in (5.9) for $\tilde\Gamma$-q-tuples $\mathbf{i}$ shows that to prove that $\mathrm{Map}(\tilde\Gamma,\mathbb{C})$ is maximally perturbable it suffices to prove that for each $\mathbf{i}$ and for each function f on $\tilde\Gamma\backslash\tilde G$ a function $h_{\mathbf{i}}\in\mathrm{Map}(\tilde G,\mathbb{C})$ such that for all $\tilde\Gamma$-q-tuples $\mathbf{j}$:

$$h_{\mathbf{i}}|(\alpha_{\mathbf{j}(1)}-1)\cdots(\alpha_{\mathbf{j}(q)}-1) \;=\; \delta_{\mathbf{i},\mathbf{j}}\cdot f\,. \tag{8.7}$$

To construct such functions we choose a *strict fundamental domain* $\mathfrak{F}_{\tilde\Gamma} \subset \tilde G$ for $\tilde\Gamma\backslash\tilde G$, *i.e.*, a set meeting each $\tilde\Gamma$-orbit exactly once. Such a fundamental domain can be constructed from a strict fundamental domain $\mathfrak{F}_{\mathfrak{H}}$ for $\Gamma\backslash\mathfrak{H}$, by taking

$$\begin{aligned}\mathfrak{F}_{\tilde\Gamma} &= \{(z,\vartheta) \,:\, z \in \mathfrak{F}_{\mathfrak{H}}\,, 0 \le \vartheta \le \pi/n_z\}\,,\\ n_z &= \min\{n \in \mathbb{N} \,:\, \text{there is } \gamma \in \tilde\Gamma \text{ fixing } z \text{ in } \mathfrak{H} \text{ conjugate to } k(\pi/n)\}\,.\end{aligned}$$

So $n_z = 1$ for all $z \in \mathfrak{F}_{\mathfrak{H}}$, except for the elliptic fixed points $z_1, \dots, z_{n_{\rm ell}}$ in $\mathfrak{F}_{\mathfrak{H}}$. These are conjugate to a fixed point of ε_j and $n_{z_j} = v_j$.

A choice for the sought function $h_{\mathbf{i}}$ is then

$$h_{\mathbf{i}}(\gamma g) = \mathbf{f}_{\mathbf{i}}(\gamma) f(g) \qquad \gamma \in \Gamma\,,\ g \in \mathfrak{F}_{\tilde\Gamma}\,. \tag{8.8}$$

With the characteristic function ψ of $\mathfrak{F}_{\tilde\Gamma}$, we can write this as

$$h_{\mathbf{i}}(g) = \sum_{\gamma\in\tilde\Gamma} \mathbf{f}_{\mathbf{i}}(\gamma)\, f(g)\, \psi(\gamma^{-1} g)\,. \tag{8.9}$$

□

8.2. **Higher order invariants in smooth functions on $\tilde G$.** We will use essentially the same construction as in the last section to prove that

Proposition 8.2. *The $\tilde\Gamma$-module $C^\infty(\tilde G)$ is maximally perturbable.*

Proof. In order to show that $C^\infty(\tilde G)$ is a maximally perturbable $\tilde\Gamma$-module, we need to have (8.7) with $h_{\mathbf{i}} \in C^\infty(\tilde G)$ for each $f \in C^\infty(\tilde\Gamma\backslash\tilde G)$. Lemma A.1 in Appendix A shows that we can find functions $\psi \in C^\infty(\mathfrak{H}\times\mathbb{R})$ such that $\sum_{\gamma\in\tilde\Gamma} \psi(\gamma^{-1}(z,\vartheta)) = 1$ for all $(z,\vartheta) \in \mathfrak{H}\times\mathbb{R}$ as a locally finite sum. If we define (8.9) with such a function ψ and $f \in C^\infty(\tilde\Gamma\backslash\tilde G)$, then the sum is locally finite, and the $h_{\mathbf{i}}$ are smooth. □

8.3. **Higher order invariants and generalised weight.** Set

$$C^\infty_k(\tilde G) = \{f \in C^\infty(\tilde G)\,, \text{of generalised weight } k\}\,. \tag{8.10}$$

Proposition 8.3. *Let $k \in 2\mathbb{Z}$. Then the $\tilde\Gamma$-module $C^\infty_k(\tilde G)$ is maximally perturbable.*

Proof. As with the previous proofs, our approach is to show that for every $\tilde\Gamma$-q-tuple $\mathbf{i} = (\mathbf{i}', t(\Gamma), \dots, t(\Gamma))$ with exactly m occurrences of $t(\Gamma)$ at the end and for every $f \in C^\infty_k(\tilde\Gamma\backslash\tilde G)$ there exists $h_{\mathbf{i}} \in C^\infty_k(\tilde G)$ satisfying equation (8.7) for all $\tilde\Gamma$-q-tuples $\mathbf{j}$. We note that, by Theorem 6.4, the $\tilde\Gamma$-invariance of f implies that its weight k is strict, *i.e.*, $f(gk(\vartheta)) = f(g)e^{ik\vartheta}$.

We will define the function $h_{\mathbf{i}}$ by an analogue of (8.9). We first define for each $g \in \tilde G$ the point $w(g) = \mathrm{pr}(g)\, i \in \mathfrak{H}$ and the real number $\Theta(g) \in \mathbb{R}$ such that $g = (w(g), \Theta(g)) \in \tilde G = \mathfrak{H}\times\mathbb{R}$. We also recall that $\Gamma = \tilde\Gamma/\tilde Z$. Since the group homomorphism ϕ_0 defined in the proof of Proposition 8.1 is trivial on $\tilde Z = \langle\zeta\rangle$, it induces a homomorphism on Γ. Now we take $\psi(z,\vartheta) = \psi_0(z)$, with ψ_0 as in Part ii) of Lemma A.1. So the function $(z,\vartheta) \mapsto \psi(\gamma^{-1}(z,\vartheta))$ obtained by left translation depends only on the image of $\gamma \in \tilde\Gamma$ in $\Gamma \cong \tilde\Gamma/\tilde Z$. Let, as in the proof of Proposition 8.1, $\psi_{t(\Gamma)}$ be the function $\tilde\Gamma \to \mathbb{R}$ such that $\psi_{t(\Gamma)}(\alpha_{j'}) = \delta_{t(\Gamma),j'}$. For a given $\gamma \in \tilde\Gamma$ we have $\psi_{t(\Gamma)}(\zeta\gamma) = \psi_{t(\Gamma)}(\gamma) + 1$ and $\Theta((\zeta\gamma)^{-1}g) = \Theta(\gamma^{-1}g) - \pi$. So $\psi_{t(\Gamma)}(\gamma) + \Theta(\gamma^{-1}g)/\pi$ is well-defined on $\Gamma = \tilde\Gamma/\tilde Z$. We can therefore set

$$h_{\mathbf{i}}(g) = \sum_{\gamma\in\Gamma} \mathbf{g}_{\mathbf{i}'}(\varphi_0(\gamma))\, Q_m(\psi_{t(\Gamma)}(\gamma) + \Theta(\gamma^{-1}g)/\pi)\, f(g)\, \psi(\gamma^{-1}g)\,. \tag{8.11}$$

The support property of the partition of unity ψ ensures convergence; it is even a locally finite sum with a bounded number of non-zero terms. All factors depend smoothly on g. So $h_{\mathbf{i}} \in C^\infty(\tilde{G})$.

We consider $(\mathbf{W} - ik)h_{\mathbf{i}}$. Since $\mathbf{W}\psi = 0$, we need only consider

$$\begin{aligned}
&\Big(\partial_\vartheta - ik\Big)Q_m\Big(\psi_{t(\Gamma)}(\gamma) + \Theta(\gamma^{-1}gk(\vartheta))/\pi\Big)\, f(gk(\vartheta)) \\
&\quad = \; Q_m\Big(\psi_{t(\Gamma)}(\gamma) + \Theta(\gamma^{-1}gk(\vartheta))/\pi\Big)\Big(\partial_\vartheta - ik\Big) f(gk(\vartheta)) \\
&\qquad + \; f(gk(\vartheta))\; \partial_\vartheta Q_m\Big(\psi_{t(\Gamma)}(\gamma) + \Theta(\gamma^{-1}g)/\pi + \vartheta/\pi\Big) \\
&\quad = \; 0 + \pi^{-1}Q_m'\Big(\psi_{t(\Gamma)}(\gamma) + \Theta(\gamma^{-1}g)/\pi + \vartheta/\pi\Big)\, f(gk(\vartheta))\,.
\end{aligned} \tag{8.12}$$

Repeating this we obtain

$$(\mathbf{W} - ik)^{m+1}Q_m\Big(\psi_{t(\Gamma)}(\gamma) + \Theta(\gamma^{-1}g)/\pi\Big)\, f(g) = \pi^{-m-1}Q_m^{(m+1)}(\cdots)\cdots \;=\; 0 \quad , \tag{8.13}$$

since the degree of Q_m is m. So $h_{\mathbf{i}} \in C_k^\infty(\tilde{G})$. □

In a similar (but much simpler) way, one shows that, if Γ acts on $C^\infty(\mathfrak{H})$ via (4.2) and $f \in C^\infty(\mathfrak{H})^\Gamma$, then the function in $C^\infty(\mathfrak{H})$ $h_{\mathbf{i}}(z) := \sum_{\gamma\in\Gamma} \mathbf{g}_{\mathbf{i}}(\varphi_0(\gamma))\, f(z)\psi_0(\gamma^{-1}z)$ satisfies (8.7) for all $(t(\Gamma)-1)$-tuples of elements of $\{1,\dots,t(\Gamma)-1\}$. This gives an alternative proof of

Proposition 8.4 (Prop. 4.1, [10])**.** *Let $k \in 2\mathbb{Z}$. Then the Γ-module $C^\infty(\mathfrak{H})$ is maximally perturbable.*

In fact, since ψ_0 is bounded, if f has polynomial growth at all cusps, then so does $h_{\mathbf{i}}$ thus proving that the submodule of $C^\infty(\mathfrak{H})$ of functions with polynomial growth is also maximally perturbable.

8.4. **Higher order invariants with support conditions.** We shall first discuss the motivation for the introduction of the invariants we will be dealing with. If Definition 6.3 of the space $\tilde{\mathcal{E}}_k(\tilde{\Gamma},\lambda)$ did not include a growth condition at the cusps, we could consider $\tilde{\mathcal{E}}_k(\tilde{\Gamma},\lambda)$ as the kernel $\mathcal{K}$ in the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow C_k^\infty(\tilde{G}) \xrightarrow{\omega-\lambda} C_k^\infty(\tilde{G})$$

With exponential growth, one might want to try to replace $C_k^\infty(\tilde{G})$ by its subspace $C_l^\infty(\tilde{\Gamma})^{\mathrm{eg}}$ of functions with exponential growth at the cusps of $\tilde{\Gamma}$. This would lead to an exact sequence

$$0 \longrightarrow \tilde{\mathcal{E}}_k(\tilde{\Gamma},\lambda) \longrightarrow C_k^\infty(\tilde{\Gamma})^{\mathrm{eg}} \xrightarrow{\omega-\lambda} C_k^\infty(\tilde{\Gamma})^{\mathrm{eg}}$$

for which we might try to show that for each $q \in \mathbb{N}$

$$0 \longrightarrow \tilde{\mathcal{E}}_k(\tilde{\Gamma},\lambda)^{\tilde{\Gamma},q} \longrightarrow (C_k^\infty(\tilde{\Gamma})^{\mathrm{eg}})^{\tilde{\Gamma},q} \xrightarrow{\omega-\lambda} (C_k^\infty(\tilde{\Gamma})^{\mathrm{eg}})^{\tilde{\Gamma},q}$$

is exact. For this to be of use it seems that we need surjectivity of the map $\omega-\lambda : (C_k^\infty(\tilde{\Gamma})^{\mathrm{eg}})^{\tilde{\Gamma}} \to (C_k^\infty(\tilde{\Gamma})^{\mathrm{eg}})^{\tilde{\Gamma}}$, which we did not succeed in proving, and which may not hold. For this reason we will instead work with other better behaved subspaces of the spaces appearing in the exact sequence. We will therefore define subspaces $C_k, \mathcal{D}_k(\lambda) \subset C_k^\infty(\tilde{G})$ and $\mathcal{E}_k'(\lambda) \subset \mathcal{E}_k(\tilde{\Gamma},\lambda)$ related by an exact sequence

$$0 \longrightarrow \tilde{\mathcal{E}}_k'(\lambda) \longrightarrow \mathcal{D}_k(\lambda) \xrightarrow{\omega-\lambda} C_k. \tag{8.14}$$

8.4.1. *The spaces C_k.* For each cusp $\kappa = \tilde{g}_\kappa\infty$, and each $a > 0$ we call

$$D_\kappa(a) \;=\; \tilde{g}_\kappa\{(z,\vartheta) \;:\; \operatorname{Im} z \geq a,\ \vartheta \in \mathbb{R}\} \tag{8.15}$$

a *horocyclic set*. There is a number A_Γ such that for each $a \geq A_\Gamma$ the $D_\kappa(a)$ are disjoint for different cusps. The sets

$$\tilde{G}_a = \{(z,\vartheta) \in \mathfrak{H}\times\mathbb{R} \;:\; \forall\kappa\ (z,\vartheta) \notin D_\kappa(a)\} \tag{8.16}$$

satisfy $\tilde{\Gamma}\tilde{G}_a = \tilde{G}_a$. This follows from the fact that the g_κ have been chosen so that

$$\gamma\tilde{\Gamma}_\kappa\tilde{g}_\kappa = \tilde{g}_{\gamma\kappa}\tilde{\Gamma}_\infty \tag{8.17}$$

for all cusps κ and for $\gamma \in \tilde{\Gamma}$. Here $\tilde{\Gamma}_\kappa := \mathrm{pr}^{-1}\Gamma_\kappa = \{\gamma \in \tilde{\Gamma} \;:\; \gamma\kappa = \kappa\}$.

Definition 8.5. Let $k \in 2\mathbb{Z}$. The space C_k consists of the $f \in C_k^\infty(\tilde{G})$ supported in $\tilde{G}_a$ for some $a \geq A_\Gamma$. (The a may depend on f).

So C_k consists of the smooth functions with generalised weight k whose supports project to compact subsets of $\Gamma\backslash\mathfrak{H}$. Clearly, the space C_k is $\tilde{\Gamma}$-invariant. If we apply the construction of $h_{\mathbf{i}}$ in the proof of Proposition 8.3 to functions $f \in C_k^{\tilde{\Gamma}} \subset C_k^\infty(\tilde{\Gamma}\backslash\tilde{G})$ then the support of each $h_{\mathbf{i}}$ is contained in the same set $\tilde{G}_a$ that contains $\mathrm{Supp}(f)$. This implies:

Proposition 8.6. *Let $k \in 2\mathbb{Z}$. Then the $\tilde{\Gamma}$-module C_k is maximally perturbable.*

8.4.2. *The spaces $\mathcal{D}_k(\lambda)$.* The construction of $\mathcal{D}_k(\lambda)$ and the proof of its maximal perturbability is much lengthier that those for C_k. We will define $\mathcal{D}_k(\lambda)$ essentially as the space of functions that accept higher-order analogues of Fourier expansions at the cusps. To make this formal we study spaces of functions defined on regions of the form

$$S(y_0) \;=\; \{(x+iy,\vartheta) \in \mathfrak{H}\times\mathbb{R} \;:\; y > y_0\}, \tag{8.18}$$

with $y_0 > 0$.

Definition 8.7. Let $k \in 2\mathbb{Z}$, $\lambda \in \mathbb{C}$, and $y_0 > 0$. We denote by $\mathcal{E}_k(y_0,\lambda)$ the space of $f \in C^\infty(S(y_0))$ that satisfy $\omega f = \lambda f$, $(\mathbf{W} - ik)^n f = 0$ for some $n \in \mathbb{N}$, and have at most exponential growth as $y \to \infty$, uniform for x and ϑ in compact sets. We denote by $\mathcal{E}_k^{\mathrm{hol}}(y_0)$ the space of holomorphic functions on $S(y_0)$ with generalised weight k and at most exponential growth as $y \to \infty$

Proposition 8.8. *Let $k \in 2\mathbb{Z}$, $s \in \mathbb{C}$ and $y_0 > 0$. The spaces $\mathcal{E}_k(y_0,\lambda_s)$ and $\mathcal{E}_k^{\mathrm{hol}}(y_0)$ are maximally perturbable $\tilde{\Delta}$-modules.*

Let $q \in \mathbb{N}$. Each $f \in \mathcal{E}_k(y_0,\lambda_s)^{\tilde{\Delta},q}$ has an absolutely convergent expansion

$$f(z,\vartheta) \;=\; \sum_{n\in\mathbb{Z}} f_n(z,\vartheta) \tag{8.19}$$

on $S(y_0)$ with $f_n \in \mathcal{V}_k(n,s)^{\tilde{\Delta},q}$ for all n, and $f_n \in \mathcal{V}_k^0(n,s)^{\tilde{\Delta},q}$ for almost all n.

Each $f \in \mathcal{E}_k^{\mathrm{hol}}(y_0)^{\tilde{\Delta},q}$ has an absolutely convergent expansion on $S(y_0)$ of the form

$$f(z,\vartheta) \;=\; \sum_{\mathbf{m},\, m_1+m_2<q} \sum_n c_{\mathbf{m}}^n\, \eta_k^{\mathbf{m}}(n;z,\vartheta) \tag{8.20}$$

where the inner sum ranges from some, possible negative, integer to infinity.

Proof. We start with the holomorphic case. Let $f \in \mathcal{E}_k^{\mathrm{hol}}(y_0)^{\tilde{\Delta}}$. Then the function $z \mapsto y^{-k/2} f(z,0)$ is holomorphic on $\{z \in \mathfrak{H} \ : \ y > y_0\}$ with period 1. So it has a finite to the left expansion of the form $\sum_n a_n e^{2\pi i n z}$ converging absolutely on $y > y_0$. For each $y_1 > y_0$ we have $a_n = \mathrm{O}(e^{2\pi n y_1})$ as $n \to \infty$.

Hence $f(z,\vartheta) = \sum_n a_n \eta_k(n;z,\vartheta)$ converges absolutely on $y > y_0$, and

$$f^{\mathbf{m}}(z,\vartheta) \ := \ \sum_{n \geq -N} a_n \eta_k^{\mathbf{m}}(n;z,\vartheta)$$

converges absolutely on $S(y_0)$, and the convergence is uniform on any set $y \geq y_1$ with $y_1 > y_0$, with x and ϑ in compact sets. These functions satisfy $f^{\mathbf{m}}|(\tau - 1) = f^{(m_1,m_2-1)}$, $f^{\mathbf{m}}|(\zeta - 1) = f^{(m_1-1,m_2)}$ and $f^{(0,0)} = f$, since all $\eta_k^{\mathbf{m}}$ have this property. Thus $f^{\mathbf{m}}$, with $\mathbf{m}$ such that $m_1 + m_2 < q$ is a perturbation of type $\mathbf{m}$ and we deduce that $\mathcal{E}_k^{\mathrm{hol}}(y_0)$ is maximally perturbable. An arbitrary element $h \in \mathcal{E}_k^{\mathrm{hol}}(y_0)^{\tilde{\Delta},q}$ can be written as a finite linear combination of such $f^{\mathbf{m}}$, which all have expansions of the type given in (8.20).

For $f \in \mathcal{E}_k(y_0,\lambda_s)^{\tilde{\Delta}}$ we proceed similarly. By Theorem 7.1 in combination with Remark 7.2 and the integrality of k, there is an absolutely converging Fourier expansion

$$f(z,\vartheta) \ = \ \sum_{n \in \mathbb{Z}} f_n(z,\vartheta)$$

on $S(y_0)$ with $f_n \in \mathcal{W}_k(n,s)$. By the exponential growth, $f_n \in \mathcal{W}_k^0(n,s)$ for $|n| > N$, for some $N \in \mathbb{N}$.

For $|n| > N$ we have $f_n = a_n \omega_k(n,s)$, and from (7.5) we conclude that $a_n = \mathrm{O}(e^{2\pi|n|y_1})$ as $|n| \to \infty$ for each $y_1 > y_0$. So by (7.5) the series

$$\sum_{n,\,|n|>N} a_n \omega_k^{\mathbf{m}}(n,s)$$

converges absolutely on $S(y_0)$, uniformly on each set $y \geq y_1$ with $y_1 > y_0$, and gives an exponentially decreasing function as $y \to \infty$. It is a λ_s-eigenfunction of ω, since the decay allows differentiation inside the sum. To produce a perturbation $f^{\mathbf{m}}$ of f we pick $f_n^{\mathbf{m}} \in \mathcal{V}_k(n,s)^{\tilde{\Delta},m_1+m_2+1}$ such that $f_n^{\mathbf{m}}|(\tau - 1) = f_n^{(m_1,m_2-1)}$, $f_n^{\mathbf{m}}|(\zeta - 1) = f_n^{(m_1-1,m_2)}$ and $f_n^{(0,0)} = f_n$ for the finitely many n with $|n| \leq N$. The estimate (7.8) shows that the growth of these terms is at most of the order $\mathrm{O}(e^{(2\pi N+\delta)y})$ as $y \to \infty$ for each $\delta > 0$. Thus we get (non-uniquely) a perturbation of type $\mathbf{m}$

$$f^{\mathbf{m}} \ = \ \sum_{n,\,|n|\leq N} f_n^{\mathbf{m}} + \sum_{n,\,|n|>N} a_n \omega_k^{\mathbf{m}}(n,s)$$

in $\mathcal{E}_k(y_0,\lambda_s)$. Thus we get (8.19) and the maximal perturbability of $\mathcal{E}_k(y_0,\lambda_s)$. □

We are now ready to define $\mathcal{D}_k(\lambda)$ and $\mathcal{D}_k^{\mathrm{hol}}$.

Definition 8.9. Let $k \in 2\mathbb{Z}$, and $\lambda \in \mathbb{C}$. We define $\mathcal{D}_k(\lambda)$ as the space of functions $f \in C_k^\infty(\tilde{G})$ (hence with generalised weight k) for which there exist $b \geq A_\Gamma$, $a \in \mathbb{R}$, and $q \in \mathbb{N}$ such that for each cusp κ of $\tilde{\Gamma}$ the function $(z,\vartheta) \mapsto f(\tilde{g}_\kappa(z,\vartheta))$ is an element of $\mathcal{E}_k(b,\lambda)^{\tilde{\Delta},q}$, and satisfies a bound $\mathrm{O}(e^{ay})$ as $y \to \infty$.

We define $\mathcal{D}_k^{\mathrm{hol}}$ similarly, with $(z,\vartheta) \mapsto f(\tilde{g}_\kappa(z,\vartheta))$ in $\mathcal{E}_k^{\mathrm{hol}}(b)^{\tilde{\Delta},q}$, with bound $\mathrm{O}(e^{ay})$.

Remark 8.10. The numbers a, b and q may depend on the function f.

Remark 8.11. Definition 8.7 of $\mathcal{E}_k(b,\lambda)$ implies that elements of $\mathcal{D}_k(\lambda)$ are λ-eigenfunctions of ω on the set $\bigsqcup_\kappa D_\kappa(b)$. Similarly, elements of $\mathcal{D}_k^{\mathrm{hol}}$ are holomorphic functions on $\bigsqcup_\kappa D_\kappa(b)$. In both cases we have exponential growth at each cusp. The definition requires that the order of this exponential growth stays bounded when we vary the cusp.

Remark 8.12. In the definition we impose $\tilde\Delta$-invariance of bounded order near all cusps. This is a bit artificial, but serves our purpose.

The space C_k is contained in $\mathcal{D}_k(\lambda)$ and in $\mathcal{D}_k^{\mathrm{hol}}$. Indeed, for given $f \in C_k$ we can take b large so that $\bigsqcup_\kappa D_\kappa(b)$ is outside the support of f.

Elements f of $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde\Gamma}$ restricted to $D_\kappa(b)$ induce elements $(z,\vartheta) \mapsto f(\tilde g_\kappa(z,\vartheta))$ in $\mathcal{E}_k(b,\lambda)^{\tilde\Delta}$ for each cusp κ, and similarly in the holomorphic case. Hence

$$\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde\Gamma} \subset \mathcal{D}_k(\lambda)^{\tilde\Gamma}, \qquad \mathcal{H}_k(\tilde\Gamma)^{\tilde\Gamma} \subset (\mathcal{D}_k^{\mathrm{hol}})^{\tilde\Gamma}. \tag{8.21}$$

Maximal perturbability of $\mathcal{D}_k(\lambda)$ and $\mathcal{D}_k^{\mathrm{hol}}$. We first need a technical lemma in order to relate $\tilde\Delta$-invariants to $\tilde\Gamma$-invariants.

We first note that if ∞ is a cusp of $\tilde\Gamma$ and if $\tilde g_\infty = 1$, then $\tilde\Delta = \tilde\Gamma_\infty$. In general the group $\tilde\Gamma_\kappa$ can be conjugated to $\tilde g_\kappa^{-1}\tilde\Gamma_\kappa\tilde g_\kappa = \tilde\Delta$ in $\tilde g_\kappa^{-1}\tilde\Gamma\tilde g_\kappa$. So we can assume here that $\tilde\Delta \subset \tilde\Gamma$.

The abelian group $\tilde\Delta$ is free on the generators $\tau = n(1)$ and $\zeta = k(\pi)$. The dimension of $\mathrm{Map}(\tilde\Delta,\mathbb{C})^{\tilde\Delta,q+1}$ is $(q+1)(q+2)/2$ with an explicit basis described as follows. Define a sequence of maps on $\tilde\Delta$ by setting

$$\begin{aligned} \varphi^{(l,m)}|(\zeta-1) &= \varphi^{(l-1,m)} \\ \varphi^{(l,m)}|(\tau-1) &= \varphi^{(l,m-1)} \end{aligned} \tag{8.22}$$

and

$$\begin{aligned} \varphi^{(0,0)} &= 1, \quad \varphi^{(l,m)} = 0 \text{ for } l \text{ or } m \text{ negative} \\ \text{and } \varphi^{(l,m)}(1) &= 0 \quad \text{for } l,m \geq 0,\ l+m>0. \end{aligned}$$

Then

$$(\mathsf{m}_q\varphi^{(l,m)})\,((\zeta^r-1)(\tau^s-1)) = \delta_{l,r}\delta_{m,s}$$

for $l+m=r+s=q$, and therefore the $\varphi^{(l,m)}$ with $l,m\geq 0, l+m\leq q$ is a basis of $\mathrm{Map}(\tilde\Delta,\mathbb{C})^{\tilde\Delta,q+1}$.

Let R be a system of representatives of $\tilde\Gamma/\tilde\Delta$; so $R\subset\tilde\Gamma$. Consider the system $\{\mathbf{f_j}\}_{|\mathbf{j}|=q} \subset \mathrm{Map}(\tilde\Gamma,\mathbb{C})^{\tilde\Gamma,q+1}$ in the proof of Proposition 8.1. If $|\mathbf{j}|=q$, then, for every $\gamma\in\tilde\Gamma$, $\delta\mapsto \mathbf{f_j}(\gamma\delta)$ is a function on $\tilde\Delta$ of order at most $q+1$. Hence there are functions $a^{\mathbf{j}}_{l,m}$ on R such that for all $\rho\in R$ and $\delta\in\tilde\Delta$

$$\mathbf{f_j}(\rho\delta) = \sum_{l,m\geq 0\,,\ l+m\leq q} a^{\mathbf{j}}_{l,m}(\rho)\,\varphi^{(l,m)}(\delta). \tag{8.23}$$

Lemma 8.13. *Let $a^{\mathbf{j}}_{l,m}$ be as in* (8.23)*, and suppose that we have functions $\psi^{(l,m)} \in \mathrm{Map}(\tilde\Delta,\mathbb{C})$ satisfying*

$$\begin{aligned} \psi^{(0,0)} &= 0, \\ \psi^{(l,m)}|(\tau-1) &= \psi^{(l-1,m)} \quad \textit{for } l\geq 1, \\ \psi^{(l,m)}|(\zeta-1) &= \psi^{(l,m-1)} \quad \textit{for } m\geq 1. \end{aligned} \tag{8.24}$$

Then

$$f(\rho\delta) = \sum_{l,m\geq 0\,,\ l+m\leq q} a^{\mathbf{j}}_{l,m}(\rho)\,\psi^{(l,m)}(\delta) \quad (\rho\in R,\ \delta\in\tilde\Delta) \tag{8.25}$$

defines an element of $\mathrm{Map}(\tilde\Gamma,\mathbb{C})^{\tilde\Gamma,q}$.

Proof. We proceed by induction in $q = |\mathbf{j}|$. If $q = 0$, then $m = n = 0$, so $f(\rho\delta) = a^{\mathbf{j}}_{0,0}(\rho) \cdot \psi^{(0,0)} = 0 \in \mathrm{Map}(\tilde\Gamma, \mathbb{C})^{\tilde\Gamma,0} = \{0\}$.

It is clear that (8.25) gives a well-defined map on $\tilde\Gamma$. It suffices to prove that, for every generator α_j of $\tilde\Gamma$, $f|(\alpha_j - 1) \in \mathrm{Map}(\tilde\Gamma, \mathbb{C})^{q-1}$. Suppose first that $\mathbf{j} = (j, \mathbf{j}')$. For each $\rho \in R$ there are unique $\rho_1 \in R$ and $\delta_1 \in \tilde\Delta$ such that $\alpha_j \rho = \rho_1 \delta_1$. From (8.23) it follows that

$$\begin{aligned} &\mathbf{f}_{\mathbf{j}}|(\alpha_{\mathbf{j}(1)} - 1)(\rho\delta) \\ &\quad = \sum_{l,m\geq 0\,,\ l+m\leq q} a^{\mathbf{j}}_{l,m}(\rho_1)\, \varphi^{(l,m)}|(\delta_1 - 1)(\delta) \quad + \sum_{l,m\geq 0\,,\ l+m\leq q} (a^{\mathbf{j}}_{l,m}(\rho_1) - a^{\mathbf{j}}_{l,m}(\rho))\, \varphi^{(l,m)}(\delta)\,. \end{aligned} \tag{8.26}$$

By (8.5), the left-hand side equals $\sum_{l,m\geq 0\,,\ l+m\leq q-1} a^{\mathbf{j}'}_{l,m}(\rho)\, \phi^{(l,m)}(\delta)$. The function $\varphi^{(l,m)}|(\delta_1 - 1)$ is a linear combination, depending on ρ, of $\varphi^{(a,b)}$ with $0 \leq a \leq l$, $0 \leq b \leq m$ and $a + b \leq q - 1$. Thus we get an expression that expresses the $a^{\mathbf{j}'}_{l,m}(\rho)$ in the $a^{\mathbf{j}}_{l,m}(\rho)$. The form of this expression depends on the relations (8.22) but not on the specific value of the constant basis element $\varphi^{(0,0)}$. The relations of (8.22) hold for $\psi^{(l,m)}$ too. Therefore, the right hand side of (8.26), upon replacement of ϕ by ψ, equals

$$\sum_{l,m\geq 0\,,\ l+m\leq q-1} a^{\mathbf{j}'}_{l,m}(\rho)\, \psi^{(l,m)}(\delta) \quad (\rho \in R\,,\ \delta \in \tilde\Delta)\,,$$

which, by induction, is in $\mathrm{Map}(\tilde\Gamma, \mathbb{C})^{\tilde\Gamma,q-1}$. Since, from (8.25), it follows that the right hand side of (8.26) with φ replaced by ψ equals $f|(\alpha_j - 1)$ too, we deduce that $f|(\alpha_j - 1) \in \mathrm{Map}(\tilde\Gamma, \mathbb{C})^{\tilde\Gamma,q-1}$.

In the same way, we deduce that $f|(\alpha_j - 1) \in \mathrm{Map}(\tilde\Gamma, \mathbb{C})^{\tilde\Gamma,q-1}$ when $j = t(\Gamma)$ or $j < t(\Gamma)$ and $j \neq \mathbf{j}(1)$. □

Proposition 8.14. *The $\tilde\Gamma$-modules $\mathcal{D}_k(\lambda)$ and $\mathcal{D}^{\mathrm{hol}}_k$ are maximally perturbable for all $k \in 2\mathbb{Z}$ and $\lambda \in \mathbb{C}$.*

Proof. It suffices to construct for a given $f \in \mathcal{D}_k(\lambda)^{\tilde\Gamma}$, a given $q \in \mathbb{N}$ and a given $\tilde\Gamma$-q-tuple $\mathbf{i}$, an element $\eta_{\mathbf{i}} \in \mathcal{D}_k(\lambda)$ such that $\eta_{\mathbf{i}}|(\alpha_{\mathbf{i}'(1)} - 1) \cdots (\alpha_{\mathbf{i}'(q)} - 1) = \delta_{\mathbf{i},\mathbf{i}'} f$ for all $\tilde\Gamma$-q-tuples $\mathbf{j}$.

We will write $f = f_{\mathrm{cpt}} + \sum_\kappa f_\kappa$, with κ running over a set C of representatives of the $\tilde\Gamma$-orbits of cusps, where $f_{\mathrm{cpt}} \in (C_k)^{\tilde\Gamma}$, $f_\kappa \in \mathcal{D}_k(\lambda)^{\tilde\Gamma}$, and will produce perturbations for each of these components.

We choose a strict fundamental domain $\mathfrak{F}_{\tilde\Gamma}$ for $\tilde\Gamma\backslash\tilde G$ so that

$$\mathfrak{F}_{\tilde\Gamma} \cap D_\infty(b) \;=\; \{(x + iy, \vartheta) \;:\; 0 \leq x < 1\,,\ y \geq b\,,\ 0 \leq \vartheta < \pi\}\,.$$

Definition 8.9 provides $b \geq A_\Gamma$ and $r \in \mathbb{N}$ such that $v_\kappa(z, \vartheta) = f(\tilde g_\kappa(z, \vartheta))$ is in $\mathcal{E}_k(b, \lambda)^{\tilde\Delta, r}$ for each cusp κ. Furthermore, b can be chosen large enough for the sets $\mathfrak{F}_{\tilde\Gamma} \cap D_\kappa(b)$ ($\kappa \in C$) to be pairwise disjoint. Since f is $\tilde\Gamma$-invariant, we even have $v_\kappa \in \mathcal{E}_k(b, \lambda)^{\tilde\Delta}$. We choose a function $\chi \in C^\infty(0, \infty)$ that is equal to 0 on $(0, b + \frac{1}{2}]$ and equal to 1 on $[b + 1, \infty)$, and define for $\kappa \in C$

$$f_\kappa(z, \vartheta) \;=\; \begin{cases} 0 & (z, \vartheta) \in \mathfrak{F}_{\tilde\Gamma} - D_\kappa(b) \\ \chi(\mathrm{Im}(z_1))\, v_\kappa(z_1, \vartheta_1) & (z, \vartheta) = \tilde g_\kappa(z_1, \vartheta_1) \in \mathfrak{F}_{\tilde\Gamma} \cap D_\kappa(b) \end{cases} \tag{8.27}$$

Extend to $\tilde G$ by $\tilde\Gamma$-linearity. So $f_\kappa = 0$ outside $\tilde\Gamma D_\kappa(b)$ and equal to f on $\tilde\Gamma D_\kappa(b+1)$. We check in Definition 8.9 that $f_\kappa \in \mathcal{D}_\kappa(\lambda)$. The function

$$f_{\mathrm{cpt}} = f - \sum_{\kappa \in C} f_\kappa$$

is $\tilde\Gamma$-invariant and vanishes on $D_\kappa(b+1)$ for all cusps κ, hence $f_{\mathrm{cpt}} \in C_k^{\tilde\Gamma}$.

Proposition 8.6 implies that there is $h_{\mathbf{i}} \in C_k \subset \mathcal{D}_k(\lambda)$ satisfying the conditions $h_{\mathbf{i}}|(\alpha_{\mathbf{i}(1)}-1)\cdots(\alpha_{\mathbf{i}(q)}-1) = f_{\mathrm{cpt}}$ and $h_{\mathbf{i}'}|(\alpha_{\mathbf{i}'(1)}-1)\cdots(\alpha_{\mathbf{i}'(q)}-1) = 0$ for $\tilde\Gamma$-q-tuples $\mathbf{i}' \neq \mathbf{i}$. So we can restrict our attention to the f_κ.

Since the supports of the f_κ with $\kappa \in C$ are disjoint, we can consider each of the f_κ separately. Without loss of generality we can assume that ∞ is a cusp of $\tilde\Gamma$ with $\tilde g_\kappa = 1$ and take $\infty \in C$. Conjugation by the original $\tilde g_\kappa$ then gives the same result for a general $\kappa \in C$.

The function v_∞ used in (8.27) is an element of $\mathcal{E}_k(b,\lambda)^{\tilde\Delta}$. The proof of Proposition 8.8 shows that for each $\mathbf{m} \in \mathbb{N}_0^2$ there is a perturbation $v_\infty^{\mathbf{m}} \in \mathcal{E}_k(b,\lambda)^{\tilde\Delta, m_1+m_2+1}$ of $(z,\vartheta) \mapsto f_\infty(z,\vartheta)$ of type $\mathbf{m}$. We define $\eta_{\mathbf{i}}$ by $\eta_{\mathbf{i}} = 0$ on $\tilde G_b$ and on all $\tilde\Gamma D_\kappa(b)$ for all $\kappa \in C \setminus \{\infty\}$, and

$$\eta_{\mathbf{i}}(\rho(x+iy,\vartheta)) \;=\; \sum_{l,m\geq 0\,,\; l+m\leq q} \chi(y)\, a_{l,m}^{\mathbf{i}}(\rho)\, v_\infty^{(l,m)}(x+iy,\vartheta) \tag{8.28}$$

for $y \geq b$ and ρ in a system of representatives R of $\tilde\Gamma/\tilde\Delta$. The functions $a_{l,m}^{\mathbf{i}}$ are as in (8.23). Since the sets $\rho D_\infty(b)$ are disjoint, this defines a smooth function, which can be checked to be an element of $\mathcal{D}_k(\lambda)$.

For each fixed $g = (x+iy,\vartheta)$ with $y \geq b$ the function $\delta \mapsto v_\infty^{(l,m)}(\delta g)$ on $\tilde\Delta$ satisfies the same relations as $\delta \mapsto \varphi^{(l,m)}(\delta)\, v_\infty(g)$ in (8.22). So, their difference, as a function of δ, satisfies (8.24).

Ignoring smoothness for a moment we have $f_\infty \in \mathrm{Map}(\tilde G,\mathbb{C})^{\tilde\Gamma}$. Equation (8.8) gives a function $h_{\mathbf{i}}$ on $\tilde G$ such that $h_{\mathbf{i}}|(\alpha_{\mathbf{i}'(1)}-1)\cdots(\alpha_{\mathbf{i}'(q)}-1) = \delta_{\mathbf{i},\mathbf{i}'}\, f_\infty$ for all $\tilde\Gamma$-q-tuples $\mathbf{i}'$. With our choice of fundamental domain, and using (8.23), we find for $\rho \in R$, $\delta \in \tilde\Delta$ and $g = (x+iy,\vartheta)$ with $y \geq b$:

$$h_{\mathbf{i}}(\rho\delta g) \;=\; \sum_{l,m\geq 0\,,\; l+m\leq q} a_{l,m}^{\mathbf{i}}(\rho)\, \varphi^{(l,m)}(\delta)\, \chi(y)\, v_\infty(g)\,. \tag{8.29}$$

Outside $\tilde\Gamma D_\infty(b)$ the functions f_∞, $h_{\mathbf{i}}$ are zero. With Lem. 8.13 we conclude that the function induced by

$$(\eta_{\mathbf{i}} - h_{\mathbf{i}})(\rho\delta g) = \sum_{l,m\geq 0\,,\; l+m\leq q} a_{l,m}^{\mathbf{i}}(\rho)\chi(y)(v_\infty^{(l,m)}(\delta g) - \varphi^{(l,m)}(\delta)\, v_\infty(g)) \tag{8.30}$$

is in $\mathrm{Map}(\tilde G,\mathbb{C})^{\tilde\Gamma,q}$. This implies that

$$\eta_{\mathbf{i}} \in (h_{\mathbf{i}} + \mathrm{Map}(\tilde G,\mathbb{C})^{\tilde\Gamma,q}) \cap \mathcal{D}_k(\lambda) = \mathcal{D}_k(\lambda)^{\tilde\Gamma,q+1}\,,$$

and behaves in the desired way under $(\alpha_{\mathbf{i}'(1)}-1)\cdots(\alpha_{\mathbf{i}'(q)}-1)$ for all $\tilde\Gamma$-q-tuples $\mathbf{i}'$. Thus, we have proved that $\mathcal{D}_k(\lambda)$ is maximally perturbable.

Everywhere in this proof we can replace $\mathcal{E}_k(b,\lambda)$ by $\mathcal{E}_k^{\mathrm{hol}}(b)$, and $\mathcal{D}_k(\lambda)$ by $\mathcal{D}_k^{\mathrm{hol}}$. In that way we also obtain the maximal perturbability of $\mathcal{D}_k^{\mathrm{hol}}$, thus completing the proof of Proposition 8.14. □

8.4.3. *Relations between the spaces C_k and $\mathcal{D}_k(\lambda)$.* By Remark 8.11, for each $f \in \mathcal{D}_k(\lambda)$ the support of $(\omega-\lambda)f$ is contained in some set $\tilde G_b$, hence $(\omega-\lambda)f \in C_k$. So the differential operator $\omega-\lambda$ maps $\mathcal{D}_k(\lambda)$ to C_k. Since the operator ω commutes with the action of $\tilde\Gamma$, we have $(\omega-\lambda)\mathcal{D}_k(\lambda)^{\tilde\Gamma,q} \subset C_k^{\tilde\Gamma,q}$ for all $q \geq 1$. Similarly, $\mathbf{E}^-(\mathcal{D}_k^{\mathrm{hol}})^{\tilde\Gamma,q} \subset C_{k-2}^{\tilde\Gamma,q}$ for all $q \geq 1$.

Proposition 8.15. *Let $\lambda \in \mathbb{C}$ and $k \in 2\mathbb{Z}$. The following maps are surjective:*
i. $\omega-\lambda : \mathcal{D}_k(\lambda)^{\tilde\Gamma} \to C_k^{\tilde\Gamma}$ *and*
ii. $\mathbf{E}^- : (\mathcal{D}_k^{\mathrm{hol}})^{\tilde\Gamma} \to C_{k-2}^{\tilde\Gamma}$

Proof. §8.4.4 and §8.4.5. □

Corollary 8.16. *For each $q \geq 1$ the maps $\omega-\lambda :\ \mathcal{D}_k(\lambda_s)^{\tilde\Gamma,q} \longrightarrow C_k^{\tilde\Gamma,q}$ and $\mathbf{E}^- : (\mathcal{D}_k^{\mathrm{hol}})^{\tilde\Gamma,q} \to C_k^{\tilde\Gamma,q}$ are surjective.*

Proof. Proposition 8.15 gives the case $q = 1$. The rows in the following commutative diagram are exact by Proposition 8.6 and 8.14. See (5.13) for m_q.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}_k(\lambda)^{\tilde\Gamma,q} & \longrightarrow & \mathcal{D}_k(\lambda)^{\tilde\Gamma,q+1} & \xrightarrow{\mathsf{m}_q} & (\mathcal{D}_k(\lambda)^{\tilde\Gamma})^{n(\tilde\Gamma,q)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \omega-\lambda} & & \downarrow{\scriptstyle \omega-\lambda} & & \downarrow{\scriptstyle \omega-\lambda} & & \\ 0 & \longrightarrow & C_k^{\tilde\Gamma,q} & \longrightarrow & C_k^{\tilde\Gamma,q+1} & \xrightarrow{\mathsf{m}_q} & (C_k^{\tilde\Gamma})^{n(\tilde\Gamma,q)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & \operatorname{coker}(\omega-\lambda) & & 0 & & \end{array} \tag{8.31}$$

The third column is exact by Proposition 8.15. With the exactness of the first column as induction hypothesis, we obtain the vanishing of $\operatorname{coker}(\omega - \lambda)$ and thus the surjectivity of $\omega - \lambda : \mathcal{D}_k(\lambda)^{\tilde\Gamma,q+1} \to C_k^{\tilde\Gamma,q+1}$ by the Snake Lemma.

The case of $\mathbf{E}^- : (\mathcal{D}_k^{\mathrm{hol}})^{\tilde\Gamma,q} \to C_k^{\tilde\Gamma,q}$ is similar. □

8.4.4. *Proof of Proposition 8.15(i).* We first note that the spaces $\mathcal{D}_k(\lambda)^{\tilde\Gamma}$ and $C_k^{\tilde\Gamma}$ are invariant under $\tilde Z$. Hence the weight k is strict and we are dealing with functions on $G = \mathrm{PSL}_2(\mathbb{R})$. (See the first statement in Theorem 6.4.) We use the spectral theory of automorphic forms to prove Proposition 8.15.

We work with the space of square integrable functions on $\tilde\Gamma\backslash\tilde G = \Gamma\backslash G$ of strict weight $k \in 2\mathbb{Z}$, where $G = \mathrm{PSL}_2(\mathbb{R})$. We can view the elements of the Hilbert space $H_k = L^2(\tilde\Gamma\backslash\tilde G)_k = L^2(\Gamma\backslash G)_k$ as functions $z \mapsto f(z, 0)$ on $\mathfrak{H}$, transforming according to weight k as indicated in (4.7). The inner product in H_k is given by

$$(f, f_1) \;=\; \int_{\mathfrak{F}} f(z,0)\,\overline{f_1(z,0)}\,\frac{dx\,dy}{y^2}\,.$$

Here $\mathfrak{F}$ can be any fundamental domain for $\Gamma\backslash\mathfrak{H}$. We take it so that for each $b > A_\Gamma$ it has a decomposition

$$\mathfrak{F} \;=\; \mathfrak{F}_b \sqcup \bigsqcup_{\kappa\in C} V_\kappa\,, \quad V_\kappa \;=\; \{g_\kappa(x+iy) \;:\; x_\kappa \le x \le x_\kappa + 1\,,\; y \ge b\}\,, \tag{8.32}$$

with C a system of representatives of the Γ-orbits of cusps, and $x_\kappa \in \mathbb{R}$ depending on $\mathfrak{F}$ and on the earlier choice of the g_κ. The set $\mathfrak{F}_b$ has compact closure in $\mathfrak{H}$.

The differential operator $\omega_k = -y^2\partial_y^2 - y^2\partial_x^2 + iky\partial_x$ in (4.8) determines a densely defined self-adjoint operator A_k in H_k. The spectral theory of automorphic forms gives the decomposition of this operator A_k in terms of Maass forms. One may consult Chapters 4 and 7 in [12] for weight 0. For other weights the proofs are almost completely similar. (See [20].) There is a subspace H_k^{discr} with an at most countable orthonormal basis $\{\psi_k^\ell\}$ of Maass forms, indexed by some subset of $\mathbb{Z}$. The ψ_k^ℓ are square integrable elements of the space of Maass forms $E_k(\Gamma, \lambda^\ell)$ with $\lambda^\ell \ge \frac{k}{2}(1 - \frac{k}{2})$. We denote the eigenspace associated to λ (which is known to be finite-dimensional) by $H_k(\lambda)$. If $k = 0$ the eigenvalue 0 occurs with multiplicity one, corresponding to constant functions, and all other λ^ℓ, if any, are positive. If $k \ne 0$, then H_k^{discr} may be zero. If $k \ge 2$ and the space $S_k(\Gamma)$ of holomorphic cusp forms of weight k is non-zero, then there are $\psi_k^\ell \in H_k^{\mathrm{discr}}$ of the form $\psi_k^\ell(z, 0) = y^{k/2}\,h(z)$ with $h \in S_k(\Gamma)$. The corresponding eigenvalues are $\lambda^\ell = \frac{k}{2}(1 - \frac{k}{2})$, which is negative if $k \ge 4$. There may also be elements obtained by differentiation of holomorphic cusp forms of weights between 2 and $k - 2$. Similarly, for negative k there may be eigenfunctions corresponding to antiholomorphic cusp forms.

The orthogonal complement H_k^{cont} of H_k^{discr} in H_k is isomorphic to a sum of n_{par} copies of $L^2((0,\infty),dt)$, where n_{par} is the number of Γ-orbits of cusps. The spectral decomposition gives the *Parseval formula*

$$(f,f_1) \;=\; \sum_{\ell} a_k^{\ell}(f)\,\overline{a_k^{\ell}(f_1)} + \sum_{\kappa}\frac{1}{2\pi}\int_0^{\infty} e_k^{\kappa}(f;it)\,\overline{e_k^{\kappa}(f_1;it)}\,dt\,, \tag{8.33}$$

with κ running through a set of representatives of the cuspidal orbit. For each $f\in H_k$ we have $a_k^{\ell}(f)=(f,\psi_k^{\ell})$. If f is sufficiently regular, then the functions $e_k^{\kappa}(f;\cdot)$ are obtained by integration against the Eisenstein series $E_k^{\kappa}(it)$ at the cusp κ.

The space $C_k^{\tilde\Gamma}$ is contained in H_k. For $f\in C_k^{\tilde\Gamma}$ the functions $e_k^{\kappa}(f;\cdot)$ are given by

$$e_k^{\kappa}(f;s) \;=\; \int_{\mathfrak{F}} f(z,0)\,\overline{E_k^{\kappa}(-\bar s;z)}\,\frac{dx\,dy}{y^2} \;=\; \int_{\mathfrak{F}} f(z,0)\,E_{-k}^{\kappa}(-s;z)\,\frac{dx\,dy}{y^2}\,,$$

for all s at which the meromorphic continuation of the Eisenstein series

$$E_k^{\kappa}(s;z) := \sum_{\gamma\in\Gamma_\kappa\backslash\Gamma} \mathrm{Im}\,(g_\kappa^{-1}\gamma z)^{\frac12+s}\,e^{-ik\arg(j(g_\kappa^{-1}\gamma,z))}$$

is holomorphic. In particular, $e_k^{\kappa}(f;s)$ is holomorphic at points of the line $i\mathbb{R}$.

On the square integrable Maass forms and on the Eisenstein series the self-adjoint operator A_k is given by ω_k in (4.8). For $f\in H_k$ in the domain of A_k, the self-adjointness of A_k together with the eigenproperty of ψ_k^l imply $a_k^{\ell}(\omega_k f)=\lambda^{\ell}a_k^{\ell}(f)$ and $e_k^{\kappa}(\omega_k f;t)=(\frac14+t^2)\,e_k^{\kappa}(f;t)$. This implies that the spectral data of elements $f\in H_k$ such that $A_k^n f$ is well defined for all $n\in\mathbb{N}$, are quickly decreasing. The convergence in L^2-sense of the Parseval formula in (8.33) is very fast for functions of this type, since the summands and integrands in the expansion are those of $(A_k^n f, A_k^n f_1)$ divided by $(\lambda^{\ell})^n$, respectively $(\frac14+t^2)^n$ for each $n\in\mathbb{N}$. (If there is a term with $\lambda^{\ell}=0$ we treat it separately; it does not influence the convergence.)

The central point of the proof of Proposition 8.15 is that we transform the equation $(A_k-\lambda)f_1=f$ with unknown $f_1\in H_k$ for a given $f\in C_k^{\tilde\Gamma}$ to the spectral decomposition. Application of $A_k-\lambda$ to $f\in C_k^{\tilde\Gamma}$ amounts to multiplying $a_k^{\ell}(f)$ by $\lambda^{\ell}-\lambda$ and multiplying $e_k^{\kappa}(f;t)$ by $\frac14+t^2-\lambda$. This suggests the following

Definition 8.17. Let $\lambda\in\mathbb{C}$. We denote by $C_k(\Gamma,\lambda)$ the space of $f\in C_k^{\tilde\Gamma}$ such that the following conditions are satisfied.

i) $a_k^{\ell}(f)=0$ if $\lambda_\ell=\lambda$,
ii) $e_k^{\kappa}(f;it_\lambda)=0$ for all κ, if $\lambda=\frac14+t_\lambda^2$ $(t_\lambda\in\mathbb{R}\setminus\{0\})$,
iii) for all κ, the map $t\mapsto e_k^{\kappa}(f;it)$ has a double zero at $t=0$ if $\lambda=\frac14$.

Note that, for each λ, the conditions i), ii), iii) impose finitely many linear conditions, so $C_k(\Gamma,\lambda)$ has finite codimension in $C_k^{\tilde\Gamma}$. If λ is not in the spectrum of A_k, then $C_k(\Gamma,\lambda)$ is equal to $C_k^{\tilde\Gamma}$.

Case I: $f\in C_k(\tilde\Gamma,\lambda)$. In this case, if we have the spectral decomposition

$$f \;=\; \sum_{\ell} a_k^{\ell}(f)\psi_k^{\ell} + \sum_{\kappa}\frac{1}{2\pi}\int_0^{\infty} e_k^{\kappa}(f;it)\overline{E_k^{\kappa}(it;-)}dt, \tag{8.34}$$

then a solution of $(A_k-\lambda)f_1=f$ is given by

$$f_1 := \sum_{\ell}\frac{a_k^{\ell}(f)}{\lambda^{\ell}-\lambda}\psi_k^{\ell} + \sum_{\kappa}\frac{1}{2\pi}\int_0^{\infty}\frac{e_k^{\kappa}(f;it)}{\frac14+t^2-\lambda}\overline{E_k^{\kappa}(it;-)}dt. \tag{8.35}$$

If λ is not in the spectrum of A_k, then the convergence of this L^2-expansion is better than that in (8.34), since $\lambda^\ell \to \infty$ and hence the denominators improve the convergence. If λ is in the spectrum of A_k, condition i) ensures that the $a_k^\ell(f)$ with $\lambda^\ell = \lambda$ vanish, and that, by the other conditions, the simple or double zero of $t \mapsto \frac{1}{4} + t^2 - \lambda$ at $t = t_\lambda$ is canceled by the zeros at $s = it_\lambda$ of the holomorphic functions $s \mapsto e_\kappa^\kappa(f; s)$. The same reasoning shows that the obtained f_1 is in H_k and, in fact, in the domain of A_k. We have $(A_k - \lambda) f_1 = f$. Therefore the relation $(\omega - \lambda) f_1 = f$ holds in distribution sense. There is $b \geq A_\Gamma$ such that the support of f is contained in $\tilde{G}_b$. So on each $D_\kappa(b)$ we have $(\omega - \lambda) f_1 = 0$. Since ω_k determines an elliptic differential operator on $\mathfrak{H}$, elliptic regularity implies that $(\omega - \lambda) f_1 = 0$ holds as a relation for real-analytic functions on each $D_\kappa(b)$. Further, the square integrability implies that f_1 must have less than exponential growth at the cusps and hence it is an element of $\mathcal{D}_k(\lambda)^{\tilde{\Gamma}}$. We have shown:

Lemma 8.18. *For each $\lambda \in \mathbb{C}$, the space $C_k(\Gamma, \lambda)$ is contained in $(\omega - \lambda)\, \mathcal{D}_k(\lambda)^{\tilde{\Gamma}}$.*

Case II: $f \in C_k^{\tilde{\Gamma}} \smallsetminus C_k(\Gamma, \lambda)$ for λ in the spectrum of A_k. The following result enables us to pick representatives h of $C_k^{\tilde{\Gamma}}/C_k(\Gamma, \lambda)$ for which we can solve $(\omega - \lambda) f_1 = h$ directly. This procedure can be carried out by singling out one cusp κ, which we fix for the proof of Case II.

Lemma 8.19. *Let κ be the cusp that we keep fixed. Suppose that λ is in the spectrum of A_k. Then there is a finite set $X \subset \mathbb{Z}$ such that, for each $n \in X$, there exist $h_n \in C_k^{\tilde{\Gamma}}$ of the form*

$$h_n(\gamma \tilde{g}_\kappa(z, \vartheta)) = \begin{cases} e^{2\pi i n x} \chi_n(y)\, e^{ik\vartheta} & \text{on } \tilde{\Gamma} D_\kappa(A_\Gamma) \\ 0 & \text{elsewhere} \end{cases} \tag{8.36}$$

for some $\chi_n \in C_c^\infty(A_\Gamma, \infty)$, such that $\{h_n + C_k(\Gamma, \lambda)\}_n$ spans $C_k^{\tilde{\Gamma}}/C_k(\Gamma, \lambda)$.

If we can solve $(A_k - \lambda) f_1 = h_n$ in another way for all $n \in X$, this lemma enables us to reduce the proof of Proposition 8.15(i) to Lemma 8.18.

Proof of Lemma 8.19. We shall examine each of the three cases for the eigenvalues of A_k on H_k separately:

- $\lambda = \frac{1}{4} - s^2 \notin [\frac{1}{4}, \infty)$. Assume $s > 0$. There are finitely many indices $\ell_1, \ldots, \ell_m$ such that $\lambda_{\ell_j} = \lambda$. The $\psi_k^{\ell_j}$ form a basis of $\ker(A_k - \lambda)$. Each of these m linearly independent square integrable automorphic forms is given by its Fourier expansion at the fixed cusp κ. By Proposition 7.1, the Fourier terms of non-zero order are multiples of $\omega_k(n, s)$. The Fourier term of order zero is a multiple of $y^{\frac{1}{2}-s} e^{ik\vartheta}$. We choose a set X of m elements in $\mathbb{Z}$ such that the $m \times m$-matrix whose columns are the n-th Fourier coefficients of $\psi_k^{\ell_j}$ $(1 \leq j \leq m)$ with $n \in X$ is invertible. We choose the $\chi_n \in C_c^\infty$, $n \in X$, in the statement of the lemma, so that $\int_{A_\Gamma}^\infty \chi_n(y)\, \overline{\omega_k(n, s)(iy, 0)}\, \frac{dy}{y^2} \neq 0$, respectively $\int_{A_\Gamma}^\infty \chi_n(y)\, \overline{y^{\frac{1}{2}-s}}\, \frac{dy}{y^2} \neq 0$. Consider the linear form on the space $A_k^2(\lambda)$ of square integrable automorphic forms with eigenvalue λ given by

$$\begin{aligned} \psi \mapsto (h_n, \psi) &= \int_{\mathfrak{F}} h_n(z, 0) \overline{\psi(z, 0)} \frac{dxdy}{y^2} \\ &= \int_{A_\Gamma}^\infty \int_{-1/2}^{1/2} \chi_n(y) e^{2\pi i n x} \bar{a}_0 y^{1/2 - \bar{s}} \frac{dxdy}{y^2} + \sum_{m \neq 0} \bar{a}_m \int_{A_\Gamma}^\infty \int_{-1/2}^{1/2} \chi_n(y) e^{2\pi i n x} \overline{\omega_k(m, s)(iy, 0)} \frac{dxdy}{y^2}. \end{aligned}$$

  This depends only on the Fourier coefficient of ψ of order n in the expansion at κ. Therefore, the $m \times m$-matrix with the scalar product $(h_n, \psi_k^{\ell_j})$ at position (j, n) is invertible. (Here j runs from 1

to m, and n runs through X.) Hence there are complex numbers $b_{j,p}$ (with $1 \le j \le m$, $p \in X$) such that $\sum_{n\in X} b_{j,n}\,(h_n, \psi_k^{\ell_{j'}}) = \delta_{j,j'}$. Setting, for $f \in C_k^{\tilde\Gamma}$, $c_n(f) = \sum_{j'=1}^m (f, \psi_k^{\ell_{j'}})\, b_{j',n}$, we obtain for $1 \le j \le m$:

$$\sum_{n\in X} c_n(f)\,(h_n, \psi^{\ell_j}) \;=\; (f, \psi_k^{\ell_j})\,.$$

So $f - \sum_n c_n(f)\,h_n$ is indeed in $C_k(\Gamma, \lambda)$.

- $\lambda = \frac14 + t^2$, $t \in \mathbb{R} \setminus \{0\}$. A basis of $\ker(A_k - \lambda)$ in this case consists of Eisenstein series $E_k^\nu(it, \cdot)$ ($\nu \in C$) and possibly cusp forms $\psi_k^{\ell_j}$ with $\lambda_{\ell_j} = \lambda$. The proof of the previous case can be applied with the obvious adjustments (e.g. replacing scalar products by integrals for the terms corresponding to E_k^ν) to give the result. The only essential modification is that we have to use the space $A_k^*(\lambda)$ of automorphic forms with polynomial growth and eigenvalue λ in place of $A_k^2(\lambda)$ because the Eisenstein series are not square integrable. This can be done because (conjugates of) elements of $A_k^*(\lambda)$ appear only integrated against elements of $C_k^{\tilde\Gamma}$ which have compact support modulo $\tilde\Gamma$.
- $\lambda = \frac14$. Now we have the condition that $e_k^\kappa(f - \sum_n h_n; it)$ should have a double zero at $t = 0$ or, equivalently, that the first two terms of the Taylor expansion at $s = 0$ should vanish. Since the first two Taylor terms of $E_k^\kappa(-; z)$ are linearly independent from the other functions in $A_k^*(1/4)$, a choice of χ_n with the desired properties is again possible. □

Now we turn to the task to solve $(\omega - \lambda_s) f_1 = h_n$ with $f_1 \in \mathcal{D}_k(\lambda)^{\tilde\Gamma}$ for h_n as in Lemma 8.19. We aim at f_1 with support in $\tilde\Gamma D_\kappa(A_\Gamma)$. Writing $f_1(\tilde g_\kappa(z, \vartheta)) = e^{2\pi i n x}\, h(y)\, e^{ik\vartheta}$, the differential equation $(\omega - \lambda) f_1 = h_n$ becomes

$$-y^2 h''(y) + (4\pi^2 n^2 y^2 - 2\pi n k y - \frac14 + s^2)\, h(y) \;=\; \chi_n(y)\,.$$

(Compare (7.3).) This ordinary differential equation is regular on $y \ge A_\Gamma$. It has a unique solution for the initial conditions $h(A_\Gamma) = h'(A_\Gamma) = 0$. It is zero below the support of χ_n. Since χ_n has compact support, the function h thus obtained is a solution of the homogeneous equation (7.3) on (b, ∞) for some $b > A_\Gamma$ depending on $\mathrm{Supp}(\chi_n)$. Thus we see that $(z, \vartheta) \mapsto f_1(\tilde g_\kappa(z, \vartheta))$ is an element of $\mathcal{W}_k(n, s)$. Hence it may have exponential growth of order $e^{(2\pi|n|+\delta)y}$. This is the point where the need to work with exponentially growing functions arises.

We extend f_1 by $\tilde\Gamma$-invariance, and check that it is an element of $\mathcal{D}_k(\lambda_s)$. This completes the proof of the first statement in Proposition 8.15.

8.4.5. *Proof of Proposition 8.15 ii.* For the surjectivity of $\mathbf{E}^- : (\mathcal{D}_k^{\mathrm{hol}})^{\tilde\Gamma} \to C_{k-2}^{\tilde\Gamma}$ we first note that, on an eigenfunction of ω in weight $k-2$ with eigenvalue λ the operator $\mathbf{E}_k^- \mathbf{E}_{k-2}^+$ acts as multiplication by $-4(\lambda - \frac{k}{2} + \frac{k^2}{4})$. See (5.7). We will use $\mathbf{E}_{k-2}^+$ to "invert" $\mathbf{E}_k^-$.

Let H_{k-2}^a denote the kernel of $\mathbf{E}_{k-2}^+$ in H_{k-2}^{discr}. It is finitely dimensional and it contains the constant functions if $k = 2$, and the functions corresponding to antiholomorphic cusp forms if $k \le 0$.

On the orthogonal complement of H_{k-2}^a in H_{k-2}^{discr} the factor $-4(\lambda - \frac{k}{2} + \frac{k^2}{4})$ is negative and stays away from 0 for all λ in the spectrum of A_k. Likewise, we denote by H_k^h the finite dimensional kernel of $\mathbf{E}_k^-$ in H_k^{discr}. Its elements correspond to square integrable holomorphic automorphic forms of weight k.

Let $(\psi_{k-2}^\ell)_\ell$ be an orthonormal basis of the orthogonal complement $H_{k-2}^{\mathrm{discr}} \ominus H_{k-2}^a$ consisting of eigenfunctions of ω_{k-2} with eigenvalue λ^ℓ. The relation $(\mathbf{E}_k^- v_1, v_2) = -(v_1, \mathbf{E}_{k-2}^+ v_2)$ for suitably differentiable

elements of H_k and H_{k-2} (see Lemma 6.1.4 of [2]) implies that $\mathbf{E}_k^-(H_k^{\mathrm{discr}} \ominus H_k^h) \subset H_{k-2}^{\mathrm{discr}} \ominus H_{k-2}^a$ and hence $(\psi_k^\ell)_\ell$ with $\psi_k^\ell = \frac{1}{\sqrt{4\lambda^\ell - 2k + k^2}} \mathbf{E}_{k-2}^+ \psi_{k-2}^\ell$ is an orthonormal system spanning $H_k^{\mathrm{discr}} \ominus H_k^h$.

For a given $f \in C_{k-2}^{\tilde\Gamma}$ orthogonal to H_{k-2}^a we set

$$f_1 \quad := \quad -\sum_\ell \frac{a_{k-2}^\ell(f)}{\sqrt{4\lambda^\ell - 2k + k^2}} \frac{\mathbf{E}_{k-2}^+ \psi_{k-2}^\ell}{\sqrt{4\lambda^\ell - 2k + k^2}} \quad - \quad \sum_\kappa \frac{1}{2\pi} \int_0^\infty \frac{e_{k-2}^\kappa(f; it)}{\sqrt{4t^2 + (k-1)^2}} \frac{\mathbf{E}_{k-2}^+ E_{k-2}^\kappa(it; -)}{\sqrt{4\lambda^\ell - 2k + k^2}} dt\,.$$

We have $f_1 \in H_k \ominus H_k^h$ and $\mathbf{E}^- f_1 = f$. A reasoning as in the previous case shows that $f_1 \in \mathcal{D}_k^{\mathrm{hol}}(\lambda)^{\tilde\Gamma}$.

So we have solved the problem for a subspace of $C_{k-2}^{\tilde\Gamma}$ with finite codimension. A general element of $C_{k-2}^{\tilde\Gamma}$will not be orthogonal to H_{k-2}^a. We proceed as in the first case in the proof of Lemma 8.18. Instead of $\psi_k^{\ell_j}$ we now use an orthogonal basis of H_{k-2}^a, and form functions h_n as in Lemma 8.18, corresponding to a set X of Fourier term orders such that elements of H_{k-2}^a are determined by the Fourier coefficients in X. Solving $\mathbf{E}_k^- f_1 = h_n$ leads to the differential equation

$$(-2iy\partial_x + 2y\partial_y - k)e^{2\pi i n x}\varphi(y) \;=\; \chi(y)\,, \quad \varphi(y_0) \;=\; \varphi'(y_0) = 0\,,$$

with which we proceed as in the previous case.

This establishes the surjectivity of $\mathbf{E}^- : (\mathcal{D}_k^{\mathrm{hol}})^{\tilde\Gamma} \to C_{k-2}^{\tilde\Gamma}$ in Proposition 8.15.

8.5. **Higher order invariants and Maass forms.** We now will derive the main results of this paper, Theorems 6.5 and 6.8, from the following result:

Proposition 8.20. *The $\tilde\Gamma$-modules*

$$\tilde{\mathcal{E}}_k'(\lambda) := \ker(\omega - \lambda : \mathcal{D}_k(\lambda) \longrightarrow C_k) \tag{8.37}$$

and

$$\mathcal{H}_k' := \ker(\mathbf{E}^- : \mathcal{D}_k^{\mathrm{hol}} \longrightarrow C_{k-2}) \tag{8.38}$$

are maximally perturbable.

Proof. We have the following extension of the commutative diagram (8.31):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathcal{E}}_k'(\lambda)^{\tilde\Gamma,q} & \longrightarrow & \tilde{\mathcal{E}}_k'(\lambda)^{\tilde\Gamma,q+1} & \xrightarrow{\mathsf{m}_q} & (\tilde{\mathcal{E}}_k'(\lambda)^{\tilde\Gamma})^{n(\tilde\Gamma,q)} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{D}_k(\lambda)^{\tilde\Gamma,q} & \longrightarrow & \mathcal{D}_k(\lambda)^{\tilde\Gamma,q+1} & \xrightarrow{\mathsf{m}_q} & (\mathcal{D}_k(\lambda)^{\tilde\Gamma})^{n(\tilde\Gamma,q)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \omega-\lambda} & & \downarrow{\scriptstyle \omega-\lambda} & & \downarrow{\scriptstyle \omega-\lambda} & & \\
0 & \longrightarrow & C_k^{\tilde\Gamma,q} & \longrightarrow & C_k^{\tilde\Gamma,q+1} & \xrightarrow{\mathsf{m}_q} & (C_k^{\tilde\Gamma})^{n(\tilde\Gamma,q)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{8.39}$$

The exactness of the columns follows from the definition of $\tilde{\mathcal{E}}_k'(\lambda)$, (3.2), the left-exactness of the functor $\hom_{\mathbb{C}[\Gamma]}(I^q\backslash\mathbb{C}[\Gamma], -)$ and Corol. 8.16. Propositions 8.3 and 8.14 imply that the second and third row are exact. The Snake Lemma then implies that the first row is exact and that $\mathsf{m}_q : \tilde{\mathcal{E}}_k'(\lambda)^{\tilde\Gamma,q+1} \to (\tilde{\mathcal{E}}_k'(\lambda)^{\tilde\Gamma})^{n(\tilde\Gamma,q)}$ is surjective.

Replacing in this diagram the space $\tilde{\mathcal{E}}_l'(\lambda)$ by $\mathcal{H}_k'$ and the map $\omega - \lambda$ by $\mathbf{E}^-$, we obtain the maximal perturbability of $\mathcal{H}_k'$. □

Proof of Theorems 6.5 and 6.8. The $\tilde\Gamma$-module $\tilde{\mathcal{E}}'_k(\lambda)$ is contained in $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)$. See Definition 6.3. It is a smaller space than $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)$ since elements of $\mathcal{D}_k(\lambda)$ have a special structure near the cusps. With (8.21), $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde\Gamma}$ is a subspace of $\tilde{\mathcal{E}}'_k(\lambda)^{\tilde\Gamma}$. Therefore $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde\Gamma}=\tilde{\mathcal{E}}'_k(\lambda)^{\tilde\Gamma}$ and thus

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathcal{E}}'_k(\lambda)^{\tilde\Gamma,q} & \longrightarrow & \tilde{\mathcal{E}}'_k(\lambda)^{\tilde\Gamma,q+1} & \longrightarrow & (\tilde{\mathcal{E}}'_k(\lambda)^{\tilde\Gamma})^{n(\tilde\Gamma,n)} & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde\Gamma,q} & \longrightarrow & \tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde\Gamma,q+1} & \longrightarrow & (\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde\Gamma})^{n(\tilde\Gamma,q)} & & \end{array}$$

with exact rows. Induction with respect to q and the Snake Lemma show that $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)^{\tilde\Gamma,q}$ is equal to $\tilde{\mathcal{E}}'_k(\lambda)^{\tilde\Gamma,q}$ for all q. Hence the space $\tilde{\mathcal{E}}_k(\tilde\Gamma,\lambda)$ is maximally perturbable.

The proof of Theorem 6.8 is completely similar. □

Appendix A. Partition of unity

The following technical lemma gives partitions of unity that are adapted to $\tilde\Gamma\backslash\tilde G$ and $\Gamma\backslash\mathfrak{H}$.

Lemma A.1. *i) For a given cofinite discrete $\tilde\Gamma\subset\tilde G$ containing $\tilde Z$ there are $\psi\in C^\infty(\tilde G)$ such that*
- *a) ψ is a bounded function.*
- *b) There is $N\in\mathbb{N}$ such that for each $g\in\tilde G$ the number of $\gamma\in\tilde\Gamma$ with $\psi(\gamma^{-1}g)\neq 0$ is bounded by N.*
- *c) $\sum_{\gamma\in\tilde\Gamma}\psi(\gamma^{-1}g)=1$ for all $g\in\tilde G$.*

ii) For a given cofinite discrete $\Gamma\subset\mathrm{PSL}_2(\mathbb{R})$ there are $\psi_0\in C^\infty(\mathfrak{H})$ such that
- *a) ψ_0 is bounded.*
- *b) There is $N\in\mathbb{N}$ such that for each $z\in\mathfrak{H}$ the number of $\gamma\in\Gamma$ with $\psi_0(\gamma^{-1}z)\neq 0$ is bounded by N.*
- *c) $\sum_{\gamma\in\Gamma}\psi_0(\gamma^{-1}z)=1$ for all $z\in\mathfrak{H}$.*

Proof. We fix a strict fundamental domain $\mathfrak{F}_{\mathfrak{H}}$ for $\tilde\Gamma\backslash\mathfrak{H}$ of the following form, based on the choice of a real number $a>A_\Gamma$, as in §8.4.1. The set $\mathfrak{F}_{\mathfrak{H}}$ is bounded by finitely many geodesic segments and half-lines such that

$$\begin{aligned}\mathfrak{F}_{\mathfrak{H}} &= C_a\sqcup\bigsqcup_{\kappa}V_\kappa(a),\\ V_\kappa(a) &= \{g_\kappa(x+iy)\ :\ y\geq a,\ x_\kappa\leq x<x_\kappa+1\},\end{aligned}\tag{1.1}$$

where C_a is relatively compact in $\mathfrak{H}$, and is contained in the image of $\tilde G_a$ under the projection $\tilde G\to\mathfrak{H}$. The disjoint union is over the set C of cusps κ in the closure $\overline{\mathfrak{F}}_{\mathfrak{H}}$ of $\mathfrak{F}_{\mathfrak{H}}$ in $\mathfrak{H}\cup\partial\mathfrak{H}$. We take $\mathfrak{F}_{\mathfrak{H}}$ such that C forms a system of representatives for the $\tilde\Gamma$-orbits of cusps. By taking the parameter a sufficiently large we arrange that all orbits of elliptic fixed points intersect $\mathfrak{F}_{\mathfrak{H}}$ in C_a. These points are necessarily on the boundary of $\mathfrak{F}_{\mathfrak{H}}$.

We take a strict fundamental domain for $\tilde\Gamma\backslash\tilde G$ of the form

$$\mathfrak{F}=\{(z,\vartheta)\ :\ z\in\mathfrak{F}_{\mathfrak{H}},\ \vartheta\in[0,\pi/v(z))\},$$

where $v(z)\in\mathbb{N}$ is the order of the subgroup Γ_z fixing z, or equivalently $\tilde\Gamma_z$ is conjugate in $\tilde G$ to the group $\{k(n\pi/v(z))\ :\ n\in\mathbb{Z}\}$. So $v(z)$ is in general equal to 1, and only larger if z is an elliptic fixed point of Γ.

i. We first define a function on $\tilde G$ satisfying a) and c), and a variant of b).

Let $\omega : \tilde{G} \to \{0,1\}$ be the characteristic function of $\mathfrak{F}$. It satisfies conditions a)–c) in part i) of the lemma, but is not smooth. To make it smooth we convolve it with a function $\psi \in C_c^\infty(\tilde{G})$ with $\psi \geq 0$ such that $\int_{\tilde{G}} \psi(g)\, dg = 1$ for a choice dg of a Haar measure on $\tilde{G}$ and such that $\mathrm{Supp}(\psi)$ is a compact neighborhood of the unit element in $\tilde{G}$.

Since ω is measurable, the integral

$$\varphi_0(g) \;=\; \int_{\tilde{G}} \omega(g_1)\,\psi(g_1^{-1}g)\, dg_1 \;=\; \int_{\tilde{G}} \omega(gg_1^{-1})\,\psi(g_1)\, dg_1$$

defines a smooth function φ_0 on $\tilde{G}$ with values in $[0,1]$ and with support contained in the neighborhood $\mathfrak{F} \cdot \mathrm{Supp}(\psi)$ (multiplication in $\tilde{G}$) of $\mathfrak{F}$. From the second form of the convolution integral we see that $\sum_{\gamma\in\tilde{\Gamma}} \varphi_0(\gamma^{-1}g) = 1$ for all $g \in \tilde{G}$. This smooth function φ_0 satisfies conditions a) and c) in part i) of the lemma. Condition b) is not satisfied, since although the support of φ_0 is contained in a neighborhood of $\mathfrak{F}$ of the form $\mathfrak{F}\,\mathrm{Supp}(\psi)$, this neighborhood may meet near the cusps infinitely many $\tilde{\Gamma}$-translates of $\mathfrak{F}$. We will construct two functions, one "away from the cusps" and another "close to the cusps" satisfying all conditions a), b), c) on overlapping regions. A suitable combination of these two functions will produce the sought function on $\tilde{G}$.

• The first function is simply the restriction of φ_0 to $\tilde{G}_b$ for any $b \geq a$. We will show that this function satisfies condition b) (and thus all conditions). First we note that the projections $p_1 : \tilde{G} \to \mathfrak{H}$ and $p_2 : \tilde{G} \to \mathbb{R}$ given by $p_1(z,\vartheta) = z$ and $p_2(z,\vartheta) = \vartheta$ are continuous. Next we note that $\mathfrak{F}\,\mathrm{Supp}(\psi) \cap \tilde{G}_b$ is contained in a compact set, and hence has compact image in $\mathfrak{H}$ under p_1. So

$$p_1(\mathfrak{F}\,\mathrm{Supp}(\psi) \cap \tilde{G}_b) \;\subset \bigsqcup_{\delta\in E} \delta\,\mathfrak{F}_{\mathfrak{H}}$$

for some finite subset E of Γ.

Fix a $g \in \tilde{G}_b$. We will show that there is a finite number (independent of g) of $\gamma \in \tilde{\Gamma}$ with $\varphi_0(\gamma\, g) \neq 0$. Indeed, for each such γ we have $\gamma g \in \mathfrak{F}\,\mathrm{Supp}(\psi) \cap \tilde{G}_b$, hence $p_1(\gamma g) = \mathrm{pr}(\gamma)\, p_1(g) \in \bigsqcup_{\delta\in E} \delta\,\mathfrak{F}_{\mathfrak{H}}$. This leaves finitely many possibilities for the image $\mathrm{pr}(\gamma)$:

$$\mathrm{pr}(\gamma) = \delta\delta_0^{-1} \qquad \text{with } \delta \in E\,.$$

for some $\delta_0 \in \Gamma$. We conclude that $\gamma = \widetilde{\delta\delta_0^{-1}k(\pi m)}$ with $m \in \mathbb{Z}$.

On the other hand, the image $p_2(\mathfrak{F}\,\mathrm{Supp}(\psi) \cap \tilde{G}_b)$ is contained in a compact set, hence it is contained in a set $[-B,B] \subset \mathbb{R}$. For the $\gamma = \widetilde{\delta\delta_0^{-1}k(\pi m)}$ with $\varphi_0(\gamma g) \neq 0$ we conclude from (5.3) that $p_2(\widetilde{\delta\delta_0^{-1}k(\pi m)}g) = p_2(\widetilde{\delta\delta_0^{-1}}g) + m\pi$. This leaves only finitely many possibilities for the integer m. This shows that condition b) is satisfied by the restriction of φ_0 to $\tilde{G}_b$ ($b \geq a$).

• We now start the construction of another function φ_1 with the desired properties near the cusps. We take a compactly supported smooth partition β of unity for $\mathbb{R}/\mathbb{Z}$, *i.e.*, $\beta \in C_c^\infty(\mathbb{R})$ with values in $[0,1]$ such that $\sum_{k\in\mathbb{Z}} \beta(x+k) = 1$ for all $x \in \mathbb{R}$. (For instance take a smooth function υ in $C^\infty(\mathbb{R})$ with value 0 on a neighborhood of 0 and value 1 on a neighborhood of $\frac{1}{2}$. Then

$$\beta(x) \;=\; \begin{cases} 0 & \text{if } x < 0\,, \\ \upsilon(x) & \text{if } 0 \leq x < \frac{1}{2}\,, \\ 1 & \text{if } \frac{1}{2} \leq x < 1\,, \\ 1-\upsilon(x-1) & \text{if } 1 \leq x < \frac{3}{2}\,, \\ 0 & \text{if } x \geq \frac{3}{2}\,. \end{cases}$$

defines such a partition of unity.) We define a function φ_1 on $\tilde{G}$ in the following way.

$$\begin{aligned}\varphi_1(\tilde{g}_\kappa(z,\vartheta)) &= \beta(x-x_\kappa)\beta(\vartheta/\pi) \quad \text{if } \kappa\in C,\ y>a\,,\\ \varphi_1 &= 0 \quad \text{elsewhere}\,,\end{aligned}$$

with x_κ as in (1.1).

The function φ_1 is smooth on $\tilde{G}\setminus\tilde{G}_a$ and bounded there. By the definition and (8.17), it is clear that the only $\tilde{\Gamma}$-translates of $\mathfrak{F}$ intersecting the support of φ_1 are the $\tilde{\Gamma}_\kappa$-translates. The definition of β then implies that at most four $\tilde{\Gamma}_\kappa$-translates can intersect $\mathrm{Supp}(\varphi_1)$, implying (ii). Likewise, the definitions of φ_1,β and (8.17) imply that $\sum_{\gamma\in\tilde{\Gamma}}\varphi_1(\gamma^{-1}g)=1$ for $g\in\tilde{G}\setminus\tilde{G}_a$.

We choose a (bounded) function $\chi\in C^\infty(\tilde{\Gamma}\backslash\tilde{G})$ equal to 0 on $\tilde{G}_a$ and equal to 1 on $\tilde{G}\setminus\tilde{G}_{a+1}$. Put

$$\psi=\chi\cdot\varphi_1+(1-\chi)\cdot\varphi_0\,,$$

where φ_0 is as constructed above with b equal to $a+1$. Since χ vanishes on $\tilde{G}_a$ the product $\chi\cdot\varphi_1$ is smooth on $\tilde{G}$. Similarly, $(1-\chi)\cdot\varphi_0$ is smooth. So $\psi\in C^\infty(\tilde{G})$. Conditions a)–c) are easily checked to hold for ψ.

ii. We turn to $\Gamma=\tilde{\Gamma}/\tilde{Z}$ and start with ψ as in part i). The sum $\psi_1(z,\vartheta)=\sum_{m\in\mathbb{Z}}\psi(z,\vartheta-m\pi)$ is locally finite and defines a smooth function with values in $[0,1]$ that is invariant under left translation by elements of $\tilde{Z}$. So $\psi_1(\gamma^{-1}(z,\vartheta))=\psi_1(\tilde{\gamma}^{-1}(z,\vartheta))$ is well defined for $\gamma\in\Gamma$, and

$$\sum_{\gamma\in\Gamma}\psi_1(\gamma^{-1}(z,\vartheta)) \;=\; \sum_{\gamma\in\Gamma}\sum_{m\in\mathbb{Z}}\psi((\tilde{\gamma}k(m\pi))^{-1}(z,\vartheta)) \;=\; 1$$

for all (z,ϑ). Since the support of ψ meets only finitely many $\tilde{\Gamma}$ translates of $\mathfrak{F}\subset\mathfrak{F}_{\mathfrak{H}}\times[0,\pi)$, the support $\mathrm{Supp}(\psi)\cdot\tilde{Z}$ of ψ_1 meets only finitely many $\tilde{\Gamma}$-translates of $\mathfrak{F}_{\mathfrak{H}}\times\mathbb{R}$. Set

$$\psi_0(z)=\frac{1}{\pi}\int_0^\pi\psi_1(z,\vartheta)d\vartheta.$$

It clearly satisfies (i). For condition c) we note that

$$\begin{aligned}\sum_{\gamma\in\Gamma}\psi_0(\gamma z) &= \frac{1}{\pi}\sum_{\gamma\in\Gamma}\int_0^\pi\psi_1(\gamma z,\vartheta)\,d\vartheta\\ &= \frac{1}{\pi}\sum_{\gamma\in\Gamma}\int_{\arg(j(\gamma,z))}^{\pi+\arg(j(\gamma,z))}\psi_1(\gamma z,\vartheta)\,d\vartheta \qquad \text{by the } \pi\text{-periodicity of } \psi_1\\ &= \frac{1}{\pi}\sum_{\gamma\in\Gamma}\int_0^\pi\psi_1(\gamma(z,\vartheta))\,d\vartheta\\ &= 1\end{aligned}$$

The support of ψ_0 is contained in the image $p_1(\mathrm{Supp}(\psi_1))\subset\mathfrak{H}$. Since $\mathrm{Supp}(\psi_1)$ is contained in finitely many $\tilde{\Gamma}$-translates of $\mathfrak{F}_{\mathfrak{H}}\times\mathbb{R}$, we conclude that condition b) is satisfied as well. □

Appendix B. Index of commonly used notation

$\mathcal{E}_k(y_0,\lambda), \mathcal{E}_k^{\mathrm{hol}}(y_0)$	Defn. 8.7	$\eta_r(n;z,\vartheta)$	(7.17)	P_i	§3.2.1
$\tilde{\mathcal{E}}_k(\tilde{\Gamma},\lambda)$	Defn. 6.3	$\eta_k(n)$	(7.18)	pr, pr_2	§5.1
$\tilde{E}_r(\tilde{\Gamma},\chi,\lambda)$	Defn. 6.3	$\eta_k^{\mathbf{m}}(n;z,\vartheta)$	(7.19)	π_i	§5.4
$E_k(\Gamma,\lambda)$	Defn. 4.1	$k(\vartheta)$	§5.1	Q_n	(7.10)
$\mathbf{f_i}$	(8.1)	κ_i	§5.4	s	§5.3
g_κ	(4.1)	L_k	(4.3)	$S(y_0)$	(8.18)
$\tilde{G}$	Defn. 5.1	$L(z,\vartheta)$	(6.3)	t	§5.3
$\tilde{G}_a$	(8.16)	m_q	(3.3)	$t(\Gamma)$	§3.2.1
$h_{\mathbf{i}}$	(8.8), (8.9)	$\mathcal{M}_k(\Gamma,\lambda)$	Defn. 4.1	$\mathcal{V}_k(n,s), \mathcal{V}_k^0(n,s)$	Defn. 7.3
H_i	§3.2.1	$M_k(\Gamma,\lambda)$	Defn. 4.1	$\mathcal{W}_r(\nu,s)$	(7.2)
$\mathbf{H}$	§5.2	$\mathcal{M}_k^{\mathrm{hol}}(\Gamma,\lambda_k)$	§4.2	$\mathbf{W}$	§5.2
$h_k^{\mathbf{m}}(n,s)$	(7.11)	μ_f	(3.5)	$\mathbf{X}$	§5.2
$\mathcal{H}_k(\tilde{\Gamma}), \mathcal{H}_k^p(\tilde{\Gamma}), \mathcal{H}_k^c(\tilde{\Gamma})$	Defn. 6.3	$n_{\mathrm{ell}}, n_{\mathrm{par}}$	§3.2.1	ζ	§5.3
η_i	§5.4	$n(x)$	§5.1	ω	§5.2
		$n(\Gamma,q)$	(3.4)	$\omega_r, \hat{\omega}_r$	(7.4)

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