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Chapter 1
A Modest Review of a Great Deal of Work

PAUL D. HUMKE

It is both an honor and a pleasure and, I might say, somewhat daunting to write a paper describing the mathematical career of a mathematician who has made such major contributions to both the scientific literature and to his chosen profession. I met Jan Lipiński in 1978 at the International Congress in Helsinki. He came to a session at which I presented a poster and was very encouraging. I recall vividly; it meant a great deal to the beginner that I was. Two days later we met again at a dinner we'd arranged for several real analysts attending the conference. Jan was there as was my wife Bonnie, Andy and Judy Bruckner, Mik Laczkovich, Krishna Garg, Ladislav Mišík, and Dan and Mudite Waterman. All of us have remained in close contact throughout our lives, and all have certainly enriched my own life immeasurably both scientifically and personally.

In 1987 a group of us here in the United States applied for a series of grants to invite several European real analysts to a Special Real Analysis Session at the Annual Joint Meeting of the A.M.S. in San Antonio, Texas. Jan was among those invited, as was one of Jan's former students, Władek Wilczyński. After the meeting Jan and I flew back to Minnesota where he stayed with me and my family in Minnesota for a week. We talked a great deal of mathematics during that week and got to know each other pretty well. Jan enchanted our children with stories and "tricks;" his *separating finger trick* is still a favorite of our eldest son, Eric who has demonstrated it to all of his nieces and nephews and now to his own children.

Jan was a first student of Zygmunt Zahorski and much of Jan's mathematical work reflects the style and delicacy of the Zahorski school. Although Zahorski wrote his dissertation under the direction of Tadeusz Ważewski, he had early worked with both Mazurkiewicz and Banach and his professional life reflects a devotion to working on hard problems and developing deep understanding of the intricacies of sets and functions. An insightful reflection on Zahorski's life and work was written by Władek Wilczyński, see [62], to whom I am indebted for introducing me to Zahorski during the Warsaw International Congress in 1983.

Several of Jan's earliest papers solve problems left unresolved in papers published by Zahorski, but he quickly branched out, successfully attacking problems in allied areas of real analysis and set theoretic topology. The forty seven papers reflected in this review reveal a sparkling energy and creative spirit that characterize their author. Jan has fathered eight mathematical children, twenty nine mathematical grand children and thirty eight mathematical great grandchildren. It is a wonderfully diverse and dedicated family of professional mathematicians who continue to push back the frontiers of real analysis.

In the sections that follow I have chosen four categories within which to discuss Jan's scientific work and most of that work is at least touched upon in the sequel. Any division of a body of intellectual work is artificial and, in some sense can detract from the overall vision of the whole. Still, I found it helpful to make some categorization and I hope it is not distractive to the reader. Inevitably, there is overlap between the sections, but frequently the overlap reflects Jan's new ways of looking at old ideas. We'll begin where Jan himself began in the early 1950's by looking at Jan's contributions to understanding derivatives.

1.1 Derivatives

The papers I've categorized as Jan's derivative body of work[1] is substantial and his interest in derivatives is clearly career long. And a solid portion of this work involves the hierarchy of classes of F_σ sets and of Baire 1 functions introduced by Zahorski in his celebrated 1950 paper, *Sur la première dérivée*, [64] . I'll first give the briefest of descriptions of these classes; a characteristically elegant and complete treatment can be found as Chapter 6 of Andy Bruckner's book, *Differentiation of Real Functions*, [2]. I'll use that treatment here, para-

[1] See [39], [34], [44], [32], [28], [27], [24], [23], [19], [16], [17], [14].

phrasing and giving enough background within which to place some of Jan's contributions.

Definition 1.1. Let $\emptyset \neq E \in F_\sigma$. Then E is said to belong to class

M_0 if every point of E is a bilateral accumulation point of E;
M_1 if every point of E is a bilateral condensation point of E;
M_2 if every one-sided neighborhood of each point of E intersects E in a set of positive measure;
M_3 if for each $x \in E$ and each sequence of closed intervals, $\{I_n\}$ converging to x but not containing x such that $\lambda(I_n \cap E) = 0$ for each n, we have $\lim_{n\to\infty} \lambda(I_n)/\mathrm{dist}(x, I_n) = 0$;
M_4 if there exists a sequence of closed sets, $\{K_n\}$ and a sequence of positive numbers η_n such that $E = \bigcup K_n$ and for each $x \in K_n$ and each $c > 0$ there is a number $\varepsilon = \varepsilon(x,c)$ such that if h and k satisfy $hk > 0$, $h/k < c$, $|h+k| < \varepsilon$, then

$$\frac{\lambda(E \cap (x+h, x+h+k))}{|k|} > \eta_n.$$

M_5 if every point of E is a point of density of E.

These classes of sets give rise to corresponding classes of Baire 1 functions in a most natural way.

Definition 1.2. A function $f \in M_i$ if every associated set of f is in class M_i for $i = 0, 1, \ldots, 5$.

By associated sets of f we mean the sets of the form $\{x : f(x) < \alpha\}$ or $\{x : f(x) > \alpha\}$.

Let $\mathcal{D}B_1$ be the class of Darboux Baire 1 functions and $\mathcal{C}_{ap}$ - the class of the approximately continuous functions. Two foundational results concerning the Zahorski Classes (see [2], Theorem 1.3, Corollary 2.4 and Theorem 2.5) are the following:

Theorem 1.3.

$$\mathcal{C}_{ap} = \mathcal{M}_5 \subsetneq \mathcal{M}_4 \subsetneq \mathcal{M}_3 \subsetneq \mathcal{M}_2 \subsetneq \mathcal{M}_1 = \mathcal{M}_0 = \mathcal{D}B_1.$$

In the next theorem, Δ' denotes the set of derivatives and $b\Delta'$ denotes the set of bounded derivatives.

Theorem 1.4.

$$\Delta' \subsetneq \mathcal{M}_3 \textit{ and } b\Delta' \subsetneq \mathcal{M}_4.$$

David Preiss in [58] and David and Maria Tartaglia in [57] also had a good deal to say about this story and a relatively complete list of related papers can be found by searching for papers citing Zahorski, [64].

Zahorski defined the classes described in the paragraph above and among many other results, showed that $\Delta' \subsetneq \mathcal{M}_2$ and $b\Delta' \subsetneq \mathcal{M}_3$. Several questions remained, however and Jan answered one of these in [14] where he proves the following theorem.[2]

Theorem 1.5. *There is a set $E \in M_3$ such that E is not an associated set for any finite derivative.*

The condition required for a set to belong to the class M_4 is rather complicated, but is not dissimilar to the original formulation of the M_3 condition. The latter was simplified to that given above, and Zahorski asked whether a similar simplification could be made for M_4 by taking the η to be dependent only on x and not on n. In [16] Jan showed that this is not possible via the following theorem.

Theorem 1.6. *Un condition nécessaire qu'un ensemble linéaire E soit identique á l'ensemble des pointes en lesquels une fonction dérivée, bornée en module, prend une valeur finie donnée est que E soit un G_δ contenant tous les points d'accumulation en mesure.*

Several other Zahorski type derivative results can be found the following Section 1.2. But now I'll turn to derivative results of different types.

In 1957 Jan published a paper in *Colloquium Mathematicum*, [17] in which he studies monotone jump functions.

Definition 1.7.

1. If $f : \mathbb{R} \to \mathbb{R}$, then $D_\infty(f') = \{x : f'(x) = \infty\}$.
2. If $f : \mathbb{R} \to \mathbb{R}$ is bounded and non-decreasing, then f is called a *jump function* provided
   a. $\sum_{\mathbb{R}} f(x+0) - f(x-0) < \infty$, and
   b. $f(b-0) - f(a+0) = \sum_{a<x<b} f(x+0) - f(x-0)$ whenever $a < b$.

He proves the following two theorems.

Theorem 1.8. *If f and g are non-decreasing jump functions and $\phi = f - g$, then there is an F_σ null set E such that $D_\infty(f') \subset E$.*

Theorem 1.9. *If E is any F_σ null set, then there is a non-decreasing jump function, f such that $f'(x) = \infty$ for every $x \in E$.*

[2] Here and in other places I restate the actual result using the terminology of this paper.

In particular, these results show that a necessary and sufficient condition for a set $E \subset D_\infty(f')$ for some jump function f is that E is a subset of an F_σ null set. This provided a complete answer to a question posed by E. Marczewski in 1955 in [56].

But there is a bit more to this story for in 1971, R. Sikorski learned of a theorem by A. J. Lohwater, [52] where Lohwater uses the theory of cluster sets and analytic functions to prove the following theorem for real functions.

Theorem 1.10 (Lohwater). *If $f : \mathbb{R} \to \mathbb{R}$ is singular, then there exists an F_σ null set, E for which $D_\infty(f') \subset E$.*

Here, *singular* means f is of bounded variation and $f'(x) = 0$ almost everywhere.

Jan wanted a purely real analysis proof of this theorem in the spirit of Theorem 1.8 and an example in the spirit of Theorem 1.9 above. In 1972, in [34] he succeeds, proving both the Lohwater Theorem 1.10 and also the following, using only classical real analysis techniques.

Theorem 1.11. *If E is a subset of an F_σ null set, then there is a singular function $f : \mathbb{R} \to \mathbb{R}$ with $E \subset D_\infty(f')$.*

This last result is then coupled with Theorem 1.10 to prove the following.

Theorem 1.12. *A necessary and sufficient condition for a set $E \subset D_\infty(f')$ for some singular function f is that E is a subset of an F_σ null set.*

In the following, $D(f)$ denotes the set of points of discontinuity of a function f and $Z(f) = \{x : f'(x) = 0\}$. In this work, Jan is tackling several problems posed by Solomon Marcus in [55] and [54]. In 1940, in [61], Tolstov had shown that if E is any null set, then there is a continuous, non-decreasing, differentiable function f with $E \subset D_\infty(f)$. Krishna Garg and Solomon Marcus wrote several papers investigating the relationships between the sets $D(f)$, $D_\infty(f)$, $Z(f')$ and others. In [28], Jan answers two of the Marcus questions in proving the following quite conclusive theorem.

Theorem 1.13. *Let f be a non-increasing everywhere differentiable function. For sets A and B to be the sets $D(f)$ and $D_\infty(f)$ respectively, it is necessary and sufficient that A is denumerable, B is a G_δ null set, and $A \subset B$.*

In [54] Marcus defines $f : [a,b] \to \mathbb{R}$ to be a *Pompeiu function* if f has a bounded derivative, $Z(f')$ is dense in $[a,b]$ and f is constant on no interval. He then investigates various properties of Pompeiu functions and poses a number of questions concerning their behavior. Independently, Andy Bruckner and Jan solved many of these questions. Two results of Jan, [32] are the following.

Theorem 1.14. *For Pompeiu functions f the sets $Z(f')$ are characterized as dense boundary M_4 sets.*

Theorem 1.15. *There exist Pompeiu functions for which $Z(f')$ is of measure zero.*

I'll finish this section with a related result Jan proved in 1963. In Theorem 1.13 that the set A must denumerable follows from the fact that the discontinuity set of any differentiable function is denumerable. Z. Zahorski, in [63] asked what could be said if "derivative" was replaced by "approximate derivative" and "continuity" by "approximate continuity." In 1963 Jan answers this in his paper [27].

Theorem 1.16. *Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is approximately differentiable everywhere. Then the set of points where f is approximately discontinuous is a first category null set.*

In the next section I'll focus on the many examples and counterexamples found in Jan's work. Some, I've already discussed in the context of derivatives, but I've reserved many of the best.

1.2 Examples and Counterexamples

One of my favorite mathematical quotes is attributed to Poincaré in a letter he wrote in 1913; it is paraphrased here:

> Heretofore when a new function was invented, it was for some practical end; today they are invented expressly to put at fault the reasonings of our fathers, and one will never get from them anything more than that.

Legend has it that this was written in a frustrated response to Dirichlét's publication of the function we now label with his name. But examples, perhaps particularly pathological ones, help hone and sharpen our understanding. These have played an important role in Jan's research career and I decided to highlight this portion of his work with a section of its own.[3]

In [16] Jan constructed a single set revealing that answers to two separate questions by Choquet and Zahorski were negative. I described the Zahorski conjecture in Section 1.1; Andy Bruckner, [2] describes the situation surrounding Choquet's question as follows:

[3] See, for example [43], [39], [47], [13], [38], [34], [35], [45], [27], [26], [16], [14].

> Choquet defined a notion of accumulation of measure of a set E at a point x_o. This condition is one satisfied by each level set of a bounded derivative. Choquet's condition is also sufficient for a set of type G_δ to be a level set of some bounded derivative, provided certain auxiliary conditions are met....

Choquet asked whether these "auxiliary conditions" were actually necessary or whether a general result was true. Jan's example showed additional conditions were indeed required.

Jan returned to the Zahorski classes in 1990 when studying real valued functions of two real variables, [43]. Among other results are two constructions. Here, if $f : \mathbb{R}^2 \to \mathbb{R}$, then the sections of f are denoted by $f_x(y)$ and $f^y(x)$.

Theorem 1.17.

1. *There is a bounded non-measurable function* $f : \mathbb{R}^2 \to \mathbb{R}$ *such that* $f_x, f^y \in \mathcal{M}_3$ *for every* $x, y \in \mathbb{R}$.
2. *There is a bounded non-Borel measurable function* $f : \mathbb{R}^2 \to \mathbb{R}$ *such that* $f_x, f^y \in \mathcal{M}_4$ *for every* $x, y \in \mathbb{R}$.

In 1963, [27] Jan answered yet another question of Zahorski, this time asked in 1948, in [63]. Here he first proves the following theorem.

Theorem 1.18. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is approximately differentiable everywhere, either finite or infinite. Then the set at which* f *is approximately discontinuous is of the first Baire category and of measure zero.*

Subsequently he shows that there is an approximately differentiable function which is approximately discontinuous at a set of cardinality $2^{\aleph_0}$. Andy Bruckner and Casper Goffman jointly authored a very nice survey focusing on approximate differentiation; this appeared in the *Real Analysis Exchange* in 1981, see [1].

Jan's career long interest in questions of continuity and connectedness are well represented in this section of examples and counterexamples. In 1972, in [35] he proves the existence of a pathological function of two real variables that has quite tame sections.

Theorem 1.19. *There is a (Lebesgue) non-measurable function* $f : [0,1]^2 \to \mathbb{R}$ *whose sections,* $f_x(y)$ *and* $f^y(x)$ *are both Baire Class 1 and Darboux.*

In reviewing Jan's paper, [38] for the *Mathematical Reviews* Andy Bruckner writes

> This provides an elegant answer to a question raised by the reviewer and J. Ceder.

The *elegant* answer Andy was referring to began with the following theorem.

Theorem 1.20. *Let g be a Darboux function which is not constant on any subinterval, I of its domain such that the set $A = \{\alpha \in \mathbb{R} : g^{-1}(\alpha)$ is perfect$\}$ is dense in $g(I)$. Then there exists a homeomorphism h from I onto I such that for every countable, dense set $D \subset \mathbb{R}$ there is a function d which takes an every real value in every subinterval of I such that D is the range of $(g \circ h) + d$ and $(g \circ h) + d$ is constant almost everywhere.*

Then, by taking g to be continuous, one obtains an example of a continuous function $f = g \circ h$ with the property that to each countable set $D \subset \mathbb{R}$ corresponds a measurable Darboux function d so that $f + d$ has range D. This provided the elegant answer Andy referred to.

1.3 Point Sets

Of course there is a great overlap between this section and the others, but with the integrated nature of research, such overlap is inevitable. So rather than attempt to avoid it in some artificial way, I chose section topics that highlight particular strengths in Jan's overall research program. And I hope you'll agree that the view of his work from the perspective of general *functions and point sets* is a particularly useful view. There are fifteen papers[4] from which I've extracted results for this section.

Early, in 1961 there were two papers in [25], [41] which portended future investigations of distinguishing sets via function behavior. Here, he first defines two properties of pairs of sets in $\mathbb{R}$.

Definition 1.21. If F_1 and F_2 are $F_{\sigma\delta}$ sets, then the pair (F_1, F_2) is said to have property

- $\mathcal{P}$ provided there are two disjoint F_σ sets E_1 and E_2 such that $F_1 \subset E_1$ and $F_2 \subset E_2$.
- $\mathcal{Q}$ provided there is a sequence of continuous functions $\{f_n : \mathbb{R} \to \mathbb{R}\}_{n=1}^{\infty}$ such that $F_1 = \{x : f_n(x) \to +\infty\}$ and $F_2 = \{x : f_n(x) \to -\infty\}$.

The paper is devoted to showing that conditions $\mathcal{P}$ is equivalent to $\mathcal{Q}$.

In 1973 Krishna Garg in [12] published a substantial paper investigating level sets of Darboux functions. In some regards this was an extension of earlier work by Sierpiński, [59]. Jan had already developed the intuition and technical expertise to tackle the open questions Krishna published in [12]. We need a modicum of notation to understand the dynamics here.

[4] See [49], [39], [47], [46], [37], [36], [34], [40], [44], [29], [26], [25], [41], [19], [14].

Definition 1.22.

1. We define a set $E \in G_\delta^+$ if there is an $A \in G_\delta$ and a countable set B such that $E = A \cup B$.
2. If $f : \mathbb{R} \to \mathbb{R}$ is a function and S is a collection of sets, then

$$Y_S(f) = \{y : f^{-1}(y) \in S\}.$$

For example, let P denote the perfect sets and C denote the continuous functions. Then $E \in \{Y_P(f) : f \in C\}$ simply means there is a continuous function f such that $E = \{y : f^{-1}(y) \text{ is perfect}\}$. In [59], Sierpiński proved:

Theorem 1.23 (W. Sierpiński, [59]). *If $f \in C$, then each $E \in Y_P(f)$ is an $F_{\sigma\delta}$ set; that is, $\{Y_P(f) : f \in C\} \subset F_{\sigma\delta}$.*

As one among many such theorems, Garg then proved the following theorem.

Theorem 1.24 (K. Garg, [12]). *If D denotes the class of Darboux functions, then $\{Y_P(f) : f \in D\} \subset G_\delta^+$.*

Each of Krishna's theorems was, like the earlier Sierpiński result, not a characterization, but rather a set inequality. It was the purpose of [37] to show that each of those inequalities was actually an equality and thus a characterization of the class in question. For example, in [37], Corollary 4, Jan shows:

Theorem 1.25. $\{Y_P(f) : f \in D\} = \{Y_P(f) : f \in C\} = G_\delta^+$.

In 1977, Jan couples similar techniques and ideas with his long held interests in continuity and connectedness to confirm a conjecture of Jack Ceder. The main result of [47] is the following.

Theorem 1.26. *If $C \subset D \subset \mathbb{R}$ are G_δ sets, then there is a Baire 2 function, f for which C is the set of continuity points of f and D is the set of Darboux points of f.*

An additional paper I'll include in this portion on functions and point sets is [29]. This paper contains a nice example, and so perhaps could have been included in Section 1.2. However, the theorem he proved in that paper together with the example leaves a bit of room for further work, so I've kept them side-by-side to emphasize the fact that there is something left to do. To begin, let $f_n : [0,1) \to [0,1)$ be defined by $f_n(x) = nx - [nx]$ where $[y]$ means the integer part of y. For a measurable set $E \subset [0,1)$, let $E^n = f_n^{-1}(E)$. Jan proves the following two theorems.

Theorem 1.27. *If $\{E_n\}$ is a sequence of measurable sets such that $\lambda(E_n) > \delta > 0$ for infinitely many $n \in \mathbb{N}$, then for every increasing sequence of integers, $\{i_n\}$, $\lambda(\limsup E_n^{i_n}) = 1$.*

Theorem 1.28. *For every fixed increasing sequence of integers, $\{i_n\}$ and each $\varepsilon > 0$ there is a sequence of measurable sets, $\{E_n\}$ such that*

1. $\sum \lambda(E_n) = +\infty$
2. $\lambda(\bigcup E_n^{i_n}) < \varepsilon$, *and*
3. $\lambda(\limsup E_n^{i_n}) = 0$.

There are three additional papers I'll make some remarks about here and the first of these, [46] concerns transfinite limits either of functions.

Here Jan considers families of mappings, $F = \{f : E \to Y\}$ where E is a fixed set and Y is a metric space.

Definition 1.29. Such a family is *closed* with respect to transfinite convergence if whenever $\{f_\alpha : \alpha < \omega_1\} \subset F$ converges pointwise to a function f, then $f \in F$.

Under the assumption of CH, Jan then shows several classical families of mappings to be closed in this sense. These include the bounded functions, increasing functions, differentiable functions and several others.

The second, [26] is a contribution to the general topic of the algebra of continuous functions, $C[a,b] = \{f : [a,b] \to [a,b]\}$. Given $f \in C[a,b]$, define $S(f) = \{g \in C[a,b] : f \circ g = g \circ f\}$. Jan proves the following.

Theorem 1.30. *For every $f \in C[a,b]$, $S(f)$ is infinite. Moreover, if f is strictly increasing, then the cardinality of $S(f)$ is $2^{\aleph_o}$.*

Theorem 1.31. *There is a function $f : [a,b] \to [a,b]$ with two simple discontinuities for which $S(f) = \emptyset$.*

Finally, in [40] Jan and Tibor Šalát define a general Banach Indicatrix function and study its measurability. More specifically, let X and Y be arbitrary sets and $f : X \to Y$.

Definition 1.32. The Banach Indicatrix of f is $\tau_f : Y \to \mathbb{N} \cup \{\infty\}$ defined as

$$\tau_f(y) = \begin{cases} \mathrm{card}(f^{-1}(y)), & \text{if } f^{-1}(y) \text{ is finite,} \\ \infty, & \text{otherwise.} \end{cases}$$

Among the theorems proved are the following.

Theorem 1.33. *If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then τ_f is in Baire class 2.*

Theorem 1.34. *If $f : \mathbb{R} \to \mathbb{R}$ is a Baire function, then τ_f is Lebesgue measurable.*

1.4 Generalized Continuity

Jan has written more than ten papers[5] concerning various notions of generalized continuity. Too, this topic proved a rich source for his active collaboration with other real analysts, and his interest in notions of generalized continuity can be seen as threading a good portion of his research career, beginning with [33] in 1968 and extending through [48] in 1993. As with other sections of this paper, the setting is not always the real line $\mathbb{R}$, but sometimes a general topological space; I'll try to keep the more general notions somewhat separate from those specifically related to those of the real line, $\mathbb{R}$, but, of course, this is not always completely possible.

In [33], Jan began an investigation of the relationship between functions that are continuous and functions that preserve connectedness. This relationship is important for a variety of reasons not the least of which is that derivatives of functions $f : \mathbb{R} \to \mathbb{R}$ preserve connectedness, but need not be continuous. Here are the definitions and theorems Jan proves in 1971.

Definition 1.35. Suppose that X is a topological space and $f : X \to \mathbb{R}$. Then,

1. f is said to have property (G) if there exists a dense set $Y \subset \mathbb{R}$, such that the set $f^{-1}(y)$ is closed for each $y \in Y$.
2. f is said to have property (D) if it maps connected sets onto connected sets.

Theorem 1.36. *If X is locally connected, then f is continuous if and only if f possesses both properties (G) and (D).*

Theorem 1.37. *If, for every open subspace $A \subset X$, every $f : A \to \mathbb{R}$ and possessing the properties (G) and (D) is continuous, then X is locally connected.*

A great deal has been done in this area since 1971 and I will mention several more of Jan's contributions in the next few paragraphs, but before listing those I would like to insert some purely real analysis references. Mike Evans and I published a general audience paper on the subject in the *Monthly* in 2009, [6] and a more technical paper with a reasonable bibliography in [7]. A wonderful introduction to the subject can be found in Andy Bruckner's classic, [2] while a paper illustrating the significance of the study is Jan Malý's paper, [53]. The list of real analysts who have written on this topic is both broad and long, yet much still needs to be discovered.

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[5] See [4], [5], [8], [9], [11], [18], [20], [33], [48], [51].

Three of the last five papers Jan published in the area of generalized continuity were coauthored with Janina Ewert, [9], [8], [11], one was coauthored with Tibor Šalát, [51] and the final paper he wrote by himself, [48]. In these papers, written over a period spanning two decades, Jan and his coauthors study the relationship between points of continuity, points of quasicontinuity and points of cliquishness. Definitions follow.

For *quasicontinuity*, let X and Y be general topological spaces and let $x \in X$.

Definition 1.38. A function $f : X \to Y$ is said to be *quasicontinuous at x* if for every pair of neighborhoods, U of x and V of $f(x)$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$.

For *cliquish*, let X be a topological spaces, Y be a metric space with metric, ρ and let $x \in X$.

Definition 1.39. A function $f : X \to Y$ is said to be *cliquish at x* if for every $\varepsilon > 0$ and neighborhood U of x there is a nonempty open set $W \subset U$ such that whenever $w_1, w_2 \in W$, then $\rho(f(w_1), f(w_2)) < \varepsilon$.

Definition 1.40. The sets of points of continuity, quasicontinuity and cliquishness are denoted $C(f)$, $E(f)$, and $A(f)$ respectively.

It is clear from the definitions that $C(f) \subset E(f) \subset A(f)$. Two results from the Lipiński-Šalát paper, [51] relate the nature of the individual sets $E(f)$ and $A(f)$.

Theorem 1.41. *Let X be a topological space and Y be a metric space. Then, for an arbitrary function $f : X \to Y$ $A(f)$ is a closed subset of X.*

If both X and Y are metric spaces, they have a full characterization, namely:

Theorem 1.42. *If X and Y are metric spaces, then there exists a function $f : X \to Y$ with $A(f) = A$ if and only if A is closed.*

Further, if $X = \mathbb{R}^n$ is a Euclidean space and $Y = \mathbb{R}$, then a similar characterization is proved for $E(f)$, namely:

Theorem 1.43. *If $X = \mathbb{R}^n$ and $Y = \mathbb{R}$, then a set $E \subset X$ is the set of quasicontinuity points for a function $f : X \to Y$ if and only if $\operatorname{int}(\overline{E}) \setminus E$ is a set of the first Baire category.*

In [11], Ewert and Lipiński build on these results using the fact that $A(f) \setminus C(f)$ is of the first Baire category to prove the following theorem.

Theorem 1.44. *Suppose $X = \mathbb{R}^n$ and $Y = \mathbb{R}$ (or alternately that X and Y are real normed linear spaces and Y is a Baire space). Then whenever C, E, and A are sets with*

1. *$C \subset E \subset A = \overline{A}$, and*
2. *$A \backslash C$ is of the first Baire category,*

then there is a function $f : X \to Y$ such that $C = C(f)$, $E = E(f)$, and $A = A(f)$.

The next two papers in this program, also by Ewert and Lipiński, [9], [8] are more technical, but continue the investigation of the relationship between the sets $C(f)$, $E(f)$ and $A(f)$. Too, they tend to assume more specific conditions on the underlying spaces X and Y. But they are interesting and reveal insight into several classical theorems. I'll give you a taste.

Theorem 1.45. *Let X be a topological space which is the union of two disjoint dense subsets, and let Y be a metric space with at least one accumulation point. Then for each decreasing sequence $\{W_n : n = 1, 2, \dots\}$ of open subsets of X and each E satisfying the inclusions*

$$C = \bigcap_{n=1}^{\infty} W_n \subset E \subset \bigcap_{n=1}^{\infty} \overline{W_n} = A$$

there is a function $f : X \to Y$ such that $C = C(f)$, $E = E(f)$, and $A = A(f)$.

The applications presented are using $Y = \mathbb{R}$ with the usual topology and $X = \mathbb{R}$ with the density topology. In this instance "quasicontinuity" is referred to as "density-quasicontinuity" etc. They first note the following theorem.

Theorem 1.46. *A function $f : \mathbb{R} \to \mathbb{R}$ is measurable if and only if f is density-cliquish.*

Using Theorem 1.45 and the fact that "approximate continuity" is equivalent to "density-continuity" they give a simple proof of Denjoy's Theorem.

Denjoy's Theorem *A function is Lebesgue measurable if and only if it is approximately continuous almost everywhere.*

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PAUL HUMKE

Department of Mathematics, St. Olaf College

Northfield, Minnesota 45701

E-mail: humke@stolaf.edu