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Ph.D. Thesis

On Inflationary Cosmological Models

S.S.D. MAT/07

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To my family

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ABSTRACT

The most common mechanisms leading to inflation are based on models of gravity minimally coupled to a scalar field ϕ rolling on a suitable potential $V(\phi)$. We discuss such a model defined by the action $I = \int \sqrt{-g}\left[R - 2(\partial\phi)^2 - V(\phi)\right] d^4x$, in order to find exact general isotropic and homogeneous cosmological solutions displaying an inflationary behavior at early times and a power-law expansion at late times.
We also study the effect of the inclusion of matter (in the form of a perfect fluid): in this case, we do not find exact solutions because of the non-integrability of the field equations, but we can investigate their global properties (and hence their stability) by means of methods of the theory of dynamical systems.

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INTRODUCTION

It is well known that the history of ideas on the structure and origin of the universe shows that humankind has always put itself at the center of creation. The sixteenth and seventeeth centuries, in fact, were characterized by the advent of the new heliocentric cosmology; until that time, actually, it was belevied that the earth was at the centre of the universe, when, in 1543, Nicholas Copernicus proposed a revolutionary theory with the sun, in place of the earth, at the centre of the cosmic system. The first true astrophysicist was actually Johannes Kepler, who, at the beginning of the seventeeth century, discovered that all the planets revolve around the sun in ellipses and the other two laws which describe mathematically how the planets move through the sky. Kepler indeed suggested that all the planets were kept in orbit by a force emanated by the sun; this idea was later proved by Isaac Newton, who even discovered gravitational force.

Obviously, religion played a fundamental role in the work of these scientists. For example, Kepler wrote that through his efforts God was being celebrated in astronomy, Newton has been described as a fanatic religious and he wanted only that his cosmology would help to convince people of the existence of God. These facts show that at the beginning science and religion were not at war with one another: this belief arose at the end of the nineteenth century, after the publication of Darwin's theory of evolution. The most important clash between science and religion is caused by Galileo, who strengthened the idea that the earth revolves around the sun, rather than the sun circling the earth.

The aim of cosmology is to explain the origin and evolution of the entire contents of the universe, the underlying physical processes, and to obtain a deeper understanding of the laws of physics assumed to hold throughout the universe. In fact, in the last decades one of the points that has emerged from cosmological studies is that the universe is not simply a random collection of irregularly distributed matter, but it is a single entity. This is the view taken in the standard models: we may have to modify these assertions when considering the inflationary models.

Unfortunately, for a very long time, the subject of cosmology consisted in a speculative approach to metaphysical issues: this radically changed only in the first half of the twentieth century with the advent of the theory of general relativity, which enabled us for the first time in history to come up with a testable theory of the universe and to understand many observations starting with galaxies receding. In the second half of the twentieth century, the hot big bang model was formulated, including the description of physical processes that occur in this expanding space-time, and the associated thermal history of the universe.
Considering the large-scale structure of the universe, the basic constituents can be taken to be galaxies, which are congregations of about 10^{11} stars bound together by their mutual gravitational attraction. Galaxies tend to occur in groups called *clusters*, and each cluster contains anything from a few to a few thousand galaxies. Observations indicate that on average galaxies are spread uniformly throughout the universe at any given time. This means that if we consider a portion of the universe which is large compared to the distance between typical nearest galaxies, then the number of galaxies in that portion is roughly the same as the number in another portion with the same volume at any given time. Moreover, the distribution of galaxies appears to be isotropic, namely it is the same, on the average, in all directions from us.
E. P. Hubble discovered (1929) that the distant galaxies are moving away from us. The velocity of recession (discovered by studying their redshifts) follows Hubble's law, according to which the velocity is proportional to distance. This rule is approximate because it does not hold for galaxies which are very near nor for those which are very far. The very distant galaxies, for instance, show departures from Hubble's law partly because light from there was emitted billions of years ago and the systematic motion of galaxies in those epochs may have been significantly different from that of the present epoch. By studying the departure from Hubble's law of the very distant galaxies one can get useful information about the overall structure and evolution of the universe.
The other important feature of our universe is homogeneity. Isotropy and homogeneity lead us to make an assumption about the model universe, known as *Cosmological Principle* (see Section 1.1), which was then validated by many observations, for example by redshift surveys (which suggest that the universe has those peculiarities only on $\sim$100 Mpc scales, while on smaller scales there exist large inhomogeneities such as galaxies, clusters and superclasters) and by the isotropy of the cosmological microwave background (CMB). This principle allows us to simplify the study of the large-scale structure of the universe and, moreover, it implies that the distance between any two typical galaxies has a universal factor, $a(t)$, called *scale factor*. One of the major current problems of cosmology is to determine the exact form of this function of time.
It is believed [1] that between 10 and 20 billion years ago there was a universal explosion, at every point of the universe, called the *Big Bang*. The explosion could have been at every point of an infinite or a finite universe. In the latter case the universe would have started from zero volume. An infinite universe remains infinite in spatial extent all the time down to the initial moment; as in the case of the finite universe, the matter becomes more and more dense and hot as one traces the history of the universe to the initial moment,

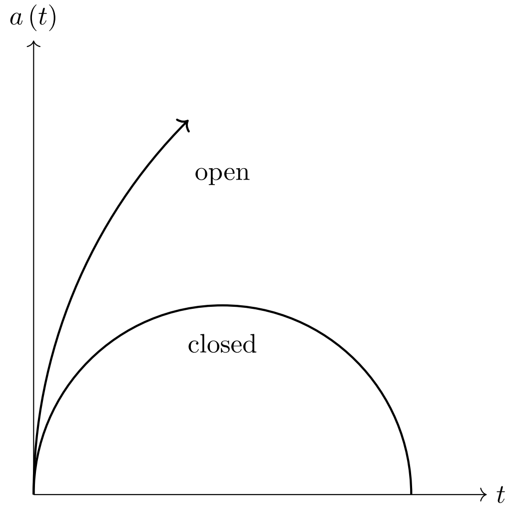


Figure 1: Evolution of the scale factor with time

which is a *space-time simgularity*. The universe is expanding now because of the initial explosion, and the motion of galaxies can be explained as a remnant of the initial impetus. The recession is slowing down because of the gravitational attraction of different parts of the universe to each other, at least in the simplest models. This is not necessarily true in models with a cosmological constant.

The expansion of the universe may continue forever, as in the *open* models, or it may halt at some future time and contraction set in, as in the *closed* models, in which case the universe will collapse at a finite time later into a space-time singularity with infinite or near infinite density (see Fig. 1). In the Friedmann models the open universes have infinite spatial extent whereas the closed model are finite (this is not valid for the Lemaître models).

There is an important piece of evidence apart from the recession of the galaxies that the contents of the universe in the past must have been in a highly compressed form. This is the *Cosmic Background Radiation* (CBR), which was discovered by Penzias and Wilson in 1965 and confirmed by many observations later. The existence of this radiation can be explained as follows. As we trace the history of the universe backwards to higher densities, at some stage galaxies could not have had a separate existence, but must have been merged together to form one great continuous mass. Due to the compression the temperature of the matter must have been very high. There is reason to believe that there must also have been present a great deal of electromagnetic radiation, which at some stage was in equilibrium with the matter. The spectrum of the radiation would thus correspond to a black body of high temperature. There should be a remnant of this radiation, still with black-body

spectrum, but corresponding to a much lower temperature.
The standard big bang model of the universe has had three major successes. Firstly, it predicts that something like Hubble's law of expansion must hold for the universe. Secondly, it predicts the existence of the microwave background radiation (MBR). Thirdly, it predicts successfully the formation of light atomic nuclei from protons and neutrons a few minutes after the big bang. Other problems remain in the standard model, like the so-called *horizon problem* and *flatness problem*.
To deal with these problems, Alan Guth (1981) proposed a model of the universe, known as the inflationary model, which does not differ from the standard model after a fraction of a second or so, but from about 10^{-45} to 10^{-30} seconds it has a period of accelerated expansion (inflation), during which time typical distances (the scale factor) increase by a factor of about 1050 more than the increase that would obtain in the standard model. Although the inflationary models (there have been variations of the one put forward by Guth originally) solve some of the problems of the standard models, they throw up problems of their own, which have not all been dealt with in a satisfactory manner.
According to inflationary theory, the universe continues to be homogeneous and isotropic over distances larger than 3000 Mpc (that is the order of the observable patch of the universe), but it becomes highly inhomogeneous on much larger scales. This behavior dampens our hope of comprehending the entire universe, and questions like "What portion of the whole universe is like the part we find ourselves in? What fraction is spatially flat, accelerating or decelerating?" are difficult to answer and hard to put in a mathematically precise way. Moreover, even if a suitable mathematical definition can be found, it is difficult to verify empirically any theoretical predictions concerning scales greatly exceeding the observable universe. On the contrary, any cosmological model must be consistent with established facts. While the standard big bang model accomodates most known facts, a physical theory is also judged by its predictive power: inflationary theory answered to these requests.

This thesis is organized as follows:
In Chapter 1 we make a brief description of our universe, focusing on its most important geometrical features: homogeneity and isotropy, which are clearly manifest if one considers a convenient coordinate system that gives rise to the so-called Friedmann-Lemaître-Robertson-Walker metric. Then, we derive the kinematical and dynamical properties of this metric, and we deal with some solutions of it, for example the Friedmann-Lemaître and the de Sitter universe. At the end of this chapter, we make a brief excursus on some alternative theories to general relativity, in particular we give a general description of scalar-gravity theories, Kaluza-Klein theory and higher-derivative theory.
In Chapter 2 we present cosmic inflation as solution of flatness and horizon problems, the most important incongruities of the standard model. We analyze, in particular, inflation caused by a single scalar field ϕ (the inflaton), minimally coupled to gravity, rolling on a scalar potential. This kind of inflation occurs when the universe is dominated by ϕ and then the universe is driven into a de Sitter expansion. At the end of this period of

accelerated expansion, the inflaton, rolling down its potential, reaches the minimum of the potential and the inflation ends. Finally, we also describe the simplest inflationary scenarios, catalogued in three classes: old inflation, new inflation and chaotic inflation.
Chapter 3 is reserved to the discussion about dynamical systems, starting from the basics of this theory and then presenting a general application to Einstein's equations.
Chapter 4 contains my original contributions, concerning a model of gravity coupled to a scalar field that admits exact cosmological solutions with an exponential behavior at early times and a power-law expansion at late times. In the case of inclusion of matter, exact solutions are not available, and hence we apply the theory discussed in Chapter 3 in order to study the asymptotic behaviours of the solutions.

CHAPTER 1

THE HOMOGENEOUS AND ISOTROPIC UNIVERSE

1.1 Some geometrical aspects of our universe

According to the standard big bang model [2], our universe's birth happened about 15 billion of years ago with a homogeneous and isotropic distribution of matter in three-dimensional space, at very high temperature and density (a statement called *Cosmological* or *Copernican Principle*), and has been expanding and cooling since then. Actually, thanks to the isotropy on large scales space-time it has a spherical symmetry: combining this observational fact with the Copernican principle, we may conclude that the universe must be homogeneous on large scales.
It is clear that the cosmological principle considerably simplifies the study of the large-scale structure of the universe: it implies, among other things, that the distance between any two galaxies has a universal factor. In order to show this, let us consider three galaxies separated by distances so large that local irregularities are ignored [3]. If the universe is expanding in a homogeneous and isotropic way, the triangle defined by the three galaxies must at all times remain similar to the original triangle. This means that the length of each side has to scale by the same factor, say $a\,(t)$, as the universe expands. By extending the net to a fourth galaxy and so on, we see that $a\,(t)$ has to be a universal scale factor. Thus the distance between two galaxies satisfies

$$l\,(t) = l_0\,a\,(t)\ ,$$

where l_0 is independent of time. In other words, the large-scale structure and behaviour of the universe can be described by the function of time $a\,(t)$, called the *scale factor* of the universe. We shall see later its physical meaning.

In this scenario, the unique expansion law compatible with homogeneity and isotropy is the *Hubble law*, a statement of a direct correlation between the distance r from the observer to a galaxy and its recessional velocity v as determined by the redshift. It can be stated as

$$v = H_0 (t) r , \tag{1.1.1}$$

where the Hubble parameter $H_0 (t)$ depends only on the time t and it measures the expansion rate. Its value has varied widely over the years, but it is known today as about 65-80 km/(s Mpc). The physical meaning of Hubble's law is that the universe is expanding: this message overwhelmed Einstein because he had until then firmly believed in a static universe. This marked the beginnig of modern cosmology and it sets the main requirement on theory.
Equation (1.1.1) shows that [4] the Hubble parameter has the dimension of inverse time; in this way, a characteristic timescale for the expansion of the universe is the *Hubble time* $\tau_H = H_0^{-1}$, while the size scale of the observable universe is the *Hubble radius* $R_H = \tau_H c$, where c is the speed of light. In the following, we consider $c = 1$.
We can geometrically represent Hubble expansion by mean of the two-dimensional surface of an expanding sphere uniformly covered on its surface with dots that represent galaxies. As the sphere expands, all dots move away from each other with speeds proportional, at any given time, to the distance. Let us consider any two points A and B on the surface of the sphere of radius $a (t)$, and θ_{AB} the angle subtended at the centre by the two points. This angle remains unchanged as its radius increases and the distance r_{AB} between the dots is given by $r_{AB} = \theta_{AB} a (t)$. Therefore, the Hubble law reads

$$v_{AB} = \dot{r}_{AB} = \dot{a} (t) \theta_{AB} = \frac{\dot{a}}{a} r_{AB} , \tag{1.1.2}$$

where dot denotes a derivative with respect to time t and $H (t) \equiv \dot{a}/a$. Integrating this equation we obtain

$$\vec{r}_{AB} (t) = a (t) \chi_{AB} , \tag{1.1.3}$$

where

$$a (t) = \exp \left(\int H (t) \, dt \right) \tag{1.1.4}$$

is the scale factor mentioned above and is the analogue of the radius of the 2-sphere. The integration constant χ_{AB} can be interpreted as the distance between points A and B at some particular moment of time: t is called the *comoving* coordinate of B, assuming a coordinate system centered at A.
Moreover, Hubble's law implies arbitrarily large velocities of the galaxies as the distance increases indefinitely. There is thus an apparent contradiction with special relativity which can easily be solved: the redshift z is defined as $z = (\lambda_r - \lambda_i) / \lambda_i$, where λ_i is the original wavelength of the radiation given off by galaxy and λ_r is the wavelength of this radiation when we received it. As soon as the velocity of the galaxy approaches that of light, $z \to \infty$, thus we cannot observe higher velocities than that of light.

From the receding rate of the galaxies [1], it can be deduced that all galaxies must have been very close to each other at the same time in the past. With regard to the geometrical analogy, it means that the sphere must have started with vanishing radius and at this initial time all points must have been on top of each other. At this moment, as we told in the introduction, the big bang took place and the matter becomes more dense and hot, while the universe continues to expand because of the explosion. This expansion may continue forever (open universe) or not (closed universe): in the latter case, the universe will collapse at a finite time later into a space-time singularity with infinite or near infinite density. Unfortunately, it is not known at present if the universe is open or closed, even if there are several ways by which this could be determined, even if they contain many uncertainties and they don't definitely solve the problem. However, one of these methods is to measure the present average density of the universe and compare it with a certain critical density: if the density is above the critical density, the attractive force of different parts of the universe towards each other will be enough to halt the recession eventually and to pull the galaxies together. If the density is below the critical density, the attractive force is insufficient and the expansion will continue forever.
Another way of determing if the universe will expand forever is to measure the rate at which the expansion of the universe is slowing down, using hte deceleration parameter q_0. Theoretically in the simpler models, in suitable units, the deceleration parameter is half the ratio Ω of the actual density to the critical density. If $\Omega < 1$, the density is subcritical and the universe will expand forever.
Another way is to determine the precise age of the universe and compare it with the Hubble time, defined as the time elapsed since the big bang until now if the rate of expansion had been the same as at present. Looking at Fig. 1.1, if ON represents the present time t_0, then PN is the scale factor $a\left(t_0\right)$. In this way,

$$\tan\alpha = PN/NT = \dot{a}\left(t_0\right) \implies NT = a\left(t_0\right)/\dot{a}\left(t_0\right) = H_0^{-1}\,. \tag{1.1.5}$$

In other words, the Hubble time is the reciprocal of Hubble's constant. Knowing the value of the Hubble constant, we can obtain a rough estimate for the age of the universe. For the actual value of the Hubble constant, t_0 is about 15 billion years: the exact value for the age of the universe differs from this rough estimate by a factor of order unity, depending on the composition and curvature of the universe (see Sec. 1.2.9).
The assumption that our universe is homogeneous and isotropic means that its evolution can be represented as a time-ordered sequence of three-dimensional space-like hypersurfaces, each of which is homogeneous (i.e. every point of any given hypersurfaces has the same physical conditions) and isotropic (i.e. the physical conditions are identical in all directions from a given point on the hypersurface). Obviously, homogeneity does not necessarily imply isotropy because, for example, a homogeneous yet anisotropic universe can contracts in one direction and expands in the other two directions.
Homogeneous and isotropic spaces have the largest possible symmetry group, or, following Weinberg ([5]), are *maximally symmetric*. Maximally symmetric spaces are uniquely determined by the curvature constant and the signature of the metric (number of positive and

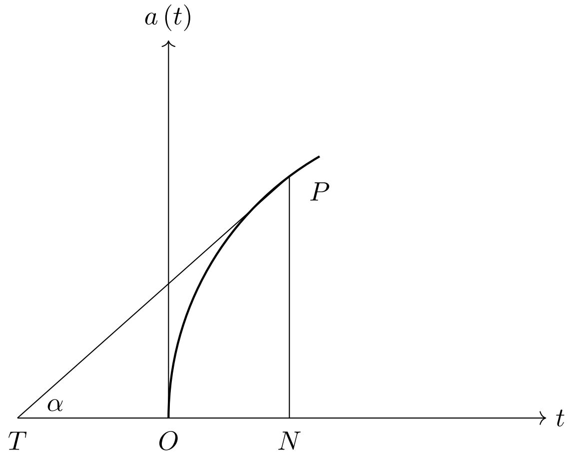


Figure 1.1: Definition of Hubble time

negative terms in the diagonal form). In a n-dimensional space, the number of such symmetries is equal to $n(n+1)/2$. For example, Minkowski space-time is 4D manifold with constant curvature, and it admits 10 isometries that correspond to the 10 transformations of Poincaré group.
Obviously, these symmetries strongly restrict the admissible geometry for such spaces. There exist only three types of homogeneous and isotropic spaces with simple topology:

(a) flat space,

(b) a three-dimensional sphere of constant positive curvature,

(c) a three-dimensional hyperbolic space of constant negative curvature.

The only way to preserve the homogeneity and isotropy of space and yet incorporate time evolution is to allow the curvature scale, characterized by a, to be time-dependent. The scale factor $a(t)$ thus completely describes the time evolution of a homogeneous, isotropic universe. There exist, however, preferred coordinate systems in which the symmetries of the universe are clearly manifest. In one of the most convenient of such coordinate systems, the interval takes the form

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \tag{1.1.6}$$

called *Friedmann-Lemaître-Robertson-Walker* (FLRW) metric, where t is the cosmic time, r the adimensional comoving radial coordinate, θ and ϕ the comoving angular coordinates. Comoving means that a particle at rest in these coordinates remains at rest (i.e., constant

coordinates). The scale factor has units of length.
When $k = 1$ the metric (1.1.6) represents the universe with positive spatial curvature whose spatial volume is finite. In fact, introducing a new coordinate ψ such that $r = \sin\psi$, the metric becomes

$$ds^2 = -dt^2 + a^2(t)\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \qquad (1.1.7)$$

and we can embed its spatial part

$$d\sigma_1^2 = a^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \qquad (1.1.8)$$

in a 4D Euclidean space Σ with coordinates (x, y, z, w) and in which the metric is given by

$$d\Sigma^2 = dx^2 + dy^2 + dz^2 + dw^2\,. \qquad (1.1.9)$$

Considering a surface in this space given by

$$x = a\cos\psi\,,\ y = a\sin\psi\sin\theta\cos\phi\,,\ z = a\sin\psi\sin\theta\sin\phi\,,\ w = a\sin\psi\cos\theta\,, \qquad (1.1.10)$$

we obtain

$$x^2 + y^2 + z^2 + w^2 = a^2\,. \qquad (1.1.11)$$

It is a simple exercise to get the spatial part of (1.1.7) from (1.1.10).
Since the rotations in the 4D embedding space (which can be affected by a 4×4 orthogonal matrix) can move any point and any direction on the 3-sphere into any other point and direction respectively, all points and directions on a 3-sphere in a 4D Euclidean space are equivalent. Moreover, these rotations leave the metric (1.1.9) and the equation of the 3-sphere (1.1.11) unchanged. This shows that the metric (1.1.6) is homogeneous and isotropic. The entire surface is swept by the coordinate range $0 \le \psi \le \pi\,,\ 0 \le \theta \le \pi\,,\ 0 \le \phi \le 2\pi$ and the volume is

$$\int (a d\psi)\,(a\sin\psi d\theta)\,(a\sin\psi\sin\theta d\phi) = 2\pi^2 a^3\,, \qquad (1.1.12)$$

which is finite.
If $k = 0$, the spatial metric is given by

$$d\sigma_2^2 = a^2\left[d\psi^2 + \psi^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]\,, \qquad (1.1.13)$$

where $0 \le \psi \le \infty\,,\ 0 \le \theta \le \pi\,,\ 0 \le \phi \le 2\pi$. The spatial volume is infinite and it represents the universe with zero spatial curvature.
When $k = -1$, the spatial part of this metric cannot be embedded in a 4D Euclidean space, but it can be embedded in a 4D Minkowski space with metric

$$ds^2 = dx^2 - dy^2 - dz^2 - dw^2\,. \qquad (1.1.14)$$

Its spatial part takes the form

$$d\sigma_3^2 = a^2\left[d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]\,. \qquad (1.1.15)$$

Following the same approach of the unitary curvature case, we can transform to a Minkowski space parametrized by

$$x = a\cosh\psi\,,\; y = a\sinh\psi\sin\theta\cos\phi\,,\; z = a\sinh\psi\sin\theta\sin\phi\,,\; w = a\sinh\psi\cos\theta\,. \quad (1.1.16)$$

In this case, the surface obtained by fixing $\psi = \psi_0$ is the 2-sphere given by

$$y^2 + z^2 + w^2 = a^2\sinh^2\psi_0\,, \quad (1.1.17)$$

whose surface area is $4\pi a^2 \sinh^2\psi_0$, which keeps on increasing indefinitely as ψ_0 increases. Since the radius of the sphere is $a\psi_0$, the surface area is larger than that of a sphere of radius $a\psi_0$ in Euclidean space. The surface is swept by $0 \le \psi \le \infty\,,\; 0 \le \theta \le \pi\,,\; 0 \le \phi \le 2\pi$ and the spatial volume is infinite.

Because of the time dependence of the scale factor, an important kinematic property of FLRW space-times is that light gets redshifted as it travels. In few words, light, emitted at time t_1 with frequency ν_1 from an object at rest in the comoving coordinate system, with radial coordinate r_1, and received at time t_0 wth frequency ν_0 by an observer located at $r_0 = 0$, travels along geodesics with $ds = 0$ following

$$\frac{dt}{a\,(t)} = \frac{dr}{\sqrt{1-kr^2}}\,. \quad (1.1.18)$$

Since the time delay δt between two crests is the inverse frequency, we have

$$\frac{\delta t_0}{a\,(t_0)} = \frac{\delta t_1}{a\,(t_1)} \quad \Rightarrow \quad \frac{\nu_1}{\nu_0} = \frac{a\,(t_0)}{a\,(t_1)} = 1 + z\,, \quad (1.1.19)$$

where the last equality defines the redshift z. This quantity only depends on the ratio of the scale factor at reception to the scale factor at emission.

In general relativity [6], the dynamical variables characterizing the gravitational field are the components of the metric $g_{\mu\nu}$ and they obey the Einstein equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \quad (1.1.20)$$

which are derived from the *Einstein-Hilbert* (EH) *action*

$$S = \int \left[\frac{1}{16\pi G}R + \mathcal{L}_M\right]\sqrt{-g}\,d^4x \quad (1.1.21)$$

proposed in 1915 by David Hilbert. In equations (1.1.20) and (1.1.21):

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} - \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\mu\rho} \quad (1.1.22)$$

is the Ricci tensor in terms of the Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\gamma\rho}\left(\frac{\partial g_{\mu\gamma}}{\partial x^\nu} + \frac{\partial g_{\nu\gamma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\gamma}\right)\,, \quad (1.1.23)$$

while R is the Ricci scalar curvature, Λ is the cosmological constant and $\mathcal{L}_M$ describes any matter fields appearing in the theory. Matter is incorporated in Einstein's equations through the symmetric, zero-divergent energy-momentum tensor $T_{\mu\nu}$.
On large scales, matter can be approximated as a perfect fluid characterized by energy density ρ, pressure p and 4-velocity u^μ: its energy-momentum tensor is

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p g_{\mu\nu}\,, \tag{1.1.24}$$

where the equation of state $p = p(\rho)$ depends on the properties of matter and must be specified. In many cosmologically interesting cases, $p = w\rho$, where w is constant. In the Section 1.2.3 we will derive in detail the conservation law and the equation of state from the Friedmann equations.
While the FLRW metric incorporates the symmetry properties and the kinematics of space-time, the Einstein equations provide the dynamics, namely the manner in which the matter and the space-time considered are affected by the forces present in the universe.

1.2 Cosmological models: Friedmann-Lemaître cosmologies

In this section we will deal with FLRW model of cosmology, probably the only paradigm based on Friedmann and Lemaître (FL) equations and Robertson-Walker (RW) metric, which takes both energy density and pressure to be functions of time in a Copernican universe. Einstein universe and Einstein-de Sitter universe are two of the solutions of FLRW model, even if they are now known to be unrealistic, while the currently accepted solution is the FL universe, which includes a positive cosmological constant.
With a substitution of the metric (1.1.6) and of energy-momentum tensor (1.1.24) into the Einstein equation (1.1.20), taking into account the conservation law (see Section 1.2.3 for a further derivation)

$$\dot{\rho} = -3H\,(\rho + p) \tag{1.2.1}$$

(with H the Hubble parameter) and integrating, we obtain two distinct dynamical relations for $a\,(t)$:

$$\begin{aligned} \frac{\dot{a}^2 + k}{a^2} &= \frac{8\pi G}{3}\rho\,, \\ \frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} &= -8\pi G p\,. \end{aligned} \tag{1.2.2}$$

These equations were derived in 1922 by Friedmann, seven years before Hubble's discovery, when Einstein did not believe in his own equations because they did not allow the universe to be static. Also Friedmann's equation did not gain recognition until after his death, when

they were confirmed by an independent derivation by Georges Lemaître in 1927.
Subtracting the first equation from the second one in (1.2.2) we obtain

$$\begin{aligned} H^2 + \frac{k}{a^2} &= \frac{8\pi G}{3}\rho\,, \\ \ddot{a} &= -\frac{4\pi}{3} G\,(\rho + 3p)\,a\,. \end{aligned} \tag{1.2.3}$$

Actually, the first is called the *Friedmann Equation*, and the second is called the *Raychaudhuri Equation*.
The Friedmann equation relates the change of the scale factor of the universe to its energy density, spatial curvature and cosmological constant. If the universe is assumed to be flat ($k = 0$), the only presence of energy will cause the universe to expand ($H > 0$) or to contract ($H < 0$).
From the Raychaudhuri equation, any form of matter such that $\rho + 3p < 0$ will cause an acceleration of the scale factor if it dominates the energy of the universe. The energy density is always positive, but in some cases the pressure can be negative and the previous inequality may be realized.
The combination of these two equations, supplemented by the equation of state $p = p\,(\rho)$, forms a complete system of equations that determines the two unknown functions $a\,(t)$ and $\rho\,(t)$. The solutions, and hence the future of the universe, depend both on the geometry and on the equation of state.
At our present time t_0 when the mass density is ρ_0, the cosmic scale is 1, the Hubble parameter is H_0 and the density parameter is $\Omega_0 \equiv \rho_0/\rho_c = 8\pi G\rho_0/3H_0^2$, Friedmann equation takes the form

$$\dot{a}_0^2 = \frac{8}{3}\pi G\rho_0 - k = H_0^2\Omega_0 - k\,, \qquad k = H_0^2\,(\Omega_0 - 1)\;. \tag{1.2.4}$$

From this equation, the relation between the RW curvature parameter k and the present density Ω_0 emerges: to the k values $+1$, 0 and -1 correspond $\Omega_0 > 1$, $\Omega_0 = 1$ and $\Omega_0 < 1$, respectively. The spatially flat case with $k = 0$ is called the *Einstein-de Sitter universe*.
To find particular solutions of the Friedmann equations it is often convenient to replace the physical time t with the conformal time η, defined as

$$\eta := \int \frac{dt}{a\,(t)}\,. \tag{1.2.5}$$

If $k = -1$, the first of (1.2.2) becomes

$$\frac{\dot{a}^2 - 1}{a^2} = \frac{8\pi G}{3}\rho\,, \tag{1.2.6}$$

from which we have

$$\frac{da}{dt} = \sqrt{\frac{8\pi G\rho}{3}a^2 + 1} \Rightarrow \int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho}{3}a^2 + 1}} . \tag{1.2.7}$$

Since $dt = a\,(\eta)\,d\eta$, we obtain

$$\eta - \eta_0 = \sinh^{-1}\left(\frac{a}{a_m}\right) \Rightarrow a\,(\eta) = a_m \sinh\eta\,, \tag{1.2.8}$$

where $a_m = \sqrt{\frac{8\pi G\rho}{3}}$ and η_0 has been fixed by requiring $a\,(\eta = 0) = 0$.
With the same approach one can find that, if $k = 0$, the scale factor in conformal time is $a\,(\eta) = a_m\eta$, and $a\,(\eta) = a_m \sin\eta$ when $k = 1$.
The physical time t is expressed in terms of η by integrating the relation $dt = ad\eta$:

$$t\,(\eta) = a_m \cdot \begin{cases} (\cosh\eta - 1)\ , & k = -1\,; \\ \eta^2/2\,, & k = 0\,; \\ (1 - \cos\eta)\ , & k = +1\,, \end{cases} \tag{1.2.9}$$

As we shall see, it follows that in the most interesting case of a flat radiation-dominated universe, the scale factor is proportional to the square root of the physical time, so that the energy density ρ_r is proportional to a^{-4}.
The range of conformal time η in flat and open universe is $0 < \eta < +\infty$, regardless of whether the universe is dominated by radiation or matter. For a closed universe, η is bounded: $0 < \eta < \pi$ and $0 < \eta < 2\pi$ in the radiation- and matter-dominated universes, respectively.

1.2.1 Einstein universe

A static universe is defined by $a\,(t) = \text{const}$, $a\,(t_0) = 1$ and infinite age of the universe. Friedmann equations (1.2.2) reduce to

$$k = \frac{8\pi}{3}G\rho_0 = -8\pi G p_0\,. \tag{1.2.10}$$

In order that the mass density ρ_0 be positive today, k must be 1, and hence the pressure of matter p_0 becomes negative.
Einstein corrected this in 1917 by introducing a covariantly constant Lorentz-invariant term $\Lambda g_{\mu\nu}$ in his equation (1.1.20), where the cosmological constant corresponds to a tiny correction to the geometry of the universe. Thus, the equations (1.2.2) take the form

$$\begin{gathered} \frac{\dot{a}^2 + k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho\,, \\ \frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} - \Lambda = -8\pi G p\,. \end{gathered} \tag{1.2.11}$$

A positive value of Λ curves space-time so as to counteract the attractive gravitation of matter. Einstein adjusted Λ to give a static solution, which is called the *Einstein universe*. The pression of matter is very small (if it were not small we would observe the galaxies having random motion similar to that of molecules in a gas under pressure). With a good approximation we can set $p = 0$. In the static case, the equation (1.2.11) becomes

$$k - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho_0 \,, \tag{1.2.12}$$

from which it follows that in a spatially flat universe

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = -\rho_0 \,. \tag{1.2.13}$$

Unfortunately, Einstein did not notice that the static solution is unstable: the smallest imbalance between Λ and ρ would make $\ddot{a} \neq 0$, causing the universe to accelerate into expansion or decelerate into contraction. This problem was noticed by Eddington in 1930 [7], after Hubble's discovery of the expansion that caused Einstein to abandon his convinction of a static universe and to consider the cosmological constant.

1.2.2 Friedmann-Lemaître universe

Lemaître noted that if the physics of the vacuum looks the same to any inertial observer, its contribution to the stress-energy tensor is the same as Einstein's cosmological constant Λ. This term is a correction to the geometrical terms in $G_{\mu\nu}$, but the mathematical contents of (1.2.11) are not changed if the Λ terms are moved to the right-hand side, where they appear as correction to the stress-energy tensor $T_{\mu\nu}$. Then the physical interpretation is that of an ideal fluid with energy density $\rho_\Lambda = \Lambda/8\pi G$ and negative pressure $p_\Lambda = -\rho_\Lambda c^2$. When the cosmological constant is positive, the gravitational effect of this fluid is a cosmic repulsion counteracting the attractive gravitation of matter, whereas a negative Λ corresponds to additional attractive gravitation.

The cosmology described by (1.2.11) with a positive cosmological constant is called the *Friedmann-Lemaître universe* or the Concordance model. In this universe, the total density parameter is conveniently split into a matter term, a radiation term and a cosmological constant term,

$$\Omega_0 = \Omega_m + \Omega_r + \Omega_\Lambda \,, \tag{1.2.14}$$

where Ω_r and Ω_Λ are defined as

$$\Omega_r = \frac{\rho_r}{\rho_c} \,, \quad \Omega_\Lambda = \frac{\Lambda}{8\pi G\rho_c} = \frac{\Lambda}{3H_0^2} \,. \tag{1.2.15}$$

Ω_m, Ω_r and Ω_Λ are important dynamical parameters characterizing the universe.

1.2.3 Conservation law and equation of state

In the general case of nonvanishing pressure p, differentiating equation (1.2.2) with respect to time we obtain, after some calculations, a new equation containing only first-order time derivatives:

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{1.2.16}$$

This equation does not contain k and Λ. All terms have dimension of energy density per time, namely this equation states that the change of energy density per time is zero, so we can interpret it as the *local energy conservation law*. In a volume element dV, ρdV represents the local decrease of graviting energy due to the expansion, whereas pdV is the work done by the expansion. Energy does not have a global meaning in the curved spacetime of general relativity, whereas work does. If different forms of energy do not transform into one another, each form obeys the previous equation separately.

There is another way to derive equation (1.2.16). Let the total energy content in a comoving volume a^3 be

$$E = (\rho + p)\,a^3\,. \tag{1.2.17}$$

The expansion is adiabatic if there is no net inflow or outflow so that

$$\frac{dE}{dt} = \frac{d}{dt}\left[(\rho + p)\,a^3\right] = 0\,. \tag{1.2.18}$$

If p does not vary with time, changes in ρ and a compensate and equation (1.2.16) immediately follows.

If we know the relation between energy density and pressure, called the *equation of state* of the universe, equation (1.2.16) can easily be integrated:

$$\int \frac{\dot{\rho}(t)}{\rho(t) + p(t)}dt = -3\int \frac{\dot{a}(t)}{a(t)}dt\,. \tag{1.2.19}$$

In contrast, the law of *conservation of entropy* S is not implied by Friedmann's equations. In this case we can make an ansatz: let p be proportional to ρ with some proportionality factor w which is constant in time:

$$p = w\rho\,. \tag{1.2.20}$$

It follows:

$$\rho(a) \propto a^{-3(1+w)} = (1+z)^{3(1+w)}\,. \tag{1.2.21}$$

Although astronomers prefer to use z instead of a because it is an observable, in cosmology it is better to use a. In fact, redshift is a property of light, but freely propagating light did not exist at times when $z \gtrsim 1080$, so z is then no longer a true observable [4].

The value of the proportionality factor w in the last two equations follows from the adiabaticity condition and it is referred to three special cases:

I. A *matter-dominated* universe filled with nonrelativistic cold matter in the form of pressureless nonradiating dust for which $p = 0$. From equation (1.2.20), this corresponds to $w = 0$, and the density evolves according to

$$\rho_m(a) \propto a^{-3} = (1+z)^3 \,. \tag{1.2.22}$$

It follows that the evolution of the density parameter Ω_m is

$$\Omega_m(a) = \Omega_m \frac{H_0^2}{H^2} a^{-3} \tag{1.2.23}$$

and hence the evolution of the Hubble parameter is

$$H(a) = H_0 a^{-1} \sqrt{1 - \Omega_m + \Omega_m a^{-1}} = H_0 (1+z) \sqrt{1 + \Omega_m z} \,. \tag{1.2.24}$$

II. A *radiation-dominated* universe filled with an ultra-relativistic hot gas composed of elasically scattering particles of energy density ϵ. Statistical mechanics then tells us that the equation of state is

$$p_r = \frac{1}{3}\epsilon = \frac{1}{3}\rho_r \,. \tag{1.2.25}$$

This corresponds to $w = 1/3$, so that the radiation density evolves according to

$$\rho_r(a) \propto a^{-4} = (1+z)^4 \,. \tag{1.2.26}$$

III. The *vacuum-energy* state corresponds to a flat, static universe without dust or radiation, but with a cosmological term. From equations (1.2.2) we obtain

$$p_\Lambda = -\rho_\Lambda \,, \qquad w = -1 \,. \tag{1.2.27}$$

Thus the pressure of the vacuum energy is negative, in agreement with the definition of the vacuum-energy density as a negative quantity. In the equation of state (1.2.20), ρ_Λ and p_Λ are then scale-independent constants.

1.2.4 Early time dependence

From the above scale dependences, it follows that the curvature term in the first of (1.2.2) obeys the following inequality in the limit of small a:

$$\frac{k}{a^2} \ll \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \,, \tag{1.2.28}$$

which is always true when

$$k = +1 \,, \quad p > -\frac{1}{3}\rho \,, \quad w > -\frac{1}{3} \,, \quad \Lambda > 0 \,. \tag{1.2.29}$$

Then we can neglet the curvature term and the Λ term in the first of (1.2.2), which simplifies to

$$\frac{\dot{a}}{a} = H(t) = \left(\frac{8\pi G}{3}\rho\right)^{1/2} \propto a^{-3(1+w)/2} . \tag{1.2.30}$$

Integrating this differential equation we can find the time dependence of a:

$$a^{3(1+w)/2} \propto t \qquad \text{for } w \neq -1 , \quad \ln a \propto t \text{ for } w = -1 \tag{1.2.31}$$

and then, solving for a:

$$a(t) \propto t^{2/3(1+w)} \text{ for } w \neq -1 , \quad a(t) \propto e^{\text{const}\cdot t} \text{ for } w = -1 . \tag{1.2.32}$$

In the two epochs of matter domination and radiation domination we know the value of w. Inserting this we obtain the time dependence of a for a matter-dominated universe,

$$a(t) \propto t^{2/3} , \tag{1.2.33}$$

and for a radiation-dominated universe,

$$a(t) \propto t^{1/2} . \tag{1.2.34}$$

1.2.5 Big Bang

We find the starting value of the scale of the universe independently of the value of k in the curvature term neglected above:

$$\lim_{t\to 0} a(t) = 0 . \tag{1.2.35}$$

In the same limit the rate of change $\dot{a}$ is obtained from equation (1.2.30) with any $w > -1$:

$$\lim_{t\to 0} \dot{a}(t) = \lim_{t\to 0} a^{-1}(t) = \infty . \tag{1.2.36}$$

Hence, an early radiation-dominated universe was characterized by extreme density and pressure:

$$\begin{aligned} \lim_{t\to 0} \rho_r(t) &= \lim_{t\to 0} a^{-4}(t) = \infty , \\ \lim_{t\to 0} p_r(t) &= \lim_{t\to 0} a^{-4}(t) = \infty . \end{aligned} \tag{1.2.37}$$

The time $t = 0$ was called the *Big Bang* by Fred Hoyle, who did not like the idea of an expanding universe starting from a singularity. Since 1988 the steady state theory has been however abandoned because of the discovery of early quasars.

1.2.6 Late Einstein-de Sitter evolution

The conclusions we derived from equation (1.2.28) were true for past times in the limit of small a. However, the recent evolution and the future depend on the value of k and on the value of Λ. For $k = 0$ and $k = -1$ the expansion always continues following equation (1.2.32), and a positive value of Λ boosts the expansion further.
In a matter-dominated Einstein-de Sitter universe which is flat and has $\Omega_\Lambda = 0$, Friedmann's equation (1.2.2) can be integrated to give

$$t(z) = \frac{2}{3H_0}(1+z)^{-3/2} , \tag{1.2.38}$$

and the present age of the universe at $z = 0$ would be

$$t_0 = \frac{2}{3H_0} . \tag{1.2.39}$$

Since, with this values, one finds $t_0 = 9.27$ Gyr and the value determined from the age of the oldest known star in the galaxy is 13.5 ± 2.9 Gyr, the flat-universe model with $\Omega_\Lambda = 0$ is in trouble.

1.2.7 Evolution of a closed universe

In a closed matter-dominated universe with $k = 1$ and $\Lambda = 0$, the curvature term k/a^2 drops with the second power of a, while the density drops with the third power, so the inequality (1.2.28) is finally violated at a scale such that

$$a_{\max}^{-2} = \frac{8\pi G\rho_m}{3} , \tag{1.2.40}$$

and the expansion halts because $\dot{a} = 0$. This is called the *turnover time* $t_{\max}$. At later times the expansion turns into contraction, and the universe returns to zero size at time $2t_{\max}$. That time is usually called the *Big Crunch*. For $k = 1$ it is

$$t_{\max} = \int_0^{a_{\max}} da \left(\frac{8\pi G}{3}\rho_m(a)\,a^2 - 1\right)^{-1/2} . \tag{1.2.41}$$

We can see the qualitative behavior in the three cases $k = -1, 0, 1$ with $\Lambda = 0$ in Fig. 1.2. Here, we can note that following the curves back in time, they intersect the time axis at different times. it means that what may be called time $t = 0$ is more recent in a flat universe than in an open universe, and even more in a closed universe.

1.2.8 The radius of the universe

The spatial curvature is given by the Ricci scalar R and it can be expressed in terms of Ω:

$$R = 6H^2(\Omega - 1) . \tag{1.2.42}$$

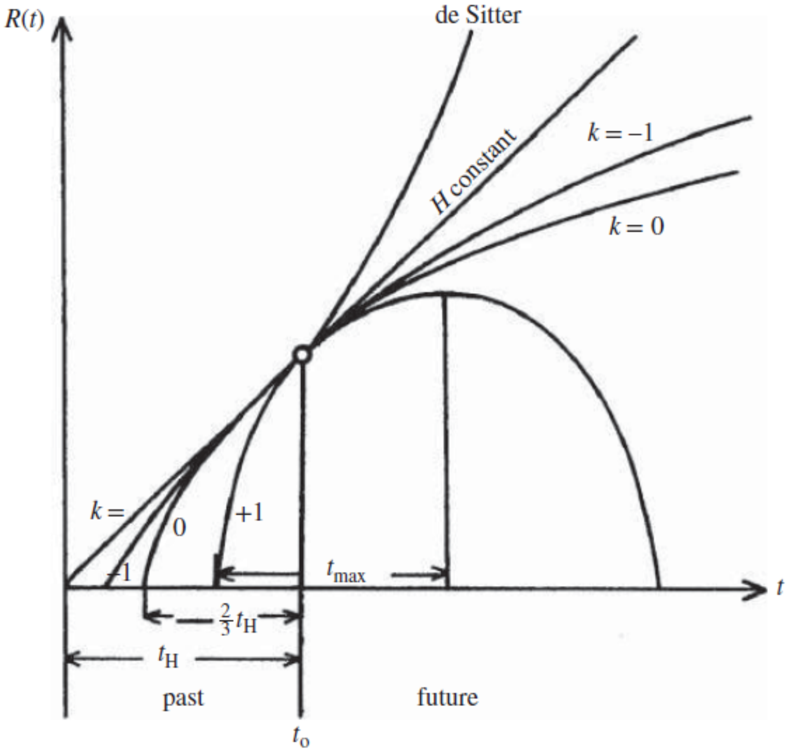


Figure 1.2: Time dependence of the cosmic scale $R(t) = a(t)$ in various scenarios. When $k = 1$, a closed universe with a total lifetime $2t_{\max}$. It started more recently than a flat universe. For $k = 0$, we have a flat universe which started $2/3t_H$ ago. When $k = -1$, an open universe started at a time $2/3t_H < t < t_H$ before the present time. de Sitter: an exponential (inflationary) scenario corresponding to a large cosmological constant. This is also called the Lemaitre cosmology. The picture is taken from [4].

R vanishes in a flat universe and it is only meaningful when it is nonnegative, as in a closed universe. It is conventional to define a radius of curvature (valid also for open universes) as

$$r_U \equiv \sqrt{\frac{6}{R}} = \frac{1}{H\sqrt{|\omega - 1|}} \,. \tag{1.2.43}$$

For a closed universe, r_U has the physical meaning of the radius of a hyper-sphere.

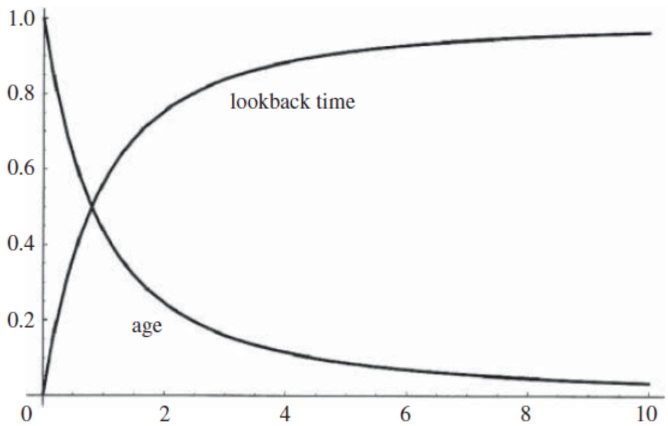


Figure 1.3: Lookback time and normalized age of the universe when $\Omega_m = 0.27$ and $\Omega_\Lambda = 1 - \Omega_m$. Credits by [4].

1.2.9 Late Friedmann-Lemaître evolution

When $\Lambda > 0$ the recent past and the future take an entirely different course. Since ρ_Λ and Ω_Λ are scale-independent constants, they will start to dominate over the matter term and the radiation term when the expansion has reached a given scale. Rewriting the second of (1.2.2) as

$$\frac{2\ddot{a}}{a} = 3H_0^2\Omega_\Lambda \,, \tag{1.2.44}$$

we can see that the expansion will independently accelerate from the value of k. In particular, a closed universe with $k = 1$ will ultimately not contract, but suffer accelerating expansion. Let us now consider the general expression for the age $\int_0^{t(a)} dt = \int_1^a \frac{da}{aH(a)}$ of a universe characterized by k and energy density components Ω_m, Ω_r and Ω_Λ and the expression

$$H\left(a\right) \equiv \frac{\dot{a}}{a} = H_0\sqrt{\left(1-\Omega_0\right)a^{-2} + \Omega\left(a\right)}\,; \tag{1.2.45}$$

inserting the Ω components into the previous equations we have

$$\frac{\dot{a}^2}{a^2} = H^2\left(t\right) = H_0^2\left[\left(1-\Omega_0\right)a^{-2} + \Omega_m\left(a\right) + \Omega_r\left(a\right) + \Omega_\Lambda\left(a\right)\right] \tag{1.2.46}$$

or

$$t\left(z\right) = \frac{1}{H_0}\int_0^{1/(1+z)} da\left[\left(1-\Omega_0\right) + \Omega_m a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2\right]^{-1/2} \,. \tag{1.2.47}$$

We remember that today $\Omega_\Lambda \approx 0.7$.
The large size of the universe, combined with the finite speed of light, produces the phenomenon known as *lookback time*: whenever we observe a distant cosmic object, we do not see it as it is now, but as it was when the light was emitted. The time elapsed between when we detect the light here on earth and when it was originally emitted by the source, is known as the lookback time. It is given by the last integral with the lower integration limit at $1/(1+z)$ and the upper limit at 1. The proper distance defined as

$$d_P = \int_1^{t_0} \frac{dt}{a(t)} \tag{1.2.48}$$

is

$$d_P(z) = t(z) \; . \tag{1.2.49}$$

In Fig. 1.3 is showed the lookback time $t(z)/t_0$ and the age of the universe $1 - t(z)/t_0$ as functions of redshifts when $\Omega_m = 0.27$ and $\Omega_\Lambda = 1 - \Omega_m$. At infinite redshift the lookback time is 1 and the age of the universe is 0.

1.2.10 De Sitter universe

The de Sitter universe [2] is another special case for which the Einstein equation can be solved exactly and, specifically, it is a spacetime with positive constant 4-curvature that is homogeneous and isotropic in both space and time. Hence, it possesses the largest possible symmetry group, as large as the symmetry group of Minkowski spacetime (ten parameters in the four-dimensional case). It plays a fundamental role in understanding the basic properties of inflation. In fact, in most scenarios, inflation is nothing more than a de Sitter stage with slightly broken time-translational symmetry. To find its metric, we use two different approaches which illustrate different mathematical aspects of this spacetime. First, we obtain the de Sitter metric as a result of embedding a constant curvature surface in a higher-dimensional surfaces. As a second approach, we obtain de Sitter spacetime as a solution to the Friedmann equations with positive cosmological constant.
Let us now consider a hyperboloid

$$x^2 + y^2 - z^2 = H_\Lambda^{-2} \, , \tag{1.2.50}$$

embedded in three-dimensional Minkowski space with the metric

$$ds^2 = dz^2 - dx^2 - dy^2 \, . \tag{1.2.51}$$

This hyperboloid has positive curvature and lies entirely outside the light cone.
Therefore, the induced metric has Lorentzian signature. To parametrize the surface of the hyperboloid, we can use x and y coordinates. The metric of the hyperboloid can then be written as

$$ds^2 = \frac{(xdx + ydy)^2}{x^2 + y^2 - H_\Lambda^{-2}} - dx^2 - dy^2 \, , \tag{1.2.52}$$

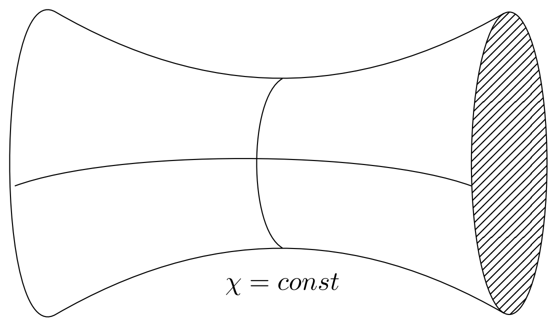


Figure 1.4: The hyperboloid with positive curvature

where $x^2 + y^2 > H_\Lambda^{-2}$. This is the metric of a two-dimensional de Sitter spacetime in x, y coordinates. Obviously, it is more convenient to use coordinates in which the symmetries of the spacetime are more explicit. The first choice is t, χ coordinates related to x, y via

$$x = H_\Lambda^{-1} \cosh\left(H_\Lambda t\right) \cos\chi\,, \quad y = H_\Lambda^{-1} \cosh\left(H_\Lambda t\right) \sin\chi\,. \tag{1.2.53}$$

These coordinates cover the entire hyperboloid for $-\infty < t < +\infty$ and $0 \le \chi \le 2\pi$ (see Fig. 1.4) and metric (1.2.52) becomes

$$ds^2 = dt^2 - H_\Lambda^{-2} \cosh^2\left(H_\Lambda t\right) d\chi^2\,. \tag{1.2.54}$$

In the four-dimensional case, this form of the metric corresponds to a closed universe with $k = +1$.
With another choice of coordinates,

$$x = H_\Lambda^{-1} \cosh\left(H_\Lambda \tilde{t}\right)\,, \quad y = H_\Lambda^{-1} \cosh\left(H_\Lambda \tilde{t}\right) \sinh\tilde{\chi}\,, \tag{1.2.55}$$

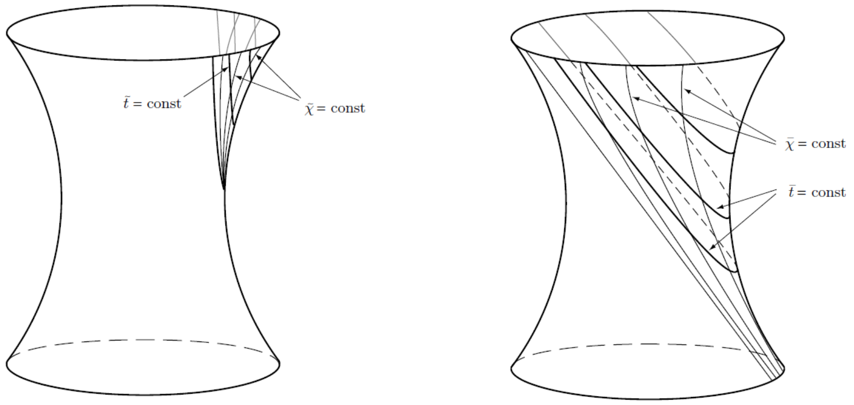
the metric (1.2.52) assumes the form corresponding to an open de Sitter universe:

$$ds^2 = d\tilde{t}^2 - H_\Lambda^{-2} \sinh^2\left(H_\Lambda \tilde{t}\right) d\tilde{\chi}^2\,. \tag{1.2.56}$$

The range of these coordinates is $0 \le \tilde{t} \le +\infty$ and $-\infty < \tilde{\chi} < +\infty$, covering only the part of de Sitter spacetime where $x \ge H_\Lambda^{-1}$ and $z > 0$ (see Fig. 1.5(a)). Moreover, the coordinates are singular at $\tilde{t} = 0$.

Finally, we consider the coordinate system defined via

$$x = H_\Lambda^{-1} \left[\cosh\left(H_\Lambda \bar{t}\right) - \frac{1}{2} \exp\left(H_\Lambda \bar{t}\right) \bar{\chi}^2\right]\,, y = H_\Lambda^{-1} \exp\left(H_\Lambda \bar{t}\right) \bar{\chi}\,, \tag{1.2.57}$$



(a) (b)

Figure 1.5: On the left, is represented the de Sitter spacetime where $x \geq H_\Lambda^{-1}$ and $z > 0$. On the right, De Sitter spacetime covered by flat coordinates [2].

where $-\infty < \bar{t}, \bar{\chi} < +\infty$. Expressing z in terms of $\bar{t}, \bar{\chi}$, one finds that only the half of the hyperboloid located at $x + z \geq 0$ is covered by these "flat" coordinates (see Fig. 1.5(b)).
The metric becomes

$$ds^2 = d\bar{t}^2 - H_\Lambda^{-2} \exp\left(2H_\Lambda \bar{t}\right) d\bar{\chi}^2 \,. \tag{1.2.58}$$

The relation between the different coordinate systems in the regions where they overlap can be obtained by comparing (1.2.53), (1.2.55), (1.2.57).
Following the second approach, a cosmological constant is equivalent to a "perfect fluid" with equation of state $p_\Lambda = -\rho_\Lambda$. From the conservation law it follows that the energy density stays constant during expansion and hence

$$\ddot{a} - H_\Lambda^2 a = 0 \,, \qquad H_\Lambda = \left(8\pi G \rho_\Lambda/3\right)^{1/2} \,. \tag{1.2.59}$$

A general solution of this equation is

$$a = C_1 \exp\left(H_\Lambda t\right) + C_2 \exp\left(-H_\Lambda t\right) \,, \tag{1.2.60}$$

where C_1, C_2 are integration constants constrained by the second Friedmann equation (1.2.2):

$$4H_\Lambda^2 C_1 C_2 = k \,. \tag{1.2.61}$$

Hence, in a flat universe one of the constants must be equal to zero. If $C_1 \neq 0$ and $C_2 = 0$, (1.2.60) describes a flat expanding de Sitter universe and we can choose $C_1 = H_\Lambda^{-1}$. If both the constants are nonzero, the time $t = 0$ can be chosen so that $|C_1| = |C_2|$. For a closed

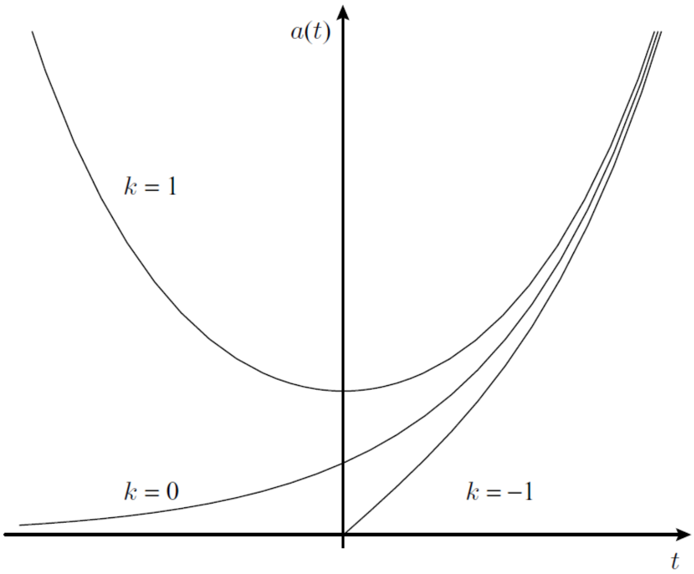


Figure 1.6: Behavior of the scale factor: when $k = 1$ it first decreases until reaching its minimum value at $t = 0$, then increases; when $k = 0$, the scale factor starts its increasing evolution from a vanishing value at early time; when $k = -1$, the scale factor behaves as in a flat universe, but vanishes at null-time [2].

universe, we have

$$C_1 = C_2 = \frac{1}{2H_\Lambda}, \tag{1.2.62}$$

while for an open universe

$$C_1 = -C_2 = \frac{1}{2H_\Lambda}. \tag{1.2.63}$$

Therefore, the three solutions are

$$ds^2 = dt^2 - H_\Lambda^{-2} \begin{pmatrix} \sinh^2(H_\Lambda t) \\ \exp(2H_\Lambda t) \\ \cosh^2(H_\Lambda t) \end{pmatrix} \left[d\chi^2 + \begin{pmatrix} \sinh^2\chi \\ \chi^2 \\ \sin^2\chi \end{pmatrix} d\Omega^2 \right] \begin{matrix} k=-1 \\ k=0 \\ k=+1 \end{matrix} \tag{1.2.64}$$

They all descibe the same physical spacetime in different coordinate systems (de Sitter spacetime is translational invariant in time). The behavior of the scale factor (see Fig.1.6) depends on the coordinate system. In a closed coordinate system, the scale factor first decreases, then reaches its minimum value, and subsequently increases. In a flat and open coordinates, $a(t)$ always increases as t grows but vanishes as $t \to -\infty$ and $t = 0$, respectively. However, the vanishing of the scale factor does not represent a real physical singularity

but simply signals that the coordinates become singular. For $t \gg H_\Lambda^{-1}$, the expansion is exponential in all coordinate systems.
There is indeed an analitical way to obtain the de Sitter metric. Consider a homogeneous flat universe with the FLRW metric in which the density of pressureless dust is constant, $\rho(t) = \rho_0$. The first of Friedmann equations (1.2.2) for the rate of expansion including the cosmological constant takes the form

$$\frac{\dot{a}(t)}{a(t)} = H \implies a(t) \propto e^{Ht}\,, \tag{1.2.65}$$

whose behaviour is drawn in Fig. 1.2 as de Sitter curve. However, H is, in this case, a constant equal to

$$H = \sqrt{\frac{8\pi}{3}G\rho_0 + \frac{\Lambda}{3}}\,. \tag{1.2.66}$$

When $k \neq 0$ we can eventually neglect the curvature term k/R^2 because the density is constant and the scale factor increases without limit.
Substituting the previous exponential solution into the FLRW metric (1.1.6) we obtain the de Sitter metric

$$ds^2 = dt^2 - e^{2Ht}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right)\,. \tag{1.2.67}$$

This solution was published by de Sitter in 1917, setting $\rho = p = 0$ and thus relating H directly to the cosmological constant Λ. The same solution is obtained in the case of vanishing cosmological constant if the density of dust ρ is constant. As we can read in [7], *Einstein's universe contains matter but no motion and de Sitter's contains motion but no matter.*
Introducing two test particles into this empty universe, they seem to recede from each other exponentially. In the case when their spatial distance is ra and $\Lambda > 0$, the equation of relative motion of the test particles is given by the second of (1.2.2) including the cosmological constant:

$$\frac{d^2(ra)}{dt^2} = \frac{\Lambda}{3}ra - \frac{4\pi}{3}G(\rho + 3p)\,ra\,. \tag{1.2.68}$$

With regard to the right-hand side, the first term is a force due to the vacuum-energy density, while the second term is the decelerating force due to the ordinary gravitationaly interaction.
If Λ is positive as in the Einstein universe, the force is repulsive and the expansion accelerates; with Λ negative, instead, the force is attractive, decelerating the expansion. The latter case is called *anti-de Sitter universe.*

1.3 Extended Gravity Theories

It is well known [8] that in general, the field equations of a physical theory correspond to the minimum of the action

$$S = \int \mathcal{L}\sqrt{-g}\,dt\,dV\,,$$

where $\mathcal{L}$ is the lagrangian and g is the determinant of the metric tensor $g_{\mu\nu}$. In other words, a physical theory corresponds to the variational principle

$$\delta \int \mathcal{L}\sqrt{-g}\, dt\, dV = 0\,.$$

For the general relativity theory, the lagrangian has the form

$$\mathcal{L}_{GR} = \frac{1}{16\pi G} R + \mathcal{L}_M\,,$$

where G is the gravitation constant and $\mathcal{L}_M$ is the lagrangian of matter. Varying over $g_{\mu\nu}$, we get the corresponding equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}\,,$$

where $T_{\mu\nu}$ is the matter's energy-momentum tensor.
According to this theory, space-time and gravitation are described by Einstein's partial differential equations, in which the only field responsible for gravitation is the metric tensor $g_{\mu\nu}$. In general relativity, the gravitational field generated by a body of mass M at a distance r is proportional to its mass M, with the gravitation constant G as the proportionality coefficient: $a \approx GM/r^2$. It turns out that the observed gravitational accelerations near several distant astronomical bodies are much larger than what is predicted based on the observable mass M_O: $a \ll GM/r^2$. The traditional approach to this problem is to conclude that, in addition to the observable masses, there are also non-observable ones; besides, to explain the observations, we can assume that on the cosmological level, 95% of the mass is formed by hypothetical non-directly-observable dark matter and dark energy.
Some physicists propose that instead of introducing such hypothetical types of matter, it is more reasonable to conclude that the parameter G that described the local strength of gravitational interactions does not have to be a universal constant: measurements of G at different points in space-time can lead, in general, to different results. Thus, in such a theory, to describe the gravitational field we need to introduce both the metric tensor $g_{\mu\nu}$ and a scalar field ϕ.
Historically, the first modification of general relativity, in which there is no need for the hypothetical dark energy and dark matter, came in the form of a modified lagrangian which only depends on the metric $g_{\mu\nu}$ but which is non-linear in the scalar curvature R. Actually, there are many ways in which general relativity could be modified, but we will consider four different possibilities:

- gravitational scalar fields
- extra spatial dimensions
- higher-order terms in the action

The most important alternative models are known as *scalar-tensor theories* of gravity, because they involve both the metric tensor $g_{\mu\nu}$ and a scalar field ϕ that couples directly to the curvature scalar (not only to the metric as the equivalence principle would seem to imply). The action can be written as a sum of a gravitational piece, a pure-scalar piece and a matter piece:

$$S = S_{fR} + S_\phi + S_M \,, \tag{1.3.1}$$

where

$$S_{fR} = \int d^4x\sqrt{-g}f\left(\phi\right)R\,,$$

$$S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}h\left(\phi\right)g^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) - V\left(\phi\right)\right]\,, \tag{1.3.2}$$

$$S_M = \int d^4x\sqrt{-g}\hat{\mathcal{L}}_M\left(g_{\mu\nu},\Psi_i\right)\,.$$

$f\left(\phi\right)$, $h\left(\phi\right)$ and $V\left(\phi\right)$ are functions that define the theory, and the matter lagrangian $\hat{\mathcal{L}}_M$ depends on the metric and a set of matter fields Ψ_i but not on ϕ.
The equations of motion for this theory include the gravitational equation and the scalar equation. The former is obtained from varying with respect to the metric: for this purpose, we consider perturbations of the metric

$$g^{\mu\nu} \;\rightarrow\; g^{\mu\nu} + \delta g^{\mu\nu}\,. \tag{1.3.3}$$

The variation of the gravitational part of the action is

$$\delta S_{fR} = \int d^4x\sqrt{-g}f\left(\phi\right)\left[G_{\mu\nu}\delta g^{\mu\nu} + \partial_\sigma\partial^\sigma\left(g_{\mu\nu}\delta g^{\mu\nu}\right) - \partial_\mu\partial_\nu\left(\delta g^{\mu\nu}\right)\right]\,, \tag{1.3.4}$$

from which double integration by parts gives us

$$\delta S_{fR} = \int d^4x\sqrt{-g}\left[f\left(\phi\right)G_{\mu\nu} + g_{\mu\nu}\Box f - \partial_\mu\partial_\nu f\right]\delta g^{\mu\nu}\,, \tag{1.3.5}$$

where $\Box = \nabla^\mu\nabla_\mu = g^{\mu\nu}\nabla_\mu\nabla_\nu$ and covariant derivatives are equivalent to partial derivatives when acting on scalars. The gravitational equation of motion, including contributions from S_ϕ and S_M, is therefore

$$G_{\mu\nu} = f^{-1}\left(\phi\right)\left(\frac{1}{2}T^M_{\mu\nu} + \frac{1}{2}T^\phi_{\mu\nu} + \partial_\mu\partial_\nu f - g_{\mu\nu}\Box f\right)\,, \tag{1.3.6}$$

where, in particular,

$$T^\phi_{\mu\nu} = h\left(\phi\right)\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}h\left(\phi\right)g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V\left(\phi\right)\right]\,. \tag{1.3.7}$$

One of the earliest scalar-tensor models is known as Brans-Dicke theory, and corresponds to the choices

$$f(\phi) = \frac{\phi}{16\pi}\,, \quad h(\phi) = \frac{\omega}{8\pi\phi}\,, \quad V(\phi) = 0\,, \tag{1.3.8}$$

where ω is the dimensionless Dicke coupling constant.
To get a full description of the scalar-tensor theory, we also nees to add, to the lagrangian, the term $\frac{(\partial_\mu\phi)(\partial_\nu\phi)}{\phi}$ describing the effective energy density of the scalar field, As a result, the scalar-tensor action takes the form

$$S_{BD} = \int d^4x\sqrt{-g}\left[\frac{\phi}{16\pi}R - \frac{\omega}{16\pi}g^{\mu\nu}\frac{(\partial_\mu\phi)(\partial_\nu\phi)}{\phi}\right]\,. \tag{1.3.9}$$

Varying over $g_{\mu\nu}$ and ϕ, we get the following Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi\right) + \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) \tag{1.3.10}$$

and

$$\Box\phi = \frac{8\pi}{3+2\omega}T\,, \tag{1.3.11}$$

where $T := T^\mu_\mu$ is the trace of the energy-momentum tensor.
In the Brans-Dicke theory, the scalar field is massless, but in the $\omega \to \infty$ limit the field becomes nondynamical and ordinary GR is recovered.
Similarly to general relativity, the Brans-Dicke theory is described by second order partial differential equations and its equations are invariant for time reversal.
A popular approach to dealing with scalar-tensor theories is to perform a conformal transformation to bring the theory into a form that looks like conventional GR. We define a conformal metric

$$\tilde{g}_{\mu\nu} = 16\pi\tilde{G}f(\phi)\,g_{\mu\nu}\,, \tag{1.3.12}$$

where $\tilde{G}$ will become Newton's constant in the conformal frame. The action S_{fR} from (1.3.6) becomes

$$S_{fR} = \int d^4x\sqrt{-g}f(\phi)\,R = \int d^4x\sqrt{-\tilde{g}}\left(16\pi\tilde{G}\right)^{-1}\left[\tilde{R} - \frac{3}{2}f^{-2}\left(\frac{df}{d\phi}\right)^2\tilde{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\right]\,, \tag{1.3.13}$$

where we have integrated by parts and discarded surface terms. In the conformal frame, therefore, the curvature scalar appears not multiplied by any function of ϕ. This frame is called the *Einstein frame* because Einstein's equations for the conformal metric $\tilde{g}_{\mu\nu}$ take on their conventional form, while the original frame with metric $g_{\mu\nu}$ is called the *Jordan frame* or sometimes *string frame* because string theory typically predicts a scalar-tensor theory rather than ordinary general relativity.
Another useful way to modify general relativity is to allow for the existence of extra spatial

dimensions: in fact, the physical consequences of extra dimensions are related to those of scalar-tensor theories. This approach means considering models in which the spacetime appears four-dimensional on large scales even though there are $4+d$ total dimensions. this may happen if the extra d dimensions are compactified on some manifold. Models of this kind are known as *Kaluza-Klein theories*.
Let l_{ab} be the metric for a $(4+d)$-dimensional spacetime with coordinates X^a, where indices a, b run from 0 to $d+3$. Thus the line element becomes

$$ds^2 = l_{ab} dX^a dX^b = g_{\mu\nu}(x)\, dx^\mu dx^\nu + b^2(x)\, h_{ij}(y)\, dy^i dy^j \,, \tag{1.3.14}$$

where the x^μ are coordinates in the four-dimensional spacetime and the y^i are coordinates on the extra-dimensional manifold, taken to be a maximally symmetric space with metric h_{ij}. The action is the $(4+d)$-dimensional Hilbert action plus a matter term:

$$S = \int d^{4+d} X \sqrt{-l} \left(\frac{1}{16\pi G_{4+d}} R[l_{ab}] + \hat{\mathcal{L}}_M \right) , \tag{1.3.15}$$

where $\sqrt{-l}$ is the square root of the determinant of l_{ab}, $R[l_{ab}]$ is the Ricci scalar of l_{ab} and $\hat{\mathcal{L}}_M$ is the matter Lagrange density with the metric determinant factored out.
The first step is to dimensionally reduce the action (1.3.15), namely to perform the integral over the extra dimensions, which is possible because we have assumed that the extra-dimensional scale factor b is independent of y^i. Therefore we can express everything in terms of $g_{\mu\nu}$, h_{ij} and $b(x)$, integrate over the extra dimensions and arrive at an effective four-dimensional theory. From the metric (1.3.14) we have

$$\sqrt{-l} = b^d \sqrt{-g} \sqrt{h} \,, \tag{1.3.16}$$

and we can evaluate the curvature scalar for this metric to obtain

$$R[l_{ab}] = R[g_{\mu\nu}] + b^{-2} R[h_{ij}] - 2d b^{-1} g^{\mu\sigma} \nabla_\mu \nabla_\sigma b - d(d-1) b^{-2} g^{\mu\sigma} (\nabla_\mu b)(\nabla_\sigma b) \,, \tag{1.3.17}$$

where ∇_μ is the covariant derivative associated with the four-dimensional metric $g_{\mu\nu}$. Denoting by $\mathcal{V}$ the volume of extra-dimensions when $b=1$,

$$\mathcal{V} = \int d^d y \sqrt{h} \,, \tag{1.3.18}$$

the four-dimensional Newton's constant G_4 is determined by evaluating the coefficient of the curvature scalar in the action so that it is related to its higher-dimensional analogue by

$$\frac{1}{16\pi G_4} = \frac{\mathcal{V}}{16\pi G_{4+d}} \,. \tag{1.3.19}$$

We have therefore

$$S = \int d^4 x \sqrt{-g} \left\{ \frac{1}{16\pi G_4} \left[b^d R[g_{\mu\nu}] + d(d-1) b^{d-2} g^{\mu\nu} (\nabla_\mu b)(\nabla_\nu b) + d(d-1)\kappa b^{d-2} \right] + \mathcal{V} b^d \hat{\mathcal{L}}_M \right\} , \tag{1.3.20}$$

where we have integrated by parts for convenience, and introduced the curvature parameter κ of h_{ij}, given by

$$\kappa = \frac{R\left[h_{ij}\right]}{d\left(d-1\right)} \,. \tag{1.3.21}$$

In order to have a more conventional expression of the action (1.3.15) we can perform a change of variables and a conformal transformation,

$$\beta\left(x\right) = \ln b \,, \qquad \tilde{g}_{\mu\nu} = e^{d\beta} g_{\mu\nu} \,, \tag{1.3.22}$$

and, to turn β into a canonically normalized scalar field,

$$\phi = \sqrt{\frac{d\left(d+2\right)}{16\pi G_4}} \,; \tag{1.3.23}$$

we get the following action:

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{16\pi G_4}R\left[\tilde{g}_{\mu\nu}\right] - \frac{1}{2}\tilde{g}^{\mu\nu}\left(\tilde{\nabla}_\mu\phi\right)\left(\tilde{\nabla}_\nu\phi\right) + \frac{1}{16\pi G_4}\kappa d\left(d-1\right)e^{-\sqrt{2(d+2)/d}\phi/\bar{M}_P} + \right.$$
$$\left. + \mathcal{V}e^{-\sqrt{2d/(d+2)}\phi/\bar{M}_P}\hat{\mathcal{L}}_M\right\} \,, \tag{1.3.24}$$

where $\bar{M}_P = \left(8\pi G_4\right)^{-1/2}$ is the reduced Planck mass. The scalar ϕ is known as the *dilaton* or *radion*, and characterized the size of the extra-dimensional manifold.

The last two terms in (1.3.24) represent a potential $V\left(\phi\right)$. Ignoring the matter term $\hat{\mathcal{L}}_M$, the behavior of the dilaton will depend only on the sign of κ. If the extra-dimensional manifold is flat ($\kappa = 0$), the potential vanishes and we have a massless scalar field; if there is curvature ($\kappa \neq 0$), the potential does not have minimum; for $\kappa > 0$ the field will roll to $-\infty$, while for $\kappa < 0$ the field will roll to $+\infty$. But $\phi \propto \ln b$, so the scale factor $b\left(x\right)$ of the extra dimensions either shrinks to zero or becomes arbitrarily large, in either case frustrating the hope for stable extra dimensions. Stability can be achieved, however, by choosing an appropriate matter Lagrangian, and an appropriate field configuration in the extra dimensions.

A different kind of alternative theory is one featuring lagrangians of more than second order in derivatives of the metric. We can write an action of the form

$$S = \int d^nx\sqrt{-g}\left(R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 g^{\mu\nu}\nabla_\mu R\nabla_\nu R + \cdots\right) \,, \tag{1.3.25}$$

where the α's are coupling constants and the dots represent every other scalar we can make from the curvature tensor, its contractions and its derivatives. Einstein's equation leads to a well-posed initial value problem for the metric, in which coordinates and momenta specified at an initial time can be used to predict future evolution. With higher-derivative terms, we would require not only those data, but also some number of derivatives of the momenta. Obviously, the character of the theory is altered.

CHAPTER 2

COSMIC INFLATION

2.1 The shortcomings of the standard Big Bang Theory

The standard big bang model [9], [10], [11] incorporates three important observations about the universe:

(a) the expansion of the universe discovered by Hubble;

(b) the discovery of the microwave background radiation (MBR) by Penzias and Wilson and its confirmation by other observers;

(c) the prediction of the abundances of various nuclei on the basis of nucleosynthesis in the early universe, in particular the abundances of helium and deuterium, which appear to conform reasonably with observations.

It is well known that matter is distributed very homogeneously and isotropically on scales larger than a few hundred megaparsecs. The CMB gives us a picture of the early universe, showing that at recombination the universe was extremely homogeneous and isotropic on all scales up to the present horizon. Since the universe evolves according to the Hubble law, we ask ourselves which initial conditions could lead to such homogeneity and isotropy. To answer to this, it is necessary to know the exact physical laws which govern the evolution of the very early universe, or even a few simple properties of them. The main question to solve is: how is the universe so homogeneous and isotropic to vast distances, extending to regions which could not have communicated with each other during the early eras? This problem is illustrated in Fig. 2.1.
Another question is why the density parameter Ω (the ratio of the energy density of the universe to the critical density) is so near unity. If the present value of Ω lying between 0.1

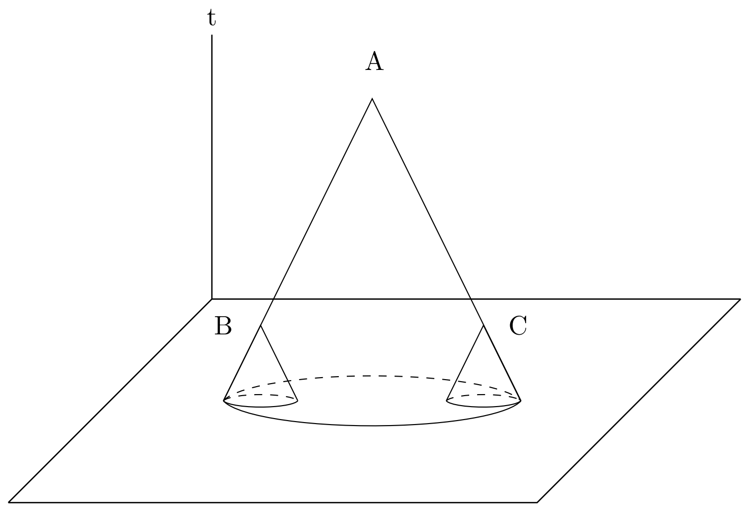


Figure 2.1: The horizon problem. The point A represents our present space-time position, the points B e C represent events at a much earlier epoch, lying in opposite spatial direction from us, but lying in our past light cone. The plane at the bottom represents the instant $t = 0$, the big bang. The past lights cones of B and C have no intersection, so these two events could not have had any casual connection.

and 2 is extrapolated to near the big bang we get the following orders of magnitude:

$$\begin{aligned} &|\Omega\,(1s) - 1| = O\left(10^{-16}\right)\,, \\ &\left|\Omega\left(10^{-43}s\right) - 1\right| = O\left(10^{-16}\right)\,, \end{aligned} \tag{2.1.1}$$

which seem difficult to explain.

The third problem is the smoothness problem, which is to explain the origin and nature of the primordial density perturbations, that is the presence of galaxies and the structure of the observable universe.

These questions may seem logically inconsistent with the standard cosmology and inflation is not the first attempt to address these shortcomings: since 1980s, cosmologists proposed alternative solutions to those problems. Inflations is the most successfull attempt to understand and describe the very early stages of our universe.

2.1.1 The flatness problem

The *flatness* problem (or problem of initial velocities) comes from considering the Friedmann equations in a universe with matter and radiation, and vanishing vacuum energy.

This problem can be reformulated in terms of the critical density $\Omega\,(t) = \rho/\rho_c$, so that the Friedmann equations (1.2.2) and (1.2.3) can be rewritten as

$$\begin{aligned} \Omega - 1 &= \frac{k}{(Ha)^2} \\ \frac{\ddot{a}}{a} &= -\frac{1}{2}H^2\Omega\,(1+3w) \end{aligned} \tag{2.1.2}$$

and, combining them with the derivative of the first, we obtain

$$\frac{d\Omega}{d\ln a} = (1+3w)\,\Omega\,(\Omega - 1)\ . \tag{2.1.3}$$

This equation is easily solved, but its most general properties are qualitatively different depending on the sign of $1 + 3w$.

A flat universe $\Omega = 1$ remains flat at all times. This is an unstable point if the strong energy condition $1 + 3w > 0$ is satisfied (from which it follows $d|\Omega - 1|/d\ln a > 0$). Any deviation from flatness is amplified by the subsequent expansion, hence the flatness of the universe at present time $\Omega_0 \sim 1$ represents an initial fine tuning problem. This is referred to as the flatness problem of standard Big Bang cosmology in which the universe is initially dominated by radiation and later matter. On the other hand, if $1 + 3w < 0$, the universe evolves towards flatness, since $d|\Omega - 1|/d\ln a < 0$. Taking into account (2.1.2), this leads to accelerated expansion. Thus, the flatness problem may be solved by introducing a period of accelerated expansion prior to radiation domination. During inflation the Hubble rate is constant, hence

$$\Omega - 1 = \frac{k}{a^2H^2} \propto \frac{1}{a^2}\,. \tag{2.1.4}$$

Moreover, in order to have a value of $\Omega_0 - 1$ of order of unity today, the initial value of $(\Omega - 1)$ at the beginning of the radiation-dominated phase must be $|\Omega - 1| \sim 10^{-60}$. Identifying the beginning of the radiation-dominated phase with the beginning of inflation, it is required

$$|\Omega - 1|_{t=t_f} \sim 10^{-60}\,, \tag{2.1.5}$$

while during inflation

$$\frac{|\Omega - 1|_{t=t_f}}{|\Omega - 1|_{t=t_i}} = \left(\frac{a_i}{a_f}\right)^2 = e^{-2N}. \tag{2.1.6}$$

Taking $|\Omega - 1|_{t=t_i}$ of order unity and defining the corresponding number of e-foldings N

$$N \equiv \ln\left(\frac{a_f}{a_i}\right) = H_I\,(t_f - t_i)\ , \tag{2.1.7}$$

one has $N \approx 70$ to solve the flatness problem. It is clear that, if the period of inflation lasts longer than 70 e-foldings, the present-day value of $\Omega_0 - 1$ will be equal to unity with a great precision. Thus, a generic prediction of inflation is that $\Omega_0 = 1$. On the other hand, inflation does not change the global geometric properties of the spacetime: if the universe is open or closed, it will remain flat or closed, independetly from inflation. As we can see in Fig. 2.2, inflation magnifies the radius of curvature so that locally the universe is flat with a great precision. Also, the current data on the CMB anisotropies confirm this prediction.

It is also possible to state the flatness problem and its solution in terms of the comoving Hubble scale $(aH)^{-1}$, from which

$$\begin{aligned} \frac{d}{dt}\,(aH)^{-1} < 0 \quad &\Rightarrow \quad \text{Expansion towards flatness} \\ \frac{d}{dt}\,(aH)^{-1} > 0 \quad &\Rightarrow \quad \text{Expansion away from flatness}\,. \end{aligned} \tag{2.1.8}$$

The first condition applies to matter and radiation, the second to a cosmological constant. The hypothesis of adiabatic expansion of the universe is connected with the flatness problem. We know that during a radiation-dominated period, since $H^2 \sim \rho_R \sim T^4/m_{Pl}^2$ and under the hypothesis of adiabaticity, $|\Omega - 1| \sim 10^{-60}$: in other words, $\Omega - 1$ is so close to zero at early epochs because of the incredibly large total entropy. The flatness problem is therefore a problem of understanding why the classical initial conditions corresponded to a universe that was so close to spatial flatness. It substuntially arise because the entropy in a comoving volume is conserved: in this sense, the problem could be solved if the cosmic expansion was non-adiabatic for some finite time interval during the early history of the universe.

2.1.2 The horizon problem

Before dealing with this hard problem, it is necessary to focus for a while on the concept of *horizons*.

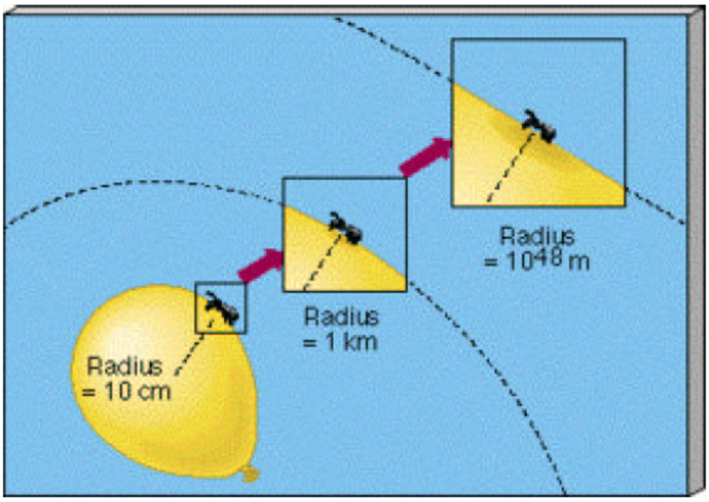


Figure 2.2: What inflation does is to magnify the radius of curvature, hence locally the universe is almost spatially flat [11].

If the universe has a finite age, then light travels only a finite distance in that time and the volume of space from which we can receive information at a given moment of time is limited. The boundary of this volume is called the *particle horizon*. Fig. 2.3 illustrates the physical meaning of this.
The maximum comoving distance light can propagate is

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt}{a}\,, \tag{2.1.9}$$

where η_i corresponds to the beginning of the universe. At time η, the information about events at $\chi > \chi_p$ is inaccessible to an observer located at $\chi = 0$. In a universe with an initial singularity, we can always set $\eta_i = 0$, but in some nonsingular spacetimes (like the de Sitter universe), it is more convenient to take the initial conformal time different from zero. Hence, we obtain the physical size of the particle horizon:

$$d_p(t) = a(t)\,\chi_p = a(t)\int_{t_i}^{t} \frac{dt}{a}\,. \tag{2.1.10}$$

On the other hand, the event horizon of the observable universe is the largest comoving distance from which light emitted now can ever reach an observer in the future. The boundary past which events cannot ever be observed is an event horizon, and it represents the

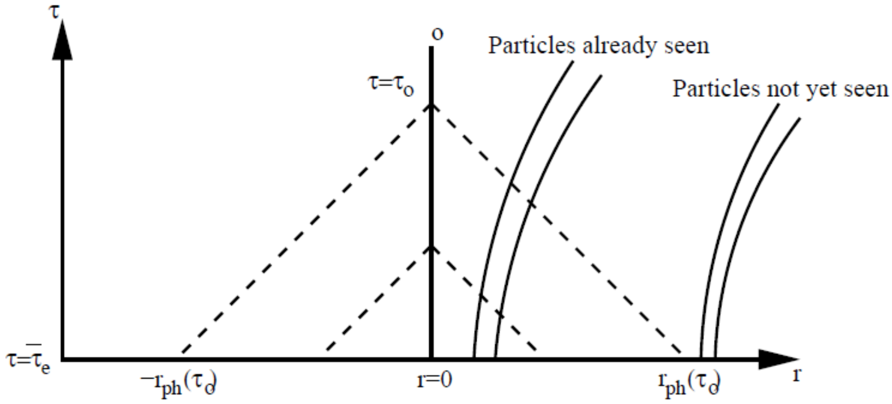


Figure 2.3: The particle horizon represents the largest comoving distance from which light emitted (e) in the past could have reached the observer (o) at a given time. For events beyond that distance, light has not had time to reach our location, even if it were emitted at the time the universe began.

maximum extent of the particle horizon. The physical size of the event horizon at time t is

$$d_e\left(t\right) = a\left(t\right) \int_t^{t_{\max}} \frac{dt}{a}\,, \tag{2.1.11}$$

where max refers to the final moment of time.
An important example is a flat de Sitter universe, where

$$d_e\left(t\right) = \exp\left(H_\Lambda t\right) \int_t^{\infty} \exp\left(-H_\Lambda t\right) dt = H_\Lambda^{-1}\,, \tag{2.1.12}$$

namely, the size of the event horizon is equal to the Hubble curvature scale.
Now, the problem of horizon arise from the finite age t_0 of the universe, that implies that photons can only have traveled a finite distance in the time since the Big Bang. Hence, the particle horizon today is finite and larger than at any earlier time. Also, the spatial width of the past light cone has grown in proportion to the longer time perspective. Thus the spatial extent of the Universe is larger than that our past light cone encloses today; with time we will become causally connected with new regions as they move in across our horizon. This renders the question of the full size of the whole Universe meaningless.
For simplicity let us imagine [8] we are in a matter-dominated universe, for which

$$a\left(t\right) \propto t^{2/3}\,.$$

The Hubble parameter is therefore given by

$$H = \frac{2}{3t} = a^{-3/2} H_0$$

and the photon travels a comoving distance

$$\Delta r = 2H_0^{-1}\left(\sqrt{a_2} - \sqrt{a_1}\right) . \tag{2.1.13}$$

The physical horizon size at any fixed value of the scale factor $a = a_*$ is

$$d_p\left(a_*\right) = a_* r_{\rm hor}\left(a_*\right) = H_*^{-1} , \tag{2.1.14}$$

so that, for any nearly-flat universe containing a mixture of matter and radiation, at any one epoch we will have

$$dp\left(a_*\right) \sim H_*^{-1} = R_H\left(a_*\right) . \tag{2.1.15}$$

The horizon problem is nothing but the fact that the CMB is isotropic to a high degree of precision [8], even though widely separated points on the last scattering surface are completely outside each others' horizons. When we look at the CMB we are observing the universe at a scale factor $a_{\rm CMB} \approx 1/1200$; from (2.1.13), the comoving distance between a point on the CMB and an observer on earth is

$$\Delta r = 2H_0^{-1}\left(1 - \sqrt{a_{\rm CMB}}\right) \approx 2H_0^{-1} .$$

However, the comoving horizon distance for such a point is

$$r_{\rm hor}\left(a_{\rm CMB}\right) = 2H_0^{-1}\sqrt{a_{\rm CMB}} \approx 6 \times 10^{-2} H_0^{-1} . \tag{2.1.16}$$

Hence, if we observe two widely-separated parts of the CMB, they will have nonoverlapping horizons; distinct patches of the CMB sky were causally disconnected at recombination. Nevertheless, they are observed to be at the same temperature to high precision. The question then is: how did they know ahead of time to coordinate their evolution in the right way, even though they were never in causal contact? (See Fig. 2.4.) We must somehow modify the causal structure of the conventional FRW cosmology.

This problem may be solved by introducing an early period of inflation prior to radiation domination [11]. To see this it can be convenient switching to conformal time η defined by (1.2.5). With these coordinates the particle horizon is conveniently given by the age of the universe in conformal time:

$$\eta = \int_0^t \frac{dt'}{a\left(t'\right)} = \int_0^a d\ln a \left(\frac{1}{aH}\right) . \tag{2.1.17}$$

The size is the width of the past light cone projected onto the surface $\tau = 0$ defined by the initial singularity. It is to note, moreover, the integral has been written in terms of the comoving Hubble scale $(aH)^{-1}$ because it is a more useful scale in inflationary cosmology than the particle horizon. For this reason, $(aH)^{-1}$ is called *the horizon* and it is about the size of the particle horizon during matter and radiation domination, but it does not hold in general. In fact, only when matter satisfies the condition $\rho + 3p > 0$, the particle horizon is usually of order of the the Hubble scale, $1/H$: hence, the terms *Hubble scale* and *particle*

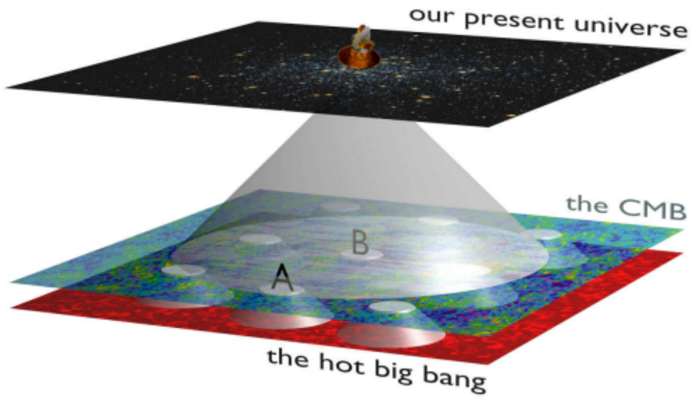


Figure 2.4: The horizon problem of hot Big Bang cosmology. The space is comoving and the time is conformal. The smaller light cones are truncated between the hot Big Bang and recombination, during which those cones are causally unrelated. The problem is: why do cones (A e B, for instance) at recombination look similar to each other? This picture is taken from [10].

horizon can be sometimes interchanged, even if they are conceptually different. Whereas the particle horizon is a scale set by kinematical considerations, the curvature scale is a dynamical scale that characterizes the rate of expansion and, for example, describes the evolution of cosmological perturbations. Moreover, the Hubble scale and particle horizon can differ by a large factor when the strong energy condition is violated, namely $\rho+3p<0$. In this case, from the second of (1.2.2), $\ddot{a}>0$ and hence the expansion is accelerating. Then, the integral in the expression

$$d_p\left(t\right)=a\left(t\right)\int^t\frac{dt}{a}=a\left(t\right)\int^a\frac{da}{a\dot{a}} \tag{2.1.18}$$

converges as $t\to\infty$ and $a\to\infty$. When t is large, the particle horizon is proportional to the scale factor a, while the curvature scale $H^{-1}=a/\dot{a}$ grows more slowly since $\dot{a}$ also increases during accelerated expansion.

Let us now classify comoving length scales λ with associated wave number $\mathcal{K}$ according to their size relative to the horizon

$$\begin{aligned}\frac{\mathcal{K}}{aH}\ll 1 &\quad\Rightarrow\quad \text{scale}\,\lambda\,\text{inside the horizon}\\ \frac{\mathcal{K}}{aH}\gg 1 &\quad\Rightarrow\quad \text{scale}\,\lambda\,\text{outside the horizon}\,.\end{aligned} \tag{2.1.19}$$

If a scale is larger than the horizon size causal physics cannot affect it. In standard Big Bang cosmology, the derivative with respect to time t of the horizon is positive so that scales which are outside the horizon at earlier times may enter it at later times. It is now clear that the horizon problem may be solved by an early period of inflation in which $d\,(aH)^{-1}/dt < 0$. In any case, during inflation the scale exits the horizon; at the end of this period, the hot Big Bang commences and the comoving horizon size starts growing so that the CMB scale reenters the horizon.

How long must inflation be in order to solve the horizon problem? Let the subscripts i and f be related to the time of beginning and end of inflation, respectively. A necessary condition to solve the horizon problem is that the observable universe today fits in the comoving Hubble radius at the beginning of inflation:

$$\frac{1}{a_0 H_0} < \frac{1}{a_i H_i}\,.$$

Let us assume that the universe was radiation dominated since the end of inflation and ignore the relatively recent matter- and dark energy-dominated epochs. Remembering that $H \sim a$ during radiation domination, we have

$$\frac{a_0 H_0}{a_f H_f} \sim \frac{a_0}{a_f}\left(\frac{a_f}{a_0}\right)^2 = \frac{a_f}{a_0} \sim \frac{T_0}{T_f} \sim 10^{-28}\,,$$

where T_f is the temperature at the end of inflation. We therefore obtain, assuming that $H_i \sim H_f$ during inflation:

$$(a_i H_i)^{-1} \sim 10^{28}\,(a_f H_f)^{-1} \qquad \Rightarrow \qquad N > 64\,, \tag{2.1.20}$$

namely the solution of the horizon problem requires about 60 e-folds of inflation.

From the considerations made above, it appears that solving the shortcomings of the standard Big Bang theory requires two basic modifications of the initial assumptions:

- the universe has to go through a non-adiabatic period to solve the flatness problem. This may give rise to the large entropy we observe today;
- the universe has to go through a primordial period during which the physical scales λ evolve faster than the horizon scale H_Λ^{-1}.

From Fig. (2.5) [11], it is clear that if there is a period during which physical length scales grow faster than H^{-1}, length scales λ which are within the horizon today, $\lambda < H^{-1}$ (such as the distance between two detected photons) and were outside the horizon for some period, $\lambda > H^{-1}$ (for instance, at the time of last-scattering when the two photons were emitted), had a chance to be within the horizon at some primordial epoch, $\lambda < H^{-1}$ again. If this happens, the homogeneity and the isotropy of the CMB can be easily explained: photons that we receive today and were emitted from the last-scattering surface from causally disconnected regions have the same temperature because they had a chance to comunicate to each other at some primordial stage of the evolution of the universe.

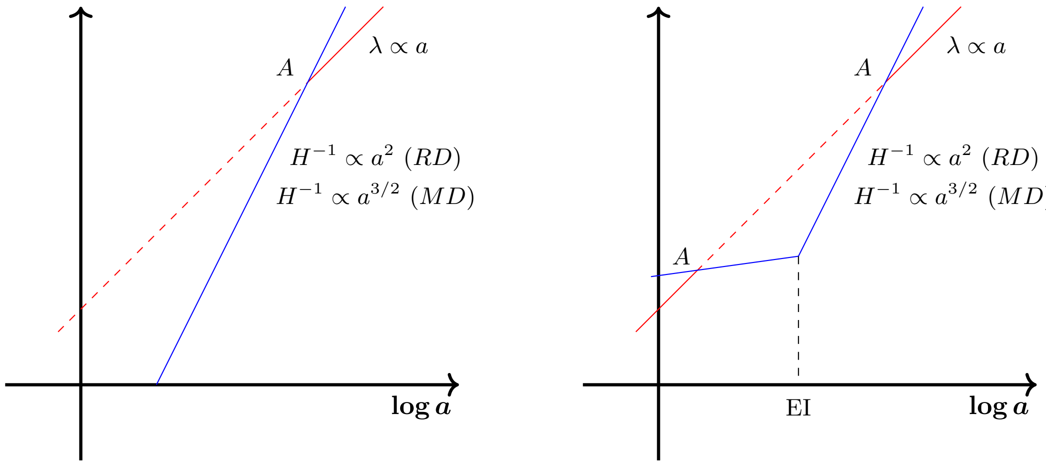


Figure 2.5: The behavior of a generic scale λ and the horizon scale H_Λ^{-1} in the standard inflationary model. Physical sizes increase as $a(t)$ in the expanding universe and the Hubble radius evolves as $R_H = H^{-1}$. In a radiation-dominated (RD) or matter-dominated (MD) universe (left panel) any physical length scale λ starts larger than R_H, then crosses the Hubble radius only once (A represents the crossing point, when $\lambda = H^{-1}$). If there was a period of early inflation when R_H increased more slowly than a (right panel), it is possible for a physical length scale to start smaller than R_H, become larger than R_H and, after inflation end, become once again smaller than R_H. The dashed line represents periods during which the scale is larger than the Hubble radius

The second condition can be easily expressed as a condition on the scale factor a. Since $\lambda \sim a$ and $H^{-1} = a/\dot{a}$, we need to impose that there is a period during which

$$\left(\frac{\lambda}{H^{-1}}\right)^{\cdot} = \ddot{a} > 0\,. \tag{2.1.21}$$

From this, we can introduce the following definition [17]: *an inflationary stage is a period of the universe during which the latter accelerates*, namely $\ddot{a} > 0$.
During this accelerating phase, the universe expands adiabatically. The non-adiabaticity condition is satisfied not during inflation, but during the phase transition between the end of inflation and the beginning of the radiation-dominated phase.

2.2 Inflation and inflaton

In this section we show that the inflationary condition can be attained by means of a single scalar field ϕ, called *inflaton*, minimally coupled to gravity. Single-field inflation occurs when the universe is dominated by the inflaton field ϕ and obeys particular conditions that we will derive step-by-step. Inflationary models assume that there is a moment when this

domination started and then drives the universe into a de Sitter expansion with quasi-zero temperature. This was named by Alan Guth an *inflationary universe* (1981).
The action of a such system is given by

$$S_\phi = -\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right], \tag{2.2.1}$$

where the function $V(\phi)$ is a potential term for the scalar field and $\sqrt{-g} = a^3$ for the FLRW metric (1.1.6). From the Eulero-Lagrange equations

$$\partial^\mu\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\partial^\mu\phi} - \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\phi} = 0 \tag{2.2.2}$$

we obtain the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,, \tag{2.2.3}$$

where the prime indicates the derivative with respect to ϕ. Indeed, the friction term $3H\dot{\phi}$ appears because a scalar field rolling down its potential suffers a friction due to the expansion of the universe.
The energy-momentum tensor of the scalar field is

$$T^{(\phi)}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left[-\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right]. \tag{2.2.4}$$

Since FRLW space-times are homogeneous and isotropic, ϕ must be homogeneous and can only depend on time. Thus, the corresponding energy density ρ_ϕ and pressure density p_ϕ are

$$\begin{aligned} T_{00} = \rho_\phi &= \frac{\dot{\phi}^2}{2} + V(\phi)\,, \\ T_{ii} = p_\phi &= \frac{\dot{\phi}^2}{2} - V(\phi)\,. \end{aligned} \tag{2.2.5}$$

Since the condition for the acceleration of the scale factor is $\rho + 3p < 0$ and since, if the gradient term in (2.2.5) were dominant, we would obtain $p_\phi = -\rho_\phi/3$ (that is not enough to drive inflation), we split the inflation field in

$$\phi(t) = \phi_0(t) + \delta\phi(\mathbf{x}, t)\,, \tag{2.2.6}$$

where ϕ_0 is the expectation value of the inflaton field on the initial isotropic and homogeneous state, while $\delta\phi$ represents the quantum fluctuations around ϕ_0. For our purpose, we focus on the evolution of the classical field ϕ_0. The energy-momentum tensor, for a homogeneous field, becomes

$$\begin{aligned} T_{00} = \rho_\phi &= \frac{\dot{\phi}^2}{2} + V(\phi)\,, \\ T_{ii} = p_\phi &= \frac{\dot{\phi}^2}{2} - V(\phi)\,. \end{aligned} \tag{2.2.7}$$

To obtain $p_\phi \approx -\rho_\phi$ (the de Sitter limit) it is necessary have $V(\phi) \gg \dot{\phi}^2$, which is the condition for inflation to take place. This means that for successfull inflation we need a scalar field that slowly rolls down its potential under certain conditions, so that its potential energy dominates over its kinetic energy. This also shows that the inflaton potential must be sufficiently flat (and hence $\ddot{\phi} = 0$), but unfortunately it is not always easy to obtain in realistic situations. This problem might be bypassed involving more scalar fields in order to have more possibilities to create inflation, but it surely looses its predictive power, expecially about cosmological perturbations, which are among the most important predictions of inflation. For this reason it is preferred to consider simple scenarios with a single inflaton, each of them leading to very similar predictions.
The period during which the scalar field slowly rolls down its potential is called *slow-roll*. In this case, being the potential flat, $\ddot{\phi}$ is negligible, thus Klein-Gordon equation (2.2.3) becomes

$$3H\dot{\phi} = -V'(\phi) \tag{2.2.8}$$

which gives $\dot{\phi}$ as a function of $V'(\phi)$. In this way, slow-roll conditions require

$$\dot{\phi}^2 \ll V(\phi) \quad \Longrightarrow \quad \frac{(V')^2}{V} \ll H^2 \tag{2.2.9}$$

and

$$\ddot{\phi} \ll 3H\dot{\phi} \quad \Longrightarrow \quad V'' \ll H^2 \,. \tag{2.2.10}$$

Liddle and Lyth [12] defined the following dimensionless *slow-roll parameters*

$$\begin{aligned} \epsilon &= -\frac{\dot{H}}{H^2} = 4\pi G \frac{\dot{\phi}^2}{H^2} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 , \\ \eta &= \frac{1}{8\pi G}\left(\frac{V''}{V}\right) = \frac{1}{3}\frac{V''}{H^2} , \\ \delta &= \eta - \epsilon = -\frac{\ddot{\phi}}{H\dot{\phi}} \,. \end{aligned} \tag{2.2.11}$$

The parameter ϵ quantifies how much the Hubble rate H changes with time during inflation. Considering the following equation

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = (1-\epsilon)\, H^2 \,, \tag{2.2.12}$$

inflation can be attained only if $\epsilon < 1$. When this condition fails, inflation ends. Slow-roll inflation is therefore attained if $\epsilon \ll 1$ and $|\eta| \ll 1$.
The exact evolution of the background (the homogeneous field and the scale factor) can

be found by solving the Friedmann and Klein-Gordon equations. However, if we are sure that the slow-roll conditions are satisfied, we can solve simply the approximate first-order equation

$$\dot{\phi} = -\frac{1}{3H}\frac{\partial V}{\partial \phi}\,, \tag{2.2.13}$$

where $\phi \equiv \phi_0$ for simplicity.
It is necessary to check that the potential allows for a sufficient number of inflationary e-folds [13], namely the total number of e-foldings between the beginning and the end of inflation (the subscripts i and f stay for the beginning and the end of inflation, respectively):

$$N \equiv \int_{t_i}^{t_f} H dt = H \int_{\phi_i}^{\phi_f} \frac{d\phi}{\dot{\phi}} \approx -3H^2 \int_{\phi_i}^{\phi_f} \frac{d\phi}{V'} \approx -8\pi G \int_{\phi_i}^{\phi_f} \frac{V}{V'} d\phi\,. \tag{2.2.14}$$

For a particular form of the potential, one can compute the value of the field ϕ_f at the end of inflation (generally, it is such that $\max\left[\epsilon, |\eta|\right] = 1$). Thus, the above relation provides a condition on the initial value ϕ_i in order to obtain a sufficient number of e-folds.
Clearly there must be a mechanism to stop this exponential expansion, otherwise it is impossible to have our present slowly expanding FL universe. This mechanism is called *graceful exit* and it depends on a convenient choice of the potential function and its temperature-dependence, $V(\phi, T)$. Inflation ends when the kinetic term $\dot{\phi}^2/2$ dominates over $V(\phi)$, namely the inflaton field arrives at the minimum $\phi = 0$ of the potential (in a time approximated to $\left(10^{-34} s\right)^{-1}$). The inflaton approchaes this minimum very slowly because of rapidly-expanding universe.
We have discussed so far the conditions the cosmic inflation needs to start and to finish. Obviously, the inflaton field has to imitate a scalar condensate in the slow-roll regime. This can be done entirely within the theory of gravity itself: in fact, Einstein gravity is only a low curvature limit of some more complicated theory whose action contains higher powers of the curvature invariants, for example the Gauss-Bonnet term. Any modification of the Einstein action introduces higher-derivative terms and the higher derivative gravity theory is conformally equivalent to Einstein gravity with an extra scalar field. However, inflation can be realized even without a potential term. It can occurr in Born-Infeld-type theories, where the action depends nonlinearly on the kinetic energy of the scalar field. These theories do not have higher-derivative terms, but they have some other peculiar properties that we do not consider in this thesis.
From a physical point of view, the simplest inflationary scenarios can be divided into three classes [2], each corresponding to the usual scalar field with a potential, higher-derivative gravity and $k-$inflation (abbreviation for kinetically driven inflation) [28]. The potential can have different shapes, as shown in Fig. 2.6. The three cases presented correspond to the so-called old, new and chaotic inflationary scenarios. The first two names refer to their historical origins.

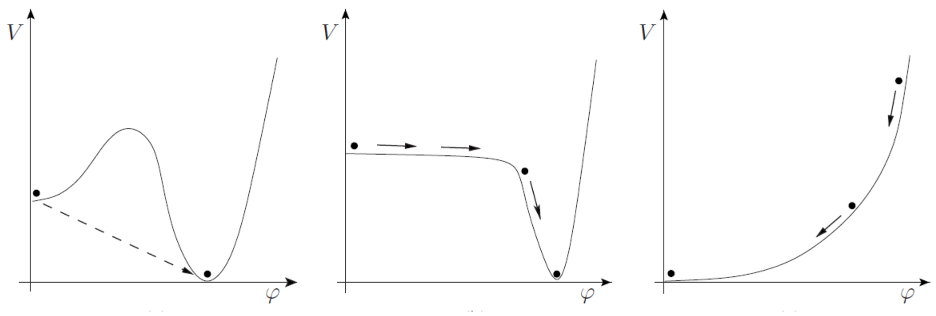


Figure 2.6: Inflation of different potentials [2].

Old inflation (see Fig. 2.6(a)) assumes that the scalar field arrives at the local minimum of the potential at $\phi = 0$ as a result of a supercooling of the initially hot universe. After that the universe undergoes a stage of accelerated expansion with a subsequent graceful exit via bubble nucleation. It was clear from the very beginning that this scenario could not provide a successful graceful exit because all the energy released in a bubble is concentrated in its wall and the bubbles have no chance to collide. This difficulty was avoided in the new inflationary scenario, a scenario similar to a successful model in higher-derivative gravity which had previously been invented.
New inflation is based on a Coleman-Weinberg type potential (Fig. 2.6(b)). Because the potential is very flat and has a maximum at $\phi = 0$, the scalar field escapes from the maximum via quantum fluctuations. It then slowly rolls towards the global minimum where the energy is released homogeneously in the whole space. Originally the pre-inflationary state of the universe was taken to be thermal so that the symmetry was restored due to thermal corrections. This was a justification for the initial conditions of the scalar field. Later it was realized that the thermal initial state of the universe is quite unlikely, and so now the original motivation for the initial conditions in the new inflationary model seems to be false. Instead, the universe might be in a self-reproducing regime.
Chaotic inflation is the name given to the broadest possible class of potentials satisfying the slow-roll conditions (Fig. 2.6(c)). The name chaotic is related to the possibility of having almost arbitrary initial conditions for the scalar field. To be precise, this field must initially be larger than the Planckian value but it is otherwise arbitrary. Indeed, it could have varied from one spatial region to another and, as a result, the universe would have a very complicated global structure. It could be very inhomogeneous on scales much larger than the present horizon and extremely homogeneous on small scales corresponding to the observable domain. In the case of chaotic inflation, quantum fluctuations lead to a self-reproducing universe.
Since chaotic inflation encompasses so many potentials, one might think to consider special cases, for example, an exponential potential. In this case, if the slow-roll conditions are

satisfied once, they are always satisfied. Therefore, it describes (power-law) inflation without a graceful exit. To arrange a graceful exit we have to damage the potential. For two or more scalar fields the number of options increases. Thus it is not helpful here to go into the details of the different models. In the absence of the underlying fundamental particle theory, one is free to play with the potentials and invent further new scenarios. In this sense the situation has changed when in the 1980s many people considered inflation a useful application of the Grand Unified Theory that was believed to be known. Besides solving the initial conditions problem, inflation also explained why we do not have an overabundance of the monopoles that are an inevitable consequence of a Grand Unified Theory. Either inflation ejects all previously created monopoles, leaving less than one monopole per present horizon volume, or the monopoles are never produced. The same argument applies to the heavy stable particles that could be overproduced in the state of thermal equilibrium at high temperatures. Many authors consider the solution of the monopole and heavy particle problems to be as important as a solution of the initial conditions problem. However, the initial conditions problem is posed to us by nature, while the other problems are, at present, only internal problems of theories beyond the Standard Model. By solving these extra problems, inflation opens the door to theories that would otherwise be prohibited by cosmology.

CHAPTER 3

DYNAMICAL SYSTEMS

In this chapter we give a brief overview of some aspects of the theory of dynamical systems, which is used [14] to study physical systems that evolve in time. This theory has its origin in the work of Poincaré at the end of the nineteenth century: he proposed that instead of trying to find particular exact solutions of a differential equation of the form

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}\,(\boldsymbol{x}) \ , \quad \boldsymbol{x} \in X \ ,$$

where $f : X \to X$ and X a phase space, one should use topological and geometrical methods to determine properties of the set of all solutions, viewed as orbits (or trajectories) in a state space. In the late 1920s, Birkhoff and others began the formal mathematical development of the theory of dynamical systems, by introducing concepts such as the flow associated with a differential equation, and the concept of an ω-limit set.
Our aim is the application of the theory of dynamical systems to the Einstein field equations in a cosmological setting, in order to obtain qualitative information about the evolution of the cosmological model under study.

3.1 Basics of dynamical theories

The most general notion of a dynamical system includes the following elements [15], [16]

(*) A *phase space* X.

(**) *Time*, which may be discrete or continuous. It may extend either only into the future (irreversible or noninvertible processes) or into the past as well as the future (reversible or invertible processes). The sequence of time moments for a reversible discrete-time process is in a natural correspondence to the set of all integers; irreversibility corresponds to considering only nonnegative integers. Similarly, for a continuous-time process, time is represented by the set of all real numbers in the reversible case and by the set of nonnegative real numbers for the irreversible case.

(***) The time-evolution law. In the most general setting this is a rule that allows us to determine the state of the system at each moment of time t from its states at all previous times. Thus, the most general time-evolution law is time dependent and has infinite memory.

The state of the system we are interested in is described by a set of quantities which are considered important about the system, and the state space is the set of all possible values of these quantities. In the case of the pendulum, for example, the position of the mass and its momentum are natural quantities to specify the state of the system. For more complicated systems like the universe as a whole, the choice of good quantities is not at all obvious and it turns out to be useful to choose convenient variables. It is possible to analyse the same dynamical system with different sets of variables, either of which might be more suitable to a particular question.

The most characteristic feature of dynamical theories, which distinguishes them from other areas of mathematics dealing with groups of automorphisms of various mathematical structures, is the emphasis on asymptotic behavior as time goes to infinity. The best way to explain what significant asymptotic properties are is to examine specific examples of dynamical systems and to determine the most characteristic features of their behavior.

There are two main types of dynamical systems: The first are continuous dynamical systems whose evolution is defined by a set of ordinary differential equations (ODEs) and the other ones are called time-discrete dynamical systems which are defined by a map or difference equations. In the context of cosmology we are studying the Einstein field equations which for a homogeneous and isotropic space result in a system of ODEs. Thus we are only interested in continuous dynamical systems. Historically, smooth continuous-time dynamical systems appeared because of Newton's discovery that the motions of mechanical objects can be described by second order ordinary differential equations. More generally, many other natural and social phenomena, such as radioactive decay, chemical reactions, population growth, dynamics of prices on the market etc. may be modeled with various degrees of accuracy by systems of ordinary differential equations.

As an example of differentiable dynamics, let us consider a *Lagrangian dynamical system.* We start with a manifold $M \subset \mathbb{R}^n$ which may be called *configuration space.* In general it is not necessary that M is compact. The phase space of the dynamical system is the tangent bundle TM. The system is described by assigning to each point $x \in M$ a potential energy $V(x)$ and to each tangent vector $v = \dot{x}$ a kinetic energy given by a positive definite quadratic form $K(v) = \frac{1}{2} k_x(v, v)$ on TM, whose coefficients in local coordinates will depend on the point, that is a scalar product depending on $x \in M$.

Now we can define a differentiable function $\mathcal{L} : TM \to \mathbb{R}$, called the *Lagrangian*

$$\mathcal{L}(x,v) = \frac{1}{2} k_x(v,v) - V(x) \ . \tag{3.1.1}$$

It satisfies a second order ordinary differential equation on M, namely, the following first order ordinary differential equation on TM

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x} \, , \tag{3.1.2}$$

which is called the *Euler-Lagrange equation*. The dynamics determined by this ordinary differential equation is independent of the local coordinate chart; it is defined for all times and determines a complete flow on TM defined for all t. In particular, the following theorem holds: *for a Lagrangian dynamical system the total energy* $H = \frac{1}{2}k_x(v,v) + V(x)$ *is invariant under the dynamics*. As a consequence, one can prove that *if the configuration space M is compact, then the Lagrange equation defines a global flow in TM*.
Let us consider, now, a particular case of Lagrangian system, corresponding to free particle motion in the configuration space. Let (M, g) be a Riemannian manifold with Riemannian metric g_x and define the Lagrangian

$$\mathcal{L}(x,v) = \frac{1}{2} g_x(v,v) \ . \tag{3.1.3}$$

The Lagrangian system on TM corresponding to this Lagrangian as well as its restriction to the unit tangent bundle SM is called the *geodesic flow* of the Riemannian manifold (M, g). It preserves the length of tangent vectors because the total energy is given by $\frac{1}{2}g_x(v,v)$. Since geodesics on a manifold are the shortest connection between any two of its points sufficiently close, the geodesic flow on any compact manifold is a complete flow.

3.2 Qualitative study

Very few ordinary differential equations have exact solutions and in this case, even if a solution can be found, the formula to obtain it is often too complicated to display clearly the principal features of the solution. The qualitative study of differential equations is concerned with how to deduce important characteristics of the solutions of differential equations without actually solving them. An important geometrical device, the phase plane, is used for obtaining directly from the differential equation such properties as equilibrium, periodicity, stability and so on.
Frequently, the appropriate formulation of mechanical, biological and geometrical problems is through a first order system of the form

$$\begin{cases} \dot{x} = X(x,y) \\ \dot{y} = Y(x,y) \ , \end{cases} \tag{3.2.1}$$

where the functions $X(x, y)$ and $Y(x, y)$ are smooth enough to make the system regular in the region of interest.
Let us recall that the n-th order system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) \tag{3.2.2}$$

is called *regular* on $\mathcal{D} \times I$, where $\mathcal{D}$ is a domain in $\mathbb{R}^n$ and I is an open interval in $\mathbb{R}$, if $\boldsymbol{f}$ is continuous and $\partial f_j / \partial x_i$ $(i, j = 1, \cdots, n)$ are continuous for $\boldsymbol{x} \in \mathcal{D}$ and $t \in I$.
The constant solutions are represented by equilibrium points obtained by solving the equations

$$\begin{cases} X(x, y) = 0 \\ Y(x, y) = 0 \, . \end{cases} \tag{3.2.3}$$

Near the equilibrium points we may make a linear approximation to $X(x, y)$, $Y(x, y)$, solve the simpler equations obtained and so determine the local character of the paths. This enables the stability of the equilibrium states to be settled and it is a starting point for global investigation of the solutions.
The system (3.2.1) is called *autonomous* because the time variable t does not appear in the right-hand side. The solutions $x(t)$, $y(t)$ of this system may be represented on a plane with cartesian axes x, y. As t increases, $(x(t), y(t))$ traces out a directed curve in the plane called a *phase path*.
The appropriate form for the initial conditions of (3.2.1) is

$$x = x_0 \, , \quad y = y_0 \qquad \text{at} \quad t = t_0 \, , \tag{3.2.4}$$

where x_0 and y_0 are the *initial values* at time t_0; by the existence and uniqueness theorem, there is one and only one solution satisfying this condition when (x_0, y_0) is an ordinary point. This does not at once mean that there is one and only one phase path through the point (x_0, y_0) on the phase diagram, because this same point could serve as the initial conditions for other starting times. Therefore it might seem that other phase paths through the same point could result: the phase diagram would then be a tangle of crisscrossed curves. We may see that this is not so by forming the differential equation for the phase paths. Since $\dot{y}/\dot{x} = dy/dx$ on a path, the required equation is

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)} \, . \tag{3.2.5}$$

Equation (3.2.5) does not give any indication of the direction to be associated with a phase path. This must be settled by reference to equations (3.2.1). The signs of X and Y at any particular point determine the direction through the point, and generally the directions at all other points can be settled by the requirement of continuity of direction of adjacent paths.
The diagram depicting the phase paths is called the *phase diagram*. A typical point (x, y) is called a *state* of the system. The phase diagram shows the evolution of the states of the

system, starting from arbitrary initial states.
Points where the right-hand side of (3.2.5) satisfies the conditions for regularity are called the *ordinary points* of (3.2.5). There is one and only one phase path through any ordinary point (x_0, y_0), no matter at what time t_0 the point (x_0, y_0) is encountered. Therefore infinitely many solutions of (3.2.1), differing only by time displacements, produce the same phase path.
Equation (3.2.5) may however have singular points, where the conditions for regularity do not hold, even though the time equations (3.2.1) have no particularity there. Such singular points occur where $X(x, y) = 0$. Points where $X(x, y)$ and $Y(x, y)$ are both zero,

$$X(x, y) = 0\,, \qquad Y(x, y) = 0\,, \tag{3.2.6}$$

are called *equilibrium points*. If x_1, y_1 is a solution of (3.2.6), then

$$x(t) = x_1\,, \qquad y(t) = y_1 \tag{3.2.7}$$

are constant solution of (3.2.1) and are degenerate phase paths. The terms *fixed points* and *critical points* are also used.
Since $dy/dx = Y(x, y)/X(x, y)$ is the differential equation of the phase paths, phase path which cut the curve $Y(x, y) = cX(x, y)$ will do so with the same slope c: such curves are known as *isoclines*. The two particular isoclines $Y(x, y) = 0$, which cut paths with zero slope, and $X(x, y) = 0$, which cut paths with infinite slope, are helpful in phase diagram sketching. The points where these isoclines intersect define the equilibrium points. Between the isoclines, $X(x, y)$ and $Y(x, y)$ must each be of one sign. For example, in a region in the (x, y) plane in which $X(x, y) > 0$ and $Y(x, y) > 0$, the phase path must have positive slopes. This will also be the case if $X(x, y) < 0$ and $Y(x, y) < 0$. Similarly, if $X(x, y)$ and $Y(x, y)$ have opposite signs in a region, then the phase paths must have negative slopes.

3.3 Linear approximation

If we have to treat a nonlinear system, approximation by linearizing it at an equilibrium point is an important and generally useful technique. If the geometrical nature of the equilibrium points can be settled in this way, the general character of the phase diagram is often clear. Let us consider the system

$$\dot{x} = X(x, y)\ , \qquad \dot{y} = Y(x, y) \tag{3.3.1}$$

and suppose that the equilibrium point to be studied has been moved to the origin by a translation of axes, if necessary, so that

$$X(0, 0) = Y(0, 0) = 0\,. \tag{3.3.2}$$

We can therefore write, by a Taylor expansion,

$$X(x, y) = ax + by + P(x, y)\ , \quad Y(x, y) = cx + dy + Q(x, y)\ , \tag{3.3.3}$$

where
$$a = \frac{\partial X}{\partial x}(0,0) \ , \ b = \frac{\partial X}{\partial y}(0,0) \ , \ c = \frac{\partial Y}{\partial x}(0,0) \ , \ d = \frac{\partial Y}{\partial y}(0,0) \tag{3.3.4}$$

and $P(x,y)$, $Q(x,y)$ are of lower order of magnitude than the linear terms as (x,y) approaches the origin $(0,0)$. The *linear approximation* to (3.3.4) in the neighbourhood at the origin is defined as the system
$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy\,. \end{cases} \tag{3.3.5}$$

We expect that the solutions of (3.3.5) will be geometrically similar to those of (3.3.1) near the origin, an expectation fulfilled in most cases.
The system (3.3.5) with constant coefficients a, b, c and d is more manageable in matrix form. By defining the column vectors

$$\boldsymbol{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} , \qquad \dot{\boldsymbol{x}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t)\ , \end{pmatrix} \tag{3.3.6}$$

the system we are considering may be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \qquad \text{with} \qquad \boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{3.3.7}$$

We shall only consider cases where there is a single equilibrium point, at the origin, the condition for this being
$$\det \boldsymbol{A} = ad - bc \neq 0\,. \tag{3.3.8}$$

If $\det \boldsymbol{A} = 0$, then one of its rows is a multiple of the other, so that $ax + by = 0$ (or $cx + dy = 0$) consists of a line of equilibrium points.
We seek a fundamental solution consisting of two linearly independent solutions of (3.3.7), having the form
$$\boldsymbol{x}_1(t) = \boldsymbol{v}_1 e^{\lambda_1 t}\,, \qquad \boldsymbol{x}_2(t) = \boldsymbol{v}_2 e^{\lambda_2 t}\,, \tag{3.3.9}$$

where λ_1, λ_2 are constants and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are constant column vectors. It is known that the general solution is given by

$$\boldsymbol{x}(t) = C_1 \boldsymbol{x}_1(t) + C_2 \boldsymbol{x}_2(t) \ , \tag{3.3.10}$$

with C_1 and C_2 arbitrary constants.
To determine λ_1, $\boldsymbol{v}_1$, λ_2, $\boldsymbol{v}_2$ in (3.3.9) we substitute

$$\boldsymbol{x}(t) = \boldsymbol{v} e^{\lambda t} \tag{3.3.11}$$

into the system equations (3.3.7). After cancelling the common factor $e^{\lambda t}$, we obtain

$$\lambda \boldsymbol{v} = A\boldsymbol{v}\,, \tag{3.3.12}$$

or

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\,\boldsymbol{v} = \boldsymbol{0}\,, \tag{3.3.13}$$

where $\boldsymbol{I}$ is the identity matrix. If we put

$$\boldsymbol{v} = \begin{pmatrix} r \\ s \end{pmatrix}\,, \tag{3.3.14}$$

equation (3.3.13) represents the pair of scalar equations

$$\begin{cases} (a-\lambda)\,r + bs = 0 \\ cr + (d-\lambda)\,s = 0 \end{cases}\,. \tag{3.3.15}$$

It is known from algebraic theory that equation (3.3.13) has nonzero solutions for $\boldsymbol{v}$ only if determinant of the matrix of the coefficients in equations (3.3.15) is zero. Therefore

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0\,, \tag{3.3.16}$$

or

$$\lambda^2 - (a+d)\,\lambda + (ad-bc) = 0\,. \tag{3.3.17}$$

This is called the *characteristic equation*, and its solutions λ_1, λ_2 are the *eigenvalues* of the matrix $\boldsymbol{A}$ defined before, or the *characteristic exponents* for the problem. In order to classify the solutions of characteristic equation, it is convenient to use the following notation:

$$\lambda^2 - p\lambda + q = 0\,, \tag{3.3.18}$$

where $p = a + d$, $q = ad - bc$. Also let us put

$$\Delta = p^2 - 4q\,. \tag{3.3.19}$$

The eigenvalues $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are given by

$$\lambda_{1,2} = \frac{1}{2}\left(p \pm \Delta^{1/2}\right)\,. \tag{3.3.20}$$

These are to be substituted successively into (3.3.15) to obtain corresponding values for the constants r and s, which represent the components of the so-called *eigenvectors* of the matrix $\boldsymbol{A}$.

There are two main classes to be considered: when the eigenvalues are real and different and when they are complex. These cases are distinguished by the sign of the discriminant Δ. We also assume that $q \neq 0$, coherently with $\det \boldsymbol{A} \neq 0$.

(I) **Time solutions with real and distinct eigenvalues**

After obtaining the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of the matrix $\boldsymbol{A}$, the general solution of (3.3.7) is given by (3.3.10):

$$\boldsymbol{x}\,(t) = C_1 \boldsymbol{v}_1 e^{\lambda_1 t} + C_2 \boldsymbol{v}_2 e^{\lambda_2 t}\,, \tag{3.3.21}$$

where C_1 and C_2 are arbitrary constants.

(II) **Time solutions with complex eigenvalues**
After putting $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ (the eigenvalues are complex conjugate), the general solution of (3.3.7) may be written in the form

$$\boldsymbol{x}(t) = \mathrm{Re}\left\{C\boldsymbol{v}e^{(\alpha+i\beta)t}\right\}, \tag{3.3.22}$$

where C is an arbitrary complex constant and $\boldsymbol{v}$ is a complex eigenvector of the matrix $\boldsymbol{A}$.

3.4 Classification of equilibrium points

For the system (3.3.5) the general character of the phase paths can be obtained from the time solutions (3.3.21) and (3.3.22).
The phase diagram patterns fall into three main classes depending on the eigenvalues, which are the solutions λ_1, λ_2 of the characteristic equation (3.3.18).

(A) The eigenvalues are real, distinct and with the same sign
Let λ_1 be the greater of the two eigenvalues, so that $\lambda_2 < \lambda_1$. In component form the general solution (3.3.21) becomes

$$x(t) = C_1 r_1 e^{\lambda_1 t} + C_2 r_2 e^{\lambda_2 t}, \quad y(t) = C_1 s_1 e^{\lambda_1 t} + C_2 s_2 e^{\lambda_2 t}, \tag{3.4.1}$$

with $C_{1,2}$ arbitrary constants and r_1, s_1 and r_2, s_2 constants obtained by solving (3.3.15) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively. From (3.4.1) we also obtain

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{C_1 s_1 \lambda_1 e^{\lambda_1 t} + C_2 s_2 \lambda_2 e^{\lambda_2 t}}{C_1 r_1 \lambda_1 e^{\lambda_1 t} + C_2 r_2 \lambda_2 e^{\lambda_2 t}}. \tag{3.4.2}$$

Let us suppose, firstly, that λ_1 and λ_2 are negative, so that

$$\lambda_2 < \lambda_1 < 0. \tag{3.4.3}$$

From (3.4.1) and (3.4.3), along any phase path, it turns out:

$$\begin{cases} x \text{ and } y \text{ approach the origin as } t \to \infty, \\ x \text{ and } y \text{ approach infinity as } t \to -\infty. \end{cases} \tag{3.4.4}$$

There are also four radial phase paths which lie along a pair of straight lines as follows:

$$C_2 = 0 \Rightarrow \frac{y}{x} = \frac{s_1}{r_1}; \qquad C_1 = 0 \Rightarrow \frac{y}{x} = \frac{s_2}{r_2}. \tag{3.4.5}$$

The dominant term are those involving $e^{\lambda_1 t}$ for large positive t and $e^{\lambda_2 t}$ for large negative t, therefore from (3.4.2) we obtain

$$\frac{dy}{dx} \to \frac{s_1}{r_1} \quad \text{as} \quad t \to \infty; \qquad \frac{dy}{dx} \to \frac{s_2}{r_2} \quad \text{as} \quad t \to -\infty. \tag{3.4.6}$$

From (3.4.4) and (3.4.5) it follows that every phase path is tangential to $y = (s_1/r_1)\,x$ at the origin, and approaches the direction of $y = (s_2/r_2)\,x$ at infinity. The radial solutions (3.4.5) are called *asymptotes* if the family of phase paths.
If the eigenvalues λ_1, λ_2 are both positive, with $\lambda_2 > \lambda_1 > 0$, the phase diagram has similar characteristics, but all the phase paths are directed outward, running from the origin to infinity.
These patterns show a new feature called a *node*, which can be *stable* or *unstable*. The conditions on the coefficients corresponding to these cases are

$$\begin{cases} \text{stable node}: \quad \Delta = p^2 - 4q > 0\,, \quad q > 0\,, \quad p < 0\,; \\ \text{unstable node}: \quad \Delta = p^2 - 4q > 0\,, \quad q > 0\,, \quad p > 0\,. \end{cases} \tag{3.4.7}$$

(B) The eigenvalues are real, distinct and of opposite signs

In this case $\lambda_2 < 0 < \lambda_1$, and the solution (3.4.1), with the formula (3.4.2), still apply. In the same way as before, we can deduce that four of the paths are straight lines radiating from the origin, two of them lying along each of the lines

$$\frac{y}{x} = \frac{s_1}{r_1} \qquad \text{and} \qquad \frac{y}{x} = \frac{s_2}{r_2}\,, \tag{3.4.8}$$

which are broken by the equilibrium point at the origin.
In this case, however, there are only two paths which approach the origin. From (3.4.1) it can be seen that these are the straight-line paths which lie along $y/x = s_2/r_2$, obtained by putting $C_1 = 0$. The other pair of straight-line paths go to infinity as $t \to \infty$, as do all the other paths. Also, every path (except for the two which lie along $y/x = s_2/r_2$) starts at infinity as $t \to -\infty$.
The pattern is like a family of hyperbolas with its asymptotes. The equilibrium point at the origin is a *saddle*. From (3.3.18) the conditions on the coefficients of the characteristic equation are

$$\text{saddle point:} \qquad \Delta = p^2 - 4q > 0\,, \quad q < 0\,. \tag{3.4.9}$$

A saddle point is always unstable.

(C) The eigenvalues are complex

Complex eigenvalues are always complex conjugate, so we can put

$$\lambda_1 = \alpha + i\beta\,, \quad \lambda_2 = \alpha - i\beta \quad (\alpha,\, \beta \quad \text{real})\,. \tag{3.4.10}$$

By separating the components of (3.3.22) we obtain for the general solution

$$x\,(t) = e^{\alpha t}\mathrm{Re}\left\{C r_1 e^{i\beta t}\right\}\,, \qquad y\,(t) = e^{\alpha t}\mathrm{Re}\left\{C s_1 e^{i\beta t}\right\}\,, \tag{3.4.11}$$

where C, r_1 and s_1 are all complex in general.
Let us suppose, firstly, that $\alpha = 0$. If we put C, r_1, s_1 in polar form:

$$C = |C|\,e^{i\gamma}\,, \quad r_1 = |r_1|\,e^{i\rho}\,, \quad s_1 = |s_1|\,e^{i\sigma}\,, \tag{3.4.12}$$

then (3.4.11), with $\alpha = 0$, becomes

$$x(t) = |C|\,|r_1|\cos(\beta t + \gamma + \rho)\ , \qquad y(t) = |C|\,|s_1|\cos(\beta t + \gamma + \sigma)\ . \tag{3.4.13}$$

The motion of the representative point $(x(t), y(t))$ in the phase plane consists of two simple harmonic components of equal circular frequency β, in the x and y directions, but they have different phase and amplitude. The phase paths therefore form a family of geometrically similar ellipses which, in general, is inclined at a constant angle to the axes.
The algebric conditions corresponding to the *centre* of the ellipses at the origin are

$$\text{centre:} \quad p = 0\,, \qquad q > 0\,. \tag{3.4.14}$$

Now let us suppose that $\alpha \neq 0$. As t increases in equations (3.4.11), the elliptical paths are modified by the factor $e^{\alpha t}$. This prevents them from closing, and each ellipse turns into a spiral: a contracting spiral if $\alpha < 0$, and an expanding spiral if $\alpha > 0$. The equilibrium point is then called a *spiral* or *focus*, stable if $\alpha < 0$, unstable if $\alpha > 0$. The directions may be clockwise or counterclockwise.
The algebric conditions are

$$\begin{cases} \text{stable spiral :} \quad \Delta = p^2 - 4q < 0\,, \quad q > 0\,, \quad p < 0\,; \\ \text{unstable spiral :} \quad \Delta = p^2 - 4q < 0\,, \quad q > 0\,, \quad p > 0\,. \end{cases} \tag{3.4.15}$$

(D) Degenerate cases

There are several degenerate cases. These occur when there is a repeated eigenvalue, and when an eigenvalue is zero.
If $q = \det \boldsymbol{A} = 0$, then the eigenvalues are $\lambda_1 = p$, $\lambda_2 = 0$. If $p \neq 0$, then as in the case (3.3.21) we have

$$\boldsymbol{x}(t) = C_1 \boldsymbol{v}_1 e^{pt} + C_2 \boldsymbol{v}_2\,, \tag{3.4.16}$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the eigenvectors. There is a line of equilibrium points given by

$$ax + by = 0 \qquad (\text{or } cx + dy = 0)\ . \tag{3.4.17}$$

The phase paths form a family of parallel straight lines. A further special case arises if $q = 0$ and $p = 0$.
If $\Delta = 0$, then eigenvalues are real and equal with $\lambda = \frac{1}{2}p$. If $p \neq 0$, it can be shown that the equilibrium point becomes a degenerate node, in which the two asymptotes have converged. We summarize all the cases studied so far in the table 3.1.
A centre may be regarded as a degenerate case, forming a transition between stable and unstable spirals. The existence of a centre depends on there being a particular exact relation, namely $a + d = 0$, between the coefficients of the system, so a centre is rather a fragile feature. Consequently, if the linear approximation to a nonlinear

	$p = a + d$	$q = ad - bc$	$\Delta = p^2 - 4q$
saddle	-	$q < 0$	$\Delta > 0$
stable node	$p < 0$	$q > 0$	$\Delta > 0$
stable spiral	$p < 0$	$q > 0$	$\Delta < 0$
unstable node	$p > 0$	$q > 0$	$\Delta > 0$
unstable spiral	$p > 0$	$q > 0$	$\Delta < 0$
centre	$p = 0$	$q > 0$	$\Delta < 0$
degenerate stable node	$p < 0$	$q > 0$	$\Delta = 0$
degenerate unstable node	$p > 0$	$q > 0$	$\Delta = 0$

Figure 3.1: Classification of equilibrium points of a dynamical system

system predicts a centre, it cannot be reliably concluded that the original system has a centre: it might have a stable, or worse, an unstable spiral. The same applies to all the degenerate cases indicated: if they are used as linear approximation then, taken alone, they are unreliable indicators.

If there exists a neighbourhood of an equilibrium point such that every phase path starting in the neighbourhood ultimately approaches the equilibrium point, this is known as an *attractor* (the term is used both for linear and nonlinear systems). The stable node and stable spirals are attractors. An attractor with all path directions reversed is a *repellor*. Unstable nodes and spirals are repellors, but a saddle point is not. The terms attractor and repellor can also be applied to limit cycles, and to less well defined attracting sets. If the eigenvalues of the linearized equation have nonzero real parts, then the equilibrium point is *hyperbolic*. It can be proved that at hyperbolic points the phase diagrams of the nonlinear equations and the linearized equations are, locally, qualitatively the same. Spirals, node and saddles are hyperbolic but the centre is not.

CHAPTER 4

AN EXACTLY SOLVABLE INFLATIONARY MODEL

4.1 Introduction

As discussed in the previous chapter, the inflationary scenario seems a natural and simple way to eliminate both the horizon and the flatness problems, which are due to the initial conditions required by the standard model of hot big bang cosmology. The standard model has a singularity at conventional time $t = 0$; in the limit $t \to 0$ the temperature T goes to infinity, so at this time we cannot define initial-value problem. In order to fix this problem, Guth proposed to begin the hot big-bang scenario at temperature $T_0 = 10^{17}$ GeV (a value below the Planck mass $M_P \equiv 1/\sqrt{G} = 1.22 \times 10^{19}$ GeV: in fact, if T_0 is of the order of M_P, the equations of the standard model are meaningless) so that one can take the description of the universe as a set of initial conditions and the equation of motion then describe the subsequent evolution. In [17] the inflationary solution has been obtained with a constant potential and it is also shown that the existence of a sufficiently long period of exponential expansion in the early universe would provide a natural solution of the above mentioned problems. Guth himself recognized that his scenario leads to some unacceptable consequences, whose solution was found in 1981-1982 with the invention of the new inflationary theory [18]. Generally, the most common mechanisms leading to inflation are based on models of gravity minimally coupled to a scalar field with suitable self-interaction potential. The new inflationary theory proposes that the scalar field starts its evolution from a value that does not minimize the effective potential and then rolls toward the minimum, inducing the inflation of the scale factor of the universe.
The exponential expansion of the universe as an example of exact solution was found by Ivanov for a nonlinear scalar field with the potential $V(\phi) = \frac{\mu}{2}\phi^2 - \frac{\lambda}{4}\phi^4$ in the spatially-flat

FRW space and has been interpretated as the universe rising from a quasivacuum state of matter.
Some exact inflationary solutions of gravity-scalar models are known for specific potentials and recently they have been classified in [19]. Actually, in some papers, for instance [20], new classes of exact solutions have been found by taking the scalar field as a function of time $\phi = \phi(t)$ and then determining the evolution of the expansion scale factor $a(t)$ and the potential $V(\phi(t))$ from it. Other authors [21] obtained the exact solutions by taking, first, the scalar factor as the function of time $a = a(t)$ and then determining the evolution of the potential $V = V(t)$ and the evolution of a scalar field $\phi = \phi(t)$, the dependence between V and ϕ being, in general, parametric. This approach was called the method of *fine tuning of the potential*.
In most cases, however, the solutions can only be obtained in an approximate way. Usually their behavior is investigated by means of the so-called slow-roll approximation. Such approximation breaks down when the scalar field is close to the minimum of the potential, where the inflation ends. The later evolution of the universe is assumed to obey the standard Friedmann-Lemaître equations and is characterized by a power-law expansion.
There exist a plenty of models that can be classified in three classes according to the features of the potential: the large-field, small-field and hybrid potentials. Large-field models are potentials typical of *chaotic* inflation scenarios, in which the scalar field is displaced from the minimum of the potential by an amount usually of order M_P. These models are characterized by $V''(\phi) > 0$ and $0 < \eta \le \epsilon$, while small-field models present a negative second-derivative potential and $0 < \eta < \epsilon$. In the latter case the field starts its evolution near an unstable equilibrium and rolls down the potential to a stable minimum. Totally different is the hybrid scenario, which frequently appears in models that incorporate inflation into supersymmetry. The inflaton field evolves toward a minimum with nonzero vacuum energy. The end of inflation arises as a result of instability in a second field. These kinds of models are characterized by $V''(\phi) > 0$ and $0 < \epsilon < \eta$. For more details see [22].
Small-field models can be realised in two different ways:

1. inflation is generated by the rolling down of the scalar field from an asymptotically constant value to a minimum, like the Starobinsky model;
2. the scalar field rolls off from a local maximum to a local minimum of a potential that is typical of spontaneous symmetry breaking and phase transitions, like quartic potentials, natural inflation models [23] and Coleman-Weinberg potentials [18].

To get a more precise description of the whole history of the universe, it would be interesting to find exact solutions that describe the transition from an exponential expansion to a later Friedmann-Lemaître behavior in a smooth way. For this purpose, it may be useful to exploit the observation [24] that cosmological solutions of models of scalar-coupled gravity can be obtained from domain-wall solutions of the same models with opposite scalar potential, simply by analytic continuing to imaginary values the time and radial coordinates. Recently,

this procedure has been applied to a model with a potential given by a sum of exponentials, which admits some exact solutions displaying an accelerated expansion at late times [25].
In the following we study an interesting application of this observation, leading to an exact solution in which the scale factor expands exponentially at early times and then evolves with a power law. This is based on a model of gravity minimally coupled to a scalar field ϕ, introduced in [26] and defined by the action

$$I = \int \sqrt{-g} \left[R - 2 \left(\partial \phi \right)^2 - V \left(\phi \right) \right] d^4 x \,, \tag{4.1.1}$$

where $R = g^{\rho\rho} R_{\rho\rho}$ is the trace of the Ricci tensor $R_{\mu\nu}$, $\sqrt{-g}$ is the determinant of the metric tensor $g_{\mu\nu}$ and the scalar potential

$$V \left(\phi \right) = -\frac{2\lambda^2}{3\gamma} \left(e^{2\sqrt{3}\beta\phi} - \beta^2 e^{2\sqrt{3}\phi/\beta} \right) \tag{4.1.2}$$

depends on two parameters λ and β, with $\gamma = 1 - \beta^2$. The analytical expression of this potential is not casual. In fact, exponential potentials for scalar field appear in several situations: compactification of extra dimensions, $f\left(R\right)$ gravity theories and low-energy effective string theory. Double exponential potential appear in the context of dimensional reduction of gravity with non-trivial four-form flux on a maximally symmetric internal space.
It was shown in [26] that this model admits solitonic solutions of the form

$$ds^2 = \hat{R}^{-2/1+3\beta^2} \left(1 + \mu \hat{R}^{-\frac{3\gamma}{1+3\beta^2}} \right)^{\frac{2\beta^2}{\gamma}} d\hat{R}^2 + \hat{R}^{\frac{2}{1+3\beta^2}} \left(1 + \mu \hat{R}^{-\frac{3\gamma}{1+3\beta^2}} \right)^{\frac{2\beta^2}{3\gamma}} \left(-d\hat{T}^2 + d\bar{s}_2^2 \right) \tag{4.1.3}$$

with μ a free parameter, and $d\bar{s}_2^2$ the line element of flat 2-space. This metric interpolates between anti-de Sitter for $\hat{R} \to 0$ and behavior for $\hat{R} \to \infty$.
Analytically continuing this solution for $\hat{R} \to iT$, $\hat{T} \to iR$, one can obtain a cosmological solution for the action (4.1.1) with the potential

$$V \left(\phi \right) = \frac{2\lambda^2}{3\gamma} \left(e^{2\sqrt{3}\beta\phi} - \beta^2 e^{2\sqrt{3}\phi/\beta} \right) \,, \tag{4.1.4}$$

that reads

$$ds^2 = -T^{-\frac{2}{1+3\beta^2}} \left(1 + \mu T^{-\frac{3\gamma}{1+3\beta^2}} \right)^{\frac{2\beta^2}{\gamma}} dT^2 + T^{\frac{2}{1+3\beta^2}} \left(1 + \mu T^{-\frac{3\gamma}{1+3\beta^2}} \right)^{\frac{2\beta^2}{3\gamma}} d\bar{s}_3^2 \,, \tag{4.1.5}$$

with $d\bar{s}_3^2$ the line element of 3-dimensional flat space. It is easy to verify that this solution behaves as a de Sitter universe for $T \to 0$ and as a Friedmann universe with power-law expansion for $T \to \infty$. It is then a promising candidate to describe the evolution of an inflationary universe. It is also interesting to note that the potential (4.1.4) was included among the exactly solvable models listed in ref. [19], but the range of parameters considered

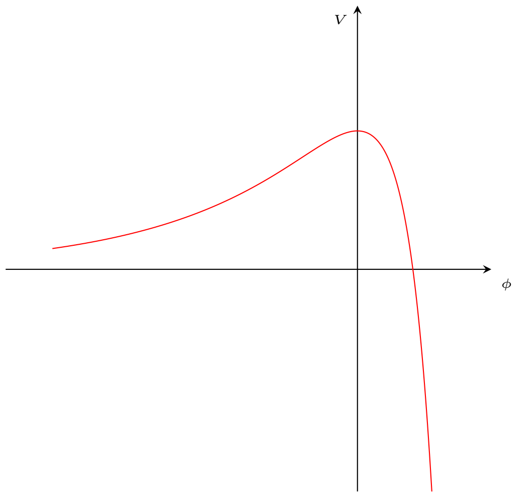


Figure 4.1: Plot of the potential (4.1.4) for $\lambda = 1$, $\beta = 3/7$.

there gave rise to a totally different behavior of the solutions from the one considered here. In the following we shall investigate the main mathematical aspects of the solutions derived from the potential (4.1.4), while for a study of their implications for observational cosmology one may refer to the work [27]. The main result of our investigation is that the solutions admitting exponential inflation, like (4.1.5), correspond to specific boundary conditions where the scalar field is initially at rest on the maximum of the potential, while generic initial conditions give rise to power-law inflation for a range of values of the parameter β. The addition of matter does not modify the picture in a substantial way, since the asymptotic behavior of the solutions is essentially unchanged.
This chapter is organized as follows: in the second section we obtain the general solutions of the system in absence of matter, and in the third section discuss their properties. In section 4 we add matter and discuss the resulting dynamical system.

4.2 General solutions

The solution (4.1.5) is not the most general cosmological solution of the model (4.1.1) with potential (4.1.4), and is therefore important to thoroughly investigate the system of equations derived from (4.1.1) in order to see if the behavior of (4.1.5) is generic and to understand if it can describe a viable cosmological model. As we shall see, the system is exactly integrable if a more suitable parametrization of the metric is used instead of (4.1.5).

The potential (4.1.4), depicted in Fig. 4.1, vanishes for $\phi \to -\infty$, has a maximum for $\phi = 0$, where it takes the value $V_0 = 2\lambda^2/3$, and goes to $-\infty$ for $\phi \to \infty$. Hence a solution with $\phi = 0$ exists, that coincides with that of pure gravity with cosmological constant $\Lambda = 2\lambda^2/3$. This is of course the de Sitter solutions with cosmological constant Λ. It is interesting to notice that the potential (4.1.4) admits a duality for $\beta \to 1/\beta$ and it is also invariant under the transformation $\beta \to -\beta$, $\phi \to -\phi$. These symmetries allow us to limit our discussion to $0 < \beta^2 < 1$. The two limiting cases $\beta = 0, 1$ correspond respectively to a pure exponential and to a potential behaving at leading order as $V = \left(2\lambda^2/3\right)\left(1 - 2\sqrt{3}\phi\right) e^{2\sqrt{3}\phi}$.
We are interested in the general isotropic and homogeneous cosmological solutions, with flat spatial sections, that we parametrize as

$$ds^2 = -e^{2a(\tau)} d\tau^2 + e^{2b(\tau)} d\Omega^2 \,, \quad \phi = \phi(\tau) \,, \tag{4.2.1}$$

with τ a time variable.
With this parametrization, the vacuum Einstein equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -T^{\phi}_{\mu\nu} \tag{4.2.2}$$

read

$$\begin{aligned} &3\dot{b}^2 = \dot{\phi}^2 + \frac{V}{2} e^{2a} \\ &2\ddot{b} + \dot{b}\left(3\dot{b} - 2\dot{a}\right) = -\dot{\phi}^2 + \frac{V}{2} e^{2a} \,, \end{aligned} \tag{4.2.3}$$

where a dot denotes a derivative with respect to τ, while the scalar field obeys the equation

$$\ddot{\phi} + \left(3\dot{b} - \dot{a}\right)\dot{\phi} = -\frac{1}{4}\frac{dV}{d\phi} e^{2a} \,. \tag{4.2.4}$$

In order to have an exactly integrable system of the previous independent equations, we can consider the gauge $b = a/3$: in fact, with this choice, equations (4.2.3) and (4.2.4) can be written in a more simple way as

$$\begin{aligned} &\frac{\dot{a}^2}{3} = \dot{\phi}^2 + \frac{1}{2} V e^{2a} \\ &\ddot{a} = \frac{3}{2} V e^{2a} \\ &\ddot{\phi} = -\frac{1}{4}\frac{dV}{d\phi} e^{2a} \,. \end{aligned} \tag{4.2.5}$$

It is now convenient to define new variables

$$\Psi = a + \sqrt{3}\beta\phi \,, \qquad \chi = a + \frac{\sqrt{3}}{\beta}\phi \,, \tag{4.2.6}$$

so that

$$a = \frac{\psi - \beta^2 \chi}{\gamma}\,, \qquad \phi = \frac{\beta\,(\chi - \psi)}{\sqrt{3}\gamma}\,. \tag{4.2.7}$$

Considering the equations (4.2.6), the third of (4.2.5) and (4.2.7), the field equations take the elementary form

$$\ddot{\Psi} = \lambda^2 e^{2\Psi}\,, \qquad \ddot{\chi} = \lambda^2 e^{2\chi}\,, \tag{4.2.8}$$

subject to the constraint

$$\dot{\Psi}^2 - \beta^2 \dot{\chi}^2 = \lambda^2 \left(e^{2\Psi} - \beta^2 e^{2\chi}\right)\,, \tag{4.2.9}$$

and are invariant under time reversal, $\tau \to -\tau$. Hence to each solution corresponds a time-reversed one.
The equations (4.2.8) admit first integrals

$$\dot{\Psi}^2 = \lambda^2 e^{2\Psi} + Q_1\,, \qquad \dot{\chi}^2 = \lambda^2 e^{2\chi} + Q_2 \tag{4.2.10}$$

with Q_1 and Q_2 integration constants. Putting the expressions (4.2.10) in the constraint (4.5.3), we obtain an algebraic relation between the two constants Q_1 and Q_2:

$$Q_1 = \beta^2 Q_2\,. \tag{4.2.11}$$

In this way, we can formally simplify the solutions of the system (4.2.8). Considering that $\dot{\Psi} = d\Psi/d\tau$ and $\dot{\chi} = d\chi/d\tau$, we can obtain the solutions depending on the sign of the integration constants. In particular, if $Q_i = q_i^2 > 0$ $(i = 1, 2)$ we have

$$\int \frac{d\Psi}{\sqrt{\lambda^2 e^{2\Psi} + q_1^2}} = \tau - \tau_1\,, \qquad \int \frac{d\chi}{\sqrt{\lambda^2 e^{2\chi} + q_2^2}} = \tau - \tau_2$$

and hence

$$\lambda^2 e^{2\Psi} = \frac{q_1^2}{\sinh^2\left[q_1\,(\tau - \tau_1)\right]}\,, \quad \lambda^2 e^{2\chi} = \frac{q_2^2}{\sinh^2\left[q_2\,(\tau - \tau_2)\right]}\,, \tag{4.2.12}$$

with τ_1, τ_2, q_1, q_2 integration constants and $q_1^2 = \beta^2 q_2^2$.
If $Q_i = -q_i^2 < 0$, from

$$\int \frac{d\Psi}{\sqrt{\lambda^2 e^{2\Psi} - q_1^2}} = \tau - \tau_1\,, \qquad \int \frac{d\chi}{\sqrt{\lambda^2 e^{2\chi} - q_2^2}} = \tau - \tau_2$$

it follows that

$$\lambda^2 e^{2\Psi} = \frac{q_1^2}{\sin^2\left[q_1\,(\tau - \tau_1)\right]}\,, \quad \lambda^2 e^{2\chi} = \frac{q_2^2}{\sin^2\left[q_2\,(\tau - \tau_2)\right]}\,, \tag{4.2.13}$$

Finally, if $Q_1 = Q_2 = 0$, with an elementary integration we obtain

$$\lambda^2 e^{2\Psi} = \frac{1}{(\tau - \tau_1)^2}\,, \quad \lambda^2 e^{2\chi} = \frac{1}{(\tau - \tau_2)^2}\,. \tag{4.2.14}$$

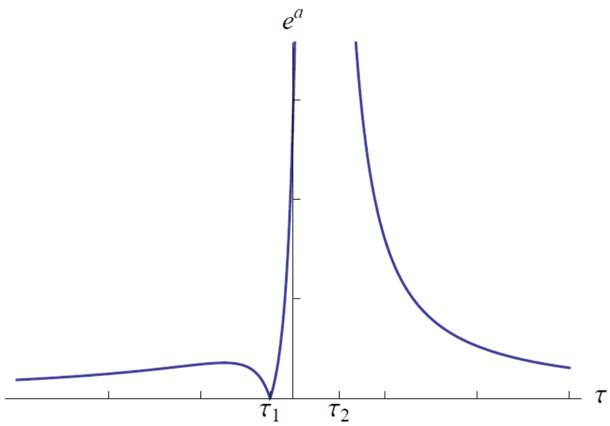


Figure 4.2: The solution (4.2.17) for $e^{a(\tau)}$ when $\tau_1 \neq \tau_2$, displaying the three branches $-\infty < \tau < \tau_1$, $\tau_1 < \tau < \tau_2$ and $\tau_1 < \tau < \infty$. The solution (4.2.15) has the same qualitative behavior.

In the following we define $q = q_2 = q_1/\beta$.
The solutions of the Friedmann equations can be obtained starting from (4.2.7) and using the equations (4.5.6)-(4.2.14)-(4.2.13), and are the following:

- If $Q_i > 0$,

$$e^{2a} = \frac{q^2}{\lambda^2} \frac{|\sinh\left[q\left(\tau-\tau_2\right)\right]|^{2\beta^2/\gamma}}{|\beta^{-1}\sinh\left[\beta q\left(\tau-\tau_1\right)\right]|^{2/\gamma}}\,, \quad e^{2\sqrt{3}\phi/\beta} = \left|\frac{\sinh\left[\beta q\left(\tau-\tau_1\right)\right]}{\beta\sinh\left[q\left(\tau-\tau_2\right)\right]}\right|^{2/\gamma}\;; \qquad (4.2.15)$$

- If $Q_i < 0$,

$$e^{2a} = \frac{q^2}{\lambda^2} \frac{|\sin\left[q\left(\tau-\tau_2\right)\right]|^{2\beta^2/\gamma}}{|\beta^{-1}\sin\left[\beta q\left(\tau-\tau_1\right)\right]|^{2/\gamma}}\,, \quad e^{2\sqrt{3}\phi/\beta} = \left|\frac{\sin\left[\beta q\left(\tau-\tau_1\right)\right]}{\beta\sin\left[q\left(\tau-\tau_2\right)\right]}\right|^{2/\gamma}\;; \qquad (4.2.16)$$

- If $Q_i = 0$

$$e^{2a} = \frac{1}{\lambda^2} \frac{\left|\tau-\tau_2\right|^{2\beta^2/\gamma}}{\left|\tau-\tau_1\right|^{2/\gamma}}\,, \quad e^{2\sqrt{3}\phi/\beta} = \left|\frac{\tau-\tau_1}{\tau-\tau_2}\right|^{2/\gamma}\,. \qquad (4.2.17)$$

The solution (4.2.17) is depicted in Fig. 4.2. Its behavior is qualitatively identical to (4.2.15). Notice that (4.2.17) is nothing but the solution (4.1.5) in different coordinates. In fact, the two solutions are related by the change of variable $\tau = \mathrm{const} \times T^{-3\gamma/\left(1+3\beta^2\right)}$. We also remark that the actual value of the parameter q is not important, since it only sets the scale of the time parameter τ, while the physically relevant time parameter is given by the cosmic time, that we introduce in the next section.

4.3 Properties of the solutions

To give a physical interpretation of the solutions, it is useful to define the cosmic time t such that $dt = \pm e^a d\tau$. The possibility of choosing the plus or minus sign derives from the invariance of the field equations under time reversal. To each expanding solution therefore corresponds an unphysical contracting solution, that we shall disregard. In the new parametrization, the line element reads

$$ds^2 = -dt^2 + e^{2b(t)} d\Omega^2 \,. \tag{4.3.1}$$

Unfortunately, the solutions obtained in the previous section cannot be written in terms of elementary functions of t, except when $Q_i = 0$, $\tau_1 = \tau_2$. In this case, from (4.2.17) we have

$$t = \pm \frac{1}{\lambda} \int \frac{d\tau}{|\tau - \tau_1|}$$

and hence

$$\lambda \, (t - t_0) = \pm \log |\tau - \tau_1| \,, \tag{4.3.2}$$

with t_0 an arbitrary integration constant. Choosing the minus sign in the previous expression, it is clear that $\lambda^2 e^{2a} = e^{\lambda(t-t_0)}$ and, taking into account the gauge $b = a/3$, one obtains an expanding universe, with $e^{2b} = e^{2\lambda(t-t_0)/3}$ and $\phi = 0$, namely a de Sitter spacetime with vanishing scalar field. This is of course the unstable solution corresponding to the scalar sitting at the top of the potential.

In the general case, the solutions have a single acceptable branch if $\tau_1 = \tau_2$, or three qualitatively different if $\tau_1 \neq \tau_2$. We are only interested in those branches where t is a monotonic function of τ and the universe expands. Studying their behavior for $\tau \to \tau_{1,2}$ and $\tau \to \pm\infty$, we obtain the following physically acceptable solutions, besides the one discussed above:

- If $Q_i > 0$, $\tau_1 = \tau_2$, the first of (4.2.15) becomes

  $$e^{2a} = \frac{q^2}{\lambda^2} \frac{|\sinh[q(\tau - \tau_1)]|^{2\beta^2/\gamma}}{|\sinh[\beta q(\tau - \tau_1)]|^{2/\gamma}} \,,$$

  so that, when τ tends to τ_1 (or τ_2), it simplifies as

  $$e^a \sim \frac{1}{\lambda(\tau - \tau_1)} \,,$$

while, when τ tends to $\pm\infty$, we have

$$e^a \sim \frac{1}{\lambda} e^{\pm\beta q(\tau-\tau_1)/(1+\beta)} .$$

The cosmic time assumes the form $t \sim \pm\frac{1}{\lambda}\ln|\tau-\tau_1| + t_0$, which tends, in the limit $\tau \to \tau_1$, to $\pm\infty$, whereas in the limit $\tau \to \pm\infty$ it tends to t_0. From this results we obtain the asymptotic behavior of the metric functions e^{2b} and $e^{2\sqrt{3}\phi/\beta}$, namely

$$\begin{aligned} &e^{2b} \sim t^{2/3}\,,\ e^{2\sqrt{3}\phi/\beta} \sim t^{2/\beta} \qquad \text{for} \quad t \to t_0 \\ &e^{2b} \sim e^{2\lambda t/3}\,,\ e^{2\sqrt{3}\phi/\beta} \sim \text{const} \qquad \text{for} \quad t \to \infty \end{aligned}$$

This solution describes a universe starting at $t = t_0$ with a power-law behavior and presenting an exponential expansion for late times.

- If $Q_i < 0\,, \tau_1 = \tau_2$, the study of the behavior is similar to the previous one, but in this case we have two branches. The first:

$$\begin{aligned} &e^{2b} \sim e^{2\lambda t/3}\,,\ e^{2\sqrt{3}\phi/\beta} \sim \text{const} \qquad \text{for} \quad t \to -\infty \\ &e^{2b} \sim (t-t_0)^{2\beta^2/3} \to 0\,,\ e^{2\sqrt{3}\phi/\beta} \sim (t-t_0)^{-2} \qquad \text{for} \quad t \to t_0 \end{aligned}$$

The other branch behaves as

$$e^{2b} \sim (t-t_0)^{2\beta^2/3}\ ,\ e^{2\sqrt{3}\phi/\beta} \sim (t-t_0)^{-2}$$

both at $t = t_0$ and at a later finite time, describing an universe that initially expands and then recollapses.

If $\tau_1 \neq \tau_2$, the solutions are more complicated and in general present three branches:

- For $Q_i = 0$ one branch has behavior

$$e^{2b} \sim e^{2\lambda t/3}\,,\ e^{2\sqrt{3}\phi/\beta} \sim \text{const} \qquad \text{for} \quad t \to -\infty\,,$$

while

$$e^{2b} \sim t^{2/3\beta^2}\ ,\ e^{2\sqrt{3}\phi/\beta} \sim t^{-2/\beta^2} \qquad \text{for} \quad t \to \infty\,.$$

Another branch behaves as

$$e^{2b} \sim (t-t_0)^{2\beta^2/3}\ ,\ e^{2\sqrt{3}\phi/\beta} \sim (t-t_0)^{-2} \qquad \text{for} \quad t \to t_0$$

with

$$e^{2b} \sim t^{2/3\beta^2}\ ,\ e^{2\sqrt{3}\phi/\beta} \sim t^{-2/\beta^2} \qquad \text{for} \quad t \to \infty.$$

In the first case the universe begins with an exponential expansion for $t \to -\infty$ and gradually turns to a power-law behavior for $t \to \infty$. In the second case, for $\beta^2 > 2/3$, we have an initial phase of power-law inflation. Also a solution that recontracts in an infinite time is present, but we shall not consider it.

- For $Q_i > 0$, in the first branch

$$e^{2b} \sim (t - t_0)^{2\beta^2/3} \, , \, e^{2\sqrt{3}\phi/\beta} \sim (t - t_0)^{-2} \qquad \text{for} \quad t \to t_0 \, ,$$

while

$$e^{2b} \sim t^{2/3} \, , \, e^{2\sqrt{3}\phi/\beta} \sim t^{2/\beta} \qquad t \to \infty \, .$$

In the second branch:

$$e^{2b} \sim (t - t_0)^{2\beta^2/3} \, , \, e^{2\sqrt{3}\phi/\beta} \sim (t - t_0) \qquad \text{for} \quad t \to t_0 \, ,$$

while

$$e^{2b} \sim t^{2/3\beta^2} \, , \, e^{2\sqrt{3}\phi/\beta} \sim t^{-2/\beta^2} \qquad \text{for} \quad t \to \infty \, .$$

A third branch recontracts in a finite time, but we do not discuss it.

- For $Q_i < 0$, the first branch has behavior

$$e^{2b} \sim (t - t_0)^{2\beta^2/3} \, , \, e^{2\sqrt{3}\phi/\beta} \sim (t - t_o)^{-2} \qquad t \to t_0 \, ,$$

while

$$e^{2b} \sim t^{2/3\beta^2} \, , \, e^{2\sqrt{3}\phi/\beta} \sim t^{-2/\beta^2} \qquad t \to \infty \, .$$

Another branch behaves as

$$e^{2b} = (t - t_0)^{2\beta^2/3} \, , \, e^{2\sqrt{3}\phi/\beta} \sim (t - t_0)^{-2}$$

both at $t = t_0$ and at a later finite time.

The metric function b and the scalar field ϕ of the expanding solutions corresponding to $Q_i > 0$ are depicted in Figs. 4.3 and 4.4 as functions of the cosmic time t.

The most interesting solutions from a physical point of view are those that behave exponentially for $t \to -\infty$ and as a power law for $t \to \infty$. These are obtained for $Q_i = 0$, $\tau_1 \neq \tau_2$. They correspond to an initial configuration where the scalar field is in the unstable equilibrium configuration at the top of the potential and then rolls down to $\phi = \infty$. The late behavior of these solutions is given by a $2/3\beta^2$ power law.
The exponential expansion lasts until the acceleration of the expansion (i.e. the second derivative of the scale factor) is positive: it happens when $\tau = \tau_f \sim 1/q\beta$, namely $tt_0 = t_f t_0 \sim 1/\lambda$. At such time the scale factor e^{2b} is of order $\lambda^{2/3}$. Denoting t_i the time at which the inflation starts, the scale factor therefore inflates by a factor $e^{-2\lambda(t_i - t_o)/3}$. Choosing $t_i - t_0$ negative, one can then obtain the desired amount of inflation.
It must be noticed that for $\beta^2 > 2/3$, also solutions that present power-law inflation exist. Moreover, for $Q > 0$ and $\tau_1 = \tau_2$, solution starting with power-law expansion and then turning to an exponential behavior can also be found.

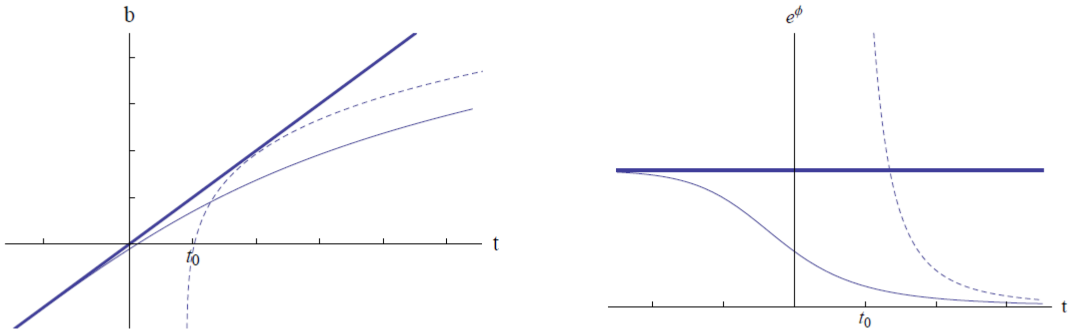


Figure 4.3: The solutions with $Q_i = 0$ in terms of the cosmic time t. In the left panel is plotted the function b, i.e. the logarithm of the scale factor, while in the right panel is plotted the function e^{ϕ}. The thick line corresponds to the de Sitter solution with $\tau_1 = \tau_2$, while the continuous and the dashed curve are the two expanding branches of the $\tau_1 \neq \tau_2$ solutions.

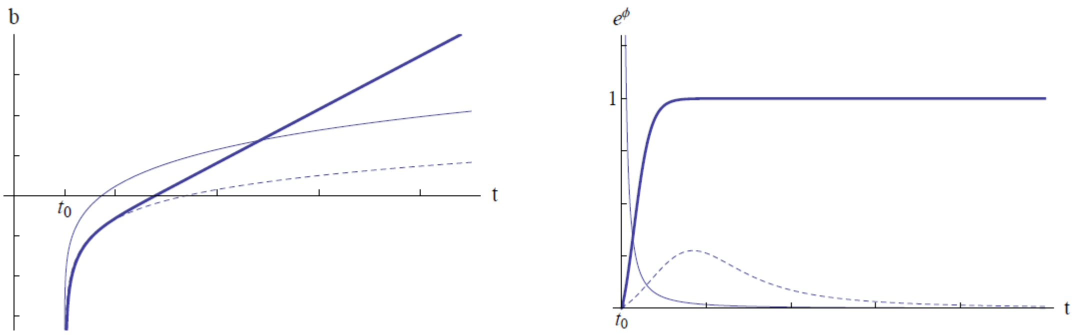


Figure 4.4: The solutions with $Q_i \neq 0$ in terms of the cosmic time t. In the left panel is plotted the function b, i.e. the logarithm of the scale factor, while in the right panel is plotted the function e^{ϕ}. The thick line corresponds to the solution with $\tau_1 = \tau_2$, while the continuous and the dashed curve are the two expanding branches of the $\tau_1 \neq \tau_2$ solutions.

4.4 Solutions with matter

We consider now the effects of ordinary matter on the exact solutions obtained above. In this case one cannot obtain solutions in analytic form, but can nevertheless discuss their global properties by means of methods of the theory of dynamical systems dealt with in the previous chapter. This will allow us to discuss the stability of the solutions.
Introducing matter in the form of a perfect fluid, the Einstein equations become

$$\frac{\dot{a}^2}{3} - \dot{\phi}^2 = \frac{1}{2} V e^{2a} + \frac{1}{2} \rho e^{2a} \tag{4.4.1}$$

$$\ddot{a} = \frac{3}{2} V e^{2a} + \frac{3}{4} (\rho - p) e^{2a}, \tag{4.4.2}$$

while the equation of the scalar field is unchanged. Deriving with respect to time the equation (4.4.1) and combining the new equation and (4.4.2) one obtains as usual the continuity equation

$$\dot{\rho} + (3p + \rho) \dot{a} = 0, \tag{4.4.3}$$

which, integrated for the equation of state $p = \omega\rho$, with the state parameter $\omega \geq -1$, gives

$$\rho = \rho_0 e^{-(1+\omega)a}. \tag{4.4.4}$$

If $\omega = 0$ (dust), in terms of the variables (4.2.6), the field equations become

$$\begin{aligned} \ddot{\Psi} &= \lambda^2 e^{2\Psi} + \frac{3}{4} \rho_0 e^{\left(\Psi - \beta^2 \chi\right)/\gamma}, \\ \ddot{\chi} &= \lambda^2 e^{2\chi} + \frac{3}{4} \rho_0 e^{\left(\Psi - \beta^2 \chi\right)/\gamma} \end{aligned} \tag{4.4.5}$$

subject to the constraint

$$\dot{\Psi}^2 - \beta^2 \dot{\chi}^2 = \lambda^2 \left(e^{2\Psi} - \beta^2 e^{2\chi}\right) + \frac{3}{2} \rho_0 \gamma e^{\left(\Psi - \beta^2 \chi\right)/\gamma}. \tag{4.4.6}$$

In the general case $\omega \neq 0$, one has

$$\begin{aligned} \ddot{\Psi} &= \lambda^2 e^{2\Psi} + \frac{3}{4} \rho_0 (1 - \omega) e^{(1-\omega)\left(\Psi - \beta^2 \chi\right)/\gamma}, \\ \ddot{\chi} &= \lambda^2 e^{2\chi} + \frac{3}{4} \rho_0 (1 - \omega) e^{(1-\omega)\left(\Psi - \beta^2 \chi\right)/\gamma} \end{aligned} \tag{4.4.7}$$

subject to the constraint

$$\dot{\Psi}^2 - \beta^2 \dot{\chi}^2 = \lambda^2 \left(e^{2\Psi} - \beta^2 e^{2\chi}\right) + \frac{3}{2} \rho_0 \gamma e^{(1-\omega)\left(\Psi - \beta^2 \chi\right)/\gamma}. \tag{4.4.8}$$

In general, these systems cannot be solved exactly. Therefore, to investigate their properties, we put them in the form of a dynamical system, defining

$$X = \frac{1}{2\gamma\,(1-\omega)}\left(\dot{\psi} - \beta^2\dot{\chi}\right)\,,\ Y = \dot{\chi}\,,\ Z = \lambda e^{\chi}\,,\ W = \sqrt{\lambda}e^{\left(\psi-\beta^2\chi\right)/2\gamma}\,. \tag{4.4.9}$$

With this choice of variables, we obtain a first order differential system, in particular the independent equations (4.4.7) become:

- If $\omega = 0$:

$$\begin{aligned}\dot{X} &= \frac{\alpha}{2}W^2 - \frac{\beta^2}{2\gamma}Z^2 + \frac{1}{2\gamma}W^{4\gamma}Z^{2\beta^2}\\ \dot{Y} &= Z^2 + \alpha W^2\\ \dot{Z} &= YZ\end{aligned} \tag{4.4.10}$$

  with $\alpha = 3\rho_0/\left(4\lambda\right)$ and W implicitly defined as

$$2\alpha W^2 + \frac{1}{\gamma}Z^{2\beta^2}W^{4\gamma} = \frac{\beta^2}{\gamma}Z^2 + 4\gamma X^2 + 4\beta^2 XY - \beta^2 Y^2\,. \tag{4.4.11}$$

- If $\omega \neq 0$:

$$\begin{aligned}\dot{X} &= \frac{\alpha}{2}W^2 - \frac{\beta^2}{2\gamma}Z^2 + \frac{1}{2\gamma}W^{4\gamma(1-\omega)}Z^{2\beta^2}\\ \dot{Y} &= Z^2 + \alpha W^2\\ \dot{Z} &= YZ\end{aligned} \tag{4.4.12}$$

  and W implicitly defined as

$$\frac{2\alpha}{1-\omega}W^2 + \frac{1}{\gamma}Z^{2\beta^2}W^{4\gamma(1-\omega)} = \frac{\beta^2}{\gamma}Z^2 + 4\gamma X^2 + 4\beta^2 XY - \beta^2 Y^2\,. \tag{4.4.13}$$

It can also be useful to note that $\dot{W} = (1-\omega)\,XW$.
In the following we do not study the system (4.4.10) because it is a particular case of the system with non-zero ω. At the end of this section we will give some solutions for this case. The global properties of the solutions of the system (4.4.12), or, equivalently, the structure

of the phase space, can be deduced from the study of its behavior near the critical points. They are the points where the trajectories of the solutions start or end, and their position is determined by the condition that the derivatives of all the phase space variables vanish there [29]. Due to the complicated relation between the cosmic time t and the variable τ, the limit $t \to \pm\infty$ can correspond either to $\tau \to \infty$ or to $\tau \to \tau_0$, for some τ_0 where the funcions Z and W diverge, and hence to critical points at finite distance or at infinity in phase space.

In particular, the critical points at finite distance correspond to the limit $\tau \pm \infty$: in other words, they are placed at the intersection of the quadric defined by (4.4.13) and lie on two straight lines on the plane $Z = 0$,

$$X_0 = \frac{\pm\beta Y_0}{2\,(1 \pm \beta)}\,, \quad Z_0 = 0\,. \tag{4.4.14}$$

The system (4.4.12) can be linearized around its critical points when

$$\frac{1}{2} < \beta^2 < \frac{3 - 4\omega}{4\,(1-\omega)}\,,$$

otherwise in the jacobian matrix Z and W appear with negative exponents and it would not be defined in the $Z = 0$ plane. However, the eigenvalues of the linearized system are $0, 0, Y_0$. Since X_0 and Y_0 are the values of X and Y (respectively) at one of these critical points, integrating the equation $\dot{W} \sim XW$ we obtain $W \sim X_0\tau$. Taking into account that $e^a \sim W^2$, we have $a \sim 2X_0\tau$. Similarly, with an elementary integration of $\dot{Z} = YZ$ one obtain $Z \sim e^{Y_0\tau}$. Since $e^{\sqrt{3}\phi/\beta} \sim Z/W^2 \sim e^{Y_0\tau}/e^{2X_0\tau}$, we obtain $\frac{\sqrt{3}\phi}{\beta} \sim \tau\,(Y_0 - X_0)$. Let us remind that the cosmic time t is defined as $dt = \pm e^a d\tau$. In this way, one can deduce that the asymptotic behavior of the functions a and ϕ near the critical points with $Y_0 \neq 0$ is

$$a \sim \frac{\pm\beta Y_0}{1 \pm \beta}\tau \sim \log t\,, \quad \frac{\sqrt{3}\phi}{\beta} \sim \frac{Y_0}{1 \pm \beta}\tau\,, \tag{4.4.15}$$

with $\tau \to \mp\infty$. It follows that the cosmic time t vanishes in this limit and that at all these points

$$e^{2b} \sim t^{2/3}\,, \qquad e^{2\sqrt{3}\phi/\beta} \sim t^{2/\beta}\,, \qquad \text{with} \quad t \to 0\,. \tag{4.4.16}$$

If the critical point is the origin, instead,

$$e^a \sim \tau^{-1} \sim e^{\mu t}\,, \quad e^{2b} \sim e^{2\mu t/3} \to 0\,, \quad e^{2\sqrt{3}\phi/\beta} \sim \text{const}\,, \qquad \text{with} \quad t \to \pm\infty \tag{4.4.17}$$

depending on the sign of the integration constant μ.

The remaining critical points lie on the surface at infinity of the phase space. They can be studied by defining new variables

$$u = \frac{1}{X}\,, \quad y = \frac{Y}{X}\,, \quad z = \frac{Z}{X}\,, \quad w = \frac{W}{X}\,, \tag{4.4.18}$$

and considering the limit $u \to 0$. This is attained for $\tau \to \tau_0$, where τ_0 is a finite constant. From this definition we get

$$\dot{X} = -\frac{u'}{u^3}\,, \qquad \dot{Y} = \frac{y'}{u^2} - y\frac{u'}{u^3}\,, \qquad \dot{Z} = \frac{z'}{u^2} - z\frac{u'}{u^3}\,, \tag{4.4.19}$$

so that the field equations become

$$\begin{aligned} u' &= -\left(\frac{\alpha}{2}w^2 - \frac{\beta^2}{2\gamma}z^2 + \frac{1}{2\gamma}v^2\right)u \\ y' &= -\left(\frac{\alpha}{2}w^2 - \frac{\beta^2}{2\gamma}z^2 + \frac{1}{2\gamma}v^2\right)y + z^2 + \alpha w^2 \\ z' &= -\left(\frac{\alpha}{2}w^2 - \frac{\beta^2}{2\gamma}z^2 + \frac{1}{2\gamma}v^2\right)z + yz \end{aligned} \tag{4.4.20}$$

where a prime denotes $ud/d\tau$ and

$$v := z^{\beta^2} w^{2\gamma/(1-\omega)} u^{-\gamma(1+\omega)/(1-\omega)}\,. \tag{4.4.21}$$

The constraint (4.5.3) takes the form

$$\frac{1}{\gamma}v^2 = \frac{\beta^2}{\gamma}z^2 + 4\gamma + 4\beta^2 y - \beta^2 y^2 - \frac{2\alpha}{1-\omega}w^2\,. \tag{4.4.22}$$

The terms proportional to v^2 in the previous equations must be considered carefully, because the divergence of $u^{-2\gamma(1+\omega)/(1-\omega)}$ can be compensated by zeros of z and w. In particular, to have a regular system, these terms must either vanish or go to constant for $u \to \infty$.
One finds the following critical points:

1. $z = 0$, $y = y_0 = 2\left(1 \pm \beta^{-1}\right)$, with $w = v = 0$ and eigenvalues are $(0, 0, y_0)$. These are the endpoints of the lines containing the critical points at finite distance and are not endpoints of trajectories lying at finite distance.
2. $z = 0$, $y = 2$, with $w^2 = \dfrac{2(1-\omega)}{\alpha}$, $v = 0$ and eigenvalues $(\omega - 1, \omega - 1, 1 - \omega)$. These are the endpoints of trajectories lying at finite distance.
3. $z^2 = \dfrac{4\gamma^2}{\beta^4}$, $y = -\dfrac{2\gamma}{\beta^2}$, with $w = v = 0$ and eigenvalues $\left(\dfrac{2\gamma}{\beta^2}, \dfrac{2\gamma}{\beta^2}, \dfrac{4\gamma}{\beta^2}\right)$.
4. $z^2 = 4$, $y = 2$, with $w = 0$, $v^2 \to 4$ and eigenvalues $(-2, 2, -4)$.
5. $z = y = 0$, with $w = 0$, $v^2 \to 4\gamma^2$ and eigenvalues $(-2\gamma, -2\gamma, -2\gamma)$.

Proceeding as for the critical points at finite distance, one can deduce the asymptotic behavior of the metric functions near the critical points at infinity, i.e. for $\tau \to \tau_0$. In particular, we can note that

$$X \sim \frac{1}{u_0\,(\tau-\tau_0)}\,, \quad W \sim |\tau-\tau_0|^{1/u_0}\,, \quad Z \sim |\tau-\tau_0|^{y_0/u_0}\,, \tag{4.4.23}$$

so that, defining $\eta = \frac{1}{2\gamma(1-\omega)}\left(\psi - \beta^2\chi\right)$,

2) $e^\chi \sim Z \sim |\tau-\tau_0|^{-2/(1-\omega)}$, $e^\eta \sim e^{a/2} \sim W \sim |\tau-\tau_0|^{-1/(1-\omega)}$, and hence $t \sim |\tau-\tau_0|^{-(1+\omega)/(1-\omega)}$, $e^{2b} \sim t^{2/[3(1+\omega)]}$, $e^{2\sqrt{3}\phi/\beta} \sim t^{-2\omega/(1+\omega)}$.

3) $e^\chi \sim |\tau-\tau_0|^{-1}$, $e^\eta \sim |\tau-\tau_0|^{\beta^2/2\gamma}$, and hence $t \sim |\tau-\tau_0|^{1/\gamma}$, $e^{2b} \sim t^{2\beta^2/3}$, $e^{2\sqrt{3}\phi/\beta} = t^{-2}$.

4) $e^\chi \sim |\tau-\tau_0|^{-1}$, $e^\eta \sim |\tau-\tau_0|^{-1/2}$, and hence $t \sim \pm\log|\tau-\tau_0|$, $e^{2b} \sim e^{\pm 2\mu t/3}$, $e^{2\sqrt{3}\phi/\beta} =$ const.

5) $e^\chi \sim$ const, $e^\eta \sim |\tau-\tau_0|^{-1/2\gamma}$, and hence $t \sim |\tau-\tau_0|^{-\beta^2/\gamma}$, $e^{2b} \sim t^{2/3\beta^2}$, $e^{2\sqrt{3}\phi/\beta} = t^{-2/\beta^2}$.

All points correspond to the limit $t \to \infty$, except point 3) that corresponds to $t \to 0$. Point 2), which is not present in absence of matter, displays the late-time behavior of ordinary cosmological models, with no scalar coupling ($\lambda = 0$), while points 3)-5) have the behavior of the exact solutions found in the previous sections. In particular, point 3) corresponds to finite t, and hence can be the origin of trajectories describing cosmological models, while points 4) and 5) correspond to possible asymptotic behaviors for $t \to \infty$.

It follows that the addition of matter does not change much the behaviour of the solutions. With respect to the phase space in absence of matter, a new critical point 2) arises, that corresponds to the classical Friedmann-Lemaître solutions in absence of scalar fields. This point is however unstable, and attracts only trajectories with $\lambda = 0$. Also point 4), corresponding to exponential behavior of the metric function is unstable. Stable critical points at infinity are 5), which attracts most of the trajectories that do not end at finite distance, and 3), that corresponds to $t \to$ const and is the origin of most trajectories.

For $\omega = 0$ the solutions 2) behave asymptotically as $e^{2b} \sim t^{2/3}$, $e^{2\sqrt{3}\phi/\beta} \sim$ const for $t \to \infty$, while for $\omega = 1/3$ (radiation) they behave as $e^{2b} \sim t^{1/3}$, $e^{2\sqrt{3}\phi/\beta} \sim t^{-1/2}$.

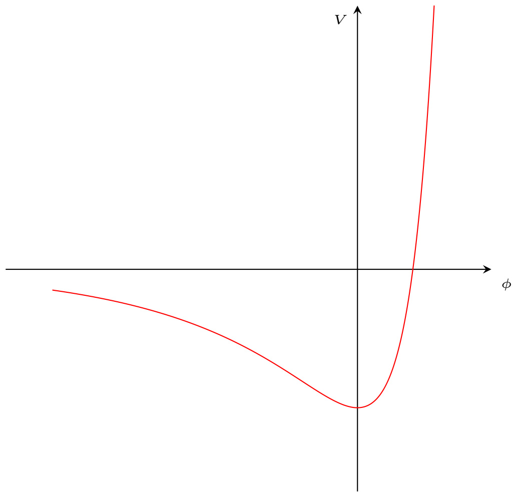


Figure 4.5: Plot of the potential (4.5.1) for $\lambda = 1$, $\beta = 3/7$.

4.5 The potential $V(\phi) \leftrightarrow -V(\phi)$

Let us now consider the case when the inflaton field rolls on the potential (4.1.2), namely

$$V(\phi) = -\frac{2\lambda^2}{3\gamma}\left(e^{2\sqrt{3}\beta\phi} - \beta^2 e^{2\sqrt{3}\phi/\beta}\right) . \qquad (4.5.1)$$

Since in this case the potential $V(\phi)$ has opposite sign respect to (4.1.4), it has a stable minimum for $\phi = 0$ and goes to infinity for $\phi \to \infty$ (see Fig. (4.5)). However, the symmetries described for the potential (4.1.4) hold also in this case and the field equations are still valid if one substitutes λ^2 with $-\lambda^2$. Only the solutions have completely different properties. In fact, the Einstein's equations read

$$\ddot{\Psi} = -\lambda^2 e^{2\Psi} , \qquad \ddot{\chi} = -\lambda^2 e^{2\chi} , \qquad (4.5.2)$$

subject to the constraint

$$\dot{\Psi}^2 - \beta^2\dot{\chi}^2 = -\lambda^2\left(e^{2\Psi} - \beta^2 e^{2\chi}\right) , \qquad (4.5.3)$$

and their first integrals read

$$\dot{\Psi}^2 = -\lambda^2 e^{2\Psi} + Q_1 , \qquad \dot{\chi}^2 = -\lambda^2 e^{2\chi} + Q_2 \qquad (4.5.4)$$

with Q_1 and Q_2 integration constants. These constants must be positive, so that we can write (4.5.4) as

$$\dot{\Psi}^2 = -\lambda^2 e^{2\Psi} + q_1^2 \, , \qquad \dot{\chi}^2 = -\lambda^2 e^{2\chi} + q_2^2 \tag{4.5.5}$$

and the integration constants satisfy $q_1^2 = \beta^2 q_2^2$.
Defining $q = q_1 = q_2/\beta$ we obtain

$$\lambda^2 e^{2\Psi} = \frac{q^2}{\cosh^2\left[q\left(\tau - \tau_1\right)\right]} \, , \quad \lambda^2 e^{2\chi} = \frac{\beta^2 q^2}{\cosh^2\left[\beta^2 q_2\left(\tau - \tau_2\right)\right]} \, , \tag{4.5.6}$$

while the solutions of the Friedmann equations are

$$e^{2a} = \frac{q^2}{\lambda^2}\left(\frac{\beta \cosh^{\beta^2}\left[q\left(\tau - \tau_2\right)\right]}{\cosh\left[\beta q\left(\tau - \tau_1\right)\right]^{2/\gamma}}\right)^{2/\gamma} \, , \quad e^{2\sqrt{3}\phi/\beta} = \left(\frac{\cosh\left[\beta q\left(\tau - \tau_1\right)\right]}{\beta \cosh\left[q\left(\tau - \tau_2\right)\right]}\right)^{2/\gamma} \, . \tag{4.5.7}$$

Contrary to the solutions found in the previous sections, these solutions are regular everywhere, since the hyperbolic cosine has no zeros.
In general, we have a first branch with $e^{2b} \sim$ const, $e^{2\sqrt{3}\beta\phi} \sim$ const and a second branch that behaves as $e^{2b} \sim (t - t_0)^{2/3}$, $e^{2\sqrt{3}\phi/\beta} \sim t^{2\beta}$. In other words, they represent universes startng with a bg bang at $t = t_0$ and recontracting after a finite time.

4.6 A more general form of the inflation potential

In [27], the authors construct the most general model in which inflation is generated by a scalar field slowly rolling off from a de Sitter maximum of the potential $V(\phi)$ in the action

$$I = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{16\pi}R - \frac{1}{2}\left(\partial\phi\right)^2 - V\left(\phi\right)\right) \, . \tag{4.6.1}$$

This class of models is very natural from a physical point of view because inflation can be thought as an instability of the de Sitter spacetime generated by a scalar perturbation.
The general form of the inflation potential considered in the paper is

$$V\left(\phi\right) = \Lambda^2\left(a_1 e^{b_1 h\phi} + a_2 e^{b_2 h\phi}\right) \, , \tag{4.6.2}$$

where $\Lambda^{-1/2}$ and μ give, respectively, the height and the curvature of the maximum of the potential function. These two length scales determine the two physical scales relevant for inflation: the vacuum energy E_V at the beginning of inflation and the inflaton mass squared M_I^2:

$$M_I^2 = V''\left(0\right) = -2\Lambda^2 h^2, \qquad E_V = \left[V\left(0\right)\right]^{1/4} \, . \tag{4.6.3}$$

Moreover, $a_{1,2}$, $b_{1,2}$ are some dimensionless constants characterising the model and they are constrained by

$$V\left(0\right) > 0, \qquad V'\left(0\right) = 0, \qquad V''\left(0\right) < 0 \tag{4.6.4}$$

giving

$$a_1 + a_2 > 0, \qquad a_1 b_1 = -a_2 b_2, \qquad a_1 b_1^2 + a_2 b_2^2 < 0\,, \tag{4.6.5}$$

from which

$$a_1 > 0, \qquad a_2 < 0, \qquad b_2 > 0, \qquad b_1 < 0, \qquad \frac{a_1}{a_2} = -\frac{b_2}{b_1} = -\beta^2\,. \tag{4.6.6}$$

β is a purely dimensionless parameter and quantifies the deviation of the potential from a pure exponential behavior attained for β near to 0.
The parameter rescaling $\Lambda^2 \to 2\Lambda^2/\left(3a_2\gamma\right)$, $h \to \sqrt{3/\left(b_1 b_2\right)}h$ brings the potential in the form

$$V\left(\phi\right) = \frac{2\Lambda^2}{3\gamma}\left(e^{\sqrt{3}\beta h\phi} - \beta^2 e^{\sqrt{3}h\phi/\beta}\right)\,. \tag{4.6.7}$$

When $h = 4\sqrt{\pi} l_P$ we obtain the model we studied in the previous sections, while for other values of h we obtain the cosmological equations discussed in this section.
The most important purpose of the paper [27] is to investigate the cosmology of the general model in the slow-roll approximation. In this regime the potential energy of the scalar field dominates over the kinetic energy and the universe has a quasi-exponential accelerated expansion as the scalar field slowly rolls off from the maximum of the potential. The authors introduce the following slow-roll parameters:

$$\epsilon = \frac{M_P^2}{16\pi}\left(\frac{V'}{V}\right)^2, \qquad \eta = \frac{M_P^2}{8\pi}\frac{V''}{V} - \epsilon\,, \tag{4.6.8}$$

where $0 \leq \epsilon < 1$. For $\epsilon = 0$ the solution is exactly de Sitter, whereas inflation ends when $\epsilon = 1$.
The branch under study is $0 \leq \phi < \infty$ because the other one $(-\infty < \phi \leq 0)$ cannot be made compatible with observations. In particular, the most interesting cosmological solutions one can obtain for the exactly solvable model with $h = 1$ are defined in this uncompatible branch.
Introducing the variable $Y = e^{\sqrt{3}\gamma h\phi/\beta}$ so that $Y \geq 1$, the slow-roll parameters, as a function of Y, take the form

$$\epsilon = \frac{\beta^2}{h^2}\left(\frac{1-Y}{1-\beta^2 Y}\right)^2, \qquad \eta = \frac{2}{h^2}\frac{\beta^2 - Y}{1-\beta^2 Y} - \epsilon\,. \tag{4.6.9}$$

The slow-roll parameter ϵ is zero on the maximum of the potential at $\phi = 0$ $(Y = 1)$, whereas $0 \leq \epsilon \leq 1$ for $1 \leq Y \leq Y_0$, where

$$Y_0 = \frac{\beta + h}{\beta\left(1 + \beta h\right)}\,. \tag{4.6.10}$$

To satisfy the condition $\epsilon \ll 1$, during inflation it is convenient to check that $Y_0 < 1/\beta^2$, whereas the simplest way to satisfy the condition $|\eta| \ll 1$ is to choose $h \gtrsim 10$. Unfortunately, the model discussed in the previous section does not satisfy this condition because it is characterized by $h = 1/\sqrt{3}$ and the exact solutions found above are not compatible with the observations [27]. For this reason, it might be useful to generalize the potential (4.1.4) with a positive parameter h, so that

$$V(\phi) = \frac{2\lambda^2}{3\gamma}\left(e^{2\sqrt{3}\beta h\phi} - \beta^2 e^{2\sqrt{3}h\phi/\beta}\right) . \tag{4.6.11}$$

The action (4.1.1) and the parametrization (4.2.1) are unchanged.
Defining two new variables

$$\Psi = \sqrt{3}\beta h\phi + 3b\,, \qquad \chi = \frac{\sqrt{3}h\phi}{\beta} + 3b\,, \tag{4.6.12}$$

the vacuum Einstein equations read

$$\ddot{\Psi} = \frac{\lambda^2}{\gamma}\left[\left(1-h^2\beta^2\right)e^{2\Psi} - \beta^2\left(1-h^2\right)e^{2\chi}\right] , \quad \ddot{\chi} = \frac{\lambda^2}{\gamma}\left[\left(1-h^2\right)e^{2\Psi} - \left(\beta^2-h^2\right)e^{2\chi}\right] \tag{4.6.13}$$

subject to the constraint

$$\left(1-h^2\beta^2\right)\beta^2\dot{\chi}^2 + \left(\beta^2-h^2\right)\dot{\Psi}^2 - 2\beta^2\left(1-h^2\right)\dot{\chi}\dot{\Psi} = \lambda^2 h^2\gamma\left(e^{2\Psi} - \beta^2 e^{2\chi}\right) . \tag{4.6.14}$$

It is to note that these equations are invariant under time reversal, $\tau \to -\tau$.
A special exact solution can be obtained when $\chi = \Psi$. In this case one gets

$$\ddot{\Psi} = \frac{\lambda^2}{\gamma}\left[\left(1-h^2\beta^2\right)e^{2\Psi} - \beta^2\left(1-h^2\right)e^{2\Psi}\right] \tag{4.6.15}$$

whose first integral is

$$\dot{\Psi}^2 = \lambda^2 e^{2\Psi} + Q\,, \tag{4.6.16}$$

where Q is an integration constant.
In this case, the solutions of the field equations depend on the sign of Q and they coincide with solutions obtained in the case $h = 1$:
If $Q = q^2 > 0$

$$\lambda^2 e^{2\Psi} = \frac{q^2}{\sinh^2\left[q\left(\tau-\tau_0\right)\right]}\,, \tag{4.6.17}$$

with τ_0 an integration constant.
If $Q = 0$

$$\lambda^2 e^{2\Psi} = \frac{1}{\left(\tau-\tau_0\right)^2}\,. \tag{4.6.18}$$

If $Q = -q^2 < 0$

$$\lambda^2 e^{2\Psi} = \frac{q^2}{\sin^2\left[q\left(\tau - \tau_0\right)\right]}\,. \tag{4.6.19}$$

The solutions of the Friedmann equations are therefore:

- If $Q_i > 0$,

$$e^{2a} = \frac{q^2}{\lambda^2}\frac{\left|\sinh\left[q\left(\tau-\tau_2\right)\right]\right|^{2\beta^2/\gamma}}{\left|\beta^{-1}\sinh\left[\beta q\left(\tau-\tau_1\right)\right]\right|^{2/\gamma}}\,,\qquad e^{2\sqrt{3}\phi/\beta} = \left|\frac{\sinh\left[\beta q\left(\tau-\tau_1\right)\right]}{\beta\sinh\left[q\left(\tau-\tau_2\right)\right]}\right|^{2/\gamma}\;; \tag{4.6.20}$$

- If $Q_i < 0$,

$$e^{2a} = \frac{q^2}{\lambda^2}\frac{\left|\sin\left[q\left(\tau-\tau_2\right)\right]\right|^{2\beta^2/\gamma}}{\left|\beta^{-1}\sin\left[\beta q\left(\tau-\tau_1\right)\right]\right|^{2/\gamma}}\,,\qquad e^{2\sqrt{3}\phi/\beta} = \left|\frac{\sin\left[\beta q\left(\tau-\tau_1\right)\right]}{\beta\sin\left[q\left(\tau-\tau_2\right)\right]}\right|^{2/\gamma}\;; \tag{4.6.21}$$

- If $Q_i = 0$

$$e^{2a} = \frac{1}{\lambda^2}\frac{\left|\tau-\tau_2\right|^{2\beta^2/\gamma}}{\left|\tau-\tau_1\right|^{2/\gamma}}\,,\qquad e^{2\sqrt{3}\phi/\beta} = \left|\frac{\tau-\tau_1}{\tau-\tau_2}\right|^{2/\gamma}\,. \tag{4.6.22}$$

In general, however, the system is not exactly integrable: it can be studied therefore with the methods of the dynamical system. Defining the variables

$$X = \dot{\Psi}\,,\qquad Y = \dot{\chi}\,,\qquad Z = \lambda e^{\Psi}\,,\qquad W = \lambda e^{\chi} \tag{4.6.23}$$

the dynamical system becomes

$$\begin{aligned}\dot{X} &= \frac{1}{\gamma}\left[\left(1-h^2\beta^2\right)Z^2 - \beta^2\left(1-h^2\right)W^2\right]\\ \dot{Y} &= \frac{1}{\gamma}\left[\left(1-h^2\right)Z^2 + \left(h^2-\beta^2\right)W^2\right]\\ \dot{Z} &= XZ\,.\end{aligned} \tag{4.6.24}$$

and W is implicitly defined by

$$\left(h^2-\beta^2\right)X^2 + 2\beta^2\left(1-h^2\right)XY - \beta^2\left(1-h^2\beta^2\right)Y^2 = h^2\gamma\left(Z^2-\beta^2W^2\right)\,. \tag{4.6.25}$$

The global properties of the solutions of the system (4.6.24) can be deduced from the study of their behaviour near the critical points of the phase space. As in the case $h = 1$, the cosmic time t and the variable τ are related by a differential relation, thus the limit $t \to \pm\infty$

can correspond either to $\tau \to \infty$ or to $\tau \to \tau_0$, where τ_0 is a point where the functions Z and W diverge, and hence to critical points at finite distance or at infinity in phase space. The critical points at finite distance, in the limit $\tau \to \infty$, lie on two straight lines on the plane $Z = 0$,

$$Y_0 = \frac{\beta \pm h}{\beta\,(1 \pm h\beta)} X \,. \tag{4.6.26}$$

The system (4.6.24) can be linearized around its critical points and its eigenvalues are $0, 0, Y_0$. One can then deduce, following the calculation shown in the previous section, that the asymptotic behaviour of the functions a and ϕ near the critical points with $Y_0 \neq 0$ is

$$a \sim \frac{\pm\beta Y_0}{1 \pm \beta}\tau \sim \log t \,, \qquad \frac{\sqrt{3}\phi}{\beta} \sim \frac{Y_0}{1 \pm \beta}\tau \,, \tag{4.6.27}$$

with $\tau \to \mp\infty$. It follows that the cosmic time t vanishes in this limit and that at all these points $e^{2b} \sim t^{2/3}$, $e^{2\sqrt{3}h\phi/\beta} \sim t^{2/\beta}$ as $t \to 0$. If the critical point is the origin, we have $e^{a} \sim \tau^{-1} \sim e^{\mu t}$, and $e^{2b} \sim e^{2\mu t/3} \to 0$, $e^{2\sqrt{3}h\phi/\beta} \sim$ const, with $t \to \pm\infty$ depending on the sign of the integration constant μ.
The remaining critical points lie on the surface at infinite of the phase space. They can be studied, as usual, by defining new variables

$$u = \frac{1}{X} \,, \qquad y = \frac{Y}{X} \,, \qquad z = \frac{Z}{X} \,, \qquad w = \frac{W}{X} \tag{4.6.28}$$

and considering the limit $u \to \infty$. As in the particular case $h = 1$ discussed above, this limit is attained for $\tau \to \tau_0$, where τ_0 is a finite constant. Following the (4.4.18), the field equations become

$$\begin{aligned} u' &= -\frac{1}{\gamma}\left[\left(1 - h^2\beta^2\right) z^2 - \beta^2\left(1 - h^2\right) w^2\right] u \\ y' &= -\frac{1}{\gamma}\left[\left(1 - h^2\beta^2\right) z^2 - \beta^2\left(1 - h^2\right) w^2\right] y + \frac{1}{\gamma}\left[\left(1 - h^2\right) z^2 + \left(h^2 - \beta^2\right) w^2\right] \\ z' &= -\frac{1}{\gamma}\left[\left(1 - h^2\beta^2\right) z^2 - \beta^2\left(1 - h^2\right) w^2 - \gamma\right] z^2 \,, \end{aligned} \tag{4.6.29}$$

where a prime denotes $ud/d\tau$. The constraint which defines w^2 as a function of the other variables is

$$h^2 - \beta^2 + 2\beta^2\left(1 - h^2\right) y - \beta^2\left(1 - h^2\beta^2\right) y^2 = h^2\gamma\left(z^2 - \beta^2 w^2\right) \,. \tag{4.6.30}$$

Solving the system (4.6.29) with vanishing derivative, one finds the following critical points:

1. $y = y_0 = \frac{\beta^2-h^2}{\beta^2(1-h^2)}$, $z = w = 0$ with eigenvalues $0, 0, y_0$. These are the endpoints of the lines containing the critical points at finite distance and are not endpoints of trajectories lying at finite distance.
2. $y = \frac{\beta^2-h^2}{\beta^2(1-h^2)}$, $z^2 = 0$, $w^2 = \frac{\gamma\left(h^2-\beta^2\right)}{\beta^4(1-h^2)}$ with eigenvalues $\frac{h^2-\beta^2}{\beta^2(1-h^2)}, \frac{h^2-\beta^2}{\beta^2(1-h^2)}, \frac{h^2\gamma}{\beta^2(1-h^2)}$.
3. $y = \frac{1-h^2}{1-\beta^2 h^2}$, $w^2 = 0$, $z^2 = \frac{\gamma}{1-h^2\beta^2}$ with eigenvalues $-1, -1, -2$.
4. $z^2 = w^2 = y = 1$, with eigenvalues $-1, -1, -\frac{2}{\gamma}\left(1 - h^2\beta^2\right)$.

Remembering that $W \sim |\tau - \tau_0|^{1/u_0}$ and $Z \sim |\tau - \tau_0|^{y_0/u_0}$, we can deduce the asymptotic behaviour of the metric functions near the critical points at infinity ($\tau \to \tau_0$):

2) $e^\chi \sim |\tau - \tau_0|^{-1}$, $e^\Psi \sim |\tau - \tau_0|^{\beta^2\left(1-h^2\right)/\left(h^2-\beta^2\right)}$, and hence $t \sim |\tau - \tau_0|^{h^2/\left(h^2-\beta^2\right)}$, $e^{2b} \sim t^{2\beta^2/3h^2}$, $e^{2\sqrt{3}h\phi/\beta} \sim t^{-2}$.

3) $e^\chi \sim |\tau - \tau_0|^{-\left(1-h^2\right)/\left(1-h^2\beta^2\right)}$, $e^\Psi \sim |\tau - \tau_0|^{-1}$, and hence $t \sim |\tau - \tau_0|^{-h^2\beta^2/\left(1-h^2\beta^2\right)}$, $e^{2b} \sim t^{2\beta^2/3h^2}$, $e^{2\sqrt{3}h\phi/\beta} = t^{-2/\beta^2}$.

4) $e^\chi \sim e^\Psi = |\tau - \tau_0|^{-1}$, and hence $t \sim \pm \log |\tau - \tau_0|$, $e^{2b} \sim e^{\pm 2\mu t/3}$, $e^{2\sqrt{3}h\phi/\beta} \sim$ const.

For $h \gg \beta$, $h > 1$, point 4) corresponds to the limit $t \to \infty$, while point 2) corresponds to $t \to 0$. Point 3) can correspond either to $t \to 0$ or $t \to \infty$ depending on $h\beta$ being greater or less than 1. If $h\beta > 1$, the only possible behaviour for $t \to \infty$ is exponential, otherwise a power-law behaviour is possible as well.

Point 3) is stable when $h\beta < 1$, so in this case it attracts most trajectories for $t \to \infty$. Of these, the one starting at the origin represents a solution with an exponential behaviour for $t \to -\infty$ and a power-law behaviour for $t \to \infty$, analogous to the exact solitonic solution found in the case $h = 1$.

By comparing the $h \neq 1$ phase space with the one corresponding to $h = 1$, we see that for most values of h they are not very different, whereas the numerical behaviour of the solution can be quite varied depending on the value of h. This allows the possibility of obtaining more realistic solutions for the expansion of the universe, as discussed in [27].

CONCLUSIONS

In this thesis we have reviewed the theory of cosmological inflation, from its proposal as a possible solution to the flatness and horizon problems of the standard cosmological models to more recent developments based on models of scalar fields coupled to gravity, aiming to explain the formation of structure in the universe. In particular, we have discussed the conditions needed for inflationary expansion to be a solution of the field equations.
In general, however, the transition from the exponential inflation to the late time power-law expansion of the universe cannot be explicitly described through the models usually adopted. For this reason, we have introduced an exactly solvable model of gravity minimally coupled to a scalar field subject to a doubly-exponential potential, that has the peculiarity of admitting a solution displaying an initial period of exponential inflation smoothly evolving to power-law expansion. Theis solution corresponds to the decay from an initial configuration where the scalar field is in an unstable vacuum state, and may originate from a quantum fluctuation.
The aim of our investigation has been to study the general solutions of such model, to see if a realistic description of the evolution of the universe were possible in that framework and if the solution discussed above were stable. It turns out that the exponential solution corresponds to very specific initial conditions, while generic solutions present instead power-law inflation.
The solutions discussed till now were obtained in vacuum, but it is interesting to add matter in the form of dust or radiation to obtain more realistic models and to check the stability of the solutions. Addition of matter to the model spoils its exact solvability, and therefore we have studied the qualitative behavior of the solutions including matter by means of the techniques of the theory of dynamical systems. It resulted that the asymptotic behaviour of the solutions is not essentially affected by the presence of matter, but the exponential solution is unstble.
In our discussion, we did not deal with the case where the potential is defined with opposite

sign, and hence presents a minimum. Physically, this case is not very interesting, even if it possesses exact cosmological solutions as well. Contrary to the solutions found in Chapter 4, they are regular everywhere and represent universes starting with a big bang at $t = 0$ and recontracting after a finite time.
Obviously, we have limited our investigations to the mathematical aspects of the solutions. Their relevance for physical cosmology depends on the possibility that the inflationary mechanism provided by our model gives correct predictions on the evolution of perturbations and on observable cosmological data. In this context, the existence of exact solutions might be helpful for calculations. These topics have been investigated in [27], where it was shown that this is not the case, unless one generalizes the potential by adding a positive parameter in the exponential. In this case the system is no longer exactly integrable, and we have studied it resorting to the theory of dynamical systems. In general, its behaviour is similar to the previous case, but can differ for a range of values of the new parameter, allowing for the possibility of obtaining more realistic predictions. Some exact solutions however are still available for specific initial conditions, which essentially coincide with those found in the previous case.

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