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Operators of harmonic analysis in weighted spaces with non-standard growth

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Abstract

Last years there was increasing an interest to the so-called function spaces with non-standard growth, known also as variable exponent Lebesgue spaces. For weighted such spaces on homogeneous spaces, we develop a certain variant of Rubio de Francia's extrapolation theorem. This extrapolation theorem is applied to obtain the boundedness in such spaces of various operators of harmonic analysis, such as maximal and singular operators, potential operators, Fourier multipliers, dominants of partial sums of trigonometric Fourier series and others, in weighted Lebesgue spaces with variable exponent. There are also given their vector-valued analogues.

Keywords: Variable exponent; Generalized Lebesgue space; Maximal operator; Extrapolation theorems; Weighted estimates; Fourier multipliers; Potential operators; Singular operators; Fefferman–Stein Function

1. Introduction

During last years a significant progress was made in the study of maximal and singular operators and potential type operators in the generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent, known also as the spaces with non-standard growth. A number of mathematical problems leading to such spaces with variable exponent arise in applications to partial differential equations, variational problems and continuum mechanics (in particular, in the theory of the so-called electrorheological fluids), see E. Acerbi and G. Mingione [1,2], X. Fan and D. Zhao [20], M. Ružička [62], V.V. Zhikov [75,76]. These applications stipulated a significant interest to such spaces in the last decade.

The most advance in the study of the classical operators of harmonic analysis in the case of variable exponent was made in the Euclidean setting, including weighted estimates. We refer in particular to the surveying articles L. Diening, P. Hästö and A. Nekvinda [16], V. Kokilashvili [33], S. Samko [73] and papers D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Perez [10], D. Cruz-Uribe, A. Fiorenza and C.J. Neugebauer [11], L. Diening [13–15], L. Diening and M. Ružička [17], V. Kokilashvili, N. Samko and S. Samko [38], V. Kokilashvili and S. Samko [41–43,45], A. Nekvinda [58], S. Samko [70–72], S. Samko, E. Shargorodsky and B. Vakulov [74] and references therein.

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Recently there also started the investigation of these classical operators in the spaces with variable exponent in the setting of metric measure spaces, the case of constant p in this setting having a long history, we refer, in particular to the papers A.P. Calderón [6], R.R. Coifman and G. Weiss [7,8], R. Macías and C. Segovia [52], books D.E. Edmunds, V. Kokilashvili and A. Meskhi [18] and I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec [22], J. Heinonen [26] and references therein. The non-weighted boundedness of the maximal operator on homogeneous spaces was proved by P. Harjulehto, P. Hästö and M. Pere [25] and Sobolev embedding theorem with variable exponents on homogeneous spaces with variable dimension was proved in P. Harjulehto, P. Hästö and V. Latvala [24].

In the present paper we give a development of weighted estimations of various operators of harmonic analysis in Lebesgue spaces with variable exponent $p(x)$. We first give theorems on the weighted boundedness of the maximal operator on homogeneous spaces (Theorems 2.11 and 2.12). Next, in Section 3 we give a certain $p(\cdot) \to q(\cdot)$-version of Rubio de Francia's extrapolation theorem [61] within the frameworks of weighted spaces $L^{p(\cdot)}_\varrho$ on metric measure spaces. Proving this version we develop some ideas and approaches of papers [10,12].

By means of this extrapolation theorem and known theorems on the boundedness with Muckenhoupt weights in the case of constant p, we obtain results on weighted $p(\cdot) \to q(\cdot)$- or $p(\cdot) \to p(\cdot)$-boundedness—in the case of variable exponent $p(x)$—of the following operators: potential type operators, Fourier multipliers (weighted Mikhlin, Hörmander and Lizorkin-type theorems, Section 4.2), multipliers of trigonometric Fourier series (Section 4.3), majorants of partial sums of Fourier series (Section 4.4), Zygmund and Cesaro summability for trigonometric series (Section 4.5), singular integral operators on Carleson curves and in Euclidean setting (Sections 4.6 and 4.7), Fefferman–Stein function and some vector-valued operators (Section 4.8).

2. Definitions and preliminaries

2.1. On variable dimensions in metric measure spaces

In the sequel, (X, d, μ) denotes a metric space with the (quasi)metric d and non-negative measure μ. We refer to [18,22,26] for the basics on metric measure spaces. By $B(x,r) = \{y \in X\colon\ d(x,y) < r\}$ we denote a ball in X. The following standard conditions will be assumed to be satisfied:

(1) all the balls $B(x,r) = \{y \in X\colon\ d(x,y) < r\}$ are measurable,
(2) the space $C(X)$ of uniformly continuous functions on X is dense in $L^1(\mu)$.

In most of the statements we also suppose that

(3) the measure μ satisfies the doubling condition:

$$\mu B(x,2r) \leqslant C\mu B(x,r),$$

where $C > 0$ does not depend on $r > 0$ and $x \in X$.

A measure satisfying this condition will be called doubling measure.

For a locally μ-integrable function $f : X \to \mathbb{R}^1$ we consider the Hardy–Littlewood maximal function

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} \big|f(y)\big|\, d\mu(y).$$

By $A_s = A_s(X)$, where $1 \leqslant s < \infty$, we denote the class of weights (locally almost everywhere positive μ-integrable functions) $w : X \to \mathbb{R}^1$ which satisfy the Muckenhoupt condition

$$\sup_B \left(\frac{1}{\mu B}\int\limits_B w(y)\, d\mu(y)\right)\left(\frac{1}{\mu B}\int\limits_B w^{-\frac{1}{s-1}}(y)\, d\mu(y)\right)^{s-1} < \infty$$

in the case $1 < s < \infty$, and the condition

$$\mathcal{M}w(x) \leqslant Cw(x)$$

for almost all $x \in X$, with a constant $C > 0$, not depending on $x \in X$, in the case $s = 1$. Obviously, $A_1 \subset A_s$, $1 < s < \infty$.

As is known, see [6,52], the weighted boundedness

$$\int_X \left(\mathcal{M}f(x)\right)^s w(x)\,d\mu(x) \leqslant C \int_X \left|f(x)\right|^s w(x)\,d\mu(x), \quad 1 < s < \infty,$$

holds, if and only if $w \in A_s$.

Definition 2.1. By $\mathcal{P}(\Omega)$, where Ω is an open set in X, we denote the class of μ-measurable functions on Ω, such that

$$1 < p_- \leqslant p_+ < \infty, \tag{2.1}$$

where $p_- = p_-(\Omega) = \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ = p_+(\Omega) = \operatorname{ess\,sup}_{x\in\Omega} p(x)$.

Definition 2.2. By $L^{p(\cdot)}_\varrho(\Omega)$ we denote the weighted Banach function space of μ-measurable functions $f : \Omega \to \mathbb{R}^1$, such that

$$\|f\|_{L^{p(\cdot)}_\varrho} := \|\varrho f\|_{p(\cdot)} = \inf\left\{\lambda > 0\colon \int_\Omega \left|\frac{\varrho(x)f(x)}{\lambda}\right|^{p(x)} d\mu(x) \leqslant 1\right\} < \infty. \tag{2.2}$$

Definition 2.3. We say that a weight ϱ belongs to the class $\mathfrak{A}_{p(\cdot)}(\Omega)$, if the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}_\varrho(\Omega)$.

Definition 2.4. A function $p : \Omega \to \mathbb{R}^1$ is said to belong to the class $\mathrm{WL}(\Omega)$ (weak Lipshitz), if

$$\left|p(x) - p(y)\right| \leqslant \frac{A}{\ln \frac{1}{d(x,y)}}, \quad d(x,y) \leqslant \frac{1}{2},\ x, y \in \Omega, \tag{2.3}$$

where $A > 0$ does not depend on x and y.

The notion of lower and upper local dimension of X at a point x introduced as

$$\underline{\dim} X(x) = \varliminf_{r\to 0} \frac{\ln \mu B(x,r)}{\ln r}, \qquad \overline{\dim} X(x) = \varlimsup_{r\to 0} \frac{\ln \mu B(x,r)}{\ln r}$$

is known, see e.g. [19]. We will use different notions of local lower and upper dimensions, inspired by the notion of the index numbers $m(w)$, $M(w)$ of almost monotonic functions w, see their definition in (2.17). These indices studied in [63–65], are versions of Matuzewska–Orlicz index numbers used in the theory of Orlicz spaces, see [53,54]. The idea to introduce local dimensions in terms of these indices by the following definition was borrowed from the papers [66,67].

Definition 2.5. The numbers

$$\underline{\mathfrak{dim}}(X;x) = \sup_{r>1} \frac{\ln\left(\varliminf_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)}\right)}{\ln r}, \qquad \overline{\mathfrak{dim}}(X;x) = \inf_{r>1} \frac{\ln\left(\varlimsup_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)}\right)}{\ln r} \tag{2.4}$$

will be referred to as local lower and upper dimensions.

Observe that the "dimension" $\underline{\mathfrak{dim}}(X;x)$ may be also rewritten in terms of the upper limit as well:

$$\underline{\mathfrak{dim}}(X;x) = \sup_{0<r<1} \frac{\ln\left(\varlimsup_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)}\right)}{\ln r}. \tag{2.5}$$

Since the function

$$\mu_0(x,r) = \varlimsup_{h\to 0} \frac{\mu B(x,rh)}{\mu B(x,h)} \tag{2.6}$$

is semimultiplicative in r, that is, $\mu_0(x, r_1r_2) \leqslant \mu_0(x, r_1)\mu_0(x, r_2)$, by properties of such functions ([47, p. 75]; [48]) we obtain that $\underline{\mathfrak{dim}}(X; x) \leqslant \overline{\mathfrak{dim}}(X; x)$ and we may rewrite the dimensions $\underline{\mathfrak{dim}}(X; x)$ and $\overline{\mathfrak{dim}}(X; x)$ also in the form

$$\underline{\mathfrak{dim}}(X; x) = \lim_{r\to 0} \frac{\ln \mu_0(x, r)}{\ln r}, \qquad \overline{\mathfrak{dim}}(X; x) = \lim_{r\to\infty} \frac{\ln \mu_0(x, r)}{\ln r}. \tag{2.7}$$

Remark 2.6. Introduction of dimensions $\underline{\mathfrak{dim}}(X; x)$ and $\overline{\mathfrak{dim}}(X; x)$ just in form (2.5)–(2.7) is caused by the fact that they arise naturally when dealing with Muckenhoupt condition for radial type weights on metric measure spaces. They seem may not coincide with dimensions $\underline{\dim}X(x)$, $\overline{\dim}X(x)$. There is an impression that probably for different goals different notions of dimensions may be useful.

We will mainly work with the lower bound

$$\underline{\mathfrak{dim}}(\Omega) := \operatorname*{ess\,inf}_{x\in X} \underline{\mathfrak{dim}}(\Omega; x)$$

of lower dimensions $\underline{\mathfrak{dim}}(X; x)$ on an open set $\Omega \subseteq X$.

In case where Ω is unbounded, we will also need similar dimensions connected in a sense with the influence of infinity. Let

$$\mu_\infty(x, r) = \overline{\lim_{h\to\infty}} \frac{\mu B(x, rh)}{\mu B(x, h)}. \tag{2.8}$$

We introduce the numbers

$$\underline{\mathfrak{dim}}_\infty(X; x) = \lim_{r\to 0} \frac{\ln \mu_\infty(x, r)}{\ln r}, \qquad \overline{\mathfrak{dim}}_\infty(X; x) = \lim_{r\to\infty} \frac{\ln \mu_\infty(x, r)}{\ln r} \tag{2.9}$$

and their bounds

$$\underline{\mathfrak{dim}}_\infty(\Omega) = \operatorname*{ess\,inf}_{x\in\Omega} \underline{\mathfrak{dim}}_\infty(X; x), \qquad \overline{\mathfrak{dim}}_\infty(\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} \overline{\mathfrak{dim}}_\infty(X; x). \tag{2.10}$$

It is not hard to see that $\underline{\mathfrak{dim}}(\Omega)$, $\underline{\mathfrak{dim}}_\infty(\Omega)$, and $\overline{\mathfrak{dim}}_\infty(\Omega)$ are non-negative. In the sequel, when considering these bounds of dimensions we always assume that $\underline{\mathfrak{dim}}(\Omega), \underline{\mathfrak{dim}}_\infty(\Omega), \overline{\mathfrak{dim}}_\infty(\Omega) \in (0, \infty)$.

2.2. *Classes of the weight functions*

We consider, in particular, the weights

$$\varrho(x) = \left[1 + d(x_0, x)\right]^{\beta_\infty} \prod_{k=1}^{N} \left[d(x, x_k)\right]^{\beta_k}, \quad x_k \in X,\ k = 0, 1, \ldots, N, \tag{2.11}$$

where $\beta_\infty = 0$ in the case where X is bounded. Let $\Pi = \{x_0, x_1, \ldots, x_N\}$ be a given finite set of points in X. We take $d(x, y) = |x - y|$ in all the cases where $X = \mathbb{R}^n$.

Definition 2.7. A weight function of form (2.11) is said to belong to the class $V_{p(\cdot)}(\Omega, \Pi)$, where $p(\cdot) \in C(\Omega)$, if

$$-\frac{\underline{\mathfrak{dim}}(\Omega)}{p(x_k)} < \beta_k < \frac{\underline{\mathfrak{dim}}(\Omega)}{p'(x_k)} \tag{2.12}$$

and, in the case Ω is infinite,

$$-\frac{\underline{\mathfrak{dim}}_\infty(\Omega)}{p_\infty} < \beta_\infty + \sum_{k=1}^{N} \beta_k < \underline{\mathfrak{dim}}_\infty(\Omega) - \frac{\overline{\mathfrak{dim}}_\infty(\Omega)}{p_\infty}. \tag{2.13}$$

Note that when the metric space X has a constant dimension s in the sense that $c_1r^s \leqslant \mu B(x, r) \leqslant c_2r^s$ with the constants $c_1 > 0$, $c_2 > 0$, not depending on $x \in X$ and $r > 0$, the inequalities in (2.12), (2.13) and (2.19) turn into

$$-\frac{s}{p(x_k)} < \beta_k < \frac{s}{p'(x_k)}, \qquad -\frac{s}{p_\infty} < \beta_\infty + \sum_{k=1}^{N} \beta_k < \frac{s}{p'_\infty} \tag{2.14}$$

and

$$-\frac{s}{p(x_k)} < m(w) \leqslant M(w) < \frac{s}{p'(x_k)}, \quad k = 1, 2, \ldots, N. \tag{2.15}$$

In fact, we may admit a more general class of weights

$$\varrho(x) = w_0\big[1 + d(x_0, x)\big] \prod_{k=1}^{N} w_k\big[d(x, x_k)\big] \tag{2.16}$$

with “radial” weights, where the functions w_0 and w_k, $k = 1, \ldots, N$, belong to a class of Zygmund–Bary–Stechkin type, which admits an oscillation between two power functions with different exponents.

By $U = U([0, \ell])$ we denote the class of functions $u \in C([0, \ell])$, $0 < \ell \leqslant \infty$, such that $u(0) = 0$, $u(t) > 0$ for $t > 0$ and u is an almost increasing function on $[0, \ell]$. (We recall that a function u is called *almost increasing* on $[0, \ell]$, if there exists a constant $C(\geqslant 1)$ such that $u(t_1) \leqslant Cu(t_2)$ for all $0 \leqslant t_1 \leqslant t_2 \leqslant \ell$.) By $\widetilde{U}$ we denote the class of function u, such that $t^a u(t) \in U$ for some $a \in \mathbb{R}^1$.

Definition 2.8. (See [4].) A function v is said to belong to the Zygmund–Bary–Stechkin class Φ_δ^0, if

$$\int_0^h \frac{v(t)}{t}\, dt \leqslant cv(h) \quad \text{and} \quad \int_h^\ell \frac{v(t)}{t^{1+\delta}}\, dt \leqslant c\frac{v(h)}{h^\delta},$$

where $c = c(v) > 0$ does not depend on $h \in (0, \ell]$.

It is known that $v \in \Phi_\delta^0$, if and only if $0 < m(v) \leqslant M(v) < \delta$, where

$$m(w) = \sup_{t>1} \frac{\ln\big(\underline{\lim}_{h\to 0} \frac{w(ht)}{w(h)}\big)}{\ln t} \quad \text{and} \quad M(w) = \sup_{t>1} \frac{\ln\big(\overline{\lim}_{h\to 0} \frac{w(ht)}{w(h)}\big)}{\ln t} \tag{2.17}$$

(see [29,63,65]).

For functions w defined in the neighborhood of infinity and such that $w(\frac{1}{r}) \in \widetilde{U}([0, \delta])$ for some $\delta > 0$, we introduce also

$$m_\infty(w) = \sup_{x>1} \frac{\ln\big[\underline{\lim}_{h\to\infty} \frac{w(xh)}{w(h)}\big]}{\ln x}, \qquad M_\infty(w) = \inf_{x>1} \frac{\ln\big[\overline{\lim}_{h\to\infty} \frac{w(xh)}{w(h)}\big]}{\ln x}. \tag{2.18}$$

Generalizing Definition 2.7, we introduce also the following notion.

Definition 2.9. A weight function ϱ of form (2.16) is said to belong to the class $V_{p(\cdot)}^{\mathrm{osc}}(\Omega, \Pi)$, where $p(\cdot) \in C(\Omega)$, if

$$w_k(r) \in \widetilde{U}\big([0, \ell]\big), \quad \ell = \operatorname{diam}\Omega \quad \text{and} \quad -\frac{\underline{\mathfrak{dim}}(\Omega)}{p(x_k)} < m(w_k) \leqslant M(w_k) < \frac{\underline{\mathfrak{dim}}(\Omega)}{p'(x_k)}, \tag{2.19}$$

$k = 1, 2, \ldots, N$, and (in the case Ω is infinite)

$$w_0\left(\frac{\ell^2}{r}\right) \in \widetilde{U}\big([0, \ell]\big)$$

and

$$-\frac{\underline{\mathfrak{dim}}_\infty(\Omega)}{p_\infty} < \sum_{k=0}^{N} m_\infty(w_k) \leqslant \sum_{k=0}^{N} M_\infty(w_k) < \frac{\underline{\mathfrak{dim}}_\infty(\Omega)}{p'_\infty} - \Delta_{p_\infty}, \tag{2.20}$$

where $\Delta_{p_\infty} = \frac{\overline{\mathfrak{dim}}_\infty(\Omega) - \underline{\mathfrak{dim}}_\infty(\Omega)}{p_\infty}$.

Observe that in the case $\Omega = X = \mathbb{R}^n$ conditions (2.19) and (2.20) take the form

$$w_k(r) \in \widetilde{U}\big(\mathbf{R}_+^1\big) := \left\{w\colon\ w(r), w\left(\frac{1}{r}\right) \in \widetilde{U}\big([0, 1]\big)\right\} \tag{2.21}$$

and

$$-\frac{n}{p(x_k)} < m(w_k) \leqslant M(w_k) < \frac{n}{p'(x_k)}, \qquad -\frac{n}{p_\infty} < \sum_{k=0}^{N} m_\infty(w_k) \leqslant \sum_{k=0}^{N} M_\infty(w_k) < \frac{n}{p'_\infty}. \tag{2.22}$$

Remark 2.10. For every $p_0 \in (1, p_-)$ there hold the implications

$$\varrho \in V_{p(\cdot)}(\Omega, \Pi) \quad \Longrightarrow \quad \varrho^{-p_0} \in V_{(\widetilde{p})'(\cdot)}(\Omega, \Pi)$$

and

$$\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi) \quad \Longrightarrow \quad \varrho^{-p_0} \in V^{\mathrm{osc}}_{(\widetilde{p})'(\cdot)}(\Omega, \Pi),$$

where $\widetilde{p}(x) = \frac{p(x)}{p_0}$.

2.3. The boundedness of the Hardy–Littlewood maximal operator on metric spaces with doubling measure, in weighted Lebesgue spaces with variable exponent

The following statements are valid.

Theorem 2.11. *Let X be a metric space with doubling measure and let Ω be bounded. If $p \in \mathcal{P}(\Omega) \cap \mathrm{WL}(\Omega)$ and $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi)$, then $\mathcal{M}$ is bounded in $L^{p(\cdot)}_{\varrho}(\Omega)$.*

Theorem 2.12. *Let X be a metric space with doubling measure and let Ω be unbounded. Let $p \in \mathcal{P}(\Omega) \cap \mathrm{WL}(\Omega)$ and let there exist $R > 0$ such that $p(x) \equiv p_\infty = \mathrm{const}$ for $x \in \Omega \setminus B(x_0, R)$. If $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\Omega, \Pi)$, then $\mathcal{M}$ is bounded in $L^{p(\cdot)}_{\varrho}(\Omega)$.*

The Euclidean version of Theorems 2.11 and 2.12 was proved in [13] in the non-weighted case and in [38,40] in the weighted case; in [40] there were also proved the corresponding versions of Theorems 2.11 and 2.12 for the maximal operator on Carleson curves (a typical example of metric measure spaces with constant dimension). The proof of Theorems 2.11 and 2.12 in the general case in main is similar, being based on the approaches used in the proofs for the case of Carleson curves.

Theorem 2.13. *Let Ω be a bounded open set in a doubling measure metric space X, let the exponent $p(x)$ satisfy conditions* (2.1), (2.3). *Then the operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}_{\varrho}(\Omega)$, if*

$$\big[\varrho(x)\big]^{p(x)} \in A_{p_-}(\Omega).$$

We refer to [44] for Theorem 2.13, its detailed proof for the case where X is a Carleson curve is given in [40], the proof for a doubling measure metric space being in fact the same.

3. Extrapolation theorem on metric measure spaces

In the sequel $\mathcal{F} = \mathcal{F}(\Omega)$ denotes a family of ordered pairs (f, g) of non-negative μ-measurable functions f, g, defined on an open set $\Omega \subset X$. When saying that there holds an inequality of type (3.3) for all pairs $(f, g) \in \mathcal{F}$ and weights $w \in A_1$, we always mean that it is valid for all the pairs, for which the left-hand side is finite, and that the constant c depends only on p_0, q_0 and the A_1-constant of the weight.

In what follows, by p_0 and q_0 we denote positive numbers such that

$$0 < p_0 \leqslant q_0 < \infty, \qquad p_0 < p_- \quad \text{and} \quad \frac{1}{p_0} - \frac{1}{p_+} < \frac{1}{q_0} \tag{3.1}$$

and use the notation

$$\widetilde{p}(x) = \frac{p(x)}{p_0}, \qquad \widetilde{q}(x) = \frac{q(x)}{q_0}. \tag{3.2}$$

Remark 3.1. The extrapolation Theorem 3.2 with variable exponents in the non-weighted case $\varrho(x) \equiv 1$ and in the Euclidean setting was proved in [10]. For extrapolation theorems in the case of constant exponents we refer to [23,61].

Observe that the measure μ in Theorem 3.2 is not assumed to be doubling.

Theorem 3.2. *Let X be a metric measure space and Ω an open set in X. Assume that for some p_0 and q_0, satisfying conditions* (3.1) *and every weight $w \in A_1(\Omega)$ there holds the inequality*

$$\left(\int\limits_\Omega f^{q_0}(x)w(x)\,d\mu(x)\right)^{\frac{1}{q_0}} \leqslant c_0\left(\int\limits_\Omega g^{p_0}(x)\big[w(x)\big]^{\frac{p_0}{q_0}}\,d\mu(x)\right)^{\frac{1}{p_0}} \tag{3.3}$$

for all f, g in a given family $\mathcal{F}$. Let the variable exponent $q(x)$ be defined by

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \left(\frac{1}{p_0} - \frac{1}{q_0}\right), \tag{3.4}$$

let the exponent $p(x)$ and the weight $\varrho(x)$ satisfy the conditions

$$p \in \mathcal{P}(\Omega) \quad \text{and} \quad \varrho^{-q_0} \in \mathfrak{A}_{(\widetilde{q})'}(\Omega). \tag{3.5}$$

Then for all $(f, g) \in \mathcal{F}$ with $f \in L^{p(\cdot)}_\varrho(\Omega)$ the inequality

$$\|f\|_{L^{q(\cdot)}_\varrho} \leqslant C\|g\|_{L^{p(\cdot)}_\varrho} \tag{3.6}$$

is valid with a constant $C > 0$, not depending on f and g.

Proof. By the Riesz theorem, valid for the spaces with variable exponent in the case $1 < p_- \leqslant p_+ < \infty$ (see [46,69]), we have

$$\|f\|^{q_0}_{L^{q(\cdot)}_\varrho} = \big\|f^{q_0}\varrho^{q_0}\big\|_{L^{\widetilde{q}(\cdot)}} \leqslant \sup \int\limits_\Omega f^{p_0}(x)h(x)\,d\mu(x),$$

where we assume that f is non-negative and sup is taken with respect to all non-negative h such that $\|h\varrho^{-q_0}\|_{L^{(\widetilde{q})'(\cdot)}} \leqslant 1$. We fix any such a function h. Let us show that

$$\int\limits_\Omega f^{q_0}(x)h(x)\,d\mu(x) \leqslant C\|g\varrho\|^{q_0}_{L^{q(\cdot)}} \tag{3.7}$$

for an arbitrary pair (f, g) from the given family $\mathcal{F}$ with a constant $C > 0$, not depending on h, f and g. By the assumption $\varrho^{-q_0} \in \mathfrak{A}_{(\widetilde{q})'}(\Omega)$ we have

$$\big\|\varrho^{-q_0}\mathcal{M}\varphi\big\|_{L^{\widetilde{q}'(\cdot)}(\Omega)} \leqslant C_0\big\|\varrho^{-q_0}\varphi\big\|_{L^{\widetilde{p}'(\cdot)}(\Omega)} \tag{3.8}$$

where the constant $C_0 > 0$ does not depend on φ.

We make use of the following construction which is due to Rubio de Francia [61]

$$S\varphi(x) = \sum_{k=0}^{\infty}(2C_0)^{-k}\mathcal{M}^k\varphi(x), \tag{3.9}$$

where $\mathcal{M}^k$ is the k-iterated maximal operator and C_0 is the constant from (3.8) (one may take $C_0 \geqslant 1$). The following statements are obvious:

(1) $\varphi(x) \leqslant S\varphi(x)$, $x \in \Omega$ for any non-negative function φ,

(2) $$\big\|\varrho^{-q_0}S\varphi\big\|_{L^{(\widetilde{q})'}(\Omega)} \leqslant 2\big\|\varrho^{-q_0}\varphi\big\|_{L^{(\widetilde{q})'}(\Omega)}, \tag{3.10}$$

(3) $\mathcal{M}(S\varphi)(x) \leqslant 2C_0S\varphi(x)$, $x \in \Omega$,

so that $S\varphi \in A_1(\Omega)$ with the A_1-constant not depending on φ. Therefore $S\varphi \in A_{q_0}(\Omega)$.

By (1), for $\varphi = h$ we have

$$\int_\Omega f^{q_0}(x)h(x)\,d\mu(x) \leqslant \int_\Omega f^{q_0}(x)Sh(x)\,d\mu(x). \tag{3.11}$$

By the Hölder inequality for variable exponent, property (2) and the condition $f \in L^{q(\cdot)}_\varrho$, we have

$$\int_\Omega f^{q_0}(x)Sh(x)\,d\mu(x) \leqslant k\big\| f^{q_0}\varrho^{q_0}\big\|_{L^{\widetilde{q}(\cdot)}} \cdot \big\|\varrho^{-q_0}Sh\big\|_{L^{(\widetilde{q})'(\cdot)}}$$

$$\leqslant C\|f\varrho\|^{q_0}_{L^{q(\cdot)}} \cdot \big\|h\varrho^{-q_0}\big\|_{L^{(\widetilde{q})'(\cdot)}} \leqslant C\|f\varrho\|^{q_0}_{L^{q(\cdot)}} < \infty.$$

Consequently, the integral $\int_\Omega f^{q_0}(x)Sh(x)\,d\mu(x)$ is finite, which enables us to make use of condition (3.3) with respect to the right-hand side of (3.11). Condition (3.3) being assumed to be valid with an arbitrary weight $w \in A_1$, is in particular valid for $w = Sh$. Therefore,

$$\int_\Omega f^{q_0}(x)Sh(x)\,d\mu(x) \leqslant C\bigg(\int_\Omega g^{p_0}(x)\big[Sh(x)\big]^{\frac{p_0}{q_0}}\,d\mu(x)\bigg)^{\frac{q_0}{p_0}}.$$

Applying the Hölder inequality on the right-hand side, we get

$$\int_\Omega f^{q_0}(x)Sh(x)\,d\mu(x) \leqslant C\Big(\big\|g^{p_0}\varrho^{p_0}\big\|_{L^{\frac{p(\cdot)}{p_0}}}\big\|(Sh)^{\frac{p_0}{q_0}}\varrho^{-p_0}\big\|_{L^{(\widetilde{p})'(\cdot)}}\Big)^{\frac{q_0}{p_0}}.$$

Thus

$$\int_\Omega f^{q_0}(x)Sh(x)\,d\mu(x) \leqslant C\|\varrho g\|^{q_0}_{L^{p(\cdot)}}\big\|\varrho^{-p_0}(Sh)^{\frac{p_0}{q_0}}\big\|^{\frac{q_0}{p_0}}_{L^{(\widetilde{p})'(\cdot)}}. \tag{3.12}$$

From (3.4) we easily obtain that $(\widetilde{p})'(x) = \frac{q_0}{p_0}(\widetilde{q})'(x)$ and then

$$\big\|\varrho^{-p_0}(Sh)^{\frac{p_0}{q_0}}\big\|^{\frac{q_0}{p_0}}_{L^{(\widetilde{p})'(\cdot)}} = \big\|\varrho^{-q_0}Sh\big\|_{L^{\widetilde{q}'(\cdot)}}.$$

Consequently,

$$\int_\Omega f^{q_0}(x)Sh(x)\,d\mu(x) \leqslant C\|\varrho g\|^{q_0}_{L^{p(\cdot)}}\big\|\varrho^{-q_0}Sh\big\|_{L^{\widetilde{q}'(\cdot)}}. \tag{3.13}$$

To prove (3.7), in view of (3.13) it suffices to show that $\|\varrho^{-q_0}Sh\|_{L^{\widetilde{q}'(\cdot)}}$ may be estimated by a constant not depending on h. This follows from (3.10) and the condition $\|h\varrho^{-q_0}\|_{L^{(\widetilde{q})'(\cdot)}} \leqslant 1$ and proves the theorem. □

Remark 3.3. It is easy to check that in view of Theorem 2.13 the condition

$$\big[\varrho(y)\big]^{q_1(y)} \in A_s, \quad \text{where } q_1(y) = \frac{q(y)(q_+ - q_0)}{q(y) - q_0} \text{ and } s = \frac{q_+}{q_0}, \tag{3.14}$$

is sufficient for the validity of the condition $\varrho^{-q_0} \in \mathfrak{A}_{(\widetilde{q})'}(\Omega)$ of Theorem 3.2.

By means of Theorems 2.11 and 2.12, we obtain the following statement as an immediate consequence of Theorem 3.2 in which we denote

$$\gamma = \frac{1}{p_0} - \frac{1}{q_0}.$$

Theorem 3.4. *Let X be a metric space with doubling measure and Ω an open set in X. Let also the following be satisfied*

(1) $p \in \mathcal{P}(\Omega) \cap \mathrm{WL}(\Omega)$, *and in the case Ω is an unbounded set, let $p(x) \equiv p_\infty = \mathrm{const}$ for $x \in \Omega \setminus B(x_0, R)$ with some $x_0 \in \Omega$ and $R > 0$;*
(2) *there holds inequality* (3.3) *for some p_0 and q_0 satisfying the assumptions in* (3.1) *and all $(f, g) \in \mathcal{F}$ from some family $\mathcal{F}$ and every weight $w \in A_1(\Omega)$.*

Then

(I) *there holds inequality* (3.6) *for all pairs (f, g) from the same family $\mathcal{F}$, such that $f \in L^{p(\cdot)}_\varrho(\Omega)$ and weights ϱ of form* (2.16) *where*

$$\left(\gamma - \frac{1}{p(x_k)}\right)\underline{\mathfrak{dim}}(\Omega) < m(w_k) \leqslant M(w_k) < \left(\frac{1}{p'(x_k)} - \frac{1}{p'_0}\right)\underline{\mathfrak{dim}}(\Omega) \tag{3.15}$$

and, in case Ω is unbounded,

$$\delta + \left(\gamma - \frac{1}{p_\infty}\right)\underline{\mathfrak{dim}}(\Omega) < \sum_{k=0}^{N} m(w_k) \leqslant \sum_{k=0}^{N} M(w_k) < \left(\frac{1}{p'_\infty} - \frac{1}{p'_0}\right)\underline{\mathfrak{dim}}(\Omega), \tag{3.16}$$

where

$$\delta = \left[\overline{\mathfrak{dim}}_\infty(\Omega) - \underline{\mathfrak{dim}}_\infty(\Omega)\right]\left(\frac{1}{p_0} - \frac{1}{p_\infty}\right);$$

(II) *in case inequality* (3.3) *holds for all $p_0 \in (1, p_-)$, the term $\frac{1}{p'_0}$ in* (3.15) *and* (3.16) *may be omitted and δ may be taken in the form $\delta = [\overline{\mathfrak{dim}}_\infty(\Omega) - \underline{\mathfrak{dim}}_\infty(\Omega)](\frac{1}{p_-} - \frac{1}{p_\infty})$.*

4. Application to problems of the boundedness in $L^{p(\cdot)}_\varrho$ of classical operators of harmonic analysis

4.1. Potential operators and fractional maximal function

We first apply Theorem 3.2 to potential operators

$$I^\gamma_X f(x) = \int_X \frac{f(y)\, d\mu(y)}{\mu B(x, d(x, y))^{1-\gamma}} \tag{4.1}$$

where $0 < \gamma < 1$. We assume that $\mu X = \infty$ and the measure μ satisfies the doubling condition. We also additionally suppose the following conditions to be fulfilled:

$$\text{there exists a point } x_0 \in X \text{ such that } \mu(x_0) = 0 \tag{4.2}$$

and

$$\mu\big(B(x_0, R) \setminus B(x_0, r)\big) > 0 \quad \text{for all } 0 < r < R < \infty. \tag{4.3}$$

The following statement is valid, see for instance [18, p. 412].

Theorem 4.1. *Let X be a metric measure space with doubling measure satisfying conditions* (4.2)–(4.3), $\mu X = \infty$, *let $0 < \gamma < 1$, $1 < p_0 < \frac{1}{\gamma}$ and $\frac{1}{q_0} = \frac{1}{p_0} - \gamma$. The operator I^γ_X admits the estimate*

$$\left(\int_X \left|v(x) I^\gamma_X f(x)\right|^{q_0} d\mu\right)^{\frac{1}{q_0}} \leqslant \left(\int_X \left|v(x) f(x)\right|^{p_0} d\mu\right)^{\frac{1}{p_0}}, \tag{4.4}$$

if the weight $v(x)$ satisfies the condition

$$\sup_B \left(\frac{1}{\mu B}\int_B v^{q_0}(x)\, d\mu\right)^{\frac{1}{q_0}} \left(\frac{1}{\mu B}\int_B v^{-p'_0}(x)\, d\mu\right)^{\frac{1}{p'_0}} < \infty \tag{4.5}$$

where B stands for a ball in X.

By means of Theorem 4.1 and extrapolation Theorem 3.2 we arrive at the following statement.

Theorem 4.2. *Let X satisfy the assumptions of Theorem* 4.1, *let* $p \in \mathcal{P}$, $0 < \gamma < 1$ *and* $p_+ < \frac{1}{\gamma}$. *The weighted estimate*

$$\left\| I_X^\gamma f \right\|_{L_\varrho^{q(\cdot)}} \leqslant C \| f \|_{L_\varrho^{p(\cdot)}} \tag{4.6}$$

with the limiting exponent $q(\cdot)$ *defined by* $\frac{1}{q(x)} = \frac{1}{p(x)} - \gamma$, *holds if*

$$\varrho^{-q_0} \in \mathfrak{A}_{(\frac{q(\cdot)}{q_0})'}(X) \tag{4.7}$$

under any choice of $q_0 > \frac{p_-}{1-\gamma p_-}$.

Proof. By Theorem 4.1, inequality (4.4) holds under condition (4.5). As is known, inequality (3.3) with $f = I^\alpha g$ holds for every weight w satisfying the $1 < p_0 < \infty$ and $\frac{1}{q_0} = \frac{1}{p_0} - \gamma$. Condition (4.5) is satisfied if $v^{q_0} \in A_1$. Consequently, inequality (3.3) with $f = I^\alpha g$ holds for every $w \in A_1$. Then (4.6) follows from Theorem 3.2. □

Let

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}}. \tag{4.8}$$

Corollary 4.3. *Let* $p \in \mathcal{P}$, *let* $0 < \alpha < n$ *and* $p_+ < \frac{n}{\alpha}$. *The weighted Sobolev theorem*

$$\left\| I^\alpha f \right\|_{L_\varrho^{q(\cdot)}} \leqslant C \| f \|_{L_\varrho^{p(\cdot)}} \tag{4.9}$$

with the limiting exponent $q(\cdot)$ *defined by* $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$, *holds if*

$$\varrho^{-q_0} \in \mathfrak{A}_{(\frac{q(\cdot)}{q_0})'}\left(\mathbb{R}^n\right) \tag{4.10}$$

under any choice of $q_0 > \frac{np_-}{n-\alpha p_-}$.

Remark 4.4. Since Theorems 2.11 and 2.12 provide sufficient conditions for the weight ϱ to satisfy assumption (4.10), we could write down the corresponding statements on the validity of (4.9) in terms of the weights used in Theorems 2.11 and 2.12. In the sequel we give results of such a kind for other operators. For potential operators in the case $\Omega = \mathbb{R}^n$ we refer to [74] and [68], where for power weights of the class $V_{p(\cdot)}(\mathbb{R}^n, \Pi)$ and for radial oscillating weights of the class $V_{p(\cdot)}^{\mathrm{osc}}(\mathbb{R}^n, \Pi)$, respectively, there were obtained estimates (4.9) under assumptions more general than should be imposed by the usage of Theorem 2.12.

4.2. Fourier multipliers

A measurable function $\mathbb{R}^n \to \mathbb{R}^1$ is said to be a Fourier multiplier in the space $L_\varrho^{p(\cdot)}(\mathbb{R}^n)$, if the operator T_m, defined on the Schwartz space $S(\mathbb{R}^n)$ by

$$\widehat{T_m f} = m \widehat{f},$$

admits an extension to the bounded operator in $L_\varrho^{p(\cdot)}(\mathbb{R}^n)$.

We give below a generalization of the classical Mikhlin theorem ([55], see also [56]) on Fourier multipliers to the case of Lebesgue spaces with variable exponent.

Theorem 4.5. *Let a function* $m(x)$ *be continuous everywhere in* $\mathbb{R}^n$, *except for probably the origin, have the mixed distributional derivative* $\frac{\partial^n m}{\partial x_1 x_2 \cdots x_n}$ *and the derivatives* $D^\alpha m = \frac{\partial^{|\alpha|} m}{\partial x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ *of orders* $|\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant n-1$ *continuous beyond the origin and*

$$|x|^{|\alpha|} \left| D^\alpha m(x) \right| \leqslant C, \quad |\alpha| \leqslant n,$$

where the constant $C > 0$ *does not depend on* x. *Then under conditions* (3.5) *and* (3.1) *with* $\Omega = \mathbb{R}^n$, *m is a Fourier multiplier in* $L_\varrho^{p(\cdot)}(\mathbb{R}^n)$.

Proof. Theorem 4.5 follows from Theorem 3.2 under the choice $\Omega = X = \mathbb{R}^n$ and $\mathcal{F} = \{T_m g, g\}$ with $g \in S(\mathbb{R}^n)$, if we take into account that in the case of constant $p_0 > 1$ and weight $\varrho \in A_{p_0}$ $(\supset A_1)$, a function m, satisfying the assumptions of Theorem 4.5, is a Fourier multiplier in $L^{p_0}_\varrho(\mathbb{R}^n)$. The latter was proved in [49], see also [34]. □

Corollary 4.6. *Let m satisfy the assumptions of Theorem* 4.5 *and let the exponent p and the weight ϱ satisfy the assumptions*

(i) $p \in \mathcal{P}(\mathbb{R}^n) \cap \mathrm{WL}(\mathbb{R}^n)$ *and* $p(x) = p_\infty = \mathrm{const}$ *for* $|x| \geqslant R$ *with some* $R > 0$,
(ii) $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\mathbb{R}^n, \Pi)$, $\Pi = \{x_1, \dots, x_N\} \subset \mathbb{R}^n$.

Then m is a Fourier multiplier in $L^{p(\cdot)}_\varrho(\mathbb{R}^n)$. In particular, assumption (ii) *holds for weights ϱ of form*

$$\varrho(x) = \bigl(1 + |x|\bigr)^{\beta_\infty} \prod_{k=1}^{N} |x - x_k|^{\beta_k}, \quad x_k \in \mathbb{R}^n, \tag{4.11}$$

where

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)}, \quad k = 1, 2, \dots, N, \tag{4.12}$$

$$-\frac{n}{p_\infty} < \beta_\infty + \sum_{k=1}^{N} \beta_k < \frac{n}{p'_\infty}. \tag{4.13}$$

Proof. It suffices to observe that conditions on the weight ϱ imposed in Theorem 4.5, are fulfilled for $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\mathbb{R}^n, \Pi)$ which follows from Remark 2.10 and Theorem 2.12. In the case of power weights, conditions defining the class $V^{\mathrm{osc}}_{p(\cdot)}(\mathbb{R}^n, \Pi)$ turn into (4.12)–(4.13). □

Theorem 4.7. *Let a function $m : \mathbb{R}^n \to \mathbb{R}^1$ have distributional derivatives up to order $\ell > \frac{n}{p_-}$ satisfying the condition*

$$\sup_{R>0} \biggl(R^{s|\alpha|-n} \int_{R<|x|<2R} \bigl|D^\alpha m(x)\bigr|^s \, dx \biggr)^{\frac{1}{s}} < \infty$$

for some s, $1 < s \leqslant 2$ and all α with $|\alpha| \leqslant \ell$. If conditions (3.5), (3.1) *with $\Omega = X = \mathbb{R}^n$ on p and ϱ are satisfied, then m is a Fourier multiplier in $L^{p(\cdot)}_\varrho(\mathbb{R}^n)$.*

Proof. Theorem 4.7 is similarly derived from Theorem 3.2, if we take into account that in the case of constant p_0 the statement of the theorem for Muckenhoupt weights was proved in [50]. □

Corollary 4.8. *Let a function $m : \mathbb{R}^n \to \mathbb{R}^1$ satisfy the assumptions of Theorem* 4.7 *and let p and ϱ satisfy conditions* (i) *and* (ii) *of Corollary* 4.6. *Then m is a Fourier multiplier in $L^{p(\cdot)}_\varrho(\mathbb{R}^n)$.*

Proof. Follows from Theorem 4.7 since conditions on the weight ϱ imposed in Theorem 4.5, are fulfilled for $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\mathbb{R}^n, \Pi)$ by Theorem 2.12 and Remark 2.10. □

In the next theorem by Δ_j we denote the interval of the form $\Delta_j = [2^j, 2^{j+1}]$ or $\Delta_j = [-2^{j+1}, -2^j]$, $j \in \mathbb{Z}$.

Theorem 4.9. *Let a function $m : \mathbb{R}^1 \to \mathbb{R}^1$ be representable in each interval Δ_j as*

$$m(\lambda) = \int_{-\infty}^{\lambda} d\mu_{\Delta_j}, \quad \lambda \in \Delta_j,$$

where μ_{Δ_j} *are finite measures such that* $\sup_j \operatorname{var} \mu_{\Delta_j} < \infty$. *If conditions* (3.5), (3.1) *with* $\Omega = X = \mathbb{R}^n$ *on* p *and* ϱ *are satisfied, then* m *is a Fourier multiplier in* $L^{p(\cdot)}_{\varrho}(\mathbb{R}^1)$.

Proof. To derive Theorem 4.9 from Theorem 3.4, it suffices to refer to the boundedness of the maximal operator in the space $L^{p(\cdot)}_{\varrho}(\mathbb{R}^1)$ by Theorem 2.12 and the fact that in the case of constant p the theorem was proved in [51] (for $\varrho \equiv 1$) and [34,35] (for $\varrho \in A_p$). □

Corollary 4.10. *Let* m *satisfy the assumptions of Theorem* 4.9 *and the exponent* p *and weight* ϱ *fulfill conditions* (i) *and* (ii) *of Corollary* 4.6 *with* $n = 1$. *Then* m *is a Fourier multiplier in* $L^{p(\cdot)}_{\varrho}(\mathbb{R}^1)$.

The "off-diagonal" $L^{p(\cdot)}_{\varrho} \to L^{q(\cdot)}_{\varrho}$-version of Theorem 4.9 in the case $q(x) > p(x)$ is covered by the following theorem.

Theorem 4.11. *Let* $p \in \mathcal{P}(\mathbb{R}^1) \cap \mathrm{WL}(\mathbb{R}^1)$ *and* $p(x) \equiv p_\infty = \mathrm{const}$ *for large* $|x| > R$, *and let a function* $m : \mathbb{R}^1 \to \mathbb{R}^1$ *be representable in each interval* Δ_j *as*

$$m(\lambda) = \int_{-\infty}^{\lambda} \frac{d\mu_{\Delta_j}(t)}{(\lambda - t)^{\alpha}}, \quad \lambda \in \Delta_j,$$

where $0 < \alpha < \frac{1}{p_+}$ *and* μ_{Δ_j} *are the same as in Theorem* 4.9. *Then* T_m *is a bounded operator from* $L^{p(\cdot)}_{\varrho}(\mathbb{R}^1)$ *to* $L^{q(\cdot)}_{\varrho}(\mathbb{R}^1)$, *where* $\frac{1}{q(x)} = \frac{1}{p(x)} - \alpha$ *and* ϱ *is a weight of form* (4.11) *whose exponents satisfy the conditions*

$$\alpha - \frac{1}{p(x_k)} < \beta_k < \frac{1}{p'(x_k)}, \quad k = 1, 2, \ldots, N, \quad \text{and} \quad \alpha - \frac{1}{p_\infty} < \beta_\infty + \sum_{k=1}^{N} \beta_k < \frac{1}{p'_\infty}. \tag{4.14}$$

Proof. In [36] there was proved that the operator T_m is bounded from $L^{p_0}_v(\mathbb{R}^1)$ into $L^{q_0}_v(\mathbb{R}^1)$ for every $p_0 \in (1, \infty)$, $0 < \alpha < \frac{1}{p_0}$, $\frac{1}{q_0} = \frac{1}{p_0} - \alpha$, and an arbitrary weight v satisfying the condition

$$\sup_I \left(\frac{1}{|I|} \int_I v^{q_0}(x)\, dx\right)^{\frac{1}{q_0}} \left(\frac{1}{|I|} \int_I v^{-p'_0}(x)\, dx\right)^{\frac{1}{p'_0}}, \tag{4.15}$$

where the supremum is taken with respect to all one-dimensional intervals. Condition (4.15) is satisfied if $v^{q_0} \in A_1$. Then inequality (3.3) with $f = T_m g$ holds for every $w \in A_1$. Then the statement of the theorem follows immediately from part (II) of Theorem 3.4, conditions (3.15)–(3.16) turning into (4.14) since $\underline{\mathfrak{dim}}(\Omega) = \underline{\mathfrak{dim}}_\infty(\Omega) = 1$, $m(w_k) = M(w_k) = \beta_k$, $k = 1, \ldots, N$, and $m(w_0) = M(w_0) = \beta_\infty$. □

All the statements in the following subsections are also similar direct consequences of the general statement of Theorem 3.4 and Theorems 2.11 and 2.12 on the maximal operator in the spaces $L^{p(\cdot)}_{\varrho}$, so that in the sequel for the proofs we only make references to where these statements were proved in the case of constant p and Muckenhoupt weights.

4.3. Multipliers of trigonometric Fourier series

With the help of Theorem 3.4 and known results for constant exponents, we are now able to give a generalization of theorems on Marcinkiewicz multipliers and Littlewood–Paley decompositions for trigonometric Fourier series to the case of weighted spaces with variable exponent.

Let $\mathbb{T} = [\pi, \pi]$ and let f be a 2π-periodic function and

$$f(x) \sim \frac{a_0}{2} + \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{4.16}$$

Theorem 4.12. *Let a sequence* λ_k *satisfy the conditions*

$$|\lambda_k| \leqslant A \quad \text{and} \quad \sum_{k=2^{j-1}}^{2^j-1} |\lambda_k - \lambda_{k+1}| \leqslant A, \tag{4.17}$$

where $A > 0$ *does not depend on* k *and* j. *Suppose that*

$$p \in \mathcal{P}(\mathbb{T}) \quad \text{and} \quad \varrho^{-p_0} \in \mathfrak{A}_{(\widetilde{p})'}(\mathbb{T}), \quad \text{where } \widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0} \tag{4.18}$$

with some $p_0 \in (1, p_-(\mathbb{T}))$. *Given* $f \in L^{p(\cdot)}_\varrho$, *there exists a function* $F(x) \in L^{p(\cdot)}_\varrho(\mathbb{T})$ *such that the series* $\frac{\lambda_0 a_0}{2} + \sum_{k=0}^\infty \lambda_k(a_k \cos kx + b_k \sin kx)$ *is Fourier series for* F *and*

$$\|F\|_{L^{p(\cdot)}_\varrho} \leqslant cA\|f\|_{L^{p(\cdot)}_\varrho}$$

where $c > 0$ *does not depend on* $f \in L^{p(\cdot)}_\varrho(\mathbb{T})$.

Corollary 4.13. *Theorem* 4.12 *remains valid if condition* (4.18) *is replaced by the assumption, sufficient for* (4.18), *that* $p \in \mathcal{P}(\mathbb{T}) \cap \mathrm{WL}(\mathbb{T})$ *and* ϱ *has form*

$$\varrho(x) = \prod_{k=1}^N w_k\big(|x - x_k|\big), \quad x_k \in \mathbb{T}, \tag{4.19}$$

where

$$w_k \in \widetilde{U}\big([0, 2\pi]\big) \quad \text{and} \quad -\frac{1}{p(x_k)} < m(w_k) \leqslant M(w_k) < \frac{1}{p'(x_k)}. \tag{4.20}$$

Theorem 4.14. *Let*

$$A_k(x) = a_k \cos kx + b_k \sin kx, \quad k = 0, 1, 2, \ldots, \quad A_{2^{-1}} = 0. \tag{4.21}$$

Under conditions (4.18) *there exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$c_1\|f\|_{L^{p(\cdot)}_\varrho} \leqslant \left\| \left(\sum_{j=0}^\infty \left| \sum_{k=2^{j-1}}^{2^j-1} A_k(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}_\varrho} \leqslant c_2\|f\|_{L^{p(\cdot)}_\varrho} \tag{4.22}$$

for all $f \in L^{p(\cdot)}_\varrho(\mathbb{T})$.

In the case of constant p and $\varrho \in A_p$ this theorem was proved in [49].

Corollary 4.15. *Inequalities* (4.22) *hold for* $p \in \mathcal{P}(\mathbb{T}) \cap \mathrm{WL}(\mathbb{T})$ *and weights* ϱ *of form* (4.19)–(4.20).

4.4. Majorants of partial sums of Fourier series

Let

$$S_*(f) = S_*(f, x) = \sup_{k \geqslant 0} \big|S_k(f, x)\big|,$$

where $S_k(f, x) = \sum_{j=0}^k A_j(x)$ is a partial sum of Fourier series (4.16).

Theorem 4.16. *Under conditions* (4.18)

$$\big\|S_*(f)\big\|_{L^{p(\cdot)}_\varrho} \leqslant c\|f\|_{L^{p(\cdot)}_\varrho}, \tag{4.23}$$

for all $f \in L^{p(\cdot)}_\varrho(\mathbb{T})$, *where the constant* $c > 0$ *does not depend on* f.

In the case of constant p and $\varrho \in A_p$, Theorem 4.16 was proved in [27].

Corollary 4.17. *Inequality* (4.23) *is valid for* $p \in \mathcal{P}(\mathbb{T}) \cap \mathrm{WL}(\mathbb{T})$ *and weights* ϱ *of form* (4.19)–(4.20).

4.5. Zygmund and Cesaro summability for trigonometric series in $L^{p(\cdot)}_{\varrho}(\mathbb{T})$

Under notation (4.16) and (4.21) we introduce the Zygmund and Cesaro means of summability

$$Z_n^{(2)}(f,x)=\sum_{k=0}^{n}\left[1-\left(\frac{k}{n+1}\right)^2\right]A_k(x)$$

and

$$\sigma_n(f,x)=\frac{1}{n+1}\sum_{k=0}^{n}S_k(f,x),$$

respectively. By

$$\Omega_{p,\varrho}(f,\delta)=\sup_{0<h<\delta}\left\|(I-\tau_h)f\right\|_{L^{p(\cdot)}_{\varrho}}$$

we denote the continuity modulus of a function f in $L^{p(\cdot)}_{\varrho}(\mathbb{T})$ with respect to the generalized shift (Steklov mean)

$$\tau_h f(x)=\frac{1}{2h}\int_{x-h}^{x+h}f(t)\,dt.$$

Theorem 4.18. *Under conditions* (4.18) *there hold the estimates*

$$\left\|f(\cdot)-Z_n^{(2)}(f,\cdot)\right\|_{L^{p(\cdot)}_{\varrho}}\leqslant C\Omega_{p,\varrho}\left(f,\frac{1}{n}\right) \tag{4.24}$$

and

$$\left\|f(\cdot)-\sigma_n(f,\cdot)\right\|_{L^{p(\cdot)}_{\varrho}}\leqslant Cn\Omega_{p,\varrho}\left(f,\frac{1}{n}\right). \tag{4.25}$$

Proof. We make use of the estimate

$$\left\|f(\cdot)-S_n(f,\cdot)\right\|_{L^{p(\cdot)}_{\varrho}}\leqslant C\Omega_{p,\varrho}\left(f,\frac{1}{n}\right) \tag{4.26}$$

proved in [28] under assumptions (4.18). For the difference $S_n(f,x)-Z_n^{(2)}(f,x)$ we have

$$\left\|S_n(f,\cdot)-Z_n^{(2)}(f,\cdot)\right\|_{L^{p(\cdot)}_{\varrho}}=\left\|\sum_{k=1}^{n}\left(\frac{k}{n+1}\right)^2A_k(\cdot)\right\|_{L^{p(\cdot)}_{\varrho}}. \tag{4.27}$$

Keeping in mind that

$$f(x)-\tau_h f(x)\sim\sum_{k=1}^{\infty}\left(1-\frac{\sin kh}{kh}\right)A_k(x), \tag{4.28}$$

we transform (4.27) to

$$\left\|S_n(f,\cdot)-Z_n^{(2)}(f,\cdot)\right\|_{L^{p(\cdot)}_{\varrho}}=\left\|\sum_{k=1}^{n}\lambda_{k,n}\left(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}\right)A_k(\cdot)\right\|_{L^{p(\cdot)}_{\varrho}}$$

where

$$\lambda_{k,n}=\begin{cases}\dfrac{(\frac{k}{n+1})^2}{1-\frac{\sin\frac{k}{n}}{\frac{k}{n}}}, & k\leqslant n,\\ 0, & k>n.\end{cases}$$

It is easy to check that $\lambda_{k,n}$ satisfies assumptions (4.17) of Theorem 4.12 with the constant A in (4.17) not depending on n. Therefore, by Theorem 4.12 we get

$$\left\| S_n(f,\cdot) - Z_n^{(2)}(f,\cdot)\right\|_{L_\varrho^{p(\cdot)}} \leqslant C \left\| \sum_{k=1}^{\infty} \left(1 - \frac{\sin \frac{k}{n}}{\frac{k}{n}}\right) A_k(\cdot) \right\|_{L_\varrho^{p(\cdot)}} = C \| f - \tau_h f \|_{L_\varrho^{p(\cdot)}}$$

by (4.28). Then in view of (4.26) estimate (4.24) follows.

Estimate (4.25) is similarly obtained, with the multiplier $\lambda_{k,n}$ of the form

$$\begin{cases} \dfrac{\frac{k}{n+1}}{n(1-\frac{\sin\frac{k}{n}}{\frac{k}{n}})}, & k \leqslant n, \\ 0, & k > n. \end{cases} \qquad \square$$

Corollary 4.19. *Estimates* (4.24), (4.25) *are valid for* $p \in \mathcal{P}(\mathbb{T}) \cap \mathrm{WL}(\mathbb{T})$ *and weights* ϱ *of form* (4.19)–(4.20).

Remark 4.20. When $p > 1$ is constant, estimates (4.24), (4.25) in the non-weighted case were obtained in [32].

4.6. Cauchy singular integral

We consider the singular integral operator

$$S_\Gamma f(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)\, d\nu(\tau)}{\tau - t},$$

where Γ is a simple finite Carleson curve and ν is an arc length.

Theorem 4.21. *Let*

$$p \in \mathcal{P}(\Gamma) \quad \textit{and} \quad \varrho^{-p_0} \in \mathfrak{A}_{(\widetilde{p})'}(\Gamma) \tag{4.29}$$

for some $p_0 \in (1, p_-)$, *where* $\widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0}$. *Then* S_Γ *is bounded in* $L_\varrho^{p(\cdot)}(\Gamma)$.

For constant p and $\varrho^p \in A_p(\Gamma)$, Theorem 4.21 by different methods was proved in [31] and [5]. (As is known, $\varrho^{-p_0} \in \mathfrak{A}_{(\widetilde{p})'}(\Gamma) \Longleftrightarrow \varrho^p \in A_{\frac{p}{p_0}}(\Gamma)$ for an arbitrary Carleson curve in the case of constant p, see [31] and [5], so that the conditions $\varrho^{-p_0} \in \mathfrak{A}_{(\widetilde{p})'}(\Gamma)$ and $\varrho^p \in A_p(\Gamma)$ are equivalent in the sense that the former always yields the latter for every $p_0 > 1$ and the latter yields the former for some $p_0 > 1$.)

Corollary 4.22. *The operator* S_Γ *is bounded in the space* $L_\varrho^{p(\cdot)}(\Gamma)$, *if* $p \in \mathcal{P}(\Gamma) \cap \mathrm{WL}(\Gamma)$ *and the weight* ϱ *has the form*

$$\varrho(t) = \prod_{k=1}^{N} w_k\big(|t - t_k|\big), \quad t_k \in \Gamma, \tag{4.30}$$

where

$$w_k \in \widetilde{U}\big([0, \nu(\Gamma)]\big) \quad \textit{and} \quad -\frac{1}{p(t_k)} < m(w_k) \leqslant M(w_k) < \frac{1}{p'(t_k)}. \tag{4.31}$$

In the case of power weights, the statement of Corollary 4.22 was proved in [37], where the case of an infinite Carleson curve was also dealt with.

4.7. Multidimensional singular type operators

We consider a multidimensional singular operator

$$Tf(x) = \lim_{\varepsilon\to 0} \int\limits_{y\in\Omega:\ |x-y|>\varepsilon} K(x,y) f(y)\,dy, \quad x \in \Omega \subseteq \mathbb{R}^n, \tag{4.32}$$

where we assume that the singular kernel $K(x, y)$ satisfies the assumptions:

$$\big|K(x,y)\big| \leqslant C|x-y|^{-n}, \tag{4.33}$$

$$\big|K(x',y) - K(x,y)\big| \leqslant C\frac{|x'-x|^\alpha}{|x-y|^{n+\alpha}}, \quad |x'-x| < \frac{1}{2}|x-y|, \tag{4.34}$$

$$\big|K(x,y') - K(x,y)\big| \leqslant C\frac{|y'-y|^\alpha}{|x-y|^{n+\alpha}}, \quad |y'-y| < \frac{1}{2}|x-y|, \tag{4.35}$$

where α is an arbitrary positive exponent,

$$\text{there exists} \quad \lim_{\varepsilon\to 0} \int\limits_{y\in\Omega:|x-y|>\varepsilon} K(x,y)\,dy, \tag{4.36}$$

$$\text{operator (4.32) is bounded in } L^2(\Omega). \tag{4.37}$$

Theorem 4.23. *Let the kernel $K(x, y)$ fulfill conditions* (4.33)–(4.37). *Then under the conditions*

$$p \in \mathcal{P}(\Omega) \quad \textit{and} \quad \varrho^{-p_0} \in \mathfrak{A}_{(\widetilde{p})'}(\Omega) \quad \textit{with } \widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0} \tag{4.38}$$

the operator T is bounded in the space $L^{p(\cdot)}_\varrho(\Omega)$.

In the case of constant p and $\varrho \in A_p(\mathbb{R}^n)$, Theorem 4.23 was proved in [9].

Corollary 4.24. *Let $p \in \mathcal{P}(\Omega) \cap \mathrm{WL}(\Omega)$ and let $p(x) \equiv p_\infty = \mathrm{const}$ outside some ball $|x| < R$ in case Ω is unbounded. The operator T with the kernel satisfying conditions* (4.33)–(4.37) *is bounded in the space $L^{p(\cdot)}_\varrho(\Omega)$ with a weight ϱ of the form*

$$\varrho(x) = \prod_{k=1}^N w_k\big(|x-x_k|\big), \quad x_k \in \Omega, \tag{4.39}$$

where $w_k \in \widetilde{U}(\mathbb{R}^1_+)$ and

$$-\frac{1}{p(x_k)} < m(w_k) \leqslant M(w_k) < \frac{1}{p'(x_k)} \quad \textit{and} \quad -\frac{n}{p_\infty} < \sum_{k=1}^N m_\infty(w_k) \leqslant \sum_{k=1}^N M_\infty(w_k) < \frac{n}{p'_\infty}.$$

In the case of variable $p(\cdot)$, the statement of Corollary 4.24 was proved in [17] in the non-weighted case, and in [39] in weighted case (4.41) for bounded sets Ω.

Let

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x), \quad x \in \mathbb{R}^n,$$

be a commutator generated by operator (4.32) with $\Omega = \mathbb{R}^n$ and a function $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Theorem 4.25. *Let the kernel $K(x, y)$ fulfill assumptions* (4.33)–(4.37) *and let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then under the conditions*

$$p \in \mathcal{P}\big(\mathbb{R}^n\big) \quad \textit{and} \quad \varrho^{-p_0} \in \mathfrak{A}_{(\widetilde{p})'}\big(\mathbb{R}^n\big) \quad \textit{with } \widetilde{p}(\cdot) = \frac{p(\cdot)}{p_0} \tag{4.40}$$

the commutator $[b, T]$ is bounded in the space $L^{p(\cdot)}_\varrho(\mathbb{R}^n)$.

In the case of constant p and $\varrho \in A_p(\mathbb{R}^n)$, $1 < p < \infty$, Theorem 4.25 was proved in [59]. In the case of variable $p(\cdot)$, the non-weighted case of Theorem 4.25 was proved in [30] under the assumption that $1 \in \mathfrak{A}_{p(\cdot)}(\mathbb{R}^n)$.

Corollary 4.26. *Let the kernel $K(x, y)$ fulfill conditions* (4.33)–(4.37) *and let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then the commutator $[b, T]$ is bounded in the space $L^{p(\cdot)}_{\varrho}(\mathbb{R}^n)$ if*

(i) *$p \in \mathcal{P}(\mathbb{R}^n) \cap \mathrm{WL}(\mathbb{R}^n)$ and $p(x) \equiv p_\infty = \mathrm{const}$ outside some ball $|x| < R$,*
(ii) *the weight ϱ has the form*

$$\varrho(x) = w_0\big(1 + |x|\big) \prod_{k=1}^{N} w_k\big(|x - x_k|\big), \quad x_k \in \mathbb{R}^n, \tag{4.41}$$

with the factors w_k, $k = 0, 1, \dots, N$, satisfying conditions (2.21)–(2.22).

For a pseudo-differential operator $\sigma(x, D)$ defined by

$$\sigma(x, D) f(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{2\pi i (x,\xi)} \hat{f}(\xi)\, d\xi$$

we arrive at the following result.

Theorem 4.27. *Let the symbol $\sigma(x, \xi)$ satisfy the condition*

$$\big|\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)\big| \leqslant c_{\alpha\beta}\big(1 + |\xi|\big)^{-|\alpha|}$$

for all the multiindices α and β. Then under condition (4.40) *the operator $\sigma(x, D)$ admits a continuous extension to the space $L^{p(\cdot)}_{\varrho}(\mathbb{R}^n)$.*

In the case of constant p and $\varrho \in A_p$ Theorem 4.27 was proved in [57].

Corollary 4.28. *Let $p \in \mathcal{P}(\mathbb{R}^n) \cap \mathrm{WL}(\mathbb{R}^n)$ and $p(x) \equiv p_\infty = \mathrm{const}$ outside some ball $|x| < R$ and let $\varrho \in V^{\mathrm{osc}}_{p(\cdot)}(\mathbb{R}^n, \Pi)$.*

For variable $p(\cdot)$ the statement of Corollary 4.28 by a different method was proved in the non-weighted case in [60].

4.8. Fefferman–Stein function and vector-valued operators

Let f be a measurable locally integrable function on $\mathbb{R}^n$, B an arbitrary ball in $\mathbb{R}^n$, $f_B = \frac{1}{|B|}\int_B f(x)\,dx$ and let

$$\mathcal{M}^{\#} f(x) = \sup_{B \in X} \frac{1}{|B|} \int_B \big|f(x) - f_B\big|\, dx$$

be the Fefferman–Stein maximal function.

Theorem 4.29. *Under condition* (4.40)*, the inequality*

$$\|\mathcal{M} f\|_{L^{p(\cdot)}_{\varrho}(\mathbb{R}^n)} \leqslant C \big\|\mathcal{M}^{\#} f\big\|_{L^{p(\cdot)}_{\varrho}(\mathbb{R}^n)} \tag{4.42}$$

is valid, where $C > 0$ does not depend on f.

In the case of constant p and $\varrho \in A_p$ inequality (4.42) was proved in [21].

Corollary 4.30. *Inequality* (4.42) *is valid under the conditions*:

(i) $p \in \mathcal{P}(\mathbb{R}^n) \cap \mathrm{WL}(\mathbb{R}^n)$ *and* $p(x) \equiv p_\infty = \text{const}$ *outside some ball* $|x| < R$,
(ii) $\varrho \in V^{\text{osc}}_{p(\cdot)}(\mathbb{R}^n, \Pi)$.

Let $f = (f_1, \dots, f_k, \dots)$, where $f_i : \mathbb{R}^n \to \mathbb{R}^1$ are locally integrable functions.

Theorem 4.31. *Let* $0 < \theta < \infty$. *Under conditions* (4.40)*, the inequality*

$$\left\| \left(\sum_{j=1}^{\infty} (\mathcal{M} f_j)^\theta \right)^{\frac{1}{\theta}} \right\|_{L^{p(\cdot)}_\varrho(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_{j=1}^{\infty} |f_j|^\theta \right)^{\frac{1}{\theta}} \right\|_{L^{p(\cdot)}_\varrho(\mathbb{R}^n)} \tag{4.43}$$

is valid, where $c > 0$ *does not depend on* f.

In the case of constant p and $\varrho \in A_p$ weighted inequalities for vector-valued functions were proved in [34–36], see also [3].

Corollary 4.32. *Inequality* (4.43) *is valid under the conditions*

(i) $p \in \mathcal{P}(\mathbb{R}^n) \cap \mathrm{WL}(\mathbb{R}^n)$ *and* $p(x) \equiv p_\infty = \text{const}$ *outside some ball* $|x| < R$,
(ii) $\varrho \in V^{\text{osc}}_{p(\cdot)}(\Omega, \Pi)$.

Remark 4.33. The corresponding statements for vector-valued operators are also similarly derived from Theorem 3.4 in the case of singular integrals, commutators, Fefferman–Stein maximal function, Fourier-multipliers, etc.

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