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Behaviour of the extended Volterra lattice ☆

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ABSTRACT

We investigate the behaviour of solutions of the recently proposed extended Volterra lattice. A variety of methods are used to determine the effects of the new terms on small amplitude equations, and, following approximation of the partial differential delay equations by PDEs we also determine similarity reductions.

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1. Introduction

The integrability of the basic or standard Volterra lattice $u_t(n,t) = u(n,t)[u(n+1,t) - u(n-1,t)]$ has been known for some time, and important properties such as Miura maps, modified systems, Hamiltonian structures, Lax pairs, Bäcklund transformations, Hirota bilinear form and N-soliton solutions have been derived [8,12,16,5,20,9,10,6]. Integrable extensions of the Volterra lattice have also been discovered, such as $(2+1)$-dimensional versions [15] or $(1+1)$-dimensional non-isospectral extensions [11,21].

In [2,3] an integrable non-isospectral $(2+1)$-dimensional extension of the Volterra lattice hierarchy was constructed, this consisting of a sequence of equations in $u(n,t,y)$ with n being discrete and t and y continuous. The consideration of reductions to ordinary difference equations allowed the derivation of discrete Painlevé hierarchies. Other reductions of this $(2+1)$-dimensional hierarchy included an extended $(1+1)$-dimensional Volterra lattice hierarchy, the members of which are evolution equations in $u(x,t)$, both x and t being continuous but with the equations involving derivatives with respect to x as well as shifts in x. The autonomous versions of such equations were subsequently placed within a suitable modification of the usual algebraic structure associated with completely integrable evolution equations (recursion operators and hierarchies of commuting flows) in [4], this in turn giving rise to a new interpretation of the lattice hierarchies which appear in the literature within the broader context of differential-delay hierarchies.

In this paper we investigate this extended $(1+1)$-dimensional Volterra lattice hierarchy, to determine the types of solution it supports, and how these compare with those supported by the basic Volterra lattice. We focus on the first equation of the hierarchy.

In Section 2 we introduce the basic Volterra lattice equation and show how it may be approximated. Section 3 describes the extended Volterra differential-delay system. In Section 4 we formulate a family of partial differential equations (PDEs)

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which approximate the extended Volterra lattice in certain continuum limits [17–19]. The behaviour of these approximating PDEs is then described and discussed. The similarity solutions of these are also explored. One equation arising in this context is a nonisospectral extended KdV equation, whose solutions merit further investigation. These are detailed in Section 5, where we also explain connections to integrable systems. Finally Section 6 concludes the paper, where we summarise the results and discuss their wider implications.

2. The Volterra equation

The basic Volterra equation is

$$u_t(x,t) = u(x,t)[u(x+1,t) - u(x-1,t)] \tag{2.1}$$

which, as commented earlier, is completely integrable. Here we take as independent variables x and t instead of n and t since later we will be considering the integrable $(1+1)$-dimensional non-isospectral extension of (2.1) introduced in [2], this being an evolution equation in two continuous variables x and t involving both derivatives with respect to x and shifts in x. We begin here by considering the standard Volterra lattice (2.1).

In the following sections we will consider asymptotic approximations of the extended Volterra equation. Here we analyse the basic Volterra Equation (2.1), and outline some results of the type which we might expect to derive later for the extended system. This section also serves as an introduction to the techniques we use later. Whilst these methods are more commonly applied to continuous systems, they are applicable for discrete problems see, for example, Remoissenet [14].

Initially we consider the evolution of small amplitude disturbances from some uniform background state, that is, we assume $u(x,t) = U(1+\epsilon^2 w(x,t))$ where $\epsilon \ll 1$ and $U, w = \mathcal{O}(1)$. The leading order equation for w is then

$$w_t = U(w(x+1,t) - w(x-1,t)), \tag{2.2}$$

We obtain the dispersion relation for this equation by seeking a solution of the form $w = \mathrm{e}^{i(kx-\omega t)}$ where $\omega = \omega(k)$, whence we obtain

$$\omega(k) = -2U\sin k. \tag{2.3}$$

This relationship determines the behaviour of linear waves.

Clearly Eq. (2.1) is *non*linear and so should be expected to have a more complex family of solutions than (2.2), which is a linear. Nonlinear terms become relevant at higher order in ϵ, and also over long timescales and at large space scales. To investigate these latter two regimes, we rescale space and time using the scalings $x = \epsilon^{-1}y$ and $t = \epsilon^{-1}\tau$. The substitution $u(x,t) = U(1+\epsilon^2 w(y,\tau))$ yields a partial differential equation (PDE) which approximates (2.1), namely

$$w_\tau = 2Uw_y + \frac{1}{3}\epsilon^2 U w_{yyy} + 2\epsilon^2 U w w_y. \tag{2.4}$$

Here we have retained both leading order and first correction terms, neglected terms are $\mathcal{O}(\epsilon^4)$.

As a side issue, we confirm that small amplitude waves in (2.4) exhibit the same behaviour as those in (2.2) provided that the wave number is small. This condition is equivalent to requiring $k \ll 1$ in (2.3). If we seek the dispersion relation of (2.4) using $w = \delta \mathrm{e}^{iKy - i\Omega\tau}$ with $\Omega = \Omega(K)$ and $\delta \ll 1$, then we find

$$\Omega = -2KU + \frac{1}{3}\epsilon^2 UK^3. \tag{2.5}$$

Noting that $\Omega\tau = \Omega\epsilon t = \omega t$ and $\kappa y = kx$, we write $k = \kappa\epsilon$ and $\omega = \epsilon\Omega$. Thus the dispersion relation (2.5) is simply the small wavenumber limit ($k \ll 1$) of (2.3).

However, to consider the evolution of weakly nonlinear waves in (2.2) we return to (2.4) and consider the effects of all the terms. The leading order terms of (2.4) have the form of a simple wave equation $w_\tau = 2Uw_y$, which has the solution $w = w(y - c\tau)$ where the speed is given by $c = -2U$. The evolution of the shape of the wave is due to terms of higher order in ϵ. To assess the rôle of these terms, we introduce a moving coordinate frame $z = y - c\tau$ and rescale time *via* $T = \epsilon^2\tau = \epsilon^3 t$. Together with the substitution $w(y,\tau) = v(z,T)$, these transformations lead to

$$v_T = 2Uvv_z + \frac{1}{3}Uv_{zzz}, \tag{2.6}$$

which is the familiar KdV equation. Thus we expect small amplitude solutions of (2.2) to include both linear waves and solitary pulse waves, which interact elastically, as well as similarity solutions.

3. The extended Volterra lattice

Eqs. (93) and (94) of [2] introduces the nonisospectral extension of the Volterra lattice hierarchy

$$\begin{aligned}\frac{u_t(x,t)}{u(x,t)} &= [u(x,t)w(x,t) + u(x-1,t)w(x-1,t) - u(x,t)w(x+1,t) - u(x+1,t)w(x+2,t)] + \alpha[u(x-1,t) - u(x+1,t)] \\ &\quad + \beta_0[(x-1)u(x-1,t) - (x+1)u(x+1,t)] - \beta_0[u(x,t) + u(x-1,t)] + \beta_1,\end{aligned} \tag{3.1}$$

$$\kappa(\log u)_x = \kappa\frac{u_x(x,t)}{u(x,t)} = w(x+1,t) - w(x,t). \tag{3.2}$$

The case of (3.1) and (3.2) with $\beta_0 = 0 = \beta_1$, $\alpha = -1$, $\kappa = 0$, corresponds to the basic Volterra Equation (2.1). Setting $\beta_0 = 0 = \beta_1$ removes terms from (3.1). Taking the limit $\kappa \to 0$, gives an equation for w, which we solve by $w(x) = W$. These simplifications mean that (3.1) reduces to

$$\frac{u_t(x,t)}{u(x,t)} = (W+\alpha)[u(x-1,t) - u(x+1,t)]. \tag{3.3}$$

The factor $W + \alpha$ can be removed by rescaling time, t, which yields the basic Volterra Equation (2.1).

In the derivation of the integrable hierarchy, Eq. (3.2) arises as a compatibility condition between linear systems. The quantity $w(\cdot)$ can be thought of as a potential which mediates other nonlocal interactions in $u(x,t)$. As will be seen in the next section, nonzero κ values introduce new behaviour into the system, through $w(x)$ being non-constant.

We will work with a shifted $w(\cdot)$ function, $\widehat{w}(\cdot)$, and suppress the t argument and the x argument when it is not shifted, so that

$$\Delta\widehat{w} = \widehat{w}\left(x+\frac{1}{2}\right) - \widehat{w}\left(x-\frac{1}{2}\right) = \kappa\frac{u_x}{u} = \kappa(\log u)_x. \tag{3.4}$$

A more complex redefinition of variables of the form $u(x) = \widetilde{u}(x-\frac{1}{2}) = \widetilde{u}(\widetilde{x}), w(x) = \widetilde{w}(x-1), \widetilde{x} = x - \frac{1}{2}$ simplifies the equation slightly, by making the functions premultiplying terms involving β_0 symmetric. However, we will not work with such a formulation of the system here, since the added complexity only slightly simplifies the resulting PDEs.

We use the relation (3.2) to simplify (3.1), which can be rewritten as

$$\begin{aligned}\frac{u_t}{u} = {} & \beta_1 - \alpha[u(x+1) - u(x-1)] - \beta_0[(x+1)u(x+1) - (x-1)u(x-1)] - \beta_0[u + u(x-1)] \\ & - \kappa[u_x(x+1) + u_x + u_x(x-1)] + u(x-1)\widehat{w}\left(x-\frac{1}{2}\right) - u(x+1)\widehat{w}\left(x+\frac{1}{2}\right).\end{aligned} \tag{3.5}$$

We would like to test stability against a variety of perturbations, in a similar fashion to finding the dispersion relation of waves, by substituting $u(x,t) = U + \epsilon e^{ikx - i\omega(k)t} + c.c.$ for some constant U, and hence find stability. However, this is not possible due to the β_0 term which is nonautonomous. Hence we use the simpler and less general stability analysis, considering $u(x,t) = U + v(t)$. For $U = \beta_1/4\beta_0$ we obtain $v_t = -\beta_1 v - 4\beta_0 v^2$, which for $v \ll U$ implies the linear equation $v_t = -\beta_1 v$ thus this is linearly stable when $\beta_1 > 0$ and unstable for $\beta_1 < 0$. Whilst for $U = 0$ we find $v_t/v = \beta_1 - 4\beta_0 v$, which has the opposite stability properties. These stability results are summarised in Fig. 1.

3.1. The special case $\beta_1 = 0 = \beta_0$

In the case where $\beta_1 = 0 = \beta_0$ then we return to the situation in which $u(x,t) = U$, with U being any constant, is a steady-state solution.

In this case the stability of small amplitude linear waves *can* be established by substituting $u(x,t) = U + \epsilon e^{ikx - i\omega t} + c.c.$, which leads to

$$\widehat{w} = \frac{\kappa k \epsilon e^{ikx - i\omega t}}{2U\sin\frac{1}{2}k} + c.c. \tag{3.6}$$

and $\omega = 2U(\alpha\sin k + \kappa k\cos k + \kappa k)$. We thus have stability for all wavenumbers k.

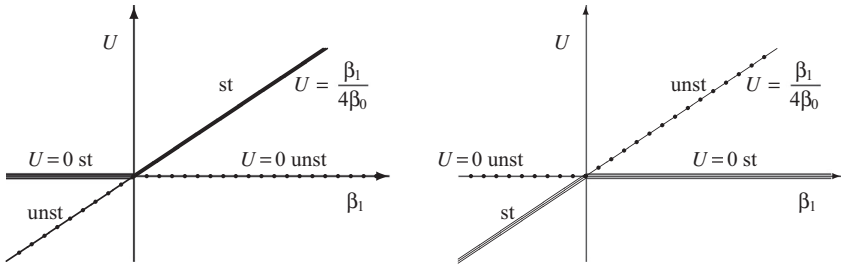

Fig. 1. Illustration of steady-state solutions of the extended Volterra equation, and their stability. The left figure is for the case $\beta_1 > 0$ and the right for $\beta_1 < 0$.

3.2. Quasi-continuum approximations

Now we consider the behaviour of solutions which are slowly varying in x and have small deviations in the u (and w) variables. Hence we start by rescaling

$$y = \epsilon x, \quad u = u_0(1 + \epsilon^2 v(y, \tau)), \tag{3.7}$$

where $\tau = \epsilon^\sigma t$ for some $\sigma > 0$ is the timescale of evolution of v, then consider the leading order problem for $v(y, \tau)$. Invoking the relationship

$$\widehat{w}\left(y + \frac{1}{2}\epsilon\right) - \widehat{w}\left(y - \frac{1}{2}\epsilon\right) = \epsilon \widehat{w}_y + \frac{1}{24}\epsilon^3 \widehat{w}_{yyy} + \frac{1}{1920}\epsilon^5 \widehat{w}_{yyyyy}, \tag{3.8}$$

together with the transformation (3.7), Eq. (3.4) becomes

$$\epsilon\partial_y\left(1 + \frac{1}{24}\epsilon^2\partial_y^2 + \frac{1}{1920}\epsilon^4\partial_y^4\right)\widehat{w} = \epsilon\partial_y\kappa \log u \sim \epsilon\partial_y\kappa\left(\epsilon^2 v - \frac{1}{2}\epsilon^4 v^2 + \frac{1}{3}\epsilon^6 v^3\right) + \mathcal{O}(\epsilon^8), \tag{3.9}$$

hence

$$\begin{aligned}\widehat{w} &= \kappa\left(1 - \frac{1}{24}\epsilon^2\partial_y^2\right)\left(\epsilon^2 v - \frac{1}{2}\epsilon^4 v^2 + \frac{1}{3}\epsilon^6 v^3 - \frac{1}{4}\epsilon^8 v^4\right),\\ &= \kappa\left(\epsilon^2 v - \frac{1}{2}\epsilon^4 v^2 + \frac{1}{3}\epsilon^6 v^3 - \frac{1}{4}\epsilon^8 v^4 - \frac{1}{24}\epsilon^4 v_{yy} + \frac{1}{24}\epsilon^6 (v v_y)_y\right) + \mathcal{O}(\epsilon^6).\end{aligned} \tag{3.10}$$

The asymptotic approximation (3.7) to the solution of $\Delta\widehat{w} = \kappa u_x/u$ will be substituted into the equation for u, namely

$$\begin{aligned}\frac{u_t}{u} = \beta_1 - \beta_0(1 + \mathrm{e}^{-\partial_x})u - (\mathrm{e}^{\partial_x} - \mathrm{e}^{-\partial_x})\alpha u - (\mathrm{e}^{\partial_x} - \mathrm{e}^{-\partial_x})\beta_0(xu) - \kappa(\mathrm{e}^{\partial_x} + 1 + \mathrm{e}^{-\partial_x})u_x + (\mathrm{e}^{-\partial_x}u)(\mathrm{e}^{-\partial_x/2}\widehat{w}) - (\mathrm{e}^{\partial_x}u)\\ \times (\mathrm{e}^{\partial_x/2}\widehat{w}).\end{aligned} \tag{3.11}$$

Various PDEs are generated, depending on the relative sizes of the parameters $\beta_0,\ \beta_1,\ \kappa,\ \alpha$; each PDE corresponding to a particular scaling of time and parameters.

4. Asymptotic reductions

Here, with $u = u_0(1 + \epsilon^2 v)$ Eq. (3.10) reduces to

$$\widehat{w} = \kappa\epsilon^2 v - \frac{1}{2}\kappa\epsilon^4 v^2 - \frac{1}{24}\kappa\epsilon^4 v_{yy} + \frac{1}{3}\kappa\epsilon^6 v^3 + \frac{1}{24}\kappa\epsilon^6 (v v_y)_y + \mathcal{O}(\epsilon^8). \tag{4.1}$$

Substituting this into (3.5) we find

$$\begin{aligned}\frac{v_t}{\epsilon u_0(1 + \epsilon^2 v)} = \frac{\beta_1 - 4\beta_0 u_0}{\epsilon^3 u_0} - \frac{2\beta_0(2v + y v_y)}{\epsilon} + \beta_0 v_y - \frac{3}{2}\beta_0\epsilon v_{yy} + \frac{1}{6}\beta_0\epsilon^2 v_{yyy} - \frac{1}{3}\beta_0\epsilon y v_{yyy} - (2\alpha + 4\kappa)v_y\\ - \left(\frac{1}{3}\alpha + \kappa\right)\epsilon^2 v_{yyy} - \kappa\epsilon^2 v v_y.\end{aligned} \tag{4.2}$$

The dominant terms are clearly those of $\mathcal{O}(\epsilon^{-3})$, which yield $u_0 = \beta_1/(4\beta_0)$, and this is what we shall assume hence forth (unless $\beta_0 = 0 = \beta_1$, when u_0 can be chosen arbitrarily, this case has been analysed already, in Section 3.1).

Clearly in (4.2), we have a variety of terms of differing magnitudes and the behaviour of the solution will depend on their relative sizes. Hence we consider a variety of scalings of the parameters and examine the evolution of the resulting leading order PDE over the relevant timescales.

4.1. The case $\beta_0 = \mathcal{O}(1)$ and $t = \mathcal{O}(1)$

In the most obvious case, we take all parameters (κ, α, β_0) to be $\mathcal{O}(1)$ and then at leading order we have

$$\frac{v_t}{u_0} = -4\beta_0 v - 2\beta_0 y v_y. \tag{4.3}$$

Here the time-evolution occurs on the $u_0 t = \mathcal{O}(1)$ scale, and so no further scaling is required. This PDE has solutions which decay to zero, *via*

$$v(y, t) = \mathrm{e}^{-4\beta_0 u_0 t} v_0(y\mathrm{e}^{-2\beta_0 u_0 t}), \tag{4.4}$$

where $v_0(y)$ is the initial data $v(y, 0)$. The solution (4.4) exhibits self-similar behaviour, with all solutions decreasing exponentially to zero. We also note that the solution and Eq. (4.3) only depends on the parameter β_0 and not on α or κ.

4.2. The case $\beta_0 = \mathcal{O}(\epsilon)$ and $t = \mathcal{O}(\epsilon^{-1})$

In the above approximating PDE (4.3) only the effects due to the parameter β_0 are evident. We consider this scenario by writing $\beta_0 = \epsilon\beta$ with $\beta = \mathcal{O}(1)$ and again assume $\kappa,\ \alpha = \mathcal{O}(1)$, then other terms enter the leading order balance, and we obtain a more complicated PDE

$$\frac{v_t}{\epsilon u_0} = -4\beta v - 2\beta y v_y - v_y(2\alpha + 4\kappa). \tag{4.5}$$

Although this does not immediately look like a consistent leading order balance, if we rescale time with $\tau = \epsilon u_0 t$ we find the LHS is simply v_τ. The solution of (4.5) is then

$$v(y,t) = \mathrm{e}^{-4\beta\epsilon u_0 t} v_0\left(\mathrm{e}^{-2\beta\epsilon u_0 t}\left(y + \frac{(\alpha+2k)}{\beta}\right)\right). \tag{4.6}$$

Thus we observe once again, an exponential decay in the solution (4.6), albeit over a slow timescale; furthermore, as shown in the example in Fig. 2, the position of the peak moves, due to the advective terms in v_y and yv_y. For the example in this figure, the function $v_0(\cdot)$ initially has a maximum at $y = 3$, and at later times, the location of the maximum occurs at

$$y_m = 3\mathrm{e}^{2\beta\epsilon u_0 t} + \frac{(\alpha+2k)(\mathrm{e}^{2\beta\epsilon u_0 t} - 1)}{\beta}. \tag{4.7}$$

The only differences between Eq. (4.5) and the previous PDE (4.3) is the rescaling of time and the additional advection term, which adds a simple Galilean shift to the solution (4.4). In contrast with (4.4), the solution (4.6) depends on all the parameters $\alpha,\ \beta_0$ and κ.

4.3. The case $\beta_0 = \mathcal{O}(\epsilon^2)$ and $t = \mathcal{O}(\epsilon^{-2})$

With the scalings $\beta_0 = \epsilon^2\beta$ and $\beta = \mathcal{O}(1),\ \alpha = \mathcal{O}(1), \kappa = \mathcal{O}(1)$, Eq. (4.2) becomes

$$\frac{v_t}{\epsilon u_0} = -4\epsilon\beta v - 2\epsilon\beta y v_y - 2\alpha v_y - 4\kappa v_y. \tag{4.8}$$

We introduce $z = y - \epsilon^{-1}s(\tau)$ and $\tau = \epsilon^2 t$ with $z,\ \tau = \mathcal{O}(1)$ to obtain

$$\epsilon u_0^{-1} v_\tau = -4\beta\epsilon v - v_z\left[2\epsilon\beta z + 2\alpha + 4\kappa + 2\beta s(\tau) - u_0^{-1}s'(\tau)\right]. \tag{4.9}$$

Hence if we choose $s(\tau)$ to be given by

$$s(\tau) = \frac{(\alpha+2\kappa)\left(\mathrm{e}^{2\beta u_0(\tau-\tau_0)} - 1\right)}{\beta}, \tag{4.10}$$

then solving $2v + zv_z + v_\tau/2\beta u_0 = 0$, which is identical in form to (4.3), hence we have

$$v(z,\tau) = \mathrm{e}^{-4\beta u_0\tau} v_0(z\mathrm{e}^{-2\beta u_0\tau}), \tag{4.11}$$

which in turn is equivalent to

$$v(y,t) = \mathrm{e}^{-4\beta\epsilon^2 u_0 t} v_0\left(y\mathrm{e}^{-2\beta\epsilon^2 u_0 t} - \frac{(\alpha+2\kappa)(1 - \mathrm{e}^{-2\beta\epsilon^2 u_0 t})}{\beta\epsilon}\right), \tag{4.12}$$

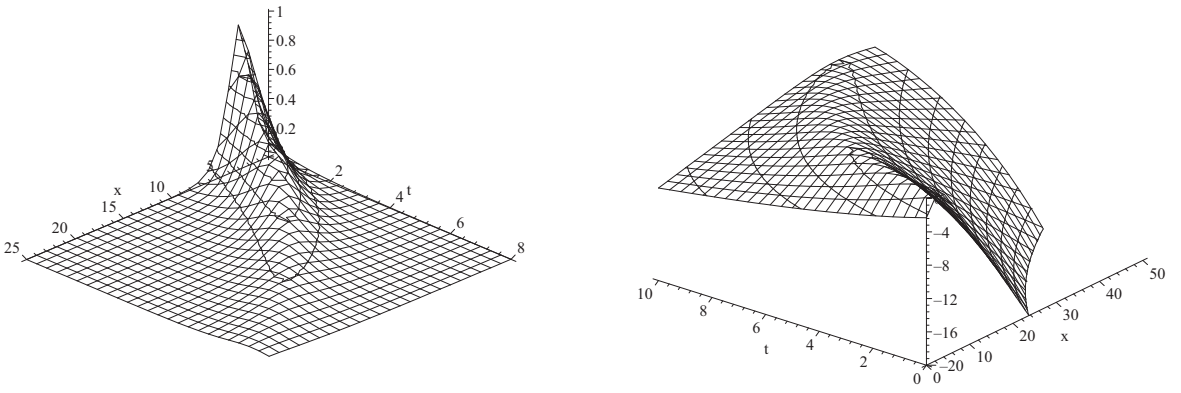

Fig. 2. Illustration of the solution (4.6) for the case $v_0(z) = \mathrm{sech}(z - 3 - \frac{\alpha+2k}{\beta})$ with $\epsilon = 0.1,\ \alpha = \kappa = \beta = 1$. Left panel, with normal axes; right panel, with a logarithmic scale on the vertical axis.

these results are similar to those obtained earlier, see equation (4.4) for example. Hence, for this choice of parameters the system exhibits similar kinetics to the examples shown earlier, only on a slower timescale due to the smaller size of β_0.

4.4. The case $\beta_0 = \mathcal{O}(\epsilon^3)$ and $t = \mathcal{O}(\epsilon^{-3})$

With the scalings $\beta_0 = \epsilon^3\beta,\ \beta = \mathcal{O}(1)$, $\kappa = \mathcal{O}(1),\ \alpha = \mathcal{O}(1)$, equation (4.2) becomes

$$\frac{v_t}{\epsilon u_0(1+\epsilon^2 v)} = -2\beta(2v + yv_y)\epsilon^2 + \beta\epsilon^3 v_y - \frac{3}{2}\beta\epsilon^4 v_{yy} + \frac{1}{6}\beta\epsilon^5 v_{yyy} - \frac{1}{3}\beta\epsilon^4 y v_{yyy} - (2\alpha + 4\kappa)v_y - \left(\frac{1}{3}\alpha + \kappa\right)\epsilon^2 v_{yyy} - \kappa\epsilon^2 v v_y. \tag{4.13}$$

Again, although the dominant terms appear to be those due to advection, that is, the v_y terms, these can be adsorbed by a change of variables, after we have multiplied through by the factor $1 + \epsilon^2 v$. We write $y = z - \epsilon^p s(\tau)$ with τ being a rescaled time variable, choosing the exponent p and timescale to simplify the equation governing the shape of the profile $v(z,\tau)$. We consider appropriate values for p in turn.

4.4.1. The subcase $p = -2$.

The transformation

$$z = y - \epsilon^{-2}s(\tau), \quad \tau = \epsilon^3 u_0 t, \tag{4.14}$$

yields, at leading order, an equation governing the motion of the wave

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = 2\beta s + 2\alpha + 4\kappa. \tag{4.15}$$

Provided that $s(\tau)$ satisfies (4.15), the next order terms give a PDE governing the evolution of the wave

$$0 = v_\tau + 4\beta v + 2\beta z v_z + (2\alpha + 5\kappa + 2\beta s(\tau))v v_z + \left(\kappa + \frac{1}{3}\alpha + \frac{1}{3}\beta s(\tau)\right)v_{zzz}. \tag{4.16}$$

Eq. (4.15) has the general solution

$$s(\tau) = C\mathrm{e}^{2\beta\tau} - \frac{(\alpha + 2\kappa)}{\beta}, \tag{4.17}$$

for some constant C. In general this gives a new coordinate frame z which is accelerating with respect to the old space variable y. There are two subcases worthy of more detailed consideration, namely when $C = 0$ and s is a constant, and the case $C = (\alpha + 2\kappa)/\beta$ so that $s(0) = 0$.

If we choose $C = 0$, then $s = -(\alpha + 2\kappa)/\beta$, and any solution is centred on $z = 0$ which, by (4.14), corresponds to $y = -(\alpha + 2\kappa)/\beta\epsilon^2$, which is in the far field ($-y \gg 1$). The shape of the profile $v(z,\tau)$ is then governed by

$$0 = v_\tau + 4\beta v + 2\beta z v_z + \kappa v v_z + \frac{1}{3}\kappa v_{zzz}. \tag{4.18}$$

We return to this case later, see Section 5.

If we choose $C = (\alpha + 2\kappa)/\beta$ then the moving coordinate frame z is related to y through (4.14), which implies

$$y = z + \frac{s(\tau)}{\epsilon^2} \sim z + 2(\alpha + 2\kappa)u_0\epsilon t + \mathcal{O}(\epsilon^4 t^2). \tag{4.19}$$

Thus, even at moderately large times, $t = \mathcal{O}(1)$ up to $t \ll \mathcal{O}(\epsilon^{-2})$, the relationship has, at leading order, the form of a travelling wave with a small speed.

Eq. (4.16) is the first leading-order equation we have derived that has the characteristic terms from the KdV equation, namely the third spatial derivative and the quadratic nonlinearity. However, we also have the complicating factors of a v and a zv_z term in the PDE, and the coefficients of the v_{zzz} and vv_z terms containing $s(\tau)$, which in turn depends on time τ in an exponential fashion. These time-dependencies make the PDE nonintegrable.

4.4.2. The subcase $p = -1$

In (4.13) we now use the transformation

$$z = y - s(\tau)/\epsilon, \quad \tau = \epsilon^3 u_0 t. \tag{4.20}$$

Both the leading order and first correction terms only involve terms of the form v_z with coefficients

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = 2\beta s + \frac{2(\alpha + 2\kappa)}{\epsilon}, \tag{4.21}$$

with solution

$$s(\tau) = Ce^{2\beta\tau} - \frac{(\alpha+2\kappa)}{\beta\epsilon}. \tag{4.22}$$

The governing equation for $v(z,\tau)$ is then

$$0 = v_\tau + (2\alpha+5\kappa)vv_z + 2\beta z v_z + 4\beta v + \left(\kappa + \frac{1}{3}\alpha\right)v_{zzz}, \tag{4.23}$$

for whatever value of C is chosen in (4.22). The PDE (4.23) is of the form considered in Section 5.

4.5. The case $\beta_0 = \mathcal{O}(\epsilon^4)$ and $t = \mathcal{O}(\epsilon^{-3})$

With the scalings $\beta_0 = \epsilon^4\beta$ and $\beta,\ \alpha,\ \kappa = \mathcal{O}(1)$, Eq. (4.2) becomes

$$\begin{aligned}\frac{v_t}{\epsilon u_0(1+\epsilon^2 v)} = &-2\beta(2v + yv_y)\epsilon^3 + \beta\epsilon^4 v_y - \frac{3}{2}\beta\epsilon^5 v_{yy} + \frac{1}{6}\beta\epsilon^6 v_{yyy} - \frac{1}{3}\beta\epsilon^5 y v_{yyy} - (2\alpha+4\kappa)v_y \\ &- \left(\frac{1}{3}\alpha+\kappa\right)\epsilon^2 v_{yyy} - \kappa\epsilon^2 vv_y.\end{aligned} \tag{4.24}$$

Only the first few orders of $\mathcal{O}(\epsilon)$ are relevant for later calculations, hence we ignore the higher order correction terms, retaining only

$$\frac{-v_t}{\epsilon u_0(1+\epsilon^2 v)} = 4\epsilon^3\beta v + 2\epsilon^3\beta y v_y + (2\alpha+4\kappa)v_y + \left(\frac{1}{3}\alpha+\kappa\right)\epsilon^2 v_{yyy} + \epsilon^2\kappa vv_y. \tag{4.25}$$

Now we multiply through by $(1+\epsilon^2 v)$, and transform to remove the v_y terms. This can be done in various ways, using $z = y - \epsilon^p s(\tau)$ and $\tau = \epsilon^3 u_0 t$, with differing values of p considered in turn below.

4.5.1. The subcase $p = -3$.

The transformation

$$z = y - \epsilon^{-3}s(\tau), \quad \tau = \epsilon^3 u_0 t, \tag{4.26}$$

gives the leading order equation

$$s = \frac{-(\alpha+2\kappa)}{\beta}, \tag{4.27}$$

and then the first correction terms give the KdV equation $0 = v_\tau + \kappa vv_z + \frac{1}{3}\kappa v_{zzz}$.

4.5.2. The subcase $p = -2$.

We now use the transformation

$$z = y - \epsilon^{-2}s(\tau), \quad \tau = \epsilon^3 u_0 t, \tag{4.28}$$

As in Case 4.4.1 we find both the leading order and the first correction terms involve only v_z, so we consider them together and obtain

$$\frac{ds}{d\tau} = 2\beta\epsilon s + 2\alpha + 4\kappa, \tag{4.29}$$

which has the solution

$$s(\tau) = Ce^{2\beta\epsilon\tau} - \left(\frac{\alpha+2\kappa}{\beta\epsilon}\right), \tag{4.30}$$

for some constant C. Note that with $C = 0$ here and (4.30) substituted into (4.28) yields the same solution as (4.27) substituted into (4.26).

The second correction terms involve many different derivatives of v, from which we obtain the leading order equation

$$0 = v_\tau + (2\alpha+5\kappa)vv_z + (\kappa+\frac{1}{3}\alpha)v_{zzz}, \tag{4.31}$$

where $\tau = \epsilon^3 u_0 t$. This has the form of the classic KdV equation, without any perturbing terms. It is also worth noting that β_0 is now so small that it does not enter the leading-order Eq. (4.31).

Although the KdV equation has similarity solutions in which $v(y,t) \to 0$ as $t \to \infty$ uniformly, it also has solutions $v(y,t)$ which do not decay, for example travelling waves in which $v(y,t) = v(y-ct)$.

The subcase $p = -1$ is almost identical to the above.

4.6. The case $\beta_0 = \mathcal{O}(\epsilon^5)$ and $t = \mathcal{O}(\epsilon^{-3})$

With the scalings $\beta_0 = \epsilon^5\beta$ and $\beta,\ \alpha,\ \kappa = \mathcal{O}(1)$, Eq. (4.2) becomes

$$\frac{v_t}{\epsilon u_0(1+\epsilon^2 v)} = -2\beta(2v + yv_y)\epsilon^4 + \beta\epsilon^5 v_y - \frac{3}{2}\beta\epsilon^6 v_{yy} + \frac{1}{6}\beta\epsilon^7 v_{yyy} - \frac{1}{3}\beta\epsilon^6 y v_{yyy} - (2\alpha + 4\kappa)v_y - \left(\frac{1}{3}\alpha + \kappa\right)\epsilon^2 v_{yyy} - \kappa\epsilon^2 v v_y. \tag{4.32}$$

As with earlier calculations presented in the previous subsections, it is not necessary to retain all these terms. Neglecting the higher order corrections, we will proceed with

$$\frac{-v_t}{\epsilon u_0(1+\epsilon^2 v)} = 4\epsilon^4\beta v + 2\epsilon^4\beta y v_y + (2\alpha + 4\kappa)v_y + \left(\frac{1}{3}\alpha + \kappa\right)\epsilon^2 v_{yyy} + \epsilon^2\kappa v v_y. \tag{4.33}$$

As above, we multiply through by $(1+\epsilon^2 v)$, and transform to a moving coordinate frame to remove the first spatial derivative terms. This can be done in various ways, using $z = y - \epsilon^p s(\tau)$ and $\tau = \epsilon^3 u_0 t$, with the differing values of p considered in turn below.

4.6.1. The subcase $p = -3$.

Using

$$z = y - \epsilon^{-3}s(\tau), \quad \tau = \epsilon^3 u_0 t, \tag{4.34}$$

combining the leading order and first correction terms, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = 2\beta\epsilon^2 s + 2\epsilon(\alpha + 2\kappa), \tag{4.35}$$

from the coefficients of v_z. This ODE is solved by

$$s = C\mathrm{e}^{2\beta\epsilon^2\tau} - \left(\frac{\alpha + 2\kappa}{\beta\epsilon}\right), \tag{4.36}$$

and whatever value for C is chosen, the equation for $v(z,\tau)$ is

$$0 = v_\tau + (2\alpha + 5\kappa)v v_z + \left(k + \frac{1}{3}\alpha\right)v_{zzz}, \tag{4.37}$$

which is the KdV equation with no perturbing terms.

4.6.2. The subcase $p = -2$.

In this case we apply

$$z = y - \epsilon^{-2}s(\tau), \quad \tau = \epsilon^3 u_0 t, \tag{4.38}$$

which yields the equation for $s(\tau)$

$$s(\tau) = 2(\alpha + 2\kappa)\tau + C. \tag{4.39}$$

The shape of the profile $v(z,\tau)$ is determined by

$$0 = v_\tau + (2\alpha + 5\kappa)v v_z + \left(k + \frac{1}{3}\alpha\right)v_{zzz} + 2\beta v_z(C + 2\alpha\tau + 4\kappa\tau), \tag{4.40}$$

which is a perturbed KdV equation, the perturbation being the final term. This can be removed by the transformation

$$v = w - \frac{2\beta(4\kappa\tau + 2\alpha\tau + C)}{2\alpha + 5\kappa}, \tag{4.41}$$

for some constant, C, which yields the integrable forced KdV equation

$$0 = w_\tau + (2\alpha + 5\kappa)w w_z + \left(\kappa + \frac{1}{3}\alpha\right)w_{zzz} - \frac{4\beta(\alpha + 2\kappa)}{(2\alpha + 5\kappa)}. \tag{4.42}$$

To demonstrate the integrability of this equation, we show how it can be transformed to the standard KdV equation. The substitution $w(z,\tau) = V(Z,\tau) - C\tau$ with $Z = z + \frac{1}{2}CN\tau^2$ maps the constantly-forced KdV equation $0 = w_\tau + Nww_z + Dw_{zzz} + C$ onto the classic form $0 = V_\tau + NVV_Z + DV_{ZZZ}$.

4.6.3. The subcase $p = -1$

Here we substitute

$$z = y - \epsilon^{-1}s(\tau), \quad \tau = \epsilon^3 u_0 t, \tag{4.43}$$

into (4.33), and by combining the leading order and first correction terms we obtain the equation

$$\epsilon\frac{\mathrm{d}s}{\mathrm{d}\tau} = 2\alpha + 4\kappa \tag{4.44}$$

for $s(\tau)$. This equation has the solution

$$s(\tau) = \frac{(2\alpha + 4\kappa)\tau}{\epsilon} + C. \tag{4.45}$$

The second order correction terms from (4.33) then yield the KdV equation

$$0 = v_\tau + (2\alpha + 5\kappa)vv_z + \left(k + \frac{1}{3}\alpha\right)v_{zzz}, \tag{4.46}$$

for the evolution of the shape of the wave. Thus there are travelling wave solutions, interacting soliton solutions, as well as more complex phonon-nonlinear wave interactions.

4.7. Summary

As β_0 takes progressively smaller values, whilst we hold $\alpha, \kappa = \mathcal{O}(1)$, we have observed a shift from the stable solution $v = 0$ which is approached exponentially in time, according to an advective PDE with a self-similar structure (4.3) and (4.5), to the KdV Eq. (4.46), which supports travelling waves and solitons which interact elastically. There is also a slowing of the kinetics due to these being dominated by the β_0 terms, whereas the KdV behaviour is caused by the α, κ terms which have a reduced impact in the long wave length limit.

Only the α terms are in the original Volterra equation, though the effect of the κ terms is to leave the form of the equation unchanged, and only to alter the strength of the nonlinearity, the size of the dispersion and the speed of the wave (in the transformation from $y = \epsilon x$ to z). However, the effect of the β terms changes the form and behaviour of the solution considerably, since the terms involving β give rise to a moving coordinate frame which accelerates exponentially in time. The effects (κ, α, and β) are balanced when one considers $\beta = \mathcal{O}(\epsilon^3)$ and timescales of $\mathcal{O}(\epsilon^{-3})$ as exemplified in equation (4.16) and (4.23).

5. Solutions of Eqs. (4.18) and (4.23)

In this section we summarise various methods for solving Eqs. (4.18) and (4.23), namely

$$0 = v_\tau + 4\beta v + 2\beta z v_z + (2\alpha + 5\kappa)vv_z + \left(\kappa + \frac{1}{3}\alpha\right)v_{zzz}. \tag{5.1}$$

As well as the trivial solution $v = 0$, there is clearly a spatially uniform solution $v(\tau) = v_0 \mathrm{e}^{-4\beta\tau}$; further general similarity solutions of this equation can be found by using Lie group methods, we also detail the derivation of more complicated solutions through the application of methods from integrable systems theory.

5.1. Lie group methods

We aim to find solutions of the general equation

$$v_t + 4\beta v + 2\beta z v_z + a v v_z + b v_{zzz} = 0, \tag{5.2}$$

where $a = 2\alpha + 5\kappa$ and $b = \kappa + \alpha/3$, through the use of Lie group techniques [13]. The infinitesimal generators are given by

$$\xi = 2c_2\beta z\mathrm{e}^{6\beta t} + c_3\mathrm{e}^{2\beta t} + c_4\mathrm{e}^{-4\beta t}, \tag{5.3}$$

$$\tau = c_1 + c_2\mathrm{e}^{6\beta t}, \tag{5.4}$$

$$\eta = -\frac{4\beta}{a}\left(\frac{3}{2}c_4\mathrm{e}^{-4\beta t} + ac_2 v\mathrm{e}^{6\beta t}\right), \tag{5.5}$$

where c_i are constants. We consider all cases in turn.

5.1.1. The case $c_2 = 0$.

Without loss of generality we can assume $c_1 = 1$. This generates the solution

$$v = \frac{3c_4\mathrm{e}^{-4\beta t}}{2a} + p(\zeta), \quad \zeta = z - \frac{c_3\mathrm{e}^{2\beta t}}{2\beta} + \frac{c_4\mathrm{e}^{-4\beta t}}{4\beta}, \tag{5.6}$$

where $p(\zeta)$ is given by the second order equation

$$2\beta\zeta p' + 4\beta p + app' + bp''' = 0. \tag{5.7}$$

This equation integrates to give the so-called P_{34} equation referred to by Ince [7].

5.1.2. The case $c_1 = 0,\ c_2 = 1$.

Here we find

$$v = \frac{c_4 e^{-10\beta t}}{a} + e^{-4\beta t} p(\zeta), \quad \zeta = e^{-2\beta t}\left(z + \frac{c_3 e^{-4\beta t}}{6\beta} + \frac{c_4 e^{-10\beta t}}{12\beta}\right), \tag{5.8}$$

where $p(\zeta)$ satisfies the equation

$$bp'' + \frac{1}{2}ap^2 - c_2 p - \frac{6\beta c_4 \zeta}{a} = K. \tag{5.9}$$

This equation is equivalent to the first Painlevé equation, P_I [7].

5.1.3. The case $c_3 = c_4 = 0,\ c_2 = 1,\ c_1 \neq 0$.

In this case we obtain the solution

$$v = \frac{p(\zeta)}{(c_1 + e^{6\beta t})^{2/3}}, \quad \zeta = z(c_1 + e^{6\beta t})^{-1/3}, \tag{5.10}$$

where

$$c_1(2\beta\zeta p' + 4\beta p) + app' + bp''' = 0, \tag{5.11}$$

which also integrates to P_{34} [7].

5.2. Mapping of (5.2) to KdV

Eq. (5.2) can be mapped onto the KdV equation *via*

$$v(z,t) = e^{-4\beta t} w(Z,T), \tag{5.12}$$

where $Z = ze^{-2\beta t},\ T = e^{-6\beta t}$, which yields

$$6\beta w_T = aww_Z + bw_{ZZZ}. \tag{5.13}$$

Hence all the integrable structures associated with the KdV equation, such as Darboux transformations, Backlund transforms apply to (5.2).

The one-soliton solution of the KdV Eq. (5.13) produces the solution

$$v = \frac{2\mu e^{-4\beta t}}{a}\operatorname{sech}^2\left(e^{-2\beta t}\sqrt{\frac{\mu}{6b}}\left(z + \frac{\mu e^{-4\beta t}}{9\beta}\right)\right), \tag{5.14}$$

where μ is an arbitrary constant. This is a special case of similarity solution (5.8) when $c_4 = 0$. However, (5.13) also has two-soliton solutions, hence (5.2) has two-soliton solutions too, for example

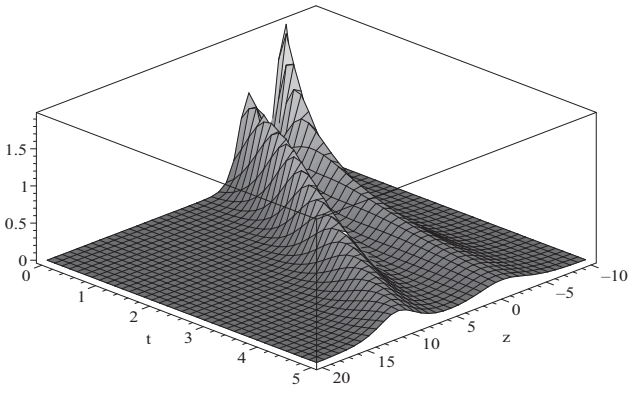

Fig. 3. Illustration of the evolution of a solution $v(z,t)$ from (5.2) using the two-soliton solution of the KdV equation. The parameters are given by $\alpha = 1,\ \kappa = 1,\ \beta = 0.1,\ k_1 = 1.5, k_2 = 2,\ \gamma_1 = 0,\ \gamma_2 = -5$.

$$
\begin{aligned}
v &= \tfrac{12be^{-4\beta t}}{a}\left[\tfrac{N}{D}\right],\\
N &= k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + 2(k_2-k_1)^2 e^{\theta_1+\theta_2} + (k_2-k_1)^2(k_2^2 e^{\theta_1} + k_1^2 e^{\theta_2})e^{\theta_1+\theta_2}/(k_2+k_1)^2,\\
D &= \left(1 + e^{\theta_1} + e^{\theta_2} + (k_2-k_1)^2 e^{\theta_1+\theta_2}/(k_2+k_1)^2\right)^2,
\end{aligned}
\tag{5.15}
$$

where $\theta_i = k_i z e^{-2\beta t} + b k_i^3 e^{-6\beta t}/6\beta - \gamma_i$ $(i = 1, 2)$. This gives rise to more complicated decay expressions for $v(z,t)$ as t increases (assuming $\beta > 0$; there are growing solutions for $\beta < 0$). An example is shown in Fig. 3.

5.3. Summary

In this section we have analysed Eq. (5.2) using a variety of techniques from classical and modern theories of differential equations. We have obtained a range of solutions, which may be relevant for various parameter ranges.

6. Conclusions

We have performed a basic stability analysis of the extended Volterra system. The parameters β_0 and β_1 impose significant new features into the system, including a unique steady-state. This is partly due to the fact that non-autonomous terms are introduced into the equations. We have considered a variety of scalings for the relative sizes of the parameters $\beta_0,\ \beta_1,\ \alpha,\ \kappa$ and derived a number of PDEs in the continuum limit. These show a range of behaviour from self-similar convergence to the zero solution to KdV kinetics. In the case of $\beta_0 = 0 = \beta_1$, more standard kinetics are observed.

These all occur for the case of large-wavelength waves, where $x \sim \epsilon^{-1}$ and $u \sim \epsilon^2$ for $\epsilon \ll 1$. We derived a variety of perturbed KdV equations many of which are integrable, and we have shown how they can be transformed to the standard KdV equation. For larger values of the parameter β, we found that the governing equation is of a simpler form, and typically permits similarity solutions.

More specifically, in this case, the α and κ terms give rise to KdV-type terms. The wave-like behaviour of this equation is changed by the β terms, which give rise to growth or decay of waves in a moving coordinate frame which is accelerating. The steady-state solution $v(x,t) =$ constant is thus stable or unstable depending on the sign of β_1. When the β terms dominate, self-similar behaviour is observed in the approach to the steady-state solution. The effects of the $\alpha,\ \beta,\ \kappa$ terms balance when we assume the asymptotic limit $y = \epsilon x = \mathcal{O}(1)$ with $\epsilon \ll 1,\ \beta_0,\ \beta_1 = \mathcal{O}(\epsilon^3)$ and the timescale of $\mathcal{O}(\epsilon^{-3})$ is considered.

We also investigated the possibility of mKdV-type scalings where $x \sim \epsilon^{-1}$ and $u \sim \epsilon$, however, no such equations could be derived, since the condition for the quadratic nonlinearity to vanish (leaving the cubic as leading-order) also causes the dispersion term to vanish.

These results have shown once again the deep connection between integrable discrete systems and integrable continuous systems. Whilst taking the continuum limit appears in many cases to preserve the properties of integrability, the reverse process remains difficult, in that 'natural' discretisations of continuous systems are rarely integrable, meaning that it is necessary to resort to sophisticated techniques to construct more complex discrete integrable systems, for example, the nonisospectral techniques used in [2]. In future work we propose to analyse the extended Toda lattice [1] in a similar fashion.

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