Ph.D. in Electronic and Computer Engineering
Dept. of Electrical and Electronic Engineering
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Consensus in Multi-agent systems with Time-delays

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Advisors: Prof. Carla Seatzu;
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XXVII Cycle
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I dedicate this dissertation to my lovely parents

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Abstract

This thesis is concerned with consensus algorithms, convergence, and stability of multi-agent systems with time-delays. The main objectives of the thesis are

- Utilize and define distributed consensus protocols for multi-agent systems with first- or second-order dynamics so that the consensus state is reached.
- Analyze the stability behavior of the designed system in the presence of delays in the system, with focus on communication delays. The stability of the whole system must be determined in a distributed manner, i.e., it must rely only on some general properties of the corresponding communication network topology of system such as algebraic connectivity.
- Reduce the amount of the communications between the pairs of the agents by using a sampled-data communication strategy. We suppose that the samplings are aperiodic, and we provide some proofs for the stability and consensus of the system.

For that purpose, this thesis is divided is three main parts:

- The first part, including Chapters 2, 3, and 4, aims at providing a sufficiently detailed state of the art of the representation and stability analysis of consensus problems, time-delay systems, and sampled-data systems.
- The second part, including Chapters 5, 6, 7, and 8 consists in a presentation of several results that demonstrate the main contributions of this thesis.
- Finally, the third part, including Chapter 9 concludes the thesis and addresses the future directions and the open issues of this research.

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Introduction and structure of the thesis

> *"Coming together is a beginning; Keeping together is progress; Working together is success."*
>
> – Henry Ford

In the following chapter, we introduce the challenges that we face in distributed control systems under different information exchange regimes. In Section 1.1, multi-agent systems are introduced. In Section 1.2, we discuss power networks and distributed lightning systems as two motivating applications. Finally, in Section 1.3, we outline the thesis.

1.1 Multi-agent systems

During the last decades, inspired by advances in small size computation, communication, sensing, and actuation, a growing interest of the control theory community in distributed control has witnessed. Recent developments in control engineering, embedded computing, and communication networks, have made it feasible to have a large group of autonomous systems working cooperatively to perform complex tasks. These technological advances require new ways of managing and decision making over the information flow generated by the single units. Especially, the design of control systems, i.e. a decision making process, has shifted from *centralized* approaches, where all the information available is flooded in the network in a neighbor-to-neighbor data exchange in some point of time and space and then decisions are dispatched through the network, to *decentralized* and *distributed* approaches, where the information locally gathered by the units (agents) is processed in locus and control decisions are taken cooperatively by the agents with no supervision. Figure 1-1 illustrates how the information flows through the units in centralized, decentralized, and distributed networked systems.

In order to describe the interactions among the different units in large scale systems, the notion of *multi-agent systems*(MASs) has been introduced. Each agent, indeed, is assumed to have some peculiar dynamics, and the network or interconnections among the agents are then described by a graph called *communication topology graph*(CTG). In a CTG each vertex indicates an agent, and the two agents that can exchange information are being connected by an edge. Cooperative MASs can be found in numerous applications like aircraft and satellite formations, intelligent transportation infrastructures, flexible structures, and forest fire monitoring. A basic common feature of multi-agent control systems is that they are composed of several subsystems coupled through their dynamics, decision-making process, or performance objectives. When designing these systems, it is often necessary to adopt a distributed architecture, in which the decision maker (e.g., controller, network manager, social planner) is composed of several interconnected units. Each local decision maker can only access a subset of the global information (e.g., sensor measurements, model parameters) and actuate on a subset of the inputs in its neighborhood. This distributed architecture is typically imposed since the central decision maker with full access to information might become very complex and not possible to implement, or because different subsystems may belong to competing entities that wish to retain a level of autonomy.

Generally, the studies of MASs are oriented in the following directions:

1. Consensus and the like problems (synchronization and rendezvous).

2. Distributed formation and the like (flocking).

3. Distributed optimization.

4. Distributed estimation and control.

The above problems are not independent but actually may have overlapping in some contexts.

One of the most attractive problems that appears in distributed control of MAS, especially in coordination-type problems, is the consensus problem. The study of distributed control of MASs was first motivated by the work in distributed computing (Lynch et al., 2008), management science (DeGroot, 1974), and statistical physics (Vicsek et al., 1995). For example, robots need to arrive at an agreement so as to accomplish some complicated tasks. Flocks of birds tend to synchronize during migration in order to resist aggression and reach their destinations. Investigations of such problems are of significance in both theory and engineering applications. A critical problem for coordinated control is to design appropriate protocols and algorithms such that the group of agents can reach consensus on the shared information. The idea behind consensus serves as a fundamental principle for the design of distributed multi-agent coordination algorithms. The aim is, given initial values (scalar or vector) of agents, establish conditions under which, through local interactions and computations, agents asymptotically agree upon a common value, or reach a consensus. Due to its broad spectrum of applications, in the past years, a large attention has been devoted to the consensus problem in MAS (Qin et al., 2011; Ren et al., 2005a; Yu et al., 2010; Zareh et al., 2013a). Sensor networks (Yu et al., 2009; Olfati-Saber and Shamma, 2005), automated highway systems (Ren et al., 2005a), mobile robotics (Khoo et al., 2009), satellite alignment (Ren, 2007a) and several more, are some of the potential areas in which a consensus problem is taken into account. In the other words, Consensus is a state of a networked MAS in which all the agents reach agreement on a common value by only sharing information locally, namely with their neighbors. Several algorithms, often called *consensus protocols*, have been proposed that lead a MAS to consensus. As an illustrative example, the coordination

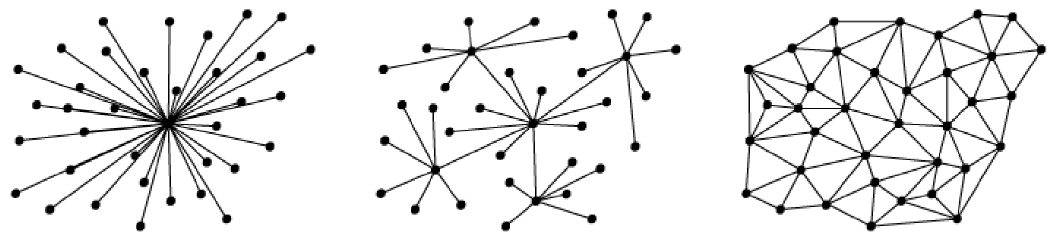


Figure 1-1: A schematic view of different communication networks in MASs.

problem of mobile robots finds several applications in the manufacturing industry in the context of automated material handling. The consensus problem in the context of mobile robots consists in the design of local state update rules which allow the network of robots to rendezvous at some point in space or follow a leading robot exploiting only measurements of speeds and relative positions between neighboring robots. To bridge the gap between the study of consensus algorithms and many physical properties inherited in practical systems, it is necessary and meaningful to study consensus by considering many practical factors, such as actuation, control, communication, computation, and vehicle dynamics, which characterize some important features of practical systems. This is the main motivation to study consensus. An overview of the research progress in the study of consensus is given in the next section regarding stochastic network topologies and dynamics, complex dynamical systems, delay effects, and quantization, which they were published mainly after 2006. Several milestone results prior to 2006 can be found in Olfati-Saber and Murray (2004); Jadbabaie et al. (2003); Moreau (2005); Tsitsiklis et al. (1986); Fax and Murray (2004); Ren et al. (2005b); Lin et al. (2005). A full review of the related works is given in the next chapters.

Time-delays exist in many real world processes due to the period of time it takes for the events to occur. Delays are particularly evident in networks of interconnected systems, such as supply chains and systems controlled over communication networks. In these control problems, taking the delays into account is particularly important for performance evaluation and control system's design. It has been shown, indeed, that delays in a controlled system (for instance, a communication delay for data acquisition) may have a very complicated nature: they may stabilize the system, or, in the contrary, they may lead to deterioration of the closed-loop performance or even instability, depending on the delay value and the system parameters. It is a fact that delays have stabilizing effects, but this is clearly conflicting for human intuition. Therefore, specific analysis techniques and design methods are to be developed to satisfactorily take into account the presence of delays at the design stage of the control system. On the other hand, time delay is ubiquitous in biological, physical, chemical, and electrical systems (Bliman and Ferrari-Trecate, 2008; Tian and Liu, 2008). In biological and communication networks, time delays are usually inevitable due to the possible slow process of interactions among agents. It has been observed from numerical experiments that consensus algorithms without considering time delays may lead to unexpected instability. In Bliman and Ferrari-Trecate (2008); Tian and Liu (2008), some sufficient conditions are derived for the first-order consensus in delayed multi-agent systems. In Mazenc and Malisoff (2014), framework to prove stability for nonlinear systems that may have delays and discontinuities, is studied. In this thesis, we try to mathematically formulate the effects of such time-delays in distributed control of complex networked systems.

In MAS, heavy computational loads can interrupt the sampling period of a certain controller. A scheduled sampling period can be used to deal with this problem. In such a case robust stability analysis with respect to the changes in the sampling time is necessary. For interesting contributions in this area we address the reader to Ackermann (1985); Fridman (2010); Zutshi et al. (2012) and the references therein. We also mention the work by Fridman et al. (2004) who exploited an approach for time-delay systems and obtained the sufficient stability conditions based on the Lyapunov-Krasovskii functional method. Seuret (2012) and Fridman (2010) proposed methods with better upper bounds to the maximum allowed sampling. Shen et al. (2012) studied the sampled-data synchronization control problem for dynamical networks. Qin et al. (2010) and Ren and Cao (2008) studied the consensus problem for networks of double integrators with a constant sampling period. In the latter two papers, even though the authors use the sampled-data notion to introduce their novelty, they suppose that the communication and the local sensing occur simultaneously and this simplifies the problem into a discrete state consensus problem. Xiao and Chen (2012) and Yu et al. (2011) studied second-order consensus in multi-agent dynamical systems with sampled *position* data. A comprehensive review of the works published in the framework of consensus problems is given in the next chapter.

Now we give some examples to illustrate the importance of consensus problems in practice.

1.2 Illustrative Examples

In this section we briefly introduce some examples to demonstrate the main problems considered in the thesis. We revisit these examples in the subsequent to illustrate the importance of the theoretical findings which will be developed in this thesis.

Commercial Lighting Control

In this section we introduce an example proposed by Sandhu et al. (2004). The application of wireless sensor networks to commercial lighting control provides a practical application that can benefit directly from artificial intelligence techniques. This application requires decision making in the face of uncertainty, with needs for system self-configuration and learning. Such a system is particularly well-suited to the evaluation of multi-agent techniques involving distributed learning. Generally, two-thirds of generated electricity is for commercial buildings, and lighting consumes 40 percent of this. An additional 45 percent energy savings are possible through the use of occupant and light sensors (Wen and Agogino, 2008). The goal in this domain is to leverage wireless sensor networks to create an intelligent, economical solution for reducing energy costs, and overall societal energy usage, while improving individual lighting comfort levels. There are also so many works in intelligent lighting control involving building control that focuses on HVAC (heating, ventilation, and air-conditioning), security or other aspects of building management. Several groups have examined the use of MAS for building control.

The proposed system consists of wireless sensor nodes located throughout the physical environment for purposes of sensing (light, temperature, and occupancy), actuation, and communication. Multiple sensors per node may be necessary for practical deployment; since a particular node may not need to use all sensors, or because it may simply act as a communication relay - dynamic resource allocation may be needed. All actuation will occur in ceiling-mounted, dimmable lighting ballasts. Primary design requirements are the inclusion of individual user preferences and the ability for the user to override the intelligent system. The most desirable automatic daylighting systems control overhead lighting but allow users to manually adjust desktop lighting (Yozell-Epstein, 2003). In order to maintain a practical system it will be necessary to encode user preferences into the system and provide methods for modifying these preferences.

The overall system for a building will functionally be decomposed into many smaller pseudo-static subnets since only local sensing affects local lighting actuation (Figure 1-2). With a single agent per node, these subnets still present multi-agent coordination problems. Within this framework, single nodes may belong to multiple adjacent subnets. While much sensor network literature predicts future networks on the order of hundreds or thousands of nodes, practical solutions to the presented problem can be accomplished with tens of nodes per subnet. At the same time that this scale makes the problem presently tractable, it also provides barriers to successful use of probabilistic techniques.

The primary goal of an MAS-based approach is to emulate the success of the decisions in a distributed manner. In particular, the interaction among the agents must emulate sensor validation and fusion techniques. Additionally, the decision making process must account for factors such as user preferences and variable electricity pricing. There are many challenges to the design and implementation of a successful MAS for this application. Many of the stated challenges are more generally applicable to designing MAS solutions for wireless sensor network problems. Simple agents are necessary because of the limited memory and processing associated with each sensor node. Limited radio communication among the nodes is necessary to conserve power.

Location awareness and reconfiguration are necessary aspects of a robust system. The system must be able to handle latency and time asynchronicity gracefully, due to communication constraints.

Agent interaction is an essential aspect of this architecture. Because of the communication and power constraints of sensor networks, agent interaction must be highly efficient. Multiple agents will contribute to the control of a given lighting actuator. In continuous domains such as this, control can be achieved by averaging agent actions or taking the median of their actions. Additionally, confidence values can be used to attenuate the global effects of aberrant local actions. When it is only necessary for the actuator to take on a fixed number of values control can be achieved by voting on what action to take. These methods allow a solution to be formed based on information from multiple sensors in disparate locations. They also add redundancy and noise reduction allowing the system to overcome faulty sensors. Many have used online learning techniques in automated building control systems, though the solutions tend to require significant computation and consequently centralized support (See for example Barnes (1995); Sharples et al. (1999); Chang and Mahdavi (2002)). In order to avoid the need for centralization, this system must be able to learn in a distributed manner; depending on the information available to the agents, supervised and reinforcement learning are the two major classes of learning that apply to this environment.

Synchronization in Power Networks

Consider the power network composed of two generators shown in Figure 1-3 from Kundur et al. (1994) and Ghandhari (2000). We can model this power network as

$$\begin{aligned} \dot{\delta}_1(t) = & \ \omega_1(t), \\ \dot{\omega}_1 = & \ \frac{1}{M_1}\left[(P_1(t)+\omega_1(t)) - K_{12}^{-1}\sin(\delta_1(t)-\delta_2(t)) - K_1^{-1}sin(\delta_1(t)) - D_1\omega_1(t)\right] \end{aligned} \tag{1.1}$$

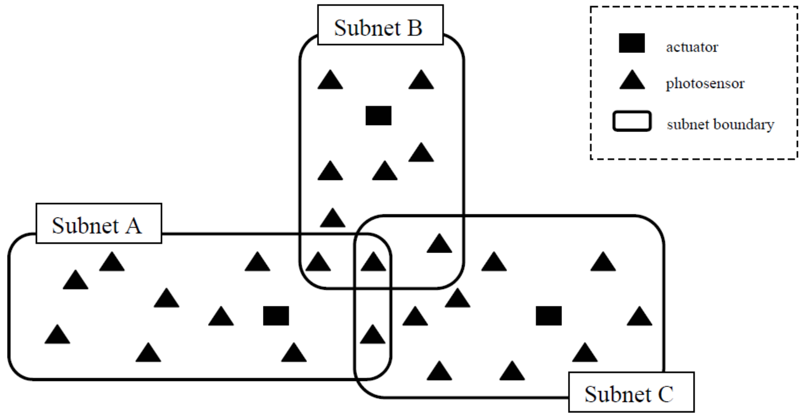


Figure 1-2: A subnet may rely on sensors that simultaneously belong to neighboring subnets. Each subnet is characterized by an MAS controlling a dimmable lighting ballast.

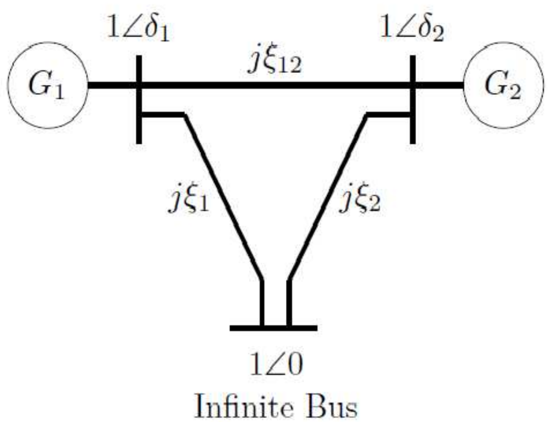


Figure 1-3: Schematic diagram of the power network

and

$$\begin{aligned} \dot{\delta}_2(t) = &\ \omega_2(t), \\ \dot{\omega}_2 = &\ \frac{1}{M_2}\left[(P_2(t) + \omega_2(t)) - K_{12}^{-1}\sin(\delta_2(t) - \delta_1(t)) - K_2^{-1}sin(\delta_1(t)) - D_2\omega_2(t)\right] \end{aligned} \tag{1.2}$$

where $\delta_i(t)$, $P_i(t)$, and $\omega_i(t)$ are the phase angle of the terminal voltage, the rotation frequency, the input mechanical power, and the exogenous input of generator i, respectively. We assume that $P_1(t) = P_{01} + M_1 v_1(t)$ and $P_2(t) = P_{02} + M_2 v_2(t)$, where $v_1(t)$ and $v_2(t)$ are the continuous-time control inputs of this system, and P_{01}

and P_{02} are constant references. Now, we can find the equilibrium point (δ_1^*, δ_2^*) of the system and linearize it around this equilibrium. Furthermore, let us discretize the linearized system by applying Euler's constant step scheme with sampling time ΔT, which results in

$$x(k+1) = Ax(k) + Bu(k) + H\omega(k), \tag{1.3}$$

where

$$x(k) = \begin{bmatrix} \Delta\delta_1(k) \\ \Delta\omega_1(k) \\ \Delta\delta_2(k) \\ \Delta\omega_2(k) \end{bmatrix} \quad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \quad \omega(k) = \begin{bmatrix} \omega_1(k) \\ \omega_2(k) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \Delta T & 0 & 0 \\ a_{21} & 1 - \dfrac{\Delta T D_1}{M_1} & a_{23} & 0 \\ 0 & 0 & 1 & \Delta T \\ \dfrac{a_{23} M_1}{M_2} & 0 & a_{21} & \end{bmatrix},$$

where

$$a_{21} = \frac{-\Delta T(K_{12}^{-1}\cos(\delta_1^* - \delta_2^*) + K_1^{-1}\cos(\delta_1^*))}{M_1},$$

$$a_{23} = \frac{\Delta T K_{12}^{-1}\cos(\delta_1^* - \delta_2^*)}{M_1}.$$

and

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 0 \\ 1/M_1 & 0 \\ 0 & 0 \\ 0 & 1/M_2 \end{bmatrix}.$$

Here, $\Delta\delta_1(k)$, $\Delta\delta_2(k)$, $\Delta\omega_1(k)$ and $\Delta\omega_2(k)$ denote the deviation of the corresponding parameters from their equilibrium points at time instances $t = k\Delta T$.

It is interesting to achieve the optimal control of this power network. Whenever we restrict our considerations to linear time-invariant controllers, the closed-loop performance measure is given by

$$J = \|T_{y\omega}(z)\|_2^2,$$

where $T_{y\omega}$ denotes the closed-loop transfer function from the exogenous input $\omega(k)$ to output vector $y(k) = [x(k)^T \quad u(k)^T]^T$ in which z is the symbol for the one time-step forward shift operator. Through minimizing such a cost function, we guarantee that the frequency of the generators stays close to its nominal value without wasting too much energy. For the design of nonlinear controllers, we consider the cost function

$$J = \lim_{T\to\infty} \sum_{k=0}^{T-1} x(k)^T x(k) + u(k)^T u(k).$$

This cost function is equal to the H_2-norm of the closed-loop transfer function for linear time-invariant systems excited by exogenous inputs that are elements of a sequence of independently and identically distributed Gaussian random variables with zero mean and unit covariance.

Let us assume that the impedance of the lines that connect each generator to the infinite bus in Figure 1-3 varies over time. Define α_i, $i = 1, 2$, as the deviation of the admittance K_i^{-1} from its nominal value. Notice that α_i only appears in the model of subsystem i. When designing the control laws, assume that the information regarding the value of parameter α_i is only available in the design of controller for subsystem i. One motivation for this can be that the generators are physically far apart from each other.

The synchronization of coupled nonlinear power generators is a closely related topic to the consensus of MASs. In the pioneering work by Pecora and Carroll (1990), the synchronization phenomenon of two master-slave chaotic systems was observed and applied to secure communications. Pecora and Carroll (1990) and Pecora and Carroll (1998) addressed the synchronization stability of a network of generator by using the master stability function method. Due to nonlinear dynamics, usually, only sufficient conditions can be given for verifying the synchronization.

1.3 Organization of the Dissertation

The dissertation is organized as follows: In Chapter 2, we study the consensus problem of multi-agent systems. A comprehensive review of the related works up to the present is given. We discuss the first-order, the second-order, and the higher-order consensus problems separately, and an overview to some more complicated problems is also given. Some main results and progress in distributed multi-agent coordination, focusing on papers published in major control systems and robotics journals since 2006. Distributed coordination of multiple vehicles, including unmanned aerial vehicles, unmanned ground vehicles, and unmanned underwater vehicles, has been a very active research subject studied extensively by the systems and control community. The recent results in this area are categorized into several directions, such as consensus, formation control, optimization, and estimation. After the review, a short discussion section is included to summarize the existing research and to propose several promising research directions along with some open problems that are deemed important for further investigations.

The purpose of Chapter 3 is to survey the recent results developed to analyze the asymptotic stability of time-delay systems. Both delay-independent and delay-dependent results are reported in this chapter. Special emphases are given to the issues of conservatism of the results and computational complexity. Connections of certain delay-dependent stability results are also discussed.

In Chapter 4, we consider the problem of stability of sampled-data systems. Sampled-data systems are a form of hybrid model which arises when discrete measurements and updates are used to control continuous-time plants. In this chapter, we use a recently introduced Lyapunov approach to derive stability conditions for both the case of fixed sampling period (synchronous) and the case of a time-varying sampling period (asynchronous). This approach requires the existence of a Lyapunov function which decreases over each sampling interval. To enforce this constraint, we use a form of slack variable which exists over the sampling period, may depend on the sampling period, and allows the Lyapunov function to be temporarily increasing.

The resulting conditions are enforced using a new method of convex optimization of polynomial variables known as Sum-of-Squares.

In Chapter 5, we address the problem of deriving sufficient conditions for asymptotic consensus of second order multi-agent systems with slow switching topology and time delays. A PD-like protocol is proposed based on local interaction protocol and the stability analysis is based on the Lyapunov-Krasovskii functional method. The approach is based on the computation of a set of parameters that guarantee stability under any network topology of a given set. A significant feature of this method is that it does not require to know the possible network topologies but only a bound on their second largest eigenvalue (algebraic connectivity).

In Chapter 6, we study the stability property of a consensus on the average algorithm in arbitrary directed graphs with respect to communication/sensing time-delays. The proposed algorithm adds a storage variable to the agents' states so that the information about the average of the states is preserved despite the algorithm iterations are performed in an arbitrary strongly connected directed graph. We prove that for any network topology and choice of design parameters the consensus on the average algorithm is stable for sufficiently small delays.

In Chapter 7, consensus in second-order multi-agent systems with a non-periodic sampled-data exchange among agents is investigated. The sampling is random with bounded inter-sampling intervals. It is assumed that each agent has exact knowledge of its own state at all times. The considered local interaction rule is PD-type. The characterization of the convergence properties exploits a Lyapunov-Krasovskii functional method, sufficient conditions for stability of the consensus protocol to a time-invariant value are derived.

Chapter 8 studies consensus in second-order multi-agent systems with a non-periodic sampled-data exchange among agents is investigated in this chapter. Sampling is random with bounded inter-sampling intervals, and each agent has exact knowledge of its own state at any time instant. A constant communication delay among agents is also considered. A local PD-type protocol is used to bring the system into an agreement state. Under the assumption that only the connectivity of the graph modeling the network topology is known, sufficient conditions for the stability of the consensus protocol to a time-invariant value are derived based on LMIs.

Chapter 9 summarizes the contributions and explains the open issues.

In Appendices, the eigenvalue properties of Laplacian matrix, perturbation bounds on matrix eigenvalues, and the eigenvalue properties of weighted adjacency matrix is addressed

2

Consensus problems

"Those who know that the consensus of many centuries has sanctioned the conception that the earth remains at rest in the middle of the heavens as its center, would, I reflected, regard it as an insane pronouncement if I made the opposite assertion that the earth moves."

– Nicolaus Copernicus

At the first glance, the word *consensus* may bring political issues into mind. According to Merriam-Webster, *consensus*, is a general agreement about something, an idea or opinion that is shared by all the people in a group. So two key features determine the definition of it: first it happens among a group, and second shared opinions among the group are necessary. We see that in the systems framework, the same features must be held.

In this chapter, the consensus problem is introduced. In the following section we introduce the main definitions of a consensus problem. In Section 2.2, we review the existing literature of consensus problem in systems whose dynamical equations are of first-order. Similarly in Section 2.3, consensus problems in systems with second-order dynamics, and in Section 2.4 systems with dynamics of an order higher than two, are reviewed. In Section 2.5, consensus in systems with complex dynamics (generally complexity indicates nonlinearity), is skimmed.

2.1 Introduction

By the help of embedded computational resources in autonomous vehicles, many civilian and military applications profit enhanced operational capability and greater efficiency through cooperative teamwork compared to those in which the vehicles perform single tasks. Some examples of such applications include space-based interferometers, surveillance, and reconnaissance systems, and distributed sensor networks. In order to cover all these applications, various cooperative control capabilities need to be developed, rendezvous, attitude alignment, flocking, foraging, task and role assignment, payload transport, air traffic control, and cooperative search. Generally, cooperative control for MAS can be categorized as either formation control problems like the control protocols used for mobile robots, unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs), satellites, spacecraft, and automated highway systems, or non-formation based cooperative control problems such as task assignment, role assignment, air traffic control, timing, and search. There are several challenges in theoretical and practical in implementation of cooperative control in MAS. An effective cooperative control strategy must take into account numerous issues, including the definition and management of shared information among a group of agents to facilitate the coordination of these agents. Generally the shared information may take the form of common objectives, common control algorithms, relative position and velocity information, or an image. Information exchange among the agents, which is necessary for cooperation, can be shared in a variety of ways, e.g., relative position sensors may enable vehicles to construct state information for other vehicles, knowledge may be communicated between vehicles using a wireless network, or joint knowledge might be preprogrammed into the vehicles before a mission begins. Obviously, several unpredicted issues may disturb the system, and hence in an effective cooperative control strategy, a team of agents must be able to respond to the new conditions that are sensed as a cooperative task. As the environment changes, the agents on the team must be in agreement as to what changes took place.

Cooperative control of multiple autonomous vehicles poses significant theoretical and practical challenges. First, the research objective is to develop a system of subsystems rather than a single system. Second, the communication bandwidth and connectivity of the team are often limited, and the information exchange among vehicles may be unreliable. It is also difficult to decide what to communicate and when and with whom the communication takes place. Third, arbitration between team goals and individual goals needs to be negotiated. Fourth, the computational resources of each individual vehicle will always be limited. Recent years have seen significant interest and research activity in the area of coordinated and cooperative control of multiple autonomous vehicles (e.g.,Anderson and Robbins (1998); Balch and Arkin (1998); Beard et al. (2001)). Much of this work assumes the availability of global team knowledge, the ability to plan group actions in a centralized manner, and/or perfect and unlimited communication among the vehicles. A centralized coordination scheme relies on the assumption that each member of the team has the ability to communicate to a central location or share information via a fully connected network. As a result, the centralized scheme does not scale well with the number of vehicles. The centralized scheme may result in a catastrophic failure of the overall system due to its single point of failure. Also, real-world communication topologies are usually not fully connected. In many cases, they depend on the relative positions of the vehicles and on other environmental factors and are therefore dynamically changing in time. In addition, wireless communication channels are subject to multi- path, fading and drop-out. Therefore, cooperative control in the presence of real-world communication constraints becomes a significant challenge.

When multiple vehicles agree on the value of a variable of interest, they are said to have reached consensus. Information consensus guarantees that vehicles sharing information over a network topology have a consistent view of information that is critical to the coordination task. To achieve consensus, there must be a shared variable of interest, called the information state, as well as appropriate algorithmic methods for negotiating to reach consensus on the value of that variable, called the consensus algorithms. The information state represents an instantiation of the coordination variable for the team. Examples include a local representation of the center and shape of a formation, the rendezvous time, the length of a perimeter being monitored, the direction of motion for a multi-vehicle swarm. By necessity, consensus algorithms are designed to be distributed, assuming only neighbor-to-neighbor interaction between vehicles. Vehicles update the value of their information states based on the information states of their neighbors. The goal is to design an update law so that the information states of all of the vehicles in the network converge to a common value.

Consensus algorithms have a historical perspective by Borkar and Varaiya (1982); Chatterjee and Seneta (1977); DeGroot (1974); Gilardoni and Clayton (1993); Lynch (1996); Tsitsiklis et al. (1986), to name a few, and have recently been studied extensively in the context of cooperative control of multiple autonomous vehicles (Fax and Murray, 2004; Jadbabaie et al., 2003; Lin et al., 2004; Moreau, 2005; Olfati-Saber and Murray, 2004; Ren et al., 2005b). Some results in consensus algorithms can be understood in the context of connective stability (Šiljak, 1974). Consensus algorithms have applications in rendezvous (Beard et al., 2006; Dimarogonas and Kyriakopoulos, 2007; Lin et al., 2004; Lin and Jia, 2011; Martınez et al., 2005; Sinha and Ghose, 2006; Smith et al., 2005, 2007), formation control (Fax and Murray, 2004; Lafferriere et al., 2005; Lawton et al., 2003; Lin et al., 2005; Marshall et al., 2006; Porfiri et al., 2007; Ren, 2007b), flocking (Cucker and Smale, 2007; Dimarogonas et al., 2006; Lee and Spong, 2007; Moshtagh and Jadbabaie, 2007; Olfati-Saber, 2006; Regmi et al., 2005; Tanner et al., 2007; Veerman et al., 2005), attitude alignment (Lawton and Beard, 2002; Ren, 2007a,c; Ren and Beard, 2004), perimeter monitoring (Casbeer et al., 2006), decentralized task assignment (Alighanbari and How, 2005), and sensor networks (Yang et al., 2008; Olfati-Saber, 2005; Olfati-Saber and Shamma, 2005; Spanos et al., 2005; Xiao et al., 2005). The basic idea of a consensus algorithm is to impose similar dynamics on the information states of each vehicle. If the communication network among vehicles allows continuous communication or if the communication bandwidth is sufficiently large, then the information state update of each vehicle is modeled using a differential equation. On the other hand, if the communication data arrive in discrete packets, then the information state update is modeled using a difference equation.

2.2 First-order Consensus

This section overviews fundamental consensus algorithms in which a scalar information state is updated by each vehicle using, respectively, a first-order differential equation and a first-order difference equation.

Suppose that there are n vehicles in the team. The team's communication topology can be represented by directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes (vehicles) and $\mathcal{E} \subseteq \{\mathcal{V} \times \mathcal{V}\}$ is the set of edges. An edge $(i, j) \in \mathcal{E}$ exists if there is a communication channel between vehicles i and j. Self loops (i, i) are not considered. The set of neighbors of agent i is denoted by $\mathcal{N}_i = \{j \ : \ (j, i) \in \mathcal{E}; j = 1, \ldots, n\}$. Let $\delta_i = |\mathcal{N}_i|$ be the degree of agent i which represents the total number of its neighbors.

The topology of graph $\mathcal{G}$ is encoded by the so-called *adjacency matrix*, an $n \times n$ matrix A_d whose (i, j)-th entry is equal to 1 if $(i, j) \in \mathcal{E}$, 0 otherwise. Obviously in an undirected graph matrix A_d is symmetric.

We denote $\Delta = diag(\delta_1, \ldots, \delta_n)$ the diagonal matrix whose non-null entries are the degrees of the nodes. Denote Δ_{in} and Δ_{out}, corresponding to in- and out- degree matrices respectively, in a directed graph. We now define the Laplacian matrix as $L = \Delta - A$. The in-Laplacian and out-Laplacian matrices of a directed graph are defined as $L_{in} = \Delta_{in} - A_d$ and $L_{out} = \Delta_{out} - A_d$. Due to the Gershgorin Circle Theorem applied to the rows of the in-Laplacian or the columns of the out-Laplacian it is possible to show that both matrices have eigenvalues with non-negative real part for any graph $\mathcal{G}$. By construction matrices L_{in} and L_{out} have at least one null eigenvalue because either the row sum or the column sum is zero. Furthermore, let $\mathbf{1}_n$ and $\mathbf{0}_n$ be respectively the n-elements vectors of ones and zeros, then $L_{in}\mathbf{1} = \mathbf{0}$ and $\mathbf{1}^T L_{out} = \mathbf{0}^T$. If $\mathcal{G}$ is strongly connected, i.e., there exists a directed path that connects any pair of nodes in $\mathcal{V}$, then the algebraic multiplicity of the null eigenvalue of both L_{in} and L_{out} is one. More details about the characteristics of Laplacian matrix is given in Appendix A.

Let x_i be the information state of the i th agent. The information state represents information that needs be coordinated among agents (Ren et al., 2005a). The information state may be agent position, velocity, oscillation phase, decision variable

The system considered in this section is similar to the one presented by Ren et al. (2005b). There are n agents each with state vectors $x_i \in \mathbb{R}$. for agents $i = 1, \ldots, n$ having single integrator dynamics:

$$\dot{x}_i(t) = u_i(t). \tag{2.1}$$

As described by Olfati-Saber and Murray (2004), a continuous-time consensus protocol can be summarized as

$$\dot{x}_i(t) = u_i(t) = -\sum_{j \in \mathcal{N}_i(t)} \gamma_{ij}(t) \left(x_i(t) - x_j(t)\right) \tag{2.2}$$

where $\mathcal{N}_i(t)$ represents the set of agents whose information is available to agent i at time t and $\gamma_{ij}(t)$ denotes a positive time-varying weighting factor. In other words, the information state of each agent is driven toward the states of its (possibly time-varying) neighbors at each time. Note that some agents may not have any information exchange with other agents during some time intervals. The continuous-time linear consensus protocol (2.2) can be written in matrix form as $\dot{x}(t) = -Lx(t)$, where L is the graph Laplacian and $x = [x_1, \ldots, x_n]^T$.

Similarly, the discrete-time form of the equation, as used by Ren (2007a) can be given as

$$x_i(k+1) = -\sum_{j \in \mathcal{N}_i(k) \cup i} \beta_{ij}(k) x_j(k) \tag{2.3}$$

where $\sum_{j \in \mathcal{N}_i(k) \cup i} \beta_{ij}(k) = 1$, and $\beta_{ij} > 0$ for $j \in \mathcal{N}_i(k) \cup i$. In other words, the next state of each agent is updated as the weighted average of its current state and the current states of its (possibly time-varying) neighbors. Note that an agent simply maintains its current state if it has no information exchange with other agents at a certain time step. The discrete-time linear consensus protocol (2.3) can be written in matrix form as $x(k+1) = P(k)x(k)$, where $P(k)$ is a stochastic matrix with positive diagonal entries.

A MAS with n agents is said has reached consensus if $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| \to 0$, for $\forall i \neq j$.

In the following section a review of some first-order consensus problems with different conditions, are reviewed and the convergence properties are given.

Convergence Analysis

Below, we briefly review the existing results on well known first order consensus problems.

- Time-invariant Information Exchange Topology

  Under a time-invariant information exchange topology, it is assumed that if one agent can access another agent's information at one time, it can obtain information from that agent all the time. For the continuous-time consensus protocol (2.2), it is straightforward to see that $L\mathbf{1} = \mathbf{0}$ and all eigenvalues of the Laplacian matrix L have non-negative real parts from Gershgorin's disc theorem. If zero is a simple eigenvalue of L , it is known that $x(t)$ converges to the kernel of L , that is, $span\{\mathbf{1}\}$, which in turn implies that $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| \to 0$.

  It is well-known that zero is a simple eigenvalue (Chung, 1997). However, this is only a sufficient condition rather than a necessary one. We have the formal statement that zero is a simple eigenvalue of the Laplacian matrix if and only if its digraph has a spanning tree. This conclusion was shown by Ren et al. (2005a) by an induction approach while the same result is proven independently by Lin et al. (2005) by a constructive approach. As a result, under a time-invariant information exchange topology, the continuous-time protocol achieves consensus asymptotically if and only if the information exchange topology has a spanning tree.

  For the discrete-time consensus protocol (2.3), it can be shown that all eigenvalues of D that are not equal to one are within the open unit circle from Gershgorin's disc theorem. If one is a simple eigenvalue of P and all other eigenvalues have module less than one, it is known that $\lim_{k \to \infty} P^k = \mathbf{1}\nu^T$, where ν is a column vector. This implies that $\lim_{k \to \infty} \|x_i(k) - x_j(k)\| \to 0$.

The well-known Perron-Frobenius theorem states that one is a simple eigenvalue of a stochastic matrix if the graph of the matrix is strongly connected. Similar to the continuous-time case, this is only a sufficient condition rather than a necessary one. Horn and Johnson (2012) showed that for a nonnegative matrix with identical positive row sums, the row sum of the matrix is a simple eigenvalue if and only if the digraph of the matrix has a spanning tree. In other words, a matrix may be reducible but retains its spectral radius as a simple eigenvalue. Furthermore, if the matrix has a spanning tree and positive diagonal entries, it is shown that the spectral radius of the matrix is the unique eigenvalue of maximum modulus. We have the formal statement that one is a unique eigenvalue of modulus one for the stochastic matrix P if and only if its digraph has a spanning tree (Lafferriere et al., 2005). As a result, under a time-invariant information exchange topology, the discrete-time protocol achieves consensus asymptotically if and only if the information exchange topology has a spanning tree.

- Time-varying Information Exchange Topology

  Consider an MAS of n agents that communicate with each other and need to agree upon a certain objective of interest or perform synchronization. Due to the fact that the nodes of the network are moving, it is easy to imagine that some of the existing communication links can fail simply due to the existence of an obstacle between two agents. The opposite situation can arise when new links between two agents are created because the agents come to an effective range of detection with respect to each other. In terms of the network topology, this means that a certain number of edges are added or removed from the graph. Here, we are interested to investigate this in case of a network with switching topology G, whether it is still possible to reach a consensus, or not.

  Based on a valid common Lyapunov function for the disagreement dynamics, Olfati-Saber and Murray (2004) proved that, for any arbitrary switching signal, solution of the switching system (2.2) globally asymptotically converges to the average of the initial value (i.e., average-consensus is reached).

- Communication delay

  In the case that information is exchanged between agents through communications, time delays of the communication channels need to be considered. Let τ_{ij} denote the time delay for information communicated from agent j to agent i. The continuous-time consensus protocol is now denoted by:

$$\dot{x}_i(t) = u_i(t) = -\sum_{j\in\mathcal{N}_i(t)} \gamma_{ij}(t)\left(x_i(t-\tau_{ii}) - x_j(t-\tau_{ij})\right). \tag{2.4}$$

  As it is shown by Olfati-Saber and Murray (2004), in the case $\tau_{ij} = \tau_{ii} = \tau \in \mathbb{R}^{>0}$, if the communication topology is fixed, undirected, and connected, average-consensus is achieved if and only if $\tau \in [0, \; \frac{\pi}{2\delta_{max}}]$, where δ_{max} denote the maximum degree of the corresponding communication topology graph. Consider another case where the time delay only affects the information state that is being transmitted. This implies that $\tau_{ii} = 0$ in (2.4). Now if $\tau_{ij} = \tau \in \mathbb{R}^{>0}$, and the communication topology is directed and switching, the consensus result for switching topologies described previously is still valid for an arbitrary time delay τ.

2.3 Second-Order Consensus

All the previously mentioned references focus on consensus protocols that take the form of first-order dynamics. In reality, equations of motion of a broad class of vehicles require second-order dynamic models. For example, some vehicle dynamics can be feedback linearized as double integrators, e.g. mobile robot dynamic models. In the case of first-order consensus protocols, the final consensus value is a constant. In contrast to the constant final consensus value, it might be proper to derive second-order consensus protocols such that some information states converge to a consistent value (e.g. position of the formation center) while others converge to another consistent value (e.g. velocity of the formation center). However, the extension of consensus protocols from first order to second order is nontrivial. In the paper of Ren (2007c), formation keeping algorithms taking the form of second-order dynamics are addressed to guarantee attitude alignment, agreement of position deviations and velocities, and/or collision avoidance in a group of vehicles.

In a very general form, a second-order MAS can be described by the following dynamics:

$$\begin{aligned} \dot{x}_i(t) &= \quad v_i(t) \\ \dot{v}_i(t) &= \quad u_i(t). \end{aligned} \tag{2.5}$$

Second-order consensus in the multi-agent system (2.5) is said to be achieved if for any initial conditions it holds:

$$\left.\begin{aligned} \lim_{t\to\infty} |x_i(t) - x_j(t)| = 0 \\ \lim_{t\to\infty} |v_i(t) - v_j(t)| = 0 \end{aligned}\right. \quad \forall i \neq j. \tag{2.6}$$

2.4 Higher-order consensus

Recently, increasing interest has turned to MASs with high-order or/and heterogeneous dynamical agents. Wang et al. (2008) and Seo et al. (2009) discussed the solvability of the consensus seeking problem for systems of identical agents in networks without communication delays, and proved that for such systems the consensus problem is solvable if the interconnection topology has a globally reachable node. Arcak (2007) developed a general framework based on passivity theory for the design of group coordination control of systems with nonlinear dynamical agents. Using the small-gain method, Lee and Spong (2006) proposed a sufficient consensus condition for high-order heterogeneous systems with diverse communication delays.

Based on the general Nyquist stability criteria and an S-hull technique, Lestas and Vinnicombe (2010) also considered high-order heterogeneous systems with diverse communication delays, and proposed frequency-domain conditions which are less conservative than small-gain-like or passivity-like results. It should be noted that only the constant-consensus problem has been considered in the above-mentioned references on high-order heterogeneous MASs, and the main focus of these references is on the stability instead of the existence of the set of consensus solutions. Actually, the existence of a constant consensus depends only on the connectivity of the interconnection topology of MASs (Ren et al., 2005a). The values of self delays introduced by agents in consensus protocols may lead to instability of the consensus solution (see Papachristodoulou et al. (2010)) but they do not influence the existence of a constant consensus solution.

However, it is possible for second-order or high-order MASs to reach not only constant consensus solutions but also dynamical consensus solutions. Such dynamical consensus solutions will be also called high-order consensus solutions in this section. An interesting problem for high-order MASs is under which condition the high-order consensus solution exists. The problem has not been fully addressed in currently existing literature. It can be shown that an inappropriate value of self-delay may lead to the in-existence of a high-order consensus solution. To guarantee the existence of high-order consensus solutions, currently existing consensus protocols introduce self-delays which are exactly equal to the corresponding communication delays (see, e.g., Hu et al. (2007)). In practice, however, communication delays can be estimated only approximately. Therefore, a high-order consensus protocol which does not depend on exact values of communication delays is of great importance for practical application of the consensus theory.

2.5 Consensus in Complex systems

In the mathematical modeling of physical systems, it is an unavoidable dilemma: use a more accurate model which is harder to manage, or work with a simpler model which is easier to manipulate but with less confidence? A complex system is a damped, driven system (for example, a harmonic oscillator) whose total energy exceeds the threshold for it to perform according to classical mechanics but does not reach the threshold for the system to exhibit properties according to chaos theory.

A topic that is closely related to the consensus of MAS, is the synchronization of coupled nonlinear oscillators. In the pioneering work by Pecora and Carroll (1990), the synchronization phenomenon of two master-slave chaotic systems was observed and applied to secure communications. Lu and Chen (2006) and Pecora and Carroll (1998) addressed the synchronization stability of a network of oscillators by using the master stability function method. Recently, the synchronization of complex dynamical networks, such as small world and scale-free networks, has been widely studied (see Chen (2008); Duan et al. (2009); Kocarev and Amato (2005); Lu et al. (2008); Porfiri et al. (2008); Wang and Chen (2002); Wu et al. (2009)) and the references therein). Due to nonlinear node dynamics, usually, only sufficient conditions can be given for verifying the synchronization.

Below, we present some examples which show some applications of consensus algorithms in complex dynamical systems.

Example 2.1 *(Bullo et al., 2009) The following models of control systems are commonly used in robotics, beginning with the early works of Dubins (1957), and Reeds and Shepp (1990). Figure 2-1(left) show a two-wheeled vehicle and a four-wheeled vehicle, respectively. The two-wheeled planar vehicle is described by the dynamical system*

$$\dot{x}(t) = v\cos\theta(t) \quad \dot{y}(t) = v\sin\theta(t) \quad \dot{\theta}(t) = \omega(t), \tag{2.7}$$

with state variables $x \in \mathbb{R}, y \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$ describing the planar position and orientation of the vehicle, and with controls v and ω, describing the forward linear velocity and the angular velocity of the vehicle.

A group of such robots as shown in Figure 2-1 (right), is an example of MAS. ■

Example 2.2 *(Bullo et al., 2009) Communication congestion: Omni-directional wireless transmissions interfere. Clear reception of a signal requires that no other signals*

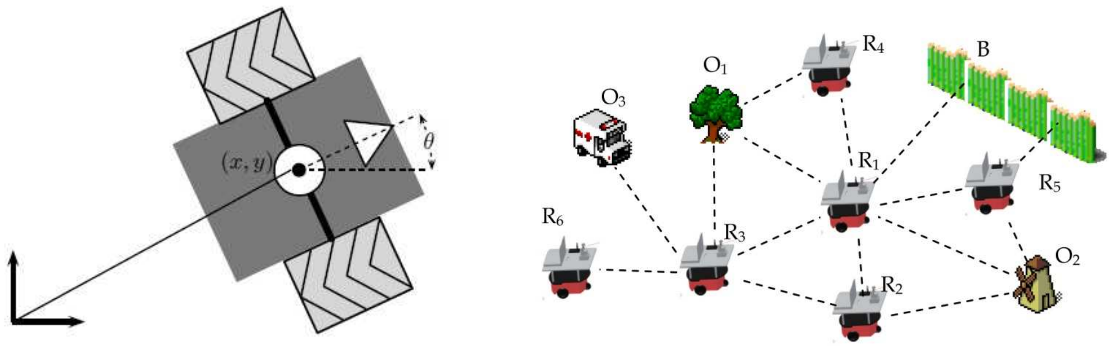


Figure 2-1: Two-wheeled robots in a plane in Example 2.2 (left). A multi-robot networked system (right).

are present at the same point in time and space. In an ad hoc network, node i receives a message transmitted by node j only if all other neighbors of i are silent. In other words, the transmission medium is shared among the agents. As the density of agents increases, so does wireless communication congestion. The following asymptotic and optimization results are known.

First, for ad hoc networks with n uniformly randomly placed nodes, it is known (Gupta and Kumar, 2000) that the maximum-throughput communication range $r(n)$ of each node decreases as the density of nodes increases; in d dimensions, the appropriate scaling law is $r(n) \in \Theta(((log(n)/n)))^{\frac{1}{d}}$. This is referred to as the connectivity regime in percolation theory and statistical mechanics. Using the k-nearest neighbor graph over uniformly placed nodes, the analysis by Xue and Kumar (2004) suggests that the minimal number of neighbors in a connected network grows with $log(n)$. Second, a growing body of literature (Santi, 2005; Lloyd et al., 2005) is available on topology control, that is, on how to compute transmission power values in an ad hoc network so as to minimize energy consumption and interference (due to multiple sources), while achieving various graph topological properties, such as connectivity or low network diameter. ■

Several authors have devised new strategies to address different consensus problems, but still there are so many open problems left in this area. In this thesis, we study first- and second-order MASs. Our focus is especially on consensus problems in systems with time-delays and sampled-data communications. We study both directed and undirected communications. In the next two chapters, we study the time-delay systems and sampled-systems and we describe the tools we use in order to analyze the MASs.

2.6 Conclusions

In this chapter we reviewed different existing consensus problems and characterized some important notions in multi agent system control framework. The agreement and stability conditions for a diversity of conditions in first order, second order, and high order and complex systems were skimmed. Some illustrative example showed the applications and importance of MASs.

3

Time-delay systems

"You may delay, but time will not."

– Benjamin Franklin

Delay, is defined as *a situation in which something happens later than it should* in Merriam-Webster. Delay is unavoidable in almost every real-world phenomena. It is well known that even a very small delay may cause big disasters. Some milliseconds of delay would be enough to happen a big car crash in highway. Delay, also can have stabilizing effects. For example, in wild water canoeing, the athlete should not immediately react to every sudden change. Instead, his or her reactions must be with a delay so that the water's behavior is better predictable. It is now clear that why delay analysis is so important. In this chapter, we provide useful tools to analyze the stability of time-delay systems.

3.1 Introduction

Time-delay systems (TDSs) belong to the class of functional differential equations, as opposed to ordinary differential equations, and represent a class of infinite-dimensional systems widely used to describe propagation and transport phenomena or population dynamics. They are also called hereditary or with memory, deviating arguments, aftereffects, post actions, dead-time, or time-lag (Hammarstrom and Gros, 1980). Time delays exist in various engineering systems such as long transmission lines in pneumatic systems, nuclear reactors, rolling mills, hydraulic systems and manufacturing processes. In economics, delays appear in a natural way due to decisions and effects (investment policy, commodity markets evolution: price fluctuations, trade cycles) are separated by some (needed analysis) time interval. In communication, data transmission is always accompanied by a non-zero time interval between the initiation- and the delivery-time of a message or signal. In other cases, the presence of a delay in a system may be the result of some essential simplification of the corresponding process model.

A famous example of the TDSs can be seen in regulating hot water on the shower. Suppose that someone is under the shower aiming at having a pleasant water temperature T_d. Due to the dynamics, it would take a while until the guy can see the effect of faucet change on the temperature after few seconds. Indeed the person receives the information with a delay and this can cause some unwanted actions like too warm or too cold water, or , if we look at it as a system theory, this may lead the system to an unstable condition. A simple solution to encounter this specific problem, can be to wait for few seconds. However, generally in more complex systems, some more comprehensive might be needed to avoid instability. Let $T(t)$ denote the water temperature in the mixer output and let $\bar{\tau}$ be the constant time needed by the water to go from the mixer output to the person's head . Assume that the change of the temperature is proportional to the angle of rotation of the handle, whereas the rate of rotation of the handle is proportional to $T(t) - T_d$. At time t the person feels the water temperature leaving the mixer at time $t - \tau$, which results in the following equation with the constant delay τ:

$$\dot{T}(t) = -k(T(t-\tau) - T_d), \quad k \in \mathbb{R}. \tag{3.1}$$

Due to its complexity, the problem of stability analysis and control of TDS has attracted much attention during the past years, which is of both practical and theoretical importance. Various types of TDS have been investigated and a great number of results on TDSs have been reported in the literature (see, e.g. Chen and Latchman (1995), Chu (1997), Hui and Hu (1997), Cao et al. (1998a), Su and Chu (1999), Hmamed (2000), Shi et al. (2000), Park (2001),Fridman and Shaked (2002), Lu et al. (2003),Niu et al. (2005), Zhou and Li (2005), Chen et al. (2006), Shi et al. (2007), Chen et al. (2010a), Gouaisbaut and Ariba (2011),Goebel et al. (2011),Chesi et al. (2012), Bekiaris-Liberis and Krstic (2013a,b), Feyzmahdavian et al. (2014), Mazenc and Malisoff (2014), and the references cited therein).

In the following section, we briefly describe a history about TDSs which has been taken mainly from Schoen (1995).

3.2 History

Studying retarded elasticity effects, Boltzman in 1874, presented one of the earliest studies of TDS. His publication, however, did not point out clearly the need of the past states for a realistic modeling of retarded elasticity effects. In the early 1900's a controversy arose over the necessity of specifying the earlier history of a system in order to predict its future evolution. This view stood in contradiction with the Newtonian tradition which claimed that the knowledge of the present values of all relevant variables should suffice for prediction. Picard in 1907 took the view that the past states are important for a realistic modeling. He analyzed a system with essential hidden variables, not themselves accessible to observation. He claimed that the prediction of that system requires also the knowledge of the earlier values of the hidden variables. His paradigm for that situation was a pendulum clock whose descending weight is encased. As long as we cannot observe the present position of the weight and its rate of descent, a prediction of the future motion of the clock hand requires the knowledge of when the clock was last wound. Systematic work with mathematical models on medicine and biology began with the epidemiological studies of Ross in 1911. Ross was laying the equations. His results were extended and improved in the 1920's. The need for delays was emphasized both by Sharpe and Lotka (1978), who discussed the discrete delays due to the incubation times in the Ross malaria epidemic model. From the very beginning of their ecological investigations, Lotka realized that, in order to achieve some degree of realism, delayed effects had to be explicitly taken into account. Lotka's main previous interest had been in physical chemistry, with special emphasis on the oscillations of chemical reactions. He had also dealt with demographic problems and with evolutionary theory. Volterra's previous interests were mostly in mechanics, including irreversible phenomena and elasticity. The latter had led him to develop the theory of functionals and integral-differential equations, for which he became well known. He also attempted to introduce a concept of energy function to study the asymptotic behavior of the system in the distant future. Minorsky (1943) pointed out very clearly the importance of the delay considerations in the feedback mechanism. The great interest in control theory during those and later years has certainly contributed significantly to the rapid development of the theory of differential equations with dependence on the past state.

While it became clear a long time ago that retarded systems could be handled as infinite dimensional problems, the paper of Myshkis (1949) gave the first correct mathematical formulation of the initial value problem. Furthermore he later introduced a general class of equations with delayed arguments and laid the foundation for a general theory of linear systems.

Subsequently, several books appeared which presented the current knowledge on the subject and which greatly influenced later developments. In their monograph at the Rand Corporation (Bellman et al., 1953) pointed out the diverse applications of equations containing past information to other areas such as biology and economics. They also presented a well organized theory of linear equations with constant coefficients and the beginnings of stability theory. A more extensive development of these ideas is contained in the book of Bellman and Cooke (1963). Some important results were supplied also by Krasovskii, who studied stability and optimal control problems for time-delay systems (Krasovskii, 1962). Further important works have been written by Elsgolts and Norkin (1973) and Hale (1971). In recent years several books have been published on this topic (Hino et al., 1991; MacDonald and MacDonald, 2008; Hammarstrom and Gros, 1980; Neudecker and Magnus, 1988; Stépán, 1989).

The stability analysis for TDSs can be divided into two main groups: Eigenvalue-based analysis and Lyapunov or energy-based methods. Eigenvalue-based methods, using the characteristic equation of the system, usually provide a necessary and sufficient conditions under which a TDS remains stable. Basically, these methods are used when a constant delay exist, aiming to find the interval/s in the delay space where the stability of the system is guaranteed. Lyapunov-based methods provide *sufficient* conditions for the stability of TDSs. Even though, it is not always so trivial to find the necessary conditions for the stability of such systems. The Lyapunov based tools are typically used to investigate the stability of such systems. Out of them Lyapunov-Razumikhin theory, Lyapunov-Krasovskii theory are used widely. The Lyapunov based methods can be also classified into two types: delay-dependent and delay-independent stability conditions; the former include the information on the size of the delay, while the latter do not. Generally speaking, delay-independent stability conditions are simpler to apply, while delay-dependent stability conditions are less conservative especially in the case when the time delay is small. The main objectives of the study of the delay-dependent stability problem are:

- to develop delay-dependent conditions to provide a maximal allowable delay as large as possible,
- to develop delay-dependent conditions by using as few as possible decision variables while keeping the same maximal allowable delay.

However, none of these basic concepts represents applicable stability tests in terms of the system matrices. The stability tests obtained can be categorized into four groups, depending on how much information concerning the delays is required for these tests:

- Delay-independent stability criteria: The length of the delay need not be known for the application of these stability tests. The delays may be state-dependent and/or time variable. The only assumption needed is that the delays are continuous and bounded.

- Stability criteria independent of constant delays: In the second group it is assumed that the delays of the system are constant; no further information on the delays is necessary.

- Stability criteria independent of a delay constant: This type of stability criteria presumes that the delays are constant and that the ratios of size of the delays are known.

- Delay-dependent stability criteria: This group includes exact algebraic stability criteria depending on the delay and on the system constants and stability criteria which yield an upper bound of the admissible delay. The need for delay-independent (and related) stability tests is obvious, since in practice the delays are difficult to estimate, especially those that are time variable and state dependent. While algebraic stability tests independent of delays are suitable to apply, exact algebraic stability conditions depending on the delay and the system constants are known only in some special cases. In this context a method is presented to achieve some extensions. The method permits the investigation of the stability of systems which are general enough to demonstrate the differences among the four types of stability tests. The stability of general, linear time-delay systems, however, can be checked exactly only by eigenvalue considerations.

In the literature, various approaches have been proposed to obtain delay-dependent stability conditions, among which the linear matrix inequality (LMI) approach is the most popular and has played an important role due to the fact that LMIs can be cast into a convex optimization problem which can be handled efficiently by resorting to recently developed numerical algorithms for solving LMIs (Boyd et al., 1994). Another reason that makes LMI conditions appealing is their frequent readiness to solve the corresponding synthesis problems once the stability (or other performance) conditions have been established, especially when state feedback is employed.

In the following section, we study different methods of analyzing TDSs.

3.3 Stability analysis

Before stating these methods, some notations must be introduced.

For a given scalar $\bar{\tau} > 0$, let $\mathcal{C}_n = \mathcal{C}([-\bar{\tau}, 0], \mathbb{R}^n)$ be the Banach space of continuous vector functions mapping the interval $[-\bar{\tau}, 0]$ into $\mathbb{R}^n$. For any $\phi \in \mathcal{C}^n$, its norm is defined by

$$\|\phi\|_c = \sup_{-\bar{\tau} \leq s \leq 0} \|\phi(s)\|, \tag{3.2}$$

where $\|\phi(s)\|$ denotes the Euclidean norm of $\phi(s) \in \mathbb{R}^n$. Define a set

$$\mathcal{C}_n^a = \{\phi \in \mathcal{C}_n | \|\phi\|_c < a\},$$

for some $a > 0$.

In a general form, a TDS can be illustrated by the following differential- difference-difference equation:

$$\dot{x}(t) = f(t, x_t), \ t \geq t_0, \tag{3.3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, and x_t is defined as:

$$x_t = x(t + \theta), \ -\bar{\tau} \leq \theta \leq 0.$$

Assume that the function $f : \mathbb{R}^+ \times \mathbb{C}_n \to \mathbb{R}^n$ is continuous and $f(t, 0) = 0$ holds for all $t \in \mathbb{R}$. The initial condition of the system is given by the following equation:

$$x_{t_0}(\theta) = \phi(\theta), \ -\bar{\tau} \leq \theta \leq 0. \tag{3.4}$$

We assume that for any $\phi \in \mathcal{C}_n$ and for any $t_0 \in \mathbb{R}$, the system in (3.3) with the initial condition (3.4) has a unique solution. We also assume that $f(t, 0) = 0$, which guarantees that (3.3) possesses a *trivial solution* $x(t) \equiv 0$.

Eigenvalue based methods

In this part, we discuss linear TDSs, characteristic equations, and location of eigenvalues of the system, as well as effects of delays on stability.

Consider a scalar retarded TDS

$$\dot{x}(t) = ax(t) + bx(t - \bar{\tau}), \tag{3.5}$$

with real constant coefficients and constant delay $\bar{\tau} > 0$. Substituting $x(t) = e^{st}$ into (3.5) we find that the solution satisfies the equation if s is the root of the characteristic equation

$$\Delta(s) = s - a - be^{-\bar{\tau}s} \tag{3.6}$$

Dissimilar to systems without delays, the transcendental equation $\Delta(s) = 0$ generally, has an infinite number of solutions. This also reflects the infinite-dimensional nature of TDSs. However, since $\Delta(s)$ is an entire function,[1] it cannot have an infinite number of zeros within any compact set $|s| \leq M, \forall M > 0$. Therefore, most of the characteristic roots go to infinity. To understand the location of the characteristic roots, i.e., of the solutions of the characteristic equation we note that

$$|s| \leq |a| + |b|e^{-\bar{\tau}Re(s)} \tag{3.7}$$

When $|s| \to \infty$, the left-hand side of the above equation approaches to ∞, thus, the right-hand side, i.e., $e^{-\bar{\tau}Re(s)}$ approaches infinity as well. This means that

$$\lim_{|s| \to \infty} Re(s) = -\infty.$$

Hence, $\forall \alpha \in \mathbb{R}$ there is a finite number of characteristic roots with real parts greater than α. Therefore, the location of the characteristic roots has a nice property

[1] In complex analysis, an *entire function*, also called an integral function, is a complex-valued function that is holomorphic over the whole complex plane. Typical examples of entire functions are polynomials and the exponential function, and any sums, products and compositions of these, such as the trigonometric functions sine and cosine and their hyperbolic counterparts sinh and cosh, as well as derivatives and integrals of entire functions such as the error function.

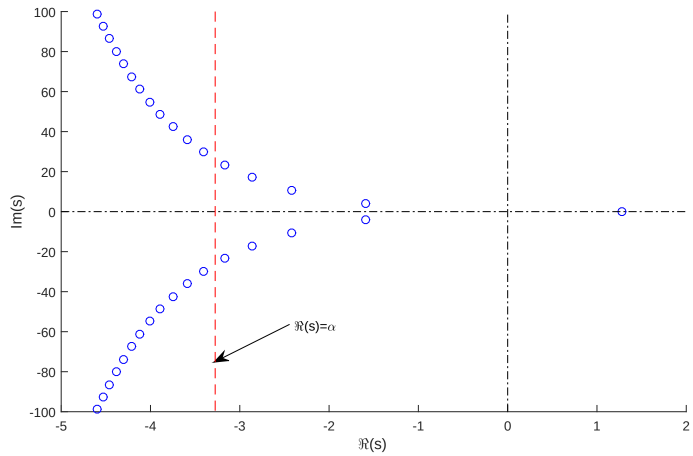


Figure 3-1: Roots of characteristics equation.

that the number of the roots on the right hand side of any vertical line, is finite. Figure 3-1 depicts this property.

An LTI system with N discrete delays and with a distributed delay has a form:

$$\dot{x}(t) = \sum_{k=0}^{N} A_k x(t - \tau_k) + \int_{-\tau_d}^{0} A(\theta) x(t + \theta) d\theta, \tag{3.8}$$

with the initial condition

$$x(t_0 + \theta) = \phi(\theta), \quad \theta \in [\bar{\tau}, 0], \quad \phi \in \mathcal{C}[\bar{\tau}, 0], \tag{3.9}$$

where $0 = \tau_0 < \tau_1 < \ldots < \tau_N, x(t) \in \mathbb{R}$, A_k are constant matrices and $A(\theta)$ is an integrable matrix function, and $\bar{\tau} = \max\{tau_d, \tau_N\}$. The characteristic equation of this system is given by

$$\det \left[sI - \sum_{k=0}^{N} A_k e^{-s\tau_k} - \int_{-\tau_d}^{0} A(\theta) e^{s\theta} \right] = 0. \tag{3.10}$$

Equation (3.10) is transcendental having infinite number of roots. Similar to what was said about the scalar case in (3.5), since here the left hand side of (3.10) is an entire function, it cannot have an infinite number of zeros within any compact set $|s| < M, \forall M > 0$. The LTI system has exponential solutions of the form $e^{st}\nu$, where s is a characteristic root and $\nu \in R^n$ is an eigenvector of the matrix inside the determinant in (3.10). The latter can be verified by substituting $e^{st}\nu$ into (3.10). Moreover, if s is a characteristic root of multiplicity m, then $t^m e^{st}\nu$ is the solution of (3.10). Hence, solutions of (3.10) are given by $x(t) = \sum_l p_l(t)e^{s_l t}$, where s_l are the characteristic roots and $p_l(t)$, are polynomials.

As mentioned above, the location of the characteristic roots has a nice property: there is a finite number of roots to the right of any vertical line. Using this fact, the following statement holds (Hale, 1993; Bellman and Cooke, 1963).

Theorem 3.1 *(Fridman, 2014) For any $\alpha \in \mathbb{R}$, there are only a finite number of characteristic roots (poles) with real parts greater than α. Let s_i be characteristic roots and $\alpha_0 = max_i \Re(s_i)$. Then $\forall \alpha > \alpha_0$ there exists $K \geq 1$ such that for any $\phi \in \mathcal{C}[-\bar{\tau}, 0]$ the solution of (3.10) with $x_0 = \phi$ satisfies the inequality*

$$|x(t)| \leq K e^{\alpha t} \|\phi\|_c, \quad t \geq 0. \tag{3.11}$$

TDS (3.10) is called exponentially stable if for any $\phi \in \mathcal{C}[-\bar{\tau}, 0]$ there exist $\alpha < 0$ and $K \geq 1$ such that the solution initialized by (3.9) satisfies (3.11).

Corollary 3.1 *Retarded TDS in (3.8) is exponentially stable* iff *all the roots of its characteristic quasi-polynomial in (3.10) have negative real parts.*

Stability of single delay characteristic equation

Note that the for the case of having a single delay, Equation (3.10) becomes

$$\det \left[sI - \sum_{k=0}^{1} A_k e^{-s\tau_k} \right] = 0, \tag{3.12}$$

where $\tau_0 = 0$ and $\tau_1 = \tau > 0$. By manipulating, one can get the following quasi-polynomial equation

$$L(s) = P(s) + Q(s)e^{-s\tau} = 0, \tag{3.13}$$

where P and Q are polynomials

$$P(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0,$$

$$Q(s) = b_m s^m + b_{m-1}s^{m-1} + \ldots + a_0, \quad n > m.$$

It is assumed that P and Q have no common imaginary roots $j\omega \; \forall\omega \in \mathbb{R}$ (otherwise $L(j\omega) = 0$), and that $a_0 + b_0 \neq 0$ (otherwise $L(0) = 0$). The key property of the quasi-polynomial (3.13) is the continuity of its roots as functions of positive τ. This means that as τ changes, the characteristic roots may transit from the LHP to the RHP (i.e., become unstable) and vice versa (i.e., become stable) by crossing the imaginary axis only. Thus, the analysis steps are as follows: locate the roots of $P(s) + Q(s)$, increase τ and check for the imaginary axis crossings of roots (for the corresponding crossing frequencies ω_c).

If at some τ roots of $L(s)$ cross the imaginary axis, we have $P(j\omega) + Q(j\omega)e^{-j\omega\tau} = 0$ and, thus, $P(j\omega) \neq 0$ since otherwise $P(j\omega) = Q(j\omega) = 0$, which contradicts the assumption that P and Q have no common imaginary roots $j\omega$ for all $\omega \in \mathbb{R}$. Hence

$$-Q(j\omega)/P(j\omega) = e^{j\omega\tau}$$

This leads to phase equations as follows

$$\omega\tau = \arg(-\frac{Q(j\omega)}{P(j\omega)}) + 2\pi k, \quad k = 0, 1, \ldots, \tag{3.14}$$

and the magnitude equation as

$$|P(j\omega)|^2 - |Q(j\omega)|^2 = 0, \tag{3.15}$$

respectively, where (with no loss of generality) we assume that $\arg(.) \in [0, 2\pi)$. The magnitude equation (3.15) is delay-independent and can be rewritten as

$$P(j\omega)P(-j\omega) - Q(j\omega)Q(-j\omega) = 0 \tag{3.16}$$

which is polynomial equation in ω^2. As a consequence, a finite number of crossing frequencies may be determined by solving this equation. It is clear that for any positive real solution ω_c of (3.15) there always exists a $\tau > 0$ (actually, a family of delays of the form $\tau_0 + \frac{2\pi}{\omega_c}k$) such that (3.14) holds for $\omega = \omega_c$ as well. If there are no positive real solutions of (3.15), no poles migrate from left to right or vice versa as τ varies and the stability (or instability) of the roots of (3.13) is delay-independent (does not depend on τ). Thus, if (for $\tau = 0$) $P(s) + Q(s)$ is stable and

$$\left|\frac{Q(j\omega)}{P(j\omega)}\right| < 1, \quad \forall\omega > 0 \tag{3.17}$$

the characteristic quasi-polynomial is delay-independently stable. Note that

$$\left|\frac{Q(j\omega)}{P(j\omega)}\right| > 1, \quad \forall\omega > 0 \tag{3.18}$$

does not hold since $n > m$. Another possibility for delay-independent stability is the stability of $P(s)$ (corresponding to $\tau = \infty$) together with (3.17).

Now, we introduce some useful definitions that determine the behavior of the roots and in turn the stability of an LTI TDS.

Definition 3.1 *Root tendency (RT): At each crossing frequency ω_c, is defined as*

$$RT = sign\left(\mathrm{Re}(\frac{ds}{d\tau})\right) \tag{3.19}$$

■

Indeed, RT indicates that the root loci of (3.13) tends to either LHP ($RT < 0$) or to RHP ($RT > 0$) at the crossings when τ increases.

Definition 3.2 *(Fridman, 2014) The sensitivity function is defined as*

$$\sigma(\omega_c) = \frac{d}{d\omega}\left(|P(j\omega)|^2 - |Q(j\omega)|^2\right)_{\omega=\omega_c}, \quad \omega_c > 0, \tag{3.20}$$

which is independent of τ. ■

Proposition 3.1 *(Fridman, 2014) If* $\sigma(\omega_c) > 0$*, a root crosses the axis from left to right(RT* > 0*); if* $\sigma(\omega_c) < 0$*, a root crosses from right to left(RT* < 0*); and, if* $\sigma(\omega_c) = 0$ *there is a touch of the roots with the imaginary axis.*

Example 3.1 *Consider a scalar TDS*

$$\dot{x}(t) = -x(t - \tau). \tag{3.21}$$

The system without delay is stable and its quasi-polynomial is given by $L(s) = s + e^{-\tau s}$*. Then, the magnitude equation* (3.15) $\omega^2 - 1 = 0$ *has a unique positive solution* $\omega_c = 1$*, where the sensitivity function*

$$\sigma(\omega_c) = \frac{d}{d\omega}[\omega^2 - 1]_{\omega_c} = 2,$$

is positive, which indicates that the characteristic roots crossing at $\omega_c = 1$ *move from LHP to RHP. The phase equation* (3.14) *has the form*

$$\tau_k = \arg(-1/j) + 2\pi k = \pi/2 + 2\pi k, k = 0, 1, \ldots.$$

Therefore, the equation is (exponentially) stable for $\tau \in [0, \frac{\pi}{2})$ *and is unstable for* $\tau > \frac{\pi}{2}$*. Moreover, for each* $k \geq 0$ *two characteristic roots move to RHP at* $\tau = \tau_k$*.*

Now, time consider

$$\dot{x}(t) = -bx(t - \tau), \quad b > 0. \tag{3.22}$$

by changing the time $\bar{t} = bt$ *we get,*$t - \tau = (\bar{t} - bt)/b$*, and denoting* $\bar{x}(t) = x(\bar{t}/b)$*, we arrive at*

$$\dot{\bar{x}}(\bar{t}) = -\bar{x}(\bar{t} - b\tau), \quad b > 0.$$

which is exponentially stable for $b\tau < \frac{\pi}{2}$ *and unstable for* $b\tau > \frac{\pi}{2}$*.*

■

Example 3.2 *Consider a TDS*

$$\dot{x}(t) = -ax(t) - bx(t-\tau), \quad a+b>0. \tag{3.23}$$

The system without delay is stable and its quasi-polynomial is given by

$$L(s) = s + a + be^{-\tau s}.$$

Since $a+b>0$, the system is stable at $\tau = 0$. The magnitude equation has the form

$$\omega^2 + a^2 - b^2 = 0.$$

It may have a nontrivial solution only when $|a| < |b|$, yielding the positive crossing frequency $\omega_c = \sqrt{b^2 - a^2}$. Clearly, this is possible only when either $a > 0$, $b > 0$, or $a < 0$, $b > 0$. Moreover, the sensitivity function

$$\sigma(\omega_c) = \frac{d}{d\omega}[\omega^2 + a^2 - b^2]_{\omega=\omega_C} = 2\omega_c > 0,$$

indicates that the characteristic roots crossing at $\omega_c = \sqrt{b^2 - a^2}$ move to RHP If $a > 0$, $b > 0$,

$$\begin{aligned}
\sqrt{b^2 - a^2}\tau_k &= \arg(-\frac{b}{j\sqrt{b^2 - a^2}} + a) + 2\pi k \\
&= \arg(a - j\sqrt{b^2 - a^2}) + 2\pi k \\
&= \pi - \arccos(\frac{a}{b}) + 2\pi k, \quad k = 0, 1, \ldots.
\end{aligned}$$

As a result, the first crossing happens at

$$\tau_0 = \frac{\pi - \arccos(b/a)}{\sqrt{b^2 - a^2}},$$

which implies the exponential stability for $\tau \in [0, \tau_0)$ and instability for $\tau > \tau_0$.

If $a < 0,\ b > 0$*, then*

$$\sqrt{b^2 - a^2}\tau_k = \arg(-\frac{b}{j\sqrt{b^2 - a^2}} + a) + 2\pi k$$
$$= \arccos(\frac{a}{b}) + 2\pi k, \quad k = 0, 1, \ldots.$$

and thus, the system is stable for $\tau \in [0, \tau_0)$ *and unstable for* $\tau > \tau_0$ *with*

$$\tau_0 = \frac{\arccos(b/a)}{\sqrt{b^2 - a^2}}.$$

On the other hand, the system is delay-independently exponentially stable if and only if $a \geq |b|$ *(provided* $a + b > 0$*), i.e., iff* $a \geq b > 0$ *or* $a > -b \geq 0$*. Indeed, the condition* $a \geq |b|$ *guarantees that*

$$\left|\frac{Q(j\omega)}{P(j\omega)}\right| = \left|\frac{b}{j\omega + a}\right| < 1, \quad \forall\omega > 0$$

. ■

Lyapunov based methods

This section presents generalizations of the direct Lyapunov method to TDSs. First, for general TDSs, the stability notions are defined, and Lyapunov-Krasovskii and Lyapunov-Razumikhin stability theorems are stated. Then delay-independent and delay-dependent stability conditions for linear TDSs are derived. Sufficient conditions are derived in terms of LMIs. Some of the presented ideas may be useful in the nonlinear case and Lyapunov-based necessary stability conditions for LTI retarded TDSs.

Note that the direct Lyapunov method is also called the second Lyapunov method, whereas the first one establishes the stability of a nonlinear system on the basis of the exponential stability of the linearized system. In order to have a better understanding of the notations, following definition of the concept of stability is given

Definition 3.3 *The trivial solution of* (3.3) *is*

- *uniformly (in t_0) stable if $\forall t_0 \in \mathbb{R}$ and $\forall \epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|x_{t_0}\|_c < \delta(\epsilon)$ implies $|x(t)| < \epsilon$ for all $t \geq t_0$;*

- *uniformly asymptotically stable if it is uniformly stable and there exists a $\delta_a > 0$ such that for any $\eta > 0$ there exists a $T(\delta_a, \eta)$ such that $\|x_{t_0}\|_C < \delta_a$ implies $|x(t)| < \eta$ for all $t \geq t_0 + T(\delta_a, \eta)$ and $t_0 \in \mathbb{R}$.*

- *globally uniformly asymptotically stable if δ_a can be an arbitrary large, finite number.*

The system is uniformly asymptotically stable if its trivial solution is uniformly asymptotically stable. ■

Note that the stability notions are not different from their counterparts for systems without delay (Khalil and Grizzle, 2002).

Now, we are in a position to present the method of Lyapunov-Krasovskii functionals.

Theorem 3.2 *Krasovskii Stability Theorem:(Hale, 1971) Suppose that the function f in (3.3) takes bounded sets of C_n in bounded sets of $\mathbb{R}^n$, and u, v, w: $\mathbb{R}^+ \to \mathbb{R}^+$ are continuous, non-decreasing functions satisfying $u(0) = v(0) = 0$ and $u(s), v(s) > 0$ for $s > 0$. If there exists a continuous function $V : R \times C_n \to R^+$ such that*

(a) $u(|x|) \leq V(t, x_t) \leq v(|x_t|_c)$.

(b) The derivative of $V(t, x_t)$ along the solution of (3.3) and (3.4), defined as

$$\dot{V}(t, x_t) = \lim_{s \to 0^+} \sup \frac{1}{s}(V(t+s, x_{t+s}) - V(t, x_t)),$$

satisfies $\dot{V}(t, x_t) \leq w(|x|)$, then the trivial solution $x = 0$ of the time-delay system in (3.3) and (3.4) is uniformly stable.

If $\lim_{s \to \infty} u(s) = \infty$, the solutions of the time-delay system in (3.3) and (3.4) are uniformly bounded.

If $w(s) = 0$ for $s = 0$, then the solution $x = 0$ is uniformly asymptotically stable.

Example 3.3 *(Fridman, 2014) Consider the nonlinear autonomous scalar equation*

$$\dot{x}(t) = -ax^3(t) - bx^3(t-h); \quad a > 0; b \in \mathbb{R}. \tag{3.24}$$

Let $|b| < a$ *and consider the following functional*

$$V(\phi) = \frac{\phi^4(0)}{2a} + \int_{-h}^{0} \phi^6(s)ds,$$

in which, ϕ *is the initial function as defined in* (3.4). *Then*

$$V(\phi) = \frac{x^4(t)}{2a} + \int_{-h}^{0} \phi^6(t+s)ds = \frac{x^4(t)}{2a} + \int_{t-h}^{0} \phi^6(s)ds.$$

derivation gives

$$\begin{aligned} \dot{V}(x_t) &= \tfrac{d}{dt}V(x_t) = \frac{2x^3(t)}{a}\dot{x}(t) + x^6(t) - x^6(t-h) \\ &= -[x^6(t) + \tfrac{2b}{a}x^3(t)x^3(t-h) + x^6(t-h)] \\ &= [x^3(t) \quad x^3(t-h)][\begin{array}{cc} -1 & -\frac{b}{a} \\ -\frac{b}{a} & -1 \end{array}] \leq \alpha |x(t)|^6. \end{aligned}$$

for some $\alpha > 0$. *Thus, the system is delay-independently asymptotically stable if* $|b| < a$. *Note that the linear equation*

$$\dot{x}(t) = -ax(t) - bx(t-h), \quad a + b > 0$$

is delay-independently asymptotically stable iff $|b| < a$. ∎

We now recall one of the widely used theorems in TDSs.

Theorem 3.3 *Razumikhin Stability Theorem:(Fridman, 2014) Suppose that the function* f *in* (3.3) *takes bounded sets of* $\mathcal{C}_n$ *in bounded sets of* $\mathbb{R}^n$ *and suppose that* $u, v, w : \mathbb{R}^+ \times \mathbb{R}^+$ *are continuous, nondecreasing functions,* $u(s)$, $v(s)$, $w(s)$ *are positive*

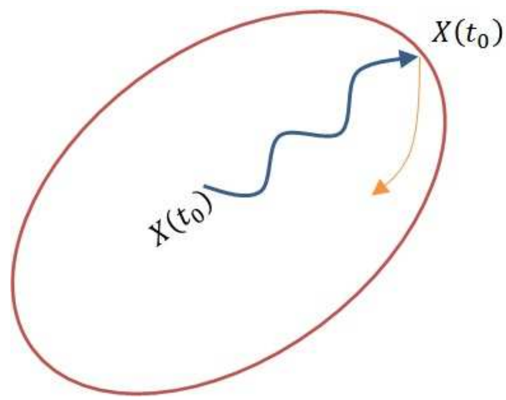


Figure 3-2: The idea of Razumikhin approach

for $s > 0$, $u(0) = v(0) = 0$. *Let* $p : \mathbb{R}^+ \times \mathbb{R}^+$ *be a continuous non-decreasing function satisfying* $p(s) > s$ *for* $s > 0$. *If there exists a continuous function* $V : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ *such that*

(a) $u(|x|) \leq V(t, x) \leq v(|x|), \forall t \in \mathbb{R}, x \in \mathbb{R}^n$.

(b) The derivative of $V(t, x)$ *along the solution of* (3.3) *and* (3.4)*, defined as*

$$\dot{V}(t, x_t) = \lim_{s \to 0^+} \sup \frac{1}{s}(V(t+s, x_{t+s}) - V(t, x_t)),$$

satisfies

$$\dot{V}(t, x_t) \leq w(|x|)$$

if

$$V(t+\theta, x_{t+\theta}) < p(V(t, x_t)), \quad \forall \theta \in [-h, 0].$$

then the trivial solution $x = 0$ *of the time-delay system in* (3.3) *and* (3.4) *is uniformly stable. Furthermore, if* $u(s) \to \infty$ *as* $s \to \infty$*, then the trivial solution is globally uniformly asymptotically stable.*

The idea of the Razumikhin method can be explained as follows for the typical choice of (quadratic) Lyapunov functions of the form $V(x) = x^T P x, \quad P > 0$ (see Fig. 3-2). If a solution begins inside the ellipsoid $V(t+\theta, x_{t+\theta}) = x_{t+\theta}^T P x_{t+\theta} < \delta, \forall \theta \in [-h \;\; 0]$, and not for any $x(t+\theta)$. This guarantees the stability of the system.

So, the solution will not leave the ellipsoid $x^T(t)Px(t) \leq \delta$ if $\frac{d}{dt}V(x(t)) < 0$ along $\dot{x}(t) = f(t, x_t)$ for all $x_t = x(t+\theta)$, $\theta \in [-h \;\; 0]$ satisfying the Razumikhin condition

$$V(x(t+\theta)) \leq V(x(t)), \quad \theta \in [-h \;\; 0]$$

The following theorem also plays an important role in the stability analysis of time-delay systems.

Theorem 3.4 *Halanay theorem(Xu and Lam, 2008) Consider that scalars k_1 and k_2 satisfy $k_1 > k_2 > 0$, and $x(t)$ governed by equation (3.3) is a non-negative continuous function on $[t_0 - \tau, t_0]$ satisfying*

$$\dot{x}(t) \leq -k_1 x(t) + k_2 \bar{x}(t), \tag{3.25}$$

for $t \geq t_0$, where $\tau \geq 0$ and

$$\bar{x}(t) = \sup_{t-\tau \leq s \leq t} x(s)$$

Then, for $t > t_0$, we have

$$x(t) \leq x(t_0)e^{-\alpha(t-t_0)},$$

where $\alpha > 0$ is the unique solution to the following equation:

$$\alpha = k_1 - k_2 e^{\alpha\tau}$$

Both Theorems 3.3 and 3.4 can be used to derive stability conditions for the case when the delay is time-varying, which is continuous but not necessarily differentiable. It is also worth pointing out that Theorem 3-2 can be used to obtain delay-dependent stability conditions for time-delay systems, which will be shown in the next section.

3.4 An LMI Approach to Stability

The direct Lyapunov method for linear ordinary differential equations leads to stability conditions in terms of LMIs. Most of the earlier works on stability of linear systems via Lyapunov method were formulated in terms of Lyapunov equations and algebraic Riccati equations. This is mostly because of the unavailability of efficient numerical algorithms for the general form of LMI. Solutions of some matrix inequalities have appeared in 1960 (see, e.g., Fridman (2014)). The realization that LMI is a convex optimization problem and the development of the efficient interior point method led to formulation of many control problems and their solutions in the form of LMIs (Boyd et al., 1994). The LMI approach is capable to provide the desired stability/performance analysis and design in spite of significant model uncertainties. Among the model uncertainties may be, e.g., uncertain delays.

There are efficient numerical methods to determine whether an LMI is feasible , or to solve a convex optimization problem with LMI constraints. Many optimization problems in control theory, system identification and signal processing can be formulated using LMIs. Also LMIs find application in Polynomial Sum-Of-Squares. The prototypical primal and dual semidefinite program is a minimization of a real linear function respectively subject to the primal and dual convex cones governing this LMI. The solution of LMIs is a part of convex programming. There exist various packages that provide efficient solutions to LMIs, e.g., MATLAB provides an LMI toolbox.

We will review the LMI techniques in deriving stability results for the single-delay case. However, the LMI techniques presented in the following can be extended to the multiple-delay case in a straightforward manner. We consider a class of TDSs with time-varying delays as follows

$$\dot{x}(t) = Ax(t) + Bu(t - \tau(t)) \tag{3.26}$$

$$x(t) = \phi(t), \quad \forall t \in [-\bar{\tau} \quad 0] \tag{3.27}$$

where $x(t) \in \mathbb{R}^n$ is the state; $\phi(t)$ is the continuous initial condition. $\tau(t)$ is the time-varying delay of system (3.26), which is assumed to be continuous and satisfies

$$0 < \tau(t) \leq \bar{\tau}, \tag{3.28}$$

A and B are known real constant matrices.

Note that stability results on (3.26) with a constant delay obtained by the method of Lyapunov-Krasovskii functionals can be easily extended to systems with differentiable time-varying delays. Considering this, time-delay systems with differentiable time-varying delays are not considered, and attention will be focused on the review of the LMI techniques in deriving both delay independent and delay-dependent stability conditions for the time-delay systems with constant and time-varying delay.

Generally, the functionals which are used as candidate Lyapunov ones are achieved by summing up the following terms (Richard, 2003)

$$\begin{aligned}
V_1(x(t)) &= x^T(t)Px(t), \\
V_2(x_t) &= x^T(t)\int_{-\tau_i}^{0} Q_i x(t+\theta)d\theta, \\
V_3(x_t) &= \int_{-\tau_i}^{0} x^T(t+\theta)S_i x(t+\theta)d\theta, \\
V_4(x_t) &= \int_{-\tau_i}^{0}\int_{t+\theta}^{t} x^T(\theta)R_i x(\theta)d\theta ds, \\
V_5(x_t) &= x^T(t)\int_{-\tau_i}^{0} P_i(\eta)x(t+\eta)d\eta, \\
V_6(x_t) &= \int_{-\tau_i}^{0}\int_{\tau_i}^{0} P_i(\eta,\theta)x(\theta)d\eta d\theta,
\end{aligned} \tag{3.29}$$

Delay-Independent Conditions for Linear TDSs

For the time-delay system (3.26) with a constant time delay $\tau(t) = \bar{\tau}$, by choosing a Lyapunov-Krasovskii functional as

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-\bar{\tau}}^{t} x^T(s)Qx(s)ds \tag{3.30}$$

and putting it into Theorem 3.2, the following stability condition can be obtained

Theorem 3.5 *(Richard, 2003) The TDS (3.26) is asymptotically stable if there exist matrices $P > 0$ and $Q > 0$ such that*

$$\begin{bmatrix} PA + A^T P + Q & PB \\ * & -Q \end{bmatrix} < 0. \tag{3.31}$$

Remark 3.1 *Hereafter,* * *in* ij*-element indicates the transpose of the* ji*-element of the same matrix.*

Note that for the general case in (3.26), since the time-varying delay $\tau(t)$ may not be differentiable, the Lyapunov-Krasovskii functional similar to (3.30) as

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(s)Qx(s)ds, \tag{3.32}$$

cannot be used to derive a stability condition. If we suppose that $\tau(t)$ is a differentiable function with $\dot{\tau}(t) < d_\tau < 1$, Fridman (2014) provided the following theorem.

Theorem 3.6 *The TDS* (3.26) *is asymptotically stable if there exist matrices* $P > 0$ *and* $Q > 0$ *such that*

$$\begin{bmatrix} PA + A^T P + Q & PB \\ * & -(1 - d_\tau)Q \end{bmatrix} < 0. \tag{3.33}$$

If $\tau(t)$ is not differentiable, however, we can use Theorem 3.3 to give a delay independent stability condition. Here, we choose a Lyapunov function as

$$V(t, x_t) = x^T(t)Px(t), \tag{3.34}$$

By setting the following conditions

$$p(s) = \delta s, \quad w(s) = \epsilon s, \tag{3.35}$$

where $\delta > 1$ and $\epsilon > 0$ are scalars, we have the following result.

Theorem 3.7 *TDS in (3.26) is asymptotically stable if there exists a matrix $P > 0$ such that*

$$\begin{bmatrix} PA + A^T P + P & PB \\ * & -P \end{bmatrix} < 0. \tag{3.36}$$

It is easy to see that the LMI condition in Theorem 3.7 is a special case of that in Theorem 3.5. Thus, Theorem 3.7 is more conservative than Theorem 3.5. However, it is worth pointing out that Theorem 3.7 can be applied to the case when the delay is time-varying and continuous, which may not be differentiable, while in the time-varying delay case, the use of Theorem 3.5 usually requires the considered delay being differentiable.

Now, we introduce some useful inequalities for TDS. Notice that from (3.26) one has $\dot{x}(t) - Ax(t) + Bu(t - \tau(t)) = 0$. Therefore, for any matrices Y, W and S with appropriate dimensions, the following equalities hold:

$$\dot{x}^T(t)Y[\dot{x}(t) = Ax(t) + Bu(t - \tau(t))] = 0 \tag{3.37}$$

$$x^T(t)W[\dot{x}(t) = Ax(t) + Bu(t - \tau(t))] = 0 \tag{3.38}$$

$$x^T(t - \tau(t))S[\dot{x}(t) = Ax(t) + Bu(t - \tau(t))] = 0 \tag{3.39}$$

Indeed the above equations add some degrees of freedom to the equations and provide a wider decision variable space. By noting these and using the Lyapunov function (3.34), we can obtain the following delay-independent stability result for the time-delay system (3.26) and (3.27).

Theorem 3.8 *The time-delay system (3.26) is asymptotically stable if there exist matrices $P > 0$, Y, W, and S such that*

$$\begin{bmatrix} WA + A^T W^T & WB + A^T S^T & A^T Y^T + P - W \\ * & SB + B^T S^T & B_1^T Y^T - S \\ * & * & -Y - Y^T \end{bmatrix} < 0 \tag{3.40}$$

Delay-dependent stability conditions

In this section we consider the linear TDSs (3.26) and (3.27). The feasibility of the delay-independent conditions in (3.33) and (3.36) implies that A and $A \pm B$ are Hurwitz. It means that these conditions cannot be applied for stabilization of unstable plants by a feedback with delay. For such systems, the stability depends on the delay.

In this section, LMI techniques in deriving delay-dependent stability conditions will be reviewed. Generally, these techniques can be divided into two main groups, i.e., the model transformation techniques and the bounding techniques. The aim of using these transformations and boundings is to achieve some LMIs that are dependent on delay. Now, we introduce some usual transformation and bounding techniques.

Transformations and boundings

One of the most used techniques in delay-dependent LMIs is Newton-Leibniz transformation. Using Newton-Leibniz formula, one gets

$$x(t-\bar{\tau}) = x(t) - \int_{t-\bar{\tau}}^{t} \dot{x}(s)ds,$$

$$= x(t) - \int_{t-\tau(t)}^{t} [Ax(s) + Bx(t-\tau(t))]ds$$

Replacing $x(t-\bar{\tau})$ in (3.26) gives us

$$\dot{x}(t) = (A+B)x(t) - B\int_{t-\tau}^{t} [Ax(s) + Bx(t-\tau)]ds \tag{3.41}$$

Note that the asymptotic stability of the TDS in (3.41) implies that of the system in (3.26) and (3.27). For this reason, we now turn to study the stability of (3.41). For a constant time-delay $\tau = \bar{\tau}$, we choose a Lyapunov-Krasovskii functional candidate as follows:

$$V(t,x_t) = x^T(t)P^{-1}x(t) + \int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} x^T(s)B^TQ_1^{-1}Bx(s)dsd\theta \\ +x^T(t)P^{-1}x(t) + \int_{-\bar{\tau}}^{0}\int_{t-\bar{\tau}+\theta}^{t} x^T(s)B^TQ_2^{-1}Bx(s)dsd\theta \tag{3.42}$$

in which $P, Q_1, Q_2 > 0$. Then, by Theorem 3.2, the stability condition for (3.41) is obtained in the following theorem.

Theorem 3.9 *(Cao et al., 1998b) The TDS in (3.41) is asymptotically stable for any delay τ satisfying $0 < \tau \leq \bar{\tau}$ if there exist matrices $P > 0$, $Q_1 > 0$ and $Q_2 > 0$ such that*

$$\begin{bmatrix} \Psi & \bar{\tau}PA^T & \bar{\tau}PB^T \\ * & -Q_1 & 0 \\ * & * & -Q_2 \end{bmatrix}, \tag{3.43}$$

where $\Psi = (A+B)P + P(A+B)^T + B(Q_1+Q_2)B^T$.

Using the Lyapunov function in (3.34) a system with a time varying delay $\tau(t)$, the following result is achieved.

Theorem 3.10 *(Cao et al., 1998a) The time-delay system in (13) is asymptotically stable for any delay $\tau(t)$, satisfying $0 < \tau(t) < \bar{\tau}$ if there exist matrices $X_1, X_2, X_3 > 0$ such that*

$$(A+B)X_1 + X_1(A+B)^T + \bar{\tau}B(X_2+X_3)B^T + 2\bar{\tau}X_1 < 0, \\ \begin{bmatrix} X_1 & X_1A^T \\ * & X_2 \end{bmatrix} \geq \\ \begin{bmatrix} X_1 & X_1B^T \\ * & X_3 \end{bmatrix} \geq 0.$$

By the Newton-Leibniz formula, we can also change system (3.26) to

$$\dot{x}(t) = (A + B)x(t) - B \int_{t-\bar{\tau}}^{t} \dot{x}(s)ds, \tag{3.44}$$

and

$$\frac{d}{dt}\left[x(t) + B \int_{t-\bar{\tau}}^{t} x(s)ds\right] = (A + B)x(t.) \tag{3.45}$$

Remark 3.2 *All the time-delay systems in (3.41), (3.44), and (3.45) are transformed from the time-delay system in (3.26) by using the Newton-Leibniz formula. However, all of them are not equivalent to (3.26). Compared with (3.26), additional dynamics are introduced in (3.41), (3.44), and (3.45) (Gu (2000); Kharitonov and Melchor-Aguilar (2003a,b)), which may cause conservatism as the delay-dependent conditions are derived based on the transformed systems.*

One of the main purposes in the study of delay-dependent stability for time-delay systems is to find methods to reduce conservatism of existing delay-dependent stability conditions. It is known that the finding of better bounds on some weighted cross products arising in the analysis of the delay-dependent stability problem plays a key role in reducing conservatism. Note that the delay-dependent stability results reported by Li and De Souza (1997) and Cao et al. (1998a,b) were obtained by using the well-known inequality on upper bound for the inner product of two vectors, that is,

$$-2a^T b \leq a^T X a + b^T X^{-1} b, \tag{3.46}$$

where $a, b \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n \times n}$. In order to reduce the conservatism in the delay-dependent stability results of Li and De Souza (1997) and Cao et al. (1998a,b), an improved inequality was proposed by Park (1999) which is re-stated as follows:

Lemma 3.11 *(Park's Inequality)(Park, 1999) Assume that $a(\alpha) \in \mathbb{R}^{n_a}$, and$b(\alpha) \in \mathbb{R}^{n_b}$ are given for $\alpha \in \Omega$. Then, for any $X \in \mathbb{R}^{n_a \times n_a}$ with $X > 0$ and any matrix $M \in \mathbb{R}^{n_a \times n_a}$, we have*

$$
\begin{aligned}
&-2\int_{\Omega} a^T(\alpha)b(\alpha)d\alpha \\
&\leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & XM \\ M^TX & (M^TX+I)X^{-1}(M^TX+I)^T \end{bmatrix} \\
&\times \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha
\end{aligned} \tag{3.47}
$$

Now, we present another important inequality, which is also effective in the derivation of delay-dependent stability conditions.

Lemma 3.12 *Jensen's Inequality (Gu, 2000): For any constant matrix $M \in \mathbb{R}^{m\times m}$ with $M > 0$, scalars $b > a$, vector function $\omega : [a, b] \to R^m$ such that the integrations in the following are well-defined, then*

$$
\begin{aligned}
&(b-a)\int_a^b \omega^T(\beta)M\omega(\beta)d\beta \\
&\geq \left[\int_a^b \omega(\beta)d\beta\right]^T M \left[\int_a^b \omega(\beta)d\beta\right]^T.
\end{aligned} \tag{3.48}
$$

3.5 Conclusions

In this chapter, we reviewed the stability analysis of TDSs. Eigenvalue-based methods give quite precise and satisfying results when the delay is constant and the system is LTI. However, when the delay becomes time-varying these methods cannot be used easily. Instead, Lyapunov-based methods can provide some sufficient conditions for the stability of TDSs. Nevertheless, one has to use these methods is most cases.

Among the Lyapunov based-methods, the delay-independent ones are usually cannot be straightforwardly achieved. Many researches have been devoted to improve the conservativeness of the delay-dependent methods at the expense of increased complexity of the resulting LMIs.

Due to its importance in our work, in the next chapter, we separately study sampled-data systems as a special case of time-varying TDSs.

4

Sampled-data systems

"Equipped with his five senses, man explores the universe around him and calls the adventure Science."

– Edwin Powell Hubble, The Nature of Science, 1954

In this chapter we consider sampled-data systems (SDS) with zero-order hold (ZOH). We start with preliminaries on main approaches to sampled-data control. We also review some recent time-dependent Lyapunov functionals in the framework of the delayed system approach. Indeed, the corresponding TDS to SDS can be considered as a system with a piecewise-continuous time-varying delay.

Introduction

SDSs have been extensively studied over the past years (Chen and Latchman (1995); Fridman (2010); Liu and Fridman (2012) and the references therein). Two main approaches have been used for the sampled-data control of linear uncertain systems leading to conditions in terms of Linear Matrix Inequalities (LMIs) (Boyd et al., 1994). The first one is the input delay approach, where the system is modeled as a continuous-time system with the delayed control input (Miheev et al., 1988). The second approach is based on the representation of the sampled-data system in the form of impulsive model (see e.g., Hespanha et al. (2008)). The input delay approach became popular in the networked control systems literature, being applied via time-independent Lyapunov-Krasovskii functionals or Lyapunov-Razumikhin functions to analysis and design of linear uncertain systems under uncertain sampling with the known upper bound on the sampling intervals (Fridman et al., 2004; Gao et al., 2008). In this chapter, we are going to focus on the delayed system approach.

Modern control employs digital technology for implementation. SDSs are dynamical systems that involve both a continuous-time dynamics and a discrete-time control. Consider the linear system

$$\dot{x}(t) = Ax(t) + \bar{B}u(t) \tag{4.1}$$

where A and $\bar{B}$ are constant matrices. The control signal is assumed to be generated by a zero-order hold (ZOH) function

$$u(t) = u(t_k), \quad t_k \leq t \leq t_{k+1} \tag{4.2}$$

with a set of hold times $\{t_0, t_1, \ldots\}$

$$0 < t_0 < t_1 < \ldots < \lim_{k\to\infty} t_k = \infty, \tag{4.3}$$

where u_d is a discrete-time ZOH control signal. The sampling interval can be either constant $t_{k+1} - t_k = \bar{\tau}$ or variable with time-varying $t_{k+1} - t_k = \bar{\tau}$. In the context of NCSs (e.g., due to packet dropout) the sampling interval may be variable and uncertain. Hereafter, we assume that the samplings happen in a bounded time, i.e.,

$$t_{k+1} - t_k \leq \bar{\tau}, \quad \bar{\tau} \in \mathbb{R}, k \in \mathbb{Z}_+.$$

Consider a state-feedback controller $u(t_k) = Kx(t_k)$. Regarding (4.1) and (4.2), we arrive at

$$\dot{x}(t) = Ax(t) + Bx(t_k), \quad t_k \leq t < t_{k+1}, \tag{4.4}$$

where $B = \bar{B}K$. For the periodic sampling case with $t_{k+1} - t_k = T$, the solution is achieved

$$x(t) = e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-\theta)}Bx(t_k)d\theta, \quad t_k \leq t < t_{k+1}, k \in \mathbb{Z}^+. \tag{4.5}$$

Finding the value of $x(t_{k+1})$ leads us to the following discrete-time system

$$x(t_{k+1}) = Dx(t_k), \quad D = e^{AT} + \int_0^T e^{A\bar{\tau}}Bd\theta, \quad k \in \mathbb{Z}^+, \tag{4.6}$$

System (4.4) is asymptotically stable iff the eigenvalues of D are located inside the unitary circle (Schur stable matrix). Under variable sampling, the closed-loop system (4.4) is converted into a linear time-varying discrete-time system

$$x(t_{k+1}) = D_k x(t_k), \quad D_k = e^{AT_k} + \int_0^{T_k} e^{AT_k}Bd\theta, \quad k \in \mathbb{Z}_+, \tag{4.7}$$

Assuming $T_k = t_{k+1} - t_k \leq \bar{\tau}$, the following bound follows from (4.6):

$$|x(t)| \leq \gamma|x(t_k)|, \quad t_k \leq t \leq t_{k+1}, \quad k \in \mathbb{Z}_+ \tag{4.8}$$

where

$$\gamma = \max_{\theta\in[0,\bar{\tau}]} |e^{A\theta}| + \max \lim_{\theta\in[0,\bar{\tau}]} \int_0^{\theta} e^{A\zeta}d\zeta,$$

Therefore, the stability of the discrete-time linear system (4.7) is equivalent to the stability of the continuous-time system (4.4).

4.1 Stability analysis

SDS in (4.4) can be considered as a continuous-time system with a piecewise-linear time-varying delay as (3.3)

$$\dot{x}(t) = Ax(t) + \bar{B}x(t-\tau(t)), \tau(t) = t - t_k, \quad t_k \leq t < t_{k+1}. \tag{4.9}$$

See Fig. 4-1 for the plot of a sawtooth delay corresponding to a variable sampling.

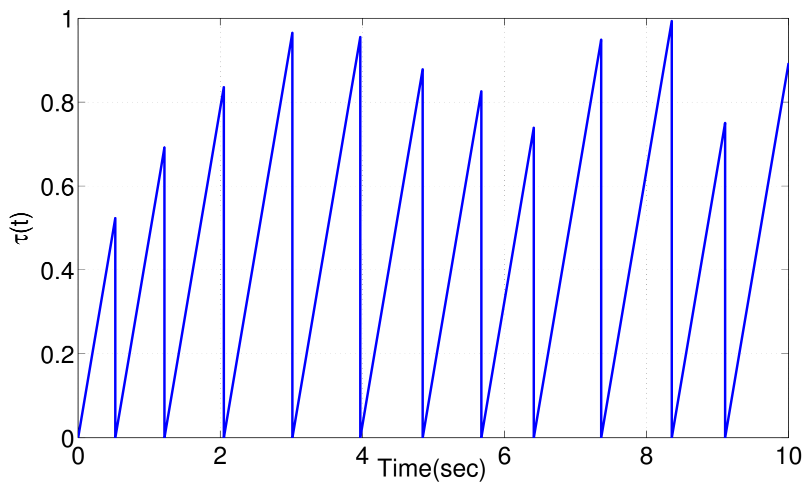


Figure 4-1: Looking an SDS systems as a time-varying TDS with $\bar{\tau} = 1$.

As for the general time-varying delay with $\tau(t) \leq \bar{\tau}$, if the LTI system without delay (i.e., the continuous-time system) is asymptotically stable, then for small enough $\bar{\tau}$ the sampled-data system preserves the stability.

Example 4.1 *Consider a simple system as follows*

$$\dot{x}(t) = -x(t_k), \quad t_k \leq t < t_{k+1}, k = 0, 1, \ldots. \tag{4.10}$$

The corresponding continuous-time system $\dot{x}(t) = -x(t)$ *is exponentially stable. It is well known (see Fridman and Shaked (2003); Fridman (2014)) that the equation* $\dot{x}(t) = -x(t - \tau(t))$ *with a constant delay* τ *is asymptotically stable for* $\tau < \frac{\pi}{2}$ *and unstable for* $\tau > \frac{\pi}{2}$*, whereas for the fast varying delay it is stable for* $\tau(t) < 1.5$ *and there exists a destabilizing delay with an upper bound greater than* 1.5.

For the constant periodic sampling case, D *in the corresponding discrete-time system* (4.6) *is given by* $D = 1 - T$*. Since the eigenvalues of* D *must be inside the unitary circle, the system remains asymptotically stable for all constant samplings less than 2 and becomes unstable for samplings greater than 2. Consider now the variable sampling with* $t_{k+1} - t_k = T_k$*, where the corresponding discrete-time system is given by* (4.6) *with* $D_k = 1 - T_k$*. For any small* $\epsilon > 0$ *and* $T_k \leq 2 - \epsilon$ *we have* $|D_k| = |1 - T_k| \leq 1 - \epsilon$*. Hence, the discrete-time (and, thus, the continuous-time*

SDS) system is asymptotically stable for $T_k \leq 2 - \epsilon \;\; \forall \epsilon > 0$. ■

In the above example, the maximum interval for the sampling that preserves the asymptotic stability is the same under the constant and the variable sampling intervals. Usually a maximum upper bound on the uncertain variable sampling that preserves the stability is smaller than the one for the constant sampling.

Example 4.2 *(Constant and time-varying sampling)*

Consider System in (4.4) with

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -6 \\ -0.6 & -3.6 \end{bmatrix}.$$

Note that, for a constant delay, τ, if $x(t_k)$ is changed by $x(t-\tau)$, the above closed-loop system is asymptotically stable for the constant delay $\tau < 0.19$ and becomes unstable for $\tau > 0.19$ (using the phase (3.14) and the magnitude equation (3.15), and considering that all the eigenvalues of the system must be located in the LHP). In the case of a constant sampling, the equivalent discrete-time system is asymptotically stable for the constant sampling interval $t_{k+1} - t_k = T$ for $T \in [0 \;\; 0.5937]$. Therefore, for the constant sampling intervals $T_1 = 0.18$ or $T_2 = 0.54$ the system is asymptotically stable (see Figures 4-2, 4-3). However, if we sample using a sequence of sampling intervals $T_1 \to T_2 \to T_1 \to \ldots$ the system becomes unstable (see Figure 4-4 with the plot of the state).

In the second case, the equivalent discrete-time system over two sampling instants can be presented as

$$x_{k+2} = D_{k+1} D_k x_k, \quad k = 0, 1, 2, l \ldots,$$

One can see that the system becomes LTV, and therefore the analysis for LTI systems are not valid anymore. Using the Razumikhin approach and convex embeddings, Fiter et al. (2012) found the following upper bound on the variable sampling was achieved $\bar{\tau} = 0.4683$.

■

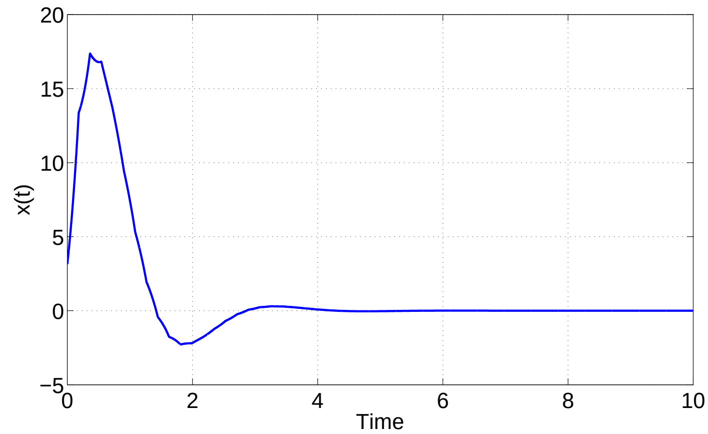


Figure 4-2: The system in Example 4.2 with a constant sampling $T_1 = 0.18$.

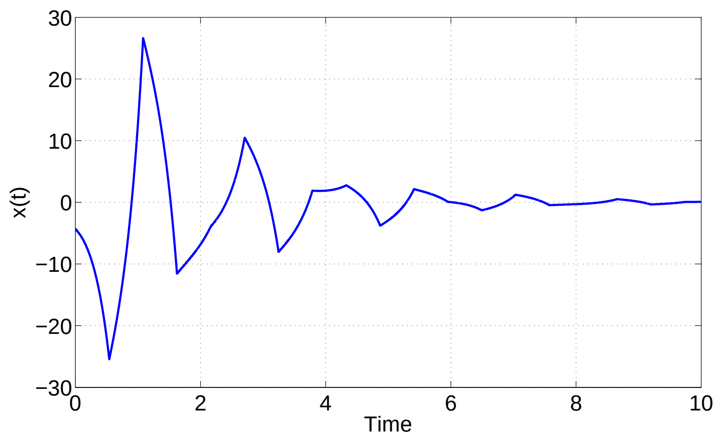


Figure 4-3: The system in Example 4.2 with a constant sampling $T_1 = 0.54$.

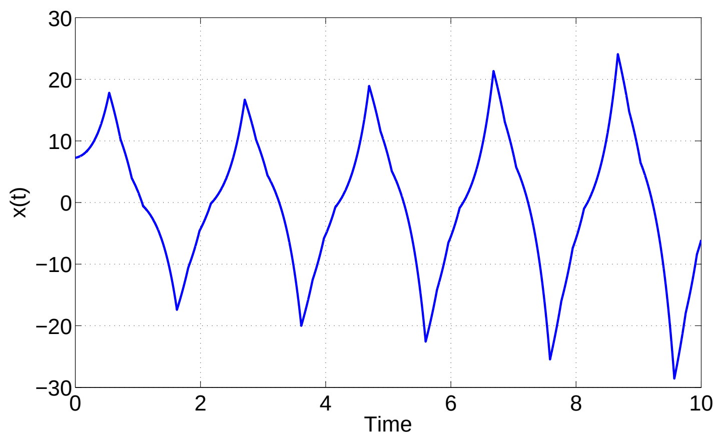


Figure 4-4: The system in Example 4.2 with a switched sampling $T_1 \to T_2 \to T_1 \ldots$.

Generally, three main approaches are used for the SDSs, i.e., the discrete-time, the time-delay, and the impulsive/hybrid system approach.

In the discrete-time approach the system is discretized (Ichikawa and Katayama, 2001). If the SDS is linear time invariant, the discretization is achieved from (4.6) that leads to the discrete-time system (4.6). The advantage of the above discretization is in the simplicity of the stability conditions. Moreover, for LTI systems these conditions are necessary and sufficient for the stability under the constant and known sampling rate. However, it becomes complicated for systems with uncertain matrices or/and uncertain variable sampling period. The main drawback is that discretization loses the knowledge about the inter-sampling behavior. It can hardly be used to performance analysis, to control and tracking of nonlinear systems. A special lifting technique was introduced by Yamamoto (1990) and Bamieh et al. (1991) for sampled-data H_∞ control.

The second approach, converts an SDS to a system with an input delay so that (4.4) is modeled as a continuous-time system (4.9) with the delayed control input (Fridman, 2014; Seuret, 2009). Robust control of SDS was started by Fridman et al. (2004) via Lyapunov-Krasovskii functionals proposed by Fridman and Shaked (2003) for systems with fast-varying delays (here $\dot{\tau} = 1$ almost everywhere). The time-delay approach became popular in NCSs, being applied to uncertain systems under uncertain sampling and network induced delay (Gao et al., 2008; Kim et al., 2010).

The third one is impulsive system approach which has been described by Naghshtabrizi et al. (2008, 2010). In this thesis this approach is not being utilized in this thesis, and we only mention it to complete the discussion.

Consider the augmented system state $\xi(t) = [x^T(t) \;\; u^T(t)]$, and

$$\dot{u}(t) = 0, \quad t \neq t_k, \quad u(t_k) = Kx(t_k^-),$$

With this we arrive at the following impulsive model

$$\begin{aligned} \dot{\xi}(t) &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \xi(t), t \neq t_k, \\ \xi(t) &= \begin{bmatrix} x(t_k^-) \\ Kx(t_k^-) \end{bmatrix}, t = t_k. \end{aligned} \tag{4.11}$$

The impulsive approach was extended to the case of variable sampling with a known upper bound, where a discontinuous Lyapunov function method was introduced (Naghshtabrizi et al., 2008). The latter method improved the existing results, based on the input delay approach via time-independent Lyapunov functionals, and gave an insight to time-dependent Lyapunov functionals suggested by Fridman (2010).

In the next section stability analysis based on Lyapunov functional is discussed.

4.2 Lyapunov based time-dependent methods

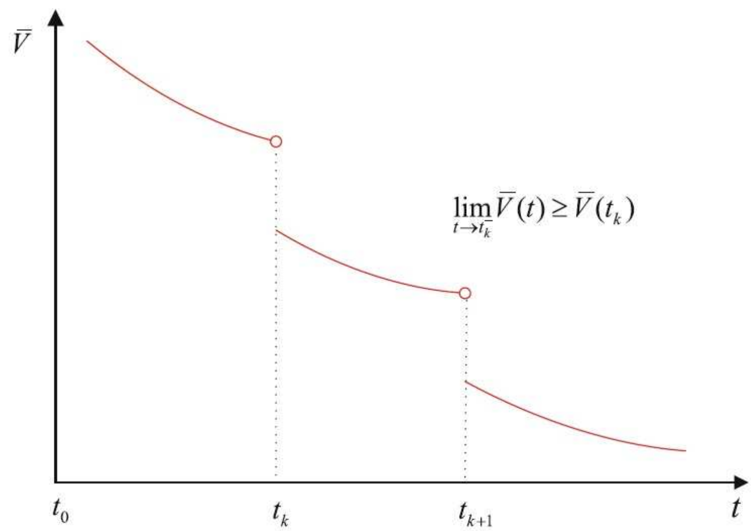
One of the earliest works in this framework is the paper of Fridman et al. (2004), in which Lyapunov functionals for stability analysis of (4.4) with external disturbance and in the case of fast-varying delay is addressed. Naghshtabrizi et al. (2008) introduced a Lyapunov function which depends on t_k for the corresponding finite-dimensional system with jumps. Following Fridman (2010), we employ below a time-dependent Lyapunov functional which may be discontinuous in time, but it is not allowed to grow in the jumps.

Lemma 4.1 *(Fridman, 2014) Consider a general SDS (4.9). Assume that there exist positive numbers α, β and a functional $V : \mathbb{R}^+ \times W[-\bar{\tau}, 0] \times L_2(-\bar{\tau}, 0) \to \mathbb{R}^+$ such that*

$$\alpha|\phi(0)|^2 \leq V(t, \phi, \dot{\phi}) \leq \beta\|\phi\|_W, \tag{4.12}$$

where ϕ indicates the space of functions $\phi : [\bar{\tau}, 0] \to \mathbb{R}$, which are absolutely continuous on $[\bar{\tau}, 0)$, have a finite $\lim_{\theta \to 0^-} \phi(\theta) = 0$, and have square integrable first order derivatives

Figure 4-5: Discontinuous Lyapunov functional



is denoted by $W[a, b)$ with the norm

$$\|\phi\|_W = \max_{\theta \in [a,b]} |\phi(\theta)| + \left[\int_a^b \dot{\phi}^2(s) ds \right]^{1/2}. \tag{4.13}$$

Consider the function $\bar{V}(t) = V(t, x_t, \dot{x}_t)$, which is continuous from the right for $x(t)$ satisfying (4.9), locally absolutely continuous on $t \in [t_k, t_{k+1}), k = 1, 2, \ldots$ and which satisfies

$$\lim_{t \to t_k^-} \bar{V}(t) \geq \bar{V}(t_k), \tag{4.14}$$

This has been shown in Figure 4-5. Given α, if along (4.9)

$$\frac{d}{dt} \bar{V}(t) + 2\alpha \bar{V}(t) \leq 0, \tag{4.15}$$

for almost all t, then (4.9) is exponentially stable with a decay rate α.

In the next section, we introduce the looped-functional method for SDSs.

4.3 Looped functional Method

Looped-functionals have been introduced by Briat and Seuret (2012a), Seuret (2012) and Seuret et al. (2014) for the analysis of sampled-data systems. The main aim was to reformulate a discrete-time condition into another condition devoid of exponential terms, allowing then for the consideration of uncertain time-varying systems and nonlinear systems (Peet and Seuret, 2014). They have been further considered for the analysis of impulsive systems (Briat and Seuret, 2014; Hespanha et al., 2008). The key idea behind the use of looped-functionals is to encode a discrete-time stability condition in a condition that is convex in terms of the matrices of the systems. Due to the convexity property, the resulting conditions can be extended to uncertain systems and linear time-varying systems, unlike the discrete-time stability conditions that are non-convex in the matrices of the system due to the presence of exponential terms.

In the papers of Briat and Seuret (2012b, 2013), the considered looped-functional led to sufficient conditions for the feasibility of a certain discrete-time stability criterion characterizing the stability of impulsive and switched systems. They show here that this very same looped-functional is complete in the sense that the resulting criterion is actually equivalent to the discrete-time stability condition aimed to be represented in a convex way. This result is proved for a larger class of systems, referred to as pseudo-periodic systems, encompassing periodic systems, impulsive systems, sampled-data systems and switched systems, proving then the sufficiency and the necessity of the conditions obtained by Briat and Seuret (2013).

The definition of a looped-functional is given below (Briat and Seuret, 2012a).

Definition 4.1 *(Looped-functional) A functional $f : [0, T_2] \times \mathbb{K}[T_1, T_2] \times [T_1, T_2] \to \mathbb{R}$, where $\epsilon \leq T_1 \leq T_2\ \infty, \epsilon > 0$, is said to be a looped functional if the following conditions are satisfied*

(i) the equality $f(0, z, T) = f(T, \mathcal{Z}, T)$ holds for all functions $\mathcal{Z} \in \mathcal{C}([0, T], \mathbb{R}^n) \subset \mathbb{K}[T_1, T_2]$ and all $T \in [T_1, T_2]$, and

(ii) it is differentiable with respect to the first variable with the standard definition

of the derivative.

The idea for proving stability of (4.9) *is to look now for a positive definite quadratic function, such that the discrete sequence is monotonically decreasing. This is formalized as the following theorem.*

■

Theorem 4.2 *Let* $0 < T_1 \leq T_2$ *be two scalars and* $V : \mathbb{R}^n \to \mathbb{R}^+$ *be a differentiable function for which there exist positive scalars* $\mu_1 < \mu_2$ *and* p *such that*

$$\forall x \in \mathbb{R}^n, \mu_1 |x|^p \leq V(x) \leq \mu_2 |x|^p. \tag{4.16}$$

Then the following statements are equivalent.

(i) The absolute value of the Lyapunov function strictly decreases $\forall k \in \mathbb{N}$ *and* $T_k \in [T_1, T_2]$, *or equivalently*

$$\Delta V(k) = V(x(t_k)) - V(x(0)) < 0$$

(ii) There exists a continuous and differentiable functional $V_0 : [0, T_2] \times \mathbb{K} \to \mathbb{R}$ *which satisfies for all* $z \in \mathbb{K}$

$$\forall T \in [T_1, T_2] \quad V_0(T, z(.)) = V_0(0, z(.)), \tag{4.17}$$

and such that, $\forall t \in [0T_k]$,

$$W_0(\tau, x(t_k)) = \frac{d}{dt}[V(x(t_k)) + V_0(t, x(t_k))] < 0. \tag{4.18}$$

If one of the above statements is satisfied, then the the system in (4.9) *is asymptotically stable.*

Now we introduce Wirtinger's inequality, which has a very important role in determining a less conservative upper bound in Lyapunov based methods in SDSs. This inequality allows the Lyapunov functionals to have a negative term.

4.4 Wirtinger based Lyapunov functionals

In mathematics, historically Wirtinger's inequality for real functions was an inequality used in Fourier analysis. It was named after Wilhelm Wirtinger. It was used in 1904 to prove the isoperimetric inequality. A variety of closely related results are today known as Wirtinger's inequality.

Wirtinger's inequality is an alternative of Jensen's inequality in delay-dependent stability analysis of linear systems with constant discrete and distributed delays or with discrete time-varying delays via Lyapunov functionals.

Lemma 4.3 *(Liu and Fridman, 2012) For all absolutely continuous functions* $\omega : [a,b] \to \mathbb{R}^n$ *with* $\omega \in L_2(a,b)$*. and all* $n \times n$ *matrices* $W > 0$ *the following holds*

$$\int_a^b \omega^T(\theta) W \omega(\theta) d\theta \leq \frac{4(b-a)}{\pi^2} \int_a^b \dot{\omega}^T(\theta) W dot\omega(\theta) d\theta. \tag{4.19}$$

The Wirtinger's inequality can help to decrease the conservativeness of the results in the context of the stability analysis of time delay systems using discrete Lyapunov-Krasovskii functionals. In this way, the following additive term was suggested by Liu and Fridman (2012) for SDSs with a constant communication delay $\eta \in \mathbb{R}^+$:

$$V_W = \bar{\tau}^2 \int_{t_k}^{t} \dot{x}^T(s) W \dot{x}(s) ds - \frac{\pi^2}{4} \int_{t_k-\eta}^{t-\eta} [x(s) - x(t_k - \eta)]^T W [x(s) - x(t_k - \eta)] ds, \tag{4.20}$$

where $W > 0, t_k \leq t \leq t_{k+1}$. According to Wirtinger's inequality, in spite of having a negative term, we get $V_W \geq 0$ for $t_k \leq t \leq t_{k+1}$. By derivation with respect to time, a negative term appears which removes the effect of some positive terms and reduces the conservativeness of the results.

4.5 Conclusions

In this chapter, some Lyapunov based stability conditions for the SDSs were studied. Main discussions in this chapter were based on time-dependent functionals, looped-functionals, and Wirtinger based functionals. The corresponding TDS to SDS was considered as a system with a piecewise-continuous time-varying delay.

5

Consensus in second-order multi-agent systems with time-delay and slow switching topology

"The scientist is not a person who gives the right answers, he's one who asks the right questions."

– Claude Lévi-Strauss

In this chapter, based on the results of Zareh et al. (2013b), we investigate the problem of deriving sufficient conditions for asymptotic consensus of second order multi-agent systems with slow switching topology and time delays. The proposed local interaction protocol is PD-like and the stability analysis is based on the Lyapunov-Krasovskii functional method. Our approach is based on the computation of a set of parameters that guarantee stability under any network topology of a given set. A significant feature of this method is that it does not require to know the possible network topologies but only a bound on their second largest eigenvalue (algebraic connectivity).

5.1 Introduction

As mentions in Chapter 2, in the past years a significant attention has been given to the consensus problem in multi-agent systems due to its broad spectrum of applications to sensor networks, automated highway systems, mobile robotics, satellite alignment and several more. The objective of a consensus algorithm is to drive the state variables of all the agents in a networked system toward a common value. This particular network state is called *consensus* state.

Motivated by the requirement to consider more complex agent dynamics, some researchers now study the consensus problem for second-order systems. This makes the consensus problem more complex and its stability properties depend not only on the interconnection topology, but also on the parameters of the local interaction protocols. In the work of Tian and Liu (2008), the case of heterogeneous multi-agent systems is investigated by means of frequency-domain analysis. Lin and Jia (2009) proposed a control strategy for consensus over a group of agents with discrete-time second-order dynamics, operating under a time delayed communication/sensing structure.

Another challenge of interest is the topology switching problem caused by intermittent and time-varying communication links or sensing capabilities. A switching network topology may result in instability even if all the topologies produce stable systems (Liberzon, 2003; Liberzon and Morse, 1999). Xie and Wang (2006); Jia et al. (2011) investigated second order multi-agent systems with switching topology are .

Despite the considerable number of contributions in second order multi-agent systems where time delays and switching topology are considered separately, to the best of our knowledge very few works have investigated both issues simultaneously.

In this chapter we extend the results of Cepeda-Gomez and Olgac (2011a) which deals with systems with communication/sensing delay but static topology, to systems with delay and slow switching topology. We provide sufficient conditions under which the consensus state is reached by agents modeled by double integrator dynamics affected by a communication/sensing time delay for any network topology with algebraic connectivity greater then a given bound. The proposed method is based on the solution of a set of LMIs and allows to infer stability for slow switching topologies by ensuring the existence of a minimum dwell time. The computation of the minimum dwell time that ensures consensus will be the object of our future research in this topic.

The next sections ar organized as follows. In Section 5.2 the problem statement is formalized. In Section 5.3 sufficient conditions based on LMIs for stability of second-order multi-agent systems with time delays are given. In Section 5.4 the main results are presented. It is a method to solve the LMIs required to infer stability of the networked system in such a way that they are independent from the network topology, thus greatly reducing the computational burden. In Section 5.5 simulations are presented to corroborate the theoretical results. In Section 5.6 concluding remarks are given and future works are discussed.

5.2 Problem statement

Consider a group of n autonomous agents with double integrator dynamics

$$\ddot{x}_i(t) = u_i(t), \qquad i = 1 \cdots n.$$

In the case of mobile robots $x_i \in \mathbb{R}$ can be considered as a scalar position and $u_i \in \mathbb{R}$ as the control law that governs their acceleration.

For simplicity, the motion of each agent is supposed to be one dimensional, but since the protocol makes use of only relative positions and velocities the results that follow can be trivially extended to higher dimensions.

Objective of the control action is to achieve the *consensus* state asymptotically, i.e.,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \quad \forall\, i, j \in V.$$

Each agent i is supposed to exchange information with a subset $\mathcal{N}_i \subset V$ of agents, called *neighbors*. The cardinality of $\mathcal{N}_i$ is denoted δ_i which is referred to as the *degree* of agent i.

Let us assume that all the interactions between the agents have a constant non-null delay τ, thus agent i at the generic time t knows the position and the velocity of its informers at time $t-\tau$.

Finally, we assume a PD-like local interaction control logic that makes the dynamics of the generic i-th agent of the form:

$$\begin{aligned}\ddot{x}_i(t) = u_i(t) = k_p & \left(\sum_{j \in N_i} \frac{x_j(t-\tau)}{\delta_i} - x_i(t) \right) \\ +k_d & \left(\sum_{j \in N_i} \frac{\dot{x}_j(t-\tau)}{\delta_i} - \dot{x}_i(t) \right) \end{aligned} \tag{5.1}$$

where $k_p, k_d \in \mathbb{R}^+$ are design parameters.

Cepeda-Gomez and Olgac (2011a) provided conditions on $k_p, k_d, \tau \in \mathbb{R}^+$ under which, if the network topology is connected, all agents reach consensus. Note that their protocol differs from all previously proposed schemes, e.g., Gao et al. (2009); Luo et al. (2010); Meng et al. (2010) in the fact that the time delay affects the information coming from all the other agents, but not the state of the i-th agent itself.

In the next section, we extend the results of Cepeda-Gomez and Olgac (2011a) and assume that the set of informers may change during the system evolution, namely the topology of the network is time-variant.

The following subsection recalls some equivalence transformations that will be useful in the rest of this chapter (Cepeda-Gomez and Olgac, 2011a).

Equivalence transformations

Let A_d be the $n \times n$ adjacency matrix the elements of which are $a_{ij} = a_{ji} = 1$ if the corresponding edge $(i, j) \in E$ exists and $a_{ij} = a_{ji} = 0$ otherwise. Let Δ be a diagonal $n \times n$ matrix the elements of which are $\Delta_{ii} = \delta_i$ the degrees of the corresponding agents.

The network dynamics of the multi-agent system, where each agent has dynamics given in equation (5.1), can be written in a compact form as:

$$\dot{x}(t) = Ax(t) + Bx(t - \tau) \tag{5.2}$$

where $x(t) = [x_1(t), \dot{x}_1(t), \cdots, x_n(t), \dot{x}_n(t)] \in \mathbb{R}^{2n}$ is the state vector,

$$\begin{aligned} A &= I_n \otimes A', \quad A' = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}, \\ B &= \Delta^{-1} A_d \otimes B', \quad B' = \begin{bmatrix} 0 & 0 \\ k_p & k_d \end{bmatrix}. \end{aligned} \tag{5.3}$$

In (5.3), $\otimes$ denotes Kronecker product, A_d is the adjacency matrix of graph $\mathcal{G}$ and I_n is the n-th order identity matrix.

A_d is a real symmetric matrix. If $\mathcal{G}$ is connected then Δ is invertible and matrix $\Delta^{-1}A_d$, a weighted adjacency matrix, is symmetrizable (Sergienko et al., 2003). Therefore, $\Delta^{-1}A_d$ is diagonalizable and has n linearly independent eigenvectors. Thus, there exists a matrix T such that $T^{-1}(\Delta^{-1}A_d)T = \Lambda$, where Λ is a diagonal matrix whose non-zero entries are the eigenvalues of $\Delta^{-1}A_d$,

$$T^{-1}(\Delta^{-1}A_d)T = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}. \tag{5.4}$$

To achieve a diagonal realization we choose as state transformation $x(t) = (T \otimes I_2)\xi(t)$ in (5.2), with

$$\xi(t) = \left[\xi_1(t), \dot{\xi}_1(t), \cdots, \xi_n(t), \dot{\xi}_n(t)\right].$$

From (5.2) and (5.3), the system dynamics in the new state coordinates becomes:

$$\begin{aligned}\dot{\xi}(t) &= (T^{-1} \otimes I_2)\left(I_n \otimes A'\right)(T \otimes I_2)\,\xi(t) \\ &+ (T^{-1} \otimes I_2)\left(\Delta^{-1} A_d \otimes B'\right)(T \otimes I_2)\,\xi(t-\tau).\end{aligned} \tag{5.5}$$

Using the features of the $\otimes$ operation, we obtain:

$$\dot{\xi}(t) = \left(I_n \otimes A'\right)\xi(t) + \left(\Lambda \otimes B'\right)\xi(t-\tau). \tag{5.6}$$

Since I_n and Λ are diagonal matrices, equation (5.6) represents a set of n decoupled second-order blocks of the form:

$$\dot{y}_i(t) = A' y_i(t) + \lambda_i B' y_i(t-\tau) \tag{5.7}$$

where

$$y_i(t) = [\xi_i(t), \dot{\xi}_i(t)]^T, \qquad i = 1, \ldots, n.$$

Now, from basic integral properties, it holds:

$$\int_{-\tau}^{0} \dot{\xi}_i(s+t)ds = \xi_i(t) - \xi_i(t-\tau)$$

or equivalently

$$\xi_i(t-\tau) = \xi_i(t) - \int_{-\tau}^{0} \dot{\xi}_i(s+t)ds. \tag{5.8}$$

By substituting (5.8) in (5.7) we obtain:

$$\dot{y}_i(t) = \bar{A}_i y_i(t) + \bar{B}_i y_i(t-\tau) + \bar{C}_i \int_{-\tau}^{0} y_i(s+t)ds \tag{5.9}$$

where

$$\bar{A}_i = \begin{bmatrix} 0 & 1 \\ -k_p(1-\lambda_i) & -k_d \end{bmatrix},$$
$$\bar{B}_i = \begin{bmatrix} 0 & 0 \\ 0 & k_d\lambda_i \end{bmatrix}, \quad \bar{C}_i = \begin{bmatrix} 0 & 0 \\ 0 & -k_p\lambda_i \end{bmatrix}. \tag{5.10}$$

Each time function $y_i(t)$ is called a *mode* of the system.

Using Gershgorin circle theorem on matrix $\Delta^{-1}A_d$ it is easy to show that $\lambda_i \in [-1, 1]$ and $\lambda_i = 1$ always exists because the matrix is row stochastic. Henceforth, without loosing generality, in the following we consider $y_1(t)$ as the mode corresponding to $\lambda_i = 1$.

Switching dynamics

We assume that the topology of the network is time-variant, consequently the adjacency matrix and Δ change with time. As a consequence, equation (5.2) can be rewritten as:

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma x(t-\tau) \tag{5.11}$$

where $\sigma : \mathbb{R}_{\geq 0} \to \Omega$ is the switching signal and $\Omega = 1, \cdots, N$ is the index set of all possible topologies.

In the following the subscript σ is used everywhere to make explicit the dependence on σ. As an example the adjacency matrix becomes a function of σ and is denoted as $A_{d,\sigma}$. Analogously, the diagonal matrix Δ becomes Δ_σ, matrices $\bar{A}_i$, $\bar{B}_i$, and $\bar{C}_i$ defined in (5.10) become $\bar{A}_{\sigma,i}$, $\bar{B}_{\sigma,i}$, and $\bar{C}_{\sigma,i}$, respectively. Finally, the i-th mode $y_i(t)$ in equation (5.7) also becomes a function of σ and is denoted $y_{\sigma,i}(t)$.

5.3 Stability analysis

In this section we prove the main result of this chapter. In particular, three are the main steps towards the derivation of conditions on k_p, k_d and τ that guarantee consensus to a common position in a finite point of the state space, under arbitrary switchings, provided that switchings occur sufficiently slowly.

- Firstly, we prove that under appropriate conditions on k_p, k_d and τ, the mode corresponding to the eigenvalue $\lambda_1 = 1$, namely, $y_1(t) = y_{\sigma,1}(t)$, common to all topologies, regardless of the switching signal σ, is a non oscillating stable mode.

- Secondly, we prove that under appropriate conditions on k_p, k_d and τ, all modes $y_{\sigma,i}(t)$, for $i = 2, \dots, n$, are asymptotically stable for any network topology with algebraic connectivity grater than a given bound. This implies that the stability is also guaranteed for sufficiently slow switching topologies. However, as already pointed out in the Introduction, the computation of the minimum dwell time that guarantees this, is still an open issue.

- Finally, we prove that if the conditions of the two items above are satisfied, all agents reach consensus both in terms of position and velocity.

The above three points are dealt in the following three subsections separately.

Stability of the common mode

In this subsection we firstly recall some results for the stability analysis of time delayed linear time invariant (LTI) systems that have been firstly proved by Olgac and Sipahi (2002), and later used by Cepeda-Gomez and Olgac (2011a, 2012) in the framework of multi agent systems.

Consider a generic system whose dynamics is expressed by equation (5.2). Its characteristic equation is equal to:

$$\det(sI_n - A - Be^{-\tau s}) = 0 \tag{5.12}$$

or

$$\prod_{i=1}^{n} \Big(s^2 + (k_d s + k_p)(1 - \lambda_i e^{-\tau s}) \Big) = 0, \tag{5.13}$$

or even equivalently

$$s^2 + k_d s + k_p - (k_d s + k_p)\lambda_i e^{-\tau s} = 0 \tag{5.14}$$

for $i = 1, \ldots, n$. The above transcendental equations obviously have an infinite number of roots.

Olgac and Sipahi (2002) proved that the number of imaginary characteristic roots are finite. Let $\Omega_c = \{\omega_{c1}, \omega_{c2}, \ldots, \omega_{cm}\}$ be the set of crossing frequencies corresponding to the roots on the imaginary axis. The number of such frequencies depends on matrices A and B. Moreover to each of such frequencies there correspond infinitely many values of τ that are periodically spaced. We denote $\Upsilon_l = \{\tau_{l0}, \tau_{l1}, \ldots, \tau_{l\infty}\}$ the infinite set of τ's associated with ω_{cl}, $l = 1, \ldots, m$.

A key parameter in this stability analysis is the *root tendency* defined as:

$$RT_l = sign\Big(Re\Big(\frac{ds}{d\tau}\Big)\Big)_{s=j\omega_{cl}}$$

where $sign$ denotes the sign operator and Re the real part.

It represents the direction of transition of the roots at ω_{cl} as τ increases from $\tau_{lk} - \varepsilon$ to $\tau_{lk} + \varepsilon$, $0 < \varepsilon << 1$, for any $\tau_{lk} \in \Upsilon_l$. In particular, if $RT_l = -1$, the root $j\omega_{cl}$ moves to the left half plane, stabilizing the system, whereas if $RT_l = 1$, the root moves to the right half plane, causing instability. Note that, since Olgac and Sipahi (2002) proved that for each crossing frequency ω_{cl}, RT_l is invariant with respect to the element in the set Υ_l, Olgac and Sipahi (2002) simply propose to analyze the smallest value of τ for each crossing frequency.

Now, the following equation provides an easy procedure to compute the number of unstable roots as τ varies from 0 to ∞, for a given couple of k_p and k_d:

$$N_U(\tau) = N_U(0) + \sum_{l=1}^{m} \Gamma\Big(\frac{\tau - \tau_l}{\Delta\tau_l}\Big) U(\tau, \tau_{l0}) RT_l \tag{5.15}$$

where $N_U(\tau)$ denotes the number of unstable roots corresponding to a generic delay τ, $N_U(0)$ is equal to the number of unstable roots for $\tau = 0$, τ_{l0} indicates the smallest positive delay related to ω_l, the function $\Gamma(x)$ gives the smallest integer greater than or equal to x, $\Delta\tau_l = \dfrac{2\pi}{\omega_{cl}}$, and $U(\tau, \tau_{l0})$ is the step function in τ with the step taking place at τ_{l0}:

$$U(\tau, \tau_{l0}) = \begin{cases} 0 & 0 < \tau < \tau_{l0} \\ 1 & \tau > \tau_{l0} \text{ and } \omega_{cl} = 0 \\ 2 & \tau > \tau_{l0} \text{ and } \omega_{cl} \neq 0 \end{cases}$$

Now, the following considerations and results can be achieved.

- Since we are interested in studying the stability of the mode common to all topologies, namely the one corresponding to $\lambda_i = 1$, we only look at the crossing frequencies of the transcendental equation (5.14) for $i = 1$, i.e., $\omega_1 = 0$ and $\omega_2 = \sqrt{2k_p}$. For more details we address to Olgac and Sipahi (2002). Basically we simply need to impose $s = j\omega$ in equation (5.14) and impose that both sides of the resulting equation in ω have the same magnitude and phase.

- The value of ω_1 corresponds to a root in the origin that prevents asymptotical stability. In particular it generates a non-oscillating mode that stabilizes in a point different from the origin. The other value of the crossing frequency may either lead to stability or instability, depending on the value of τ. In particular, as explained above, the values of τ that lead to stability can be computed using equation (5.15) considering that the number of unstable roots at $\tau = 0$ is $N_U(0) = 1$. Indeed, for $\tau = 0$, the system has two roots in the origin: one is stable at the limit and the other one is unstable.

- It is easy to show that $RT_1 = -1$ for infinitesimally small values of τ, while it is $RT_2 = 1$ relatively to $\omega_2 = \sqrt{2k_p}$. This means that mode $y_{\sigma,1}(t)$ is stable, but not asymptotically stable, for $\tau \in (0 \;\; \tau_{20}]$ where τ_{20} is the smallest positive delay corresponding to ω_2 that is equal to

$$\tau_{20} = \frac{1}{\omega_2} \arctan\left(\frac{-k_d\omega_2^3}{k_p^2 + \omega_2^2(k_d^2 + k_p)}\right) \tag{5.16}$$

as proved by Cepeda-Gomez and Olgac (2011a). For all the other values of τ

the mode is unstable.

Note that in the following $y_{\sigma,1}(t)$ is more simply denoted as $y_1(t)$ to emphasize that it does not depend on the switching signal $\sigma(t)$.

Asymptotic stability of the remaining modes

In this section we introduce a criterion based on LMIs that enables us to prove the asymptotical stability of all modes $y_{\sigma,i}(t)$ for all $i = 2, \ldots, n$ and any fixed value of $\sigma \in \Omega$ to which it corresponds a network topology with a sufficiently large algebraic connectivity.

Theorem 5.1 *Consider the multi-agent system (5.11) consisting of n agents with a time-invariant time delay $\tau > 0$. Consider the $n-1$ modes $y_{\sigma,i}(t)$ for $i = 2, \ldots, n$ obtained via the equivalence transformation $x(t) = (T_\sigma \otimes I_2)\xi(t)$, and relative to a given $\sigma \in \Omega$. If there exist three positive definite matrices P, Q and S of appropriate dimensions such that the following LMI*

$$M_{\sigma,i} = \begin{bmatrix} \frac{1}{\tau}(P\bar{A}_{\sigma,i} + \bar{A}_{\sigma,i}^T P + Q) + S & P\bar{B}_{\sigma,i} & P\bar{C}_{\sigma,i} \\ \bar{B}_{\sigma,i}^T P & -\frac{1}{\tau}Q & 0 \\ \bar{C}_{\sigma,i}^T P & 0 & -S \end{bmatrix} < 0 \tag{5.17}$$

holds for any $\sigma \in \Omega$, then all modes $y_{\sigma,i}(t)$ with $i = 2, \ldots, n$ are asymptotically stable for any topology Ω.

Proof Let us denote as $y_{\sigma,i}(t)$ the generic i-th mode of the system obtained via the equivalence transformation $x(t) = (T_\sigma \otimes I_2)\xi(t)$ assuming that $\sigma(t) = \sigma$ for any $t \geq 0$.

Consider the following candidate Lyapunov-Krasovskii functional:

$$\begin{aligned} V_{\sigma,i}(t) = \;& y^T_{\sigma,i}(t)Py_{\sigma,i}(t) + \int_{-\tau}^{0} y^T_{\sigma,i}(s+t)Qy_{\sigma,i}(s+t)ds \\ & + \int_{-\tau}^{0}\int_{\theta}^{0} y^T_{\sigma,i}(r+t)Sy_{\sigma,i}(r+t)drd\theta. \end{aligned}$$

Derivation with respect to the time gives

$$\begin{aligned} & \dot{V}_{\sigma,i}(t) = y^T_{\sigma,i}(t)(P\bar{A}_{\sigma,i} + \bar{A}^T_{\sigma,i}P + Q + \int_{\tau}^{0} Sd\theta)y_{\sigma,i}(t) \\ & +2y^T_{\sigma,i}(t)P\bar{B}_{\sigma,i}y_{\sigma,i}(t-\tau) + 2y^T_{\sigma,i}(t)P\bar{C}_{\sigma,i}\int_{-\tau}^{0} y^T_{\sigma,i}(s+t) \\ & -y^T_{\sigma,i}(t-\tau)Qy_{\sigma,i}(t-\tau) - \int_{-\tau}^{0} y^T_{\sigma,i}(r+t)Sy_{\sigma,i}(r+t)dr \\ & = \int_{-\tau}^{0} z^T_{\sigma,i}(t)M_{\sigma,i}z_{\sigma,i}(t)d\theta \end{aligned}$$

where $z_{\sigma,i}(t) = [y_{\sigma,i}(t), \quad y_{\sigma,i}(t-\tau), \quad y_{\sigma,i}(t+\theta)]$. Obviously if the condition in (5.17) holds, then all modes $y_{\sigma,i}(t)$ for $i = 2, \ldots, n$ are asymptotically stable regardless of the value of σ.

In simple words $V_{\sigma,i}(t)$ is a Lyapunov function for all $i = 2, \ldots, n$ for any network topology Ω in which the LMI in eq. (5.17) holds. □

The above LMI has been introduced by Richard (2003) in a more general form. Clearly, the requirement that matrices P, Q and S exist for any network topology in Ω is a very computational demanding task. The dependence on the network topology in eq. (5.17) consists in a different set of eigenvalues for every topology. In Section 5.4 we show how to extend this approach to avoid the verification of the LMI in eq. (5.17) for any network topology in Ω.

Obviously, the asymptotic stability of the above modes corresponding to a static topology does not imply in general the asymptotically stability of the switched system, in particular under the assumption of arbitrary switching. However, for sure there exists a minimum dwell time that ensures this (Liberzon, 2003). We conjecture that such a dwell time may be computed appropriately defining a common Lyapunov function starting from the considered Lyapunov-Krasovskii function.

Consensus agreement

Theorem 5.2 *Consider the multi-agent system* (5.11) *consisting of* n *agents with a time-invariant time delay* $\tau > 0$ *where* $\sigma \in \Omega$ *is constant. Assume that all modes* $y_{\sigma,i}(t)$ *for* $i = 2, \ldots, n$ *are asymptotically stable and that the mode* $y_1(t)$ *corresponding to the eigenvalue* λ_1*, common to all topologies in* Ω *by construction, is stable. Then the consensus state is achieved asymptotically*

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \quad \forall i, j \in V.$$

Proof The asymptotic stability assumption for all modes $i = 2, \ldots, n$ implies that

$$\lim_{t\to\infty} y_{\sigma,i}(t) = 0, \quad i = 2, \ldots, n.$$

Moreover, being by definition $y_{\sigma,i}(t) = [\xi_{\sigma,i}(t),\ \dot{\xi}_{\sigma,i}(t)]$, it is

$$\lim_{t\to\infty} \xi_i(t) = 0, \quad i = 2, \ldots, n$$

and

$$\lim_{t\to\infty} \dot{\xi}_i(t) = 0, \quad i = 2, \ldots, n.$$

Now, being by definition $x_\sigma(t) = (T_\sigma \otimes I_2)\xi_\sigma(t)$, it holds

$$[x_{\sigma,1}(t) \ldots, x_{\sigma,n}(t)] = T_\sigma[\xi_{\sigma,1}(t), \ldots, \xi_{\sigma,n}(t)]$$

therefore

$$\begin{aligned} \lim\nolimits_{t\to\infty}[x_{\sigma,1}(t) \ldots, x_{\sigma,n}(t)] &= T_\sigma[L_{\xi_1}, 0, \ldots, 0] \\ &= L_{\xi_1}T[1, 0, \ldots, 0] = L_{\xi_1}Te_1 \end{aligned} \tag{5.18}$$

where

$$L_{\xi_1} = \lim_{t\to\infty} \xi_{\sigma,1}(t) \tag{5.19}$$

and $e_1 = [1, 0, \ldots, 0]$. Note that we removed the dependence on σ in L_{ξ_1} since it is related to the mode common to all topologies associated with $\lambda_1 = 1$. Moreover, the limit in (5.19) exists and is finite since by assumption the first mode is a non oscillating stable mode.

The term $T_\sigma e_1$ returns the first column of T_σ or equivalently the eigenvector associated to $\lambda_i = 1$ that is equal to $[1, \ldots, 1]$. This means that equation (5.18) can be rewritten as

$$\lim_{t\to\infty}[x_{\sigma,1}(t) \ldots, x_{\sigma,n}(t)] = L_{\xi_1}[1, \ldots, 1], \tag{5.20}$$

i.e., all $x_{\sigma,i}(t)$, for $i = 1, \ldots, n$, reach the same value equal to L_{ξ_1}, thus proving the statement.

□

From the above theorem the next result follows.

Corollary 5.1 *Consider the multi-agent system (5.11) consisting of n agents with a time-invariant communication delay $\tau \in (0, \tau_{20}]$ where τ_{20} is defined as in equation (5.16). Assume that all conditions of Theorem 5.1 are satisfied. Then, all the agents reach consensus.* □

5.4 LMI computation

In this section we provide a method to solve the LMI introduced in Theorem 5.1 which is independent from the network topology. To this aim, let us first observe that $M_{\sigma,i}$ can be rewritten as

$$M_{\sigma,i} = \tilde{M}_{\sigma,i} + \hat{M}_{\sigma,i}$$

where

$$\tilde{M}_{\sigma,i} = \begin{bmatrix} \frac{1}{\tau}(P\bar{A}_{\sigma,i} + \bar{A}_{\sigma,i}^T P + Q) + S & 0 & 0 \\ 0 & -\frac{1}{\tau}Q & 0 \\ 0 & 0 & -S \end{bmatrix} \tag{5.21}$$

and

$$\hat{M}_{\sigma,i} = \begin{bmatrix} 0 & P\bar{B}_{\sigma,i} & P\bar{C}_{\sigma,i} \\ \bar{B}_{\sigma,i}^T P & 0 & 0 \\ \bar{C}_{\sigma,i}^T P & 0 & 0 \end{bmatrix}. \tag{5.22}$$

Obviously, $M_{\sigma,i}$ is negative definite if and only if $\tilde{M}_{\sigma,i} + \hat{M}_{\sigma,i}$ is negative definite. Now, substituting $\bar{B}_{\sigma,i}$ and $\bar{C}_{\sigma,i}$ in (5.22), we can rewrite (5.22) as

$$\hat{M}_{\sigma,i} = \begin{bmatrix} 0 & 0 & 0 & k_d p_{12} & 0 & -k_p p_{12} \\ 0 & 0 & 0 & k_d p_{22} & 0 & -k_p p_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_d p_{12} & k_d p_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_p p_{12} & -k_p p_{22} & 0 & 0 & 0 & 0 \end{bmatrix} \lambda_{\sigma,i} \tag{5.23}$$

where p_{ij} is the entry of P corresponding to row i and column j. Since $\hat{M}_{\sigma,i}$ is symmetric, $\|\hat{M}_{\sigma,i}\|_2 = \rho(\hat{M}_{\sigma,i})$, where $\rho(\hat{M}_{\sigma,i})$ is its spectral radius. The eigenvalues η of matrix $\hat{M}_{\sigma,i}$ are the solutions of equation

$$\det(\hat{M}_{\sigma,i} - \eta I_2) = 0.$$

Since $\lambda_{\sigma,i}$ is a multiplying scalar, we can neglect it and simply solve the following equation with respect to η:

$$\det \begin{bmatrix} -\eta & 0 & 0 & k_d p_{12} & 0 & -k_p p_{12} \\ 0 & -\eta & 0 & k_d p_{22} & 0 & -k_p p_{22} \\ 0 & 0 & -\eta & 0 & 0 & 0 \\ k_d p_{12} & k_d p_{22} & 0 & -\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\eta & 0 \\ -k_p p_{12} & -k_d p_{22} & 0 & 0 & 0 & -\eta \end{bmatrix} = 0.$$

This can be solved analytically by exploiting the Laplace rule to compute the determinant. In particular there are 4 null eigenvalues plus the following two non null eigenvalues

$$\eta_{1,2} = \pm\lambda_{\sigma,i}\sqrt{(k_d^2 + k_p^2)(p_{12}^2 + p_{22}^2)}.$$

Therefore,

$$\|\hat{M}_{\sigma,i}\|_2 = |\lambda_{\sigma,i}|\sqrt{(k_d^2 + k_p^2)(p_{12}^2 + p_{22}^2)}.$$

Now, let us observe that,

$$\tilde{M}_{\sigma,i} + \hat{M}_{\sigma,i} \leq \tilde{M}_{\sigma,i} + \|\hat{M}_{\sigma,i}\|_2 I_6 \tag{5.24}$$

thus, if we prove that

$$\tilde{M}_{\sigma,i} + \|\hat{M}_{\sigma,i}\|_2 I_6 < 0,$$

we can be sure that $\tilde{M}_{\sigma,i} + \hat{M}_{\sigma,i} < 0$ as well, or equivalently, $M_{\sigma,i} < 0$.

Now, equation (5.24) can be rewritten as

$$\begin{cases} \frac{1}{\tau}(P\bar{A}_{\sigma,i} + \bar{A}_{\sigma,i}^T P + Q) + S + \|\hat{M}_{\sigma,i}\|_2 I_2 & < 0 \\ -\frac{1}{\tau}Q + \|\hat{M}_{\sigma,i}\|_2 I_2 & < 0 \\ -S + \|\hat{M}_{\sigma,i}\|_2 I_2 & < 0 \end{cases} \tag{5.25}$$

where the last two equations are always verified if

$$\begin{cases} \tau \|\hat{M}_{\sigma,i}\|_2 I_2 & < Q \\ \|\hat{M}_{\sigma,i}\|_2 I_2 & < S \end{cases} \tag{5.26}$$

therefore, as a particular case, they are satisfied by

$$Q = \tau \left(\|\hat{M}_{\sigma,i}\|_2 + \varepsilon \right) I_2 \tag{5.27}$$

and

$$S = \left(\|\hat{M}_{\sigma,i}\|_2 + \varepsilon \right) I_2 \tag{5.28}$$

for any $\varepsilon > 0$.

Let $Q = \tau|\lambda_{\sigma,i}|\alpha I_2$ and $S = |\lambda_{\sigma,i}|\alpha I_2$ with $\alpha > \sqrt{(k_d^2 + k_p^2)(p_{12}^2 + p_{22}^2)}$. We need to solve with respect to P the first inequality in (5.25), that becomes equal to

$$P\bar{A}_{\sigma,i} + \bar{A}_{\sigma,i}^T P + 3\tau|\lambda_{\sigma,i}|\alpha I_2 < 0. \tag{5.29}$$

Our objective is to prove that inequality in equation (5.29) holds for any $\lambda_{\sigma,i} \in \left[-1, \bar{\lambda}_2\right]$. Since matrix $P\bar{A}_{\sigma,i} + \bar{A}_{\sigma,i}^T P + |\lambda_{\sigma,i}| (3\tau\alpha) I_2$ is symmetric, if its eigenvalues are negative then it is a negative definite matrix and equation (5.29) holds.

We now choose a set of parameters of interest $k_p, k_d, \tau, \bar{\lambda}_2$. We solve the inequality in equation (5.17) for this set to determine candidate matrix P. We choose a value $\alpha > \sqrt{(k_d^2 + k_p^2)(p_{12}^2 + p_{22}^2)}$.

To verify that the eigenvalues of matrix $P\bar{A}_{\sigma,i} + \bar{A}_{\sigma,i}^T P + |\lambda_{\sigma,i}|3\tau\alpha$ are negative we compute its determinant and trace and verify that they are respectively positive and negative.

Its trace corresponds to

$$T = -2p_{12}k_p(1 - \lambda_{\sigma,i}) + 6|\lambda_{\sigma,i}|\tau\alpha + 2p_{12} - 2p_{22}k_d \tag{5.30}$$

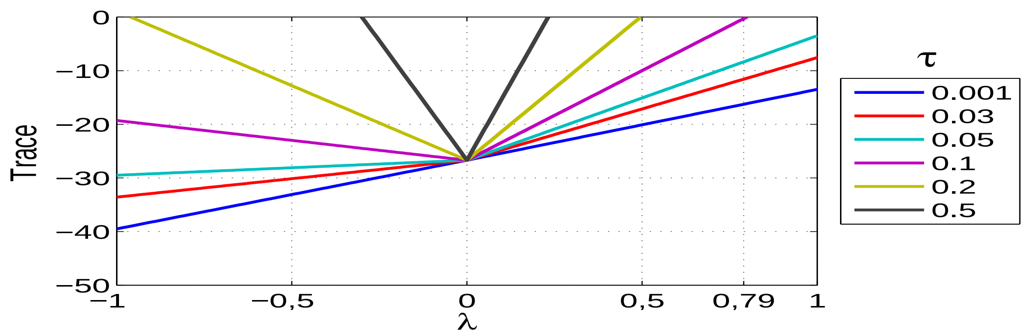


Figure 5-1: Trace in equation (5.30) versus $\lambda_{\sigma,i}$ for different values of τ

and its determinant corresponds to

$$\begin{aligned}
&\Delta = \\
&\lambda_{\sigma,i}^2 \left(3\tau\alpha^2 - p_{22}^2 kp^2 + 2p_{12}k_p 3\tau\alpha\right) \\
&+\lambda_{\sigma,i}\left(2p_{22}^2 k_p^2 - 2p_{12}k_p p_{22}k_d + 4p_{12}^2 k_p - 2p_{11}p_{22}k_p\right) \\
&+|\lambda_{\sigma,i}|\left(-2p_{12}k_p 3\tau\alpha - 4p_{12}^2 k_p + 6\tau\alpha p_{12} - 6\tau\alpha p_{22}k_d\right) \\
&+2p_{12}k_p p_{22}k_d - p_{11}^2 + 2p_{12}k_d p_{11} + 2p_{11}p_{22}k_p \\
&-p_{12}^2 kd^2 - p_{22}^2 k_p^2.
\end{aligned} \tag{5.31}$$

The above quantities can be evaluated numerically for $\lambda_{\sigma,i} \in [-1, 1]$. Thus, for any value of τ constraints on the spectrum for any network topology can be given. For a sufficiently small τ a constraint involving only the algebraic connectivity $\bar{\lambda}_2$ can be computed so that the proposed consensus protocol is stable for all network topologies with algebraic connectivity smaller than $\bar{\lambda}_2$. As an example in Figure 5-1 and Figure 5-2 the determinant and trace given in equation (5.30) and (5.31) are computed versus $\lambda_{\sigma,i} \in [-1, 1]$ for different values of τ. Simulations are performed with parameters $k_p = 10, k_d = 50$ and a candidate $P = [35\ 0.65; 0.65\ 0.15]$. Consider as an example the simulation with $\tau = 0.1$: the trace is negative for all $\lambda_{\sigma,i} \in [-1, 0.79]$ while the determinant is positive for all $\lambda_{\sigma,i} \in [-1, 0.54]$. This implies that the proposed consensus protocol is stable for any network topology with $\bar{\lambda}_2 = 0.54$. Thus, there exists a minimum dwell time such that consensus is achieved even with slow arbitrary switchings between any topology satisfying such constraint.

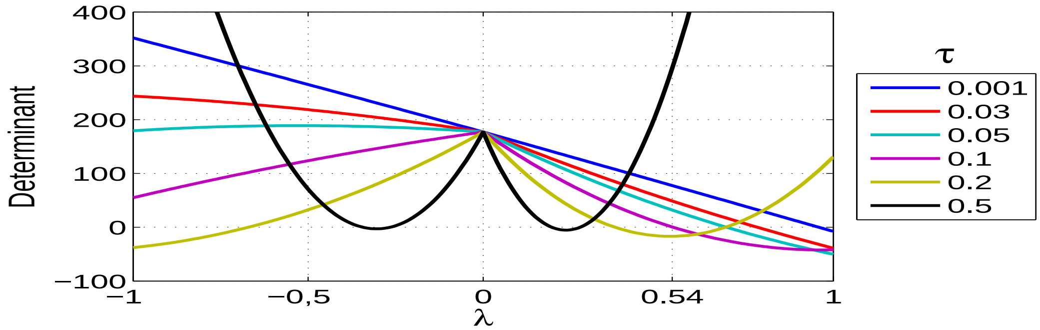


Figure 5-2: Determinant in equation (5.31) versus $\lambda_{\sigma,i}$ for different values of τ

5.5 Simulations

In this section we present a simulation of the consensus protocol in (5.1). We consider a network of six agents with control parameters $k_p = 10$, $k_d = 50$ and a time delay $\tau = 0.1$. As shown in Section 5.4 these parameters guarantee stability of the consensus protocol for any topology with second largest eigenvalue (algebraic connectivity) smaller than $\bar{\lambda}_2 \leq 0.54$. Furthermore, the chosen value of τ guarantees the stability of the common mode as explained in Section 5.3, in fact to the above parameters it corresponds a value $\tau_{20} = 0.6827$ as in equation (5.16). In Figure 5-3 we consider a network that switches randomly among 6 randomly generated connected network topologies which satisfy the bound on the algebraic connectivity. In this case the simulation shows that with a dwell time of one second the system remains stable.

5.6 Conclusions

In this chapter we investigated the consensus problem for networks of agents with double integrator dynamics affected by time-delay in their coupling. We provided a stability result based on the Lyapunov-Krasovskii functional method and a numerical procedure based on an LMI condition which depends only on the algebraic connectivity of the considered network topologies, thus reducing greatly the computational complexity of the procedure. Obviously, this result implies the existence of a minimum dwell time such that the proposed consensus protocol is stable for slow

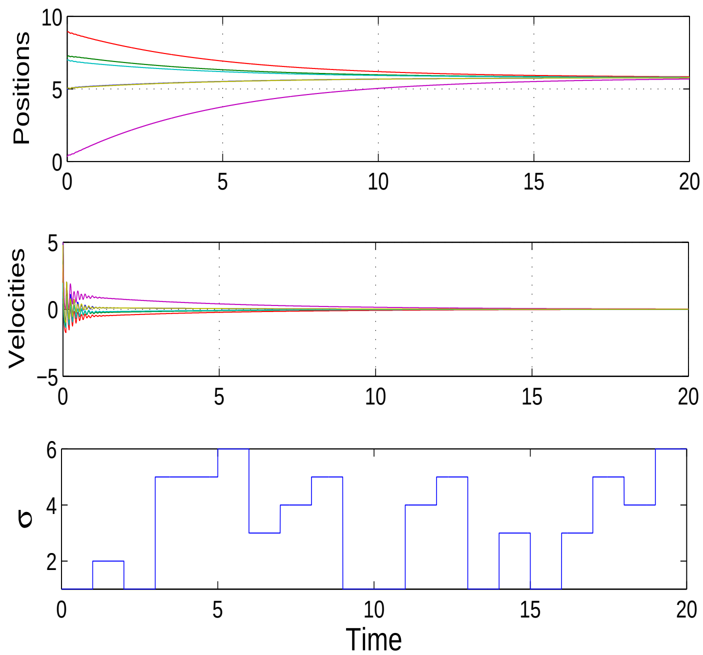


Figure 5-3: Simulation of the consensus protocol for a switching network topology

switchings between network topologies with sufficient algebraic connectivity. Future work will involve actually computing this dwell time by adopting a multiple Lyapunov function method and evaluating the worst case convergence rate. Furthermore we will evaluate novel consensus protocols that consider only delayed relative measurements instead of delayed absolute values of the neighbors' state variables.

6

Average consensus in arbitrary directed networks with time-delay

"The noblest pleasure is the joy of understanding"

– Leonardo da Vinci

In this chapter, based on the results of Zareh et al. (2013a), we study the stability property of a consensus on the average algorithm in arbitrary directed graphs with respect to communication/sensing time-delays. The proposed algorithm adds a storage variable to the agents' states so that the information about the average of the states is preserved despite the algorithm iterations are performed in an arbitrary strongly connected directed graph. We prove that for any network topology and choice of design parameters the consensus on the average algorithm is stable for sufficiently small delays. We provide simulations and numerical results to estimate the maximum delay allowed by an arbitrary unbalanced directed network topology.

6.1 Introduction

The consensus problem in multi-agent systems consists in the design of a coupling law between dynamical systems (agents) such that the state of each one converges to the same value in absence of external reference signals. Multi-agent systems are considered to be *complex* systems since the pattern of interconnections between agents is often arbitrary and unknown at the controller design stage. This clearly makes challenging the design of interaction rules between agents that exploit only local information. For these reasons agents modeled by simple single integrators or second order systems are usually investigated. One of the major works from which we take inspiration is the one by Olfati-Saber and Murray (2004) where the consensus problem for networks of first order agents for switching topologies or time-delays is investigated. In this chapter, we prove that simple averaging local interaction rules can achieve consensus on the average, i.e., the state of each agent converges to the average of the initial states only if the directed graph that encodes the network topology is strongly connected and balanced (each agent receives and sends information to the same number of agents). They also explored the consensus problem in the case of time-delays for undirected network topologies.

Since then several authors have explored ways to design consensus on the average algorithms that work on general directed graphs not necessarily balanced. In the work of Franceschelli et al. (2008, 2009) the idea to use an augment state space to add robustness to a networked system represented by an undirected graph that executes a consensus algorithm was proposed. The proposed algorithms aim at recovering the correct network average once malicious or faulty agents have been removed from the network.

Franceschelli et al. (2009, 2011) presented a discrete time consensus on the average algorithm for arbitrary strongly connected directed graph based on asynchronous state updates , based on the idea to augment the state of each agent with an additional variable to preserve the information about the initial average of the states in the network. Simulations were used to characterize the convergence properties and the performance of the algorithm.

Cai and Ishii (2012) characterized a discrete time consensus on the average algorithm based on additional state variables was in terms of a tuning parameter. It was proven that there always exist sufficiently small values of such tuning parameter so that the proposed algorithm converges to the average of the initial state in arbitrary strongly connected directed graphs.

Dominguez-Garcia et al. (2012) addressed the control of distributed energy resources by developing a consensus on the average protocol based on the so called *ratio* consensus. Their algorithm is based on two independent distributed dynamical systems, one with arbitrary initial conditions and one with predetermined initial values. The authors consider time-varying network topologies described by directed graphs and show that for each agent the ratio of the output of these two dynamical systems converges to the average of the initial states.

Chen et al. (2010b, 2011) proposed the *Corrective Consensus* algorithm. It consists in a local state update rule where each agent keeps track of several additional variables corresponding to the number of its neighbors which are used to periodically steer the average of the network state to the correct value corresponding to the average of the states at the initial instant of time.

Aysal et al. (2009) proposed the broadcast gossip algorithm . This algorithm is based upon discrete time and asynchronous state updates with directed information flow, it makes each agent agree upon a random variable whose expectation is the average of the initial states.

Most of the literature on consensus on the average in directed graphs deals with methods and techniques to achieve consensus on the average in networks of agents described by single integrators. On the other hand the literature on consensus with time-delays in directed graphs usually deals with the problem of making the state of each agent converge to the same value which can be time-varying and not related to the initial state of the network in an explicit way.

Yu et al. (2010) characterized necessary and sufficient conditions for convergence of second-order multi-agent systems with velocity feedback are given and the effect of time-delays in directed graphs while the consensus value is arbitrary.

In the work of Sun and Wang (2009) several instances of consensus problems with time-delays are investigated. In particular the cases of switching directed topologies, packet data dropouts, and finite time consensus are all characterized separately by considering the effect of time-delays for the achievement of consensus on an arbitrary value.

In this chapter, we propose a continuous time consensus algorithm inspired from the discrete time algorithms of Franceschelli et al. (2009, 2011); Cai and Ishii (2012). We consider a description in continuous time to describe a network of n vehicles with a local interaction rule that controls the instantaneous speed of each vehicle. Then we extend the proof method of Cai and Ishii (2012) to the case at hand and study the convergence properties of the resulting system considering a time-delay in the state update of each agent. We finally provide simulation results to corroborate the theoretical analysis.

The main contributions of this chapter can be summarized in the following three items.

- We provide a continuous time version of a consensus on the average algorithm for arbitrary directed strongly connected graphs derived from results of Franceschelli et al. (2009, 2011) and Cai and Ishii (2012).
- We provide a characterization of the convergence properties of the algorithm with respect to time-delays.
- We present simulations to characterize numerically the performance of the proposed protocol with respect to different time-delays and tuning parameters.

The next sections are structured as follows. In Section 6.2 we introduce a consensus on the average protocol and the corresponding model considering time-delays. In Section 6.3 we characterize the convergence properties of the proposed algorithm with respect to time-delays. In Section 6.4 we corroborate the theoretical analysis with a numerical example and simulations. Concluding remarks are finally given in Section 6.5.

6.2 Consensus on the average protocol

We now introduce a consensus protocol stated in continuous time that takes inspiration from protocols addressed by Franceschelli et al. (2011) and Cai and Ishii (2011b) in a discrete time setting. In the protocol under consideration each agent is a single integrator with an additional state variable called *surplus* or *storage*. This additional variable is used to preserve information about the average value of the agents' states at the initial instant of time, that is a time-varying quantity in directed graphs that are not balanced, i.e., graphs in which the in-degree and out-degree of each node are not necessarily equal.

The local state update rule implemented by each node is the following:

$$\begin{cases} \dot{x}_i(t) = & -\sum_{j\in\mathcal{N}_{i,in}} (x_i(t) - x_j(t)) + \varepsilon z_i(t), \\ \dot{z}_i(t) = & \sum_{j\in\mathcal{N}_{i,in}} (x_i(t) - x_j(t)) \\ & -\sum_{j\in\mathcal{N}_{i,in}} (z_i(t) - z_j(t)) \\ & -(\varepsilon - \delta_{i,in} + \delta_{i,out})\, z_i(t), \end{cases} \tag{6.1}$$

where $x_i, z_i \in \mathbb{R}$ are the states of agent i and $\varepsilon \in \mathbb{R}^+$ is a tuning parameter of the algorithm. It is clear that to implement protocol (6.1) each agent requires only relative state information with respect to variable x_i, absolute state information with respect to variable z_i, and knowledge of its own out-degree.

The network dynamics that emerges when each agent implements the local state update rule in eq. (6.1) can be formulated in matrix form as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -L_{in} & \varepsilon I \\ L_{in} & -L_{out} - \varepsilon I \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{6.2}$$

where $x = [x_1, x_2, \ldots, x_n]$ and $z = [z_1, z_2, \ldots, z_n]$ are a compact representation of the agents' state.

The proposed local interaction scheme can be interpreted as a network of n vehicles each modeled as a continuous time single integrator $\dot{x}(t)_i = u_i(t)$ where each $x_i(t)$ represents a position in space and variables $z_i(t)$ are software variables which enable the interaction scheme to converge to the initial average position.

In this chapter, we study protocol (6.2) under the assumption that communication/sensing delays affect the multi-agent system. The network dynamics are thus described by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = M(\varepsilon) \begin{bmatrix} x(t-\tau) \\ z(t-\tau) \end{bmatrix} \tag{6.3}$$

with $x(\theta) = x_0, \quad z(\theta) = z_0, \quad -\tau \leq \theta \leq 0$, where

$$M(\varepsilon) = \begin{bmatrix} -L_{in} & \varepsilon I \\ L_{in} & -L_{out} - \varepsilon I \end{bmatrix} \tag{6.4}$$

and $\tau \in \mathbb{R}^+$ denotes a time-delay. We study system (6.3) in the approximation that the delay for all the agent is the same.

6.3 Convergence properties

In this section we study the convergence properties of system (6.3).

We preliminary observe that by construction matrix $M(\varepsilon)$ satisfies

$$\left[\mathbf{1}_n^T \ \mathbf{1}_n^T\right] M(\varepsilon) = \left[\mathbf{0}_n^T \ \mathbf{0}_n^T\right],$$

for any $\varepsilon \in \mathbb{R}$. Therefore, since

$$\mathbf{1}_n^T \dot{x}(t) + \mathbf{1}_n^T \dot{z}(t) = 0, \quad \forall t \geq 0$$

it holds

$$\mathbf{1}_n^T x(t) + \mathbf{1}_n^T z(t) = \mathbf{1}_n^T x(0) + \mathbf{1}_n^T z(0), \quad \forall t \geq 0. \tag{6.5}$$

Now consider matrix $M(\varepsilon)$ for $\varepsilon = 0$, namely

$$M(0) = \begin{bmatrix} -L_{in} & 0 \\ L_{in} & -L_{out} \end{bmatrix}. \tag{6.6}$$

It is clear that since matrix $M(0)$ is a $2n \times 2n$ block lower triangular matrix it has $2n$ eigenvalues equal to the eigenvalues of matrices $-L_{in}$ and $-L_{out}$. If graph $\mathcal{G}$ is strongly connected, then $M(0)$ has one null eigenvalue with algebraic multiplicity 2 and geometric multiplicity 2, all other eigenvalues have strictly negative real part.

In the following we denote as $\lambda_i(0)$, $i = 1, \dots, 2n$, the eigenvalues of matrix $M(0)$ and assume that

$$0 = \lambda_1(0) = \lambda_2(0) > \Re(\lambda_3)(0) \geq \dots \geq \Re(\lambda_{2n}(0)).$$

Eigenvalues of matrix $M(\varepsilon)$ are denoted as $\lambda_i(\varepsilon)$, $i = 1, \dots, 2n$, and ordered as $\Re(\lambda_1)(\varepsilon) \geq \dots \geq \Re(\lambda_{2n}(\varepsilon))$.

We now prove some properties of the eigenvalues of matrix $M(\varepsilon)$ for small values of $\varepsilon > 0$, that can be derived from the results of Cai and Ishii (2011b).

Proposition 6.1 *Let matrix $M(\varepsilon)$ be defined as in eq. (6.4). If $\mathcal{G}$ is strongly connected, there exists $\bar{\varepsilon} \in \mathbb{R}^+$ such that if $\varepsilon \in (0, \bar{\varepsilon}]$ then $M(\varepsilon)$ has one null eigenvalue and $2n - 1$ eigenvalues with strictly negative real part.*

Proof: Matrix $M(\varepsilon)$ depends smoothly on parameter $\varepsilon \geq 0$, therefore if eigenvalues $\lambda_3(0), \dots, \lambda_{2n}(0)$ of $M(0)$ have strictly negative real part, there exists $\bar{\varepsilon} > 0$ such that if $\varepsilon \in [0, \bar{\varepsilon}]$ then for $i = 3, \dots, 2n$, it holds $\Re(\lambda_i(\varepsilon)) < 0$. Therefore, according to Cai and Ishii (2011b), we only have to show that for ε sufficiently small it is $\lambda_1(\varepsilon) = 0$ and $\Re(\lambda_2(\varepsilon)) < 0$.

Since the null eigenvalue of $M(0)$ is semi-simple[1] *and $Rank(M(0)) = 2n - 2$ it has two linearly independent right eigenvectors r_1, r_2 and left eigenvectors l_1, l_2 corresponding to the null eigenvalue. It holds*

$$M' = \frac{dM(\varepsilon)}{d\varepsilon} = \begin{bmatrix} 0 & I \\ 0 & -I \end{bmatrix}. \tag{6.7}$$

Then, as shown by Cai and Ishii (2011b), $d\lambda_1(\varepsilon)/d\varepsilon|_{\varepsilon=0}$ and $d\lambda_2(\varepsilon)/d\varepsilon|_{\varepsilon=0}$ are the eigenvalues of the following matrix

[1] An eigenvalue is semi-simple if its algebraic and geometric multiplicity are equal.

$$\begin{bmatrix} l_1^T M' r_1 & l_1^T M' r_2 \\ l_2^T M' r_1 & l_2^T M' r_2 \end{bmatrix}. \tag{6.8}$$

If graph $\mathcal{G}$ is strongly connected then $l_1 = \alpha_1 \mathbf{1}_{2n}$ and $r_1 = \alpha_2 \left[\mathbf{1}_n^T, \mathbf{0}_n^T\right]$ where $\alpha_1, \alpha_2 \in \mathbb{R}$ can be chosen such that $l_1^T r_1 = 1$. By substituting l_1 and r_1 in (6.8) it can be shown by simple computations that

$$d\lambda_1(\varepsilon)/d\varepsilon|_{\varepsilon=0} = 0, \quad d\lambda_2(\varepsilon)/d\varepsilon|_{\varepsilon=0} = l_2^T M' r_2.$$

The first equality enables us to conclude that for sufficiently small values of ε, it is $\lambda_1(\varepsilon) = 0$.

Now, let $\nu_{r,out}$ be the right eigenvector corresponding to the null eigenvalue of matrix L_{out} and $\nu_{l,in}$ be the left eigenvector corresponding to the null eigenvalue of matrix L_{in}. It is possible to verify by substitution that we can choose $r_2 = \left[\mathbf{0}_n^T, \nu_{r,out}^T\right]$ and $l_2 = \left[\nu_{l,in}^T, \mathbf{0}_n^T\right]$. Therefore,

$$d\lambda_2(\varepsilon)/d\varepsilon|_{\varepsilon=0} = -\nu_{l,in}^T \nu_{r,out}.$$

Since L_{in} and L_{out} are Metzler matrices (Berman and Plemmons (1979)), the eigenvectors $\nu_{l,in}$ and $\nu_{r,out}$ corresponding to the null eigenvalue have only positive elements. Therefore

$$d\lambda_2(\varepsilon)/d\varepsilon|_{\varepsilon=0} = -\nu_{l,in}^T \nu_{r,out} < 0$$

and $\lambda_2(\varepsilon) < 0$ for $\varepsilon > 0$ sufficiently small, thus proving the statement. □

We are now ready to study the stability of system (6.3) with respect to time-delays. Let $Y(s) = \left[X(s)^T \ Z(s)^T\right]^T$ denote the Laplace transform of $y(t) = \left[x(t)^T \ z(t)^T\right]^T$. Then the Laplace transform of system (6.3) is

$$Y(s) = \left(sI - M(\varepsilon)e^{-s\tau}\right)^{-1} Y(0)$$

and the stability property of system (6.3) depends upon the roots of the quasi-polynomial

$$det\left(sI - M(\varepsilon)e^{-s\tau}\right). \tag{6.9}$$

By simple manipulations it holds

$$det\left(sI - M(\varepsilon)e^{-s\tau}\right) = e^{-2ns\tau} det\left(se^{s\tau}I - M(\varepsilon)\right) \tag{6.10}$$

thus the roots of (6.9) correspond to the solutions of

$$se^{s\tau} = \lambda_i(\varepsilon), \quad i = 1, \ldots, 2n. \tag{6.11}$$

Theorem 6.2 *Let matrix $M(\varepsilon)$ be defined as in eq. (6.4) and $\varepsilon \in (0, \bar{\varepsilon}]$ as in Proposition 6.1. If $\mathcal{G}$ is strongly connected and*

$$\tau \leq \tau_c(\varepsilon) = \min_{i=2,\ldots,2n} \frac{\theta_i(\varepsilon) - \frac{\pi}{2}}{R_i(\varepsilon)}, \tag{6.12}$$

where $R_i(\varepsilon) = |\lambda_i(\varepsilon)|$ and $\theta_i(\varepsilon) = \angle\lambda_i(\varepsilon)$ with $\lambda_i(\varepsilon)$ the i-th eigenvalue of $M(\varepsilon)$, then the roots of

$$det\left(sI - M(\varepsilon)e^{-s\tau}\right) \tag{6.13}$$

have all strictly negative real part except one in $s = 0$.

Proof: By Proposition 6.1 since $\mathcal{G}$ is strongly connected by assumption, there exists $\bar{\varepsilon}$ such that for $\varepsilon \in (0, \bar{\varepsilon}]$, $M(\varepsilon)$ has a single null eigenvalue and $2n - 1$ eigenvalues with strictly negative real part. Since the roots of eq. (6.13) depend continuously on τ and for $\tau = 0$ they coincide with the roots of $M(\varepsilon)$, we compute the smallest positive value of τ, denoted as τ_c, for which at least one non-null root crosses the imaginary axis. By eq. (6.11), assuming $s = j\omega$ it holds

$$j\omega e^{j\omega\tau} = R_i(\varepsilon)e^{j\theta_i(\varepsilon)}.$$

By simple manipulations the above equation can be rewritten as

$$j\omega = R_i(\varepsilon)\ cos(\theta_i(\varepsilon) - \omega\tau) + jR_i(\varepsilon)\ sin(\theta_i(\varepsilon) - \omega\tau),$$

therefore

$$\begin{cases} R_i(\varepsilon)\ cos(\theta_i(\varepsilon) - \omega\tau) = 0, \\ \omega = R_i(\varepsilon)\ sin(\theta_i(\varepsilon) - \omega\tau). \end{cases}$$

This implies that

$$\begin{cases} \theta_i(\varepsilon) - \omega\tau = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{N} \\ \omega = R_i(\varepsilon)\ sin(\frac{\pi}{2} + k\pi) = R_i(\varepsilon)(-1)^k. \end{cases}$$

Finally, considering only the top-half of the Gauss plane, $\theta_i(\varepsilon) \in \left(\frac{\pi}{2}, \pi\right]$ *for* $i = 1, \dots, 2n$*. Thus*

$$\begin{aligned} \tau_c(\varepsilon) &= \min_{i=2,\dots,2n} \min_{k\in\mathbb{N}} \frac{\theta_i(\varepsilon) - \frac{\pi}{2} - k\pi}{R_i(\varepsilon)(-1)^k} \\ &= \min_{i=2,\dots,2n} \frac{\theta_i(\varepsilon) - \frac{\pi}{2}}{R_i(\varepsilon)}, \end{aligned} \tag{6.14}$$

proving the statement. □

Next we give bounds on the maximum length of the time delay that ensures stability as function of known network parameters computed for $\varepsilon = 0$. If the actual time delay is smaller than the proposed bound then we are sure that there exist $\varepsilon > 0$ sufficiently small such that the system is stable and achieves consensus.

Theorem 6.3 *Consider a multi-agent system that implements protocol (6.1) in graph* $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$*, with tuning parameter* $\varepsilon > 0$*, initial condition* $z(0) = \mathbf{0}_n$ *and time-delay* $\tau > 0$*. If* $\mathcal{G}$ *is strongly connected, there exists* $\tilde{\varepsilon}$ *such that if* $\varepsilon \in (0, \tilde{\varepsilon}]$ *and*

$$\tau < \tilde{\tau} = \frac{1}{2\bar{\delta}} \arctan\left(\frac{\Re\{\lambda_3(0)\}}{\bar{\delta}}\right)$$

where

$$\bar{\delta} = \max_{i\in\mathcal{V}}\{\delta_{i,in}, \delta_{i,out}\}$$

and $\lambda_3(0)$ *is the rightmost non-null eigenvalue of matrix* $M(0)$*, then*

$$\lim_{t\to\infty} x(t) = \frac{\mathbf{1}_n^T x(0)}{n}\mathbf{1}_n.$$

Proof: By definition it holds

$$M(\varepsilon) = M(0) + \varepsilon M',$$

where M' *is defined as in eq. (6.7). Since* $M(\varepsilon)$ *can be seen as a perturbation of matrix* $M(0)$ *its eigenvalues depend continuously on parameter* ε*. This implies that the ratio in eq. (6.12) can be bounded for an arbitrary small* ε *as a function of the eigenvalues of* $M(0)$*. In particular, for* $\varepsilon = 0$ *by the Gershgorin disc theorem applied to matrices* L_{in} *and* L_{out} *we have* $R_i(\varepsilon) \leq \max_{i=1,\dots,2n} |\lambda_i(\varepsilon)| \leq 2\max_{i\in\mathcal{V}}\{\delta_{i,in}, \delta_{i,out}\} = 2\bar{\delta}$*, thus it holds*

$$\begin{aligned}\min_{i=2,\dots,2n} \frac{\theta_i(\varepsilon) - \frac{\pi}{2}}{R_i(\varepsilon)} &\geq \frac{\min_{i=2,\dots,2n} \theta_i(\varepsilon) - \frac{\pi}{2}}{\max_{i=2,\dots,2n} R_i(\varepsilon)} \\ &\geq \frac{1}{2\bar{\delta}} \arctan\left(\min_{i=1,\dots,2n} \frac{\Re(\lambda_i(\varepsilon))}{\Im(\lambda_i(\varepsilon))}\right).\end{aligned}$$

Finally, since for $\varepsilon = 0$*, it is* $\Im(\lambda_2(\varepsilon)) = 0$ *and* $\max_{i=1,\dots,2n} |\Im(\lambda_i(\varepsilon))| \leq \bar{\delta}$*, it holds*

$$\min_{i=2,\dots,2n} \frac{\theta_i(\varepsilon) - \frac{\pi}{2}}{R_i(\varepsilon)} \geq \frac{1}{2\bar{\delta}} \arctan\left(\frac{\Re\{\lambda_3(\varepsilon)\}}{\bar{\delta}}\right).$$

Therefore, since by Theorem 6.2 we may conclude that for $\tau \leq \tau_c(\varepsilon)$ *all the roots of eq. (6.9) have strictly negative real part except one, this also holds for a sufficiently small value of* ε *provided that*

$$\tau < \frac{1}{2\bar{\delta}} \arctan\left(\frac{\Re\{\lambda_3(0)\}}{\bar{\delta}}\right) = \tilde{\tau} \leq \tau_c(\varepsilon).$$

Therefore, the solutions $x(t)$ *and* $z(t)$ *of system (6.3) converge to the null space of matrix* $M(\varepsilon)$*, i.e.,*

$$\lim_{t\to\infty} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = r_1 = \alpha \begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix}.$$

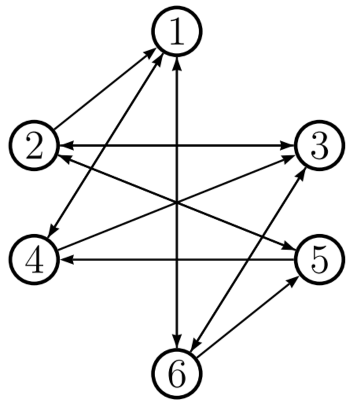


Figure 6-1: The directed graph considered in Section 6.4.

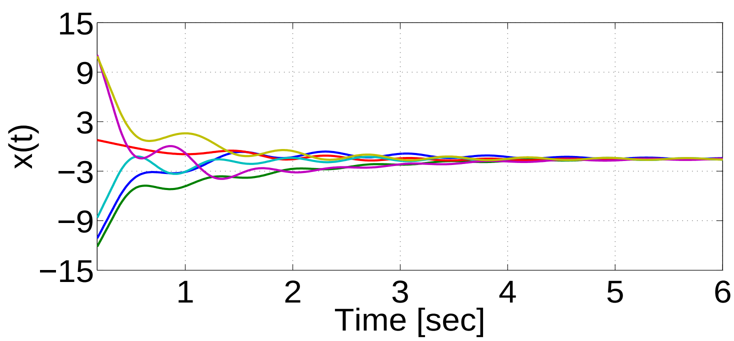


Figure 6-2: Evolution of $x(t)$ for $\varepsilon = 1.3$ and $\tau = 0.19$.

Since $\mathbf{1}_n^T x(t) + \mathbf{1}_n^T z(t) = \mathbf{1}_n^T x(0) + \mathbf{1}_n^T z(0)$ *for any* $t \geq 0$ *we have that*

$$\alpha = \frac{\mathbf{1}_n^T x(0) + \mathbf{1}_n^T z(0)}{n}.$$

Since by assumption $z(0) = \mathbf{0}_n$, *it holds*

$$\lim_{t \to \infty} x(t) = \frac{\mathbf{1}_n^T x(0)}{n} \mathbf{1}_n,$$

thus proving the statement. □

6.4 Numerical example and simulations

In this section we consider a numerical example to corroborate the theoretical results presented in the previous section.

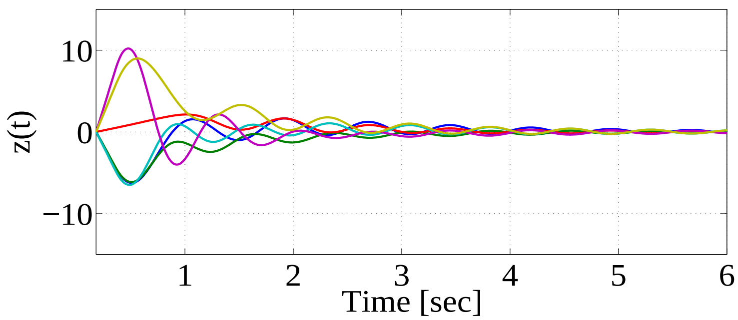


Figure 6-3: Evolution of $z(t)$ for $\varepsilon = 1.3$ and $\tau = 0.19$.

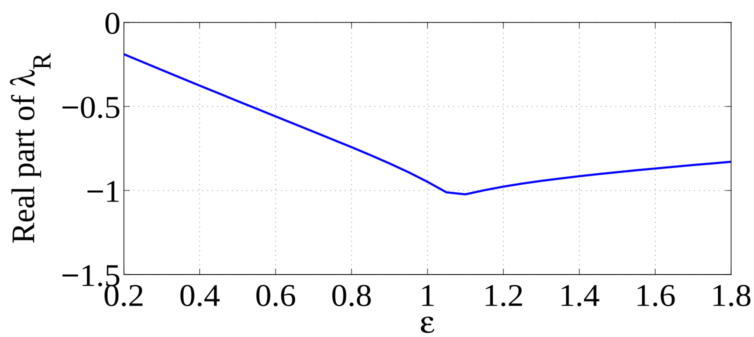


Figure 6-4: Real part of the rightmost non-null eigenvalue of matrix $M(\varepsilon)$ with respect to ε.

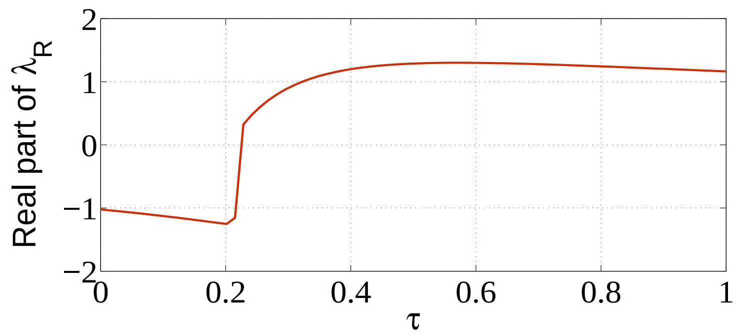


Figure 6-5: Real part of rightmost non-null root of eq. (6.9) with respect to τ, for $\varepsilon = 1.1$.

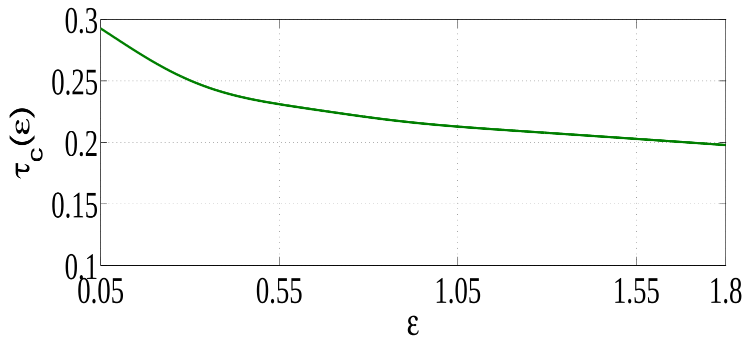


Figure 6-6: The value of $\tau_c(\varepsilon)$ with respect to ε.

We consider the network of 6 agents whose topology is shown in Fig. 6-1. Such a network is encoded by the adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{6.15}$$

The in and out-Laplacian matrices are, respectively

$$L_{in} = \begin{bmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

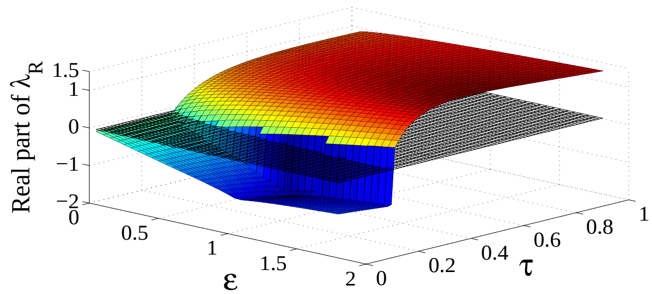


Figure 6-7: Value of the real part of the rightmost non-null root λ_R of eq. (6.9) versus increasing ε and time delay τ.

and

$$L_{out} = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & -1 \\ 0 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & -1 & 0 & 0 & 3 \end{bmatrix}.$$

Fig. 6-2 shows the evolution of system (6.3) when $\varepsilon = 1.3$ and $\tau = 0.18$. Initial conditions $x(0)$ are chosen uniformly at random while initial conditions $z(0) = \mathbf{0}_n$. Fig. 6-2 shows how consensus on the average of the initial state $x(0)$ is achieved. Fig. (6-3) presents the evolution of the storage variables $z(t)$. All storage variables are initially set to zero and then vary during the dynamical evolution of the system so that the quantity $\mathbf{1}_n^T x(t) + \mathbf{1}_n^T z(t)$ remains constant.

We now present the results of a series of numerical simulations whose aim is that of showing how the consensus achievement is related to parameters ε and τ. In particular, Fig. 6-4 shows how the rightmost non-null eigenvalue λ_R of matrix $M(\varepsilon)$ varies for $\varepsilon \in [0.2, 1.8]$. Fig. 6-4 shows that there exists an optimal value at $\varepsilon = 1.1$ for which matrix $M(\varepsilon)$ in the given example has the smallest rightmost non-null eigenvalue.

In Fig. 6-5 we show how the rightmost non-null root of eq. (6.9) varies for increasing values of the time-delay τ when $\varepsilon = 1.1$. Fig. 6-5 shows that despite the time-delay can make the system unstable, it can also improve the convergence speed to average consensus. For this example the optimal value of the time-delay is $\tau = 0.19$.

Fig. 6-6 shows the values of τ_c in eq. (6.14) for which eq. (6.9) has roots in the imaginary axis, i.e., it shows the maximum time delay sustainable by system in eq. (6.3) for the considered network topology in Fig. 6-1.

Finally, in Fig. 6-7 we show a plot of the real part of the rightmost non-null eigenvalue of eq. (6.9) for $\varepsilon \in (0, 2]$ and $\tau \in [0, 1]$. Fig. 6-7 shows how the convergence properties are affected by parameters ε and τ: there exists an optimal value at $\varepsilon = 1.1$ and $\tau = 0.19$ for which λ_R is the most negative and there exists a connected region of the plane defined by ε, τ where λ_R has strictly negative real part.

The rightmost non-null root of eq. (6.9) for a given set of (ε, τ) is computed using the spectral method with the heuristic presented by Wu and Michiels (2012).

6.5 Conclusions

The results of Zareh et al. (2013a) were addressed in this chapter. A continuous time version of a consensus on the average protocol for arbitrary strongly connected directed graphs was proposed and its convergence properties with respect to time delays in the local state update were characterized. The convergence properties of this algorithm depend upon a tuning parameter that can be made arbitrary small to prove stability of the networked system. Simulations were presented to corroborate the theoretical results and show that the existence of a small time delay can actually improve the algorithm performance. The future work will include an extension of the mathematical characterization of the proposed algorithm to consider possibly heterogeneous or time-varying delays.

7

Consensus in multi-agent systems with second-order dynamics and non-periodic sampled-data exchange

> *"In questions of science, the authority of a thousand is not worth the humble reasoning of a single individual."*
>
> – Galileo Galilei

In this chapter based on the results of Zareh et al. (2014a), consensus in second-order multi-agent systems with a non-periodic sampled-data exchange among agents is investigated. The sampling is random with bounded inter-sampling intervals. It is assumed that each agent has exact knowledge of its own state at all times. The considered local interaction rule is PD-type. The characterization of the convergence properties exploits a Lyapunov-Krasovskii functional method, sufficient conditions for stability of the consensus protocol to a time-invariant value are derived. Numerical simulations are presented to corroborate the theoretical results.

7.1 Introduction

This chapter deals with the problem of consensus in second-order MAS with a non-periodic data sending manner among the agents. We consider the case in which each agent has a perfect knowledge of its own state with almost no delay, i.e., it knows its own speed and position. Information exchanges between neighboring agents happens at discrete time intervals which are possibly non-periodic but strictly positive and bounded.

The network dynamics can thus be modeled as a *sampled-data system* (SDS), a class of systems extensively investigated in the literature.

For interesting contributions in this area we point the reader to Ackermann (1985); Fridman (2010); Zutshi et al. (2012) and the references therein. We also mention the work by Fridman et al. (2004) who exploited an approach for time-delay systems and obtained the sufficient stability conditions based on the Lyapunov-Krasovskii functional method. Seuret (2012) and Fridman (2010) proposed improved methods with better upper bounds to the maximum allowed delay. Shen et al. (2012) studied the sampled-data synchronization control problem for dynamical networks. Qin et al. (2010) and Ren and Cao (2008) studied the consensus problem for networks of double integrators with a constant sampling period. In the latter two papers, even though the authors use the sampled-data notation to introduce their novelty, they suppose that the communication and the local sensing occur simultaneously and this simplifies the problem into a discrete state consensus problem. Xiao and Chen (2012) and Yu et al. (2011) studied second-order consensus in multi-agent dynamical systems with sampled *position* data.

We propose a PD-like consensus algorithm with non-periodic sampled-data exchange among agents with bounded and strictly positive inter-sampling intervals. A characterization of the convergence properties exploiting a Lyapunov-Krasovskii functional method is provided and sufficient conditions for exponential stability of the consensus protocol to a time-invariant value are derived. Numerical simulations are presented to corroborate the theoretical results.

This chapter is organized as follows. In Section 7.2 some notation and preliminaries are introduced. In Section 7.3 the consensus problem for second order multi-agent systems with non-periodic sampled-data exchange is formalized. In Section 7.4 the convergence properties of the proposed consensus protocol are characterized. In Section 7.5 simulation results are presented to corroborate the theoretical analysis. In Section 7.6 concluding remarks and directions for future research are discussed.

7.2 Notation and Preliminaries

In this section we recall some basic notions on graph theory and introduce the notations.

The topology of bidirectional communication channels among the agents is represented by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes (agents) and $\mathcal{E} \subseteq \{\mathcal{V} \times \mathcal{V}\}$ is the set of edges. An edge $(i, j) \in \mathcal{E}$ exists if there is a communication channel between agent i and j. Self loops (i, i) are not considered. The set of neighbors of agent i is denoted by $\mathcal{N}_i = \{j \ : \ (j, i) \in \mathcal{E}; j = 1, \ldots, n\}$. Let $\delta_i = |\mathcal{N}_i|$ be the degree of agent i which represents the total number of its neighbors.

The topology of graph $\mathcal{G}$ is encoded by the so-called *adjacency matrix*, an $n \times n$ matrix A_d whose (i, j)-th entry is equal to 1 if $(i, j) \in \mathcal{E}$, 0 otherwise. Obviously in an undirected graph matrix A_d is symmetric.

We denote $\Delta = diag(\delta_1, \ldots, \delta_n)$ the diagonal matrix whose non null entries are the degrees of the nodes. Moreover, matrix $W_d = \Delta^{-1} A_d$ is the *weighted adjacency matrix* associated with $\mathcal{G}$. The following result holds.

Lemma 7.1 *If a graph $\mathcal{G}$ is connected then the eigenvalues of the weighted adjacency matrix W_d, namely $\lambda_i, \ i = 1, \ldots, n$, are all located in the interval $[-1, \ 1]$, and $\lambda_1 = 1$ is always a simple eigenvalue of W_d.*

Proof: Using Gershgorin theorem since all the diagonal elements of W_d are zero and each row sums up to 1, it immediately follows that $\lambda_i \in [-1, \ 1]$. Now, let $L = \Delta - A_d$ be the Laplacian matrix associated with the considered graph. If such a graph is connected, then the origin is a simple eigenvalue of L which implies that it is a simple eigenvalue also for $-\Delta^{-1}L = \Delta^{-1}A_d - I = W_d - I$. Consequently, if the graph is connected, $\lambda_1 = 1$ is a simple eigenvalue of the weighted adjacency matrix. $\square$

Finally, in the rest of this chapter we denote with $*$ the symmetric elements of symmetric matrices.

7.3 Problem Statement

Consider a second-order multi-agent system with an undirected communication topology. Consider the PD-type consensus protocol inspired by Cepeda-Gomez and Olgac (2011b) and Zareh et al. (2013b):

$$\begin{cases} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \frac{k_p}{\delta_i}\sum_{j\in\mathcal{N}_i} x_j(t) + \frac{k_d}{\delta_i}\sum_{j\in\mathcal{N}_i} v_j(t) \\ & \quad -k_p x_i(t) - k_d v_i(t), \end{cases} \tag{7.1}$$

where $i = 1, \ldots, n$, n denotes the number of agents, $x_i(t)$ and $v_i(t)$ are the position and the velocity of agent i, and δ_i indicates its degree.

We suppose that the local information, i.e., the information that each agent receives from its own sensors, is measured instantaneously. This obviously makes sense when the sensor dynamics are fast enough.

Moreover, we assume that the communication between the generic agent i and its set of neighbors $\mathcal{N}_i$ occurs in stochastic sampling time instants t_k, $k = 0, 1, \ldots, \infty$ that satisfy the following conditions:

$$0 < t_{k+1} - t_k \leq \bar{\tau} \in \mathbb{R}^+$$

and

$$\lim_{k\to\infty} t_k = \infty.$$

Under the above assumptions, equation (7.1) can be rewritten as:

$$\begin{cases} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \dfrac{k_p}{\delta_i}\sum_{j\in\mathcal{N}_i} x_j(t_k) + \dfrac{k_d}{\delta_i}\sum_{j\in\mathcal{N}_i} v_j(t_k) \\ & \quad -k_p x_i(t) - k_d v_i(t) \end{cases} \tag{7.2}$$

or, alternatively, doing some simple manipulations, as:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = (A \otimes I_n) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + (B \otimes W_d) \begin{bmatrix} x(t_k) \\ v(t_k) \end{bmatrix} \tag{7.3}$$

where $t \in [t_k, t_{k+1})$, $x = [x_1, x_2, \ldots, x_n]$, $v = [v_1, v_2, \ldots, v_n]$, $\Delta = diag\{\delta_1, \delta_2, \ldots, \delta_n\}$, A_d is the adjacency matrix, $W_d = \Delta^{-1}A_d$ is the weighted adjacency matrix, and matrices A and B are equal, respectively, to:

$$A = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ k_p & k_d \end{bmatrix}. \tag{7.4}$$

A MAS with an undirected communication topology and following equation (7.1), is said to converge to a *consensus state* if

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$$

and

$$\lim_{t\to\infty} |v_i(t) - v_j(t)| = 0.$$

In this chapter, given the value of the maximum admissible difference $\bar{\tau}$ between any two consecutive sampling time instants, and a communication topology with a given spectrum, we aim at finding conditions that guarantee consensus to a fixed point among agents that evolve according to equation (7.3).

We will also address the issue of evaluating an upper bound to the decay rate of convergence.

We conclude this section pointing out some differences among our problem formulation and the ones by Xiao and Chen (2012) and Yu et al. (2011). The most important difference is that we assume that each agent receives a message containing its neighbors' positions and velocities in a sampled-data basis. On the contrary, Xiao and Chen (2012) and Yu et al. (2011) supposed that the agents gather the sampled positions of their neighbors and their own at the same time instants.

7.4 Convergence properties

In the following subsection we first introduce a state variable transformation to decouple the dynamics of modes associated with the eigenvalues of the weighted adjacency matrix. Then, the stability of such modes is analyzed in detailed.

Stability analysis

Apply the following change of variables:

$$x(t) = Tz(t) \tag{7.5}$$

to eq. (7.3). Then, it holds:

$$\begin{aligned}(I_2 \otimes T)\begin{bmatrix}\dot{z}(t)\\ \ddot{z}(t)\end{bmatrix} &= (A \otimes T)\begin{bmatrix}z(t)\\ \dot{z}(t)\end{bmatrix}\\ &\quad +(B \otimes W_d T)\begin{bmatrix}z(t_k)\\ \dot{z}(t_k)\end{bmatrix}\end{aligned} \tag{7.6}$$

and eq. (7.3) can be rewritten as:

$$\begin{aligned}\begin{bmatrix}\dot{z}(t)\\ \ddot{z}(t)\end{bmatrix} &= (A \otimes I_n)\begin{bmatrix}z(t)\\ \dot{z}(t)\end{bmatrix}\\ &\quad +(B \otimes T^{-1} W_d T)\begin{bmatrix}z(t_k)\\ \dot{z}(t_k)\end{bmatrix}.\end{aligned} \tag{7.7}$$

Since W_d is a symmetrizable matrix, then it is also diagonalizable (Cepeda-Gomez and Olgac, 2011b), and the transformation matrix T can be chosen such that

$$\Lambda = T^{-1} W_d T = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

are the eigenvalues of the weighted adjacency matrix W_d. As a result, eq. (7.7) can be rewritten as:

$$\begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = (A \otimes I_n) \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} + (B \otimes \Lambda) \begin{bmatrix} z(t_k) \\ \dot{z}(t_k) \end{bmatrix},$$

or alternatively, as

$$\begin{bmatrix} \dot{z}_i(t) \\ \ddot{z}_i(t) \end{bmatrix} = A \begin{bmatrix} z_i(t) \\ \dot{z}_i(t) \end{bmatrix} + \lambda_i B \begin{bmatrix} z_i(t_k) \\ \dot{z}_i(t_k) \end{bmatrix} \tag{7.8}$$

where $i = 1, \ldots, n$, and $z_i(t)$ is the i-th element of vector $z(t)$.

Now, if we define

$$y_i(t) = [z_i(t) \;\; \dot{z}_i(t)]^T \tag{7.9}$$

the i-th mode of the system, we can say that its dynamics follows equation:

$$\dot{y}_i(t) = Ay_i(t) + \lambda_i B y_i(t_k). \tag{7.10}$$

Moreover, assuming $\tau(t) = t - t_k$, the above equation can be rewritten as:

$$\dot{y}_i(t) = Ay_i(t) + \lambda_i B y_i(t - \tau(t)). \tag{7.11}$$

The above SDS is a special case of a time varying delayed system where the delay $\tau(t)$ is upper bounded by $\bar{\tau}$, and its derivative is $\dot{\tau}(t) = 1$, while the delay switches at times $t = t_k$, $k = 0, 1, \ldots, \infty$.

In the rest of this chapter we assume that the graph $\mathcal{G}$ describing the communication topology is *connected*. By Lemma 7.1 this implies that its largest eigenvalue is $\lambda_1 = 1$. We call *unitary eigenvalue mode* (UEM) the mode associated with $\lambda_1 = 1$.

The following lemma characterizes the dynamics of the UEM. In particular it shows that the UEM converges asymptotically to a vector whose first entry $z_1(t)$ is equal to a constant value and the second entry $\dot{z}_1(t)$ is null.

Lemma 7.2 *Consider a system whose dynamics in the time interval $t \in [t_k, t_{k+1})$, $k = 0, 1, \ldots, \infty$, follows eq. (7.10) with $\lambda_i = 1$. Assume $t_{k+1} - t_k > 0$ for any $k = 0, 1, \ldots, \infty$. It holds*

$$\lim_{k \to \infty} z_1(t_k) = \gamma, \quad \gamma \in \mathbb{R}. \tag{7.12}$$

Proof: To prove this lemma we observe that by eq. (7.10) and by definition of matrices A and B, it follows that

$$\ddot{z}_1(t) + k_d \dot{z}_1(t) + k_p z_1(t) = k_d \dot{z}_1(t_k) + k_p z_1(t_k), \tag{7.13}$$

for $t \in [t_k \quad t_{k+1}]$. We consider two cases separately.

Case A

The characteristic polynomial associated with eq. (7.13) has two distinct roots. This corresponds to

$$\sigma = \frac{k_d^2}{4} - k_p \neq 0.$$

In such a case the solution of the above ordinary linear differential equation is equal to:

$$\begin{aligned} z_1(t) &= c_1 \dot{z}_1(t_k) e^{s_1(t-t_k)} - c_2 \dot{z}_1(t_k) e^{s_2(t-t_k)} \\ &\quad + z_1(t_k) + \frac{k_d}{k_p} \dot{z}_1(t_k), \end{aligned} \tag{7.14}$$

where

$$s_{1,2} = \frac{-k_d}{2} \pm \sqrt{\frac{k_d^2}{4} - k_p},$$
$$c_1 = \frac{1}{s_1 - s_2}(1 + \frac{k_d}{k_p}s_2),$$
$$c_2 = \frac{1}{s_1 - s_2}(1 + \frac{k_d}{k_p}s_1).$$

Now, let $T_k = t_{k+1} - t_k$. From (7.14) we can compute $z_1(t_{k+1})$ and $\dot{z}_1(t_{k+1})$ as:

$$\begin{bmatrix} z_1(t_{k+1}) \\ \dot{z}_1(t_{k+1}) \end{bmatrix} = M(T_k) \begin{bmatrix} z_1(t_k) \\ \dot{z}_1(t_k) \end{bmatrix} \tag{7.15}$$

where

$$M(T_k) = \begin{bmatrix} 1 & \mu_k \\ 0 & \beta_k \end{bmatrix}, \tag{7.16}$$

$$\mu_k = c_1 e^{s_1 T_k} - c_2 e^{s_2 T_k} + \frac{k_d}{k_p}, \tag{7.17}$$

and

$$\beta_k = c_1 s_1 e^{s_1 T_k} - c_2 s_2 e^{s_2 T_k}. \tag{7.18}$$

Therefore for all $k > 0$ it holds:

$$\begin{bmatrix} z_1(t_k) \\ \dot{z}_1(t_k) \end{bmatrix} = \bar{M}_k \begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \end{bmatrix}$$

where

$$\begin{aligned} \bar{M}_k &= M(T_k)M(T_{k-1})\dots M(T_0) \\ &= \begin{bmatrix} 1 & \sum_{m=0}^{k} \mu_m \prod_{j=0}^{m-1} \beta_j \\ 0 & \prod_{j=0}^{k} \beta_j \end{bmatrix}. \end{aligned} \tag{7.19}$$

We now prove that $|\beta_k| < 1$ where β_k is defined as in eq. (7.18).

Let

$$s_1 = \frac{-k_d}{2} + \sqrt{\sigma}, \quad s_2 = \frac{-k_d}{2} + \sqrt{\sigma}, \quad \sigma = \frac{k_d^2}{4} - k_p.$$

We consider separately the case of $\sigma > 0$ and $\sigma < 0$.

Case A1: $\sigma > 0$

In this case it is trivial to show that $s_1, s_2 \in \mathbb{R}$ and $s_2 < s_1 < 0$. Furthermore, we have $e^{s_2 T_k} < e^{s_1 T_k}$ and $c_1 s_1 - c_2 s_2 = 1$. We can also show that:

$$\begin{aligned} c_1 s_1 &= \frac{1}{s_1 - s_2}(s_1 + \frac{k_d}{k_p} s_1 s_2) \\ &= \frac{1}{2\sqrt{\sigma}}(\frac{k_d}{2} + \sqrt{\sigma}) > 0 \end{aligned}$$

and

$$\begin{aligned} c_2 s_2 &= \frac{1}{s_1 - s_2}(s_2 + \frac{k_d}{k_p} s_1 s_2) \\ &= \frac{1}{2\sqrt{\sigma}}(\frac{k_d}{2} - \sqrt{\sigma}) > 0. \end{aligned}$$

Let $\omega = \sqrt{\sigma}$ and $\nu = k_d/2 = \sqrt{\omega^2 + k_p}$. We get:

$$\beta_k = \frac{(\nu + \omega)e^{\omega T_k} - (\nu - \omega)e^{-\omega T_k}}{2\omega e^{\nu T_k}}. \tag{7.20}$$

Moreover, since $\sigma > 0$, it is $\omega \in (0,\ \infty)$ and therefore $\nu \in (\sqrt{k_p},\ \infty)$. For any $k_p > 0$ we obtain:

$$\begin{aligned} \lim_{\omega \to 0} \quad \beta_k &= \frac{1 + \sqrt{k_p} T_k}{e^{\sqrt{k_p} T_k}}, \\ \lim_{\omega \to \infty} \quad \beta_k &= 1. \end{aligned}$$

Hence due to the continuity in (7.20), for any value of k_p and k_d such that $\sigma > 0$, knowing that $T_k > 0$, we achieve

$$\beta_k \in \left(\frac{1 + \sqrt{k_p}}{e^{\sqrt{k_p}}},\ 1 \right)$$

thus prove the statement.

Case A2: $\sigma < 0$

In such a case s_1 and s_2 are complex conjugate numbers and

$$\begin{aligned}\beta_k = \;& (c_1 s_1 - c_2 s_2) e^{-T_k k_d/2} \cos(\sqrt{\sigma} T_k) + \\ & j(c_1 s_1 + c_2 s_2) e^{-T_k k_d/2} \sin(\sqrt{\sigma} T_k).\end{aligned}$$

Being $c_1 s_1 + c_2 s_2 = 0$ and $c_1 s_1 - c_2 s_2 = 1$ the second term vanishes and we get:

$$\beta_k = e^{-T_k k_d/2} \cos(\sqrt{\sigma}) < 1 \tag{7.21}$$

This leads us to

$$\lim_{k\to\infty} \prod_{j=0}^{k} \beta_j = 0.$$

Therefore, due to the fact that for all $m > 0$ the norm of μ_m is bounded by some $\bar{\mu} < \infty$, we can conclude that the term $\sum_{m=0}^{k} \mu_m \prod_{j=0}^{m-1} \beta_j$, which is obtained multiplying bounded numbers and exponentially decreasing products gets a constant bounded value $\bar{\Pi}$. Hence $\lim_{k\to\infty} z_1(t_k) = \lim_{t\to\infty} (z_1(0) + \bar{\Pi}\dot{z}_1(0))$ and $\lim_{k\to\infty} \dot{z}_1(t_k) = 0$ which in turn implies that there exists $\gamma \in \mathbb{R}$ such that:

$$\lim_{k\to\infty} z_1(t_k) = \gamma. \tag{7.22}$$

Case B

The characteristic polynomial of (7.13) has a single real root $s = -k_d/2$ with multiplicity 2.

In such a case the solution of eq. (7.13) is:

$$\begin{aligned} z_1(t) \;=\;& d_1 \dot{z}_1(t_k) t e^{s_1(t-t_k)} - d_2 \dot{z}_1(t_k) e^{s_2(t-t_k)} \\ & + z_1(t_k) + \frac{k_d}{k_p} \dot{z}_1(t_k), \end{aligned} \tag{7.23}$$

where

$$d_1 = \left(1 + \frac{k_d}{k_p}s\right) = 0$$
$$d_2 = \left(t_k + \frac{k_d}{k_p}t_k s + \frac{k_d}{k_p}\right) = \frac{2}{k_d}.$$

Therefore it is

$$\begin{bmatrix} z_1(t_{k+1}) \\ \dot{z}_1(t_{k+1}) \end{bmatrix} = M'(T_k) \begin{bmatrix} z_1(t_k) \\ \dot{z}_1(t_k) \end{bmatrix}, \tag{7.24}$$

where

$$M'(T_k) = \begin{bmatrix} 1 & \mu'_k \\ 0 & \beta'_k \end{bmatrix},$$

with $\mu'_k = \frac{k_d}{k_p}(1 - e^{sT_k})$, and $\beta'_k = -e^{sT_k}$. Since for any $T_k > 0$, it is $|\beta_k| < 1$, then, repeating the same reasoning as in Case A, we conclude that there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{k \to \infty} z_1(t_k) = \gamma. \tag{7.25}$$

□

We now characterize the conditions on the design parameters $k_p, k_d, \bar{\tau}$ under which the modes $y_i(t)$, $i = 2, \ldots, n$, defined in eq. (7.9) are exponentially stable.

To do this we provide the following lemma, whose proof is inspired by Seuret (2012).

Lemma 7.3 *Consider the generic mode $y_i(t)$ defined in eq. (7.9) whose dynamics follows eq. (7.11). Matrices A, B are defined as in eq. (7.4), $\tau(t) = t - t_k$, $k = 0, 1, \ldots, \infty$, and $\lambda_i \in [-1, 1)$.*

Assume that the difference between any two consecutive sampling times is smaller than a given $\bar{\tau}$, i.e., it holds $t_{k+1} - t_k \leq \bar{\tau}$ for all $k = 0, 1, \ldots, \infty$.

If there exist symmetric positive definite matrices P_i, R_i, $S_i \in \mathbb{R}^{2\times 2}$, a matrix $Q_i = \begin{bmatrix} Q_{i,1} \\ Q_{i,2} \end{bmatrix} \in \mathbb{R}^{4\times 2}$ and a constant value $\alpha > 0$ such that the following inequalities are satisfied:

$$\begin{bmatrix} \Psi_{i,11}(\bar{\tau}, \alpha) & \Psi_{i,12}(\bar{\tau}, \alpha) \\ * & \Psi_{i,22}(\bar{\tau}, \alpha) \end{bmatrix} < 0, \tag{7.26}$$

$$\begin{bmatrix} \Psi_{i,11}(0,\alpha) & \Psi_{i,12}(0,\alpha) & \bar{\tau}Q_{i,1} \\ * & \Psi_{i,22}(0,\alpha) & \bar{\tau}Q_{i,2} \\ * & * & -\bar{\tau}(1-2\alpha\bar{\tau})R_i \end{bmatrix} < 0 \tag{7.27}$$

where

$$\begin{aligned} \Psi_{i,11}(\bar{\tau},\alpha) = {} & P_iA + A^TP_i - S_i - Q_{i,1} - Q_{i,1}^T \\ & +\bar{\tau}(S_iA + A^TS_i + A^TR_iA + 2\alpha S_i) \\ & +2\alpha P_i - 2\alpha R_i, \end{aligned}$$

$$\begin{aligned} \Psi_{i,12}(\bar{\tau},\alpha) = {} & \lambda_iP_iB + S_i + 2\alpha R_i + Q_{i,1} - Q_{i,2}^T \\ & +\bar{\tau}(-A^TS_i + \lambda_iS_iB + \lambda_iA^TR_iB - 2\alpha S_i), \end{aligned}$$

$$\begin{aligned} \Psi_{i,22}(\bar{\tau},\alpha) = {} & -S_i - 2\alpha R_i + Q_{i,2} + Q_{i,2}^T \\ & -\bar{\tau}(\lambda_iB^TS + \lambda_iS_iB - \lambda_i^2B^TR_iB + 2\alpha S_i), \end{aligned}$$

then mode $y_i(t)$ is exponentially stable with decay rate α.

Proof: Consider the following functional:

$$\begin{aligned} V_i(t, y_i(t), y_i(t_k)) \;=\;& y_i^T(t)P_iy_i(t) \\ & + (\bar{\tau} - \tau(t))\,\xi_i^T(t)S_i\xi_i(t) \\ & + (\bar{\tau} - \tau(t))\int_{t_k}^{t} \dot{y}_i{}^T(s)R_i\dot{y}_i(s)ds, \end{aligned} \tag{7.28}$$

where

$$\xi_i(t) = y_i(t) - y_i(t_k). \tag{7.29}$$

Obviously $\dot{\xi}_i(t) = \dot{y}_i(t)$. Note that the second and the third term of the functional vanish during the jump due to the fact that $\lim_{t\to t_k} y_i(t) = y_i(t_k)$ which leads to $\lim_{t\to t_k} V(t) \leq V(t_k^-)$. Hence we should look the functional only inside the intervals without being worried about the jumps.

Derivating eq. (7.28) with respect to time we get:

$$\begin{aligned}
\dot{V}_i(t, y_i(t), y_i(t_k)) &= y_i^T(t)\Big(P_iA + A^TP_i - S_i \\
&+ \big(\bar{\tau} - \tau(t)\big)(S_iA + A^TS_i + A^TR_iA)\Big)y_i(t) \\
&+ 2y_i^T(t)\Big(\lambda_iP_iB + S_i + \big(\bar{\tau} - \tau(t)\big)(S_iA \\
&+ A^TS_i + A^TR_iA)\Big)y_i(t_k) \\
&+ y_i^T(t_k)\Big(-S_i - \big(\bar{\tau} - \tau(t)\big)(\lambda_iB^TS_i + \lambda_iS_iB \\
&- \lambda_i^2B^TR_iB\Big)y_i(t_k) - \int_{t_k}^{t}\dot{y}_i^T(s)R_i\dot{y}_i(s)ds.
\end{aligned} \tag{7.30}$$

Now consider the following candidate functional:

$$\begin{aligned}
&W_i(t, y_i(t), y_i(t_k), \alpha) \\
&= \dot{V}(t, y_i(t), y_i(t_k)) + 2\alpha V_i(t, y_i(t), y_i(t_k)) \\
&= y_i^T(t)\Big(P_iA + A^TP_i - S_i + 2\alpha P_i \\
&+ \big(\bar{\tau} - \tau(t)\big)(S_iA + A^TS_i + A^TR_iA + 2\alpha S_i)\Big)y_i(t) \\
&+ 2y_i^T(t)\Big(\lambda_iP_iB + S_i + \big(\bar{\tau} - \tau(t)\big)(S_iA \\
&+ A^TS_i + A^TR_iA - 2\alpha S_i)\Big)y_i(t_k) \\
&+ y_i^T(t_k)\Big(-S_i - \big(\bar{\tau} - \tau(t)\big)(\lambda_iB^TS_i + \lambda_iS_iB \\
&- \lambda_i^2B^TR_iB + 2\alpha S_i\Big)y_i(t_k) \\
&- (1 - 2\alpha(\bar{\tau} - \tau(t)))\int_{t_k}^{t}\dot{y}_i^T(s)R_i\dot{y}_i(s)ds.
\end{aligned} \tag{7.31}$$

To ensure the exponential stability of mode $y_i(t)$ with decay rate α it is sufficient to prove that:

$$W_i(t, y_i(t), y_i(t_k), \alpha) < 0.$$

We manipulate the integral term

$$-\left(1 - 2\alpha(\bar{\tau} - \tau(t))\right)\int_{t_k}^{t}\dot{y}_i^T(s)R_i\dot{y}_i(s)ds \tag{7.32}$$

to achieve a bound on that based on a function of $y_i(t)$ and $y_i(t_k)$. To this aim, we rewrite the above term as the summation of two terms

$$-(1-2\alpha\bar{\tau})\int_{t_k}^{t}\dot{y}_i^T(s)R_i\dot{y}_i(s)ds \tag{7.33}$$

and

$$-2\alpha\tau(t)\int_{t_k}^{t}\dot{y}_i^T(s)R_i\dot{y}_i(s)ds \tag{7.34}$$

and provide an upper bound to each term separately.

To provide an upper bound to (7.33), we introduce the following inequality for two vectors ω_1 and ω_2 and an arbitrary matrix Γ with compatible dimensions:

$$2\omega_1^T\omega_2 \leq \omega_1^T\Gamma^{-1}\omega_1 + \omega_2^T\Gamma\omega_2.$$

Rewriting the above inequality assuming $\omega_1 = Q_i^T\begin{bmatrix} y_i(t) \\ y_i(t_k) \end{bmatrix}$, $\omega_2 = \dot{y}_i(s)$ and $\Gamma = (1-2\alpha\bar{\tau})R_i$, we get:

$$\begin{gathered} 2[y_i^T(t) \quad y_i^T(t_k)]Q_i\dot{y}_i(s) \leq \\ \begin{bmatrix} y_i^T(t) & y_i^T(t_k) \end{bmatrix} Q_i\frac{R_i^{-1}}{1-2\alpha\bar{\tau}}Q_i^T\begin{bmatrix} y_i(t) \\ y_i(t_k) \end{bmatrix} \\ +(1-2\alpha\bar{\tau})\dot{y_i}^T(s)R_i\dot{y}_i(s). \end{gathered}$$

Integrating it in the interval $[t_k,\ t]$ in which $\dot{y}_i(t)$ is continuous we obtain:

$$\begin{gathered} -(1-2\alpha\bar{\tau})\int_{t_k}^{t}\dot{y}_i^T(s)R\dot{y}_i(s)ds \leq \\ -2[y_i^T(t) \quad y_i^T(t_k)]Q_i\xi_i(t) \\ +\tau(t)\begin{bmatrix} y_i^T(t) & y_i^T(t_k) \end{bmatrix} Q_i\frac{R_i^{-1}}{1-2\alpha\bar{\tau}}Q_i^T\begin{bmatrix} y_i(t) \\ y_i(t_k) \end{bmatrix}. \end{gathered} \tag{7.35}$$

To provide an upper bound to (7.34) we use Jensen integral inequality (Xu and

Lam, 2008):

$$
\begin{aligned}
&-2\alpha\tau(t)\int_{t_k}^{t}\dot{y}_i^T(s)R_i\dot{y}_i(s)ds \leq \\
&-2\alpha\int_{t_k}^{t}\dot{y}_i^T(s)dsR_i\int_{t_k}^{t}\dot{y}_i(s)ds \\
&= -2\alpha(y_i(t)-y_i(t_k))^T R_i(y_i(t)-y_i(t_k))
\end{aligned}
\tag{7.36}
$$

Introducing inequalities (7.35) and (7.36) in (7.31), the following inequality is achieved for $t \in [t_k,\ t_{k+1})$:

$$
\begin{aligned}
&W_i(t, y_i(t), y_i(t_k)) \leq [y_i^T(t) \quad y_i^T(t_k)] \\
&\Bigg(\begin{bmatrix} \Psi_{i,11}(\bar{\tau}-\tau(t),\alpha) & \Psi_{i,12}(\bar{\tau}-\tau(t),\alpha) \\ * & \Psi_{i,22}(\bar{\tau}-\tau(t),\alpha) \end{bmatrix} \\
&+\frac{\tau(t)}{1-2\alpha\bar{\tau}}Q_i R_i^{-1}Q_i^T\Bigg)\begin{bmatrix} y_i(t) \\ y_i(t_k) \end{bmatrix}.
\end{aligned}
\tag{7.37}
$$

The above inequality corresponds to an LMI that is linear with respect to $\tau(t)$. Therefore, according to Scherer and Weiland (2000), in order to be sure that it holds for all $\tau(t) \in [0,\ \bar{\tau}]$ we only need to check it at the boundary of the interval, namely for $\tau(t) = 0$ and $\tau(t) = \bar{\tau}$.

Now, if we particularize eq. (7.37) with $\tau(t) = 0$ this obviously leads to the LMI in eq. (7.26).

To complete the proof we need to show that particularizing eq. (7.37) with $\tau(t) = \bar{\tau}$ we get the LMI in eq. (7.27). But this follows from the fact that

$$
\begin{bmatrix} \Psi_{i,11}(0,\alpha) & \Psi_{i,12}(0,\alpha) \\ * & \Psi_{i,22}(0,\alpha) \end{bmatrix} + \frac{\bar{\tau}}{1-2\alpha\bar{\tau}}Q_i R_i^{-1}Q_i^T
\tag{7.38}
$$

is the Schur complement of matrix $-\bar{\tau}(1-2\alpha\bar{\tau})R_i$ in eq. (7.27). Thus, if the LMI in eq. (7.27) is definite negative, also it is matrix in eq. (7.38). □

Consensus among agents

We now prove the main result, namely the consensus of agents to a common position.

Theorem 7.4 *Consider a MAS evolving according to equation (7.2) where $\bar{\tau}$ is such that $0 < t_{k+1} - t_k < \bar{\tau} < \infty$. Let λ_i, $i = 2, \ldots, n$ be the eigenvalues of the weighted adjacency matrix associated with the undirected connected graph $\mathcal{G}$ modeling the communication topology. If there exists a positive constant α such that the LMIs defined in eq. (7.26) and (7.27) are satisfied for all λ_i, $i = 2, \ldots, n$, then there exists a $\gamma \in \mathbb{R}$ such that $x(t)$ exponentially converges to $\gamma\vec{1}$ and $v(t)$ exponentially converges to $\vec{0}$. Moreover, the rate of convergence is greater than or equal to α.*

Proof: By Lemma 7.3, if the LMIs in eq. (7.26) and (7.27) hold, all modes except the UEM are stable, i.e., $\lim_{t\to\infty} y_i(t) = 0$ and thus $\lim_{t\to\infty} z_i(t) = 0$ for $i = 2, \ldots, n$ with rate of convergence of at least α. Furthermore, by Lemma 7.2, there exists a positive constant $\gamma \in \mathbb{R}$ such that $\lim_{t\to\infty} z_1(t) = \gamma$.

Now, the first column of T is the eigenvector corresponding to the unitary eigenvalue of W_d, therefore it is equal to $\vec{1} = [1 \quad 1 \quad \ldots, \quad 1]^T$. Thus, being $x(t) = T[z_1(t) \quad 0 \quad \ldots \quad 0]^T$, it is trivial to show that when $t \to \infty$ it is $x_i(t) = x_j(t)$, for all $i, j = 1, \ldots, n$. The same calculations can be repeated for the velocities, thus proving that for $t \to \infty$, it is $v_i(t) = v_j(t)$, $i, j = 1, \ldots, n$. $\square$

7.5 Simulation results

In this section we present the results of some numerical simulation that shows the effectiveness of the consensus protocol in eq. (7.3). To this aim we consider a system with 6 agents and adjacency matrix:

$$A_d = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

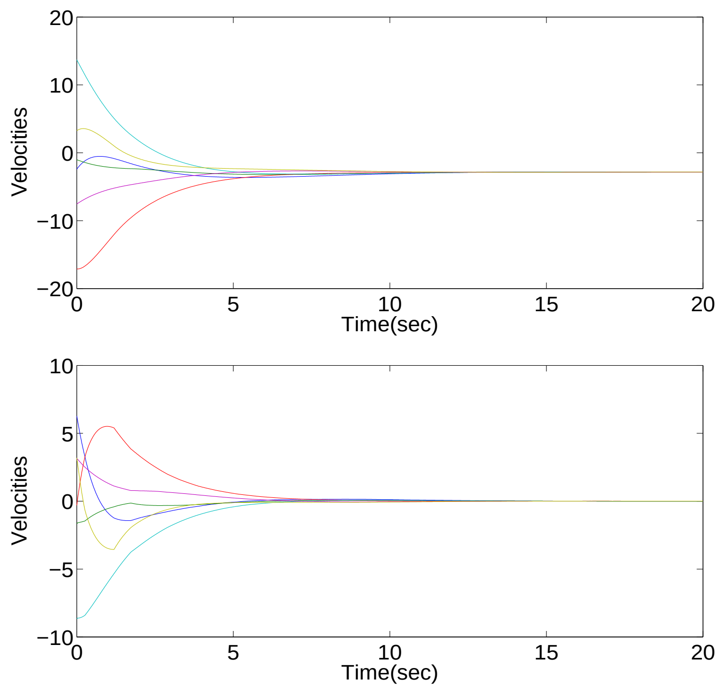


Figure 7-1: Positions and velocities when the proposed protocol is implemented.

Assume $k_p = 1$, $k_d = 2$ and $\bar{\tau} = 1$. Using the above LMIs with $\alpha = 0.38$ we can prove that the system reaches consensus to a fixed point.

Fig. 7-1 shows the evolution of positions and velocities when the proposed algorithm is implemented, while Fig. 7-2 shows the sampled positions and velocities aperiodically transmitted to neighbors by each agent.

We conclude this section presenting the results of another numerical simulation carried out under the assumption that only sampled positions are transmitted to neighbors, i.e., the second term is removed in eq. (7.2) that is equivalent to redefine B as $B' = [0\ 0; k_p\ 0]$.

It can be proved that in such a case the consensus to a fixed point is still reached, but with decay rate bounded by 0.21 that is almost the half of the previous case. Such a conclusion can also be drawn by looking at Fig. 7-3.

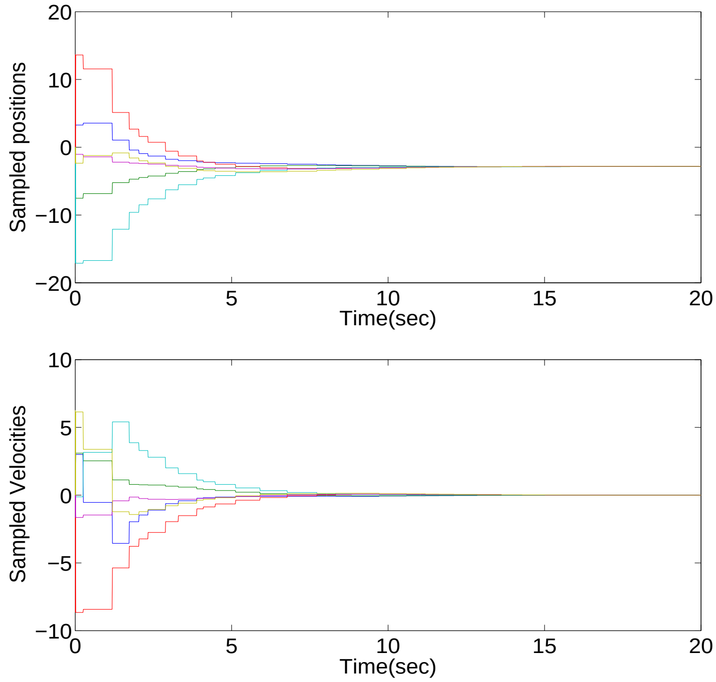


Figure 7-2: Aperiodic sampled positions and velocities when the proposed protocol is implemented.

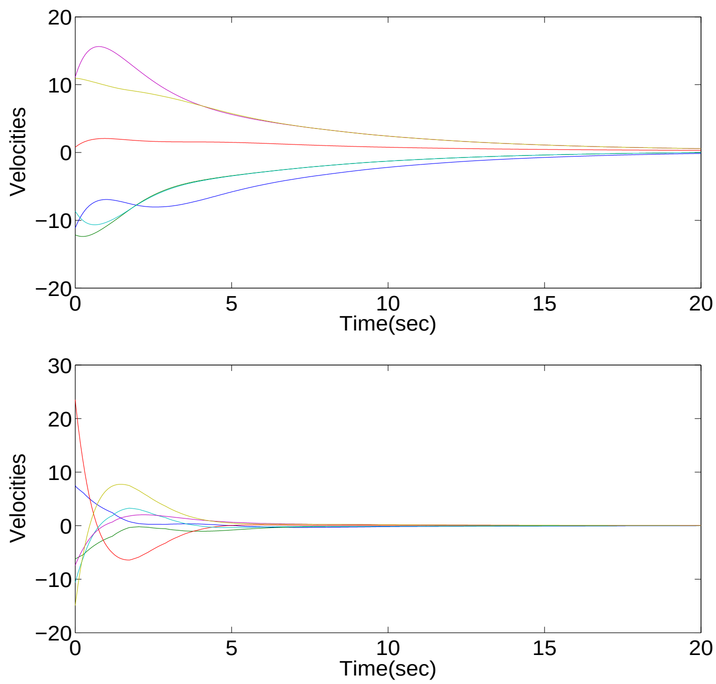


Figure 7-3: Positions and velocities when the proposed protocol is modified in order to only consider sampled positions.

7.6 Conclusions and future work

The contribution of this chapter consists in a PD-like consensus algorithm for a second-order multi-agent system where, at non-periodic sampling times, agents transmit to their neighbors information about their position and velocity, while each agent has a perfect knowledge of its own state at any time instant. Conditions have been given to prove consensus to a common fixed point, based on LMIs verification. Moreover, we also show how it is possible to evaluate an upper bound on the decay rate of exponential convergence of stable modes.

The main directions of our future research in this framework are

(i) We want to also study the case where agents do not have a perfect knowledge of their own state.

(ii) Finally, we plan to relax the assumption that all communications among agents occur simultaneously.

8

Non-periodic sampled-data consensus in second-order multi-agent systems with communication delays over an uncertain network

> *"Give me a lever long enough and a fulcrum on which to place it, and I shall move the world."*
>
> – Archimedes

In this chapter consensus in second-order multi-agent systems with a non-periodic sampled-data exchange among agents is investigated. The sampling is random with bounded inter-sampling intervals. It is assumed that each agent has exact knowledge of its own state at any time instant. The considered local interaction rule is PD-type. Sufficient conditions for stability of the consensus protocol to a time-invariant value are derived based on LMIs. Such conditions only require the knowledge of the connectivity of the graph modeling the network topology. Numerical simulations are presented to corroborate the theoretical results.

8.1 Introduction

Due to its broad spectrum of applications, in the past years, a large attention has been devoted to the consensus problem in multi-agent systems (MAS) Qin et al. (2011); Ren et al. (2005a); Yu et al. (2010); Zareh et al. (2013a). Sensor networks Yu et al. (2009); Olfati-Saber and Shamma (2005), automated highway systems Ren et al. (2005a), mobile robotics Khoo et al. (2009), satellite alignment Ren (2007a) and several more, are some of the potential areas in which a consensus problem is taken into account. Consensus is a state of a networked multi-agent system in which all the agents reach agreement on a common value by only sharing information locally, namely with their neighbors. Several algorithms, often called *consensus protocols*, have been proposed that lead a MAS to consensus. In particular, the coordination problem of mobile robots finds several applications in the manufacturing industry in the context of automated material handling. The consensus problem in the context of mobile robots consists in the design of local state update rules which allow the network of robots to rendezvous at some point in space or follow a leading robot exploiting only measurements of speeds and relative positions between neighboring robots. Robots are hereafter referred to as agents.

In MAS, heavy computational loads can interrupt the sampling period of a certain controller. A scheduled sampling period can be used to deal with this problem. In such a case robust stability analysis with respect to the changes in the sampling time is necessary. For interesting contributions in this area we address the reader to Ackermann (1985); Fridman (2010); Zutshi et al. (2012) and the references therein. We also mention the work by Fridman et al. (2004) who exploited an approach for time-delay systems and obtained the sufficient stability conditions based on the Lyapunov-Krasovskii functional method. Seuret Seuret (2012) and Fridman Fridman (2010) proposed methods with better upper bounds to the maximum allowed sampling. Shen *et al.* Shen et al. (2012) studied the sampled-data synchronization control problem for dynamical networks. Qin *et al.* Qin et al. (2010) and Ren and Cao Ren and Cao (2008) studied the consensus problem for networks of double integrators with a constant sampling period. In the latter two papers, even though the authors use the sampled-data notation to introduce their novelty, they suppose that the communication and the local sensing occur simultaneously and this simplifies the problem into a discrete state consensus problem. Xiao and Chen Xiao and Chen (2012) and Yu *et al.* Yu et al. (2011) studied second-order consensus in multi-agent dynamical systems with sampled *position* data.

In this chapter, we consider the case in which each agent has a perfect knowledge of its own state with almost no delay, i.e., it knows its own speed and position. Information exchanges between neighboring agents happen at discrete time intervals which are possibly non-periodic but strictly positive and bounded. The network dynamics can thus be modeled as a *sampled-data system* (SDS), a class of systems extensively investigated in the literature. Using PD-like algorithm we guarantee that all the agents reach consensus. We proposed such a protocol in Chapter 7 where we provided a characterization of the convergence properties exploiting a Lyapunov-Krasovskii functional method. In particular in Chapter 7 we provided sufficient conditions for exponential stability of the consensus protocol to a time-invariant value under the assumption that the spectrum of the weighted adjacency matrix is known. With respect to Chapter 7, in this chapter we relax such assumption and provide sufficient conditions for consensus under the assumption that the only information on the network topology is its connectivity, i.e., the second largest eigenvalue of the weighted adjacency matrix. This is obviously a significant improvement with respect to the previous chapter, not only because much less information on the network topology is needed, but also because, despite of Chapter 7, the number of LMIs that have to be computed does not depend on the number of agents.

The chapter is organized as follows. In Section 8.2 the consensus problem for second order multi-agent systems with non-periodic sampled-data exchange is formalized. In Section 8.3 the convergence properties of the proposed consensus protocol are characterized. In Section 8.4 simulation results are presented to corroborate the theoretical analysis. Finally, in Section 8.5 concluding remarks and directions for future research are discussed.

8.2 Problem Statement

Consider a second-order multi-agent system with an undirected communication topology. Consider the PD-type consensus protocol introduced in (7.1).

We suppose that the local information, i.e., the information that each agent receives from its own sensors, is measured instantaneously. This obviously makes sense when the sensor dynamics are fast enough.

Moreover, we assume that the communication between the generic agent i and its set of neighbors $\mathcal{N}_i$ occurs in stochastic sampling time instants t_k, $k = 0, 1, \ldots, \infty$, that satisfy the following conditions:

$$0 < t_{k+1} - t_k \leq \bar{\tau} \in \mathbb{R}^+$$

and

$$\lim_{k\to\infty} t_k = \infty.$$

Under the above assumptions, equation (7.1) can be rewritten as:

$$\begin{cases} \dot{x}_i(t) & = & v_i(t), \\ \dot{v}_i(t) & = & \dfrac{k_p}{\delta_i} \sum_{j\in\mathcal{N}_i} x_j(t_k) + \dfrac{k_d}{\delta_i} \sum_{j\in\mathcal{N}_i} v_j(t_k) \\ & & -k_p x_i(t) - k_d v_i(t) \end{cases} \tag{8.1}$$

or, alternatively, doing some simple manipulations, as:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = (A \otimes I_n) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + (B \otimes W_d) \begin{bmatrix} x(t_k) \\ v(t_k) \end{bmatrix} \tag{8.2}$$

where $x = [x_1, x_2, \ldots, x_n]$, $v = [v_1, v_2, \ldots, v_n]$, $\Delta = Diag\{\delta_1, \delta_2, \ldots, \delta_n\}$, A_d is the adjacency matrix, W_d is the weighted adjacency matrix, and matrices A and B are equal, respectively, to:

$$A = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ k_p & k_d \end{bmatrix}. \tag{8.3}$$

A MAS with an undirected communication topology and following equation (7.1), is said to converge to a *consensus state* if

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$$

and

$$\lim_{t\to\infty} |v_i(t) - v_j(t)| = 0.$$

In this chapter, given the value of the maximum admissible difference $\bar{\tau}$ between any two consecutive sampling time instants, and a communication topology whose connectivity is known to be smaller than or equal to a given value $\bar{\lambda}$, we aim at finding conditions that guarantee consensus to a fixed point among agents that evolve according to equation (8.2).

8.3 Convergence properties

In the following subsection we recall a state variable transformation, firstly introduced in Chapter 7, to decouple the dynamics of modes associated with the eigenvalues of the weighted adjacency matrix. Then, the stability of such modes is analyzed in detail.

Stability analysis

Apply the following change of variables:

$$x(t) = Tz(t) \tag{8.4}$$

to eq. (8.2). Then, it holds:

$$(I_2 \otimes T) \begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = (A \otimes T) \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} + (B \otimes W_d T) \begin{bmatrix} z(t_k) \\ \dot{z}(t_k) \end{bmatrix} \tag{8.5}$$

and eq. (8.2) can be rewritten as:

$$\begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = (A \otimes I_n) \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} + (B \otimes T^{-1} W_d T) \begin{bmatrix} z(t_k) \\ \dot{z}(t_k) \end{bmatrix}. \tag{8.6}$$

Since W_d is a symmetrizable matrix, then it is also diagonalizable, and the transformation matrix T can be chosen such that

$$\Lambda = T^{-1} W_d T = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

are the eigenvalues of the weighted adjacency matrix W_d. As a result, eq. (8.6) can be rewritten as:

$$\begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = (A \otimes I_n) \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} + (B \otimes \Lambda) \begin{bmatrix} z(t_k) \\ \dot{z}(t_k) \end{bmatrix},$$

or alternatively, as

$$\begin{bmatrix} \dot{z}_i(t) \\ \ddot{z}_i(t) \end{bmatrix} = A \begin{bmatrix} z_i(t) \\ \dot{z}_i(t) \end{bmatrix} + \lambda_i B \begin{bmatrix} z_i(t_k) \\ \dot{z}_i(t_k) \end{bmatrix} \tag{8.7}$$

where $i = 1, \ldots, n$, and $z_i(t)$ is the i-th element of vector $z(t)$.

Now, if we define

$$y_i(t) = [z_i(t) \;\; \dot{z}_i(t)]^T \tag{8.8}$$

the i-th mode of the system, we can say that its dynamics follows equation:

$$\dot{y}_i(t) = Ay_i(t) + \lambda_i B y_i(t_k). \tag{8.9}$$

Moreover, assuming $\tau(t) = t - t_k$, the above equation can be rewritten as:

$$\dot{y}_i(t) = Ay_i(t) + \lambda_i B y_i(t - \tau(t)). \tag{8.10}$$

The above SDS is a special case of a time varying delayed system where the delay $\tau(t)$ is upper bounded by $\bar{\tau}$, and its derivative is $\dot{\tau}(t) = 1$, while the delay switches at times $t = t_k$, $k = 0, 1, \ldots, \infty$.

We assume that the graph $\mathcal{G}$ describing the communication topology is *connected*. By Lemma 7.1 this implies that its largest eigenvalue is $\lambda_1 = 1$. We call *unitary eigenvalue mode* (UEM) the mode associated with $\lambda_1 = 1$.

Based on Lemma 7.2, we can characterize the dynamics of the UEM. In particular it shows that the UEM converges asymptotically to a vector whose first entry $z_1(t)$ is equal to a constant value and the second entry $\dot{z}_1(t)$ is null. In other words

$$\lim_{k \to \infty} z_1(t_k) = \gamma, \quad \gamma \in \mathbb{R}. \tag{8.11}$$

We now provide the main contribution of this chapter, i.e., we characterize the conditions on the design parameters $k_p, k_d, \bar{\tau}, \bar{\lambda}$ under which the modes $y_i(t)$, $i = 2, \ldots, n$, defined in eq. (8.8) are asymptotically stable provided that $\lambda_i \leq \bar{\lambda}$ for all $i = 2, \ldots, n$.

Theorem 8.1 *Consider the generic mode $y_i(t)$ defined in eq. (8.8) whose dynamics follows eq. (8.10) where λ_i is an uncertain parameter in $[-1, \bar{\lambda}]$, and obviously $\bar{\lambda} < 1$.*

If there exist positive definite matrices P and R and square matrices Q_1 and Q_2

such that the following inequalities hold:

$$M_1 = \begin{bmatrix} Q_1^T(A-B)+ & P - Q_1^T + \\ (A-B)^T Q_1 & (A-B)^T Q_2 \\ & \\ * & -Q_2 - Q_2^T + \bar{\tau} R \end{bmatrix} < 0 \quad (8.12)$$

$$M_2 = \begin{bmatrix} Q_1^T(A+\bar{\lambda}B)+ & P - Q_1^T + \\ (A+\bar{\lambda}B)^T Q_1 & (A+\bar{\lambda}B)^T Q_2 \\ & \\ * & -Q_2 - Q_2^T + \bar{\tau} R \end{bmatrix} < 0 \quad (8.13)$$

$$M_3 =$$

$$\begin{bmatrix} Q_1^T(A-B)+ & P - Q_1^T + & \bar{\tau} Q_1^T B \\ (A-B)^T Q_1 & (A-B)^T Q_2 & \\ & & \\ * & -Q_2 - Q_2^T & \bar{\tau} Q_2^T B \\ & & \\ * & * & -\bar{\tau} R \end{bmatrix} \quad (8.14)$$

$$< 0$$

$$M_4 =$$

$$\begin{bmatrix} Q_1^T(A+\bar{\lambda}B)+ \\ (A+\bar{\lambda}B)^T Q_1 & P - Q_1^T + \\ (A+\bar{\lambda}B)^T Q_2 & -\bar{\tau}\bar{\lambda}Q_1^T B \\ * & -Q_2 - Q_2^T & -\bar{\tau}\bar{\lambda}Q_2^T B \\ * & * & -\bar{\tau}R \end{bmatrix} \tag{8.15}$$

$$< 0$$

then the system with dynamics (8.10) is asymptotically stable.

Proof: Consider the Lyapunov function

$$\begin{aligned} V(t, y_i(t), y_i(t_k)) = \ & y_i^T(t) P y_i(t) \\ & + (\bar{\tau} - \tau(t)) \int_{t_k}^{t} \dot{y}_i{}^T(s) R \dot{y}_i(s) ds. \end{aligned} \tag{8.16}$$

It holds:

$$\begin{aligned} \dot{V}(t, y_i(t), y_i(t_k)) = 2\dot{y}_i^T(t) P y_i(t) \\ - \int_{t_k}^{t} \dot{y}_i{}^T(s) R \dot{y}_i(s) ds + \\ (\bar{\tau} - \tau(t)) \left(\dot{y}_i{}^T(t) R \dot{y}_i(t) - \dot{y}_i{}^T(t_k) R \dot{y}_i(t_k) \right). \end{aligned} \tag{8.17}$$

To provide an upper bound to (8.17) we use Jensen integral inequality:

$$\int_{t_k}^{t} \dot{y}_i^T(s) R \dot{y}_i(s) ds \leq \int_{t_k}^{t} \dot{y}_i^T(s) ds R \int_{t_k}^{t} \dot{y}_i(s) ds. \tag{8.18}$$

Define $\xi_i(t) = \dfrac{1}{\tau(t)} \displaystyle\int_{t_k}^{t} \dot{y}_i(s) ds.$

We get:

$$\int_{t_k}^{t} \dot{y}_i^T(s) R \dot{y}_i(s) ds \leq \tau(t) \xi_i^T(t) R \xi_i(t) \tag{8.19}$$

From the descriptor method (Fridman and Shaked, 2002) we know:

$$[y_i(t) \quad \dot{y}_i(t)] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \cdot \\ \cdot((A+\lambda_i B)y_i(t) - \tau(t)\xi_i(t) - \dot{y}_i(t)) = 0 \tag{8.20}$$

Adding this to the right side of the inequality in (8.17) and using the inequality (8.19) we obtain:

$$\dot{V} \leq \eta_i^T(t)\Psi(\tau(t),\lambda_i)\eta_i(t) - (\bar{\tau} - \tau(t))\dot{y}_i{}^T(t_k)R\dot{y}_i(t_k),$$

where

$$\eta{=}[y_i^T(t) \quad \dot{y}_i^T(t) \quad \xi_i^T(t)]^T$$

and:

$$\Psi(\tau(t),\lambda_i) =$$

$$\begin{bmatrix} Q_1^T\Gamma_i + \Gamma_i^T Q_1 & \begin{matrix} P - Q_1^T \\ +\Gamma_i^T Q_2 \end{matrix} & -\tau(t)\lambda_i Q_1^T B \\ * & \begin{matrix} -Q_2 - Q_2^T + \\ (\bar{\tau} - \tau(t))R \end{matrix} & -\tau(t)\lambda_i Q_2^T B \\ * & * & -\tau(t)R \end{bmatrix} \tag{8.21}$$

where

$$\Gamma_i = (A + \lambda_i B).$$

Notice that $(\bar{\tau} - \tau(t))\dot{y}_i{}^T(t_k)R\dot{y}_i(t_k)$ is always positive. Thus:

$$\dot{V} \leq \eta_i^T(t)\Psi(\tau(t),\lambda_i)\eta_i(t), \tag{8.22}$$

Hence to prove the stability one needs to prove that $\Psi(\tau(t),\lambda_i)$ is negative definite.

Now define the following matrices:

$$\Phi_{i,0}(\lambda_i) =$$

$$\begin{bmatrix} Q_1^T\Gamma_i + \Gamma_i^T Q_1 & P - Q_1^T + \Gamma_i^T Q_2 \\ \\ * & Q_2 - Q_2^T + \bar{\tau}R \end{bmatrix}. \tag{8.23}$$

and

$$\Phi_{i,\bar{\tau}} =$$

$$\begin{bmatrix} Q_1^T\Gamma_i + \Gamma_i^T Q_1 & P - Q_1^T + \Gamma_i^T Q_2 & -\bar{\tau}\lambda_i Q_1^T B \\ \\ * & Q_2 - Q_2^T & -\bar{\tau}\lambda_i Q_2^T B \\ \\ * & * & -\bar{\tau}R \end{bmatrix} \tag{8.24}$$

Define

$$\eta_i'(t) = [y_i^T(t) \;\; \dot{y}_i^T(t)]^T.$$

One can show that:

$$\begin{aligned} &\eta_i^T(t)\Psi(\tau(t),\lambda_i)\eta_i(t) = \\ &\frac{\bar{\tau}-\tau(t)}{\bar{\tau}}\eta_i'^T(t)\Phi_{i,0}\eta_i'(t) + \frac{\tau(t)}{\bar{\tau}}\eta_i^T(t)\Phi_{i,\bar{\tau}}\eta_i(t) = \\ &\frac{\bar{\tau}-\tau(t)}{\bar{\tau}}\eta_i'^T(t)\Big(\frac{\bar{\lambda}-\lambda_i}{\bar{\lambda}+1}M_1 + \frac{\lambda_i}{\bar{\lambda}+1}M_2\Big)\eta_i'(t) + \\ &\quad \frac{\tau(t)}{\bar{\tau}}\eta_i^T(t)\Big(\frac{\bar{\lambda}-\lambda_i}{\bar{\lambda}+1}M_3 + \frac{\lambda_i}{\bar{\lambda}+1}M_4\Big)\eta_i(t) \end{aligned} \tag{8.25}$$

Define $\mu_\tau = \dfrac{\bar{\tau}-\tau(t)}{\bar{\tau}}$ and $\mu_\lambda = \dfrac{\bar{\lambda}-\lambda_i}{\bar{\lambda}+1}$.

Then $\dfrac{\tau(t)}{\bar{\tau}} = 1 - \mu_\tau$, $\dfrac{\lambda_i}{\bar{\lambda}+1} = 1 - \mu_\lambda$ and

$$\begin{aligned}
&\eta_i^T(t)\Psi(\tau(t),\lambda_i)\eta_i(t) = \\
&\mu_\tau \eta_i'^T(t)\Phi_{i,0}\eta_i'(t) + (1-\mu_\tau)\eta_i^T(t)\Phi_{i,\bar{\tau}}\eta_i(t) = \\
&\mu_\tau \eta_i'^T(t)\Big(\mu_\lambda M_1 + (1-\mu_\lambda)M_2\Big)\eta_i'(t) + \\
&(1-\mu_\tau)\eta_i^T(t)\Big(\mu_\lambda M_3 + (1-\mu_\lambda)M_4\Big)\eta_i(t).
\end{aligned} \tag{8.26}$$

Since $\mu_\tau \in [0,1]$ and $\mu_\lambda \in [0,1]$, coefficients μ_τ, $1-\mu_\tau$, μ_λ, and $1-\mu_\lambda$ are positive. Moreover, by equations (8.12) to (8.15) it follows that $\Psi(\tau(t),\lambda_i)$ is negative definite and this proves the stability of the system. □

Consensus among agents

We now prove the consensus of agents to a common position.

Theorem 8.2 *Consider a MAS evolving according to equation (8.1) where $\bar{\tau}$ is such that $0 < t_{k+1} - t_k < \bar{\tau} < \infty$. Assume that the undirected connected graph $\mathcal{G}$ modeling the network topology is such that the second largest eigenvalue of its weighted adjacency matrix is smaller than or equal to $\bar{\lambda}$. If the LMIs defined in eq. (8.12) to (8.15) are satisfied, then there exists a $\gamma \in \mathbb{R}$ such that $x(t)$ asymptotically converges to $\gamma\vec{1}$ and $v(t)$ asymptotically converges to $\vec{0}$.*

Proof: By Theorem 8.1, if the LMIs in eq. (8.12) to (8.15) hold, all modes except the UEM are asymptotically stable, i.e., $\lim_{t\to\infty} y_i(t) = 0$ and thus $\lim_{t\to\infty} z_i(t) = 0$ for $i = 2, \dots, n$. Furthermore, by Lemma 7.2, there exists a positive constant $\gamma \in \mathbb{R}$ such that $\lim_{t\to\infty} z_1(t) = \gamma$.

Now, the first column of T is the eigenvector corresponding to the unitary eigenvalue of W_d, therefore it is equal to $\vec{1} = [1 \quad 1 \quad \dots, \quad 1]^T$. Thus, being $x(t) = T[z_1(t) \quad 0 \quad \dots \quad 0]^T$, it is trivial to show that when $t \to \infty$ it is $x_i(t) = x_j(t)$, for all

$i, j = 1, \ldots, n$. The same calculations can be repeated for the velocities, thus proving that for $t \to \infty$, it is $v_i(t) = v_j(t)$, $i, j = 1, \ldots, n$. □

8.4 Simulation results

In this section we present the results of some numerical simulations that show the effectiveness of the proposed consensus protocol. To this aim we consider a system with 8 agents and assume $k_p = 1$ and $k_d = 1$.

In Fig. 8-1 the area under the curve shows the stability region in the $\bar{\lambda} - \bar{\tau}$ plane. Such an area has been computed using the LMIs (8.12) to (8.15).

We now consider a graph with adjacency matrix (randomly generated) equal to:

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8.27}$$

Fig. 8-2 shows the positions and velocities of the agents, while Fig. 8-3 shows the sampled positions and velocities aperiodically transmitted to neighbors by each agent.

8.5 Conclusions and future work

In this chapter we considered a PD-like consensus algorithm for a second-order multi-agent system where, at non-periodic sampling times, agents transmit to their neighbors information about their position and velocity, while each agent has a perfect knowledge of its own state at any time instant. The main contribution consists in

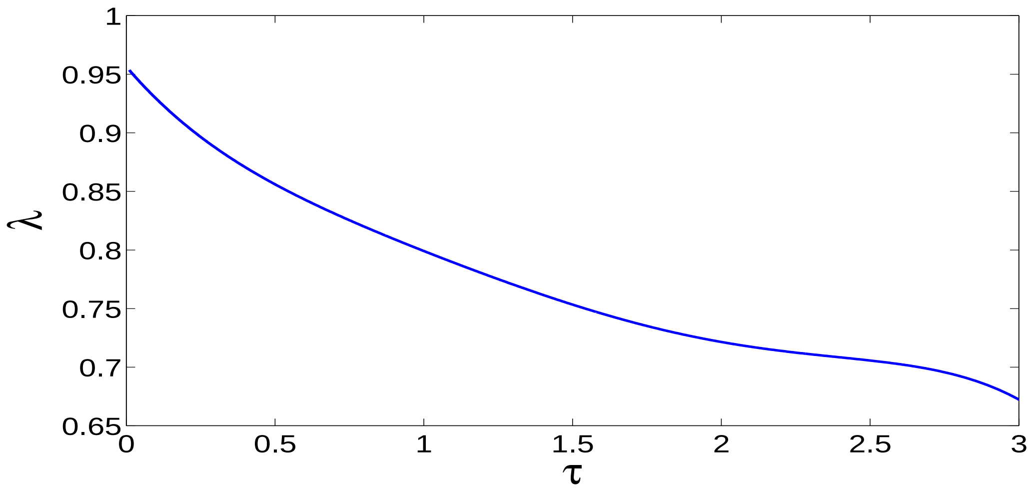


Figure 8-1: The stability area in the $\bar{\lambda} - \bar{\tau}$ plane.

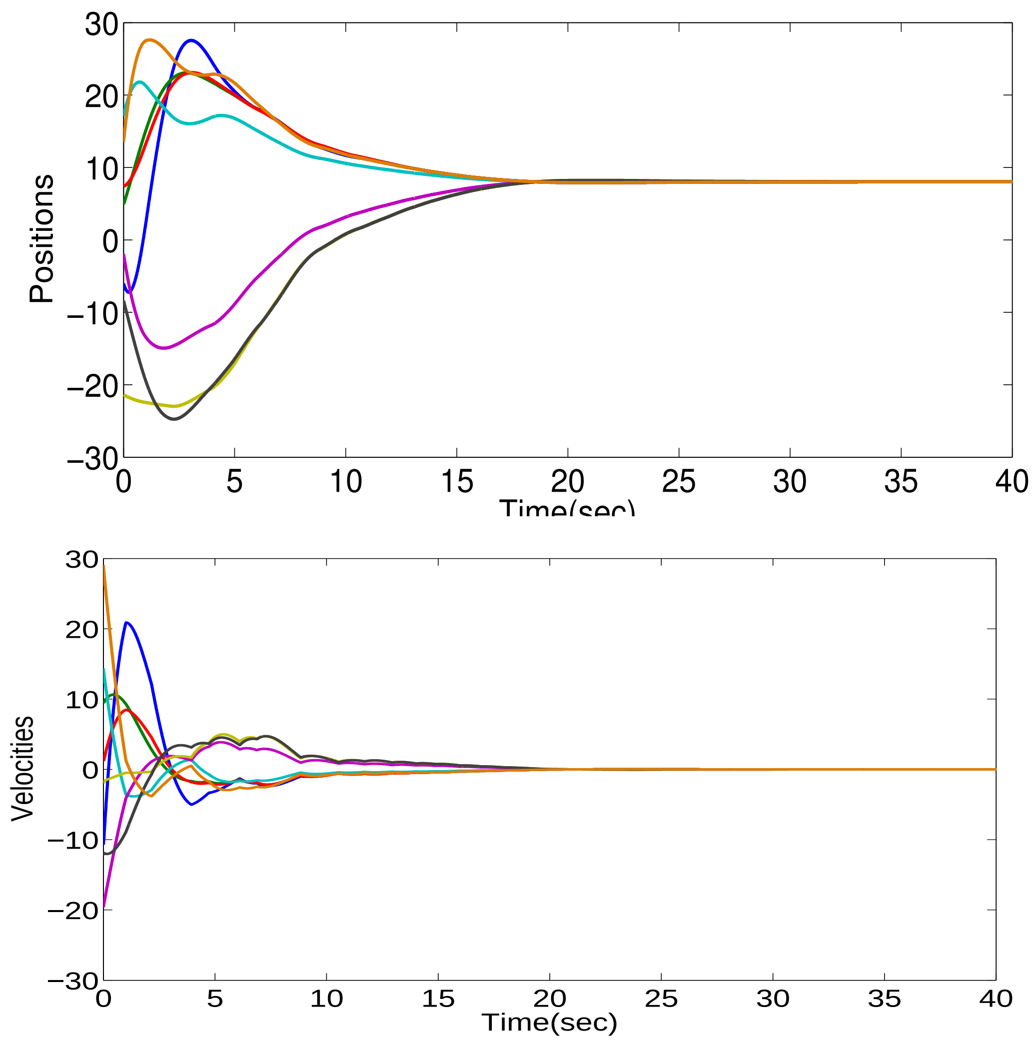
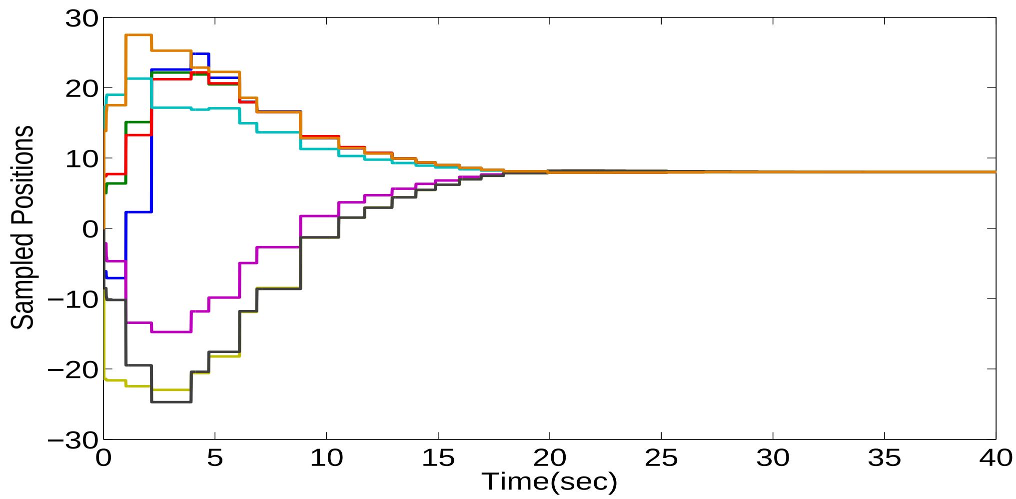


Figure 8-2: Positions and velocities when the proposed protocol is implemented.



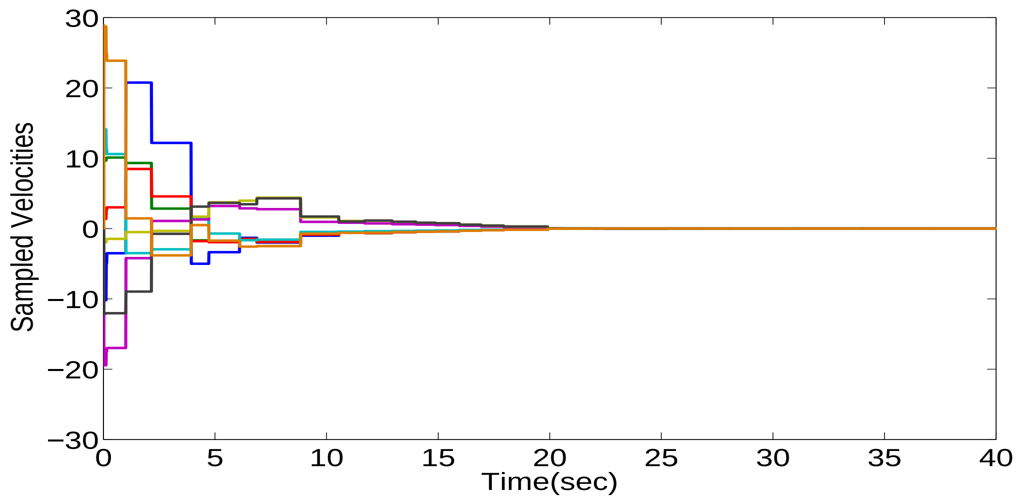


Figure 8-3: Positions and velocities when the proposed protocol is implemented.

proving consensus to a common fixed point, based on LMIs verification, under the assumption that the network topology is not known and the only information is an upper bound on the connectivity.

Two are the main directions of our future research in this framework. First, we want to compute analytically an upper bound on the value of the second largest eigenvalue of the weighted adjacency matrix that guarantees consensus, as a function of the other design parameters. Second, we plan to study the case where agents do not have a perfect knowledge of their own state.

9

Conclusions and open issues

"Give me a lever long enough and a fulcrum on which to place it, and I shall move the world."

– Martin H. Fischer

Different consensus problems in multi-agent systems have been addressed in this thesis. They represent improvements with respect to the state of the art.

In the first part of the thesis including Chapters 2, 3, and 4, the state of the art of the representation and stability analysis of consensus problems, time-delay systems, and sampled-data systems have been presented.

Novel contributions have been illustrated in Chapters 5-8. Particularly, in Chapter 5 we reported the results of Zareh et al. (2013b), where we investigated the consensus problem for networks of agents with double integrator dynamics affected by time-delay in their coupling. We provided a stability result based on the Lyapunov-Krasovskii functional method and a numerical procedure based on an LMI condition which depends only on the algebraic connectivity of the considered network topologies, thus reducing greatly the computational complexity of the procedure. Obviously, this result implies the existence of a minimum dwell time such that the proposed consensus protocol is stable for slow switchings between network topologies with sufficient algebraic connectivity. Future work will involve actually computing such a dwell time by adopting a multiple Lyapunov function method and evaluating the worst case convergence rate. Furthermore we will evaluate novel consensus protocols that consider only delayed relative measurements instead of delayed absolute values of the neighbors' state variables.

The results of Zareh et al. (2013a) were addressed in Chapter 6, in which a continuous time version of a consensus on the average protocol for arbitrary strongly connected directed graphs is proposed and its convergence properties with respect to time delays in the local state update are characterized. The convergence properties of this algorithm depend upon a tuning parameter that can be made arbitrary small to prove stability of the networked system. Simulations have been presented to corroborate the theoretical results and show that the existence of a small time delay can actually improve the algorithm performance. Future work will include an extension of the mathematical characterization of the proposed algorithm to consider possibly heterogeneous or time-varying delays.

In Chapter 7 we proposed a PD-like consensus algorithm for a second-order multi-agent system where, at non-periodic sampling times, agents transmit to their neighbors information about their position and velocity, while each agent has a perfect knowledge of its own state at any time instant. Conditions have been given to prove consensus to a common fixed point, based on LMIs verification. Moreover, we also show how it is possible to evaluate an upper bound on the decay rate of exponential convergence of stable modes.

In Chapter 8, mainly based on our paper Zareh et al. (2014b), we considered the same problem as in Chapter 7. The main contribution consists in proving consensus to a common fixed point, based on LMIs verification, under the assumption that the network topology is not known and the only information is an upper bound on the connectivity. Two are the main directions of our future research in this framework. First, we want to compute analytically an upper bound on the value of the second largest eigenvalue of the weighted adjacency matrix that guarantees consensus, as a function of the other design parameters. Second, we plan to study the case where agents do not have a perfect knowledge of their own state.

Appendices

Appendix A

Laplacian matrix

In the mathematical field of graph theory, the Laplacian matrix, sometimes called admittance matrix, Kirchhoff matrix or discrete Laplacian, is a matrix representation of a graph. Together with Kirchhoff's theorem, it can be used to calculate the number of spanning trees for a given graph. The Laplacian matrix can be used to find many other properties of the graph. Cheeger's inequality from Riemannian geometry has a discrete analogue involving the Laplacian matrix; this is perhaps the most important theorem in spectral graph theory and one of the most useful facts in algorithmic applications. It approximates the sparsest cut of a graph through the second eigenvalue of its Laplacian. Given a simple graph G with n vertices, its Laplacian matrix $L := (l_{i,j})_{n \times n}$ is defined as:

$$L = \Delta - A,$$

where Δ is the degree matrix and A is the adjacency matrix of the graph. In the case of directed graphs, either the in-degree or out-degree might be used, depending on the application.

From the definition it follows that:

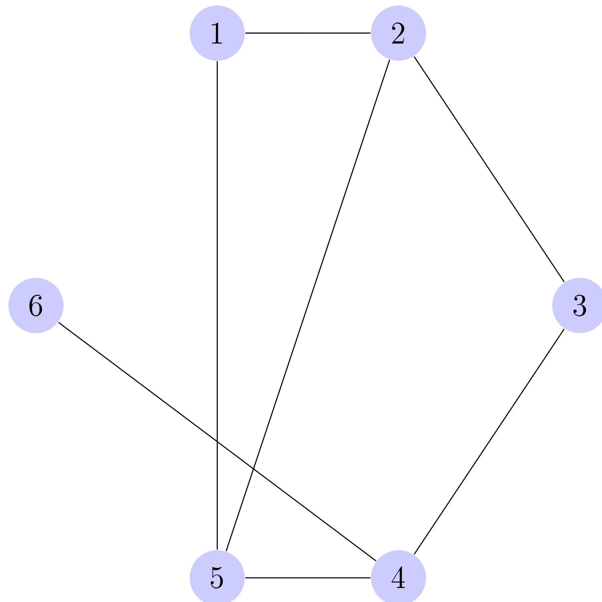
$$l_{ij} = \begin{cases} deg(v_i) & if i = j \\ -1 & i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & otherwise \end{cases}$$

where $deg(v_i)$ is degree of the vertex i.

The normalized Laplacian matrix is defined as (Bollobás, 1998):

$$\mathcal{L} := D^{-1/2} L D^{-1/2} = I - D^{-1/2} A D^{-1/2} = (\tilde{\ell}_{ij}),$$

where:



$$\tilde{\ell}_{i,j} := \begin{cases} 1 & \text{if } i = j \text{ and } \deg(v_i) \neq 0 \\ -\frac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

We note that L can be written as

$$\mathcal{L} = SS^*,$$

where S is the matrix whose rows are indexed by the vertices and whose columns are indexed by the edges of G such that each column corresponding to an edge $e = u, v$ has an entry $1/\sqrt{d_u}$ in the row corresponding to u, an entry $1/\sqrt{d_v}$ in the row corresponding to v, and has zero entries elsewhere. (As it turns out, the choice of signs can be arbitrary as long as one is positive and the other is negative.)

Here is a simple example of a labeled graph and its Laplacian matrix. Consider a 6-vertex graph as shown in fig. A

In this example the weight matrix is

$$\Delta = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

The adjacency matrix

$$A \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

and the Laplacian matrix

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Some properties of Laplacian matrix is provided below (Bollobás, 1998; Anderson Jr and Morley, 1985).

For an undirected graph G and its Laplacian matrix L with eigenvalues $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$:

- L is symmetric.
- L is positive-semidefinite (that is $\lambda_i \geq 0$ for all i). This can also be seen from

the fact that the Laplacian is symmetric and diagonally dominant.

- L is an M-matrix (its off-diagonal entries are non-positive, yet the real parts of its eigenvalues are nonnegative).
- Every row sum and column sum of L is zero. Indeed, in the sum, the degree of the vertex is summed with a "-1" for each neighbor.
- In consequence, $\lambda_0 = 0$, because the vector $\mathbf{v}_0 = (1, 1, \ldots, 1)$ satisfies $L\mathbf{v}_0 = \mathbf{0}$.
- The number of times 0 appears as an eigenvalue in the Laplacian is the number of connected components in the graph. The smallest non-zero eigenvalue of L is called the spectral gap.
- The second smallest eigenvalue of L is the algebraic connectivity (or Fiedler value) of G.
- The Laplacian is an operator on the n-dimensional vector space of functions $f : V \to \mathbb{R}$, where V is the vertex set of G, and $n = |V|$.
- When G is k-regular, the normalized Laplacian is: $\mathcal{L} = \frac{1}{k}L = I - \frac{1}{k}A$, where A is the adjacency matrix and I is an identity matrix.
- For a graph with multiple connected components, L is a block diagonal matrix, where each block is the respective Laplacian matrix for each component, possibly after reordering the vertices (i.e. L is permutation-similar to a block diagonal matrix).
- For a graph G on n vertices, we have

$$\sum_i \lambda_i \leq n.$$

  with equality holding if and only if G has no isolated vertices.
- For $n \geq 2$,

$$\lambda_1 \leq \frac{n}{n-1},$$

with equality holding if and only if G is the complete graph on n vertices. Also, for a graph G without isolated vertices, we have

$$\lambda_{n-1} \geq \frac{n}{n-1}.$$

Appendix B

Perturbation bounds on matrix eigenvalues

In this section, the goal is the exposition of bounds for the distance between the eigenvalues of two matrices A and B in terms of expressions involving $\|A - B\|$. The prototype of such bounds is H. Weyl's inequalityBhatia (2007).

For several years the most prominent conjecture on perturbation inequalities, which attracted the attention of several mathematicians, was that the inequality

$$d(eig(A), eig(B)) \leq \|A - B\|,$$

would be true for all normal matrices A and B. $d(eig(A), eig(B))$ indicates the maximum distance between the eigenvalues of matrices A and B. In 1992, J. Holbrook published a counterexample to this with 3×3 matrices. It is now known that the inequality

$$d(eig(A), eig(B)) \leq c\|A - B\|,$$

is true for all $n \times n$ normal matrices A and B with $c < 2.904$ and that the best constant c here is bigger than 1.018.

We now give a brief summary of the major inequalities which are proved (occasionally just stated) below. Let A, B be $n \times n$ Hermitian matrices with eigenvalues $\alpha_1 \geq \ldots \geq \alpha_n$ and $\beta_1 \geq \ldots \geq \beta_n$ respectively. Then

$$\max \lim_{j} |\alpha_j - \beta_j| \leq \|A - B\|.$$

Kahan (1975), showed that

$$d(eig(A), eig(B)) \leq (\gamma_n + 2)\|A - B\|,$$

where γ_n is a constant depending on the size n of the matrices. Further they showed that the optimal constant for this inequality is bounded as

$$\frac{2}{\pi}\ln(n) - 0.1 \leq \gamma_n \leq \log_2(n) + 0.038.$$

Based on the results of an extended work, can see that if A is normal and B

arbitrary then

$$d(eig(A), eig(B)) \leq (2n-1)\|A-B\|.$$

If, in addition, B is Hermitian then the factor $(2n-1)$ can be replaced by $\sqrt{2}$ in the above inequality.

When A,B are arbitrary $n \times n$ matrices the situation is not so simple. Results of this type in the general case were obtained by Ostrowski et al. (1960); Henrici (1962); Bhatia (2007). This latest result says that for A, B arbitrary $n \times n$ matrices

$$d(eig(A), eig(B)) \leq n(2M)^{1-1/n}\|A-B\|,$$

where $M = max(\|A\|, \|B\|)$.

Appendix C

Properties of weighted adjacency matrix

Let ν_1 be a right eigenvector of $W_d = \Delta^{-1}A_d$ then $\Delta^{-1}A_d\nu_1 = \lambda\nu_1$. Using the transformation $\nu_1 = \Delta^{-\frac{1}{2}}\nu_2$ one gets:

$$\Delta^{-1}A_d\Delta^{-\frac{1}{2}}\nu_2 = \lambda\Delta^{-\frac{1}{2}}\nu_2 \rightarrow \Delta^{-\frac{1}{2}}A_d\Delta^{-\frac{1}{2}}\nu_2 = \lambda\Delta^{-\frac{1}{2}}\nu_2$$

This shows that matrices $\Delta^{-\frac{1}{2}}A_d\Delta^{-\frac{1}{2}}$ and $W_d = \Delta^{-1}A_d$ have the same eigenvalues and since the former is a symmetric matrix, they possess real eigenvalues. From Courant-Fischer theorem (Horn and Johnson, 2012) the largest eigenvalue which is a simple one is achieved from the following equation:

$$\lambda_1 = max\{\nu_2^T\Delta^{-\frac{1}{2}}A_d\Delta^{-\frac{1}{2}}\nu_2\}, \quad \nu_2^T\nu_2 = 1.$$

Suppose the vector $\nu_2^\dagger$ is a solution to the above optimization problem. In order to find the second largest eigenvalue we must search in a subspace which is perpendicular to the one in which the largest eigenvalue is located:

$$\lambda_2 = max\{\nu_2^T\Delta^{-\frac{1}{2}}A_d\Delta^{-\frac{1}{2}}\nu_2\}, \quad \nu_2^T\nu_2 = 1, \nu_2^T\nu_2^\dagger = 0.$$

Since $\Delta^{-\frac{1}{2}}$ is diagonal we get:

$$\lambda_2 = max\{(\Delta^{-\frac{1}{2}}\nu_2)^T A_d(\Delta^{-\frac{1}{2}}\nu_2)\} = max\{\nu_1^T A_d\nu_1)\},$$

We know that the corresponding eigenvector of $\lambda_1 = 1$ is parallel to $\mathbb{1}$ so $\nu_2^\dagger = \Delta^{-\frac{1}{2}}\mathbb{1}$. The constraints become:

$$\sum_{i=1}^{n}\delta_i\nu_{1i}^2 = 1, \quad \sum_{i=1}^{n}\delta_i\nu_{1i} = 0.$$

Notice that:

$$\nu_1^T A_d\nu_1 = \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\nu_{1i}\nu_{1j} = \sum_{(i,j)\in E}2\nu_{1i}\nu_{1j},$$

It should be noted that $2\nu_{1i}\nu_{1j} = \nu_{1i}^2 + \nu_{1j}^2 - (\nu_{1i} - \nu_{1j})^2$. Hence:

$$\nu_1^T A_d \nu_1 = \sum_{i=1}^{n} \delta_i \nu_{1i}^2 - \sum_{(i,j)\in E} (\nu_{1i} - \nu_{1j})^2 = 1 - \sum_{(i,j)\in E} (\nu_{1i} - \nu_{1j})^2,$$

Consequently

$$\max\{\nu_1^T A_d \nu_1\} = \max\{1 - \sum_{(i,j)\in E} (\nu_{1i} - \nu_{1j})^2\} = 1 - \min\{\sum_{(i,j)\in E} (\nu_{1i} - \nu_{1j})^2\},$$

The minimum of the last term is achieved when the number of the edges set is the minimum possible which allows the graph to be connected. Trivially such a graph is a tree graph.

By looking at the matrix $\bar{W} = \Delta^{-\frac{1}{2}} A_d \Delta^{-\frac{1}{2}}$ we can see that it can be converted to the well known shape normalized Laplacian matrix, L_d, as follows:

$$L_d = I - \bar{W},$$

It can be easily observed that the eigenvalues of L_d are equal to $1-\lambda$. In order to find the second largest eigenvalue of $\bar{W}$ (or W_d) we can check the second largest eigenvalue of L_d for tree graphs.

Now we introduce the following conjecture which gives a relationship between the second largest eigenvalue of the weighted adjacency matrix and the number of agents.

Conjecture C.1 *For a given number of agents, n, the second largest eigenvalue of the weighted adjacency matrix (W_d and equivalently that of $\bar{W}$) over whole possible connected graphs is upper bounded by* $\cos(\frac{\pi}{n-1})$.

The above conjecture has been validated by many different simulations, and we are trying to find a proof for it.

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List of Publications Related to the Thesis

Published papers

- M. Zareh, D. V. Dimarogonas, M. Franceschelli, K. H. Johansson, and C. Seatzu, "Consensus in multi-agent systems with non-periodic sampled-data exchange and uncertain network topology," in *Control, Decision and Information Technologies (CoDIT), 2014 International Conference on.* IEEE, 2014, pp. 411–416

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