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Chapter 14
Lineability, algebrability and strong algebrability of some sets in $\mathbb{R}^{\mathbb{R}}$ or $\mathbb{C}^{\mathbb{C}}$

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14.1 Introduction

The algebraic properties of sets of functions have been investigated in the Real Analysis for many years. The direction example of such a research is the idea of finding maximal additive (as well as for other operations: composition, taking maximum etc.) classes for certain families of functions that has its origin in the 1930's (cf. [20], [35]). In the example in [35] it was proved that the maximal additive class for Darboux real functions is the class of constant functions (i.e. for any function f with Darboux property and any constant function c the function $f+c$ is still Darboux, but for any nonconstant function g there is a Darboux function f_g such that f_g+g does not have Darboux property). This studies are still under investigations (cf. [26], [30], [34]). In the last 10 years there appeared a new point of looking on the largeness of some sets included in the functions spaces. For example, for an algebra $\mathcal{L}$, one can call a set $A \subseteq \mathcal{L}$ **big** if $A \cup \{0\}$ contains a **nice** structure inside, i.e. contains a non-trivial vector space, an algebra or a closed vector space (if $\mathcal{L}$ has a topology). This way of thinking was coined by V. I. Gurariy (see [27], [28]) but the earliest results in this area can be found in the classical paper due to B. Levine and D. Milman from 1940 (cf. [33]). The notion we are presenting here has its origin in the works of R. M. Aron, V. I. Gurariy, D. Pérez-García, J. B. Seoane-Sepúlveda (see [5], [6], [7]).

Definition 14.1. Let κ be a cardinal number.

1. Let $\mathcal{L}$ be a vector space and a set $A \subseteq \mathcal{L}$. We say that A is κ-lineable if $A \cup \{0\}$ contains a κ-dimensional vector space;
2. Let $\mathcal{L}$ be a Banach space and a set $A \subseteq \mathcal{L}$. We say that A is spaceable if $A \cup \{0\}$ contains an infinite dimensional closed vector space;
3. Let $\mathcal{L}$ be a linear commutative algebra and a set $A \subseteq \mathcal{L}$. We say that A is κ-algebrable if $A \cup \{0\}$ contains a κ-generated algebra B (i.e. the minimal system of generators of B has cardinality κ).

Note that any linear space is a free structure in category of vector spaces. Therefore a κ-lineable set contains a free structure of κ generators. Following this observation one can ask for existence of free structures inside some set $A \cup \{0\}$. A. Bartoszewicz and S. Głąb in their work [14] introduced the notion of strong algebrability.

Definition 14.2. [14] Let κ be a cardinal number. Let $\mathcal{L}$ be a linear commutative algebra and a set $A \subseteq \mathcal{L}$. We say that A is strongly κ-algebrable if $A \cup \{0\}$ contains a κ-generated algebra that is isomorphic with a free algebra (denote by $X = \{x_\alpha : \alpha < \kappa\}$ the set of generators of this free algebra).

Remark that the set $X = \{x_\alpha : \alpha < \kappa\}$ is the set of generators of some free algebra contained in $A \cup \{0\}$ if and only if the set $\tilde{X}$ of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of elements from $\tilde{X}$ are in $A \cup \{0\}$; equivalently for any $k \in \mathbb{N}$, any nonzero polynomial P in k variables without a constant term and any distinct $y_1, ..., y_k \in X$, we have $P(y_1, ..., y_k) \neq 0$ and $P(y_1, ..., y_k) \in A$.

The main problem in this area is to consider a subset of $\mathbb{R}^{\mathbb{R}}, \mathbb{C}^{\mathbb{C}}, \mathbb{R}^{\mathbb{N}}, \mathbb{C}^{\mathbb{N}}$ consisting of functions or sequences which naturally appears in Analysis that does not have any linear property (i.e. it is not closed under taking sums) and construct inside it a vector space, an algebra or a free algebra with a large set of generators. One can also try to find a natural subset $A \subseteq \mathbb{R}^{\mathbb{R}}$ (or $\mathbb{C}^{\mathbb{C}}, \mathbb{R}^{\mathbb{N}}, \mathbb{C}^{\mathbb{N}}$) such that inside $A \cup \{0\}$ we cannot construct any **nice** structure.

Here we give a general overview on this topic presenting some general methods and recalling recent results. We will focus on the outcomes obtained by the authors of this chapter with other coworkers. It should be mentioned here that recently many other authors have considered and got interesting results in the algebrability (see [2], [3], [25]).

It is easy to check that for any cardinal number κ the following implications hold:

$$\kappa\text{-strong algebrability} \Rightarrow \kappa\text{-algebrability} \Rightarrow \kappa\text{-lineability}.$$

Moreover, since every infinite dimensional Banach space has a linear base of cardinality $\mathfrak{c}$, then

$$\text{spaceability} \Rightarrow \mathfrak{c}\text{-lineability}.$$

The following examples show that these implications cannot be reversed.

Example 14.3. c_{00} (the set of all sequences from c_0 that are equal 0 from some place) is algebrable in c_0, but it is not strongly 1-algebrable.

In fact, suppose that c_{00} contains a free algebra generated by an element $x = (x_1,\dots,x_n,0,\dots) \in c_{00}$. Since x is a generator of a free algebra, the set $\tilde{X} = \{x^n : n < \omega\}$ is linearly independent. In particular, the elements $x, x^2, \dots, x^{n+1}$ are linearly independent but in fact they are elements of $\mathbb{R}^n$, contradiction.

Example 14.4. For an unbounded interval I the set of Riemann integrable functions that are not Lebesgue integrable is lineable but not algebrable (cf. [24]).

Example 14.5. For a sequence $x \in \ell_1$, consider a set of all subsums of $\sum_{n=1}^{\infty} x(n)$ defined by

$$E(x) = \{a \in \mathbb{R} : \exists_{A\subseteq\mathbb{N}} \sum_{n\in A} x(n) = a\}.$$

It is known (cf. [29]) that for any $x \in \ell_1 \setminus c_{00}$ the set $E(x)$ is either a finite union of closed intervals, or is homeomorphic to the Cantor set, or is homeomorphic to the M-Cantorval. Hence the set ℓ_1 can be decomposed into disjoint sets c_{00} and $\mathcal{I}, \mathcal{C}, \mathcal{MC}$ (consisting of sequences $x \in \ell_1 \setminus c_{00}$ for which the set $E(x)$: is a finite union of closed intervals, is homeomorphic to the Cantor set, is homeomorphic to the M-Cantorval, respectively). In [11] T. Banakh, A. Bartoszewicz, S. Głąb and E. Szymonik proved that the set $\mathcal{C}$ is strongly $\mathfrak{c}$-algebrable and comeager but is **not** spaceable.

Generally, while proving the lineability, algerbability or strong algebrability of a set $A \subseteq \mathcal{L}$ on some level κ (i.e. constructing κ-generated vector space, algebra or free algebra, where κ is a cardinal number) we have to show the existence of κ independent generators in A. However the upper bound for the maximal level of lineability, algerbability or strong algebrability is given by $\mathrm{card}(\mathcal{L})$. Hence the main aim is to prove this property on the maximal possible level. Usually this level is at $2^{\mathfrak{c}}$ or $\mathfrak{c}$. All of the presented constructions and methods are valid in ZFC.

14.2 Notation

We will use the standard set-theoretical notation. In the sequel we will consider the following classes of functions.

Definition 14.6. A function $f : \mathbb{K} \to \mathbb{K}$ is called:

1. *Everywhere surjective* ($f \in \mathcal{ES}(\mathbb{K})$) if $f(I) = \mathbb{K}$, for every nonempty open set I;
2. *Strongly everywhere surjective* ($f \in \mathcal{SES}(\mathbb{K})$) if it takes every value $\mathfrak{c}$ times on every nonempty open set (i.e. for every $y \in \mathbb{K}$ and every nonempty open set I the set $\{x \in I : f(x) = y\}$ has cardinality $\mathfrak{c}$);
3. *Perfectly everywhere surjective* ($f \in \mathcal{PES}(\mathbb{K})$) if $f(P) = \mathbb{K}$, for every perfect set $P \subseteq \mathbb{K}$;
4. *Everywhere discontinuous Darboux* ($f \in \mathcal{EDD}(\mathbb{K})$) if it is nowhere continuous and has the Darboux property (i.e. it maps connected sets to connected sets);
5. *Everywhere discontinuous with finite range* ($f \in \mathcal{EDF}(\mathbb{K})$) if it is nowhere continuous and has finitely many values;
6. *Everywhere discontinuous compact to compact* ($f \in \mathcal{EDC}(\mathbb{K})$) if it is nowhere continuous and maps compact sets to compact sets.

Observe here that the choice of the set $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) has an influence on possible results in algebrability. In particular one can easy check that the set $\mathcal{PES}(\mathbb{R})$ cannot be algebrable. Indeed, for any function $f \in \mathcal{PES}(\mathbb{R})$ the function $f^2 \notin \mathcal{PES}(\mathbb{R})$ since it takes only nonnegative real values, hence cannot be onto $\mathbb{R}$. The similar argument works in the case of conditionally convergent series in $\mathbb{R}$ (considered with point-wise product). On the other hand the set $\mathcal{PES}(\mathbb{C})$ and the family of all conditionally convergent series in $\mathbb{C}$ are strongly $2^{\mathfrak{c}}$-algebrable and strongly $\mathfrak{c}$-algebrable, respectively (for details see the proof of Theorem 14.22 and [17]).

Everywhere surjective functions have been considered in the context of algebrability by many authors (see [1], [4], [7], [21]). Let us point out that any function $f \in \mathcal{PES}(\mathbb{K})$ has the property that a preimage of any singleton by f is a Bernstein set, hence f is nonmeasurable in many senses.

There is a nice observation that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if f maps compact sets to compact sets and f has Darboux property. This was proved and generalized by many authors (see [10] and references therein). It is clear that neither Darboux property nor mapping compact sets to compact sets does not imply continuity by itself. Furthermore, there are nowhere continuous functions which are Dabroux and nowhere continuous functions which map compact sets to compact sets. The study of these families of functions in the context of lineability was initiated in [22] where $2^{\mathfrak{c}}$-lineability of family of compact to compact functions was established.

14.3 Sierpiński-Zygmund functions

In this section we will consider a type of functions that firstly appeared in 1920's (see [38]). The classical Luzin's Theorem implies that for every Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ there is a set $S \subseteq \mathbb{R}$ with infinite measure such that $f|_S$ is continuous. In 1922 (see [19]) H. Blumberg showed that if we omit the assumption that $f : \mathbb{R} \to \mathbb{R}$ is measurable, then some weaker version of the assertion of the Luzin's Theorem remains true. More precisely H. Blumberg proved the following.

Theorem 14.7 ([19]). *Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function, then there exists a dense subset $S \subseteq \mathbb{R}$ such that $f|_S$ is continuous.*

In the proof the set S was countable so naturally one can ask if the set S can be uncountable. The partial answer to this was given by W. Sierpiński and A. Zygmund in [38].

Theorem 14.8 ([38]). *There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that for any set $Z \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$ the restriction $f|_Z$ is not a Borel map.*

Obviously under (CH) the restriction of this function to any uncountable set cannot be continuous. J. Shinoda (see [37]) proved that with $(MA) + \neg(CH)$ for every function $f : \mathbb{R} \to \mathbb{R}$ there exists an uncountable set $Z \subseteq \mathbb{R}$ such that $f|_Z$ is continuous. On the other hand Gruenhage proved that there is a model of ZFC in which $\mathfrak{c} > \omega_1$ and there is a function f such that $f|_Z$ is not Borel for any uncountable set $Z \subseteq \mathbb{R}$ (for details see [36]). By classical theorems of Luzin and Nikodym, a function from Theorem 14.8 is nonmeasurable and does not have Baire property. Moreover, although it is possible to construct it to be injective, it is nowhere monotone in the sense that its restriction to any set of cardinality $\mathfrak{c}$ is not monotone. Let us introduce the following notion.

Definition 14.9. We say that a function $f : \mathbb{K} \to \mathbb{K}$ is Sierpiński-Zygmund function if for every set $A \subseteq \mathbb{K}$ of cardinality $\mathfrak{c}$, the restriction $f|_A$ is not a Borel map.

The set of all Sierpiński-Zygmund functions $f : \mathbb{R} \to \mathbb{R}$ (denote it by $\mathcal{SZ}(\mathbb{R})$) was firstly considered, in the context of algebrability, by J. L. Gámez-Merino, G. A. Munoz-Fernández, V. M. Sánchez and J. B. Seoane-Sepúlveda in [21]. The authors proved that this set is $\mathfrak{c}^+$-lineable and, also, $\mathfrak{c}$-algebrable. This was the motivation to ask a question if one can prove $2^{\mathfrak{c}}$-algebrability of the set $\mathcal{SZ}(\mathbb{R})$ in *ZFC*. Here we recall a result due to A. Bartoszewicz, S. Głąb, D. Pellegrino and J. B. Seoane-Sepúlveda (cf. [16]). Before stating it let us recall the following notion.

Definition 14.10. Let κ be an infinite cardinal number. Let A and B be any subsets of $\mathbb{R}$ with cardinality κ. We say that A and B are almost disjoint provided that $\operatorname{card}(A \cap B) < \kappa$.

The existence of $2^{\mathfrak{c}}$ pairwise almost disjoint subsets, of size $\mathfrak{c}$, of $\mathbb{R}$ follows, for example, from (CH) or from (MA). It is also known that in *ZFC* in every set of cardinality κ there is an almost disjoint family (a family consisting of pairwise almost disjoint sets) of cardinality κ^+.

The main result in this context from [16] is the following.

Theorem 14.11. *The set $\mathcal{SZ}(\mathbb{R})$ is strongly κ-algebrable, provided there exists an almost disjoint family in $\mathbb{R}$ of cardinality κ. Moreover,* $\operatorname{card}(\mathcal{SZ}(\mathbb{R})) = 2^{\mathfrak{c}}$.

Before proving the Theorem 14.11 let us state a useful lemma.

Lemma 14.12 ([16]). *Let $\mathcal{P}$ be a family of nonzero polynomials without constant term and X be a subset of $\mathbb{R}$, both of cardinality less than $\mathfrak{c}$. Then there exists a set $Y = \{y_\xi : \xi < \mathfrak{c}\}$, of cardinality $\mathfrak{c}$, such that $P(y_{\xi_1}, ..., y_{\xi_n}) \notin X$, for any polynomial $P \in \mathcal{P}$ in n variables and any distinct $\xi_1 < ... < \xi_n < \mathfrak{c}$.*

Now we are able to prove Theorem 14.11.

Proof. Let $\{g_\alpha : \alpha < \mathfrak{c}\}$ be a numeration of all Borel functions, $\{x_\alpha : \alpha < \mathfrak{c}\}$ a numeration of $\mathbb{R}$ and $\{P_\alpha : \alpha < \mathfrak{c}\}$ a numeration of all nonzero polynomials without constant term. Let us inductively define a family $\{Y_\alpha : \alpha < \mathfrak{c}\}$ of subsets of $\mathbb{R}$ with cardinality $\mathfrak{c}$, by putting at the step $\alpha < \mathfrak{c}$ as Y_α the set which existence implies Lemma 14.12 used for $X = \{g_\lambda(x_\alpha) : \lambda \leq \alpha\}$ and $\mathcal{P} = \{P_\lambda : \lambda \leq \alpha\}$. For each $\alpha < \mathfrak{c}$ consider a numeration $Y_\alpha = \{y^\alpha_\xi : \xi < \mathfrak{c}\}$. Note that $\mathcal{Y} = \prod_{\alpha<\mathfrak{c}} Y_\alpha \subseteq \mathcal{SZ}(\mathbb{R})$. Thus $\operatorname{card}(\mathcal{SZ}(\mathbb{R})) = 2^{\mathfrak{c}}$. Let $\{N_\zeta : \zeta < \kappa\}$ be an almost disjoint family in $\mathfrak{c}$ with each member of cardinality $\mathfrak{c}$. For any $\zeta < \kappa$ let $\{\zeta(\xi) : \xi < \mathfrak{c}\}$ be an increasing numeration of N_ζ and define $f_\zeta : \mathbb{R} \to \mathbb{R}$ by the formula $f_\zeta(x_\alpha) = y^\alpha_{\zeta(\alpha)}$. We will show that $\{f_\zeta : \zeta < \kappa\}$ is a set of generators and generates a subalgebra of $\mathcal{SZ}(\mathbb{R})$.

Let P be a nonzero polymonial in n variables without a constant term (let $P = P_\beta$). Take $\zeta_1 < ... < \zeta_n < \kappa$, a Borel function g_γ and let $Z \subseteq \mathbb{R}$ be a set of cardinality $\mathfrak{c}$. Consequently, $Z' = Z \setminus \{x_\alpha : \alpha \leq \max\{\beta, \gamma\}\}$ also has cardinality $\mathfrak{c}$. Since $\mathrm{card}(N_{\zeta_k} \cap N_{\zeta_l}) < \mathfrak{c}$ for any distinct $k, l \in \{1, ..., n\}$, there is $\alpha \in \mathfrak{c} \setminus \left(\bigcup_{k \neq l} N_{\zeta_k} \cap N_{\zeta_l}\right)$ and $x_\alpha \in Z'$. Hence $f_{\zeta_1}(x_\alpha), ..., f_{\zeta_n}(x_\alpha) \in Y_\alpha$ are distinct. Since $\alpha > \gamma$ and $\alpha > \beta$, $P(f_{\zeta_1}, ..., f_{\zeta_n})$ is different from g_γ at the point x_α. Therefore $P(f_{\zeta_1}, ..., f_{\zeta_n})$ is a Sierpiński-Zygmund function. $\square$

Notice here also a simple observation.

Remark 14.13. Any additive group $\mathcal{A} \subseteq \mathcal{SZ}(\mathbb{R}) \cup \{0\}$ of cardinality κ generates an almost disjoint family in the plane $\mathbb{R} \times \mathbb{R}$ (by considering graphs of $f \in \mathcal{A}$ as members of this family).

Hence the result described above and obtained by A. Bartoszewicz, S. Głąb, D. Pellegrino and J. B. Seoane-Sepúlveda with strong algebrability on the level of cardinality of maximal almost disjoint family in $\mathfrak{c}$ is the best possible in that case. J. L. Gámez-Merino, J. B. Seoane-Sepúlveda noted in [23] that "$\kappa = 2^{\mathfrak{c}}$" is independent with *ZFC*. This implies that the sentence "$\mathcal{SZ}(\mathbb{R})$ is strongly $\mathrm{card}(\mathcal{SZ}(\mathbb{R}))$-algebrable (lineable)" is independent with *ZFC*. Moreover, it was the first time, when strong algebrability was proved in *ZFC* on the level higher than $\mathfrak{c}$ (since in *ZFC* there is an almost disjoint family of cardinality $\mathfrak{c}^+$). So the considerations took the authors to the question if there exists a free algebra of $2^{\mathfrak{c}}$ generators in $\mathbb{R}^{\mathbb{R}}$ or $\mathbb{C}^{\mathbb{C}}$. The answer to this was given by A. Bartoszewicz, S. Głąb and A. Paszkiewicz in their work [15]. We will come back to this in the Section *Method of large free algebras*.

14.4 Some general methods in algebrability and strong algebrability

We will recall some general methods of constructing algebras and free algebras of real and complex functions.

14.4.1 Independent Bernstein sets

The following general construction was described by A. Bartoszewicz, M. Bienias and S. Głąb in [12] but firstly was used by A. Bartoszewicz, S. Głąb,

D. Pellegrino and J. B. Seoane-Sepúlveda in [16] in the proof of $2^{\mathfrak{c}}$-algebrability of the set $\mathcal{PES}(\mathbb{C})$. Let us recall the idea of the method.

For a nonempty set X and $A \subseteq X$ let us denote $A^0 = X \setminus A$ and $A^1 = A$. We name a family $\mathcal{A}$ to be *$\mathcal{B}$-independent* (where $\mathcal{B}$ be is a family of subsets of X) if $A_1^{\varepsilon_1} \cap \ldots \cap A_n^{\varepsilon_n} \in \mathcal{B}$ for any distinct $A_i \in \mathcal{A}$, any $\varepsilon_i \in \{0,1\}$ for $i \in \{1,\ldots,n\}$ and $n \in \mathbb{N}$. We say that $\mathcal{A}$ is independent if it is $\mathcal{P}(X) \setminus \{\emptyset\}$-independent.

Well known theorem of Fichtenholz and Kantorovich (generalized to the case of any complete Boolean algebra, by B. Balcar and F. Franěk in [9]) says that for any infinite set of cardinality κ there is an independent family of 2^κ subsets of this set. Let us recall the well known definition of a Bernstein set.

Definition 14.14. A subset B of a Polish space is called a *Bernstein set* if $B \cap P \neq \emptyset \neq B^0 \cap P$ for every perfect subset P. Denote by $\mathcal{B}$ the family of all Bernstein sets.

We say that a family $\mathcal{A}$ is an independent family of Bernstein sets provided that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A}$ is $\mathcal{B}$-independent. Our aim is to construct an independent family of Bernstein sets of cardinality $2^{\mathfrak{c}}$. Repeating the idea from [12] consider the decomposition of $\mathbb{R}$ into $\mathfrak{c}$ pairwise disjoint Bernstein sets $\{B_\alpha : \alpha < \mathfrak{c}\}$. It is easy to check that for any $s \subseteq \mathfrak{c}$ with $s \neq \emptyset$ and $\mathfrak{c} \setminus s \neq \emptyset$, the set $\bigcup_{\alpha \in s} B_\alpha$ is Bernstein. Let $\{N_\xi : \xi < 2^{\mathfrak{c}}\}$ be an independent family in $\mathfrak{c}$ such that for every $\xi_1 < \ldots < \xi_n < 2^{\mathfrak{c}}$ and for any $\varepsilon_i \in \{0,1\}$ $(i \in \{1,\ldots,n\})$, the set $N_{\xi_1}^{\varepsilon_1} \cap \ldots \cap N_{\xi_n}^{\varepsilon_n}$ has cardinality $\mathfrak{c}$. To construct the desired family of Bernstein sets let us put $B^\xi = \bigcup_{\alpha \in N_\xi} B_\alpha$, for $\xi < 2^{\mathfrak{c}}$. Then every set B^ξ is Bernstein. Note that for every $\xi_1 < \ldots < \xi_n < 2^{\mathfrak{c}}$ and any $\varepsilon_i \in \{0,1\}$ for $i \in \{1,\ldots,n\}$ the set

$$(B^{\xi_1})^{\varepsilon_1} \cap \ldots \cap (B^{\xi_n})^{\varepsilon_n} = \bigcup_{\alpha \in N_{\xi_1}^{\varepsilon_1} \cap \ldots \cap N_{\xi_n}^{\varepsilon_n}} B_\alpha$$

is Bernstein, too. That means $\{B^\xi : \xi < 2^{\mathfrak{c}}\}$ is an independent family of Bernstein sets.

Having the independent family of Bernstein sets, we can define $2^{\mathfrak{c}}$ linearly independent functions: for $\alpha < \mathfrak{c}$, let $g_\alpha : B_\alpha \to \mathbb{C}$ (or $\mathbb{R}$) be a nonzero function defined on a Bernstein set B_α (where $\{B_\alpha : \alpha < \mathfrak{c}\}$ is the decomposition of $\mathbb{R}$ into pairwise disjoint Bernstein sets). Then for every $\xi < 2^{\mathfrak{c}}$ let us put

$$f_\xi(x) = \begin{cases} g_\alpha(x) & \text{, if } x \in B_\alpha \text{ and } \alpha \in N_\xi; \\ 0 & \text{, otherwise.} \end{cases}$$

Then the family $\{f_\xi : \xi < 2^{\mathfrak{c}}\}$ is linearly independent. Finally (cf. [12]) by spanning the algebra by the functions $\{f_\xi : \xi < 2^{\mathfrak{c}}\}$ we obtain an algebra of $2^{\mathfrak{c}}$ generators. It is worth mentioning here that the independent Bernstein sets method cannot be used to prove **strong** $2^{\mathfrak{c}}$-algebrability. Indeed, one can consider a nonzero polynomial $P(x_1,...,x_n) = \prod_{k\neq l}(x_k - x_l)\prod_k x_k$, for which, for any collection of functions $f_{\xi_1},...,f_{\xi_n}$, of the above type, we have $P(f_{\xi_1},...,f_{\xi_n}) = 0$. Hence, an algebra spanned by $\{f_\xi : \xi < 2^{\mathfrak{c}}\}$ is not a free algebra.

Using the method described above it is possible to get the $2^{\mathfrak{c}}$-algebrability of several families of functions in $\mathbb{R}^{\mathbb{R}}$ or $\mathbb{C}^{\mathbb{C}}$, namely $\mathcal{PES}(\mathbb{C}), \mathcal{SES}(\mathbb{C}) \setminus \mathcal{PES}(\mathbb{C}), \mathcal{EDD}(\mathbb{R})$. To present this method let us establish the following

Theorem 14.15. *The set $\mathcal{EDF}(\mathbb{R})$ is $2^{\mathfrak{c}}$-algebrable.*

Proof. Consider an independent family of Bernstein sets $\{B^\xi : \xi < 2^{\mathfrak{c}}\}$. For $\xi < 2^{\mathfrak{c}}$ let f_ξ be the characteristic function of the set B^ξ. We will show that the family $\{f_\xi : \xi < 2^{\mathfrak{c}}\} \subseteq \mathcal{EDF}(\mathbb{R})$ generates an algebra in $\mathcal{EDF}(\mathbb{R})$. Let P be any nonzero polynomial in n variables without a constant term and let $\xi_1 < \xi_2 < ... < \xi_n < 2^{\mathfrak{c}}$. Suppose $P(f_{\xi_1},...,f_{\xi_n})$ is nonzero. Notice that each f_{ξ_i} is constant on every set of the form $(B^{\xi_1})^{\varepsilon_1} \cap ... \cap (B^{\xi_n})^{\varepsilon_n}$ so is $P(f_{\xi_1},...,f_{\xi_n})$. Since $P(f_{\xi_1},...,f_{\xi_n})$ is nonzero, there are $\varepsilon_i \in \{0,1\}$ for $i \in \{1,...,n\}$ such that $P(f_{\xi_1},...,f_{\xi_n})|_{(B^{\xi_1})^{\varepsilon_1}\cap...\cap(B^{\xi_n})^{\varepsilon_n}} \neq 0$. Clearly $P(f_{\xi_1},...,f_{\xi_n})|_{(B^{\xi_1})^0\cap...\cap(B^{\xi_n})^0} = 0$. Therefore, $P(f_{\xi_1},...,f_{\xi_n})$ is everywhere discontinuous (the set of the type $(B^{\xi_1})^{\varepsilon_1} \cap ... \cap (B^{\xi_n})^{\varepsilon_n}$ is Bernstein so it is dense). Hence $\mathcal{EDF}(\mathbb{R})$ is $2^{\mathfrak{c}}$-algebrable. $\square$

By similar argument as in Example 14.3, it is easy to see that the set $\mathcal{EDF}(\mathbb{R})$ is not even strongly 1-algebrable. Since finite sets are compact, we have that $\mathcal{EDF}(\mathbb{R}) \subseteq \mathcal{EDC}(\mathbb{R})$, therefore the following holds.

Corollary 14.16. *The set $\mathcal{EDC}(\mathbb{R})$ is $2^{\mathfrak{c}}$-algebrable.*

One can ask if this result can be strengthen. The answer is: it cannot. T. Banakh, A. Bartoszewicz, M. Bienias and S. Głąb proved in [10] that any compact-preserving nowhere continuous function f cannot take infinitely many values on every interval. So for any function $f \in \mathcal{EDC}(\mathbb{R})$ there is an interval I, such that $f(I)$ is finite. And again the same argument as in Example 14.3 shows that this set cannot be even strongly 1-algebrable. Hence $\mathcal{EDC}(\mathbb{R})$ is **not** strongly κ-algebrable for any cardinal $\kappa > 0$. Therefore, the result from Corollary 14.16 is the best possible in the sense of algebrability.

14.4.2 The method of large free algebras

Let us come back to the question (from the Section *Sierpiński-Zygmund functions*): does there exist a free algebra of $2^{\mathfrak{c}}$ generators in $\mathbb{R}^{\mathbb{R}}$ or $\mathbb{C}^{\mathbb{C}}$? The positive answer to this question was given by A. Bartoszewicz, S. Głąb and A. Paszkiewicz in their work [15]. Moreover, they described a new general method of proving the strong $2^{\mathfrak{c}}$-algebrability, that was very useful in improving some results to the highest possible level. Here we recall some of their results. Let us start with the answer.

Theorem 14.17. *Let X be an infinite set of cardinality κ with $\kappa^{\omega} = \kappa$. Let I be a subset of $\mathbb{K}$ with a nonempty interior. Then there exists a free linear subalgebra of $\mathbb{K}^X$ with 2^{κ} generators $\{f_\xi : \xi < 2^{\kappa}\}$, such that $P(f_{\xi_1},...,f_{\xi_k})$ maps X onto $P(I^k)$ for every nonzero polynomial P in k variables without constant term, any $\xi_1,...,\xi_k < 2^{\kappa}$ and any $k \in \mathbb{N}$.*

For the reader's convenience we give a sketch of their proof.

Proof. Let $Y = ([0,1] \times \kappa)^{\mathbb{N}}$ and let $\{A_\xi : \xi < 2^{\kappa}\}$ be an independent family of subsets of κ. For each $\xi < 2^{\kappa}$ let us define a function $\bar{f}_\xi : Y \to [0,1]$ by a formula

$$\bar{f}_\xi(t_1,y_1,t_2,y_2,...) = \prod_{n=1}^{\infty} t_n^{\chi_{A_\xi}(y_n)},$$

where $t_n \in [0,1], y_n \in \kappa$ and χ_A stands for the characteristic function of a set A (assume here that $0^0 = 1$). The condition $\kappa^{\omega} = \kappa$ implies that $\kappa \geq \mathfrak{c}$ so $\mathrm{card}(Y) = \mathrm{card}(X)$. I has nonempty interior, therefore $\mathrm{card}(I) = \mathfrak{c}$ and there are bijections $\phi : X \to Y$ and $\psi : [0,1] \to I$. Let $f_\xi = \psi \circ \bar{f}_\xi \circ \phi : X \to \mathbb{K}$ for $\xi < 2^{\kappa}$. Then $\{f_\xi : \xi < 2^{\kappa}\}$ is a set of free generators in $\mathbb{K}^X$. Indeed, take any $\xi_1 < ... < \xi_k < 2^{\kappa}$. Let

$$Y_0 = \Big\{(t_1,y_1,t_2,y_2,...) \in Y : t_1,...,t_k \in [0,1], t_i = \frac{1}{2} \text{ for } i > k,$$

$$y_i \in A_{\xi_i} \setminus \bigcup_{j \neq i} A_{\xi_j} \text{ for } i \leq k \text{ and } y_i \in \bigcap_j A^0_{\xi_j} \text{ for } i > k\Big\}.$$

Consider a nonzero polynomial P in k variables without a constant term. Let $x = \phi(t_1,y_1,t_2,y_2,...) \in X_0 = \phi^{-1}(Y_0)$. Then

$$P(f_{\xi_1},...,f_{\xi_k})(x) = P(\psi \circ \bar{f}_{\xi_1},...,\psi \circ \bar{f}_{\xi_k})(\phi(x)) = P(\psi(t_1),...,\psi(t_k)).$$

Notice here that $\phi|_{X_0}$ is onto I. Since I has a nonempty interior, P is nonzero on I^k. Therefore, $P(f_{\xi_1},...,f_{\xi_k})$ is nonzero on X_0, so it is on X. To finish the proof observe that each function f_ξ is onto I. □

Let us introduce a notion that generalizes the notion of strongly everywhere surjective functions (cf. [15]).

Definition 14.18. [15] Let $\mathcal{F} \subseteq \mathcal{P}(X)$ and $I \subseteq \mathbb{K}$. We say that a function $f \in \mathbb{K}^X$ is I-strongly everywhere surjective with respect to $\mathcal{F}$ (in short $f \in \mathcal{SES}(I,\mathcal{F})$), if for every $F \in \mathcal{F}$ there are $k \in \mathbb{N}$ and polynomial P in k variables without constant term such that $f(F) = P(I^k)$ and $\operatorname{card}(\{x \in F : f(x) = y\}) = \operatorname{card}(X)$ for every $y \in f(F)$.

In particular we have that:

- for a family $\mathcal{F}$ consisting of all nonempty open subsets of $\mathbb{C}$, the set $\mathcal{SES}(\mathbb{C},\mathcal{F})$ is the set of all strongly everywhere surjective complex functions;
- for a family $\mathcal{F}$ consisting of all nonempty perfect subsets of $\mathbb{C}$, the set $\mathcal{SES}(\mathbb{C},\mathcal{F})$ is the set of all perfectly everywhere surjective complex functions.

A. Bartoszewicz, S. Głąb and A. Paszkiewicz proved the strong algebrability of $\mathcal{SES}(I,\mathcal{F})$ for some families $\mathcal{F}$ of subsets of X and $I \subseteq \mathbb{K}$. They were using a classical result due to Kuratowski and Sierpiński (see [32]).

Proposition 14.19. *(Disjoint Refinement Lemma) Let $\kappa \geq \omega$. For any family $\{P_\alpha : \alpha < \kappa\}$ of sets of cardinality κ there is a family $\{Q_\alpha : \alpha < \kappa\}$, such that for every distinct $\alpha, \beta < \kappa$*

- $Q_\alpha \subseteq P_\alpha$;
- $Q_\alpha \cap Q_\beta = \emptyset$.

The family $\{Q_\alpha : \alpha < \kappa\}$ is called a disjoint refinement of the family $\{P_\alpha : \alpha < \kappa\}$.

Thanks to Disjoint Refinement Lemma and existence of a large free subalgebra in $\mathbb{R}^X, \mathbb{C}^X$, the authors obtained the following general result.

Theorem 14.20 ([15]). *Let X be an infinite set of cardinality κ with $\kappa^\omega = \kappa$. Let I be a subset of $\mathbb{K}$ with a nonempty interior and $\mathcal{F} \subseteq \mathcal{P}(X)$. Assume that* $\operatorname{card}(\mathcal{F}) \leq \kappa$ *and* $\operatorname{card}(F) = \kappa$ *for every $F \in \mathcal{F}$. Then the set $\mathcal{SES}(I,\mathcal{F})$ is strongly 2^κ-algebrable.*

Proof. Consider a numeration $\{F_\alpha : \alpha < \kappa\}$ of $\mathcal{F}$, such that each set F occurs κ times. Let $\{Q_\alpha : \alpha < \kappa\}$ be its disjoint refinement. Without loss of generality, we may assume that $\bigcup_{\alpha<\kappa} Q_\alpha = X$. By Theorem 14.17 for every $\alpha < \kappa$ there is a free algebra $\mathcal{A}_\alpha$ of surjections from Q_α onto I with 2^κ generators.$\{f_\xi^\alpha : \xi < 2^\kappa\}$. Define for $\xi < 2^\kappa$ the function $f_\xi(x) = f_\xi^\alpha$, for $x \in Q_\alpha$. Let $\mathcal{A}$ be an algebra generated by $\{f_\xi : \xi < 2^\kappa\}$, then $\mathcal{A}$ is a free algebra contained in $\mathcal{SES}(I,\mathcal{F}) \cup \{0\}$. □

Similar proof as that of Theorem 14.20 can be applied to get the following result.

Theorem 14.21 ([15]). *Let X be an infinite set of cardinality κ with $\kappa^\omega = \kappa$. Let I be a subset of $\mathbb{K}$ with a nonempty interior and $\mathcal{F}_i \subseteq \mathcal{P}(X)$, for $i = 1,2$. Assume that* $\mathrm{card}(\mathcal{F}_i) \leq \kappa$ *and* $\mathrm{card}(F) = \kappa$ *for every $F \in \mathcal{F}_i, i = 1,2$. Suppose that there is a set $F_2 \in \mathcal{F}_2$, such that for every $F_1 \in \mathcal{F}_1$ we have* $\mathrm{card}(F_1 \setminus F_2) = \kappa$. *Then the family $\mathcal{SES}(I,\mathcal{F}_1) \setminus \mathcal{SES}(I,\mathcal{F}_2)$ is strongly 2^κ-algebrable.*

Using the method described above it is possible to improve some known results (cf. Section *Independent Bernstein sets*).

Theorem 14.22 ([15]). *The sets $\mathcal{PES}(\mathbb{C})$, $\mathcal{SES}(\mathbb{C}) \setminus \mathcal{PES}(\mathbb{C})$ and $\mathcal{EDD}(\mathbb{R})$ are strongly $2^\mathfrak{c}$-algebrable.*

Proof. We will show that the set $\mathcal{PES}(\mathbb{C})$ is strongly $2^\mathfrak{c}$-algebrable. Firstly notice that $\mathfrak{c}^\omega = \mathfrak{c}$ and consider the family $\mathcal{F}$ of all nonempty perfect sets in $\mathbb{C}$. Then it is clear that $\mathrm{card}(\mathcal{F}) = \mathfrak{c}$ and by Theorem 14.20 we get the result.

We will show that the set $\mathcal{SES}(\mathbb{C}) \setminus \mathcal{PES}(\mathbb{C})$ is strongly $2^\mathfrak{c}$-algebrable. Let $\mathcal{F}_1$ stands for the family of all nonempty open subsets of $\mathbb{C}$ and $\mathcal{F}_2$ stands for the family of all nonempty perfect subsets of $\mathbb{C}$. Fix a nowhere dense perfect set $P \in \mathcal{F}_2$. Then it is easy to see that for any $U \in \mathcal{F}_1$, we have that $U \setminus P$ is nonempty and open, so it is of cardinality $\mathfrak{c}$. By the Theorem 14.21 we get the result.

To prove strong $2^\mathfrak{c}$-algebrability of $\mathcal{EDD}(\mathbb{R})$, take $\mathcal{F}$ as a family of all perfect sets and consider $\mathcal{SES}([0,1],\mathcal{F})$. □

Theorem 14.22 closes considerations of the classes $\mathcal{PES}(\mathbb{C})$, $\mathcal{SES}(\mathbb{C}) \setminus \mathcal{PES}(\mathbb{C}), \mathcal{EDD}(\mathbb{R})$, since the result is the best possible in the notion of algebrability.

We will present also another result from [15]. It is well known that any real function is continuous on a G_δ set. Many authors considered algebrability of sets of functions with predescribed set of continuity points. F. J. García-Pacheco, N. Palmberg and J. B. Seoane-Sepúlveda proved ω-lineability of the set of functions with finite number of continuity points (see [24]). Moreover, A. Aizpuru, C. Pérez-Eslava, F. J. García-Pacheco and J. B. Seoane-Sepúlveda in [1] established that a set of functions with a fixed open set of continuity points is ω-lineable. Recently A. Bartoszewicz, M. Bienias and S. Głąb proved $2^{\mathfrak{c}}$-algebrability of the set of functions whose continuity points is a fixed closed set (see [12]). Let $G \subseteq \mathbb{R}$ be a G_δ set and consider the set $\mathcal{C}_G$ of all functions $f : \mathbb{R} \to \mathbb{R}$, for which the set of continuity points is exactly G.

Let us introduce the following notion.

Definition 14.23. We say that a set $A \subseteq \mathbb{R}$ is $\mathfrak{c}$-dense in itself if for any open interval I we have either $I \cap A = \emptyset$ or $\operatorname{card}(I \cap A) = \mathfrak{c}$.

Recall (cf. [15]) a property of $\mathfrak{c}$-dense in itself sets.

Lemma 14.24. *Let F be non closed $\mathfrak{c}$-dense in itself F_σ set. Then there are perfect sets $\{C_n : n \in \mathbb{N}\}$ such that $F = \bigcup_{n\in\mathbb{N}} C_n$ and $C_{n+1} \setminus C_n$ is $\mathfrak{c}$-dense in itself for any $n \in \mathbb{N}$.*

A. Bartoszewicz, S. Głąb and A. Paszkiewicz obtained the following characterization.

Theorem 14.25. *The following conditions are equivalent:*

- (i) *$\mathcal{C}_G$ is strongly $2^{\mathfrak{c}}$-algebrable;*
- (ii) *$\mathcal{C}_G$ is $\mathfrak{c}^+$-lineable;*
- (iii) *$\mathbb{R} \setminus G$ is $\mathfrak{c}$-dense in itself.*

Proof. The implication $(i) \Rightarrow (ii)$ is obvious.
The implication $(ii) \Rightarrow (iii)$: suppose that $\mathbb{R} \setminus G$ is not $\mathfrak{c}$-dense in itself, so there is an open interval I such that $0 < \operatorname{card}(I \setminus G) \le \omega$ (since the set $I \setminus G$ is Borel). Any function $f : I \to \mathbb{R}$ which set of continuity points is exactly $G \cap I$ is of the form $f = f|_{G\cap I} \cup f|_{I\setminus G}$. Since there are exactly $\mathfrak{c}$ continuous functions from $G \cap I$ to $\mathbb{R}$, at most $\mathfrak{c}$ functions from $I \setminus G$ to $\mathbb{R}$ and there are exactly $\mathfrak{c}$ many functions $f : I \to \mathbb{R}$ which set of continuity points is $G \cap I$. By the assumption, let $\{f_\xi : \xi < \mathfrak{c}^+\}$ be a basis of linear subspace of $\mathcal{C}_G$. By the above observation there are $\xi_1, \xi_2 < \mathfrak{c}^+$, such that $f_{\xi_1}|_I = f_{\xi_2}|_I$. But then $f_{\xi_1} - f_{\xi_2} = 0$, so it is continuous on I. On the other hand, $f_{\xi_1} - f_{\xi_2} \in \mathcal{C}_G$ and $I \setminus G \neq \emptyset$ so we obtain a contradiction.
The implication $(iii) \Rightarrow (i)$: notice that any $\mathfrak{c}$-dense in itself set is dense in itself. Consider the following cases.

Assume that $\mathbb{R} \setminus G$ is closed, hence perfect. Since $\mathbb{R} \setminus G$ is $\mathfrak{c}$-dense in itself, for any open set $U \subseteq \mathbb{R}$, the set $U \cap (\mathbb{R} \setminus G)$ is either empty or of cardinality $\mathfrak{c}$. Let $\mathcal{F} = \{U \cap (\mathbb{R} \setminus G) : U$ is an open set and $U \cap (\mathbb{R} \setminus G) \neq \emptyset\}$. By Theorem

14.20 there exists a free linear algebra $A \subseteq \mathcal{SES}(\mathbb{R}, \mathcal{F})$ with the set of generators $\{f'_\xi : \xi < 2^\mathfrak{c}\}$ being surjections from $\mathbb{R} \setminus G$ onto $\mathbb{R}$. Define for $\xi < 2^\mathfrak{c}$ a function

$$f_\xi(x) = \begin{cases} f'_\xi(x) & , \text{if } x \in \mathbb{R} \setminus G; \\ 0 & , \text{if } x \in G. \end{cases}.$$

It is easy to see that the set $\{f_\xi : \xi < 2^\mathfrak{c}\}$ generates a free subalgebra of $\mathcal{C}_G$.

Assume that $\mathbb{R} \setminus G$ is not closed. By Lemma 14.24 there exists a sequence of perfect sets $\{C_n : n \in \mathbb{N}\}$ such that C_1 and $C_{n+1} \setminus C_n$ are $\mathfrak{c}$-dense in itself and $\mathbb{R} \setminus G = \bigcup_{n\in\mathbb{N}} C_n$. Let

$$\mathcal{F}_n = \{U \cap C_n \setminus C_{n-1} : U \text{ is an open set and } U \cap C_n \setminus C_{n-1} \neq \emptyset\}$$

for $n \in \mathbb{N}$, where $C_0 = \emptyset$. By Theorem 14.20 there exist free linear algebras $A_n \subseteq \mathcal{SES}([0, \frac{1}{n}], \mathcal{F}_n)$ with the set of generators $\{f^n_\xi : \xi < 2^\mathfrak{c}\}$ being surjections from $C_n \setminus C_{n-1}$ onto $[0, \frac{1}{n}]$. Define for $\xi < 2^\mathfrak{c}$ a function

$$f_\xi(x) = \begin{cases} f^n_\xi(x) & , \text{if } x \in C_n \setminus C_{n-1}; \\ 0 & , \text{if } x \in G. \end{cases}.$$

We will show that the set $\{f_\xi : \xi < 2^\mathfrak{c}\}$ spans a free algebra $\mathcal{A}$ contained in $\mathcal{C}_G$. Let $f \in \mathcal{A}$, then there is a nonzero polynomial P in k variables without a constant term and $\xi_1 < ... < \xi_k < 2^\mathfrak{c}$ with $f = P(f_{\xi_1}, ..., f_{\xi_k})$. Observe that for any $n \in \mathbb{N}$ we have $f|_{C_n \setminus C_{n-1}} \in A_n$, so f maps every nonempty open subset of $C_n \setminus C_{n-1}$ onto $P([0, \frac{1}{n}]^k)$. Therefore f is discontinuous at each point of $C_n \setminus C_{n-1}$. Hence, take a point $x \in G$. If x is in the interior of G, then obviously f is continuous at x (since it is constant and equal to 0). So, assume that x is not in the interior of G, then there is a sequence $(x_m)_{m\in\mathbb{N}}$ of elements of $\mathbb{R} \setminus G$ convergent to x. Since $x_m \notin G$ so there is $l_m \in \mathbb{N}$ with $x_m \in C_{l_m}$. Notice that $(l_m)_{m\in\mathbb{N}}$ tends to infinity. Since $f(x_m) \in P([0, \frac{1}{l_m}]^k)$ so $f(x_m) \to 0 = f(x)$. $\square$

By the power of the above Theorem, the authors of [15] left only one question connected with the algebrability of the set $\mathcal{C}_G$: is the set $\mathcal{C}_G$ (independently with the set G) algebrable on the level of $\mathfrak{c}$? In the next section we will describe another general method that will imply a possitive answer to this.

14.4.3 Exponential like functions method

The idea of using exponential functions, for the first time, appeared in the papers of L. Bernal-González, M. O. Cabrera, P. Jiménez-Rodríguez, G. A. Muñoz-Fernández and J. B. Seoane-Sepúlveda (see [18], [31]). The concept was the following: *consider a set* $H = \{r_\xi : \xi < \mathfrak{c}\} \subseteq \mathbb{R}$ *and construct from one function* F *(of a certain type) a linearly independent set* $\{\exp(r_\xi x)F(x) : \xi < \mathfrak{c}\}$ *which will generate a vector space.*

Recently the following method has been invented by M. Balcerzak, A. Bartoszewicz and M. Filipczak in [8]. Here we present their idea.

Let us introduce the following notion.

Definition 14.26. [8] We say that a function $f : \mathbb{R} \to \mathbb{R}$ is exponential like (of range m) whenever for $x \in \mathbb{R}$

$$f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x},$$

for some distinct nonzero real numbers $\beta_1, \dots, \beta_m$ and some nonzero real numbers $a_1, \dots, a_m$. We will also consider exponential like functions (of the same form) with the domain $[0,1]$.

Observe here a simple property of exponential like functions.

Lemma 14.27. *For every positive integer* m*, any exponential like function* $f : [0,1] \to \mathbb{R}$ *of a range* m*, and each* $c \in \mathbb{R}$*, the preimage* $f^{-1}[\{c\}]$ *has at most* m *elements. Consequently,* f *is not constant in every subinterval of* $[0,1]$.

The criterion of strong $\mathfrak{c}$-algebrability is the following. This should be compared with a similar idea applied in [17], [8], [11].

Theorem 14.28 ([8]). *Let* $\mathcal{F} \subseteq \mathbb{R}^{[0,1]}$ *and assume that there exists a function* $F \in \mathcal{F}$ *such that* $f \circ F \in \mathcal{F} \setminus \{0\}$ *for every exponential like function* $f : \mathbb{R} \to \mathbb{R}$*. Then* $\mathcal{F}$ *is strongly* $\mathfrak{c}$*-algebrable. More exactly, if* $H \subseteq \mathbb{R}$ *is a set of cardinality* $\mathfrak{c}$ *and linearly independent over the rationals* $\mathbb{Q}$*, then* $\exp \circ (rF)$*,* $r \in H$*, are free generators of an algebra contained in* $\mathcal{F} \cup \{0\}$.

Proof. Let H be a set linearly independent over $\mathbb{Q}$ and of cardinality $\mathfrak{c}$. By the assumption we have that $\{\exp \circ (rF) : r \in H\} \subseteq \mathcal{F}$. To show that it is a set of free generators consider any $n \in \mathbb{N}$ and a nonzero polynomial P in n variables without a constant term. The function given by $[0,1] \ni x \mapsto P(e^{r_1 F(x)}, e^{r_2 F(x)}, \dots, e^{r_n F(x)})$ is of the form

$$\sum_{i=1}^{m} a_i \left(e^{r_1 F(x)}\right)^{k_{i1}} \left(e^{r_2 F(x)}\right)^{k_{i2}} \dots \left(e^{r_n F(x)}\right)^{k_{in}} = \sum_{i=1}^{m} a_i \exp\left(F(x) \sum_{j=1}^{n} r_j k_{ij}\right),$$

where $a_1, \dots, a_m$ are nonzero real numbers and the matrix $[k_{ij}]_{i \le m, j \le n}$ has distinct nonzero rows, with $k_{ij} \in \{0, 1, 2, \dots\}$. Since the function

$$t \mapsto \sum_{i=1}^{m} a_i \exp(t \sum_{j=1}^{n} r_j k_{ij})$$

is exponential like, the function $[0,1] \ni x \mapsto P(e^{r_1 F(x)}, e^{r_2 F(x)}, \dots, e^{r_n F(x)})$ is in $\mathcal{F} \setminus \{0\}$. $\square$

Using this method, we can formulate the answer to the question from the previous section. In the paper [15] the authors constructed the following function.

Proposition 14.29. *Let $G \subseteq [0,1]$ be a G_δ set. Consider the set $\mathcal{C}_G$ of all functions $f : [0,1] \to \mathbb{R}$ which set of continuity points is exactly G. There exists a function $F \in \mathcal{C}_G$, such that it has infinitely many limit points at each point of its discontinuity .*

Now we may apply Theorem 14.28.

Theorem 14.30. *The set $\mathcal{C}_G$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let $G \subseteq [0,1]$ be a G_δ set and $F \in \mathcal{C}_G$ be a function like in Proposition 14.29. We will show that for any range $m \in \mathbb{N}$ and any exponential like function $f : [0,1] \to \mathbb{R}$ of the range m, we have $f \circ F \in \mathcal{C}_G \setminus \{0\}$. Let $m \in \mathbb{N}$ and f be an exponential like function of the range m. Take a point $x \in [0,1]$. We have the following:

1. If F is continuous at x then clearly $f \circ F$ is also continuous at x;
2. If F is not continuous at x then there are sequences (cf. Proposition 14.29) $(t_n^{(k)})_{n \in \mathbb{N}}$ for $k \in \{1, \dots, m+1\}$ such that for all k we have $t_n^{(k)} \to x$ and $F(t_n^{(k)}) \to y^{(k)}$ with $y^{(k)} \neq y^{(l)}$ for $k \neq l$. Then $f \circ F(t_n^{(k)}) \to f(y^{(k)})$. By the Lemma 14.27 we have $f(y^{(k)}) \neq f(y^{(l)})$ for some $k \neq l$, so $f \circ F$ is not continuous at x.

Hence, for every exponential like function f we have $f \circ F \in \mathcal{C}_G \setminus \{0\}$, so by Theorem 14.28 we get a strong $\mathfrak{c}$-algebrability of $\mathcal{C}_G$. $\square$

Moreover, using the exponential like functions method, we can obtain the following.

Theorem 14.31. *The sets of*

- *differentiable functions on $\mathbb{R}$ that are nowhere monotone;*
- *continuous functions on $\mathbb{R}$ that are nowhere differentiable;*
- *Baire class one functions on $\mathbb{R}$ that does not have the Darboux property;*
- *Baire class α functions on $\mathbb{R}$ that are not Baire class β with $\beta < \alpha$*

are strongly $\mathfrak{c}$-algebrable.

The exponential like functions method has numerous applications in functions and sequences spaces. The work on this is still in progress, and it will be contained in the paper [13].

Looking at Theorem 14.28 one can ask: *is it true that if a set $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable, then there is a function $F \in \mathcal{F}$ with the property $f \circ F \in \mathcal{F} \setminus \{0\}$ for every exponential like function f.*

It turns out that the answer is negative.

Example 14.32. Consider a set $\mathcal{K} \subseteq \mathbb{R}^{\mathbb{R}}$ of all smooth functions $f : \mathbb{R} \to \mathbb{R}$ with compact support. Let

$$F(x) = \begin{cases} \exp \frac{1}{1-x^2}, & \text{if } |x| < 1; \\ 0, & \text{elsewhere.} \end{cases}.$$

Clearly $F \in \mathcal{K}$. Let H be a set of cardinality $\mathfrak{c}$ linearly independent over $\mathbb{Q}$. It is rather easy to see that $\{F(x)\exp(rx) : r \in H\}$ generates a free subalgebra of $\mathcal{K}$. Hence $\mathcal{K}$ is strongly $\mathfrak{c}$-algebrable. On the other hand, for an exponential like function $f(x) = \exp(x)$ and any $G \in \mathcal{K}$, we have $0 \neq f \circ G \notin \mathcal{K}$ (since it has a full support, in particular non compact). So the method from Theorem 14.28 does not work in that case.

14.5 Some open problems

Let us propose some open questions connected with algebrability.

Problem 14.33. 1. Is the set $\mathcal{ES}(\mathbb{R}) \setminus \mathcal{SES}(\mathbb{R})$ $2^{\mathfrak{c}}$-lineable (cf. [21])?
2. Is the set $\mathcal{ES}(\mathbb{C}) \setminus \mathcal{SES}(\mathbb{C})$ strongly $2^{\mathfrak{c}}$-algebrable (cf. [21])?
3. Are there natural classes $\mathcal{F}$ of functions in $\mathbb{R}^{\mathbb{R}}$ other than $\mathcal{SZ}(\mathbb{R})$ such that the sentence: "$\mathcal{F}$ is $\mathrm{card}(F)$-algebrable" is undecidable in ZFC?
4. We would like to know if Theorem 14.22 can be improved in the following way. Is it true that there is a family $\{f_\alpha : \alpha < 2^{\mathfrak{c}}\} \subseteq \mathcal{PES}(\mathbb{C})$ such that for

any $\alpha_1 < ... < \alpha_n$ and any entire function $G : \mathbb{C}^n \to \mathbb{C}$ (entire means analytic on the whole domain) we have $G(f_{\alpha_1}, ..., f_{\alpha_n}) \in \mathcal{PES}(\mathbb{C})$?

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