Pólya and Dienes: Two men of one mind or one culture?

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Abstract

George Pólya and Zoltán Paul Dienes played decisive roles in changing how mathematics was taught and learned in the twentieth century. Following the crucial ideas of their work and the turning points of their lives, we trace the origin and the goals of the two men, considering each 'paidagogos' in the light of the other. Contrasting their approaches and resources, both as mathematicians and teaching wizards, gives us the opportunity to compare their pedagogical backgrounds when faced with the same dominant teaching practice. The emergent questions concerning age dependent preconceptions of educational psychology not only help to resolve the differences of their attitudes, but also to recognize the complementary nature of their methods; a combination that can be utilized more widely for present-day didactic practice.

Keywords: heuristics, maverick philosophy of mathematics, embodiment, duration

Introduction

The names of George Pólya (1887-1985) and Zoltán Paul Dienes (1916-2014) are intertwined with mathematical heuristics and didactics, and the psychology and practice of problem solving. The closer we get to their work the more similarities appear. Still, even after the first look, we can detect characteristic differences in their approaches which seem more to complement one another rather than to suggest one and the same conception. Is it enough to find the common origin of their philosophy of mathematics, in contrast to Platonism, Formalism or Intuitionism, in considering mathematics as a human activity that is conceived in problem solving? Can we attribute their common views about the human side of this activity to the same point of departure, to the same educational objectives, or some other factors? Was it the milieu of their Hungarian background or, leaving it, a bi-, or multicultural identity that resulted in the apparent single mindedness of their pursuit of changing math education? Answering these questions, we compare the crucial ideas from their work and place the standard interpretations of their views into the historical context of the origin of modern heuristics.

Developers of new directions in mathematics education

Both Pólya and Dienes intended to renew the way mathematics was taught and learned throughout their life, and became world renown for their innovative efforts. They carried their Hungarian-born experiences with them in their mind-sets as seeds of their multicultural identity. Just to name a few highlights of their legacy:

Bjarnadóttir, K., Furinghetti, F., Krüger, J., Prytz, J., Smid, H. J. & Schubring, G. (Eds.) (201X). *"Dig where you stand" 5. Proceedings of the fifth International Conference on the History of Mathematics Education.* Utrecht: University of Utrecht (2019).

- Pólya and Gábor Szegő published a two-volume *Problems and Theorems in Analysis* (Pólya and Szegő, 1925 in German, 1972 in English). In the preface of the book, they claimed:

  > This book is no mere collection of problems. Its most important feature is the systematic arrangement of the material which aims to stimulate the reader to independent work and to suggest to him useful lines of thought. We have devoted more time, care and detailed effort to devising the most effective presentation of the material than might be apparent to the uninitiated at first glance.

  This book is a classic masterpiece of guided discovery, that both Pólya and Szegő were the proudest of, among their many publications. The two authors became life-long friends, and determining figures in the shaping of "Stanford Mathematics" (Royden, 1989, p. 250).
- Pólya was one of the founders of the California Mathematics Council (CMC) in 1942. The workshops and summer institutes he conducted for teachers (particularly in California and in Switzerland) throughout the years have influenced the teaching and learning of mathematics for generations of people (Taylor & Taylor, 1993, p. 92).
- Pólya's *How to Solve It* (first published in the USA in 1945 and translated into 17 languages since then) and Rózsa Péter's *Playing with Infinity* (first published in Hungary in 1957, the first translation into English by Dienes was published in 1961 in England) were written at about the same time (early 1940s). They both reflected "best practices" in Hungary (*How to Solve It:* on heuristic thinking and problem solving, and *Playing with Infinity:* on inquiry-based learning and student engagement) (Péter, 1976). Imre Lakatos translated *How to Solve It* into Hungarian (published in 1957) because he recognized its importance for Hungarian education, and had already begun work on his *Proofs and Refutations* during the 1950s. The work was first published in 1976 (after his death) as a book, based on the first three chapters of his four-chapter doctoral thesis, *Essays in the logic of mathematical discovery*, presented in 1961. It has many common roots with *Playing with Infinity* in terms of its approach and investigation of mathematical thinking.
- Dienes played a key role in the development of experimental research-based mathematics education. He had a unique position in the research community as a well-trained mathematician and psychologist who also had extended experience in working with children and with school teachers worldwide. He compiled the report of the *International Study Group for Learning Mathematics* on mathematics in primary education in 1966. This initiative of UNESCO played a crucial role in changing mathematics education around the world (International Study Group for Mathematics Learning, 1966). The report provided the most comprehensive list of educational research from the 1960's by Dienes with contributions from

Bruner, Jeeves, and Piaget, and many other major researchers of the time, such as, for example, Bartlett, Gattegno, Papy, Skemp, and Suppes. The report also included specific recommendations for teacher training that remain relevant in 2018, decades later. In addition, Dienes worked closely with Tamás Varga on "New Math" in Hungary[1] during the 1960s. They collaboratively examined, evaluated, and perfected the uses of many manipulatives in instruction, such as, for example, Vygotsky's logic blocks (Servais and Varga, 1971, pp. 38-46), Cuisenaire rods, and Dienes's Multibase Arithmetic Blocks (Servais and Varga, 1971, p. 107, Kántor, 2006).

Achievements

From our current standpoint, we attribute to Pólya the interdisciplinary 'science' of heuristics which, by now, has close links with didactics, cognitive psychology, and the contexts and methods of discovery in various fields from sociology or the history of science to computational models. After a period of more 'practical' views on 'teaching' giving rise to new interpretations he wrote:

> **Modern heuristic** endeavors to understand the process of solving problems, especially the *mental operations typically useful* in this process. […] A serious study of heuristic should take into account both the logical and the psychological background, it should not neglect what such older writers as Pappus, Descartes, Leibnitz, and Bolzano have to say about the subject, but it should least neglect unbiased experience. *Experience in solving problems and experience in watching other people solving problems must be the basis on which heuristic is built.* […] The study of heuristic has »practical« aims; a better understanding of the mental operations typically useful in solving problems could exert some good influence on teaching, especially on the teaching of mathematics (Pólya, 1945, pp. 129-130, italics ours).

Dienes placed games and the issues of multiple embodiment at the intersection of mathematics education, theories of concept formation in epistemology, and developmental psychology. He described his philosophies about the 'art and craft' of active learning in the following way:

> The mathematics I was bringing into the schools was really a Trojan horse. It was not just mathematics, it was a way to look at what learning is all about, or, even more fundamentally, what knowledge is all about. *To 'know' something surely is to know how to handle it. Handling means* action: *present action or at least past action, remembered accurately, burnt into our person as internalized action.* So if knowledge is internalized action, then learning must be the process of internalizing such action. If there is no action, then there is nothing to internalize, so no learning of any permanent nature can happen. It is philosophies such as these that climb out of the Trojan horse once it is smuggled into the educational system under the guise of essential learning, such as the learning of mathematics (Dienes, 2003 p. 317, italics ours).

[1] A renewal of the content and method of math education but in radically different sense than "New Math" was conducted in the USA.

Common points of their didactics

Their names not only intertwined with mathematical heuristics and didactics, the psychology (Dienes), and practice of problem solving (Pólya), but also there are deep commonalities in their approach to mathematics, and to learning and teaching in general.

Both of them freely and frequently turned to perceptual experiences, tools that augmented the learner's mind. They both trusted and built on the intelligence of all learners. They wanted to "meet" the students at their current level, and lead them to a deeper understanding. They never complained about any lack of "pre-requisite" knowledge of the learners. They could communicate about math with any age, gender, ethnic, and socio-economic group of learners. The learners were attracted to the tasks and challenges proposed by them.

They had lively and remarkable presence. Even one-time exposure to either of them built in many people the memory of a teaching wizard having nothing up his sleeves who taught the learners to make their own miracles. They collaborated intensively with the learners, thinking, playing, and exploring together. They viewed their role as teachers as a "play master participant", who has solid content knowledge and a clear vision of educational goals, but works with the learners, in the moment, with great presence and with great respect and admiration towards the ideas, attempts, and needs of the others (see, for example, the *Let Us Teach Guessing* video described in footnote 6 below). This sensitivity to others might have been a big part of their magic, which meant more than the kind of coaching that is so fashionable today.

The most important common conviction of the two 'paidagogoi' was, however, an attitude to mathematics that was novel compared to the contemporary western logical positivist, analytic philosophies of mathematics. They both considered mathematics (first and foremost) a 'human activity'. The legacy of this approach that comes through every aspect of their work is prominent in Imre Lakatos's philosophy of mathematics and in the recent revival of the so called 'maverick philosophies of mathematics'.[2]

Tibor Frank found the preface to a course that Pólya gave at Stanford University in the archives of the university's library:

> Start from something that is familiar or useful or challenging: From some connection with the world around us, from the prospect of some application, from an intuitive idea. Don't be afraid of using colloquial language when it is more suggestive than the conventional, precise terminology. In fact, do not introduce technical terms before the student can see the need for them. Do not enter too early or too far into the

[2] The reception of Lakatos was gradual and oft debated among mathematicians. He played an undeniably important role in twentieth century philosophy of mathematics in spite of his unexpected death in the midst of writing a book jointly with Paul Feyerabend, the first part of which came out as Feyerabend's *Against Method* (1975/1979), without the second part in which Lakatos was to "restate and defend" the rationalist position. Concerning the recent flow of papers addressing the "maverick philosophy of mathematics" cf. Cellucci (2017) and related citations.

> heavy details of a proof. Give first a general idea or just the intuitive germ of the proof. More generally, realize that the natural way to learn is to learn by stages: First, we want to see an outline of the subject, to perceive some concrete source or some possible use. Then, gradually, as soon as we can see more use and connections and interest, we take more willingly the trouble to fill in the details. (Frank, 2004, p. 32)

Pólya's "familiar or useful or challenging [situation*,* a] connection with the world around us" for Dienes meant "embodiment".[3] In Dienes's view, structure, formalization, and proof must build on experiences gained *in situ*, which, for children, meant multiple embodiment through plays (Dienes, 1963, pp. 21-32).

Common cultural background behind the commonalities

When Pólya was taking courses in philosophy he fell under the influence of Professor Bernát Alexander, the grandfather of the eminent Hungarian mathematician, Alfréd Rényi. Alexander encouraged Pólya to take additional courses in physics and mathematics (Alexanderson 2000, p. 17).

Pólya was a successful student, in general, but mathematics was neither his strength, nor his interest (Alexanderson 2000, p. 16) until he realized the necessity of studying mathematics for advancement in philosophy and simultaneously, Lipót Fejér sparked his interest in mathematics. Fejér, himself, did not like or care about mathematics until Zsigmond Maksay became his high school teacher. Maksay encouraged Fejér to work on challenging problems and submit his solutions to the high school math journal, KöMaL.[4] Fejér became one of the best problem solvers of the journal. Pólya may have been deeply inspired by Fejér's personal 'journey', but also by Fejér's style of giving a personal background and flavor to problems. According to Pólya, the essential aspect of Fejér's mathematical talent was his love for the intuitively clear details:

> It was not given to him to solve very difficult problems or to build vast conceptual structures. Yet he could perceive the significance, the beauty, and the promise of a rather concrete, not too large problem, foresee the possibility of a solution, and work at it with intensity. And, when he had found the solution, he kept on working at it with loving care, till each detail became fully intuitive and the connection of the details in a well-ordered whole fully transparent. (Pólya, 1961, p. 505)

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[3] "Situation" here not only means the way in which something is placed in relation to its surroundings, but also the sum total of internal and external stimuli that acts upon an organism and the combination of circumstances within a given time interval, a "position in life". In new or unusual state of affairs the familiar situation turns into a problem: a critical, trying challenge that calls for "awakening", implies reaction, interpretation, and reply. (Cf. ~**situation** in Webster's New Coll. Dictionary, 1974)

[4] KöMaL (the Hungarian acronym for High School Mathematics and Physics Journal) is a monthly periodical for high school students. It was established in 1894. It has been the intellectual cradle for mathematicians like von Neumann, Paul Erdős, or László Lovász. Generations of future scientists, mathematicians, and also interested students have had their first mathematical experience by trying to solve problems posed every month in the journal's yearly problem solving contest (source: www.komal.hu/home.e.shtml). The attitude of the group to mathematics and the social network of teachers (central figures: Dániel Arany, Manó Beke, Sándor Mikola, László Rátz) behind the journal was a key factor in the genesis and survival of "martian" mathematics. Cf. Marx (2001).

Pólya and Károly Polányi were the founders of a student society called Galilei Kör [Galilei Circle]. Pólya gave a lecture there on the philosophy of Ernst Mach. The Galilei Circle (1908-1918) was the meeting place of radical intellectuals, mostly Jewish college students from the up and coming Budapest families of a new bourgeoisie (Frank, 2004, p. 28).

Zoltán Paul Dienes's parents were part of the same social and intellectual environment during the first decade of the 20th century as Pólya. Paul Dienes (the father of Zoltán Paul Dienes) and Pólya knew each other from the Galilei Circle and from the radical Társadalomtudományi Társaság (Social Sciences Association) of left-liberal artists and scientists gravitating around the social science journal *Huszadik Század* (Twentieth Century, ed. in chief Oszkár Jászi). They also worked closely with G. H. Hardy during the 1920's. Paul Dienes even invited Pólya to give a series of lectures at the University College of Swansea in Wales, where he worked in 1925 (Alexanderson 2000, p. 81; Cooke, 1960, p. 251).

Valéria Dienes (née V. Geiger, 1879-1978), mother of Zoltán Paul Dienes, was a philosopher, choreographer, dance teacher, and the creator of the dance theory *orkesztika* [orchestics]. She was among the first females to graduate from a university in Hungary. Her university work focused on studying mathematics and physics. Lipót Fejér was in love with Valéria, and he introduced Paul Dienes to her, claiming that Paul Dienes was a mathematical genius (Borus, 1978, p. 14).

Valéria began her doctoral studies of philosophy by listening to Bernát Alexander's lectures, just like Pólya did. She received her degree in the same ceremony as Paul Dienes in 1905 at Pázmány Péter University, Budapest. They exchanged engagement rings during that ceremony. Paul received his doctorate in mathematics. Valéria's doctorate was in philosophy as a major subject with a first minor in mathematics and a second minor in aesthetics. Her interest in aesthetics was based on her love of composing and playing music (Borus, 1978, pp. 15-18; Boreczky, 2013, p. 56).

Reforming ways of learning and teaching mathematics

Pólya concentrated on finding beautiful solutions to well-understood problems, and on the undeniable structure that arises from previous attempts and guesses. One of his favorite problems was the Riemann Hypothesis. He believed that the proof could be found after serious attempts, and he made several attempts. Doing mathematics was meant to give the pleasure of understanding and criticizing the reasoning of others. For him, mathematics seems to be analogous to an ever-perfected building under construction, that has some remarkable "free masons" who have worked on it (like Euclid, Archimedes, Descartes, Leibniz, Euler, etc.), and learning mathematics was equivalent to studying the methods and the reasoning of these masters so that the building could be renovated, copied, and expanded in the same, or in a new, ambitious artistic style. The problems novice learners were dealing with had been masterfully solved in the past, but had access points to these learners through which they could reconstruct some minor details of the construction. As in a guided tour, the leader of the group, who in this simile is the math teacher in the classroom, was to ask good questions to make the participants aware of the why's and how's of the constructions of the masters. Otherwise, the learning of mathematics could be disturbing, as it was to Pólya:

> The author remembers […] a question that disturbed him again and again: 'Yes, the solution seems to work, it appears to be correct; but how is it possible to invent such a solution? Yes, this experiment seems to work, this appears to be a fact; but how can people discover such facts? And how could I invent or discover such things by myself?' (Pólya, 1945, p. vi)

Dienes, in many senses, also continually reflected on his own 'journey' of personal concept formation, and on building a personal view of the mathematical world from interactions with objects and environments (multiple embodiment) and with people (social interactions during games). He concentrated more on the exploration of structures than on solving pre-formulated, 'posed' problems. There was no necessarily correct (or incorrect) answer in his mind, since there were no pre-formulated theorems or problems, either. The immersion into intuitions based on a personal history of efforts was meant to provide the learner with the enjoyment of mathematics.

Dienes's childhood, just as Polya's in an earlier stage, was situated in a period of school reforms in Europe, and in Hungary. A series of teachers' societies emerged after the 'Ausgleich' (Compromise of 1867) that followed the 1848 Hungarian Revolution, and multicultural education catalyzed cross-cultural interactions between different layers, social and ethnic groups of the society, as well as transfer processes between arts and the sciences, between different disciplines, intellectual, political, and life-reform movements. The generative and fertile effects of these interactions which contributed to, and grew out from what is sometimes called the 'Hungarian Avant-garde' constituted a much more complex social development than it may look from the perspective of any single field. Its history is recently explored from the point of view of new arts, modernization, free schools, and reform pedagogy. (Beke, Németh and Vincze, 2013) Its effects in mathematics, the sciences, and on education were noted by several authors (e.g. by Frank 2012), but should be further explored as a process which relies and feeds on a considerable social network of personal and international relations as pointed out by Boreczky (2013).[5] It was a period when the rapid development of psychology and Durkheimian sociology brought new ideas for social understanding and catalyzed educational experiments facing the dominantly Herbartian pedagogy of the end of 19th century. This special intellectual era which in the end of the war concluded in the attempts of the 1918 and 1919 revolutions forced radical changes. Foreseeing, or in consequence

[5] The complex, interdisciplinary, social interaction, and network theoretic approach was initiated and taken by Ágnes Boreczky and her research group for the reconstruction and understanding of the "social and symbolic space [that] considerably went beyond the avant-garde" (Boreczky, 2013, p. 55), catalyzing modern civil attitudes. It existed until the 1930s in a politically irreconcilable situation that Zoltán left behind moving to England to his father at the age of 15.

of their failure, many thinkers who believed in a 'New Hungary' left, or had to leave the country. The following Christian-conservative turn brought about much more étatist educational conceptions and transformations.

For Dienes's parents this turn meant difficult years. After his parents left Hungary for Vienna and divorced, Zoltán went to live with his mother and brother in Nice, France, at a commune set up by Raymond Duncan, brother of dancer Isadora Duncan, who brought 'free dance' into Europe from the USA. The commune was a social experiment, where all the children were 'owned' in common (Dienes, 2003, p. 23). His mother eventually fled the commune with her children and made her way to Bavaria where Zoltán met again with his father before he returned to Hungary with his mother. His childhood and teenage travels in Europe provided him colorful social, cultural and linguistic experiences contributing to his open minded, observant attitudes. In his autobiography, Dienes reflects on his childhood experiences related to learning and discovering a new language as a way to understand learning itself (Dienes, 2003, pp. 59-61). His views are similar to Michael Polányi's (1962, pp. 92-95) conception of "tacit knowledge" that can not only be contributed to direct exchanges but to common school, and life experiences in their youth, to resources coming from reform theories of "natural learning" through excursions, body culture or dance-schools:

> We must consider for a moment the difference between *natural* and *artificial* learning, for this provides important clues to the understanding of children's difficulties when they are confronted with artificial scholastic learning situations. If a child is taken to a foreign country where his mother-tongue is not spoken, within a few months he can speak the new language as well as his newly acquired friends, whereas his parents may still be struggling with grammar years later, trying to learn the language 'properly'. It is, of course, the child who has learnt it 'properly', and the reason is that he learnt it naturally. Fortunately, it is impossible to learn skating or riding a bicycle from a book, or many people would have a try. The 'fiddling around' with the data is the only way of making sure that you do not eventually take a tumble on the ice or fall off your bicycle. Natural learning is not invariably preferable to artificial learning; it is however *a priori* probable that it will be more effective. (Dienes, 1964, p. 24)

Pólya's views on learning and teaching mathematics resemble Dienes's assertions above. They suggest that the teaching of mathematics need to involve not only demonstrative reasoning or 'artificial scholastic learning', but also plausible reasoning and fiddling around with mathematics by making guesses. This is the 'natural way' of learning mathematics:

> Mathematics has two faces. Presented in a finished form, mathematics appears as a purely demonstrative science, but mathematics in the making is a sort of experimental science. A correctly written mathematical paper is supposed to contain strict demonstrations only, but the creative work of the mathematician resembles the creative work of the naturalist: observation, analogy, and conjectural generalizations, or mere guesses, if you prefer to say so, play an essential role in both. A mathematical theorem must be guessed before it is proven. […] Older writers, as Euler

> and Laplace, did not fail to notice that the role of inductive evidence in mathematical investigation is similar to its role in physical research, but more modern writers seem to have forgotten this remark almost completely […] an ambitious teacher of mathematics […] should, more than any particular facts, teach his students two things: First, to distinguish a valid demonstration from an invalid attempt, a proof from a guess. Second, to distinguish a more reasonable guess from a less reasonable guess. […] At any rate, we should not forget an important opportunity of our profession: *Let us teach guessing*! (Pólya, 1984, p. 512-520).[6]

Dienes enjoyed listening to adults' mathematical conversations when he was a child, and bonded with his father through discussing mathematical ideas. These positive childhood experiences may have inspired him to give similarly positive experiences to other children, 'replicating the magic' that happened with him. He was capable of grasping some sophisticated mathematical concepts at an early age. Many of his educational experiments were to see whether any(!) mathematical concept could be taught to children, through providing adequate multiple embodiments and contrasts (Dienes, 1964, p. 40). These methodological points which were influenced by Piaget's structuralism and the developments of experimental psychology in the 1920's and 30's that contrast theories of formal operations after World War 2 (WW2) can be traced back to artistic theories of abstraction of modern Constructivism and set in parallel with Polányi's conception.

Contrasts and differences

Proofs, operations, structures, and embodied procedures

Pólya placed higher importance on proofs because of the higher age-group of his students. He considered exploring ways of finding different proofs for the same theorem essential and a *bona fide* result of the heuristic study of a problem. He encouraged his students to find special cases first, and then to generalize by testing various conditions and models. He often studied proofs to explore the problem-space and to widen applicability or generalizability of solutions. He extended the understanding of mathematical concepts and theorems to applications in other fields and *vice versa*. His mathematical works showed an extremely wide interest in applied mathematics, but always addressed problems that were mathematically fundamental. There are more than forty concepts, methods, and theorems named after him, and many of them are considered deep insights that opened up new fields of application (Royden, 1989, p. 251).

Operations, not only abstract ones, but perceptual ones, either as embodied actions or as manipulation of physical objects, and structures of the environment that can be experimentally experienced were of key importance for Dienes. From his childhood, he considered indirect proofs as genuine ones, while being second rate ways for understanding mathematical reality. In teaching mathematics,

[6] Pólya's lecture, demonstrating his ideas 'in action' by instructing a mathematics class for university students with the title *Let Us Teach Guessing* is the first video recording made by the Mathematical Association of America in 1966.

especially for young learners, he preferred activities that turned the features of numbers or topological properties into a human experience and used representations that could be physically perceived. He used steps, jumps, and hops as moves, for experiencing number theoretic operations, and made the learners build toy harbours and waterways at the banks of a river to provide experiences about various concepts of topological connectedness (Dienes, 2003, p. 328).[7] He used musical tones, not just to represent mathematical relations, but to transpose and transform regularities into musical experiences of abstract operations (e.g. multiplication) or relations, making their essence understandable, and amused children by teaching them to sing and express with arm positions the similarities of multi-tonal, transposed musical structures in chorus, in the form of "mirror" or "augmentation canon" in the spirit of the Kodály method (Dienes, 2007, pp. 11-12).

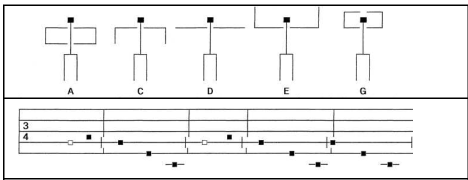

Fig. 1. Musical tones expressed by arm positions (Dienes, 2007, pp. 11-12)

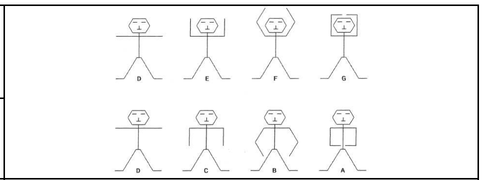

Fig. 2. Symmetricity of arm positions relative to note D (Dienes, 2007, p. 55)

Interdisciplinarity from the point of view of problem solving (Pólya) and concept formation (Dienes) belonged to different contexts with different meaning for the two heuristic thinking instructors. However, both of them were against the division of the intertwined aspects of human activities into different disciplines. For Pólya, it meant the universal presence and applicability of heuristics in all disciplines throughout historically interconnected problems. For Dienes, it meant "threads", such as time, form, contrast, meaning, or the use of symbols that "run through" different fields of human experience (e.g. mathematics, language, music, bodily movements). These threads, as embodied procedures, that establish "connexions between spatial and temporal sequences", can be *organized* (e.g. in games) so that they form patterns (Dienes, 2007, pp. 43-52, Kántor, 2006). The 'interdisciplinary threads' also provide the basis for jumping from one domain of abstraction to another discipline, understanding the same structure in different settings. It helps to operationalize, also in a computational sense, the construction of functions according to his functions principle (Dienes, 2007 p. 58, p. 68, Benedek, 2018). His slogan was: "Give me a mathematical structure and I'll turn it into a game". He could lead children from bodily movement experiences and the description of positions of chairs in the room through the playful explorations of physical transformation paths of chairs to an integrated abstract experience of operations of a 3D (or higher) transformation group, and wrote a whole book of "algebra stories" (Dienes, 2002).

[7] For a representative collection of potential influences originating from 20th century art movements that could also have an effect on Dienes's methods cf. Weibel, P. (Ed.) (2005).

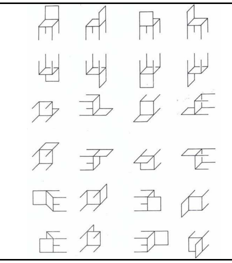		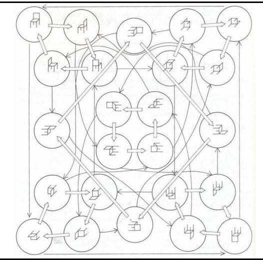
Fig. 3. The 24 position of chairs in the "Chairs-game" (Dienes, 1989, p. 153)	Fig. 4. Dienes playing the *Chair-game* with children who compete in finding the shortest inverted path between two positions reducing cycle length	Fig. 5. The discovered transformation groups and their relations (Dienes, 1989, p. 154)

Time: Duration versus Historicity

According to the standard interpretations, Dienes merged his math didactics with Piaget's psychology of concept formation, while Pólya did so with the history of mathematical thinking and personal progression of thought.

It is Dienes's system of thoughts, however, that can be traced back to Henri Bergson or the 20th-century schools of Process Philosophy, pointing towards the role that games, activities and stories play in children's thinking. One of Dienes's initial insights was that numbers, in the early concept formation of a child, within the Piagetian pre-operational developmental period, rely on the experience of succession (ordinal, rather than on cardinal number (Dienes 2007, pp. 3-4). This insight, that goes against contemporary formalisms of the 50's and set-theoretic conceptions of the 'New Math' of the 1960's, had computational consequences and originated from many sources for and against Piaget's work. Dienes's first resource included childhood experiences with time and rhythm in dance, music, and bodily movements, and later, transfers from, or rather "resonances" to, his mother's system of thought. His mother, Valéria Dienes played a major role in teaching and conceptualizing modern dance. Between 1908 and 1912, she studied in Paris with Bergson and translated Bergson's works into Hungarian. Inspired by Bergson and the dancer Isadora Duncan, she founded the school and theory of orchestics, which is the first modern dance conception that builds up a communication theory. Duration, the central concept of Bergson's philosophy of time, plays a crucial role in Valéria Dienes's dance-theory. In her theory, sequences of bodily movements and positions, are perceived and understood as quasi-grammatical semantic components in the space-time structures of communicative human expressions (Dienes, 2016). Similarly, the perceptual experience of kinaesthetic patterns is just as good for creating, as for understanding, the regularities of basic mathematical operations associated with sounds or bodily movements in Zoltán's game-based didactic methods (Holt & Dienes, 1973, p. 110). In this way, he creates personal knowledge, in Polányi's sense, that involves first perceptual experience, and afterwards, via kinaesthetic variation and sequential repetition, constitutes the abstractive structural affordances of this knowledge. Lastly he extends, in an analogical manner, the expressive time dependence of acts of meaningful motion (as those of modern dance) to experience with symbolic operations, and to mathematical thought expression as an abstractive movement of thought.

The processual aspects of action and understanding ("awakening") as fundamental features of the subjective, but communicative, and intentionally meaningful human reality on behalf of Valéria Dienes on one hand, and the perception and abstraction of the regularities of experienced phenomena via embodied operations as key components in the understanding of mathematical structures, on behalf of Zoltán Dienes, have common roots. They can be linked to the 'Systems Movement' of the 'Hungarian Avant-garde', Bergson's time conception, phenomenological music theory, and artistic Constructivism. Dienes raised certain questions that both Piaget's theory and Brunner's school struggled to answer which can be summed up in the following question: "What makes the jump between different levels of abstraction and what constitutes the transformation from pre-operational to operational stages: how can the "move" be taught and reproduced?" (Dienes 2007, pp. 15-16). He spelled out his position on the developmental stages of abstraction in math psychological terms (Dienes, 2007, 62-86, Benedek, 2018) but preferred demonstrating instead of arguing that the jump is 'teachable'.

Pólya's historical approach is linked to questions that address the role of time, and temporal order in mathematical discovery. He considers the feedback of the history of thought and temporal emergence of solutions in individual problem-solving a "hermeneutic circle" (Kiss, 2009). He connects history with the mental development of understanding: with the ways of (re)producing the processes of awakening in the arts and the sciences.[8] This was a major theme of inter-disciplinary discourse with the involvement of many fields before WW2, and an issue of the "Genetic Method" that received continued attention afterwards, partially due to Hadamard (1945). It is most concisely expressed in the 1962 Memorandum of 75 mathematicians (Pólya played an active role in its formulation)

> as "… a general principle: The best way to guide the mental development of the individual is to let him retrace the mental development of the race - retrace its great lines, of course, and not the thousand errors of detail. This *genetic principle* may safeguard us from a common confusion: If A is logically prior to B in a certain system, B may still justifiably precede A in teaching, especially if B has preceded A in history."[9]

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[8] In contrast, Polányi was interested in the relationship of ontogenetic and the phylogenetic development and the logic of emergence. (Polányi 1962, pp. 393-94, 337-39, and fn. 3, 4). Dienes, studying transfer processes between structures, looked for principles that explain the moves between stages of personal cognitive development (Dienes and Jeeves, 1970).

[9] Before the above lines it reads: "**Genetic method.** »It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when it is in the nascent state« wrote James Clerk Maxwell. There were some inspired teachers, such as Ernst Mach, who in order to explain an idea referred to its genesis and retraced the historical formation of the idea." (Memorandum, 1962)

The historical aspects of discovery and Pólya's heuristics became a basic line of thought in Lakatos's philosophy of mathematics and rational reconstructions. Their relationship, as well as Lakatos's acquaintance with Polányi's conception, is documented by now (Kiss, 2009). Their common roots are, however, less known. Similarly, the origins of Dienes's methods were the same artistic and scientific backgrounds from whence Polányi's interest emerged in the temporal process of the development of personal knowledge. They originate in the progressive counter-culture and in the continuing professionalization into which most representatives of this short period of post-feudal modernization and social-political progress took refuge from the upcoming 'Christian Course' nationalism after the war.

The 'late' Hungarian fin-de-siècle

The closing of the 19th century and the onset of the 20th up to the 1920's was a special period in the intellectual history of Hungary. It was similar to, but in many respects different, late blooming of spiritual powers from the avant-garde movements of the European fin-de-siècle to which it was strongly tied. This 'late' development, however, did not mean the same decadency, degeneration, cynicism, or pessimism as elsewhere in Europe. It was instead a rapid cultural and economic modernization that was due already in 1848 but came rather late, absorbing every intellectual movement of European thought with fresh eye and critical reception. After the feudal and agricultural backwardness of the country and in the light of the administrative and political reforms 'civilization' had a different, less pessimistic meaning, and insights about its 'degeneration' allied with critical reconsideration of the developments of industrial societies. By the arrival of millenarianism and the World Fair of 1900, Budapest showed the face of a modern capital strongly linked to Vienna, Berlin, München, Paris, London, and aspired to belonging to an integrated global whole with the cosmopolitan German, French, Anglo-Saxon new world of the twentieth century.

This blooming was not the result of a revolution exchanging the ruling political establishment but the result of the 'Ausgleich' of the Austrian and Hungarian elite in 1867, which brought about a second reform period that changed its social and cultural composition. "The rise of a new urban middle class affected the school system." (Frank, 2012, p. 358) With the failed 1848 Freedom Fight, the educational and emancipatory initiatives of the civil (but noble-led) revolution were extinguished in result of the consequent political retorsion, but by the 1849 *Entwurf der Organisation der Gymnasien und Realschulen in Oesterreich* and after 1867, Franz Joseph I. established top-down educational reforms similar to the German transformation of the elementary, secondary, and university system and the modernization of the state. It was a change that coincided with the non-state controlled modernization efforts of an emerging civil society and bourgeoisie industrialization that also kept an eye on Italian, French and Anglo-Saxon educational reforms including new approaches of Felix Klein in Germany, John Perry in England and E.H. Moore in the US. Baron Joseph Eötvös (Minister of Religion and Education during the 1848

Revolution) could return to his position and was able to put through his crucial liberal reforms concerning educational rights, obligations and emancipation. Many Hungarian teachers, such as, for example, Mór Kármán, Manó Beke, László Rátz, and Gyula Kőnig (Szénássy, 1992, pp. 217-218) and intellectuals at all levels (Béla Bartók, László Moholy-Nagy, Ferenc Molnár, just to name a few key artists) experimented with new methods, invented new approaches, and took an active part in the formation of the cultural life of Europe. Members of the professional educational circles were integrated into the European scientific and artistic elite in various ways: through Masonic lodges, scientific associations, by family relations, or as private tutors.

> [T]his modernizing group came partly from the decaying landed gentry of feudal origins and partly from intellectually aspiring members of the assimilating (predominantly German and Jewish) middle class. While creating metropolitan Budapest in the intellectual sense, they constituted themselves as a group through what proved to be a completely new and unique social and psychological experience (Frank, 2012, p. 358).

This experience reached Pólya during his university studies of philosophy, when the renewal of art and scientific life peaked. Therefore, his constructive and reconstructive heuristic movements of thought rely on the problem solving practices of a reflective age, both historically and in a developmental sense underlying a belief in reflective human attitudes. Since he left Hungary by the 1910's starting his career as a mathematician, he missed several direct consequences of Hungarian politics and of the pedagogical and life-reform movements that played decisive roles in the lifeway of Dienes's parents. This is the period of women emancipation, new cults of childhood, the growing psychological attention to children's upbringing, and of alternative school, and life experiments. The fall of the Monarchy, the 1918-19 revolutions, during which Zoltán's father, Paul Dienes, was responsible for reorganizing the university, was a period, followed by the White Terror, the *Numerus Clausus* act, and revisionist territorial nationalism, that changed the political climate. It brought for Zoltán a series of diverse experiences, since both his father and his mother, who later divorced, were forced to flee the country. While Pólya started his 'random walks' at the age of 23, (in Vienna, 1911; continued in Göttingen 1912-13 after getting his PhD in 1912 at Budapest; then in Paris 1914; and in Zurich, 1914-1940, spending one year during 1924-25 in Oxford and Cambridge with G.H. Hardy), Zoltán started his 'petite promenades' at the age of 3 (in Vienna, Nice, Paris, Bavaria, returning to Hungary, Pápa, then to Budapest) and experienced a latent family counter-culture in opposition as a child. The mentality of alternative schooling, the educated but creative upbringing, was instilled in Dienes by his parents and their social circles. These circles included Mihály Babits, Anna Lesznai, Alice Jászi (né Madzsar), Ervin Szabó, Béla Balázs, Zoltán Kodály, Lajos Bárdos, and their Austrian, French, German and English friends who helped them after World War 1 (WWI). The diversity of cultural circles may partially explain his highly developed skills of communicating with children in result of being introduced into new and new schools and children groups from Duncan's commune through Hungary to England where he joined one of the most avant-garde schools of the country (Dienes, 2003, p 97-98, 123-124). His early interest in the foundation and philosophy of mathematics and his turn to the psychological and experimental aspects of mathematical concept formation was motivated both from his father's and mother's side, though his devotion to lower age groups came probably from the latter side. Dienes's father, in a younger age also his mother and his math teacher, Zoltán Ferenczy, imprinted the legacy of Hungarian mathematics in his mind in the post-WWI period. Pólya's and Dienes' father's circles were essentially the same both in Hungary and in England including G.H. Hardy, but Zoltán Dienes had different experiences at his mother's side at different times. Pólya, just as Dienes's father, was influenced directly by Lipót Fejér and the social, scientific and artistic trends in which Austria and Hungary brought forth its pre-war intellectual achievements. In light of Zoltán Dienes's temporally consecutive, culturally manifold inheritance, the two heuristic thinkers, with their own interpretations and reflections, seem to represent different periods in the organic development of theoretical and experimental approaches to mathematical discovery and creative thinking in Hungary and Europe, torn by two wars and political cataclysms.

A most needed outcome: Connecting the two heuristics

Dienes's innovations were often confused with the New Math of the 1960s that he substantially criticized. Even in Hungary where Tamás Varga succeeded in implementing Dienes's methods at a time when Pólya's heuristics were relatively well known, several factors hindered joining their heuristics both theoretically and in praxis. Procuring this defalcation by a synthesis is more and more exigent.

Dienes's and Pólya's approaches complement one another in many ways. (1) Their (1a) embodied and (1b) reflexive heuristics can be built on one another as the age and personal development of the students grow. (2) The rich perceptual experience obtained in games, movements and embodied manipulation of the situation can be combined with the more conscious planning and exploration of the problem space (2a) turning tacit knowledge into abstract concept formation, and (2b) using invented intersubjective symbolic expressions. The joint affordances of 2a and 2b pave the way for improving guesses and the discovery of proof ideas. (3) The levels of abstraction and operational stages of early developmental and former experimental periods can be gradually turned into methods that use symbolic expressions at every level. It (3a) preserves a sense for subsuming syntactic and formal tools to case-based meaningful understanding and (3b) the learner can use operations on domain dependent structures for the development of higher semantic architectures. (4) Linking events and occurrences along the students' (4a) temporal route to personal knowledge with (4b) the reconstruction of histories that (4c) retrace the mental development of the solutions of problems within a historical subject matter integrates the selection of domain dependent heuristic methods and heuristic selection methodologies in a shared common pool.

The combination of Dienes's and Pólya's methodologies makes reflection on our heuristic generation processes possible at all levels of conceptual and methodological development, applying symbolic representation for raising awareness. Such a synthesis could embed mathematical discoveries into the whole spectrum of temporal and historical development of conscious human activity, opening new ways for its symbolic and computational representations.

We indicated that their heuristics built on the cultural and artistic reform movements of the pre-, and post-World War I. period of Hungary and Europe. These movements incorporated visions and reform pedagogies that we can extend today to social learning, computational media and technologies, and to our social life. Adopting them, we were able to implement the essence of these reforms realizing the goals of the progenitors. Adapting their principles and methods we could go beyond what their contemporary practitioners could only partially achieve because WW2 interrupted their attempt to reform industrial civil society, and its education. We believe that the reconstruction of the methodological insights and the humanistic visions behind Dienes's and Pólya's methods of teaching creative thinking proves to be as appropriate now, as ever!

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