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Relational Representation of Uncertain and Imprecise Time Assessments: An Application to Artworks Dating

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Abstract

Imprecision and uncertainty appear together in many situations of real life and therefore soft computing techniques must be studied to tackle this problem. Imprecise and uncertain values are usually expressed by means of linguistic terms, specially when they have been provided *by* a human being. This is also the case of temporal information where, in addition to handling time constraints, we may also have both uncertainty and imprecision in the description, like in the sentence *"It is very possible that Giotto's Crucifix was painted by 1289"*. To manage both uncertainty (very possible) and imprecision (by 1289) in a separate way would lead to a quite complicated computation and a lack of comprehension by the users of the system. Because of these reasons, it is very desirable that both sources of imperfection of time values are combined into a single value which appropriately describes the intended information. In this work, we extend our previous research on this topic and we study how to adapt it to relational systems in order to be useful. The final goal is obtaining normalized fuzzy values that provide an equivalent information about the described temporal fact than the original ones, for making it possible to store and manage them in a fuzzy relational database. On the other hand, there will be some situations where more than one expert opinion about a time period must be taken into account and we need to find a representative value of them all in order to be stored and managed. For the sake of simplicity, comprehensibility and the efficiency in computation (when using trapezoidal representation), the fuzzy average is used to find such a representative value.
Keywords: Fuzzy Uncertainty, Fuzzy Relational Database, Fuzzy Time Interval, Fuzzy Temporal Database

1. Introduction

Imprecision and uncertainty coexist in many applications. More often than desirable, a given datum is affected by imprecision and uncertainty at the same time. For example, if we have the information *"It is very possible that Giotto's Crucifix was painted by 1289"* two measures of imperfect information are supplied; on the one hand, the information *by 1289* is fuzzy in the sense that it could mean *from the end of 1288 to the end of 1289 but more possibly in 1289*; on the other hand, *very possible* measures to what point it is true the previous assertion. To work with both measures at the same time is quite impossible in the framework of relational temporal databases, where data (tables) take part in both relational operations (like cartesian product, selection, join, groupping,...) and fuzzy temporal operations (like before, after, between, at the same time...).

For these reasons it is specially important to solve this problem when fuzziness and uncertainty refers to time.

Since temporal data are stored in a relational way (22) (11), it is of great importance that they are normalized for the sake of simplicity in the representation, management, and understanding. Take into account that data are not only stored, but manipulated in many ways where normalization is a must.

All these considerations led us to find a mechanism (13) to:

- represent uncertainty in linguistic terms
- combine such uncertainty with the imprecision of the datum itself.
- transform the resulting value into an equivalent fuzzy normalized one.

That is, we start with an information like *"It is very possible that the picture was painted by 1289* and we want to obtain something like *"The picture was painted around 1289 "* (where *around 1289* has a wider range than *by 1289* due to the fact that we have omitted uncertainty from our assertion).

Another problem that may arise is that some experts give an opinion about the same work, originating different time intervals. Moreover, it is very common to find uncertainty in the expert assessments about the date on which a work of art was made; when the work is very old, many different assessments can be found and most of them are fuzzy and uncertain at the same time.

For example, if we look for Giotto Crucifix in Internet, the following assessments can be found: *"... the magnificent crucifix by Giotto, who, probably painted it between 1288 and 1289"*, *"wood painting... between 1287 and 1288"*, *"scholars suspect Giotto may have painted the crucifix in his early years"*, *"...has been dated to the same time as the frescoes in Padua"* where uncertainty (probably painted, may have painted) on fuzzy or crisp intervals, time labels (early years) and fuzzy time comparators (the same time as) appear. In this case, we need to find a consensus among all these time appreciations in order to be able to represent and manipulate such information properly.

It is important to note that we show in this paper an application of our theoretical results to art works dating but there are many areas where these ideas can be useful, like ancient buildings or documents, crime evidences, eyewitnesses assessment,... In summary, any problem where it is necessary jointly deal with imprecise and uncertain time values.

The paper is organized as follows. Section 2 is devoted to present all the preliminary concepts needed to understand the rest of the paper: representation of fuzzy values, information functions, fuzzy numbers transformation and the relational representation of fuzzy time. In section 3, the problem of finding a consensus when more than one time assessment is found, is addressed. We need a unique fuzzy time interval which represents all of them, in order to be stored and managed properly. Section 4 shows how to represent uncertainty and imprecision in a relational way , extending data tables suitably. Section 5 describes the implementation of these results on our prototype of Fuzzy Object-Relational DBMS, including many examples of data definition, inserting and different types of queries mainly focused on fuzzy time intervals. Finally, section 6 is devoted to show the most interesting conclusions of this work and points out some future research lines to explore.

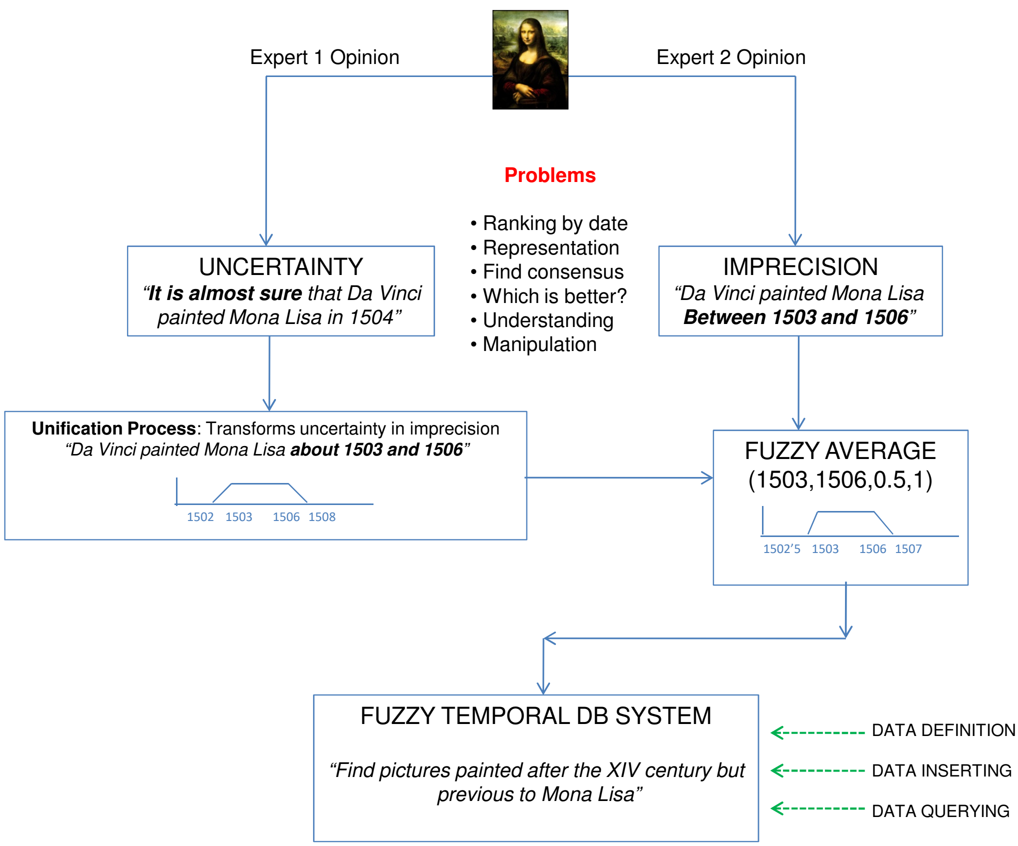

Figure 1: Summary of the whole process.

2. Preliminary Concepts

2.1. Fuzzy values representation

A fuzzy value is a fuzzy representation about the real value of a property (attribute) when it is not precisely known. We will use $\widetilde{R}$ to denote the set of fuzzy numbers, and $h(A)$ to denote the height of the fuzzy number A. For the sake of simplicity, we will use capital letters at the beginning of the alphabet to represent fuzzy numbers.

The interval $[a_\alpha, b_\alpha]$ (see figure 2) is called the α-cut of A. Therefore, fuzzy numbers are fuzzy quantities whose α-cuts are closed and bounded intervals: $A_\alpha = [a_\alpha, b_\alpha]$ with $\alpha \in (0, 1]$. The set $Supp(A) = \{x \in R \mid A(x) > 0\}$ is called the *support set of* A[1]. If there is, at least, one point x verifying $A(x) = 1$ we say that A is a *normalized* fuzzy number.

A widely used fuzzy number representation is the trapezoidal one (7) , where a fuzzy number is completely characterized by four parameters (m_1, m_2, a, b) and the height $h(A)$, as figure 2 shows. The interval $[m_1, m_2]$ (i.e, the set $\{x \in Supp(A) \mid \forall\, y \in R,\ A(x) \geq A(y)\}$) will be called *modal set*. The values a and b are called left and right *spreads*, respectively.

The basic idea is that when a fuzzy number is not normalized, the situation can be interpreted as a lack of confidence in the information provided by such a number (2).

[1] In the rest of the paper, for the sake of simplicity, $A(x)$ will stand for $\mu_A(x)$ for every A.

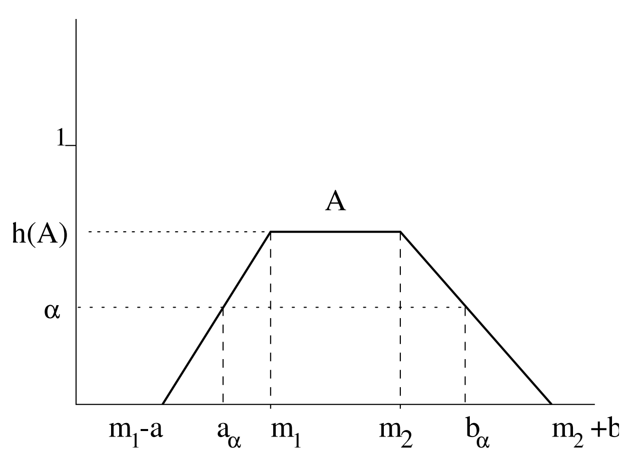

Figure 2: *Trapezoidal Fuzzy Number*

In fact, the height of the fuzzy number could be considered as a certainty degree of the represented value, and this implies that normalized fuzzy numbers represent imprecise quantities on which we have complete certainty.

2.2. Information Measure on Fuzzy Values

As pointed out in the previous section, we are going to translate fuzzy uncertainty into imprecision under certain conditions. The most important of these conditions is that the amount of information provided by the fuzzy number remains equal before and after the transformation. Therefore, the first step is to define an information function for fuzzy numbers.

In (12), we propose an axiomatic definition of information, partially inspired in the theory of generalized information given by Kampé de Fériet (16) and that can be related to the precision indexes (8) and the specificity concept introduced by Yager in (27).

Definition 1. *Let* $\mathcal{D} \subseteq \widetilde{R} \mid R \subseteq \mathcal{D}$*; we say that the application I defined as (13):*

$$I : \mathcal{D} \longrightarrow [0, 1]$$

is an **information** *on* $\mathcal{D}$ *if it verifies:*

1. $I(A) = 1, \ \forall \ A \in R$
2. $\forall \ A, B \in \mathcal{D} \ \mid \ h(A) = h(B)$ *and* $A \subseteq B \Longrightarrow I(B) \leq I(A)$.

The information about fuzzy numbers may depend on different factors, in particular, on imprecision and certainty. In this work, we focus on general types of information related only to these two factors.

To compute a measure of the imprecision contained in a fuzzy number, we will consider a measure of the imprecision of its α-cuts:

$$\forall \ A \in \widetilde{R}, \ f_A(\alpha) = \begin{cases} b_\alpha - a_\alpha \, if \ \alpha \leq h(A) \\ 0, \ otherwise \end{cases}$$

From this imprecision function on the α-cuts, we define the total imprecision of a fuzzy value as a combination of the imprecision in every level α. We will consider that $f_A(0)$ is the length of the support set.

Definition 2. *The* **imprecision** *of a fuzzy number is defined as follows (13):*

$$f : \tilde{R} \longrightarrow R_0^+$$

$$\forall\ A \in \tilde{R},\ f(A) = \int_0^{h(A)} f_A(\alpha)\ d\alpha$$

That is, the imprecision function f coincides with the area below the membership function of the fuzzy value, and can also be expressed as follows:

$$\forall\ A \in \tilde{R},\ f(A) = \int_{m_1-a}^{m_2+b} A(x)\ dx$$

With respect to the height (certainty) and the imprecision of a fuzzy value, we use the following general type of function on $\tilde{R}$ (12):

$$I_{\mathcal{F}} : \tilde{R} \longrightarrow [0,1]$$

$$I_{\mathcal{F}}(A) = \mathcal{F}(h(A), f(A))$$

which depends on the certainty (height) and the imprecision (area below the fuzzy value).

There are many ways to build information functions but, for our purpose, we are defining an information associated to a particular function. This $\mathcal{F}$-information will permit, subsequently, the definition of transformations that keep constant the amount of information a fuzzy number provides.

In (12) we used the function:

$$I_{\mathcal{F}} : \tilde{R} \longrightarrow [0,1]$$

$$\forall\ A \in \tilde{R},\ I_{\mathcal{F}}(A) = \frac{h(A)}{f(A)+1}$$

where $h(A)$ is A height and f(A) is the imprecision associated to A. This is the simplest function that verifies the mentioned properties of information functions.

We show how uncertainty can be translated (using some suitable transformations) into imprecision, taking into account that the less the uncertainty (or the more the certainty) about a fuzzy number is, the more the imprecision of such number is. This transformation is made in such a way that the amount of information provided by the fuzzy number is the same before and after the modification.

2.3. Transformation of Fuzzy Numbers

Once we have an information function on fuzzy numbers, we can use it to define transformations which preserve the information amount it provides. The idea is to find an *equivalent* representation of the considered fuzzy number in such a way that we change fuzzy uncertainty by imprecision keeping constant the relationship between them, which is determined by the information function.

The aim of the transformations is, basically, to be able to modify the height of a fuzzy number but keeping the information contained in it.

The expression for the transformation will be obtained from the condition of equality in the information and is widely explained in (12) and (13); it consists of finding the four parameters of $A^{\mathcal{T}}$ in such a way that:

$$I(A^{\mathcal{T}}) = \frac{1}{f(A^{\mathcal{T}})+1} = \frac{h}{f(A)+1} = I(A)$$

being

$$f(A^{\mathcal{T}}) = \int_0^1 (b_\alpha^{\mathcal{T}} - a_\alpha^{\mathcal{T}})d(\alpha) \text{ and } [a_\alpha^{\mathcal{T}}, b_\alpha^{\mathcal{T}}] = \{x/A^{\mathcal{T}}(x) \geq \alpha\}$$

In general, the transformation process is the following. We will note by $\mathcal{H}$ the class of trapezoidal fuzzy values on $\tilde{R}$. Let $A, B \in \mathcal{H}$ be two fuzzy values with heights $h(A) = \alpha_A$ and $h(B) = \alpha_B$, respectively (B is $A^{\mathcal{T}}$). Then,

$$I_{\mathcal{F}}(A) = I_{\mathcal{F}}(B) \Longleftrightarrow f_B(0) + f_B(\alpha_B) = f_A(0) + f_A(\alpha_A) + \frac{2}{k}\Delta(\alpha_A, \alpha_B)$$

where

$$\Delta(\alpha_A, \alpha_B) = \frac{\alpha_B - \alpha_A}{\alpha_A.\alpha_B}$$

which is immediate for trapezoidal fuzzy numbers, taking into account that $f(A) = \frac{f_A(0)+f_A(\alpha_A)}{2}\alpha_A$

In figure 3 we summarize this process in a graphical way.

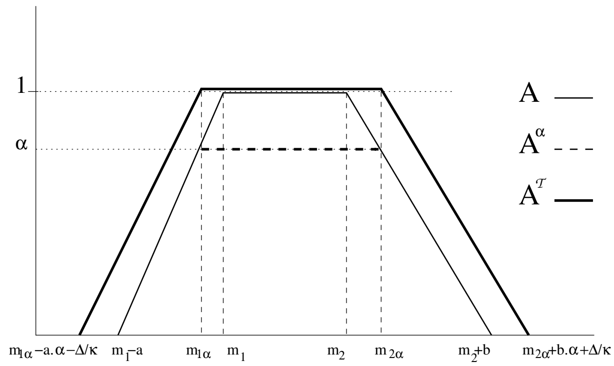

Figure 3: *Graphical representation of A,* A^{α} *and* $A^{\mathcal{T}}$

2.4. Transformation of Fuzzy Certainty in Imprecision

The linguistic qualification of certainty is not a new problem and has already been addressed by some authors under different names (17), (4). Our approach (13) starts from a different point of view, since the membership function of the fuzzy uncertainty $C(.)$ will be used to truncate the fuzzy value A in some way, as we did in the crisp uncertainty case.

As we said in the introduction, an example of the process we are carrying out could be the following: we start with an information like *"It is very possible that Giotto painted The Crucifix by 1289"* and we want to obtain something like *"Giotto painted The Crucifix around 1289"* (where *around 1289* has a wider support set than *by 1289* due to the fact that we have omitted uncertainty from our assertion).

In summary, we want to translate the information $X\ is\ A\ is\ C$ into $X\ is\ T(A, C)$ (i.e. a transformation of A depending on C).

The difficulty is now to give a suitable procedure for computing T(A,C). To do it, we will consider that, for any possible truncation level α, the membership function of the linguistic label modifies in a certain way the uncertainty level. In fact we can assume that:

$$(X\ is\ A)\ is\ C \longleftrightarrow \forall\alpha \in [0,1],\ X\ is\ A\ to\ a\ degree\ C(\alpha),\ \alpha \in [0,1]$$

Figure 4 depicts the general problem we are tackling

Consequently T(A,C) has to be defined in such a way that it summarizes the right side of the above sentence by means of some average. It should be remarked that the

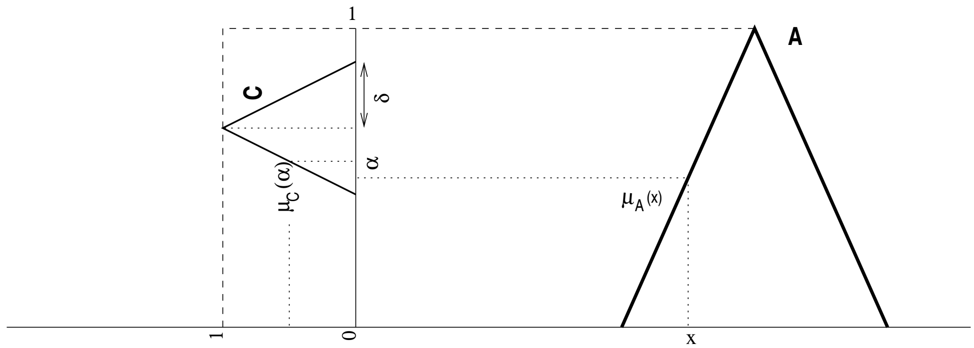

Figure 4: *Fuzzy Uncertainty on a Fuzzy Value.*

membership function $C(.)$ induces two fuzzy measures (possibility/necessity) on the [0,1] what gives rise to two choices.

In (25), Sugeno introduced the concept of fuzzy integral of a fuzzy measure as a way to compute some kind of average value of a function in terms of the underlying fuzzy measure. Obviously, fuzzy measures formally include possibility/necessity measures as special cases.

The fuzzy integral over a referential set X of a function $f(x)$ with respect to a fuzzy measure g is defined as follows:

$$\int_X f(x) \circ g(.) = sup_{\alpha \in [0,1]} \{\alpha \wedge g(A \cap F_\alpha)\}$$

where $F_\alpha = \{x | f(x) \geq \alpha\}$.

Depending on the measure, Sugeno's integral has the following expressions (24):

$$\int_X f(x) \circ g(.) = sup_{x \in A}(f(x) \wedge \mu(x)) \quad (1)$$

$$\int_X f(x) \circ g(.) = inf_{x \in A}(f(x) \vee (1 - \mu(x))) \quad (2)$$

for possibility and necessity respectively.

In this case, after truncating the fuzzy number at the level α, obtaining an non-normalized fuzzy number $A^\alpha(.)$, we apply the Sugeno's integral to the function $A^\alpha(x)$ with respect to the α variable, to compute the mean of the truncated values, and obtaining a possibly non-normalized fuzzy number with membership function S(x). This fuzzy number will be transformed into a normalized one in the step ii.

We have two choices:

1. *Possibility*

   Let $\Pi_C(.)$ stands for the possiblity measure induced by C and $T_p(.)$ stands for the mean of the truncated fuzzy numbers. Using the expression 1, we have:

$$T_p(x) = \int_{[0,1]} A^\alpha(x) \circ \Pi_C(\alpha) = sup_{\alpha\in[0,1]}(A^\alpha(x) \wedge C(\alpha)) =$$
$$= sup_{\alpha\in[0,1]}(A(x) \wedge \alpha \wedge C(\alpha)) = A(x) \wedge sup_{\alpha\in[0,1]}(\alpha \wedge C(\alpha))$$

   If $C_p = sup_{\alpha\in[0,1]}(\alpha \wedge C(\alpha))$, then we finally have:

$$T_p(x) = A(x) \wedge C_p$$

   which indicates that, in the case of the possibility measure, the mean of truncated values is the result of truncating with an specific value which only depends on the linguistic label $C(.)$.

2. *Necessity*

   Let $N_C(.)$ stands for the necessity measure induced by C and $T_n(.)$ stands for the mean of the truncated fuzzy numbers. Using expression 2, we have:

$$T_n(x) = \int_{[0,1]} A^\alpha(x) \circ N_C(\alpha) = inf_{\alpha\in[0,1]}(A^\alpha(x) \vee (1 - C(\alpha))) =$$
$$= inf_{\alpha\in[0,1]}(A(x) \wedge \alpha \vee (1 - C(\alpha))) = A(x) \wedge inf_{\alpha\in[0,1]}(\alpha \vee (1 - C(\alpha)))$$

   If $C_n = inf_{\alpha\in[0,1]}(\alpha \vee (1 - C(\alpha)))$, then we finally have:

$$T_n(x) = A(x) \wedge C_n$$

   which indicates that, also in the case of the necessity measure, the mean of truncated values is the result of truncating with an specific value which only depends on the linguistic label $C(.)$

As it happens with all dual measures, the expert can choose either to work with both of them or to decide which one is the most suitable for the purpose of the system. In figure 5 we graphically show the results obtained considering that the linguistic label C has a trapezoidal membership function with parameters (0.4,0.8,0.1,0.2).

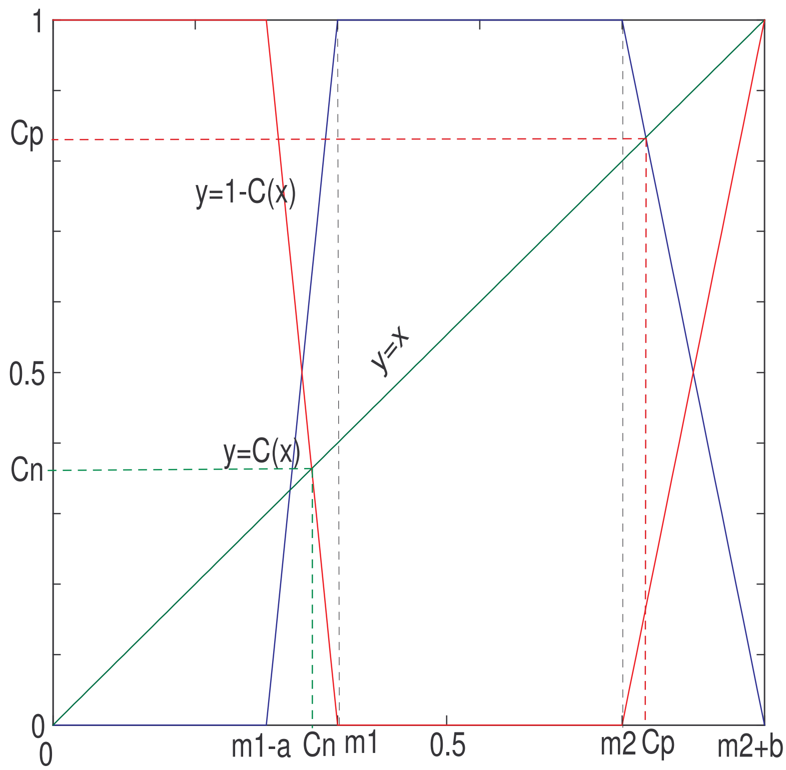

Figure 5: *Upper and lower measures*

In figure 6 we show in a graphical way the results obtained for the Pict-ID 7 *The Garden of Delights*.

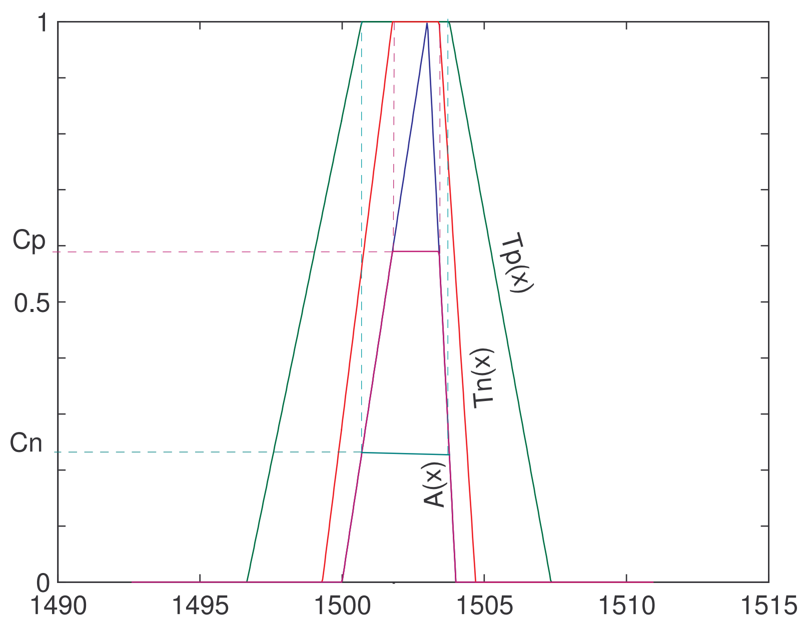

Figure 6: Computed periods for the Picture *The Garden of Delights*.

In table 1 we show the results obtained for C_n, C_p, FVP_n^τ and FVP_p^τ. The last ones are the computed periods taking into account the necessity and the possibility approaches, respectively. As expected, the periods obtained from the necessity approach are wider than the ones from the possibility, due to the fact that the truncation is made at a lower level, where the α-cut leads to a larger interval.

2.5. Relational representation of fuzzy valid time

The primary aim of temporal databases is to offer a common framework to those DB applications that need to store or handle different types of temporal data from a variety of sources, since they allow the concept of time to be unified from the point of view of meaning, representation and manipulation.

Although at first sight the incorporation of time into a DB might appear to be a direct and even simple task, it is, however, quite complex because new structures and specific

Pict-ID	C_n	C_p	FVP_n^τ	FVP_p^τ
15	1	1	(1813,1814,2,0)	(1813,1814,2,0)
87	0.5	0.77	(1284.5, 1290.5,4.5,2.5)	(1286.4,1289.7,5.7,2.6)
7	0.23	0.59	(1500.7,1503.8,4,3.6)	(1501.8,1503.4,2.5,1.3)
6579	0.33	0.9	(805.32,933.35, 4.34,3.68)	809.30,850.50,6.4,4.6)
469	1	1	(1890,1890,1,1)	(1890,1890,1,1)
72	0.33	0.58	(1473,1519,7,3,2.4)	(1473.7, 1519.4,2.5,1.3)
580	1	1	(1498,1499,0,0)	(1498,1499,0,0)
131	0.95	1	(1517,1520.2,1,2)	(1517,1520,1,2)
9	0.95	1	(1477,1480.1,1,2)	(1477,1480,1,2)

Table 1: Computed periods using the necessity and the possibility measures.

operators must be included when temporal data are present.

The time dimension may appear with many semantics according to the problem to be represented. In many situations, time periods are used to express the validity of the data representing a fact. This way of interpreting time is called **valid time**.

When valid time is represented in a relational framework, the table schema must be extended in order to include the attributes VST (Valid Start Time) and VET (Valid End Time), and a valid time relation (VTR) is obtained.

In addition, temporal information is not always as precise as desired since it is affected by imprecision due to the use of natural language or to the nature of the source. More concretely, it is not always possible for the user to give an exact but an imprecise starting/ending point for the validity period associated to a fact. In this case, the fuzzy set theory is a very suitable tool for not missing such information since fuzzy time values can be represented and managed. Several authors have dealt with the problem of imprecision on time from different points of view. In (3) a comparison of approaches to model uncertainty in time is presented. In (26) fuzziness on spatio-temporal information is represented by means of generalised constraints in an object-oriented framework, (23) presents general aspects of dealing with imperfect data in temporal databases and in (22) the concept of possibilistic valid-time period is introduced together with the extension of relational algebra operators in order to operate with it.

The more immediate solution to this problem is to soften the VST and the VET in such a way that they may contain fuzzy dates represented by means of a fuzzy number, as shown in 2.

Author	**Pict-ID**	**Pict-Name**	**...**	**VST**	**VET**
Da Vinci	145	Mona Lisa	...	$\widetilde{1475}$	$\widetilde{1519}$
Goya	15	Third of May	...	$\widetilde{1812}$	1814
...	...	...	...	...	...

Table 2: Fuzzy time relation with VST and VET

This means that, if we use the parametrical representation for fuzzy numbers, we need to store four values for the VST and four values for the VET; that is, $\widetilde{1475}$ will be internally stored as (1475,1475,3,3).

Since the meaning of the attributes VST and VET is the period of time during which the values of a tuple are valid, it is more convenient to summarize the information given by the two fuzzy dates in a fuzzy interval (from now on FVP or fuzzy validity period).In (22) the authors show two ways to transform two fuzzy dates into one single time period; the use of an only fuzzy value for representing the period of validity reduces the complexity of storage and data manipulation in queries. In (11) the authors present all necessary operators to handle FVPs on a relational framework and (18) presents a generalized model of object-relational fuzzy DB that supports the representation and handling of temporal data in a classical RDBMS (Oracle).

Let us consider the following example of the time relation WOA (Works of Art), where each tuple represents the available information about a painting. In the Table 3 an instance of WOA is shown. A cronon of year has been used for the sake of simplicity due to the nature of the time specification.

This representation has the additional advantage that, not only periods of time, but

Author	**Pict-ID**	**Pict-Name**	**...**	**FVP**
Goya	15	Third of May	...	(1813,1814,2,0)
Giotto	87	Crucifix	...	(1288,1289,7,3)
Bosch	7	The Garden of Delights	...	(1503,1503,3,1)
Castelseprio	6579	Bust of Christ	...	(810,850,7,5)
Renoir	460	Two Sisters	...	(1890,1890,1,1)
Da Vinci	72	Mona Lisa	...	(1503,1506,1,5)
Miguel Angel	580	Pieta	...	(1498,1499,0,0)
Rafael	131	La Trasfigurazione	...	(1517,1520,1,3)
Botticelli	9	The Spring	...	(1477,1480,1,2)

Table 3: Instance of WOA fuzzy valid-time relation

fuzzy dates can also be represented in a unified way. Think that a parametric representation as (m,m,a,b) represents a central time point with some imprecision at both sides, what is interpreted as a fuzzy date. In (11) the authors present a temporal extension of fuzzy SQL which permits to manage FVP in a classical relational framework. Thanks to this extension, queries like *Find pictures finished before Mona Lisa* or *Find pictures of middle sixth century* can be made.

3. Finding a Consensus

There will be some situations where various experts (decision makers) have different more or less conflicting opinions about the date of a work of art, and all of them should be taken into account at a certain point. In this case, the notion of consensus plays a key role. In (10) the authors present a review of well known fuzzy logic-based approaches to model flexible consensus, which constitute a well defined research area in the context of fuzzy group decision making (GDM). Fuzzy consensus describes the degree of agreement regarding a specific problem among multiple opinion or decision makers. Most of these indexes are useful under certain circumstances but they are often criticized due to the difficulty and complexity of the computation process. Moreover, in our case, the aim is not to know how accurate or disjoint the assessments are but to find a representative value of all the expert opinions about a time period in order to be stored. Obviously, the coherence of every assessment with respect to the others, can be studied in a previous step using the mentioned indexes in order to filter the data.

It is not the aim of this paper to study the different approaches to consensus, but to choose a suitable one for our purposes. So then, for the sake of simplicity, comprehensibility and the efficiency in computation (when using trapezoidal representation), the fuzzy average is used to find such a representative value (14) (15). Such fuzzy average is calculated based on an arithmetic average of the fuzzy time periods in the following way.

If we assume that there are `n` experts dating the same work of art, there will be `n` trapezoidal fuzzy time intervals to represent their particular assessments. Let us note these fuzzy periods by $(m1_i, m2_i, a_i, b_i)$, $i = 1...n$.

Then, the resulting average fuzzy time period (FVP_a) will be the one represented by the parameters:

$$FVP_a = (\frac{\sum_{i=1...n}^{n} m1_i}{n}, \frac{\sum_{i=1...n}^{n} m2_i}{n}, \frac{\sum_{i=1...n}^{n} a_i}{n}, \frac{\sum_{i=1...n}^{n} a_i}{n})$$

Obviously, this average can be weighted for representing the relative importance of individual experts. In particular Ordered Weighted Aggregation Operators (28) have been widely applied to address GDM problems.

4. Relational Representation of Imprecision and Uncertainty on Fuzzy Time Intervals

The main goal of this paper is to propose a unified way to represent uncertain imprecise temporal information by means of normalized fuzzy intervals that can be used as the basis for the implementation of fuzzy temporal capabilities in a conventional database system. To do that, the first step is to extend the table once again by adding the column FC (Fuzzy Certainty). The result is Table 4.

It is also possible to use linguistic labels instead of the four-parameters representation

Author	Pict-ID	Pict-Name	FVP	FC
Goya	15	Third of May	(1813,1814,2,0)	Sure
Giotto	87	Crucifix	(1288,1289,7,3)	Quite-Possible
Bosch	7	The Garden of Delights	(1503,1503,3,1)	Possible
Castelseprio	6579	Bust of Christ	(810,850,7,5)	(0.6,0.9,0.2,0)
Renoir	460	Two Sisters	(1890,1890,1,1)	Sure
Da Vinci	72	Mona Lisa	(1503,1506,1,5)	(0.4,0.5,0.2,0.2)
Miguel Angel	580	Pieta	(1498,1499,0,0)	Sure
Rafael	131	La Trasfigurazione	(1517,1520,1,3)	Almost-Sure
Botticelli	9	The Spring	(1477,1480,1,2)	Almost-Sure

Table 4: Relational representation of Fuzzy Certainty on FVP.

for the fuzzy certainty. One familiy of certainty labels is the one used in (17) which is shown in table 5.

Although we have found a solution to the problem of representation in a relational framework, there are still many problems derived mainly of the application of relational and/or temporal operators to such a database. For example, what happens if we apply the *before* operation (11) on the WOA table when tuples are affected by fuzzy uncertainty? What should we do when joining two WOA tables?. It is clear that we need a unified way to represent the whole information about time and transform the original data provided into a different format.

On the other hand, the original table WOA should be kept in the system since the information provided separately by the fuzzy interval and the fuzzy certainty could be used only to be queried or as part of a different reasoning process (for example to match the opinions of two experts about the same painting).

So then, the idea is to create a transformed table $WOA^{\mathcal{T}}$ in order to avoid all the above mentioned problems.

Once we have found the mechanisms to transform uncertainty into imprecision in a suitable way, we can now build the transformed table $WOA^{\mathcal{T}}$ as shown in table 6. In this

Label	Parameters (m_1,m_2,a,b)
Almost-Impossible	(0,0.05,0,0.03)
Slightly-Possible	(0.07,0.14,0.02,0.03)
Moderately-Possible	(0.15,0.35,0.05,0.1)
Possible	(0.35,0.55,0.1,0.1)
Quite-Possible	(0.55,0.75,0.1,0.1)
Very-Possible	(0.75,1,0.1,0)
Almost-Sure	(0.98,1,0.03,0)
Sure	(1,1,0,0)

Table 5: Family of Linguistic Certainty Values

table, we have omitted the `Pict-ID` column for the sake of space.

Author	Pict-Name	$FVP_n^{\mathcal{T}}$	$FVP_p^{\mathcal{T}}$
Goya	Third of May	(1813,1814,2,0)	(1813,1814,2,0)
Giotto	Crucifix	(1284.5,1290.51,4.5,2.49)	(1286.39,1289.69,5.69,2.61)
Bosch	The Garden of Delights	(1500.96,1503.68,3.09,2.44)	(1501.77,1503.41,2.47,1.28)
Castelseprio	Bust of Christ	(806.51,852.5,4.51,3.5)	(809.3,850.51,6.41,4.6)
Renoir	Two Sisters	(1890,1890,1,1)	(1890,1890,1,1)
Da Vinci	Mona Lisa	(1502.33,1509.35,2.36,3.67)	(1502.58,1508.1,1.3,3.62)
Miguel Angel	Pieta	(1498,1499,0,0)	(1498,1499,0,0)
Rafael	La Trasfigurazione	(1516.95,1520.15,1.01,2.9)	(1517,1520,1,3)
Botticelli	The Spring	(1476.95,1480.1,1.01,1.95)	(1477,1480,1,2)

Table 6: $WOA^{\mathcal{T}}$ table obtained after the transformation of uncertainty in imprecision.

5. Handling Imprecision and Uncertainty in Fuzzy Database Systems

This section describes how we have implemented this proposal of integration of imprecision with uncertainty on our prototype of Fuzzy Object-Relational Database Management System (FORDBMS) (6; 1). This system is developed as an extension of a market leader RDBMS (Oracle®) by using its advanced object-relational features. This strategy let us take full advantage of the host DBMS features (high performance, scalability, availability, etc.) adding the ability of representing and handling fuzzy data.

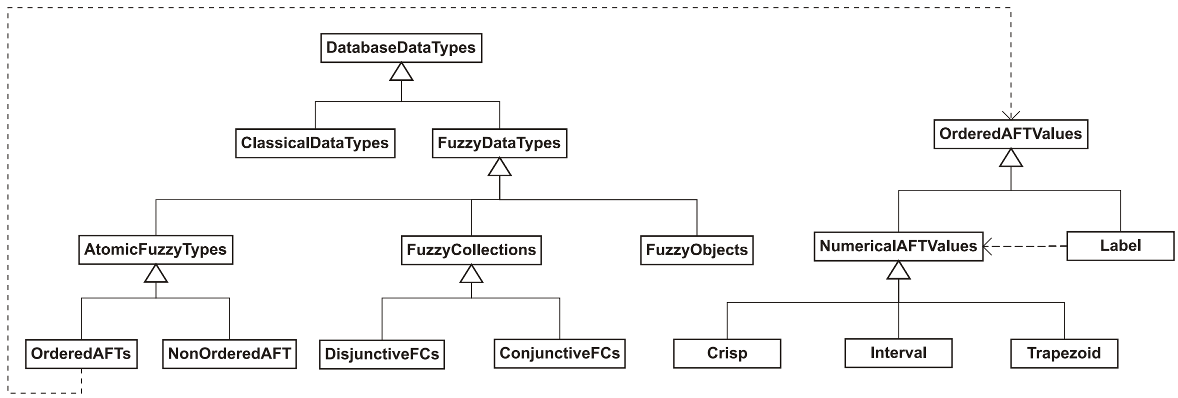

Figure 7: Data type hierarchy for the FORDBMS

The FORDBMS extension includes a set of user-defined types (shown in Fig. 7) to allow the representation of a wide variety of fuzzy data, as the following ones:

- Atomic fuzzy types (AFT), represented as possibility distributions over ordered (OAFT) or non ordered (NOAFT) domains.

- Fuzzy collections (FC), represented as fuzzy sets of objects, fuzzy or not, with conjunctive (CFC) or disjunctive (DFC) semantics.

- Fuzzy objects (FO), whose attribute types could be crisp or fuzzy, and where each attribute is associated with a degree to weigh its importance in object comparison.

All fuzzy types include a Fuzzy Equal operator (FEQ) definition that computes the degree of fuzzy equality for each pair of instances. The OAFT type implements other fuzzy comparators, based on the possibility measure, such as FGT (Fuzzy Greater Than), FGEQ (Fuzzy Greater or EQual), FLT (Fuzzy Less Than) and FLEQ (Fuzzy Less or EQual). Also, OAFT implements another version of those operators by using the necessity measure (NFGT, NFGEQ, NFLT and NFLEQ).

In (18) we present a fuzzy temporal model for fuzzy object-relational databases, and its implementation on top of the FORDBMS called FVTM (Fuzzy Valid Time Support Module). This module includes, among other things, all the operators needed to perform fuzzy temporal queries, using approaches based on possibility: fbefore, fafter, foverlaps, etc... and based on necessity: nfbefore, nfafter, nfoverlaps,...

The flexible and extensible architecture of these modules, allows to integrate our proposal to handle imprecision with uncertainty in an easy way. To do this, we only need to define and implement two new operators: `toFUP` and `toFUN`, which takes a fuzzy uncertainty value and a fuzzy interval and returns a fuzzy value that combines imprecision and uncertainty using possibility and necessity, respectively.

We have implemented several overloads of these operators, to handle fuzzy data and fuzzy temporal data, also to take into account the k factor (where $k = k_0 \cdot \lambda$, as mentioned in section 2.3).

Next, we show the main signatures of these operators:

```
FUNCTION toFUP(fval OAFT, fcval FC, lambda DEFAULT 1 NUMBER) RETURN OAFT;
FUNCTION toFUP(tfval FVP, fcval FC, lambda DEFAULT 1 NUMBER) RETURN FVP;
FUNCTION toFUN(fval OAFT, fcval FC, lambda DEFAULT 1 NUMBER) RETURN OAFT;
FUNCTION toFUN(tfval FVP, fcval FC, lambda DEFAULT 1 NUMBER) RETURN FVP;
```

where `FC` is a `OAFT` subtype defined on the underlying domain [0,1] and `lambda` is the relation between the scale we are using and the base scale.

Using `cp` expression and the normalization process shown in figure 3 we can implement the former expression in this way (using PL/SQL).

```
FUNCTION toFUP(fval OAFT,fcval FC,lambda DEFAULT 1 NUMBER) RETURN OAFT IS
cp NUMBER; k NUMBER; k0 NUMBER; m1 NUMBER; m2 NUMBER; a NUMBER; b NUMBER;
BEGIN
  -- retrieves from the catalog the k0 value for the type of fval.
  k0:= fval.getK0();
  IF k0 IS NULL THEN k0:=1;
  k:= k0*lambda;
  cp:=(fcval.m2+fcval.b)/(fcval.b+1);
  m1:=fval.m1+fval.a*(cp-1);
  m2:=fval.m2+fval.b*(1-cp);
  a:=(cp-1)/cp*k)-fval.a*cp;
```

```
  b:=fval.b*cp+(1-cp)/(cp*k);
  RETURN m1m2ab(m1,m2,a,b);
END;
```

The implementation of toFUN function is similar; it only differs in the use of cn instead of cp.

5.1. The WOA example on the Imprecise-Uncertain extension

Our FORDBMS with the FVTM and the described imprecise-uncertain extension, can store and query several kind of fuzzy data and temporal fuzzy data even affected by uncertainty. A query can combine fuzzy conditions on imprecise data and on temporal data, stored with uncertainty.

Next we are going to show some of these features using the example of the time relation WOA (Works of Art) shown in table 4.

5.1.1. Data Definition

To store the information of the pictures (including the imprecision and the uncertainty of the dates in which they were painted) we need to create a subtype of OAFT to store FC definitions and labels as this code shows:

```
-- Creation of oaft datatype to define labels for uncertainty in Table 4.
exec OAFT.extends('woa_FC');
-- Definition of labels
exec woa_FC.labelDef('Almost-Impossible',m1m2ab(0,0.05,0.03));
exec woa_FC.labelDef('Slightly-Possible',m1m2ab(0.07,0.14,0.02,0.03));
exec woa_FC.labelDef('Moderately-Possible',m1m2ab(0.15,0.35,0.05,0.1));
exec woa_FC.labelDef('Possible',m1m2ab(0.35,0.55,0.1,0.1));
exec woa_FC.labelDef('Quite-Possible',m1m2ab(0.55,0.75,0.1,0.1));
exec woa_FC.labelDef('Very-Possible',m1m2ab(0.75,1,0.1,0));
exec woa_FC.labelDef('Almost-Sure',m1m2ab(0.98,1,0.03,0));
exec woa_FC.labelDef('Sure',m1m2ab(1,1,0,0));
```

Then, the table WOA_FUT (Table 4) can be created using the following SQL sentence:

```
CREATE TABLE WOA_FUT (author VARCHAR2(30), pict_ID NUMBER PRIMARY KEY,
pic_Name VARCHAR2(30), fvp FVP, fc woa_FC);
```

Note that this table does not include the fields to store the values for `Cn`, `Cp`, `FVPn` and `FVPp` shown in Tables 1 and 6; this is because this values are computed from the `fvp` and `fc` values by means of the functions previously described: `toFUN` and `toFUP`, respectively. It is also important to note that, in this example, both λ and k_0 take value 1, and therefore, $k = 1$.

5.1.2. Data Inserting and Fuzzy Querying with Uncertainty

Now, we can insert into the WOA_FUT table the information shown in Table 4, running the following DML sentences:

```
INSERT INTO WOA_FUT VALUES ('Goya', 15,'Third of May',
FVP('1811','1813','1814','1814','yyyy'),woa_FC('Sure'));
INSERT INTO WOA_FUT VALUES ('Castelseprio', 6579,'Bust of Christ',
FVP('803','810','850','855','yyyy'),woa_FC(0.6,0.9,0.2,0));
...
INSERT INTO WOA_FUT VALUES ('Botticelli', 9,'The Spring',
FVP('1476','1477','1480','1482','yyyy'),woa_FC('Almost-Sure'));
```

The parameter `'yyyy'` in the `FVP` constructor indicates that the above four arguments are interpreted as years in four-digit format.

On this table, we can perform several kinds of queries that involve imprecision and uncertainty on the temporal domain, as the following examples illustrate:

Query 1. *"Find pictures painted after the XIV century but previous to Mona Lisa" (threshold 0.5).*

The query, in possibilistic terms, can be expressed by means of this sentence:

```
SELECT a.author,a.pic_name, toFUP(a.fvp,a.fc), CDEG(1) CDeg
FROM WOA_FUT a, WOA_FUT a WHERE b.pic_name='Mona Lisa' AND
FCOND(FzAND(FBEFORE(toFUP(a.fvp,a.fc),toFUP(b.fvp,b.fc)),
FAFTER(toFUP(a.fvp,a.fc),(1399,1399,0,0))),1)>=0.5
```

The operator `FBEFORE(fv1,fv2)` returns the possibility degree in which `fv1` is previous to `fv2`; in the same way, the operator `FAFTER(fv1,fv2)` returns the possibility degree in which `fv1` is posterior to `fv2`. The operator `CDEG(1)` shows the fulfillment degree of each tuple for the condition expressed into the `FCOND` operator. Finally, the `FzAND` operator applies the T-Norm of minimum on its arguments.

The results of this query are shown in Table 7.

Author	**Pict-Name**	$FVP_p^{\mathcal{T}}$	**CDeg**
Bosch	The Garden of Delights	(1501.77,1503.41,2.47,1.28)	.87
Da Vinci	Mona Lisa	(1502.58,1508.1,1.3,3.62)	.5
Miguel Angel	Pieta	(1498,1499,0,0)	1
Botticelli	The Spring	(1477,1480,1,2)	1

Table 7: Results of Query 1 using possibility measure

The same query expressed in terms of necessity would be:

```
SELECT a.author,a.pic_name, toFUN(a.fvp,a.fc), CDEG(1) CDeg
FROM WOA_FUT a, WOA_FUT b WHERE b.pic_name='Mona Lisa' AND
FCOND(FzAND(NFBEFORE(toFUN(a.fvp,a.fc),toFUN(b.fvp,b.fc)),
NFAFTER(toFUN(a.fvp,a.fc),(1399,1399,0,0))),1)>=0.5
```

Where the operators `NFBEFORE` and `NFAFTER` are the necessity based versions of the operators `FBEFORE` and `FAFTER`, respectively.

The execution of this query retrieves the results shown in Table 8.

Author	**Pict-Name**	$FVP_n^{\mathcal{T}}$	**CDeg**
Miguel Angel	Pieta	(1498,1499,0,0)	1
Botticelli	The Spring	(1476.95,1480.1,1.01,1.95)	1

Table 8: Results of Query 1 using necessity measure

Query 2. *"Find the pictures of the same period than 'The Garden of Delights' by El Bosco (Bosch)" (threshold 0.5).*

The query expression for possibility is:

```
SELECT a.author,a.pic_name, toFUP(a.fvp,a.fc), CDEG(1) CDeg
FROM WOA_FUT a, WOA_FUT b WHERE b.pic_name='The Garden of Delights' AND
FCOND(FOVERLAPS(toFUP(a.fvp,a.fc),toFUP(b.fvp,b.fc)),1)>=0.5
```

Where the operator `FOVERLAPS(fv1,fv2)` returns the possibility degree in which `fv1` and `fv2` are overlapped.

The results are shown in Table 9.

Author	**Pict-Name**	$FVP_p^{\mathcal{T}}$	**CDeg**
Bosch	The Garden of Delights	(1501.77,1503.41,2.47,1.28)	1
Da Vinci	Mona Lisa	(1502.58,1508.1,1.3,3.62)	1

Table 9: Results of Query 2 using possibility measure

The expression for this query using necessity measures is:

```
SELECT a.author,a.pic_name, toFUN(a.fvp,a.fc), CDEG(1) CDeg
FROM WOA_FUT a, WOA_FUT b WHERE b.pic_name='The Garden of Delights' AND
FCOND(FNOVERLAPS(toFUN(a.fvp,a.fc),toFUN(b.fvp,b.fc)),1)>=0.5
```

Where the operator `NFOVERLAPS` is the necessity based version of the operator `FOVERLAPS`

The Table 10 shows the results of this query.

Author	**Pict-Name**	$FVP_n^{\mathcal{T}}$	**CDeg**
Bosch	The Garden of Delights	(1500.96,1503.68,3.09,2.44)	.5

Table 10: Results of Query 2 using necessity measure

Query 3. *"Find pictures of the second half of XV century" (threshold 0.5).*

In a fuzzy way, we can say that this period starts in (1450,1460,5,0) and ends in (1505,1505,5,5). So, we could express the query in this way:

```
SELECT author,pic_name, toFUP(fvp,fc), CDEG(1) CDeg FROM WOA_FUT WHERE
FCOND(FzAND(FAFTER(toFUP(fvp,fc),(1450,1460,5,0)),
FBEFORE(toFUP(fvp,fc),(1505,1505,5,5))),1)>= 0.5
```

Results in Table 11

Author	**Pict-Name**	$FVP_p^{\mathcal{T}}$	**CDeg**
Bosch	The Garden of Delights	(1501.77,1503.41,2.47,1.28)	.78
Da Vinci	Mona Lisa	(1502.58,1508.1,1.3,3.62)	.62
Miguel Angel	Pieta	(1498,1499,0,0)	1
Botticelli	The Spring	(1477,1480,1,2)	1

Table 11: Results of Query 3 using possibility measure

The query for necessity measures is:

```
SELECT author,pic_name, toFUN(fvp,fc), CDEG(1) CDeg FROM WOA_FUT WHERE
FCOND(FzAND(NFAFTER(toFUN(fvp,fc),(1450,1460,5,0)),
NFBEFORE(toFUN(fvp,fc),(1505,1505,5,5))),1)>= 0.5
```

and the results are shown in Table 12.

Author	**Pict-Name**	$FVP_n^{\mathcal{T}}$	**CDeg**
Miguel Angel	Pieta	(1498,1499,0,0)	1
Botticelli	The Spring	(1476.95,1480.1,1.01,1.95)	1

Table 12: Results of Query 3 using necessity measure

6. Conclusions and Future Work

As we have seen, imprecision and uncertainty coexist in many applications. More often than desirable, a given datum is affected by imprecision and uncertainty at the same time and working with both measures is almost impossible. Though this situation may arise in any context, it is very common to find together uncertainty and imprecision when dating ancient objects and this is the reason why we have chosen this area to apply our results. Concretely, in the framework of temporal databases, where data (tables) take part in both relational operations (like cartesian product, selection, join, groupping,...) and fuzzy temporal operations (like before, after, between, at the same time...) this problem is still worse and make it specially important to solve this problem when fuzziness and uncertainty refers to time.

Thanks to the presented approach, we can integrate fuzzy uncertainty and imprecision on fuzzy time intervals into a single fuzzy value in order to solve the following drawbacks:

- *Representation problems*: As mentioned in the paper, a unified representation of uncertainty and imprecision in a relational framework saves a lot of memory and improves the query response time.

- *Manipulation*: Both relational and time operators have serious applicability problems when both measures appear together in a tuple. Concretely, it should be impossible to join two WOA tables or to apply the fuzzy before operation.

- *Comprehension*: Working with uncertainty and imprecision at the same time (if possible) is very difficult to be understood by the users: why so many degrees, what do they really mean, which of them is more important, how can we rank the data...

- *Reusability*: It is obvious the advantage of this process. Uncertainty and imprecision are included in the fuzzy interval itself and there is no need to develop or use different mechanisms from those already introduced for fuzzy validity periods (temporal reasoning, rules definition, languages like TFSQL,...). Besides, there is no need to consider levels of truth (all the values to be handled are completely true). Nevertheless, we keep in the system the original table with both measures for the interest of experts.

- *Unification criteria*: As we have mentioned in the introduction, when looking for the *Crucifix of Giotto* painting date in Internet, we can find many expert assessments; therefore, from these results, the unification of criteria is very important, not only to be able to put them in a database, but to have an only answer to this question.

Regarding the extension of these results, different research lines for future work appear:

First, the application of these results to other problems of representation where it is necessary jointly deal with imprecise and uncertain time values. This problem may appear in all cases concerning events not well referenced due to unreliable sources: databases concerning historical facts, buildings, crimes etc.

Another possible future work closely related to the situation explained above, is the consideration that different sources of information with different reliability provide particular approximations of temporal data. We intend to apply or extend this case to different forms of aggregation provided by fuzzy set theory. (5)

Finally, it should be noted that in the case of linguistic uncertainty, the result of the transformation is actually two fuzzy numbers (including one the other) which can be seen as a single interval fuzzy set (19); we propose to apply the theory of such fuzzy sets to give a better interpretation and use the results in fuzzy temporal databases.

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