

Chapter 10

On Extension Problem, Decomposing and Covering of Functions

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In this chapter, we review some problems related to extension, decomposition and covering of functions. We mainly do not give proofs of the results stated here.

10.1 An Extension Problem

If (X, \mathcal{T}) is a topological normal space, then by well known Tietze extension theorem, for each nonempty closed set $A \subset X$ and every continuous function $g : A \rightarrow [0, 1]$ there is a continuous function $f : X \rightarrow [0, 1]$ such that $f \upharpoonright A = g$.

Let $\mathcal{H} \subset \mathcal{G}$ be nonempty families of functions from X to Y and let $A \subset X$ be a nonempty subset of X . A map $W : \mathcal{G} \rightarrow \mathcal{H}$ is said to be an extension operator from A onto X if the restrictions $f \upharpoonright A$ and $W(f) \upharpoonright A$ are equal for each function $f \in \mathcal{G}$.

So, from Tietze theorem it follows that in the case of topological normal space X , for each nonempty closed set $A \subset X$, the family \mathcal{G} of all real functions with restrictions to A being continuous and the family \mathcal{H} of all continuous

functions from X to \mathbb{R} , there is an extension (from A onto X) operator $W : \mathcal{G} \rightarrow \mathcal{H}$.

We recall that a function $h : A \rightarrow \mathbb{R}$, where $A \subset \mathbb{R}$ is quasi-continuous at a point $x_0 \in A$ if for each positive real $\eta > 0$ and for each open set U containing x_0 , there is an open set $W \subset U$ such that $W \cap A \neq \emptyset$ and $h(W \cap A) \subset (h(x_0) - \eta, h(x_0) + \eta)$. The function f is quasi-continuous if it is quasi-continuous at each point (see [38]).

Denote by $L_A(f, x)$ the set of limit numbers of a function f at x over the set A (i.e. $L_A(f, x) = \{\alpha \in \tilde{\mathbb{R}} : \exists_{\{x_n\} \subset A \setminus \{x\}} x_n \rightarrow x \wedge f(x_n) \rightarrow \alpha\}$). Observe that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is quasi-continuous and the set A is dense then for all $x \in \mathbb{R}$, $f(x) \in L_A(f, x)$.

Consider the function $f(x) = \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. The function f is continuous on the set $A = \mathbb{R} \setminus \{0\}$ which is dense, but $L_A(f, 0) \subset \{-\infty, +\infty\}$ so there is no quasi-continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ with $g \upharpoonright A = f \upharpoonright A$.

Remark 10.1. Let $A \subset \mathbb{R}$ be a dense set and $f : A \rightarrow \mathbb{R}$ be a quasi-continuous function. The function f can be extended on \mathbb{R} to a quasi-continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ if and only if for every $x \in \mathbb{R} \setminus A$, $L_A(f, x) \setminus \{-\infty, +\infty\} \neq \emptyset$.

Proof. If there is a quasi-continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g \upharpoonright A = f$ then for each $x \in \mathbb{R} \setminus A$ we have $g(x) \in L_A(f, x) \setminus \{-\infty, +\infty\}$.

Now suppose that for every $x \in \mathbb{R} \setminus A$, $L_A(f, x) \setminus \{-\infty, +\infty\} \neq \emptyset$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be such that $g(x) = f(x)$ for $x \in A$ and $g(x) \in L_A(f, x) \setminus \{-\infty, +\infty\}$ for $x \notin A$. Then the function g is quasi-continuous. \square

Example 10.2. Let $A \subset \mathbb{R}$ be a dense set, \mathcal{G} be the family of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ quasi-continuous on A and such that $L_A(f, x) \setminus \{-\infty, +\infty\} \neq \emptyset$ for each $x \in \mathbb{R} \setminus A$, and $\mathcal{H} \subset \mathcal{G}$ be the family of all quasi-continuous functions. By Remark 10.1 there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(h) = h$ for $h \in \mathcal{H}$.

Remark 10.3. Let $A = \mathbb{R} \setminus \{0\}$. We put: $f_1(x) = 0$ and $f_2(x) = n$ for $x \in (\frac{1}{2n}, \frac{1}{2n-1}]$, $n = 1, 2, \dots$; $f_1(x) = n$ and $f_2(x) = 0$ for $x \in (\frac{1}{2n+1}, \frac{1}{2n}]$ for $n = 1, 2, \dots$; $f_1(x) = f_1(-x)$, $f_2(x) = f_2(-x)$ for $x \in [-1, 0)$ and $f_1(x) = f_2(x) = 0$ for $x \in (-\infty, -1) \cup (1, \infty)$. Then $f_1, f_2 : A \rightarrow \mathbb{R}$ are quasi-continuous; f_1 and f_2 can be extended onto \mathbb{R} to quasi-continuous functions (we can take $f_1(0) = 0 = f_2(0)$), but $L_A(f_1 + f_2, 0) = \{+\infty\}$ so $f_1 + f_2$ cannot be extended onto \mathbb{R} to a quasi-continuous function.

By Remark 10.3 the space \mathcal{G} from Example 10.2 is not a linear space. Now we will consider families \mathcal{G} of real functions which form linear spaces with

the natural operations of addition of functions and multiplication by reals and discuss the problem of existence of linear extension operators.

Let \mathcal{G} be some linear space of bounded real functions defined on a topological space X , with the norm $\|f\| = \sup_{x \in X} |f(x)|$ for $f \in \mathcal{G}$. Let $\mathcal{H} \subset \mathcal{G}$ and $A \subset X$ be a nonempty set. Suppose that for each function $f \in \mathcal{G}$ there is a function $g : X \rightarrow I_f$ (where $I_f = [\inf f, \sup f]$) belonging to \mathcal{H} such that $f \upharpoonright A = g \upharpoonright A$. Then there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X satisfying the condition $\|W(f)\| \leq \|f\|$, $f \in \mathcal{G}$, and such that $W(f) = f$ for every $f \in \mathcal{H}$, but this operator may be not linear.

Observe that if there are functions $f, g \in \mathcal{H} \subset \mathcal{G}$ with $f + g \in \mathcal{G} \setminus \mathcal{H}$ then there is no a linear operator $W : \mathcal{G} \rightarrow \mathcal{H}$ such that $W(h) = h$ for $h \in \mathcal{H}$. Indeed, for such f, g we have $W(f + g) = W(f) + W(g) = f + g \in \mathcal{G} \setminus \mathcal{H}$, contrary to $W(\mathcal{G}) \subset \mathcal{H}$.

Remark 10.4 ([17]). Let $A \subset \mathbb{R}$ be a nonempty nowhere dense set. For each function $f : \mathbb{R} \rightarrow \mathbb{R}$ there is a quasi-continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \upharpoonright A = g \upharpoonright A$ and g is continuous at each point $x \in \mathbb{R} \setminus cl(A)$. Moreover, if f is Lebesgue measurable (resp. of Baire α class, $\alpha \geq 1$), then g may have the same property.

Example 10.5. Let $A \subset \mathbb{R}$ be a nonempty nowhere dense set. Assume that \mathcal{G} is the family of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and \mathcal{H} is the family of all quasi-continuous functions, continuous on $\mathbb{R} \setminus cl(A)$. By Remark 10.4 there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(h) = h$ for $h \in \mathcal{H}$. Since the sum of two quasi-continuous functions may belong to $\mathcal{G} \setminus \mathcal{H}$, such operator cannot be linear.

Example 10.6. Let $A \subset \mathbb{R}$ be a nonempty nowhere dense set. Assume that \mathcal{G} is the family of all Lebesgue measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and \mathcal{H} is the family of all Lebesgue measurable quasi-continuous functions, continuous on $\mathbb{R} \setminus cl(A)$. By Remark 10.4 there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ such that $W(f) = f$ for $f \in \mathcal{H}$. Since the sum of two Lebesgue measurable quasi-continuous functions may belong to $\mathcal{G} \setminus \mathcal{H}$, such operator cannot be linear.

Example 10.7. Let $A \subset \mathbb{R}$ be a nonempty nowhere dense set. Assume that \mathcal{G} is the family of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ from Baire α class ($\alpha \geq 1$) and \mathcal{H} is the family of all quasi-continuous functions of Baire α class, that are continuous on $\mathbb{R} \setminus cl(A)$. By Remark 10.4 there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ such that $W(f) = f$ for $f \in \mathcal{H}$. Since the sum of two Baire α , quasi-continuous functions may belong to $\mathcal{G} \setminus \mathcal{H}$, such operator cannot be linear.

Example 10.8. Let \mathcal{G} be the family of all approximately continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and let \mathcal{H} be the family of all approximately continuous and quasi-continuous functions $g : \mathbb{R} \rightarrow \mathbb{R}$. Let $A \subset \mathbb{R}$ be a nonempty nowhere dense set of Lebesgue measure zero. It is known ([17]) that for every function $f \in \mathcal{G}$ there is a function $g \in \mathcal{H}$ such that $f \upharpoonright A = g \upharpoonright A$. So there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(h) = h$ for $h \in \mathcal{H}$. Since there is an approximately continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$, that is not quasi-continuous and for each function $f \in \mathcal{G}$ there are functions $f_1, f_2 \in \mathcal{H}$ with $f = f_1 + f_2$, such operator W cannot be linear.

Example 10.9. Let \mathcal{G} be the family of all Baire 1 functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and let \mathcal{H} be the family of all Baire 1 and quasi-continuous functions. Let $A \subset \mathbb{R}$ be a nonempty nowhere dense set. It is known ([17]) that for every function $f \in \mathcal{G}$ there is a function $g \in \mathcal{H}$ such that $f \upharpoonright A = g \upharpoonright A$. So there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(h) = h$ for $h \in \mathcal{H}$. Since there is a Baire 1 function that is not quasi-continuous, and since for every function $f \in \mathcal{G}$ there are functions $f_1, f_2 \in \mathcal{G}$ with $f = f_1 + f_2$, such operator W cannot be linear.

Remark 10.10. Let a nonempty Borel set $A \subset \mathbb{R}$ be such that its complement $\mathbb{R} \setminus A$ is c -dense in \mathbb{R} . Using the similar construction as applied in the articles [14],[44] we can show that for each Lebesgue measurable (resp. having Baire property) function $f : \mathbb{R} \rightarrow \mathbb{R}$ there is a Lebesgue measurable (resp. with the Baire property) Darboux function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \upharpoonright A = g \upharpoonright A$. Similarly, for each function (resp. of Baire α class, where $\alpha \geq 1$ function), $f : \mathbb{R} \rightarrow \mathbb{R}$ there is a Darboux (resp. of Darboux and Baire α class) function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \upharpoonright A = g \upharpoonright A$.

Example 10.11. Let a nonempty Borel set $A \subset \mathbb{R}$ be such that its complement $\mathbb{R} \setminus A$ is c -dense in \mathbb{R} . Assume that \mathcal{G} is the family of all Lebesgue measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and \mathcal{H} is the family of all Lebesgue measurable Darboux functions $g : \mathbb{R} \rightarrow \mathbb{R}$. Then \mathcal{G} is a linear space, but for each $f \in \mathcal{G}$ there are two functions $f_1, f_2 \in \mathcal{H}$ such that $f = f_1 + f_2$. By Remark 10.10 there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(h) = h$ for $h \in \mathcal{H}$ but it cannot be linear.

Example 10.12. Let a nonempty Borel set $A \subset \mathbb{R}$ be such that its complement $\mathbb{R} \setminus A$ is c -dense in \mathbb{R} . Assume that \mathcal{G} is the family of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ from Baire α class, where $\alpha \geq 1$, and \mathcal{H} is the family of all Darboux functions $g : \mathbb{R} \rightarrow \mathbb{R}$ of Baire α class. Then \mathcal{G} is a linear space, but for each $f \in \mathcal{G}$ there are two functions $f_1, f_2 \in \mathcal{H}$ such that $f = f_1 + f_2$. By Remark 10.10 there is

an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(h) = h$ for $h \in \mathcal{H}$, but it cannot be linear.

Example 10.13. Let a nonempty Borel set $A \subset \mathbb{R}$ be such that its complement $\mathbb{R} \setminus A$ is c-dense in \mathbb{R} . Assume that \mathcal{G} is the family of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with the Baire property and \mathcal{H} is the family of all Darboux functions $g : \mathbb{R} \rightarrow \mathbb{R}$ with the Baire property. Then \mathcal{G} is a linear space, but for each $f \in \mathcal{G}$ there are two functions $f_1, f_2 \in \mathcal{H}$ such that $f = f_1 + f_2$. By Remark 10.10 there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(h) = h$ for $h \in \mathcal{H}$ but it cannot be linear.

Example 10.14. Let a nonempty Borel set $A \subset \mathbb{R}$ be such that its complement $\mathbb{R} \setminus A$ is c-dense in \mathbb{R} . Assume that \mathcal{G} is the family of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and \mathcal{H} is the family of all Darboux functions $g : \mathbb{R} \rightarrow \mathbb{R}$. Then \mathcal{G} is a linear space, but for each $f \in \mathcal{G}$ there are two functions $f_1, f_2 \in \mathcal{H}$ such that $f = f_1 + f_2$. By Remark 10.10 there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(h) = h$ for $h \in \mathcal{H}$ but it cannot be linear.

In the next example we will consider cliquish functions. Recall that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is cliquish if the set $C(f)$ of all its continuity points is dense. As above, the following remark is true.

Remark 10.15. Let a nonempty Borel set $A \subset \mathbb{R}$ be such that its complement $\mathbb{R} \setminus A$ is c-dense in \mathbb{R} . Using a similar construction as applied in the article [14] we can show that for each cliquish function $f : \mathbb{R} \rightarrow \mathbb{R}$ there is a Darboux cliquish function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g \upharpoonright A = f \upharpoonright A$.

Example 10.16. Let a nonempty Borel set $A \subset \mathbb{R}$ be such that its complement $\mathbb{R} \setminus A$ is c-dense in \mathbb{R} . Assume that \mathcal{G} is the family of all cliquish functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and \mathcal{H} is the family of all Darboux cliquish functions $g : \mathbb{R} \rightarrow \mathbb{R}$. Then \mathcal{G} is a linear space, but for each $f \in \mathcal{G}$ there are two functions $f_1, f_2 \in \mathcal{H}$ such that $f = f_1 + f_2 \in \mathcal{G} \setminus \mathcal{H}$. By Remark 10.15 there is an extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(h) = h$ for $h \in \mathcal{H}$ but it cannot be linear.

A general construction for linear spaces

Now we consider the case in which $\mathcal{H} \subset \mathcal{G}$ are families of real functions being both linear spaces with the natural operations of addition of functions and multiplication by reals.

Proposition 10.17. *Let $\mathcal{H} \subset \mathcal{G}$ be linear spaces of real functions. Let $A \subset X$ be a nonempty set and suppose that for every function $g \in \mathcal{G}$ there is a function $f : X \rightarrow \mathbb{R}$ belonging to \mathcal{H} such that $f \upharpoonright A = g \upharpoonright A$. Then there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.*

Proof. Without loss of generality we can suppose that $\mathcal{H} \neq \mathcal{G}$ and that \mathcal{H} contains elements different from 0. Let $B(\mathcal{H})$ be a basis of the linear space \mathcal{H} and let $B(\mathcal{G})$, such that $B(\mathcal{G}) \supset B(\mathcal{H})$ is a basis of the linear space \mathcal{G} . For $f \in B(\mathcal{H})$ we put $W(f) = f$ and for $g \in B(\mathcal{G}) \setminus B(\mathcal{H})$ we take $W(g)$ to be an element of \mathcal{H} such that $g \upharpoonright A = W(g) \upharpoonright A$. Next, if $h \in \mathcal{G}$ is of the form $h = \sum_{i=1}^k r_i h_i$, where $h_i \in B(\mathcal{G})$ and $r_i \in \mathbb{R}$ for $i = 1, 2, \dots, k$, then we put $W(h) = \sum_{i=1}^k r_i W(h_i)$. Evidently, the operator W is well defined on \mathcal{G} and its values belong to \mathcal{H} . It is also linear. Since $f = 1 \cdot f$, for $f \in \mathcal{H}$, we have for such an f that $W(f) = 1 \cdot W(f) = f$. □

Example 10.18. Let X be a topological normal space and let $A \subset X$ be a nonempty closed set. Denote by \mathcal{H} the family of all continuous functions $f : X \rightarrow \mathbb{R}$ and by \mathcal{G} the family of all functions $g : X \rightarrow \mathbb{R}$, whose restrictions $g \upharpoonright A$ are continuous. Then by the Tietze Theorem and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Recall that an F -space is a vector space V over the real (or complex) numbers together with a metric $d : V \times V \rightarrow \mathbb{R}$ so that scalar multiplication in V is continuous with respect to d and the standard metric on \mathbb{R} (or on \mathbb{C}), addition in V is continuous with respect to d , the metric is translation-invariant and the metric space (V, d) is complete.

Remark 10.19. Let $\mathcal{H} \subset \mathcal{G}$ be families of real functions satisfying all requirements of Proposition 10.17. If \mathcal{G} is an F -space and \mathcal{H} has a complement in \mathcal{G} , then there is a continuous linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Proof. Let W be a linear extension operator from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$. Because $W^2 = W$, the operator W obtained in the above construction is a projection. Because \mathcal{G} is an F -space and \mathcal{H} has a complement in \mathcal{G} , we obtain that W is continuous. □

Let (X, ρ) be a metric space and $A \subset X$. We say that a function $\omega_f : [0, \infty) \rightarrow [0, \infty)$ is a modulus of continuity of the function $f : A \rightarrow \mathbb{R}$ if

$$|f(x_1) - f(x_2)| < \omega_f(t) \quad \text{for all } x_1, x_2 \in A \quad \text{with } \rho(x_1, x_2) < t$$

In [30] M. J. McShane proved the following theorem:

Theorem 10.20. (*McShane*) *If the function f defined on a subset A of metric space X has a concave modulus of continuity ω_f such that $\lim_{t \rightarrow 0} \omega_f(t) = 0$, then f can be extended to X preserving the modulus of continuity.*

Example 10.21. Let A be a nonempty set in a metric space (X, ρ) . Denote by \mathcal{H} the family of all functions $f : X \rightarrow \mathbb{R}$ for which there exists some concave modulus of continuity ω_f such that $\lim_{t \rightarrow 0} \omega_f(t) = 0$. Let \mathcal{G} be the family of all functions $g : X \rightarrow \mathbb{R}$, whose restrictions to A have a modulus of continuity ω_f which is concave and such that $\lim_{t \rightarrow 0} \omega_f(t) = 0$. Then by Theorem 10.20 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Let A be a nonempty set in a metric space (X, ρ) . We say that a function $f : A \rightarrow \mathbb{R}$ satisfies Hölder condition if

$$|f(x_1) - f(x_2)| \leq M(\rho(x_1, x_2))^\alpha \quad \text{for } x_1, x_2 \in A$$

for some constants $M > 0$ and $\alpha > 0$.

Example 10.22. Let A be a nonempty set in a metric space (X, ρ) and $0 < \alpha \leq 1$. Denote by \mathcal{H} the family of all functions $f : X \rightarrow \mathbb{R}$ satisfying Hölder condition with the exponent α and by \mathcal{G} the family of all functions $g : X \rightarrow \mathbb{R}$ whose restrictions $g \upharpoonright A$ satisfy the Hölder condition with α on the set A . By McShane Corollary 1 from [30] and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Example 10.23. Let A be a nonempty set in a metric space (X, ρ) . Denote by \mathcal{H} the family of all bounded uniformly continuous functions and by \mathcal{G} the family of all functions $g : X \rightarrow \mathbb{R}$ whose restrictions $g \upharpoonright A$ are bounded and uniformly continuous. By Corollary 2 from [30] and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

We recall that a vector $L \in \mathbb{R}^n$ is a derivative of a function $f : F \rightarrow \mathbb{R}$ at a point $a \in F \subset \mathbb{R}^n$ if either a is an isolated point of F , or

$$\frac{|f(x) - f(a) - (L, (x - a))|}{|x - a|} \rightarrow 0 \quad \text{as } x \rightarrow a, x \in F,$$

where (x, y) denotes the scalar product of vectors $x, y \in \mathbb{R}^n$. The vector L is a strict derivative of f at a point $a \in F$ if either a is an isolated point of F , or

$$\frac{|f(y) - f(x) - (L, (y - x))|}{|y - x|} \rightarrow 0 \quad \text{as } x, y \rightarrow a,$$

$(x, y \in F, x \neq y; x = a$ or $y = a$ is allowed).

In [3] the authors prove the following theorem:

Theorem 10.24 (Aversa, Laczkovich, Preiss). *Let $F \subset \mathbb{R}^n$ be a nonempty closed set, $f : F \rightarrow \mathbb{R}$ and $L : F \rightarrow \mathbb{R}^n$ be functions such that for each $a \in F$ the vector $L(a)$ is a derivative of f at a . Then f can be extended to an everywhere differentiable function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $g' = L$ on F if and only if the map $L : F \rightarrow \mathbb{R}^n$ is Baire 1.*

In [28] the authors show the following theorem.

Theorem 10.25 (Koc-Zajícék). *Let $F \subset \mathbb{R}^n$ be a nonempty closed set, $f : F \rightarrow \mathbb{R}$ and $L : F \rightarrow \mathbb{R}^n$ be functions such that for each $a \in F$ the vector $L(a)$ is a derivative of f at a . Moreover, suppose that the mapping $L : F \rightarrow \mathbb{R}^n$ is Baire 1. Then f can be extended to an everywhere differentiable function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that g is C^∞ on $\mathbb{R}^n \setminus F$, the derivative $g' = L$ on F and g' is continuous at all points $a \in F$ at which L is continuous and $L(a)$ is a strict derivative of f at a .*

These theorems are used in the next examples.

Example 10.26. Let $X = \mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be a nonempty closed set. Denote by \mathcal{H} the family of all differentiable functions $g : \mathbb{R}^n \rightarrow \mathbb{R}$, and by \mathcal{G} the family of all functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, whose restrictions $f \upharpoonright A$ are differentiable and its derivatives $(f \upharpoonright A)'$ are Baire 1. By Theorem 10.24 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Example 10.27. Let $X = \mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be a nonempty closed set. Denote by \mathcal{H} the family of all differentiable functions $g : \mathbb{R}^n \rightarrow \mathbb{R}$, that are C^∞ on $\mathbb{R}^n \setminus A$, and by \mathcal{G} the family of all functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, whose restrictions $f \upharpoonright A$ are differentiable and its derivatives $(f \upharpoonright A)'$ are Baire 1. By Theorem 10.24 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Let $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is called tangent to A if there exist $\{x_k\}_{k=1}^\infty \subset A$ and $\{\alpha_k\}_{k=1}^\infty \subset [0, \infty)$ such that $x_k \rightarrow x$ and $\alpha_k(x_k - x) \rightarrow v$. The set of all tangent vectors to A is called the contingent cone of A at x and will be denoted by $Tan(A, x)$. In [3] the authors proved that for a function $f : A \rightarrow \mathbb{R}$ differentiable at x , the derivative $f'(x)$ is determined uniquely iff $Tan(A, x)$ spans \mathbb{R}^n .

Remark 10.28 ([3]). There is a nonempty compact set $A \subset \mathbb{R}^2$ and a function $f : A \rightarrow \mathbb{R}$ such that the contingent cone $Tan(A, x)$ spans \mathbb{R}^2 for every $x \in A$ and f has a derivative everywhere on A , but the derivative is not Baire 1 and thus f cannot be extended to \mathbb{R}^2 as an everywhere differentiable function.

Theorem 10.29 ([28]). *Let $A \subset \mathbb{R}^n$ be a nonempty closed set such that $Tan(A, x)$ spans \mathbb{R}^n for every $x \in A$. If a function $f : A \rightarrow \mathbb{R}$ has a strict derivative everywhere on A , then f can be extended to \mathbb{R}^n as an everywhere differentiable function.*

Remark 10.30 ([28]). There is a nonempty compact set $A \subset \mathbb{R}^2$ and a function $f : A \rightarrow \mathbb{R}$ such that $Tan(A, x)$ spans \mathbb{R}^2 for every $x \in A$ and f has a strict derivative everywhere on A , but f cannot be extended to \mathbb{R}^2 as an everywhere continuously differentiable function.

Example 10.31. Let $A \subset \mathbb{R}^n$ be a nonempty closed set such that $Tan(A, x)$ spans \mathbb{R}^n for each $x \in A$. Denote by \mathcal{G} the set of all functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, whose restrictions $f \upharpoonright A$ have a strict derivative everywhere on A , and by \mathcal{H} the family of all differentiable functions belonging to \mathcal{G} . By Theorem 10.29 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

In the next examples we use the ordinary differentiation basis in \mathbb{R}^n (see [11] and [45]). Recall that a point $x \in \mathbb{R}^n$ is a density point of a Lebesgue measurable set $E \subset \mathbb{R}^n$ if

$$\lim_{r \rightarrow 0^+} \frac{\lambda_n(E \cap Q(x, r))}{\lambda_n(Q(x, r))} = 1,$$

where $Q(x, r)$ denotes the cube with the center x and the length of edge equal r , and λ_n denotes the Lebesgue measure in \mathbb{R}^n . Moreover, x is a density point of an arbitrary set $H \subset \mathbb{R}^n$ if there is a Lebesgue measurable set $E \subset H$ such that x is a density point of E . The family \mathcal{T}_d of all sets $M \subset \mathbb{R}^n$ such that every point $x \in M$ is a density point of M is a topology called the ordinary density topology. All sets belonging to \mathcal{T}_d are Lebesgue measurable. If \mathcal{T}_{nat} denotes the natural topology in \mathbb{R} then the ordinary approximate continuity of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the continuity of f as a map from $(\mathbb{R}^n, \mathcal{T}_d)$ into $(\mathbb{R}, \mathcal{T}_{nat})$. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, locally Lebesgue integrable at a point $x \in \mathbb{R}^n$, is said to be an ordinary derivative at x if

$$\lim_{r \rightarrow 0^+} \frac{\int_{Q(x, r)} f(t) dt}{\lambda_n(Q(x, r))} = f(x).$$

It is well known that a Lebesgue measurable function f , locally bounded at x and approximately continuous at x is an ordinary derivative at x . All ordinary derivatives and all approximately continuous functions are Baire 1.

Theorem 10.32 ([2]). *If a set $A \subset \mathbb{R}^n$ has Lebesgue measure 0 and a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is Baire 1 then there is an ordinary derivative $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and an*

approximately continuous function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f \upharpoonright A = g \upharpoonright A = h \upharpoonright A$. Moreover, if the image $f(A)$ is contained in a closed interval K then g and h can be chosen so that their images are contained in K .

Example 10.33. Let $X = \mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be a nonempty set of Lebesgue measure 0. Denote by \mathcal{G} the family of all Baire 1 functions, and by \mathcal{H} the family of all ordinary derivatives. By Theorem 10.32 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Example 10.34. Let $X = \mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be a nonempty set of Lebesgue measure 0. Denote by \mathcal{G} the family of all Baire 1 functions, and by \mathcal{H} the family of all approximately continuous functions. By Theorem 10.32 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Example 10.35. Let $X = \mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be a nonempty set of Lebesgue measure 0. Denote by \mathcal{G} the family of all bounded Baire 1 functions, and by \mathcal{H} the family of all bounded ordinary derivatives. By Theorem 10.32 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Example 10.36. Let $X = \mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be a nonempty set of Lebesgue measure 0. Denote by \mathcal{G} the family of all bounded Baire 1 functions, and by \mathcal{H} the family of all bounded approximately continuous functions. By Theorem 10.32 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Remark 10.37. Let a nonempty set $A \subset \mathbb{R}^n$ be such that its closure $cl(A)$ is of Lebesgue measure zero. From the proof of Theorem 3 in [13] it follows that for each Baire 1 function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ there is an approximately continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, that is continuous at every point $x \in \mathbb{R}^n \setminus cl(A)$ and such that $f(x) = g(x)$ for $x \in A$.

Example 10.38. Let $A \subset \mathbb{R}^n$ be a nonempty set, whose closure $cl(A)$ is of Lebesgue measure zero, \mathcal{H} be the family of all approximately continuous functions, continuous at all points $x \in \mathbb{R}^n \setminus cl(A)$ and let \mathcal{G} be the family of all Baire 1 functions. By Remark 10.37 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Remark 10.39. The above results concerning extensions of Baire 1 functions to approximately continuous functions and extensions of Baire 1 functions to

almost everywhere continuous, approximately continuous functions were generalized for functions defined on some special metric spaces with measures (see [12]).

In [24] the author considered the notion of B_1 -retracts. A subset A of a topological space X is a B_1 -retract of X if and only if for any topological space Y every continuous function $f : A \rightarrow Y$ can be extended to a Baire 1 function $g : X \rightarrow Y$.

Example 10.40. Let X be a topological space and let $A \subset X$ be a B_1 -retract of X . Denote by \mathcal{G} the family of all functions $f : X \rightarrow \mathbb{R}$, whose restrictions to A are continuous, and by \mathcal{H} the family of all Baire 1 functions $g : X \rightarrow \mathbb{R}$ whose restrictions to A are continuous. Then the families \mathcal{G} and \mathcal{H} are linear spaces and by Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Remark 10.41. Let $A \subset [a, b]$ be a nonempty set whose complement $[a, b] \setminus A$ is countable and let $f : [a, b] \rightarrow \mathbb{R}$ be a function with a bounded variation, continuous at each point $x \in A$. There is a right-continuous (resp. left-continuous) function $f_1 : [a, b] \rightarrow \mathbb{R}$ with a bounded variation and continuous at each point $x \in A$ and such that $f \upharpoonright A = f_1 \upharpoonright A$.

Proof. Since f is of bounded variation, there are two increasing functions $g, h : [a, b] \rightarrow \mathbb{R}$ such that $f = g - h$ and g, h are continuous at each point $x \in A$. If $u \in [a, b]$ is a discontinuity point of g then we put $g_1(u) = \inf\{g(t); t \in (u, b)\}$. For other points $u \in [a, b]$ we put $g_1(u) = g(u)$. Then the function g_1 is right-continuous and continuous at each point $x \in A$, where $g_1(x) = g(x)$. Similarly, if $u \in [a, b]$ is a discontinuity point of h then we put $h_1(u) = \inf\{h(t); t \in (u, b)\}$. For other points $u \in [a, b]$ we put $h_1(u) = h(u)$. Then the function h_1 is right-continuous and continuous at each point $x \in A$, where $h_1(x) = h(x)$. Moreover, the functions g_1 and h_1 are increasing. So the function $f_1 = g_1 - h_1$ has bounded variation on $[a, b]$, is right-continuous and $f_1 \upharpoonright A = f \upharpoonright A$. The case of left-continuity is similar. \square

Example 10.42. Let $A \subset [a, b]$ be a nonempty set such that its complement $[a, b] \setminus A$ is countable. Denote by \mathcal{G} the family of all functions $f : [a, b] \rightarrow \mathbb{R}$ with a bounded variation on $[a, b]$, that are continuous at each point $x \in A$. Moreover, let \mathcal{H} be the family of all right-continuous (resp. left-continuous) functions $g : [a, b] \rightarrow \mathbb{R}$ with bounded variation on $[a, b]$ and continuous at each point $x \in A$. By Remark 10.41 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

In [15] there was introduced the following classes of functions. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ has property \mathcal{A}_3 at a point x if for each positive real r and each set $U \in \mathcal{T}_d$ containing x there is an open interval I such that $C(f) \supset I \cap U \neq \emptyset$ and $|f(t) - f(x)| < r$ for $t \in I \cap U$, where $C(f)$ denotes the set of all continuity points of f . A function f has property \mathcal{A}_3 if it has property \mathcal{A}_3 at each point $x \in \mathbb{R}$. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ has property \mathcal{A}_5 if for each nonempty set $U \in \mathcal{T}_d$ there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$.

These classes were considered as some special notions of quasi-continuity and cliquishness using two topologies: natural topology \mathcal{T}_{nat} and density topology \mathcal{T}_d . All functions of these classes are almost everywhere continuous and are very useful for considerations related to measurability of functions of two variables. For example there is a nonmeasurable (in the sense of Lebesgue) function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with approximately continuous vertical sections f_x , for $x \in \mathbb{R}$, and measurable horizontal sections f^y , for $y \in \mathbb{R}$, while the property \mathcal{A}_3 of vertical sections f_x , for $x \in \mathbb{R}$, and measurability of horizontal sections f^y , for $y \in \mathbb{R}$, imply the measurability of function f .

Example 10.43. Denote by \mathcal{H} the family of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ having property \mathcal{A}_i (where $i = 3$ or $i = 5$) and by \mathcal{G} the family of all almost everywhere continuous functions $g : \mathbb{R} \rightarrow \mathbb{R}$. Let $A \subset \mathbb{R}$ be a nonempty set whose closure $cl(A)$ is of Lebesgue measure 0. Then the families \mathcal{G} and \mathcal{H} are linear spaces and by Theorem 1 in [16] and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Remark 10.44. Let X be a metric space and let $A \subset X$ be a nonempty G_δ -set. It is well known (see [29]) that for every Baire 1 function $f : A \rightarrow \mathbb{R}$ there is a Baire 1 function $g : X \rightarrow \mathbb{R}$ such that $f \upharpoonright A = g \upharpoonright A$.

Example 10.45. Let X be a metric space and let $A \subset X$ be a nonempty G_δ -set. Denote by \mathcal{G} the family of all functions $f : X \rightarrow \mathbb{R}$ whose restrictions to A are Baire 1 functions and by \mathcal{H} the family of all Baire 1 functions $g : X \rightarrow \mathbb{R}$. Then families \mathcal{G} and \mathcal{H} are linear spaces and by Remark 10.44 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

Remark 10.46 ([23]). Let X be a completely regular topological space and let A be a nonempty Lindelöf hereditarily Baire subset of X . If $f : A \rightarrow \mathbb{R}$ is a Baire 1 function then there is a Baire 1 function $g : X \rightarrow \mathbb{R}$ such that $f \upharpoonright A = g \upharpoonright A$.

Example 10.47. Let X be a completely regular topological space and let A be a nonempty Lindelöf hereditarily Baire subset of X . Denote by \mathcal{G} the family

of all functions $f : X \rightarrow \mathbb{R}$ whose restrictions to A are Baire 1 and by \mathcal{H} the family of all Baire 1 functions $g : X \rightarrow \mathbb{R}$. Then families \mathcal{G} and \mathcal{H} are linear spaces and by Remark 10.46 and Proposition 10.17 there is a linear extension operator $W : \mathcal{G} \rightarrow \mathcal{H}$ from A onto X such that $W(f) = f$ for every $f \in \mathcal{H}$.

10.2 Decomposing and Covering of Functions

Decomposing and covering of functions by continuous functions

Let X, Y be topological spaces and k be a positive integer. We say that $f : X \rightarrow Y$ is k -continuous if there exist sets $\{X_n\}_{n=1}^k$ such that $X = \bigcup_{n=1}^k X_n$ and $f \upharpoonright X_n$ is continuous for $n = 1, \dots, k$. We say that f is finitely continuous if it is k -continuous for some k . We say that f is countably continuous if it is decomposable into countably many continuous functions, i.e. if there exists a sequence $\{X_n\}_{n=1}^\infty$ such that $X = \bigcup_{n=1}^\infty X_n$ and $f \upharpoonright X_n$ is continuous for all n .

We say that f is strongly k -continuous if its graph can be covered by k graphs of continuous functions, i.e. there exist continuous functions $f_i : X \rightarrow Y$, $i = 1, \dots, k$ such that $Gr(f) \subset \bigcup_{i=1}^k Gr(f_i)$ where $Gr(f)$ denotes the graph of a function f . We say that f is strongly finitely continuous if it is strongly k -continuous for some k . We say that f is strongly countably continuous if its graph can be covered by graphs of countably many continuous functions, i.e. if there exists the sequence $\{f_i\}_{i=1}^\infty$ of continuous functions $f_i : X \rightarrow Y$ such that $Gr(f) \subset \bigcup_{i=1}^\infty Gr(f_i)$.

In [39] R. J. O'Malley introduced the class of Baire-one-star function. We say that $f \in \mathcal{B}_1^*$ if for any nonempty closed set $F \subset X$ there is an open set $U \subset X$ such that $U \cap F \neq \emptyset$ and the restriction $f \upharpoonright F \cap U$ is continuous.

If X is a complete metric space then functions from class \mathcal{B}_1^* are of first Baire class. Moreover, $f \in \mathcal{B}_1^*$ iff it is piecewise continuous, i.e. there exists a sequence of nonempty closed sets $\{X_n\}_{n=1}^\infty$ such that $X = \bigcup_{n=1}^\infty X_n$ and $f \upharpoonright X_n$ is continuous (see [22], [25]). Of course from Tietze Theorem we conclude that a piecewise continuous function is strongly countably continuous.

It is easy to see that finitely continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ have a nowhere dense graph, but of course even 2-continuous functions can be discontinuous everywhere (consider for example the Dirichlet function). In [40] R. J. Pawlak considered a nice subclass \mathcal{B}_1^{**} of 2-continuous functions (we say that f belongs to \mathcal{B}_1^{**} if is continuous or $f \upharpoonright D(f)$ is continuous, where $D(f)$ denotes the set of all discontinuity points of f). He proved that if $f \in \mathcal{B}_1^{**}$ then the set of discontinuity points of f must be nowhere dense and $\mathcal{B}_1^{**} \subsetneq \mathcal{B}_1^*$.

In [32] the author proves that the Darboux real function defined on a locally connected metric space is 2-continuous if and only if it belongs to the class \mathcal{B}_1^{**} . In this paper we give an example of 3-continuous Darboux function that is not in the first class of Baire. However, in the case $f : \mathbb{R} \rightarrow \mathbb{R}$ (similar to that for functions from \mathcal{DB}_1^*), if f is finitely continuous Darboux function then its set of discontinuity points must be nowhere dense and the set $f(\mathbb{R}) \setminus f(\text{int}(C(f)))$ must be nowhere dense too [32], [33].

We say that $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a Hamel function if its graph is a Hamel basis for \mathbb{R}^{n+k} . Many authors considered Hamel functions with some nice properties. In [10] the authors show an example of Marczewski measurable Hamel function and an example of Hamel function which is both Lebesgue measurable and with the Baire property. In [36] T. Natkaniec gives an example of a quasi-continuous Hamel function. An example of finitely continuous Hamel function is given in the paper [43].

It is well known that for every measurable function $f : I \rightarrow \mathbb{R}$ (where I is an interval) there exists a sequence of measurable sets $\{X_n\}_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty X_n$ has full measure and $f \upharpoonright X_n$ is continuous for all n . Lusin asked if any Borel function is necessarily countably continuous. The answer is negative and many authors give counterexamples (see [1], [9], [34], [46]).

An interesting counterexample is the Lebesgue measure: in the paper [26] S. Jackson and R. D. Mauldin proved that the Lebesgue measure λ considered on the space of nonempty closed subsets of the unit interval with Hausdorff metric is not countably continuous.

We say that $f : \mathbb{R} \rightarrow \mathbb{R}$ is symmetrically continuous if

$$\lim_{t \rightarrow 0} (f(x+t) - f(x-t)) = 0 \quad \text{for } x \in \mathbb{R}.$$

K. Ciesielski gives an example of symmetrically continuous function that is not countably continuous. The author uses the notion of Sierpiński function. The function $f : A \rightarrow \mathbb{R}$ (where $A \subset \mathbb{R}$) is of Sierpiński type if $f \upharpoonright Y$ is discontinuous for every set $Y \subset A$ of cardinality continuum. In the paper [7] the author observes that if there exists $A \subset \mathbb{R}$ of cardinality continuum, such that $f \upharpoonright A$ is Sierpiński-Zygmund type, then f is not countably continuous. Next the author constructed a symmetrically continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ that "contains" a Sierpiński-Zygmund type function i.e. for some $A \subset \mathbb{R}$ of cardinality continuum $f \upharpoonright A$ is a Sierpiński-Zygmund type.

In the paper [47] S. Solecki proves the following dichotomy result: a Baire one function is countably continuous or "contains" (to be explained in detail in below) some complicated function. Let $g : X_1 \rightarrow Y_1$ and $f : X_2 \rightarrow Y_2$. We write

$g \sqsubseteq f$ if there exist embeddings (i.e. open continuous injections) $\phi : X_1 \rightarrow X_2$ and $\psi : g[X_1] \rightarrow Y_2$ with $\psi \circ g = f \circ \phi$.

Recall that any ordinal number can be considered as a topological space by endowing it with the order topology. Let ω denote the first infinite ordinal. Define the Pawlikowski function $P : (\omega + 1)^\omega \rightarrow \omega^\omega$ by $p(\eta) = \gamma$, where

$$\gamma(n) = \begin{cases} 0 & \text{if } \eta(n) = \omega, \\ \eta(n) + 1 & \text{if } \eta(n) < \omega. \end{cases}$$

Theorem 10.48 ([47]). *Let X be a Souslin space, Y be a separable metric space, and $f : X \rightarrow Y$ be a Baire one function. Then either f is countably continuous or P is "contained" in f , (i.e. $P \sqsubseteq f$).*

This result was generalized by Pawlikowski and Sabok. In the paper [42] the authors prove that if f is a partial Borel function from one Polish space to another, then either f can be decomposed into countably many partial continuous functions, or f contains the countable infinite power of a bijection that maps a convergent sequence together with its limit onto a discrete space.

In [47] the authors consider also decomposition of Baire one functions into continuous functions with closed domain, i.e. piecewise continuity and prove similar dichotomy result using Lebesgue functions L_1 and L_2 defined as follows.

Let Q be the set of all points in 2^ω that are eventually equal to 1. For each $x \in Q$ fix a number $a_x > 0$ so that:

1. if $x, y \in Q, x \neq y$, then $a_x \neq a_y$,
2. $a_x < \frac{1}{3^{n_0}}$ where n_0 is the smallest natural number such that $x(n) = 1$ for $n \geq n_0$.

Let $H : 2^\omega \rightarrow [0, 1]$ be the embedding $H(x) = \frac{x(n)}{3^{n+1}}$, for $x \in 2^\omega$.

The functions $L_1, L : 2^\omega \rightarrow \mathbb{R}$ are defined by:

$$L(x) = \begin{cases} H(x) & \text{if } x \notin Q, \\ H(x) + a_x & \text{if } x \in Q, \end{cases}$$

$$L_1(x) = \begin{cases} 0 & \text{if } x \notin Q, \\ a_x & \text{if } x \in Q. \end{cases}$$

Theorem 10.49 ([47]). *Let X be a separable complete metric space, Y be a separable metric space, and $f : X \rightarrow Y$ be a Baire one function. Then either f is piecewise continuous or one of L, L_1 is "contained" in f (i.e. $L \sqsubseteq f$ or $L_1 \sqsubseteq f$).*

Strongly countably continuous functions were investigated in [20], [21], [37]. Let f be a monotone function that is discontinuous on a dense countable set (for example $f(x) = \sum_{q_n < x} \frac{1}{2^n}$ where $\{q_n : n = 1, 2, \dots\}$ is an enumeration of the rational numbers). Then f is countably continuous but it is not strongly countably continuous (see [8] and [20], Example 1). An interesting construction of an almost everywhere continuous and everywhere approximately continuous function, that is not strongly countably continuous is given in [21].

In the paper [37] Natkaniec constructed an example of additive Darboux function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is strongly countably continuous and discontinuous. The author noted that every finitely continuous and additive function is continuous.

Countably decomposable functions

We say that $f_n : X \rightarrow Y$ converges to $f : X \rightarrow Y$ discretely if for all $x \in X$ there is a positive integer $n(x)$ such that $f_n(x) = f(x)$ for $n \geq n(x)$. If X is complete metric space then the function $f : X \rightarrow \mathbb{R}$ is Baire-one-star if and only if it is a discrete limit of continuous functions (see [8]). It follows that f is a discrete limit of continuous functions if and only if it is piecewise continuous, i.e. there exists a sequence of nonempty closed sets $\{X_n\}_{n=1}^{\infty}$ such $\bigcup_{n=1}^{\infty} X_n = X$ and $f \upharpoonright X_n$ is continuous.

In the paper [18] there was considered the notion of almost monotone convergence. We say that $f_n : X \rightarrow \mathbb{R}$ (where X is a topological space) almost decreases (increases) to f if it pointwise converges to f and for all $x \in X$ there is a positive integer $n(x)$ such that $f_{n+1}(x) \leq f_n(x)$ ($f_{n+1}(x) \geq f_n(x)$) for $n \geq n(x)$. The author proves that if a sequence of continuous functions $f_n : X \rightarrow \mathbb{R}$ almost decreases (increases) to the function f then there exist closed sets $A_n \subset A_{n+1}$ for $n = 1, 2, \dots$, such that $X = \bigcup_{n=1}^{\infty} A_n$ and restricted functions $f \upharpoonright A_n$, $n = 1, 2, \dots$ are upper (lower) semicontinuous.

If X is the real line with the density topology then characteristic functions of Lebesgue measure zero sets are upper semicontinuous (with respect to \mathcal{T}_d) but if $A \subset \mathbb{R}$ is the set with measure zero but non Borel then there is no sequence of approximately continuous functions (i.e. continuous with respect to \mathcal{T}_d) that pointwise converges to its characteristic function. Moreover we have the following equivalence when $X = Y = \mathbb{R}$ with the natural topology.

Theorem 10.50 ([18]). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. The following conditions are equivalent:*

- a) there is a sequence of continuous functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, which almost decreases (increases) to f ;
- b) there is a sequence of nonempty closed sets $\{A_n\}_{n=1}^{\infty}$ such that $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$ and f_n is upper semicontinuous (lower semicontinuous) for $n \in \mathbb{N}$.

Strongly countably \mathcal{A} -functions

Let X, Y be topological spaces and \mathcal{A} be a nonempty family of functions from X to Y . A function $f : X \rightarrow Y$ is said to be a strongly countably \mathcal{A} -function if there is a sequence $(f_n)_{n=1}^{\infty}$ of functions from \mathcal{A} such that $Gr(f) \subset \bigcup_{n=1}^{\infty} Gr(f_n)$. In [19] properties of strongly countably \mathcal{A} -function for some families $\mathcal{A} \subset \mathbb{R}^{\mathbb{R}}$ are considered. In particular, it is proved that the families $\mathcal{A}_{\omega}(\mathbb{R})$ of all strongly countably \mathcal{A} -functions are closed with respect to some operations depending on the properties of the families \mathcal{A} :

Theorem 10.51 ([19]). *Let $D \subset \mathbb{R}^2$ be a nonempty open set and $F : D \rightarrow \mathbb{R}$ be a function. Assume that the family \mathcal{A} is closed with respect to the operation F , i.e. for arbitrary functions $\phi, \psi \in \mathcal{A}$ with $(\phi(x), \psi(x)) \in D$ for all $x \in \mathbb{R}$ the function $F(\phi, \psi) \in \mathcal{A}$. Then for arbitrary two functions $f, g \in \mathcal{A}_{\omega}(\mathbb{R})$ with $(f(x), g(x)) \in D$ for all $x \in \mathbb{R}$, the function $F(f, g)$ belongs to $\mathcal{A}_{\omega}(\mathbb{R})$.*

Corollary 10.52. *Assume that the family \mathcal{A} is closed with respect to addition (subtraction) [multiplication by constant] { multiplication }, then the family $\mathcal{A}_{\omega}(\mathbb{R})$ has the same property. Moreover, if for arbitrary two functions $\phi, \psi \in \mathcal{A}$ with $\psi(\mathbb{R}) \subset \mathbb{R} \setminus \{0\}$ the quotient $\frac{\phi}{\psi} \in \mathcal{A}$, then for arbitrary two functions $f, g \in \mathcal{A}_{\omega}(\mathbb{R})$ with $g(\mathbb{R}) \subset \mathbb{R} \setminus \{0\}$ the quotient $\frac{f}{g}$ belongs to $\mathcal{A}_{\omega}(\mathbb{R})$.*

Proof. Our corollary follows immediately from Theorem 10.51. For the proof of the second part it suffices to observe that the division is defined on the set $D = \mathbb{R} \times (\mathbb{R} \setminus \{0\})$. □

Similarly, from Theorem 10.51 we obtain the following

Corollary 10.53. *Assume that the family \mathcal{A} is closed with respect to the operation $F_1(x, y) = \max(x, y)$ ($F_2(x, y) = \min(x, y)$). Then the family $\mathcal{A}_{\omega}(\mathbb{R})$ has the same property.*

Theorem 10.54 ([19]). *Assume that the family \mathcal{A} is closed with respect to the superposition. Then the family $\mathcal{A}_{\omega}(\mathbb{R})$ has the same property.*

In the case where \mathcal{A} is the family of all continuous functions the above results were obtained in [20]. If \mathcal{A} is the family of all constant functions then evidently a function $f \in \mathcal{A}_\omega(\mathbb{R})$ (we will say that f is strongly countably constant) if and only if the image $f(\mathbb{R})$ is countable.

Denote by Δ the family of all differentiable functions. This family is closed with respect to the addition, subtraction product, division (if the image of the denominator is contained in $\mathbb{R} \setminus \{0\}$) and superposition. It is not closed with respect to the operations max and min. Strongly countably Δ -functions will be called strongly countably differentiable. Characteristic functions of nonmeasurable (in the sense of Lebesgue) sets are strongly countably constant, so also strongly countably differentiable. Evidently such functions are nonmeasurable (so also discontinuous).

Theorem 10.55 ([19]). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then $f \in \Delta_\omega(\mathbb{R})$ if and only if for each nonempty closed set $H \subset \mathbb{R}$ there is an open interval I such that $I \cap H \neq \emptyset$ and there is a differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \upharpoonright (H \cap I) = g \upharpoonright (H \cap I)$.*

Corollary 10.56. *If $f \in \Delta_\omega(\mathbb{R})$ then it is differentiable on an open dense set.*

Corollary 10.57. *If a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is nowhere differentiable then it is not strongly countably differentiable.*

Denote by C_{ap} the family of all approximately continuous functions from \mathbb{R} to \mathbb{R} . Strongly countably C_{ap} -functions will be called strongly countably approximately continuous. Approximate continuity of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ implies measurability (in the sense of Lebesgue) of its graph. So, the graphs of strongly countably approximately continuous functions are measurable.

Theorem 10.58 ([19]). *There is measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is not strongly countably approximately continuous.*

Theorem 10.59 ([19]). *If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is of the first Baire class then it is strongly countably approximately continuous.*

The following problem is natural.

Problem 10.60. Does there exist Baire two function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is not strongly countably approximately continuous?

The positive answer to this problem follows from Corollary 3.4 in [5] (see also [6], the foot of page 160), where authors proved that for each ordinal

$\alpha < \omega_1$ there is a function $f : \mathbb{R} \rightarrow \mathbb{R}$ of the $\alpha + 1$ Baire class for which there is no countable sequence $\{X_n\}_{n=1}^{\infty}$ such that $\mathbb{R} = \bigcup_{n=1}^{\infty} X_n$ and $f \upharpoonright X_n$ is of the α Baire class on X_n . Clearly such function can't be covered by countably many functions Baire class α .

Denote by QC the family of all quasi-continuous functions from \mathbb{R} to \mathbb{R} . Strongly countably QC-functions will be called strongly countably quasi-continuous. We can observe that the graphs of strongly countably quasi-continuous functions are of the first category on the plane \mathbb{R}^2 . Moreover, there is strongly countably quasi-continuous function, not having the Baire property, for example characteristic functions of sets without the Baire property.

Theorem 10.61 ([19]). *If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ has the Baire property then it is strongly countably quasi-continuous.*

We have observed that all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ of the first Baire class are strongly countably approximately continuous and strongly countably quasi-continuous.

Problem 10.62. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function of Baire 1 class. Is it strongly countably approximately continuous and quasi-continuous function i.e, does there exist a sequence of approximately continuous and simultaneously quasi-continuous functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $n = 1, 2, \dots$ such that $Gr(f) \subset \bigcup_{n=1}^{\infty} Gr(f_n)$?

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