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Statistics of an Autoregressive Correlated Random Walk along a Return Path

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A closed-form analytical solution is derived for the statistical outcome of a random walk along a return path. The random walk is generated from the cumulative sum of correlated samples in a Gaussian-distributed autoregressive sequence. The outcome exhibits a smaller variance compared to a one-way path of equivalent length due to cancellation of correlated steps along the return leg. Furthermore, the variance decreases towards zero as the correlation coefficient approaches unity. An example application for this general result is the modelling of cumulative errors in dead-reckoning navigation systems, e.g., Doppler velocity log-aided inertial navigation systems used commonly on underwater vehicles. In this particular application, it can be used to express and quantify the natural cancellation of correlated error components between subsequent opposing legs in a typical “lawnmower” survey pattern.

Autoregressive Correlated Random Walk: Many processes occurring in engineering can be modelled by the accumulation of samples from an autoregressive correlated sequence driven with uncorrelated (white) Gaussian noise; this is a type of random walk. Consider the outcome of a random walk of n steps from the origin $X_0 = 0$ with a mean step size μ,

$$X_n = n\mu + \sum_{m=1}^{n} U_m. \tag{1}$$

For each step m, the correlated deviation U_m from the mean is generated by the iteration [?]

$$U_m = \rho\, U_{m-1} + \left[\sigma\sqrt{1-\rho^2}\right] W_m, \tag{2}$$

where $\rho \in \mathbb{R}, [0,1]$ is the correlation coefficient (quantifying the similarity between each step) and $W_m, U_0 \sim \mathcal{N}\{0,\sigma\}$ are uncorrelated Gaussian-distributed random variables with zero mean and variance σ^2.

Statistics for a One-Way Path: The statistical distribution $\mathrm{P}(X_n)$ of outcomes for the random walk is illustrated in Figure 1(a). The mean outcome is simply $\mathrm{E}\{X_n\} = n\mu$. The variance can be derived following the approach of Bazant [?, Lecture 9] beginning with the expectation

$$\sigma^2_{X_n} = \mathrm{E}\left\{(X_n - n\mu)^2\right\} = \sum\sum_{m_1,m_2=1}^{n} \mathrm{E}\{U_{m_1} U_{m_2}\}. \tag{3}$$

The term inside the summation is the covariance C_{m_1,m_2} between correlated samples of the sequence (2) at steps m_1 and m_2. Thus, (3) can be rewritten as

$$\sigma^2_{X_n} = \sum\sum_{m_1,m_2=1}^{n} C_{m_1,m_2}. \tag{4}$$

Assuming stationarity, the covariance is dependent on the separation $m = |m_2 - m_1|$ only and the covariance values over the entire path can be described by the symmetric $n \times n$ covariance matrix

$$\mathbf{C} = \begin{pmatrix} C_0 & C_1 & \dots & C_{n-1} \\ C_1 & C_0 & \dots & C_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & \dots & C_0 \end{pmatrix}, \tag{5}$$

where the single subscript denotes the separation. The common terms along the diagonals of (5) can be grouped, allowing the double summation in (4) to be reduced to a single summation

$$\sigma^2_{X_n} = n\, C_0 + 2\sum_{m=1}^{n-1} (n-m)\, C_m. \tag{6}$$

The covariance for the autoregressive sequence (2) is given by

$$C_m = \sigma^2 \rho^m. \tag{7}$$

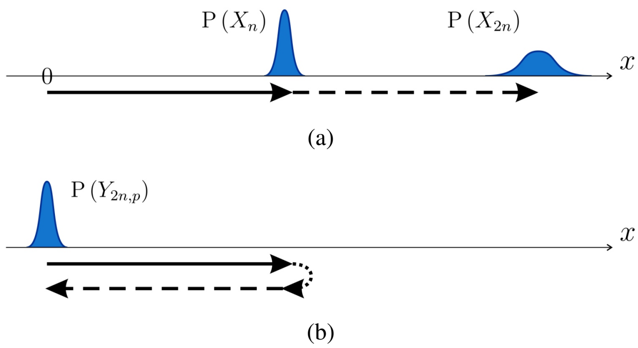


Fig. 1 *Statistical distributions of (a) positions X_n and X_{2n} after n and $2n$ steps of a random walk along a one-way path; and (b) position $Y_{2n,p}$ after two legs of n steps each along a return path and with a pause of p steps between legs.*

Substituting (7) into (6) and rearranging then gives

$$\sigma^2_{X_n} = -n\sigma^2 + 2\sigma^2 \left[n\sum_{m=0}^{n-1} \rho^m - \sum_{m=0}^{n-1} m\rho^m \right]. \tag{8}$$

Finally, the two summations can be recognised as a geometric series

$$S_n(\rho) = \sum_{m=0}^{n-1} \rho^m = \frac{1-\rho^n}{1-\rho} \tag{9}$$

and a function of its derivative

$$\rho \cdot \frac{\mathrm{d}}{\mathrm{d}\rho} S_n(\rho) = \sum_{m=0}^{n-1} m\rho^m = \rho \cdot \frac{1-\rho^n - n\rho^{n-1}(1-\rho)}{(1-\rho)^2}. \tag{10}$$

Substituting for these series and rearranging yields the closed-form solution

$$\sigma^2_{X_n} = \sigma^2 \cdot \frac{n\left(1-\rho^2\right) - 2\rho\left(1-\rho^n\right)}{(1-\rho)^2}. \tag{11}$$

Statistics for a Return Path: Now consider a random walk along a return path with n steps in one direction followed by another n) steps in the opposite direction. Consider also a possible pause of p steps between each of these legs. (For our underwater navigation application, this pause concerns a maneuver through 180 deg in preparation for the return leg.) The walk along the return path can be expressed by

$$Y_{2n,p} = \sum_{m=1}^{n} U_m - \sum_{m=n+p+1}^{2n+p} U_m \tag{12}$$

and the statistical distribution of outcomes $\mathrm{P}(Y_{2n,p})$ in the world frame is illustrated in Figure 1(b).

The mean outcome is simply a return to the origin, $\mathrm{E}\{Y_{2n,p}\} = 0$. The variance can be derived by following a procedure similar to that used for the one-way path, beginning with the expectation

$$\sigma^2_{Y_{2n,p}} = \mathrm{E}\left\{(Y_{2n,p})^2\right\}. \tag{13}$$

Expanding this gives

$$\sigma^2_{Y_{2n,p}} = \sum\sum_{m_1,m_2=1}^{n} \mathrm{E}\{U_{m_1}U_{m_2}\} + \sum\sum_{m_1,m_2=n+p+1}^{2n+p} \mathrm{E}\{U_{m_1}U_{m_2}\} - 2\sum\sum_{m_1=1,\, m_2=n+p+1}^{n,\, 2n+p} \mathrm{E}\{U_{m_1}U_{m_2}\}. \tag{14}$$

The expectations in (14) can be replaced by covariances. Moreover, each of the first two terms can be recognised as the variance for a one-way path (11), i.e.,

$$\sigma^2_{Y_{2n,p}} = 2\,\sigma^2_{X_n} - 2 \sum_{m_1=1,\, m_2=n+p+1}^{n,\, 2n+p}\sum C_{m_1,m_2}. \tag{15}$$

Again, assuming stationary, the covariance values over the entire path can be described by the symmetric $(2n+p)\times(2n+p)$ covariance matrix,

$$\mathbf{C} = \left(\begin{array}{cccc|c} & & & & \iddots \\ \hline C_{p+n} & \cdots & C_{p+2} & C_{p+1} & \\ C_{p+n+1} & \cdots & C_{p+3} & C_{p+2} & \\ \vdots & \iddots & \vdots & \vdots & \\ C_{p+2n-1} & \cdots & C_{p+n+1} & C_{p+n} & \end{array}\right). \tag{16}$$

The terms in the bottom-left $n\times n$ portion of this matrix correspond to the terms in the double summation of (15). The common terms on the diagonals can be grouped to reduce this to a single summation

$$\sigma^2_{Y_{2n,p}} = 2\,\sigma^2_{X_n} - 2\sum_{m=0}^{n-1}\left[m\,C_{p+m} + (n-m)\,C_{p+n+m}\right]. \tag{17}$$

Substituting (7) into (17) and rearranging gives

$$\sigma^2_{Y_{2n,p}} = 2\,\sigma^2_{X_n} - 2\sigma^2\rho^p\left[n\rho^n\sum_{m=0}^{n-1}\rho^m + (1-\rho^n)\sum_{m=0}^{n-1}m\rho^m\right]. \tag{18}$$

Finally, the summations in (18) can be recognised as the series (9) and (10). Making these substitutions and rearranging yields the closed-form solution

$$\sigma^2_{Y_{2n,p}} = 2\sigma^2\cdot\frac{n\left(1-\rho^2\right) - \rho\left(1-\rho^n\right)\left[2+\rho^p\left(1-\rho^n\right)\right]}{(1-\rho)^2} \tag{19}$$

for the variance of the outcome after a correlated random walk along a return path.

Results: We validate the expressions (11) for the one-way path and (19) for the return path by comparing them with Monte-Carlo simulations. We also explore the general trends with respect to the correlation properties and duration of the walk. For equivilence, we consider a total of $2n$ steps for both the one-way and return paths, as illustrated in Figure 1.

The correlation properties can be expressed independently from the sample rate by considering the exponential time constant τ, i.e.,

$$\rho = \exp\left(-\Delta t/\tau\right), \tag{20}$$

where Δt is the time period between steps. The time constant τ can then be normalised with respect to the duration of a one-way leg in the return path, i.e.,

$$\bar{\tau} = \tau/\left(n\,\Delta t\right). \tag{21}$$

The variances of the random walk outcomes can also be normalised with respect to the variance for the same random walk with uncorrelated steps, i.e.,

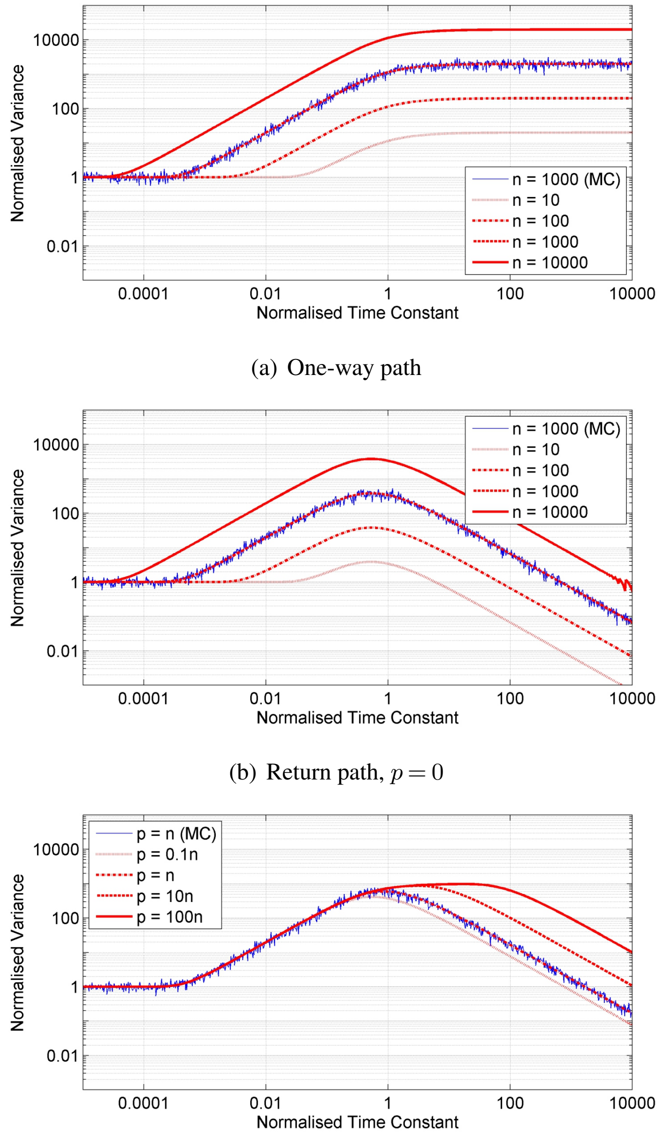
$$\bar{\sigma}^2_{X_{2n}} = \sigma^2_{X_{2n}}\,/(2n\,\sigma^2)\,, \tag{22}$$

$$\bar{\sigma}^2_{Y_{2n,p}} = \sigma^2_{Y_{2n,p}}\,/(2n\,\sigma^2)\,. \tag{23}$$

The expressions for the normalised variance are plotted in Figure 2 with respect to the normalised time constant for a varying number of steps. Monte-Carlo simulations were used to validate these results and examples are shown for $n=1000, p=0$ and $n=1000, p=1000$ to demonstrate the agreement with the closed-form expressions.

Both one-way and return paths exhibit the same trends for small time constants, tending to a normalised variance of one as $\bar{\tau}$ tends to zero. The trends are different for time constants comparable to or greater than the duration of a one-way leg. As $\bar{\tau}$ exceeds unity, the normalised variance of the one-way path approaches a limit of $2n$, whereas the variance of the return path tends towards zero.

Conclusion and Discussion: The novel contribution of this work is the closed-form analytical solution (19) for the variance of a correlated random walk along a return path. This generalised result has been visualised in Figure 2(b) and (c). It can be applied to a number of problems in engineering. However, the authors' particular interest is the modelling of error accumulation in dead-reckoning navigation systems



(a) One-way path

(b) Return path, $p=0$

(c) Return path, $n=1000$

Fig. 2 *Normalised variance in position after a random walk with correlated steps for (a) a one-way path of $2n$ steps and (b,c) a return path with two legs of n steps each and a pause of p steps between legs. The variance is normalised by an equivalent walk with uncorrelated steps and the correlation is expressed as a time constant that is normalised to the duration of one leg in the return path.*

– specifically, Doppler velocity log-aided inertial navigation systems used commonly on underwater vehicles. The expression explains the observation that correlated errors in the velocity measurements with time constants on order of the duration of a leg (or more) result in partial cancellation of the errors in subsequent opposing legs [**?**]. Furthermore, it provides a quantifiable prediction of this effect.

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