2018

Research Proceedings of BCME9

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Golding, J., Bretscher, N., Crisan, C., Geraniou, E., Hodgen J. and C. Morgan (Eds). (2018) *Research Proceedings of the 9th British Congress on Mathematics Education* (3-6 April 2018, University of Warwick, UK). Online at www.bsrlm.org.uk/bcme-9/

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Introduction

The British Congress of Mathematics Education, BCME, has its origins in an attempt by the British Society for Research into the Learning of Mathematics, BSRLM, to populate a gap in the 4-yearly international ICME conferences: BCME was originally intended to offer an opportunity for researchers in mathematics education to prepare for the international ICME, and to share their current work with interested others. Over time, BCME developed to have a rather broader remit, under the auspices of the UK Joint Mathematical Council, JMC, and in particular, to include addressing the interests and needs of those in the classroom-facing professional associations. Recent BCMEs, including BCME9, have been organised by JMC with particular input from BSRLM, ATM and MA, but supported also by other JMC participating bodies, and with a key aim of bringing together researchers and practitioners.

One strand of BCME9, held at University of Warwick 3-6 April 2018, therefore focused on current mathematics education research, and included over 50 such sessions. Post-conference, researchers were able to submit formal papers related to their conference sessions for peer review, and if accepted, to have those published in these Research Proceedings. Others opted instead to publish shorter papers in the all-comers' Informal Proceedings, now available at www.bcme9.org. The research strand featured novice researchers, some school- and some university-based, as well as welcoming those with more, sometimes substantial, research experience. One particular aim of the editorial team, though, was to particularly encourage and support those beginning their journey in mathematics education research, and differential support was available for such authors in their preparation of papers for this volume. It therefore represents the outcome of the formal peer-reviewed process for the range of accepted submissions, and it has been our very great pleasure to work with authors in preparing their papers for publication: we hope they will feel the outcome justifies their effort!

What we see exhibited here is the rude health of research in mathematics education in the UK, together with its variety - by phase of education from early years to adult, by research focus, and by theoretical and methodological framing. Papers are presented by alphabetical order of first author surname, but key themes include emerging modes of teacher education, the use of resources, including digital, in the mathematics classroom, and pathways to more effective formative assessment. English schools in particular are currently grappling with significant curriculum reform in mathematics and related areas, and we see that reflected in these papers in a constructive focus on ways to support learners in coming to achieve a deeper and more connected conceptual understanding, with well-developed mathematical reasoning and problem-solving capabilities: all are in some way addressed here, and all address issues which are of global interest in the 21st-century.

Most of the papers included in this volume report on small scale qualitative studies which, though not necessarily generalisable, offer reasonably nuanced indications of what might be achievable. They are complemented by other articles which report on emergent theoretical frameworks which have the potential to move our understanding of mathematics education in more focused, and sometimes new, directions.

We live in fast-changing times, where the broad aims of mathematics education might persist, but specific goals and the means to achieving those, as well as the applications of the resultant learning, are likely to remain fluid: challenging, but interesting, times for teachers and learners – and so of course, also for researchers. It is our belief that this volume, freely available online at www.bsrlm.org.uk/bcme-9, will have achieved much of its purpose if it is used by both researchers and practitioners as a source of continued cross-community dialogue in pursuance of our common goal: of appropriately evolving and empowering mathematics education for all.

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October 2018

Golding, J., Bretscher, N., Crisan, C., Geraniou, E., Hodgen J. and C. Morgan (Eds). (2018) *Research Proceedings of the 9th British Congress on Mathematics Education* (3-6 April 2018, University of Warwick, UK). Online at www.bsrlm.org.uk/bcme-9/

"I get better and better all the time": Impact of resources on pupil and teacher confidence

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We report on the findings from the first year of a two-year study exploring how teachers and children experience and use Pearson Abacus resources, including perceptions of impact on (teacher and children's) confidence. Abacus was designed to foster a confident learning environment for children to master mathematical concepts within the 2014 English National Curriculum. Data were collected from nine schools: from teachers and pupils in nine KS1 classes and nine KS2 classes, and from the schools' Maths Coordinators. Teachers considered Abacus impacted positively on both their own and children's confidence to work mathematically. However, some teacher confidence may not be well-founded, and the learning potential of the resources is not being harnessed, if they do not use the support provided to enhance their subject (and subject pedagogical) knowledge for teaching a richly conceptual network.

Keywords: Abacus; confidence; self-efficacy; resources; primary.

Introduction

Multiple studies have shown that young people often lack confidence in their mathematics functioning – and further that their confidence often declines with age (e.g. Hannula, 2012). Related research has often taken place within a secondary context. This paper is based on a 2016-18 study exploring the impact that the use of Pearson Abacus resources has on pupil learning/experience in a sample of English primary schools. In England, primary teachers typically teach one class across the curriculum, so are not usually mathematics specialists. Teacher confidence in teaching mathematics is therefore also often an issue (Ofsted, 2012). We therefore asked, 'What impact do the Abacus mathematics resources have on teacher and pupil confidence?'

Background

The Resources

Abacus is a set of English primary (usually age 5-11, years 1 to 6) mathematics materials, developed in line with Oates' (2014, p. 4) characterisation of effective resources. The resources are primarily accessed electronically on ActiveLearn, a digital learning space that includes a toolkit for teachers and pupil resources. This is complemented by a range of text books and progression workbooks for pupils.

As described on the website (Pearson, n.d.), Abacus has been produced to "inspire confidence and a love of maths" as well as to "help your school develop confidence in using Abacus". Based on the 2014 English National Curriculum (DfE, 2014), the Abacus objectives reflect that programme of study, mirroring a government aspiration for higher attainment in mathematics given perceived mediocre performance in international comparisons. The pupil resources aim to engage and inspire children to learn mathematics, creating a confidence-supportive environment including through support for teachers in their understanding and use of the resources.

For example, the online teacher toolkit includes a planning tool (at a variety of scales), the 'teaching tools' - whole class and interactive activities - and a variety of assessment and tracking tools and tests, together with reporting tools. There are adaptable daily, weekly or termly lesson plans that include substantial teacher support, pointing to likely misconceptions and ways to expose and address those, prerequisite knowledge, learning design and opportunities within the resources, key probing questions and valuable responses to those. These provide for varying levels of teacher experience and confidence. Examples of teaching tools include the bar modeller, '5-minute fillers', 'QuickMaths', 'Fluency Fitness', 'mastery checkpoints' and homework sheets. Accompanying these are interactive digital versions of many related physical resources, for class projection.

The literature shows resources convey specific messages about mathematics and its organisation (Raman, 2004), as well as influencing what and how mathematics should be taught (Love & Pimm, 1996), though Chevallard (2003) shows teachers often ignore suggested approaches or elements unless those are already present in their 'personal relationship' with mathematics.

Confidence and related characteristics

Affect is a key variable in students' learning (Hannula, 2012). While academic literature uses a broad range of theoretical constructs to explore self-confidence, the Oxford English Dictionary (2017) defines it as "a feeling of trust in one's abilities, qualities and judgement". Some theorists suggest that students' confidence in their own abilities is a better predictor of achievement than their current attainment (Pajares & Miller, 1994). For the purposes of this study two key constructs, academic self-concept and academic self-efficacy, will be taken as being key to understanding pupils' confidence in mathematics. The two constructs are grounded in social cognitive theory which suggests that students' potential is dependent on the relationship between their own behaviours, personal factors (e.g., thoughts, beliefs), and environmental conditions, pointing to the centrality of classroom learning environment and ethos.

Bong and Skaalvik (2003, p. 10) define academic ***self-concept*** as "knowledge and perceptions about oneself in achievement situations". This includes an individual's broad appraisal of their own competence, as perceived over an extended period of time, and is informed by frames of reference that are likely to be grounded in social comparison. In contrast, academic ***self-efficacy*** is embedded in specific contexts, even in specific tasks. It is less contingent on "what skills and abilities individuals possess", instead focusing on what students believe they can achieve with those skills and abilities. These beliefs are likely to change over time and are linked to students' previous experiences of undertaking a given task. Bong and Skaalvik (2003) show that that self-efficacy and self-concept are distinct, if related, concepts with self-efficacy feeding into students' more holistic and stable sense of self-concept.

Students' levels of ***motivation***, and of cognitive, affective and behavioural ***engagement*** are also strongly interrelated with feelings of self-efficacy, self-concept and ultimately achievement (Bandura, 2001). Motivation can be understood as either being extrinsic (based on external social factors) or intrinsic, where students are engaged in an activity chosen or pursued for its own sake. Intrinsic motivation is key to achieving meaningful learning (Schweinle, Meyer & Turner, 2006). Motivation is influenced by the nature of the task the students are set. In addition to their expectations of success, the personal value that they place on the outcomes is also important (Eccles & Wigfield, 2000). Pedagogy should therefore develop these characteristics. The development of a growth mindset, explained for Abacus teachers by Pearson (n.d.) is also important for intrinsic motivation. In contrast, there is evidence showing a 'fixed mind set' is often pervasive in English mathematics education (e.g. Ofsted, 2012).

Where digital technologies are used, they have the potential to increase mathematics students' intrinsic motivation (Calder, 2011), potentially providing another dimension to classroom learning. Mathematics-focused digital learning practices may also help primary-school-age students significantly raise their mathematics related self-efficacy (Hung et al., 2014). For example, multiple representations such as those easily afforded digitally are key to children developing deep conceptual understanding (Bryant, Nunes & Watson, 2009).

The Study

We report from the first, qualitative, year of Pearson-funded research which asked how teachers and children experience and use the Abacus resources; ethical approval and use of external researchers addressed issues of funding-related threats to validity of outcomes. We base our discussion on findings from 3 sub-questions: 1) To what extent do the resources as used engage children in mathematics? 2) Which aspects of the resources impact on their confidence? and 3) To what extent do the resources support teachers' confidence? Data were collected as shown in Table 1 and then analysed by sub- question in N-Vivo and axially coded. Coding was validated by at least one other researcher, and final interpretations and reports offered to field researchers and teacher participants for further validation.

Table 1: Summary of data collection

Fieldwork	**Methods Used**	**Data**
Autumn 2016	Standardised baseline assessment of individual and class-level characteristics. Telephone interviews: 18 class teachers + 7 (other) maths coordinators (MCs)	18 class assessment reports 25 interview (i/v) transcripts
Spring 2017 visit	Lesson observations. Class teacher (plus trainee teacher) interviews. Pupil focus groups	19 i/v transcripts Plans and observation notes for 18 lessons 18 focus group transcripts
Summer 2017	Teacher and MC interviews	25 i/v transcripts

Twelve participant schools were selected based on a variety of characteristics (type, size and inspection categories) as well as of socio-economic and geographical contexts. Additionally, schools had also bought different combinations of print or digital Abacus resources. Nine became established participants in the study; three others withdrew during early autumn 2016 due to local changes. There is no claim to generalisability from the study: rather, it aims to provide an in-depth understanding of a range of use and impact of the Abacus resources.

Findings

This first year highlighted that at least 15 of 25 teachers perceived challenges with pupil confidence in their classrooms, with many referring to fixed mind-sets:

> It's …not being scared of numbers, just having their confidence, I mean some children are really under confident when it comes to things like maths, or they don't understand it so they just shut off. (Year 5 teacher 9, Autumn interview)
>
> They're too quick to jump, because maths is a right or wrong. If they're not sure, they…think it's wrong and don't attempt it. (Year 5 teacher 4, Spring interview)

By the summer interviews, however, at least 14 of 18 class teachers reported confidence in mathematics had grown amongst the pupils. Below, we discuss the reasons given for this.

Children's engagement with the resources

Engagement is clearly a prerequisite for classroom learning, so was an aspect of initial probing in teacher interviews. Where, additionally, a classroom offers an environment fostering deep, conceptual learning then well-founded mathematics confidence can develop. Among physical resources, Abacus textbooks were the most used resource for year 5s whereas the workbooks were most popular for year 1s. Teachers widely endorsed these resources as engaging for children, pointing in particular to the colours, usability, characters and range and variety of activities.

While physical resources were highly praised in the interviews, teachers particularly noted the interactive whiteboard (IWB) front-of-class activities as a means of sustaining children's engagement in learning. All 18 class teachers pointed to at least one part of the IWB activities that children found particularly useful or engaging. Praise was primarily centred on the opportunity afforded for teachers to place learning in a different context:

> …other than me writing on the board constantly then just following along the same old sums and whatever. It just puts it in a different context and makes it a little bit more fun so it engages them a bit more I think (Year 5 teacher 1, Autumn interview)
>
> Most units I would use the interactive whiteboard activities because they are very engaging and most of the time they are super. They love the things like the number line with the dinosaurs, when they roar when it moves up and down (Year 1 teacher 4, Autumn interview)

These examples support the wider research that discusses the importance of authentic representations (e.g. Bryant et al., 2009). Dinosaurs moving up and down may not be a realistic representation of a number line, this particular activity exemplifies an authentic model and academic task that engages children in learning. Similarly, a Year 1 lesson observation illustrates the use of a digital clock tool activity. This task and activity can be applied to a real-life context, immediately underpinning the children's learning in a context they are already familiar with:

> The clock tool worked extremely well in this lesson – it is such a flexible resource that teacher could adapt. It was particularly powerful to be able to show digital alongside analogue e.g. when counting in tens: the count was visible on the digital clock; 1/2 past – the digital clock reinforced idea that half an hour is 30 mins. Children were very motivated by being able to click the button to forward the clock – the large visual image was very helpful (Year 1 lesson observation notes)

Teachers felt that context and relatability were important and therefore, the more practical they made the subject, the better. The practical activities offered by Abacus proved to be hugely popular with the pupils. Teachers claimed that involving pupils with physical, as well as digital, resources increased pupil engagement and enjoyment in lessons. Practical activities were particularly valued as they were seen to be very effective in supporting links underpinning deep conceptual understanding:

> I know it's very simple but they really love it because they can see that a number is being represented in front of them physically and I think for a lot of them it took a while. If I was to write a number on the board they knew which number it was but they didn't really fully understand what the number represented. But when I put, if it be blocks or Legos or even just a dice, they could see it in front of them and they understood then right nine means nine dots or nine or six dots or so on and so on. (Year 1 teacher 2, Summer interview)

Evidence from teacher interviews is consistent with wider research (Bryant et al., 2009) that suggests as children become more actively involved in their learning, in a variety of ways, there is an impact on engagement, motivation and maths related self-efficacy. The collection of learning resources provided in the Abacus scheme allows teachers to create an engaging and motivational learning environment that cater for a variety of learning needs. At least 14 of the 18 teachers reported that, as a result, a range of their pupils built and developed their confidence in working mathematically.

Abacus and children's confidence

Many teachers (at least ten) noted that Abacus' spiral structure and the repetition of focus over time benefited the development of children's confidence over the school year:

> I think the scheme does help in that way because of the way it's sort of cyclical revisiting things so if they didn't get it the first time you come to it another time and it's presented in a slightly different way. And they think 'oh actually I have seen this before and I think I can do this'… a lot of them are sort of emerging as more confident mathematicians. (Year 5 teacher 4, Summer interview)

> The way it goes back to each area: I think that's good for their confidence because sometimes, even after doing, say, a topic for a week, some of them might not get it or they might not be confident in the fact that they've got it. And the fact that it generally goes back to the same sort of topics over a period of weeks...does wonders for their confidence, because then they're able to keep practising. (Year 5 teacher 6, Summer interview)

Furthermore, teachers also suggested that the scaffolded progression helped pupils to visualise their own progression and achievement:

> And it does develop. They get quicker, they get more confident because the first one's easy, and then they can build it up to the harder ones towards the end. (Year 1 teacher 3, summer interview)

> I get better and better all the time. (Year 1 pupil, Spring focus group)

The differentiated and progressive approach to the activities were also mentioned as an effective means of impacting pupils' confidence:

> a lot of the children enjoy doing … the support work first before they move onto the core because usually it's the support page in the textbook gives them step by step instructions about how to solve it, whereas the core page will literally just say, here's a problem, get on with it. So usually if there's an issue with confidence, I suggest to the children, well you can do the support work first. But then you need to get onto the core...and I think that does help build their confidence (Year 5 teacher 9, Summer interview)

Finally, the ActiveLearn Games were perceived to change the way that pupils approached learning and had a measurable impact on their confidence: 'I think the online games have helped developed their confidence.' (Maths coordinator 4, Summer interview) This evidence reinforces the notion that engagement is a prerequisite to building confidence. One pupil stated: 'The ActiveLearn, it's really fun because you do the sums and the maths but you get to do a game as well, so it's fun.' (Year 5 pupil, spring focus group). At least three teachers went further, pointing to specific children who had begun the year with significant mathematics anxiety, but had progressed to being keen to the point of asking for extra mathematics tasks or games.

Teacher confidence

The Williams Review (2008) is clear about the enormous impact the teacher has on creating appropriate and confident learning environments and supporting valued learning outcomes in mathematics, even if mediated by appropriate and motivational resources. We therefore included questions about teacher knowledge, skills and affect in our interviews. A consistent theme that emerged was a recognition of the responsibility of teachers to effectively understand and implement the resources in order to best impact students, but also stories of teachers *coming to learn* how to best use Abacus. As one teacher explained:

> It is difficult because the best teacher in the world can make the worst resources look good and the worst teacher in the world can make the best resources look bad. It is how the teacher uses and delivers them that affects the motivation. (Maths coordinator 4, Autumn interview)

A positive example of this was teachers' productive use of pair work, as discussed by at least 8 teachers in interviews:

> Sometimes I'd get them to pair up because some of them are very shy. And I paired them up with somebody who was a bit more confident, a bit louder and I got them to maybe do an activity or a game together to do with what we were learning. And I found that it made them a bit more confident to speak but also more confident with numbers. (Year 1 teacher 2, Summer interview)

Teachers were also clear that pupil confidence is directly influenced by teacher confidence. Only two of the 25 teachers interviewed came from a mathematics specialist background, with many of the others (at least fifteen) describing how the Abacus resources had improved their own confidence in teaching mathematics. Of the Year 1 teachers, for example, 8 of 9 had only studied mathematics to age 16, with some even stating that they were 'maths-shy' in general. That the Abacus resources can be instrumental in shifting that confidence, then, including in the early years of teaching, is an important finding:

> I had …a student teacher, she's in Year 1 of her teaching degree and even she said to me that the session plans for Abacus are so helpful for her because they were so thorough and she could, she could take that lesson plan, read it over, and feel completely secure in delivering that to the class, which, for a Year 1 student, is quite an impressive comment (Year 1 teacher 8, Summer interview)

One interview with a trainee teacher provides an affirming example of the support Abacus provides for teachers lacking in experience and confidence:

> I love teaching it, and I really love Abacus, it's just very helpful when you're starting off with no background experience in teaching maths….Especially you know when I started with Year 1 I had no experience with Year 1, I didn't know what sort of

> level they worked… It's a really good starting point. (Trainee teacher 6, Spring interview)

Every teacher interviewed also praised the flexibility of the planning resources, usually for supporting a wide variety of teacher background and expertise, so teachers can adjust them to meet their specific needs. Two maths coordinators talked about how teachers who feel supported by the resources, and so confident in delivering the content, will create a learning environment best suited to develop pupil confidence:

> It’s given me an opportunity to feel confident in myself and to enjoy teaching it which in turn means that they will enjoy learning … it’s given me the confidence to be able to kind of deliver that securely. (Year 1 teacher 6, Spring interview)

However, observations showed that while responses to Abacus resources were almost entirely positive, many teachers were still not fully using the resource supports to their full learning potential, sometimes because of lack of familiarity. Several classroom observations pointed to occasions when resource design had been under-utilised because the teacher had a misplaced confidence in the depth of their subject knowledge, so that they missed learning opportunities factored into e.g. choice of examples. If they did not then make full use of the lesson plan guidance, children did not fully benefit from design intentions. At least 12 teachers also pointed to lack of time for teachers to get to know the resources in depth. None of the sample schools had bought in Pearson resource-specific CPD, and only two had used a CPD video included in ActiveLearn, choosing instead to come to know the resources informally and sometimes collaboratively. This last was talked about as a positive option, but might limit the depth of understanding of the intentions of the materials.

Conclusion

It is clear that the sample teachers feel that the use of Abacus, to whatever extent, significantly impacts pupil confidence. They suggested Abacus tools motivate and engage children, and so support an environment where pupils can develop their learning and build their confidence. All, but particularly the majority who are non-mathematics specialists, claimed that different facets of Abacus also impact positively on their own confidence as teachers. What the observation and other data clearly point to, however, is the importance of appropriate teacher understanding and use of the resources. Many teachers stressed the importance of this during interviews, placing onus on teacher enactment rather than on the resources themselves. When teachers are confident and effective in harnessing the resources to teach content, this in turn has a positive impact on pupil confidence.

However, lesson observations suggest that some of the teacher confidence (and so sometimes, pupil confidence) is not well-founded, as some teachers do not yet possess the deep subject (and subject pedagogical) knowledge necessary to teach for a deep conceptual network of mathematical concepts without external support. Such support is available, for example, in the lesson plan teacher notes but teachers do not always recognise the benefit of using those, so don't harness them. Critically, most teachers are not engaging with paid-for or in-package CPD provision which would point them to the benefits for children's learning of using, and acting on, those notes. Only if less specialist teachers access appropriate CPD will they be able to build and support well-grounded pupil confidence in meaningful mathematical functioning.

Acknowledgement

This research was funded by Pearson UK.

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Redesigning the assessment-feedback loop to enhance student engagement: a report of audio feedback

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The third and fourth year course Algebraic Number Theory at the University of Bristol is only assessed at the end of the course. To make sure that students are involved in continuous learning, they receive fortnightly problem sheets that they can hand in for marking and feedback. A project was developed to try and improve the problem sheets and feedback given to students. This report looks at the initial evaluation of the implementation of audio feedback, and the next step for this project.

Keywords: Assessment design; feedback; use of technology; university.

Framing this project within the learning and teaching in Higher Education (HE) literature

Up to the academic year 2016/17, Algebraic Number Theory (ANT) was a third (final year BSci) and fourth (final year MMath) year course in the School of Mathematics at the University of Bristol[1]. While the course is 100% assessed via exams for third years, and 80% exam plus 20% project for fourth years, students are also given non-assessed problem sheets throughout the course. Pre-spring 2017, the students would receive 5 problem sheets in the year (roughly fortnightly), which they could choose to do and hand it in for marking and feedback. The sheets varied in length and total marks. The marker (normally a PhD student) would mark each sheet that was handed in (based on solutions provided by the lecturer) and would provide a mark and personal feedback (not seen by the lecturer). The marker was also expected to write up general class feedback (that was seen by the lecturer) and upload it onto Blackboard[2].

Assessment can serve many purpose such as: to get students to check what they know; for the lecturer to see how effective their teaching is; to diagnose students' difficulties; to motivate students to study; and to help develop students' skills and knowledge (Kahn, 2003; Cox, 2011). As formative assessment, the problem sheets are meant to focus on the last two points. Part of this project was to redesign the problem sheets to see if they can further develop students' skills and knowledge of ANT. As a pure mathematics course, ANT relies on definitions and proofs. While many of the definitions will not have been encountered by the students before, as they are in their final year, they will already have many *concept images* built in from their previous courses. As such "one should do more than introduce the definition. One should point at the conflicts between the concept image and the formal definition and deeply discuss the weird examples" (Vinner, 2002). By learning and understanding *definitions* and

[1] In spring 2018, ANT became a fourth-year course only

[2] Blackboard is an online system used by the University of Bristol to manage courses (Virtual Learning Environment). Each course has its own page where students (registered on the course) can access lecture notes, recorded lectures, assessments, marks etc.

mathematical objects students can more successfully engage with the rest of the material, including proofs and higher concepts. Such understanding can be retroactively gained as students work through other materials, but part of the problem sheets should help students with this process of learning definition and building new concept images. Part of the problem sheets should inform the students on the type of problems lecturers expect them to solve (Biggs, 1996; Gibbs, 1999). Therefore, they can diagnose themselves on where they are compared to the learning objectives of the course. Finally, mathematics courses have many links to each other, which students should be exposed to if possible (but with the expectation that only a few students would be interested in exploring).

For these reasons, in Spring 2017, I trialled the use of three-part problem sheets: Part A: Questions that get students to think about the new definitions and theorems they have seen, including boundary cases, these questions were based on ideas by Alcock and Simpson (2009, pp. 14-16 & 30-31); Part B: Problems lined up with the learning objectives, that students should be able to solve if they understood the current material in the course; Part C: Extension questions to challenge students, indicating links between parts of the course, as well as with other courses or areas of mathematics research. Only Part B was required to be handed in for marking and feedback, but the students were encouraged to at least read Part A. I encouraged students to have a think about the questions in Part C, both in lectures (by pointing out questions that generalise a certain topic) and in their feedback.

A key element to formative assessment is the feedback process, from which students can see the gaps in their understanding and how to proceed from there (Sadler, 1989). Nicol and Macfarlane-Dick (2006) give seven principles for good feedback. One of the points they expand on is "that feedback is provided in a timely manner" (Nicol & Macfarlane-Dick, 2006, p. 9) also backed up by other researchers (Gibbs, 1989; Cox, 2011; Choy, McNickle, & Clayton, 2009). Students in the School of Mathematics, University of Bristol, feel that they do not receive good enough (prompt, detailed and useful) feedback (Higher Education Funding Council for England, 2016). On the other hand, lecturers and teachers find that there are pressed for time to give meaningful feedback and have the impression students do not take into account the given feedback in any case (University of Bristol Staffs, personal communication, December 2016). Robinson, Pope and Holyoak hypothesize that "Poor satisfaction with feedback is likely to occur if students see the feedback as an end in itself and do not work independently with the feedback provided to improving their performance." (2011, p. 261). Indeed Sadler's (1989) third point in effective feedback is the importance of *student engagement*, how does the student learn how to proceed from their current work?

Various methods of feedback have been suggested, but typically in mathematics, feedback on written work uses a mixture of "(i) short comments on scripts (ii) model answers (iii) review of common errors in class (iv) written summary of common errors (v) follow up one-to-one discussion in practical classes following the return of work" (Robinson , 2015, p. 163). Thompson and Lee remarks that "the problem with [(i)] isn't necessarily in the mark themselves, but in the disconnect between what teachers communicate and how students interpret that feedback" (2012). While (ii) is highly valued by students, "[they] may not always understand the important differences between their own work and the model solutions" (Robinson, Loch, & Croft, 2015, p. 367). Several quick informal surveys (a show of hands in various classes), suggests that a vast majority of students don't engage with (iv) (University of Bristol Students, personal communication, December 2016). When asked to expand, the most common complaint is that the class-wide feedback are impersonal and seem irrelevant to their own work. From experience, it is true that with small classes there can often be no common grounds on which to write class-wide feedback except for comments like "Q3 most students got the right idea but had trouble formulating their argument clearly" or "A lot of students seemed to have problems with Q5". This kind of feedback falls short of Sadler's definition of effective feedback. Rotheram (2008) suggested audio feedback as a way to save time spent on producing feedback, while creating high quality and effective feedback. Unfortunately, he does not back up his claim that time was saved. Arias (2014) implemented a similar idea at the University of Bristol, Department of Hispanic, Portuguese and Latin American Studies, where she recorded her feedback while marking the assignments online. She found that this saves time while producing effective feedback. In mathematics, studies have been done on the use of audio-visual to work through problems to teach mathematics (Loomes, Shafarenko, & Loomes, 2002; Kay & Kletskin, 2012; Keen, 2009). More related to feedback, Robinson, Loch and Croft (2015) evaluated the use of audio-visual class-wide feedback, by working through model answers on questions set as homework. This form of feedback was well received by the students, who prefered it to other kinds of feedback provided.

Focus of the report

In Spring 2017, I re-designed the formative assessment - feedback loop to try and improve students' understanding of ANT. For part of this project, I decided to trial audio feedback on top of the feedback already in place for this course (mark, written comments, class-wide written text). This report investigates whether audio feedback increases the students' engagement with feedback, and whether it is not too time consuming. Rotheram (2008) argues that audio feedback is time saving, a fact that is backed up by Edwards, Dujarding and Williams (2012) for essays in communication, and by Arias (2014) for Spanish language coursework. As these are essay based assignment, this project tries to see if the same result can be attained in a mathematics setting where problem sheets are often used. Closer to mathematics, O'Malley found that "no significant extra time was expended in using screencast feedback compared with the traditional format" (2011, p. 30) for chemistry first year problem sheet.

While class-wide audio-visual feedback (as done by Robinson, Loch and Croft (2015)) seems to be a sensible idea to implement, it has two main drawbacks: 1) it is not personal, and 2) it is time consuming, taking four hours for 34 minutes' worth of material on only two questions. The idea behind using personal audio feedback is that it can be tailored to each specific student, hence not only giving feedback on where they went wrong, but also how can they challenge themselves in the future.

Context of the study

Workflow of giving audio feedback

As a marker had already been assigned to the course, the implementation of the feedback was as follows. The students would hand in their work (bi-weekly) on a Monday when the marker would collect it. The marker would mark their work (based on the mark scheme I would provide them with), giving each question a mark and highlight where errors were made (when they were made). Once the marking was done, I would receive all the scripts in one go. For each script, I would read through the script, reading both what the student wrote and what the marker wrote. I would roughly think of what I should cover, then start recording the feedback. At the start, I tried to limit myself to two minutes, but as the course progressed, I aimed to record audio between three and four minutes long. The recordings were made using Mediasite, a plugin by Blackboard that allows screen casting and is often used to record lectures. Mediasite automatically uploads any recording to Blackboard. Once all the recordings were uploaded, I had to find the link pointing to the recording and copy-paste it next to the student's mark. Hence, when students access their mark on Blackboard, they could click the link next to it and listen to their audio feedback. The scripts were returned to the students at the same time as the mark were made available, on the following Monday. This was to ensure a timely feedback, with only one week between handing in the work and receiving it back. Furthermore, it gave students a week to engage with the feedback and implement any changes before the next problem sheet was due.

Audio feedback comments

As the marker's comment and the model solutions cover the first two points on Sadler's (1989) definition of effective feedback, the audio recording focused on Sadler's last point. That is, the audio feedback should prompt the students into action that will help them close the gap between their work and the expected standard. For a final year course in pure mathematics students are required to prove various facts and present clear arguments and solutions to show an understanding of the course's concept. Part B of the problem sheets reflected this by having most of the questions asking for proofs. Therefore, the audio feedback could go into more depth on the circular arguments[3], flaws and gaps in logic, as well as misconceptions in the course.

With the audio feedback, I tried to incorporate all of this. With errors of misconception or lack of understanding, I took the time to point out relevant Part A questions that the student might want to redo and pointed out where their misunderstanding could have stemmed from. I tried to supplement such comments with extra concrete examples (when possible). With errors of gaps in proofs (whether special cases missing, wrong logical steps or incomplete idea), I took the time guide the students through their error and the potential correction with comments like "take example X through your argument, where does the proof fail? Can you amend your proof to cover that gap? It might be useful to remember that…". With students who understood the material, I took the time to point out which Part C questions they might want to do to extend themselves, or ask them to think about how they would go about generalizing this idea, or does this proof work in this context, etc. All the above comments are examples of me expanding a one sentence point that the marker had made in the margin of the problem sheets.

Methodology and key results

Design of questionnaire

A questionnaire was given to students during the beginning of a lecture in week 10 (out of 12) and collected at the end. For ethical reasons, the answers were collected anonymously and there was a paragraph explaining how the data collected will be used. To increase participation, 12 multiple choice questions were asked (6 on problem sheets

[3]An argument that only works if one assumes what one wants to prove is true. This is a common mistake which can be hard to pick up on, and even harder to explain why the argument is circular.

and 6 on audio feedback), plus a general comment question at the end. For the 12 multiple choice questions, students were asked to circle their response from a list of option (to increase participation) but were also given extra space and told "[they] can also write [their] own response". Out of the 29 students registered in the course, around 20 turned up to the Monday lecture when the questionnaire was handed out. Of those, 10 filled in the questionnaire and returned it. Their answers are compiled in Figure 1 and Figure 2 contains relevant comments.

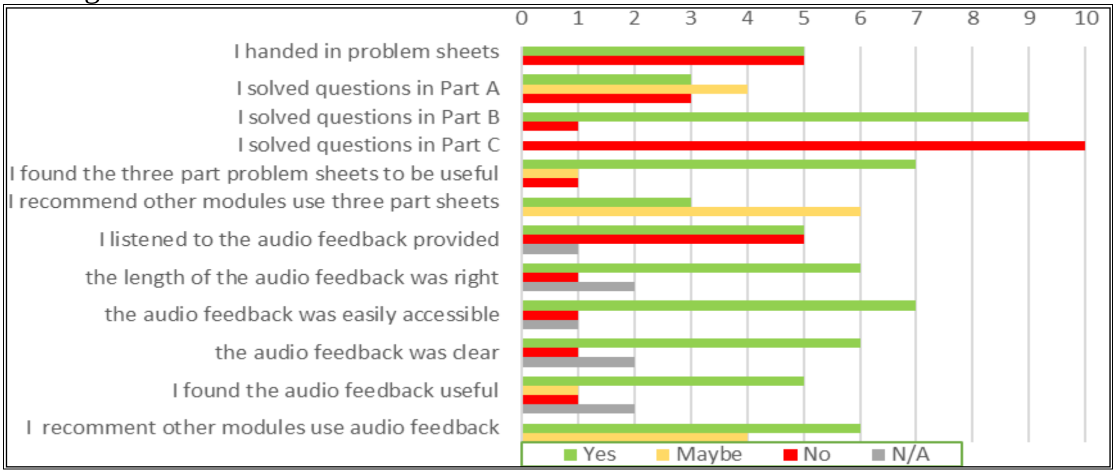

Figure 1:Students' questionnaire responses.

I like the idea [Audio Feedback] but I didn't hand in problem sheets. I keep forgetting about [the Audio Feedback], but I will listen to [the Audio Feedback]. Audio feedback is extremely helpful because it is much more detailed than what one could write on the homework. Personalised audio homework feedback an excellent and very helpful idea

Figure 2: Students' extra comments.

Data collection

For every problem sheet, I counted the number of students who handed in any work, the total time taken to record all the audio feedbacks and the number of "views" for each feedback. The total length of time taken to record all the feedback was calculated by looking at the time difference between when the first feedback and the last feedback was uploaded, and adding the length of the audio of the first feedback. Note that this is an underestimate as it does not include the time needed to set up recording that first feedback, nor does it include the time taken afterwards to make the feedback available to students (i.e., uploading the recording, and making it available to the student). As in theory students only had access to their own feedback (unless they share the link to their friends), the number of "views" each audio had can be counted as the student's engagement with their feedback.

The data was compiled twice. The first time was on the same day that the questionnaire was handed out, i.e. the same day as the audio feedback for the fourth problem sheet was made available. This could explain the low number of views (in the first instance) in the column of problem sheet 4. The second time was after the exams. This was to see if the number of views had gone up during the revision period. Table 1 shows the data collected, with the black numbers the first data compilation and blue numbers being the second data compilation (if different from the first). Unfortunately, the feedback for sheet 5 was recorded on different days and hence no estimate on how long it took could be made, hence the question mark in column 5.

Problem sheet number	1	2	3	4	5
Total number of problem sheets handed in	15	7	7	9	5
Total time taken to record all feedbacks (mins)	103	67	61	89	?
Average time taken per user (mins)	6:52	9:34	8:43	9:53	?
Average length of audio recording (mins)	3:03	3:33	3:56	3:04	2:58
Number of audios with 0 views	5	2	5 4	8 6	4
Number of audios with 1 view	8 7	5 4	2	1 3	0
Number of audios with 2+ views	2 3	0 1	0 1	0	1

Table 1: Data on audio recordings

Conclusion

While initially, I thought that the feedback would take on average 2-3 minutes per students, the data showed that it took at least (on average) 9 minutes per student. This is broken down into roughly 3-4 minutes of recording and 5-6 minutes of pre-reading and formulating thoughts. While this seems to be a lengthy process, as remarked by a student, more can be said in three minutes than can be written. It remains to be seen whether, although I believe it to be true, more can be said in three minutes than can be written in nine. Therefore, this initial report can not conclude whether individual audio feedback is a time-effective form of feedback for mathematics.

In terms of engagement, as not many students handed in problem sheets, not many students could experience the audio feedback. As the number of students handing in non-assessed assessments followed the usual pattern of starting at 50% and quickly dropping to around 25% handing in rates (Blackboard Data on Pure Mathematical Courses, personal communication, September 2016), audio feedback did not increase students' engagement of problem sheets. Of those who did hand in the sheet, after a peak of 71% listened to their personal audio feedback on Sheet 2, there is a steady decline of percentage of students who listen to their audio feedback at least once. While this would suggest a low level of engagement from students with the audio feedback, we can not compare to the level of engagement they had with their written feedback. What was interesting, although not surprising, is that the number of people listening to their feedback went slightly up. As students approached the exams, they turn to (and hopefully make us of) all the resources they can lay their hands on. Unfortunately, this kind of extra resources is only available to those students who handed in work throughout the year.

Overall, students found the implementation of the audio feedback to be good, and of those who used it, the majority found it to be useful. Furthermore, while they did not all make use of the audio feedback, the majority would recommend audio feedback to be implemented in other courses. This is in line with the literature that reports students view screencast as better than traditional feedback (Arias, 2014; Choy, McNickle, & Clayton, 2009; Edwards, Dujardin, & Williams, 2012; O'Malley, 2011; Robinson, Pope, & Holyoak, 2011; Thompson & Lee, 2012).

Future work

This initial report shows that the methodology of this project needs to be changed for the next implementation, both in terms of workflow to deliver feedback and in terms of collecting data to evaluate the use of audio feedback. Part of the problems with the current workflow stems from having decision about each course (assessment weighting, content, number of markers) are made before lecturers are assigned to courses. In future implementation, the workflow would be that I record the audio feedback while I mark the sheets. This should cut down on the 4-5 minutes I needed to think on what to say for each student. To compare speed of recording against speed of writing, I will select a few random audio feedbacks, and time how long it takes me to write down what I said. The next implementation needs to monitor more closely the use of other feedback, by enabling tracking of who view the class-wide written feedback and the model solutions. Furthermore, the questionnaire should be designed to ask questions comparing the uses of the different feedback available to the students.

While the sample size (10 students) seems to be small, the point of the project is to evaluate the changes of the assessment-feedback loop for a pure mathematical course in later years. Such courses have a relatively small number of students, hence any information gained from the ANT setting can be valid for other pure courses. When this project was presented during BCME9, a discussion followed on how to engage students with feedback. In particular, the idea of feedforwarding in the audio-feedback, by way of giving the student a specific task to concentrate on in the next sheet, was suggested as a way to measure students' engagement to feedback. An idea that emerged from the discussion (and which I had considered), is to use video as well as audio feedback. This way the student will be able to see what I write down (some maths is better communicated by hand than verbally), and furthermore will have further insight into how a mathematician thinks. Hence, they would understand more what is expected of them. Hopefully, I will be able to get the equipment to implement this next time round, and hence evaluate the full use of video-audio recording.

While this project is quite specific to the environment of one specific course in one specific university, I hope that after some tweaking of the implementation and design of the audio feedback, I will be able to recommend audio feedback (alongside other approaches) as potential method to improve the student learning experience in pure mathematical units. Audio-feedback has potential to cover the engagement aspect of feedback, which would complement the use of model solution.

Acknowledgment

The author would like to thank the reviewers for their extensive helpful comments which reshaped and improved this article.

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I can do it: Year 3 children's perceptions of mathematics lessons identified through their drawings

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This research involved children drawing themselves in a mathematics lesson, in order to access some of their perceptions about mathematics lessons. Drawings can provide a rich source of data and allow children to communicate emotional and social characteristics while focusing on other features that are important to them. The sample was 234 Year 3 pupils (7 and 8 years of age) from ten primary schools in Lincolnshire, England. The drawings were analysed for teacher-pupil interactions, pupil-pupil interactions and pupils' perceptions of themselves as learners of mathematics, using a coding system devised for a similar study in Finland. The majority of pupils indicated perceived competence in mathematics. Some gender differences were noted in terms of teacher position and teacher-pupil interactions. Teacher-pupil interactions are an important aspect of mathematics lessons which emphasise communicating reasoning, so teachers should be aware that girls and boys may perceive teacher-pupil interactions differently.

Key Words: Mathematics; competence; teacher-pupil relationships; drawings

Introduction

The National Curriculum for mathematics in England is underpinned by three aims: conceptual fluency, reasoning and problem solving (Department for Education, 2014). These involve explaining your thinking to others, such as the teacher or other pupils (Askew, 2016; Zijlstra, Wubbles, Brekelmans, & Koomen, 2013). However, many people, including some teachers, have a more limited view of mathematics as calculations and procedures that must be memorised and performed quickly, which can result in maths anxiety and negative attitudes towards mathematics (Boaler, 2016). The 2012 PISA results of English 15-year olds found higher maths anxiety in girls, along with lower confidence and motivation, even in those achieving the same scores as the boys (Organisation for Economic Co-operation and Development [OECD], 2013). There were also gender differences in English children's performance on the TIMSS and PISA international tests, with boys outperforming girls at Year 5 and 15 years old (Mullis, Martin, Foy, & Hooper, 2016; Greany, Barnes, Mostafa, Pensiero, & Swensson, 2016; Jerrim & Shure, 2016).

Links between confidence and competence have also been found with younger children and these are further related to teacher-child relationships. Stephanou's (2014) research, with 200 kindergarten children in Greece, found that the more positive children were about their relationship with the teacher, the higher their attainment in mathematics, beliefs in their own competence and motivation. Zijlstra et al. (2013), studying 828 first and second grade children and 40 teachers in Dutch primary schools, found a positive correlation between children's attainment in mathematics and their perceptions of their teacher as friendly, organised and helpful. The opposite has also been found, with negative perceptions of the teacher-child relationship associated with lower attainment; gender differences were a factor, with teachers reporting more frequent negative relationships and conflict with boys (Koepke & Harkins, 2008; McFarland, Murray, & Phillipson, 2016; White, 2016). Relationship issues between boys and teachers have also been identified in the UK (Myhill & Jones, 2006).

Pupil perceptions are often gathered through direct approaches, such as questionnaires and psychometric tests involving Likert rating scales (Stephanou, 2014; Zijlstra et al., 2013; McFarland, Murray & Phillipson, 2016), although in a comparative study Harrison, Clarke and Ungerer (2007) found that the indirect approach of asking children to draw a picture proved to be a better measure of teacher-pupil relationships. Observations have also been used but these have been found to vary depending on factors such as length and timing (Pianta & Cash, 2004).

Leitch (2008) and Hannula (2007) consider the use of children's drawings in research to provide a richer source of data and to support children in communicating both their emotional and social worlds, compared with more traditional research tools such as interviews and questionnaires. Barlow, Jolley and Hallam (2010) noted that drawings encourage children to include more details than they would in discussion, without having to ask leading questions. Drawings are a way that children share their perceptions of the world and identify aspects that are important to them, even when they struggle with the vocabulary to communicate these verbally (Papandreou, 2014; Cugmas, 2004). Within the research setting of mathematics classrooms Dahlgren and Sumpter (2010) suggest drawings may be used to support inferences regarding the pedagogical approach regularly experienced by pupils during the teaching of mathematics. These views are supported by Selwyn, Boraschi and Ozkula (2009), who also emphasise the greater opportunities that drawings give to children to express themselves, although they concede that a lack of artistic skill can be a constraint.

There have been several studies about young children's perceptions of mathematics using drawings as a research method. Perkkilä and Aarnos (2009) asked 300 six to eight-year olds in Finland to draw themselves in math land. The researchers analysed the emotions portrayed in the pictures and found that girls were more likely to display joy (53% v. 21%), whereas boys were more likely to draw sad expressions (19% v. 5%). However, it may be that the girls were conforming to stereotype pressures on girls to present themselves as cheerful rather than this indicating a greater liking of mathematics.

Towers, Takeuchi and Martin (2018) also looked at young children's emotions and mathematics, with 46 four to nine-year old children in Canada. They used semi-structured interviews, alongside asking children to complete two drawings: one which showed how they felt while doing mathematics and another that showed what mathematics is. The children in this study drew very different images of mathematics to those in Perkkilä and Aarnos' (2009) study, which were mostly outdoors and focused on real-world applications of mathematics. The children Towers et al. (2018) studied mostly drew children in school. These drawings gave access to many details about the learning environment and included features that the children had not spoken about. Towers et al. (2018) reported that the young children were generally positive about mathematics, but they also identified that children were already forming ideas about mathematics being hard or easy and whether they were able to do mathematics. They reported that both perceptions were problematic and recommended that early years teachers explore these ideas about mathematics with children.

Borthwick (2011) analysed 162 drawings completed by primary aged children from four schools in Norfolk to determine the children's perceptions about their mathematics lessons. She looked at emotions and attitudes in mathematics lessons, perceptions of peers, perceptions of the teacher and the type of mathematics presented. The drawings showed a range of emotions but, similar to Perkkilä and Aarnos (2009), there was evidence that younger boys were already showing disaffection for mathematics. A factor that led to this disaffection, determined through the drawings and interviews, was the teaching approach that had children seated in groups but working independently, although they would rather work as a group.

Foley (2015) was particularly interested in girls' perceptions of mathematics and their identity as mathematicians. She used a wide range of data collection methods with 14 eight and nine-year old girls from a single class. She was determined to ensure that the girls' voices were heard so included methods such as the children drawing themselves doing mathematics and then annotating the picture to explain what they were thinking. Similar to Towers et al. (2018), most of the drawings showed mathematics as number and calculation, taking place in a classroom at a desk. The majority of these showed mathematics to be a solitary activity, as found by Borthwick (2011).

All of these authors (Borthwick, 2011; Foley, 2015; Perkkilä & Aarnos, 2009; Towers et al., 2018) commented that children's drawings were an effective method for eliciting children's perceptions about mathematics. The children responded easily to the task of creating a drawing related to mathematics. These were often annotated by the child or followed up with interviews to assist in interpreting the drawing.

Methodology

This study was modelled on research undertaken by Tikkanen et al. (2001) from Helsinki University, about third-graders' drawings of mathematics lessons in Finland, because the Finnish team requested that a parallel study be done in England for comparative data. The core research question was: *What are children's perceptions of mathematics classrooms?* The aspects considered were: teacher-pupil interactions; pupil-pupil interactions and perceptions of mathematics. The participants were 7 and 8 year olds in Year 3 (n=234, 119 boys and 115 girls) from 10 primary schools in Lincolnshire, United Kingdom. The schools ranged in size, number of children eligible for free school meals, children with English as an additional language and children with Special Educational Needs. However, results of national testing in Year 6 showed that children from these schools showed above average attainment in mathematics. This may be due to schools being recruited through teachers who had completed the MaST (Mathematics specialist teachers) programme taught by the researchers.

Informed consent was obtained from the headteachers of the schools, the teachers involved in the study and the parents of the children. Informed consent from the children was obtained by explaining the purpose of the study orally and providing the children with the option of not submitting their drawing. Only one child chose not to take part. One of the researchers instructed the class of children:

> Draw yourself in a maths lesson. Use speech and thought bubbles to show what different people are saying or thinking. Label yourself as 'me' on the drawing.

The researcher and class teacher acted as a scribe for the speech / thought bubbles if requested. Explanations of what was happening in the picture were either written on the back by the pupil or verbalised by the pupil and then recorded by an adult. This is in accordance with suggestions that children should be given the opportunity to explain their drawing rather than it being left entirely to the adult's interpretation (Leitch, 2008; Cugmas, 2004).

The drawings were analysed using codes developed by Tikkanen et al. (2011) that related to the teacher's position in the class, teacher's interaction with pupils, interaction between pupils, perceptions of mathematics, teacher-centred and pupil-centred working methods. This resulted in some difficulties because the Finnish codes did not always fit the English context, particularly those related to working methods, which is why those aspects are not discussed in this paper. Each coding category included the option of 'non-recognisable', which accounted for a lack of evidence (e.g. no teacher drawn in the picture), an inability to interpret that aspect of the drawing (e.g. scribbles rather words in the speech bubbles) and data which did not fit other codes. Coding was done by the researcher who had gathered the data, which allowed knowledge of the setting to inform interpretations, though may introduce bias. A sample of the coded drawings was exchanged to check inter-rater reliability. Where differences occurred, these were discussed between the researchers and then clarified with the Finnish team who devised the codes. The subsequent sample check had identical coding from both researchers. Frequency tables were used to organise the data.

Figure 1 Example of drawing from a boy (I can do it / It is easy)

Figure 1 is an example of a pupil's drawing. In this picture there are three pupils, with the teacher standing at the whiteboard. There is an addition on the board with 'rainbows', which are meant to indicate that the numbers should be partitioned with the tens added together and the ones added together. Two of the children make comments related to competence ('I can do it' and 'It is easy'). The data analysis codes for this picture are: teacher position at whiteboard; teacher gives mathematical instruction through explicitly pointing at the board; several pupils separately remark / think in connection to the instruction; pupil thinks mathematics is easy; pupil can do mathematics. The type of mathematics was not coded but most drawings showed number and calculation, as found by Towers et al. (2018) and Foley (2015).

Figure 2 Example of drawing from a girl (Millie is right)

Discussion of findings

Statistical tests, including chi squared, were used to check the significance of outcomes grouped by schools and gender. Gender differences are often researched in education but this is a complex area, with questions raised about whether these promote equality or entrench stereotypes by ignoring the intersections that gender has with other factors such as race and class (Dhar, 2014). Three aspects were found to be significant with gender: teacher position $c^2(3, N = 234) = 15.39$, $p = .02$; teacher-pupil interaction $c^2(7, N = 234) = 15.9$, $p = .03$ and perceptions of mathematics (Fisher's Exact Test) $p = .01$. Boys were more likely than girls to draw the teacher away from them, at the board (Figure 1) or teacher's desk, or draw no teacher. While many girls did draw the teacher at the board, it was more common for girls to draw the teacher among the pupils (Figure 2). Grouping data by school did not prove significant, which means that the differences in the pictures result from differing perceptions of shared experiences. This suggests that interpreting the drawings as an indicator of typical practice should be considered with some caution. During a lesson it is common for teachers to move about and interact with pupils in different ways. Therefore, it is likely that there were times when the teacher was at the board and other times when the teacher was among the pupils so both perceptions could be accurate. Nevertheless, the differences in position might be an indication of what teacher position the child subconsciously perceives as more important to her or his learning.

There was a wide range of responses for teacher-pupil interaction. Both genders had a large number coded 'teacher is quiet' because they did not include speech bubbles or other indications of communication, such as pointing at mathematical instruction on the board (Figure 1). Where communication was evident, boys were most likely to show mathematical instruction or behavioural orders. Girls included even more behavioural orders but were far more likely than boys to show the teacher giving positive feedback (Figure 2) or asking questions. These findings are consistent with research into the gender differences in teacher-pupil relationships, where girls have warmer relationships (Koepke & Harkins, 2008; McFarland, Murray, & Phillipson, 2016; White, 2016). According to Papandreou (2014) drawings allow children to focus on aspects of the experience which are important to them. Therefore, it may be that boys see the teacher as a more distant figure and have a greater focus on the instructional elements, while girls may focus more on physical closeness with the teacher and emotional closeness through receiving positive feedback. However, it is also possible that children's drawings were reflecting gender stereotypes, rather than true perceptions.

The category 'Pupils are competent' was identified through what the child said in speech bubbles (e.g. I can do it.), through the teacher's praise (e.g. Well done) or through the pupil showing the correct answer to a mathematical task in the drawing. Both Figure 1 and Figure 2 show children who are confident about their mathematical ability. In the boy's picture (Figure 1) the two other children have thought bubbles which indicate competence and confidence, although the artist's own competence is unknown since there is no speech bubble or other clues. The girl (Figure 2) has demonstrated her competence by getting the right answer to the question on the board (6+6=12) and by receiving praise from the teacher. Competence in mathematics was by far the most frequent code in this category for both genders. In the discussion of the sample it was noted that the English schools participating in the study were broadly typical of English schools except for above average test results in mathematics. Therefore, the sample might be skewed towards higher competence in mathematics which would impact on the generalisability. However, the TIMSS 2015 data for Year 5 found England to be in the top ten countries for confidence, which correlated with increasing competence (Mullis et al., 2016), so this may be an accurate portrayal. It was very rare for either gender to show a child asking for help. This could be due to the high levels of competence being displayed or may relate to a classroom ethos that discourages seeking help.

There were some drawings which presented polarised views regarding confidence and competence in mathematics, with pupils identifying themselves as "good at mathematics", while identifying peers as unhappy with mathematics or unable to do questions. Such polarisation may suggest pupils are developing the common misconception that people either can or cannot do mathematics (Boaler, 2016). However, it may also indicate an attempt to emphasise their own level of competence by contrasting it with their peers' ability. Several examples of this type of drawing came from children sitting in the same row, with the drawing process accompanied by giggling, and so may have been a form of teasing rivalry rather than a serious perception of their own and their peers' abilities.

Girls were more likely than boys to comment on mathematics being difficult or easy, with nearly twice as many choosing difficult. It is not clear whether the children who rated mathematics as difficult saw this as positive (i.e. a challenge) or negative (i.e. beyond their capabilities). Boys were more likely than girls to comment on whether mathematics was fun or boring, with slightly more choosing fun. However, all 10 of the drawings which showed mathematics as boring were from boys. This may be evidence of the early disaffection in boys noted by Borthwick (2011).

Conclusions

This is a small-scale study so any conclusions must be considered with caution and should not be assumed to be generalisable. Further caution should be exercised since this study was about children's perceptions of their mathematics lessons, rather than attempting to determine what was objectively happening in these lessons. Although teachers need to be careful not to make stereotypical gender assumptions about children, gender differences were found in the data. This study found that perceptions about teacher position and teacher-child interactions differed by gender, which suggests that teachers should consider not only their physical position and interactions but also how these may be perceived by the children. There were further gender differences regarding perceptions of mathematics as easy or hard, boring or fun. Since teacher-child relationships and perceptions about mathematics have been found to impact on confidence, competence and commitment to mathematics (Towers et al., 2018; Stephanou, 2014; Zijlstra et al., 2013), teachers may benefit from exploring the perceptions their own pupils have of mathematics. The pupils' attitudes towards mathematics in this study were generally positive and the majority of pupils positioned themselves as people who could do mathematics but there was little evidence of being willing to ask for help. In order to address perceptions of mathematics being too hard, teachers might need to encourage a classroom ethos that encourages children to ask for help. This may help to develop further positive perceptions towards mathematics, including the belief that all can learn mathematics.

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How might the Numberlink Board™ be used to develop deep conceptual understanding of multiplication through exposing structure and making connections?

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In this paper I will draw upon findings from the action research I conducted, using the Numberlink Board, aimed initially at developing deep understanding and rapid recall of multiplication table facts. In particular, I examine the value of exposing the multiplicand to draw attention to the structure of repeated addition multiplication. The use of a double line representation is also explored to determine whether this encourages children to see patterns and make connections. The research then considers the extent to which children derive answers using known facts and apply the use of the distributive law, to numbers beyond multiplication table facts. The research was conducted with children in Year 4 over a period of ten months working for one morning each week. At the end of the research period, results indicated that: children had an increased awareness of the structure of multiplication and could apply the distributive law to derive multiplication table facts; children used the double line on the Numberlink Board to reason mathematically about patterns and connections between multiples; there was limited evidence of children deriving answers from known facts and applying the distributive law when multiplying with larger numbers.

Keywords: Numberlink Board; multiplication; multiplicand; structure; multiplication tables; double number line

Introduction

The extent to which deep conceptual understanding and visualisation of multiplication can be developed through exposing structure and making connections, was explored through an action research project that I conducted for my Masters Degree Thesis. At that time in my Year 4 classroom, I was using the counting stick and the array to support children in their understanding of the structure of multiplication as repeated addition. I found that the array was a very powerful representation to support children's understanding of why multiplication is commutative.

Barmby & Harris (2007) recognised the potential of the array to support reasoning in multiplication but also drew attention to some limitations. In their study, some Year 4 children (aged 8-9) had lost sight of the calculation within the array and just focussed on the total number of dots in the representation. These children were not able to use the array for multiplicative reasoning. Through discussion with children, I had found a similar situation in my classroom, particularly when the dot array was used to represent larger multiplication calculations.

I also observed many children in my class struggle with inefficient methods of recalling multiplication facts, for example, finding the answer to 9 groups of something by counting up from 1 lot. To move away from this inefficient strategy, I wanted to strengthen the awareness of the distributive law ie; a x (b + c) = (a x b) + (a x c). I wanted to encourage children to use key facts such as 10x and 5x to work out other facts, that is, to derive facts. I used the counting stick to model this strategy. However, children were not applying this when working independently. Delezar et al. (2005) studied the learning of mathematics facts in two ways, through strategies and through memorisation. They concluded that both pathways are effective for recalling facts fluently but that those who learned through strategies, for example learning 17 x 8 from 17 x 10 and then subtracting 17 x 2, were able to connect their conceptual understanding to new problems. In their research to analyse children's different approaches to arithmetic, Gray and Tall (2007; 2008) found that the children with a more secure understanding used many more derived facts, while children who were not yet secure resorted to counting to reach each answer.

With regard to multiplication facts, I do think that accurate and rapid recall of facts is important, but the recall of number facts based on structure and number sense is a far more powerful tool (Boaler, 2016). I started thinking more deeply about a representation that would support children in their understanding and application of the distributive law.

Research questions

My research questions focussed on finding how children within the year 4 class developed multiplicative reasoning. Using an inductivist approach to my research complemented my teaching style, where I actively engage children to be part of the learning process and to discover and reflect upon knowledge that is built from the social context of the classroom. Mathematics teaching pedagogies which aim to promote deep conceptual understanding and visualisation of multiplication as repeated addition were considered in relation to the data collected.

Methodology

The research was conducted with children in Year 4 over a period of ten months. I worked with the same class of children for one morning each week. Field notes were taken to describe and reflect upon each cycle of the action research process. Recorded interviews were conducted with the class teaching assistant and with a focus group of four children. The results of the research included descriptions and categorisation of: classwork and assessments completed during the period of action research, an interview with the class teaching assistant and responses to tasks undertaken during a task-based interview.

This paper provides an overview of the Numberlink Board, a representation which I designed before the research, and the extent to which it appeared to support children's understanding of the structure of multiplication and their ability to apply multiplicative reasoning.

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What is the Numberlink Board?

The Numberlink Board, as shown in figure 2 below, uses the same principle as the counting stick but differs in three main ways:

- It exposes the multiplicand – the number in the group or set. In figure 1 below the 8 times table is the focus, so the multiplicand is 8.
- It has a second line so connections can be made and patterns spotted.
- Each child has their own dry-erase Numberlink Board so learning can be personalised.

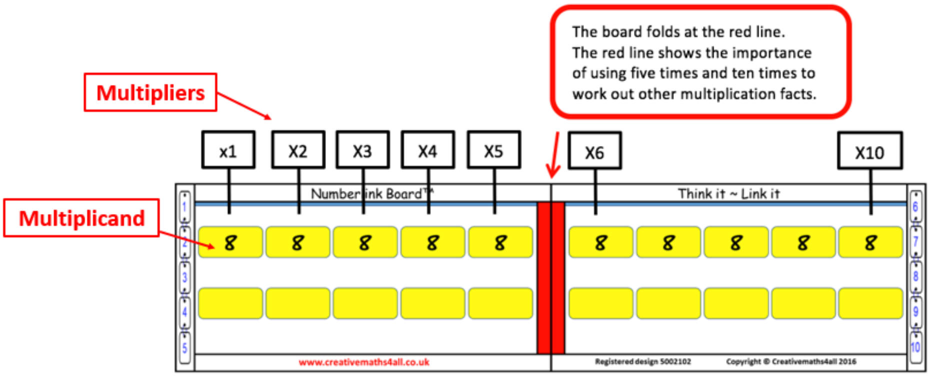
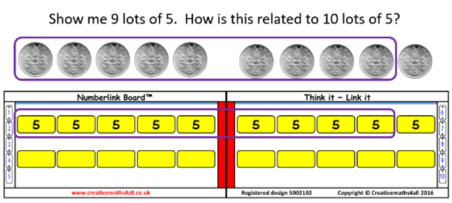


Figure 1: The design of the Numberlink Board

I designed the Numberlink Board in order that the multiplicand was a key feature of the board, to emphasise the number in the repeated group or set. In other representations, such as the counting stick, this part of the multiplicative structure was implicit; it was stored mentally rather than shown on the representation. I argue that this is the essential feature of the repeated addition structure of multiplication and, as such, needs to be shown explicitly. I worked with the children in the research project to strengthen visualisation of repeated addition with relation to multiplication. I designed the board so that the middle is represented by a large red line – a key reference point on the board to highlight where 'five times' the multiplicand is. Before thinking about products, I asked children to explore the representation of ten groups of a particular multiplicand, both cardinally and symbolically, as shown in figure 2 below, then asked questions like:



Show me 10 groups of …
Show me 9 groups of …
How did you find 9 groups so quickly?
How is 5 groups related to 10 groups?
Show me 5 groups, now show me 6 groups.

Figure 2: The Numberlink Board with multiplicands represented.

I found that spending time on the orientation of finding multiplicands in relation to the key points of '10 groups' and '5 groups' laid the foundations for using the distributive law. Children explained that 6 groups was one more group than 5 groups and 9 groups was one less group than 10 groups. When putting the products onto the Numberlink Board I did ask that the children put them on in the order 1 group, 10 groups, then 5 groups (which we discussed could be found by halving 10 groups). The children found this frustrating initially as they had been used to skip counting and wanted to put the products on 'in order'. I argue that this strategy of working out 1 group, 2 groups, 3 groups etc. is restrictive as it is reliant on an adding strategy, counting on from one number to the next; I believe it has limited use beyond learning multiplication table facts. Once key facts, 1 group, 10 groups, 5 groups, had been written on the board, other products could then be derived by looking at their relative position to these. I was keen to support depth of understanding of repeated addition so that each calculation wasn't an isolated picture but one within the picture of ten groups of the multiplicand. Figure 3 below shows how the three parts of repeated addition multiplication, the multiplicand, the multiplier and the product, are exposed on the Numberlink Board.

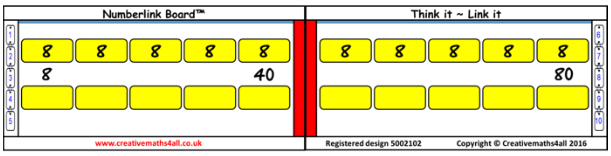


Figure 3: Numberlink Board showing the structure of multiplication and key facts.

I continued to work with the children and the Numberlink Board to support learning multiplication facts; each lesson involved mathematical reasoning using questions prompts such as "How do you know?" and "Convince me!" Research suggests teachers who make effective use of representations use them to expose mathematical structure and link mathematical concepts and processes (Mason, Stephens, & Watson, 2009; Booker et al., 2014). Children will not necessarily make the connection between the representation and the mathematics themselves. The more that children use representations alongside the teacher, the more they become familiar with their structure and the mathematics that is being exposed (Askew, 2012). Harries and Suggate (2006) also suggest that representations do not convey the mathematics without process. Attention needs to be drawn to the link between the representation and the mathematical structures involved, if understanding is to be developed, a view also supported by the work of Mason (2003).

Initially, the Numberlink Boards were used alongside counters and other resources to show cardinality, the size of the multiplicand and how it was repeated. The children soon became familiar with the simple representation of the Numberlink Board and how it exposed the multiplicand and the multiplier. The class teaching assistant commented on the impact that the simple structure had:

> It's the visual thing. They can visualise it and you can ask them to work out 6 times, 7 times and you can see them visualising 10 times, then halving it and then adding one more lot, two more groups of and they love doing it.

The children started to derive and prove multiplication facts using the Numberlink Board as a visual support, as demonstrated by these children's comments below:

> 5 x 6: The answer is 30 because $\frac{1}{2}$ of 60 is 30
>
> 9 x 6: You take away 6 from 60
>
> 6 x 6: It is 36 because 5 x 6 = 30 and it's one more 6

Another stimulus for exposing the multiplicand came when a pupil was trying to work out 98 multiplied by 3. I asked her to give me an approximate answer; she considered this for a short time and said 'about 500'. We discussed what 98 x 3 meant and, when encouraged to write down 98, 98 and 98 on her piece of paper, she then reasoned that the answer would have to be about three hundred and even clarified that it had to be a little less than three hundred as 98 was a little bit less than 100. By supporting her to focus on multiplication as repeated addition, her multiplicative reasoning had improved; her number sense was engaged.

Findings and discussion

Using known facts and the distributive law when multiplying numbers beyond known multiplication table facts up to 12 x 12

The research data collected over the course of the action research period showed that children only used known facts and applied their understanding of the distributive law to larger numbers, beyond multiplication table facts up to 12 x 12, when prompted. Part of the class assessment was to find the answer to 68 x 5 in three different ways. 87% of the children who answered the question used column multiplication as one of the ways to solve the calculation and for 96% of these children it was the first method they chose. Although some children derived the answer by halving the known fact of ten times 68 as one of their other ways of solving the calculation, for the large majority, it was not the preferred strategy.

Since completing my research project I have been using the Numberlink Board more with the multiplier as a focus. We use it to think about how we can derive 5 groups of a multiplicand by halving the known fact of 10 groups of the same multiplicand, applying the associative law of multiplication. In my experience the children find this particularly revealing when working with decimals, for example 1.8 x 5. They realise how simple it is when they consider that it is actually half of 18. I believe that time spent developing the mental calculation strategy of deriving 5 groups of something by halving 10 groups is time well spent. The distributive law can then be used efficiently by using key facts, for example finding 6 groups of something by adding 5 groups and 1 group. The data suggest a visual picture of multiplicative structure develops, from which other calculations, not just multiplication table facts, can be derived.

The significance of the double line representation in developing multiplicative reasoning.

The relationship of numbers along the line of ten boxes on the Numberlink Board has been discussed in relation to the distributive law. The second row of boxes on the Numberlink Board was added initially so that children could explore place value links and scaling by 10 or 100. For example, children explored how multiples of 8 were linked to multiples of 0.8, or 80 or 800. As the action research spiralled and the use of the Numberlink Board developed. The children were encouraged to use the second row of boxes to spot more patterns and connections between the rows, as well as along the row of products.

Using the second row of the Numberlink Board to adjust from known facts, estimate and develop number sense

I encouraged some children in the class to apply their existing understanding of multiplication facts to go deeper and think about connected facts. In the weekly multiplication fluency sessions, some children started with their base facts, for example working on the 8 times table, then they would use the second row to scale up or down and find the multiples of 80 or 0.8, as shown in figure 4 below. We spent time discussing how and why each product was ten times bigger or smaller than the 8 times table products.

Figure 4: Making connections on the second row by scaling up or down

Children started to see patterns and make links between multiples on each line. The class teaching assistant said:

> The children I worked with today were talking about your 8 times table and 80 times or your 800s. Being able to put those on there (the Numberlink Board) too, seeing the connections between the numbers helps them with place value.

I developed this idea further using the second row of boxes on the Numberlink Board to estimate products to multiplication calculations. I found that by exposing both the actual multiplicand and the rounded multiplicand, children were able to think about a reasonable answer as shown in figure 5 below.

Figure 5: Using the second row of boxes to support estimation and adjusting from known facts.

Some children were then able to go further and see how many times the multiplicand had been rounded up or down and by how much, so that they were then able to mentally work out the answer. This was shown by jottings on whiteboards; 50 x 6 = 300, 52 x 6 = 300 + 12 = 312

During the research project, I also explored using the second row of boxes to expose the structure of multiplication using procedural variation. The effect on the learner of using procedural variation is analysis of the structure of the calculation and a deeper understanding of the concept (Gu, Huang, & Marton,2004; Lai & Murray, 2012). I explored with children how the structure of the multiplication calculation changes if the multiplicand is increased or decreased by 1 and how this affects the product, for example:

> 46 x 6 = 276 how can we use this to work out the product of 47 x 6?

We also discussed what happens to the product when the multiplier is increased or decreased by 1, for example:

> 46 x 6 = 276 how can we use this to work out the product of 46 x 7?

This requires the children to use one known product and adjust the answer to reflect the change in multiplicative structure. These examples are shown on the Numberlink Board in Figure 6.

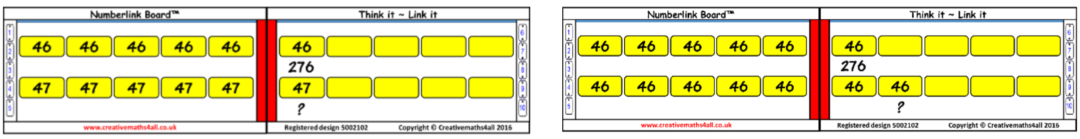


Figure 6: adjusting the multiplicand or multiplier by 1

At the end of the research project in the assessment task, 52% of children correctly answered this question:

> The product of 147 and 6 is 882. What is the product of 148 and 6?

This result indicated that the changing multiplicative structure was not secure for a large proportion of the class. Follow up work after the research project involved repeating procedural variation exercises like the one above but with smaller numbers to gradually build up the visual picture and conceptual understanding. The Numberlink Board, arrays and the area model were used as visual representations to support this concept.

Using a double number line to support multiplicative reasoning

Research conducted with secondary school students using a Double Number Line model (Brown, Hodgen, & Kuchemann, 2014) suggests that the model is useful to support students' understanding of the notion of scaling and that the students become more aware of ratio relations by looking between-the-lines. In one lesson during my action research project, we had been using the Numberlink Board to compare multiples of 3 and multiples of 6. Many patterns and connections had been discussed, for example:

> Every second multiple of 3 is a multiple of 6 because 2 groups of 3 is 6.

One child then asked what would happen if we put in 3s and 8s. We initially just wrote in the multiplicands, 3 on the top row and 8 on the bottom row. I then asked the children to think about what they thought the connection between the products would be. Children discussed whether it might have something to do with 5 since the difference between 3 and 8 is 5. The children then wrote in all the multiples of 3 and 8 on the board. After a lot of discussion, the children realised that there was a difference of 5 between the first two multiples between-the-lines, then a difference of 10, then 15 etc. Some children then went further to explain why this was. The secondary teacher who had come along to watch the lesson then mentioned the picture of equivalent fractions. This had not been the intended lesson but had become so much richer as a result of trying something different and pattern spotting. The lesson prompted the start of using the second row of boxes to explore ratio relations more explicitly.

Conclusion

In June, three months after the end of the teaching section of the action research project, I asked the children to give me some written feedback about the Numberlink Boards, which we had continued to use. Pupils were asked, 'What do you think of the Numberlink Boards?' Responses were anonymous in order to encourage frank responses and are summarized below:

- 24 of 25 children made a positive comment about the Numberlink Board
- 17 of 25 children mentioned a positive impact on their learning
- 16 of 25 children referred to the structure of the board in a positive way

. Four children also commented about an internal picture of the Numberlink Board:

> When I do my maths … at home I always think of it
>
> I imagine the Numberlink Board and get it right
>
> Gives me a picture in my head
>
> … picture it in my head

These comments suggest a link between the familiar external representation and the internal representation being accessed to apply structure to new questions and mathematical ideas.

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Working with the IMPaCT Taxonomy: Encouraging Deep and Varied Questioning in the Mathematics Classroom

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Despite a wealth of research into improving questioning in mathematics, recent research has identified the need for more effective questioning strategies which are accessible to mathematics teachers. This paper looks at the types of questions which encourage mathematical thinking, with the aim of deepening and varying mathematical thinking for learners. The research forms part of a doctoral thesis of the same title, and was conducted through an action research project, working with teachers to use a new tool developed by the author, to improve questioning in mathematics: the *Intended Mathematical Processes and Cognitive Thought (IMPaCT) Taxonomy*. The results presented in this paper demonstrate that teachers' variety and depth of questioning can be increased through working with the IMPaCT Taxonomy.

Keywords: questioning; reasoning; surface; deep; thinking; classification.

Introduction

Teachers' questioning is not always "productive for learning" (DfES, 2004, p.4) and research highlights the need to use "open, higher-level questions to develop pupils' higher-order thinking skills" (ibid, p.18). But what constitutes higher-level questions and higher-order thinking and how can these be established in the mathematics classroom? Yackel & Cobb (1996) consider social norms to be established by the teacher in the classroom which are "characterised by explanation, justification, and argumentation" (p.460). These characteristics are not specific to mathematics lessons, as learners should be expected to justify their thinking and challenge the thinking of others across the curriculum. Yackel and Cobb (1996) believe that to develop learners' mathematical thinking, norms which are unique to the learning of mathematics need to be established, which they refer to as *sociomathematical* norms. These include developing a learner's understanding of what constitutes an *acceptable mathematical explanation and justification*, as well as developing an understanding of *mathematical difference*, *mathematical sophistication*, *mathematical efficiency* and *mathematical elegance*. Yackel and Cobb (1996) explain that for learners to establish mathematical autonomy, teachers have to ensure that the sociomathematical norm of acceptable explanations and justifications involves "described actions on mathematical objects rather than procedural instructions" (p.461). Therefore, just explaining *what* they did was insufficient, of great importance was the *how* and, more importantly, justifying *why*.

The teacher plays an important role in developing this autonomy (Holster, 2006) by providing opportunities for learners to explain and justify their ideas, which are key aspects in learners developing reasoning skills in mathematics. Whitenack and Yackel

(2002) list questions that learners may start to ask themselves as they go about problem-solving in mathematics:

> Why might I use one approach over another? What information might I use to help me solve this problem? Can I solve the problem in more than one way? Are some approaches 'easier' or more efficient? (Whitenack & Yackel, 2002, p.526)

Yackel and Cobb (1996) found that sociomathematical norms can be constrained by the teacher. If a teacher only asks questions which require lower-order thinking, then learners will give a superficial answer. If, however, the teacher probes the learners' understanding, then justification becomes the norm. It is the teacher's responsibility to share with learners "what counts as an acceptable mathematical explanation and justification" (ibid, p.461) for it to become a sociomathematical norm. Black et al. (2006) identified that in order for the focus to move from teacher to learner in the classroom, teachers need support to develop such questioning strategies. However, Ofsted (2008) found that teachers need to "develop their skills in targeting questions to challenge pupils' understanding, prompting them to explain and justify their answers individually, in small groups and in whole class dialogue" (p. 7). What is needed, therefore, is "to identify and characterize more effective questioning strategies" (Orrill, 2013, p. 287) which are easily accessible to mathematics teachers.

Classification of Questioning

Since the 1950s, many researchers have attempted to produce a hierarchy for the complexity of thinking skills (Gall, 1970). However it was Bloom's Taxonomy which became widely accepted as the optimal classification of questioning (ibid) and was later updated by Anderson et al. (2001). This presents a hierarchy of thinking skills, where remembering and understanding are considered to be lower-order thinking skills, while applying, analysing, evaluating, and creating are considered higher-order skills. However, is such a hierarchy necessarily applicable to the learning of mathematics? Watson (2007) claims that Bloom's Taxonomy "underplays knowledge and comprehension in mathematics" (p.114) as these can be interpreted at different levels of mathematical thought and states that it "does not provide for post-synthetic mathematical actions, such as abstraction and objectification" (ibid). Indeed, some researchers would argue that mathematical understanding is not necessarily a linear progression (Sfard, 1991; Gray & Tall, 1991). Bloom's Taxonomy could help the teacher establish social norms for developing learners' thinking in the classroom, but does not necessarily support teachers to develop socio-mathematical norms specific for conceptual development in mathematics.

Many educational researchers have attempted to distinguish between the understanding in performing mathematics and the grasping of mathematical concepts. Skemp (1976) for example describes the difference as instrumental and relational understanding, where only relational understanding is considered to be true mathematical understanding. Michener considers this more conceptual understanding of mathematics as "an intuitive feeling for the subject, how it hangs together, and how it relates to other theories" (1978, p.1). Fan and Bokhove (2014) on the other hand contend that there is a place in mathematics learning for algorithms, as they can contribute to higher-order thinking and mathematical understanding. This is as a result of how an algorithm is used as a cognitive process. For example, simply remembering an algorithm in order to use it requires lower-order thinking skills, however understanding how and why an algorithm works and evaluating the efficiency of algorithms, can pave the way to the learner creating their own algorithms which becomes a higher-order thinking skill (ibid). According to Fan and Bokhove (2014) "[t]he problem is not in the algorithms themselves, but how to teach them effectively and, more, cognitively" (p. 491).

Marton and Saljo (1976) developed the terms *surface approach* and *deep approach* to learning at the same time as Skemp's (1976) instrumental and relational understanding. The characteristics which determine whether a learner adopts a surface or a deeper approach to learning, are in part down to the approach taken by the teacher in encouraging connections in learners' understanding, as opposed to presenting mathematical ideas as a series of unconnected concepts (Howie & Bagnall, 2013).

Perhaps it is more important to consider questions which elicit higher-order thinking as opposed to identifying higher-order questions, as according to Kawanaka and Stigler (2000), "asking more higher order questions does not simply improve student learning" (p. 255). Furthermore, questioning in mathematics and eliciting meaningful responses is impacted by the sociocultural-mathematical norms in the classroom (Mason, 2014), that is, if the teacher asks simple questions requiring low level responses then learners will not develop mathematical autonomy. Similarly, according to Mason (2014), if the teacher does not vary the type of question they pose, then learners do not learn to pose questions themselves.

The IMPaCT Taxonomy

While researching questioning in mathematics as part of my Masters, I found that the existing taxonomies were limited in their accessibility for mathematics teachers to use them as a tool to deepen and vary their questioning, and so I developed the IMPaCT Taxonomy (Figure 1) for my doctoral thesis. The IMPaCT Taxonomy determines whether questions are higher-order or lower-order, by considering whether or not they require learners to take a surface or deeper approach to their mathematical thinking. However, in the IMPaCT Taxonomy, this is considered in terms of what mathematical thinking the teacher intended, as Watson (2007) argues that what a teacher intends and what a learner perceives are not necessarily consistent.

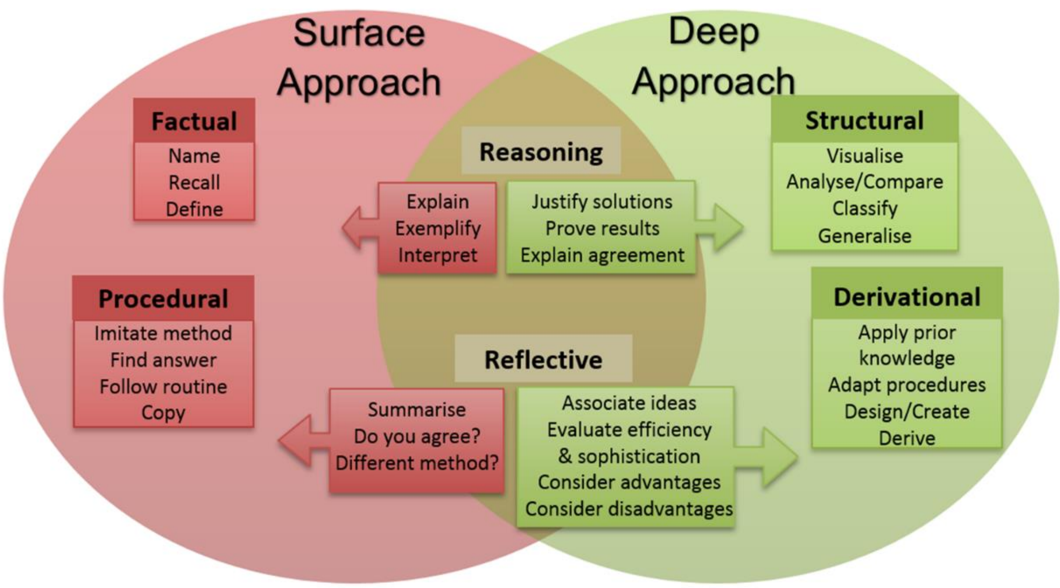

Figure 1. The IMPaCT Taxonomy

The categories in the IMPaCT Taxonomy do not form a hierarchy as such on their own, as the taxonomy considers the depth of the intended mathematical thinking in addition to the type of question, however factual and procedural questions can only be classified as surface level, and structural and derivational can only be classified as deeper level. The reflective and reasoning categories could be tackled at a surface or deeper level, for example with reasoning, a learner may have been asked to simply explain what they did in terms of following a procedure which would be considered surface level, whereas if they were asked to justify or prove their answer then a deeper level of thought would be required to reason in terms of the structure of the mathematics.

Research Design

This paper addresses the following question from the aforementioned doctoral thesis:

Does working with the IMPaCT Taxonomy affect the type and depth of questioning?

An action research strategy was employed and a mixed methods approach of both qualitative and quantitative methods in the form of lesson observations and teacher interviews was used. This paper focuses primarily on the data analysis from the lesson observations in relation to the above research question. Four teachers from a 13-18 mixed school volunteered to take part in this research. Their profiles can be seen in Table 1.

Teacher	Gender	Age range	No. of years teaching	No. of years at the school	Last lesson observation grade
P	Female	20-29	4	2	Good
Q	Female	40-49	13	5	Outstanding
R	Male	20-29	2	1	Good
S	Male	30-39	7	2	Good

Table 1. Profile of the participant teachers in the action research

Five classes were chosen; all from the same year group (Year 10 into Year 11) to eliminate the variable of the age of the learners. Four of the classes were higher attaining learners, to reduce the variable of attainment when comparing the effect that the teacher has on the type and depth of questioning employed. One of the four teachers was also observed with a lower attaining class to allow comparison between his two classes. Three one-hour lesson observations per participant class were carried out to estimate the current depth and variety of questioning used by the participant teachers. All the questions asked by the teachers were transcribed, then coded and the frequencies of the types and depth of questioning were calculated.

Following these baseline observations, the participant teachers took part in training on establishing sociomathematical norms in the classroom and using the IMPaCT Taxonomy to support planning for more varied questioning. The teachers used prompts, adapted from Watson's (2007) analytical instrument, and formative question stems, from Hodgen and Wiliam (2006), to support the classifications in the IMPaCT Taxonomy. Each teacher also received an analysis sheet of their initial three observed lessons, including a breakdown of the proportions of the question type and depth observed. This was provided both graphically and in tables. After the developmental work on the IMPaCT Taxonomy with the participant teachers, three further lesson observations per higher attaining class were conducted to compare the differences before and after the intervention. Unfortunately, due to organisational issues, it was only possible to observe one post-intervention lesson with the lower attaining class, which had to be taken into account when considering the validity and reliability of the findings. To test that the differences in proportions of both type and depth of questioning were statistically significant, the z-test was used to test the null hypothesis that any difference could be attributed to chance (Warner, 2016).

Findings and Discussion

The change in percentage of deeper questioning for each participant teacher can be seen in Table 2. Overall, the percentage of deeper level questions following working with the IMPaCT Taxonomy rose from 25.3% to 51.7%, an increase of 26.4 percentage points and with a z-score of 12.64, indicates that the percentage of deeper questions post-intervention is significantly greater than pre-intervention ($p<0.001$).

Teacher	% Deeper Pre-intervention	% Deeper Post-intervention	Actual Difference	Percentage Increase	*z*-score
P	22.8	48.2	25.4	111.4	4.657217
Q	28.0	60.6	32.6	116.4	8.725988
R	19.4	29	9.6	49.5	2.2084
S (Set 1)	32.8	55.8	23	70.1	5.181773

Table 2. Percentages of surface and deeper questioning in the post-intervention observations.

The largest percentage change in the proportion of each question type was derivational with a percentage increase of 207% and the z-test indicates that the proportion of derivational questions post-intervention is significantly greater than before ($p<0.001$) (see Figure 2).

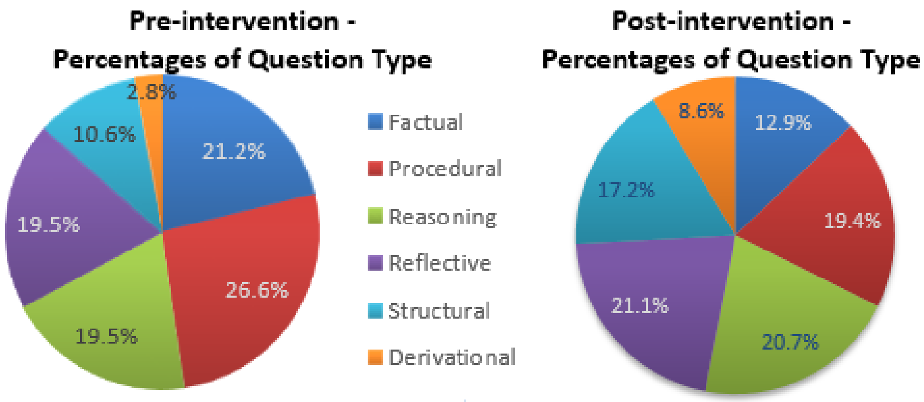

Figure 2. Overall percentages of questions type in the post-intervention observations.

41.8% of all questions posed in the post-intervention observations, appeared to intend either reflective or reasoning thinking. Although this was only a 2.8 percentage point increase since the baseline observations, the noticeable difference was the percentage of surface and deeper questions within each of these question types. The reasoning category had 63.2% deeper level questions post-intervention, compared to 34.8% in the pre-intervention observations. An even larger difference was seen in the reflective category where it rose from 26.5% deeper level at the start of the action research to more than double this figure at 60.2% post-intervention. Both of these increases are very unlikely to have occurred by chance ($p<0.001$).

The lower attaining class for Teacher S only provided one lesson of post-intervention data. As a result of this more limited data, the findings were analysed separately with a degree of caution to making generalisations due to the small sample of questions available for analysis. Figure 3 shows the increase in the variety of questions posed with the lower attaining class compared to the baseline observations. The biggest percentage increase can be seen in the proportion of opportunities for derivational thinking for the learners and a substantial decrease in the proportions of factual and procedural questioning which allowed for this. The *z*-test on these differences, indicates that the proportions post-intervention are significantly greater (p<0.001), implying that, despite the smaller sample of questions to analyse, the impact of the intervention was statistically significant with this class.

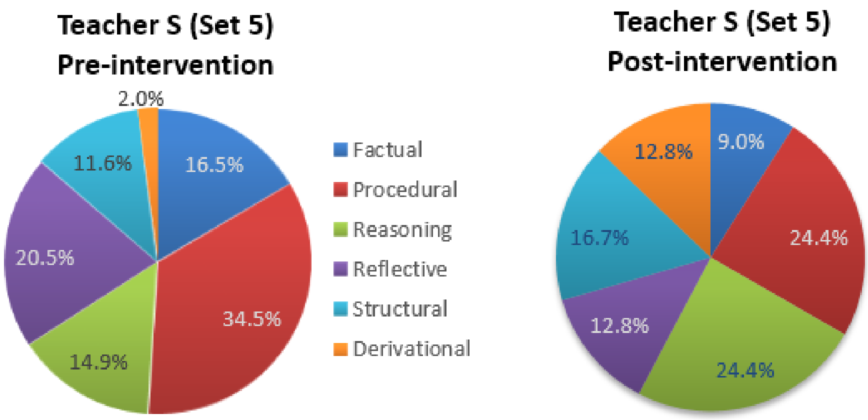

Figure 3. Change in percentages of question type for Teacher S (Set 5).

Interestingly, Teacher S followed the profile of a more experienced teacher with his higher attaining class, but closer to the profile of a less experienced teacher for the lower attaining class. Although he still made significant improvements to the depth and variety of questioning with his lower attaining group in the post-intervention observations, it was less significant than the difference made in his higher attaining group.

There was a difference post-intervention in the establishment of socio-mathematical norms, in particular those of mathematical difference, efficiency, elegance, and sophistication, and what constitutes a mathematical explanation and justification. This implies that the classifications in the IMPaCT Taxonomy supported teachers in moving from questions which established social norms, for example:

Teacher P: What could you do instead?
Teacher Q: Could you do it a different way?

to questions which established sociomathematical norms, for example:

Teacher P: Is that the same as the other suggestion?
Teacher P: Why do you think that one and not that one?
Teacher Q: Did you need to do that?
Teacher Q: Is that your most efficient method? What would be a really efficient method?

These questions mirror those listed early by Whitenack and Yackel (2002), as do these learner questions observed post-intervention:

What's the difference between methods?
How can I tell which to use and when?
What's the easiest way to do this?

The last question was answered by another student, evidencing that the focus had moved away from the teacher explaining to the learners engaging in rich mathematical discourse.

Teachers also started to put less emphasis on accepting *what* learners did as an acceptable mathematical explanation, instead putting more emphasis on the *how* and *why* and indeed, by comparing approaches in this way, established the socio-mathematical norms of efficiency and sophistication which Yackel and Cobb (1996) found lacking in their observations of teachers. There was, however, variation on the impact of the training for individual teachers, as shown by the smaller amount of progress made by Teacher R compared to the other teachers (see Table 2). This indicates that different teachers require different levels of support to develop their understanding of the IMPaCT Taxonomy. The interviews indicate, however, that the participant teachers found the IMPaCT Taxonomy straightforward to use:

> **Teacher P:** [The IMPaCT Taxonomy is] much more relevant to maths to be honest. I've always struggled with Bloom's Taxonomy.
> **Teacher Q:** It's easy to read […] I quite like the IMPaCT Taxonomy from the fact that the questions actually do overlap, but you can actually see how you can take a particular question into the deeper understanding.
> **Teacher R:** It's really clear, the Venn diagram really helps […] It made me consciously think about what questions I would have to ask.
> **Teacher S:** Very straightforward…it's clearly labelled.

These comments suggest that the IMPaCT Taxonomy could be an accessible tool for developing effective questioning strategies for teachers.

An area requiring further research is to investigate how we can close the gap between the depth of questioning experienced by higher and lower attaining classes. Watson et al. (2003) found lower attaining learners benefit from opportunities for deep mathematical thinking, however Teacher S had the same intervention to apply to both types of class and yet a statistically more significant change in the depth of his questioning was found for the higher attaining class.

Conclusion

Black et al. (2006) wrote of the need for teachers to develop effective questioning strategies in order for the focus to shift away from the teachers and towards the learners. This research has shown that the IMPaCT Taxonomy can support this process. Furthermore, Orrill (2013) stated that further research was required to "identify and characterize more effective questioning strategies" which are accessible to mathematics teachers. This research has shown that while some teachers may need some additional support, the IMPACT Taxonomy is, on the whole, an accessible and visual tool to improve the depth, variety and learner-focus of questioning in the mathematics classroom.

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(Missed) opportunities for teaching with digital resources: what and why?

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We report on teacher use and appreciation of the distinctively digital affordances of a publisher's mathematics resources for English 11-16-year-old students. The data come from the first year of our two-year study and were gathered through teacher interviews and observations. We show that, as is common with other digital resources, teachers' use is currently under-developed, and we discuss reported reasons for that. We show that, in addition to common technical and familiarity challenges, the demands of preparation for teaching a new curriculum across the age range currently marginalize other teacher development, including for effective use of resources perceived to be well-designed to support that curriculum change.

Keywords: Mathematics, technology, CPD, digital resources

Introduction

We report on part of the first year of a two-year mathematics study focused on the impact of a large publisher's mathematics resources in England. This paper focuses on the impact of the digital 'ActiveLearn' packages. These are carefully-designed digital resources intended to complement use of other elements of the 'Key Stage 3 Maths Progress' and 'GCSE Mathematics 9-1' schemes that between them offer provision for the range of students 11-16. The study therefore adds to the evidence base around teachers' use (and non-use) of digital resources in mathematics.

Background

The resources

Key Stage 3 Maths Progress (MP) and GCSE 9-1 Mathematics (GCSE) between them set out to offer (Pearson, n.d.) "a coherent set of mathematics materials for use in Key Stages 3 and 4" respectively in England, in preparation for the high-stakes GCSE examinations at 16. The resources' structure and progression are intended to be consistent with the 2014 English National Curriculum for Mathematics (DfE, 2014). This is set out in two Key Stages, and schools largely operate differentially over those. The range and scope at KS3 are intended to be common to virtually all young people, but the Key Stage 4 curriculum is conceived at distinct Foundation and Higher levels, the former consolidating and deepening the KS3 curriculum, and the latter designed to give a foundation appropriate to the study of Higher (level 3) school mathematics. The 2014 curriculum includes a renewed focus on problem solving and reasoning. Both MP and GCSE resources include differentiated textbooks and the online ActiveLearn (AL) platform, though schools can decide to buy only one part of the resources. Additionally, there are a variety of practice books and workbooks available.

This paper focuses on the digital resource AL, which has both an online toolkit for teachers and an online student interface. Figure 1 shows the four different components of the digital service (Pearson, n.d.). Schools are recommended to buy the entire package but some schools choose to purchase only a subset. The 'Front-of-

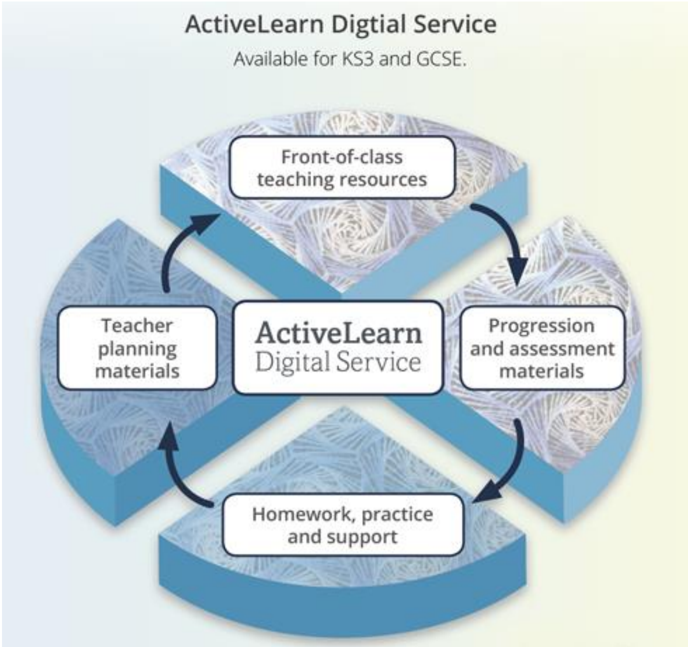

Figure 1: Components of the Pearson ActiveLearn service

class teaching resources' include a digital, interactive version of the textbook that teachers can project, as well as other resources such as videos, through which 'other experts' can be brought into the classroom. 'Homework, practice and support' is the student-facing side that students can use for homework, or extra experience or support at home or school. This component allows clear and quick communication of multiple representations (e.g. tables, graphs), access to an extended textbook (if schools opt in to this) that includes some hints towards solutions, and instant access to answers and feedback; it also allows for formative assessment as it monitors individual progress. The 'planning' and 'assessment' materials are online versions of paper ones, although in the latest update, there are now interactive, hyperlinked lesson plans. Here, we focus on the distinctive digital affordances of the front-of-class and student aspects of AL rather than the planning and assessment support.

AL is designed to meet recommended English practice as suggested by NCETM (2015), whose guidance includes:

> Careful consideration should be given as to how and when technology is used to support learning in mathematics, to ensure it does not detract from the development of essential knowledge and skills (p.4)

The digital textbook for students, while mirroring the appearance and structure of the paper version, expands learning opportunities by offering a range of digital interactions designed to enhance students' skills and understanding and gives personalised feedback. Digital calculators are only used when the focus is not on mental calculation. The digital resources also conform to other areas of NCETM guidance such as setting out to expose and address likely misconceptions and misunderstandings, offering a wide range of tasks and exercises that use deliberate variation, and addressing 'real life' uses of mathematics.

Digital technologies and student learning

There is a large body of research that suggests digital technologies can contribute to student learning, e.g. Higgins, Xiao and Katsipataki (2012) and Drijvers et al. (2016). This highlights the pivotal role of the teacher and the school for successful use, including the need for good teacher pedagogical (including technological) content knowledge. Drijvers et al. (2016, p.25) state:

> In a technology-rich classroom, the teacher will play a pivotal role in crafting effective lessons that capitalize on the affordances of technology (Yerushalmy & Bolzer, 2011). A key to planning and delivering effective lessons is to have good pedagogical content knowledge, which includes deep knowledge of students' understanding and how technology can positively influence this.

Where, and how, then, are digital technologies used to greatest effect? Clark-Wilson, Oldknow and Sutherland (2011) argue that in order to improve the UK's capacity for technological innovation and creativity, we need to focus on high quality mathematics learning - as well as other STEM subjects - with or without technology. However, there is currently limited use of digital technologies in e.g. lower secondary mathematics teaching in the UK (OECD, 2015). Ofsted (2012) also report that technology is underused in mathematics and that its potential is generally underexploited. Use is largely teacher-led and focused on presentational software such as PowerPoint and interactive white board software, which does not by itself seem to affect learning gains (Clark-Wilson et al., 2011). Aspects of AL are purely presentational e.g. the digital version of the textbook. However, AL also aims to harness the potential of technology, e.g. through hyperlinks to supplementary representations or dynamic apps, so the hope is that teachers will go beyond the presentational use when using AL. In this respect, the hyper-linked resources share characteristics of pre-prepared files created in more generic mathematics software such as GeoGebra or Autograph, that can be used to stimulate mathematical exploration and discussion (e.g. Higgins et al., 2012), though they lack the breadth and flexibility of such software. Critically, student resources also offer opportunity for immediate formative assessment of learning.

There are, though, known barriers to use. Clark-Wilson et al. (2011) focus on maths-specific digital tools and packages, including specific software such as that offered by AL, identifying as potential barriers perceptions of digital technologies as an add-on only, school-level assessment practices not accommodating the use of technologies, and inadequate guidance on how to use the tools. They particularly note that even when perception and assessment have changed, continuous professional development always remains important if the potential of digital affordances is to be realised.

This focus on professional development is supported by other research: Drijvers et al. (2016), for example, call for research-based and easily-accessible professional development for deeper teachers' pedagogical content knowledge for teaching with technology (2016, p.25). In Ertmer's (1999) and Bai and Ertmer's (2008) seminal works around first and second order barriers to technology adoption, they also stress the importance of professional development, including training, reflection and collaboration, for changing teachers' ingrained attitudes and beliefs. These form a second-order barrier, while quality of and access to the technology can be first-order barriers. It is the former that are harder to overcome.

The study

This paper reports on some early findings from a two-year Pearson-UCL Institute of Education collaboration funded by Pearson. As such, particular care was taken in ethical justification, to address potential threats to the validity of findings, e.g. by using external-to-Pearson researchers for all fieldwork. Overall, the study set out to begin to understand the motivations for adoption of MP and GCSE resources, how the resources are used and experienced in schools, and the perceptions of their effectiveness in meeting teacher and student needs. Here, we focus on findings around teachers' use of the digital resources specifically. We probed access to those and their impact on learning, asking:

- How is KS3 MP/GCSE Mathematics (9-1) being implemented in schools?
- What are the barriers, if any, for students and teachers in accessing the digital resources?
- Do teachers value the overall content, and specific features of the AL platform and CPD element?

We used a variety of methods (interviews, focus groups, lesson observations, and surveys) with both teachers and students in the first year of the study: here we draw on just the first year's (2016-17) termly interviews with teachers and Heads of Mathematics (HoMs), and Spring term lesson observations. Participant schools were recruited from those using one or both sets of resources, so as to give a variety of key school characteristics, but there is no claim to representativeness. Not all sample schools used both schemes or catered for students at both KS3 and KS4. Shrinkage reduced the original 20 schools to an active 15 from the start of 2017. In the first full year, data was drawn from at least one year 10 class in each school and/or at least one year 7 or 8 class, their teachers, and the HoM, with the intention of following those classes through to the completion of a two-year programme of study. Some HoMs also participated as either the KS3 or KS4 class teachers, and for a variety of reasons, complete intended data collection was not achieved. Table 1 gives an overview of the teacher-related data on which we draw in this paper.

	Autumn 2016	Spring 2017	Summer 2017
Teacher and HoM telephone interview transcriptions	13 KS3 teachers 21 KS4 teachers 16 HoMs		12 KS3 teachers 20 KS4 teachers 15 HoMs
Semi-structured lesson observation notes, lesson plans		13 KS3 classes 20 KS4 classes	
Teacher face-to-face interview transcriptions		11 KS3 teachers 18 KS4 teachers	

Table 1: Overview of the teacher-related data in the first year of the study

All interviews were recorded and transcribed, then analysed through a thematic analysis in N-Vivo. The overarching themes were based on the research questions (e.g. access and experience of teachers, learner progression, achievement and competence), while supplementary themes derived from open descriptive coding of the range of data. Ethical justification for the study cited evidence that participation in professionally-focused interviews with a knowledgeable other can result in deep teacher reflection and learning (e.g. Baker & Johnson, 1998), and teachers did express acknowledgement of that in interviews.

Findings

We draw on data related to teachers' use of the AL Digital Service, particularly the distinctively digital elements of the front-of-class and student aspects.

(Under)use of resources

Schools as well as individual teachers within schools reported variable use of the digital resources (and indeed, schools had purchased different subsets of the package), though the overwhelming picture was one of very limited use, illustrated by the following Head of Maths:

> A couple of teachers are taking the lead on ActiveLearn but to be honest we are not using it as much as we could because we go back to the books. We need to evaluate as a team whether or not we are getting value for money for it. (HoM 3, Autumn 2016)

AL was most frequently used for textbook projection on the board, observed in 30 of 33 lessons. In 28 observations that was the only use. Teachers felt those were fairly typical lessons, but many teachers said they would make a decision by topic. While there are interactive elements to the projection of the textbook, observations suggested these are underused, limiting the use of the resource to presentational purposes only. One teacher explained:

> I'm still learning my way around it. I haven't used it as much as I'd like. And, you know, the functionality, I haven't really had the chance apart from I, you know, sometimes use the questions and flag them up on the board so they're just there (Y10 Teacher 7, Spring 2017)

At least 20 of 33 teachers used the AL Digital Service for assigning homework – though with variable frequency. Such use was linked with mixed experiences for students, often marred by technical difficulties. On probing with the schools concerned, it appears those were largely bandwidth challenges rather than being integral to the software - but nevertheless, discouraging for both teachers and students. It also took quite some time and investment for schools to fully incorporate the system into their way of working:

> I used to do it when I first started this year on sort of paper hardcopy sheets. Now ActiveLearn has all been sorted they've got their individual logins and they now will get set weekly ActiveLearn (Y7 Teacher 5, Autumn 2016)

> What we plan to do is pilot it with a few groups in each year and then have feedback of what it is […] Generally you're more familiar with what you use at the moment so I feel like I need to get to using it, have the staff using it to have a feel to have an opinion of whether it could replace it. (Y8 Teacher 10, Spring 2017)

At least 10 out of 33 teachers sometimes used the AL videos with their students and were generally positive about them, as bringing a 'different voice' into the classroom (Y8 teacher 6, Spring 2017).

At the end of the first year of the study, teachers at 9 of the 15 schools also indicated that one of their goals for the upcoming year was to develop and encourage the use of AL in their schools, and two of the schools even bought additional digital resources. The HoMs at two schools explained:

> We haven't done ActiveLearn yet. I mean, we bought it but we haven't used it. We're going to do it in September so they can access ActiveLearn. We haven't done that yet (HoM 12, Summer 2017 interview).

> We haven't used much of the ActiveLearn part of the resources. So that's going to be a bigger part of the Key Stage 4. We want to make sure that the students can, their homework will be set on ActiveLearn as that is compatible with the content that they use in class (HoM 9, Summer 2017 interview).

Reasons for using ActiveLearn

When teachers do use the interactive elements, reasons given include their reported high quality, their ability to engage students and potential for improving student outcomes through familiarising students with different approaches and engaging them. Some particularly mentioned the videos as useful because they give the students a different authority or explanation. Online homework was also considered to be of good quality and three teachers spoke explicitly of the value they place on the integral formative assessment.

Reasons for not using ActiveLearn

The Spring 2017 interviews suggested the two main reasons for not using the digital resources were teachers' lack of familiarity with its affordances, and challenges with the software functionality (each mentioned by 12 teachers). Other reasons included problems with infrastructure (e.g. white board, internet), limited appropriateness of content (e.g. the homework was too easy/difficult), curriculum pressures of a new and more aspirational curriculum, and maintaining existing classroom habits.

While technical problems are clearly a first-order barrier (and fortunately most were addressed over the year), the lack of teacher's familiarity is a second-order barrier that is harder to overcome. Teachers often said they had not had enough time to get used to the resources. This resulted in some schools hardly using the digital service for the entire year. Teachers commonly reported going through a slow process of independent discovery, dealing with a sometimes-overwhelming choice.

Role of professional development

Professional development opportunities and a strong, solution-focused community in schools have been identified as crucial to overcome this kind of second-order barrier (e.g. Bai & Ertmer, 2008; Clark-Wilson et al., 2011; Drijvers et al., 2016). Study interviews suggested that none of the schools had bought the Pearson CPD resource-linked training, though a handful of teachers had attended some online training or recounted the demonstration of a Pearson representative (which focuses on a technical demonstration rather than pedagogical). Most sample schools, though (at least 9 of 15), claimed collaborative environments: teachers talked about working in teams who share experiences and resources. This was particularly the case as they were adapting to a new curriculum, when sharing knowledge and resources was essential to avoid the changes becoming overwhelming. Some schools had additional meetings around new GCSE topics. These kinds of collaborative sessions, however, tended not to focus on the use of the digital resources specifically, because teachers understandably prioritised new or re-focused curriculum content areas, or emerging new assessments: time for such development is always an issue, but particularly when teachers are accommodating significant other change.

During the summer 2017 interviews, teachers reflected on the development of their use of the digital resources over the first year of the study. While most teachers (at least 13 of 19 commenting) reported that they developed and increased their use of the AL, at least two started to use the AL less as the year progressed: they again gave as reasons the pressures of coming to work with the new GCSE (with first assessment Summer 2017), with this trumping other considerations.

While many teachers emphasised collaboration within the school, only a minority of teachers (about 16 of 50 involved) reported learning from external events or programmes during the year, and in all but two schools this was limited to the HoM or Key Stage coordinator. Time and costs were quoted as big constraints here. Teachers repeatedly said that given the demands of learning to teach for a new curriculum, 'getting to know' AL was not top of their priorities – but that they fully intended to invest time in getting to know it better as other pressures allowed. In many ways this seems a 'catch-22' situation: these resources are designed to support teachers in opening up more aspirational curriculum goals to students – and yet teachers say they are having difficulty finding time to explore the potential of AL for their teaching, precisely because of the pressures of learning to teach for those aspirations.

Implications and Further Research

Although this study focused on specific materials, asking how and why they were used, as well as probing their impact on learning, the findings may have implications beyond the particular resources to other digital curriculum materials, including those designed for self-supported study, and mathematics-specific apps for exploration and discussion. The study offers evidence that teachers are often not fully using the learning potential of the digital resources invested in, even though those were carefully developed to offer reported widely valued, and varied, learning opportunities. The main challenges appear to be the lack of teacher familiarity, and technical issues, resulting in a slow process of the development of teacher knowledge around their use. This might have been addressed by more external professional development, or else by more targeted internal sessions – but there is a tension with other demands on teacher time.

We suggest that to better harness the potential of such resources, schools must recognise the need to invest time in software-specific professional development – whether bought-in, using AL technical- and pedagogical-focused CPD videos, or via peer-led internal collaborative development sessions focused on the digital resources. In parallel with understanding the technical aspects of the resource, collaboration and development should focus on the pedagogical knowledge around effective use. Teachers need to be confident with the technicalities if the platform is to enhance teaching and learning, but also to reflect on the most effective ways to integrate use of AL into their teaching, if its full potential, complementing the teacher role, is to be harnessed for students' benefit. Those responsible for curriculum change also need to be aware that the introduction of a fully coherent curriculum system (Schmidt & Prawat, 2006) of intended curriculum, assessment, and resources (which in the 21st century must surely include the harnessing of digital resources) – demands for its mature and embedded enactment sustained and informed teacher learning, related to *each* of those aspects, including the effective use of resources. Without that, we have shown that the demands of preparation for teaching a new curriculum across the age range can marginalize other teacher development, including, paradoxically, for effective use of resources well-designed to support that curriculum change.

Year 2 of the study will probe the evolving extent and depth of use of KS3 Maths Progress and GCSE Mathematics 9-1 digital affordances as the new curriculum and GCSE bed down. It will further explore the ways in which, and reasons why, teachers and students use distinctively digital aspects, and the perceived impact on student learning. Additionally, it will probe what teachers consider Pearson's role should be in supporting them to make a more significant shift towards full use of the potential of AL.

Acknowledgement

This research was funded by Pearson UK.

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A tentative framework for students' mathematical digital competencies

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Abstract: New digital trends have found a place in the mathematics classroom and there is a potentially "hidden" demand for students to acquire both digital and mathematical competencies. Current frameworks often talk about one or the other. In this paper, we propose a combined framework for mathematical digital competencies based on two existing frameworks: the KOM framework for mathematical competencies, and the DigComp framework for digital competencies. We discuss the potential value of such a framework for the mathematics education community, i.e. researchers, mathematics educators and practitioners.

Keywords: Mathematical competencies; digital competencies; mathematical digital competencies.

Introduction

Although it is surely possible to distinguish between mathematical and digital competencies, it appears productive to "coin" the two in order to be able to talk about *mathematical digital competencies*, or MDCs (Geraniou & Jankvist, under review) – not least taking into consideration the large-scale embedment of digital technologies in mathematics education today. Of course, tools to do mathematics come in different forms, e.g., physical tools such as centicubes, abacuses, Cuisenaire rods, etc. not to mention rulers, compasses, spirographs, specially ruled paper and so on and so forth. Surely, technology is only *one* tool amongst many. But while several other tools serve one, or a few, purposes, a technological software such as a Dynamic Geometry System (DGS) or a Computer Algebra System (CAS) serves a multitude of purposes. As mathematical digital technologies advance, so do the demands to the competencies of their uses, both inside and outside mathematics educational contexts. However, one should not be blind to the potential pitfalls of the increasing use of technology in mathematics education (e.g., Geraniou & Mavrikis, 2015; Jankvist & Misfeldt, 2015; Jankvist, Misfeldt, & Marcussen, 2016). As well-known, digital tools can perform many of the mathematical tasks that students traditionally are expected to do. For example, the *GeoGebra* feature for constructing regular polygons. As pointed out by Niss (2016), digital technologies should not be a substitution for competencies, but an amplifier of capacities. Enforcing mathematical capacity is the positive idea of using technology as a lever potential (Dreyfus, 1994), i.e. that students may save time on tedious routine work and instead focus their mathematical efforts and increase their capacity. The pragmatic outsourcing of the lever potential, however, also black boxes the underlying mathematical processes, and may leave students dependant on the digital tool for carrying out even basic mathematical exercises (Lagrange, 2005). Surely such scenarios are not what we aim at with the notion of MDCs. Rather we are concerned with those situations where neither mathematical nor digital competencies are replaced by technology, but where the digital tools actually enforce students' capacities in an epistemic sense (e.g., Geraniou & Mavrikis, 2017).

The Danish KOM framework: mathematical competencies

In relation to mathematics and competencies, Kilpatrick (2014) states that school mathematics sometimes "is portrayed as a simple contest between knowledge and skill" while "Competency frameworks are designed to demonstrate to the user that learning mathematics is more than acquiring an array of facts and that doing mathematics is more than carrying out well-rehearsed procedures" (p. 87). As examples of such frameworks, Kilpatrick mention three: the five strands of mathematical proficiency as identified by the Mathematics Learning Study of the US National Research Council; the five components of mathematical problem-solving ability identified in the Singapore mathematics framework; and the Danish KOM project[1], which lists eight distinct yet mutually related mathematical competencies. Of these three, the KOM framework appears to be the more elaborated one concerning mathematical competencies, but also that which so far has had the most widespread influence in other countries (Niss & Højgaard, in progress). Furthermore, KOM's competencies description was implemented as the basis of the PISA mathematical framework in the years from 2000 through 2018 (e.g., see OECD, 2013).

The Danish KOM defines *mathematical competency* as (an individual's) "…well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (Niss & Højgaard, 2011, p. 49). By addressing the question of what it means to master mathematics, KOM identified eight competencies, each possessing both an analytic side and a productive side. The competencies fall into two groups (see Table 1 below).

The ability to ask and answer questions in and with mathematics	(1) mathematical thinking competency (2) problem tackling competency (3) modelling competency (4) reasoning competency
The ability to deal with mathematical language and tools	(5) representing competency (6) symbol and formalism competency (7) communicating competency (8) aids and tools competency

Table 1. The eight mathematical competencies of the KOM framework (see Niss & Højgaard, 2011).

Each of the eight competencies has both an analytic side involving understanding and examining mathematics, and a productive side involving carrying it out. For instance, the aids and tools competency, firstly consists of having knowledge of the existence and properties of the diverse sorts of relevant aids and tools employed in mathematics and of having an insight into their capabilities and limitations within different kinds of contexts. Secondly, it comprises the ability to reflectively use such

[1] KOM is short for "Kompetencer Og Matematiklæring" which is Competencies and Mathematical Learning.

aids and tools. In KOM, the general description of the aids and tools competency also covers the use of digital tools. As a consequence the digital aspects of this competency are not very elaborated.

Digital Competencies frameworks and the European DigComp framework

Living in the digital era, we are witnesses of an increasingly digitalised society, in which digital competencies are becoming 'life skills' and can be compared to skills, such as mathematics and literacy (Ferrari, 2013). A *digital competency* is "the set of knowledge, skills, attitudes […] required when using ICT and digital media to perform tasks; solve problems; communicate; manage information; collaborate; create and share content; and build knowledge effectively, efficiently, appropriately, critically, creatively, autonomously, flexibly, ethically, reflectively for work, leisure, participation, learning, socialising, consuming, and empowerment" (Ferrari, 2012, p. 43). There is a plethora of terms used to refer to digital competencies. For example, digital literacies, which in essence are the information, media and communication skills (Hockly, 2012) or media literacy or ICT literacy as identified and cited by Hatlevik and Christophersen (2013). Hague and Payton (2010) describe digital literacy across the curriculum as: "the skills, knowledge and understanding that enables critical, creative, discerning and safe practices when engaging with digital technologies in all areas of life" (p.19). Regarding the terms digital competency and digital literacy, some authors use them interchangeably (Hockly, 2012). However, referring to school students in particular, Hatlevik and Christophersen (2013) claim that there are differences:

> A concept such as digital skills focuses on dealing with the technical conditions, whereas digital competence and literacy are broader terms that emphasise what kind of skills, understandings, and critical reflections students are able to use. When analysing and discussing the terminology, the concepts seem to have gradually shifted focus from the simple use of digital tools, often linked to concepts such as digital skills, to broader terms, including the students' digital competence and literacy (p. 241).

In fact, many countries include into their curriculum digital literacies, although there is disagreement in terminology: e.g., "digital competency" (Norway); "digital media literacy" (Australia); "media literacy" (UK) (Hatlevik & Christophersen, 2013).

Digital Competencies have been used to characterise people's certain skills in different contexts; these being the workplace, everyday responsibilities or in education and schools in particular. For example, Kent et al. (2005) introduced the term techno-mathematical literacies "as a way of thinking about mathematics as it exists as part of modern, increasingly IT-based workplace practices" (p.1). Focusing though in the school context, there are certain digital literacies which we expect school students to acquire and these are referred to as school-based digital literacies:

> Students' mastery of basic tools and computer programs is only a first step towards the development of advanced knowledge, skills, and attitudes [...]. Often the development of digital competency is considered a continuum from instrumental skills into productive and strategic personal competency and cognitive skills [...]. Therefore, digital competency includes students' ability to use technology in order to consume and access information. Moreover, digital competency also includes how students make use of technology to process, acquire, and evaluate gathered information. Finally, digital competency means that students can produce and communicate information with digital tools or media (Hatlevik & Christophersen, 2013, p. 241).

There are various digital competencies frameworks currently used at schools (e.g. Hague & Payton, 2010; in wales: learning.gov.wales/resources/browse-all/digital-competence-framework/). All these different digital competencies frameworks have similarities in what skills students are required to gain. The main difference is that for each framework, these skills are grouped in different overarching categories. Upon reviewing the above mentioned different digital competencies frameworks, any of these frameworks could have been chosen as a basis for our investigation on a potential framework on MDCs. The counterargument though is that these are produced to be used to the schools in these specific countries, Norway, UK and Wales, and to our knowledge have not been used outside these countries in different contexts. We have therefore decided to choose the most internationally recognised framework on digital competencies, the DigComp Framework for Citizens by Ferrari (2013).

Like the KOM framework, the DigComp framework is structured around a number of main areas, each encompassing a number of digital competencies as shown in table 2. These though are not directly linked to the mathematical context. The digital competencies not deemed to be of relevance in relation to the development and possession of mathematical competencies have been omitted. Of the remaining ones, we briefly elaborate on those digital competencies, which are less self-explanatory than the rest. One such is (3.2) which encompasses to "modify, refine and mash-up existing resources to create new, original and relevant content and knowledge" (Ferrari, 2013, p.5). Another one is (5.1) which comprises to "identify possible problems and solve them (from trouble-shooting to solving more complex problems) with the help of digital means" (Ferrari, 2013, p.6), and not least (5.4) which has the nature of a kind of meta-competency: "To understand where [one's] own competence needs to be improved or updated, to support others in the development of their digital competence, to keep up-to-date with new developments" (Ferrari, 2013, p.6).

(1) Information	(1.1) Browsing, searching and filtering information (1.2) Evaluating information (1.3) Storing and retrieving information
(2) Communication	(2.1) Interacting through technologies (2.2) Sharing information and content […] (2.4) Collaborating through digital channels […]
(3) Content criterion	(3.1) Developing content (3.2) Integrating and re-elaborating […] (3.4) Programming
(4) Safety	[…]
(5) Problem-solving	(5.1) Solving technical problems (5.2) Identifying needs and technological responses (5.3) Innovating and creatively using technology (5.4) Identifying digital competency gaps

Table 2. The DigComp Framework for Citizens with its five main areas (Ferrari, 2013, p.12).

Exploring the potential interplay between mathematical and digital competencies

In our experiences as educators, we have noticed in several occasions what appears to be a simultaneous activation of mathematical competencies and digital competencies. From the KOM framework perspective, digital competency might fit as a minor part of the tools and aids competency, i.e. in terms of the *reflective* use of ICT, referring also to having an understanding of ICT's capabilities and limitations in given contexts. However, from a digital competency perspective, this would constitute too narrow a point of view. Considering the relevance of the eight mathematical competencies for each of the 21 digital competencies of the DigComp framework and vice versa, we have identified *two overarching themes for interplay*, which may provide structure to a potential framework for MDCs: "communication and collaboration" and "problem handling and modelling" (Table 3).

Starting from *communication and collaboration* it seems somewhat straightforward to expect learners to acquire competencies, mathematical and digital, so as to apply both and use them effectively. Digital resources for mathematical learning are designed to incorporate and map mathematical language, but for example students would not be able to share their mathematical answers in a given digital resource or medium if they did not know how to type their answers and use the keyboard effectively, save their answers, upload them on a sharing forum, etc. Being literate in both domains, the mathematical and the digital, seems necessary to achieve in either one of them. Also, in mathematics being able to represent mathematical concepts, entities, etc. is an integral part of communication and so is being able to interpret other's representations, digital or not.

Moving onto the second overarching theme for interplay, "problem handling and modelling", the ability to ask and answer questions in and with mathematics is in fact a problem handling and/or modelling capability. The digital competencies area of problem solving (cf. Table 2) involves identifying what is needed to provide technological responses or identifying one's gaps in technical knowledge. Indeed, both these competencies can reasonably be placed under the overarching umbrella of "problem handling and modelling". But, in our view, thinking about someone who possesses MDCs in terms of problem handling and modelling in the context of (educational) technologies, we have in mind those individuals who have the competencies to (i) address a mathematical problem using digital resources and media creatively and effectively; (ii) use digital resources and media to solve mathematical problems or model extra-mathematical situations, which they were unable to handle or found it more difficult to deal with without the support digital technologies offer; (iii) interpret the instant feedback given by digital technologies and decide upon the next step or action to take. "Problem handling and modelling" also involves the interplay between mathematical thinking and computational thinking, e.g., algorithms, recursion, programming, etc. (for a description of computational thinking, see e.g., Weintrop et al. 2016). Of course, one should bear in mind that "problem", whether it be handled by means of digital or mathematical competencies or an interplay of both, is still relative to the individual (cf. the KOM framework).

Suggesting a tentative framework for mathematical digital competencies

For each of the two overarching themes for interplay between mathematical and digital competencies we now attempt to "flesh out" a set of MDCs (Table 3). Of course, the division of two types of interplays into MDCs should not be thought of as a strict division. As with the KOM framework, overlap of competencies may occur. The placement and description of the MDCs has been made according to what we conceive as the competency's "center of gravity".

Communication and collaboration	(1) Mathematical digital literacy (2) Mathematical digital collaboration (3) Mathematical digital representation (4) Mathematical digital interpretation
Problem handling and modelling	(5) Mathematical digital thinking (6) Mathematical digital reasoning (7) Mathematical digital manipulation

Table 3. Two main areas and seven mathematical digital competencies.

(1) Mathematical digital literacy – Being literate digitally, but mathematically too, in order to take a critical stance to the integration of digital technologies in mathematical activities (in particular in teaching and learning situations). It involves knowing which digital tools are most applicable for different kinds of mathematics as well as different mathematical problems and modelling situations. The competency involves also being able to interpret mathematical tasks presented within a digital environment, use the mathematical language to share answers and justifications within the digital environment, but also save, revisit, edit, submit one's work.
(2) Mathematical digital collaboration – Being able to collaborate verbally and/or digitally with peers. Having the ability to build upon one's peers' contributions with the aim of producing shared problem solutions or mathematical models. Within a digital environment being able to articulate mathematical ideas accurately as well as carry out discussions using mathematically valid arguments with peers. Also ensuring that the language used is appropriate and relevant to the given task. *(3) Mathematical digital representation* – Choosing the most appropriate functionality/feature of the digital tool/medium to represent and solve a mathematical problem or build a mathematical model. Also, being creative when representing mathematical entities involved in the given task, or the task itself. And knowing how to use mathematical notation in a digital environment. *(4) Mathematical digital interpretation* – Reading and interpreting mathematically the instant (usually dynamic) feedback – this includes recognising a mathematical error and fixing it (e.g., when you get an "*x*" instead of a tick) including also being able to interpret the digital media's feedback (e.g., digital responses such as "true", "false", "undefined" etc.). Observing the animation/simulation of any constructed models and interpreting mathematically such simulations. *(5) Mathematical digital thinking* – Being able to think mathematically as well as computationally, e.g., algorithmically and/or recursively. Knowing what kinds of mathematical and extra-mathematical problems that may be dealt with by means of digital tools and which may not. Understanding and being able to apply principles of programming, and to understand what is behind the programme. *(6) Mathematical digital reasoning* – Verifying solutions and validating mathematical models with the support of the digital technology by being able to provide mathematically valid justifications (not only rely on the tool's instant feedback, e.g., getting a tick or "looking" at an image). Knowing what constitutes a valid mathematical argument or proof, and make reflective decisions about when to outsource (e.g., black box)

processes of a mathematical reasoning (i.e. a chain of arguments) to a digital tool and knowing when not to. *(7) Mathematical Digital Manipulation* – Manipulating constructed mathematical representations or features of the digital tool and identifying the mathematical rules/connections within these. Being able to manipulate mathematical expressions using a digital tool, while at the same time knowing and understanding why such manipulations are both possible and correct.

Exemplifying and discussing the tentative framework for MDCs

Taking as an example some of the embedded affordances of a widespread DGS like GeoGebra, allows us to briefly exemplify the above described combined framework for MDCs. Recall the mentioning of 'regular polygon' in the introduction. Surely, if students are to create a regular polygon in GeoGebra using the 'regular polygon' feature of the DGS, not much mathematics may be activated. However, if students are to construct a regular polygon equivalent to GeoGebra's regular polygon, i.e. one which keeps its internal structure when dragged, then the activation of both mathematical and digital competencies may be so intertwined that it no longer makes sense to distinguish the two.

For example, students may revisit their existing knowledge of mathematics and/or digital technologies, gather information while interacting with GeoGebra and decide upon a sequence of actions, which potentially changes or gets adapted based on the instant dynamic feedback they receive from the tool and their inferences of that feedback. They may decide that GeoGebra is the ideal digital tool to construct a regular polygon, which indicates the activation of the *mathematical digital literacy MDC*; or they may choose to use the GeoGebra's affordances, such as constructing line segments and circles to make their chosen regular polygon, which indicates the *mathematical digital representation MDC*; or they may decide to use their constructed polygon to construct a different polygon or solve another mathematical problem, which indicates the *mathematical digital manipulation MDC*; or they interpret GeoGebra's feedback, which indicates the *mathematical digital interpretation MDC*; and they may argue for the correctness of their construction considering their mathematical knowledge of the properties of the chosen polygon as well as its mathematical definition, which indicates the *mathematical digital reasoning MDC*.

To conclude, our argument is that there seems to be a potential in the fruitful *interplay* between mathematical and digital competencies, which perhaps is not captured efficiently using two separate frameworks, and that this interplay might be better articulated through one framework for MDCs. In a sense the sum of the whole is greater than its parts.

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Flexible autonomy: an online approach to developing mathematics subject knowledge for teachers

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This paper uses Brookfield's (2017) lenses to critically reflect upon a Subject Knowledge Enhancement Course designed and taught by the authors. Learning occurs through a synthesis of asynchronous engagement with online e-learning modules, weekly synchronous tutorials and self-reflection following formative and summative assessment opportunities. Interrogating the course design, learner feedback and observation, and tutor pedagogic choices through connectivist and social constructivist learning theory, the paper concludes that the common perceived learning gains occur through the flexibility in learning, and the supported autonomy that learners are given. Further developments in our offer should therefore aim to improve these opportunities for learners where possible.

Keywords: SKE; e-learning; connectivism; reflection; flexibility; autonomy.

Introduction

Subject Knowledge Enhancement (SKE) programmes have successfully increased prospective teachers' confidence in the mathematics skills required for today's school curriculum – students surveyed have indicated a 53% increase in confidence from the start of study to the end of the course (80% expressed a high level of understanding) (Gibson et al., 2013, p.33). The provision this paper is based on has seen a 99% student satisfaction rate regarding progression in mathematics subject knowledge, through online engagement with digital learning resources and virtual dialogues with a subject specialist tutor. We propose three reasons for this. Firstly, it is suggested that by harnessing knowledge forged via engagement with online learning materials, a 'More Knowledgeable Other' is able to increase understanding via interactive dialogues that contextualise learning within students' own personal experience and Zone of Proximal Development (Vygotsky, 1980). Secondly, it is suggested that the increasing accessibility of online learning resources changes the role of the tutor from that of the didactic pedagogue, to that of the provocateur who challenges and disrupts the understanding of the student in which to advance their knowledge (Osberg & Biesta, 2008). Thirdly, it is this combination of flexibility in learning with a sense of supported learner autonomy which threads through the different facets of the SKE course that leads to the development of learner knowledge and confidence.

Literature Review

Established pedagogic models may become increasingly obsolete as digital technology empowers students to direct their own learning. According to Siemens

(2004) and Downes (2012), online technology's capacity to facilitate networks of adaptable and accessible information empowers students to autonomously interpret data and make connections within their own learning. Learning in the digital age is therefore increasingly 'distributed across a network of connections' characterised by 'diversity, autonomy [and] openness' (p. 85), allowing students the opportunity to independently and actively engage with a variety of information in a range of different modalities. Kropf (2013) describes 21st century students as "do-it-yourself" learners who acquire information from a series of nodes (points within an online network at which a plurality of information both intersects and branches out) and become active partners in learning, equally capable of sharing their knowledge and expertise with other individuals' (p.13). Siemens (2004) and Downes (2012) call this theory of online learning *Connectivism*.

For Green et al. (2017), perceived benefits of online learning include flexible access, personalisation, agency and connectivity. Personalisation is the ability to provide 'unique learning pathways for individual students'; agency is the opportunity to allow students to 'participate in key decisions in their learning experience'; connectivity is the ability to give learners the opportunity to 'experience learning in collaboration with peers and [tutors both] locally and globally' (p.6). Online courses typically consist of a variety of multimodal interactive media to support learning. Typical online multimodal media includes online forums, blogs, collaborative spaces, electronic documents, interactive online assessments, virtual spaces, digital videos and audio files. Mills (2011) suggests that an engagement with multimodal learning enhances students' experience, reception and comprehension – what is observed is a significant pedagogical shift, in which 'students are positioned to think […] collaboratively and creatively within a community of practice' (p.2).

Developing an online strategy that forefronts notions of connectivity, diversity, autonomy and openness whilst addressing the need to develop systematic knowledge and its application to set problems must consider teacher presence (the facilitator of learning), learner presence (the one initiated and motivated to learn), cognitive presence (understanding and its development) and social presence (collaboration and communication) (Shea & Bidjerano, 2010). Social Constructivism posits the view that knowledge develops as a result of social interaction and is therefore a shared, rather than an individual, experience. According to Vygotsky (1980), students learn most effectively by interaction within a Zone of Proximal Development that allows students to scaffold their learning via communication with their peers and a More Knowledgeable Other (in our context, the tutor) within a social environment conducive to the context of their current understanding. According to Osberg and Biesta's (2008) concept of an emergentist pedagogy, this tutor is defined as a 'provocateur' who is responsible for 'continuously complicating the scene, thereby making it possible for those being educated to continue to emerge as singular beings' (p.326). By consistently challenging understanding via a range of contexts, questions and set problems, the tutor is able to move the learner beyond their comfort zone and enrich their learning.

Prospective teachers' attitude and knowledge of mathematics can be increased through a combination of e-learning and problem-based approaches which provide required knowledge whilst challenging students to reflect upon, and evaluate their understanding (Uzel & Ozdemir, 2012, p. 1157). The most effective e-learning environments combine autonomous, individual learning with a community of learning involving tutors and peers (Hung & Nichani, 2000). The traditional role of the tutor as a conduit to knowledge is obsolete for students who can immediately access information online; hence the tutor as provocateur is preferred for an activity that requires challenging and enhancing understanding. In this context, a combined connectivist and social constructivist model would seem to provide learners with the benefits of autonomy, whilst providing students with learning that is sensitive to the context of individual and practical experience.

Methodology

This paper adopts a critical reflection methodology; we attempt to uncover issues of power and hegemony (Brookfield, 2017) through using learning theory and observations and experiences of the SKE course to question or validate decisions made about the course structure and methods of learning. As our SKE course is relatively new and subject to continuous self-evaluation and revision, we choose to critically reflect through lenses of theory, student eyes, colleague (course designer) perceptions and personal (tutor) experience (Brookfield, 2017).

The authors (a blended learning specialist, a mathematics education specialist and SKE course lead) design, teach and lead the SKE course inevitably drawing upon assumptions informed by our values, knowledge and practice about how we might best serve our learners. An effective and honest self-evaluation of this course must therefore 'unearth and scrutinise' these assumptions (Brookfield, 2017, p. 9), particularly related to the effectiveness of the tutor/student relationship (thus issues of power) and the balance of synchronous and asynchronous learning (and related hegemony). We use our review of blended learning literature, student feedback (written and oral), recordings of tutorial sessions, student e-portfolio data and individual tutor reflection to inform our analysis. This analysis will increase the effectiveness of the SKE course through providing a rationale for our choices and helping us take informed actions for continual improvement (Brookfield, 2017).

There is a lot of 'newness' and pedagogical uncertainty associated with this course. Subject Knowledge Enhancement courses have existed for a number of years, but there are currently no guidelines for the level of mathematical knowledge that applicants to courses have, or expectations of course structure. As such, although enrolment, progress, completion and attainment statistics are collected and monitored as part of the improvement process, self-evaluation of the SKE course at this stage requires continual scrutiny of the course from a wide variety of vantage points. As such, our conclusions can only be secure for this specific course at this point in time, we will resist 'epistemological distortion' and claims of our findings remaining valid for further cohorts at different points in time (Brookfield, 2017). However, we attempt to look beyond the 'what, so what, now what' of reflection-in-action (Driscoll, 2007), and establish conclusions that, within the limitations of our research methods, are creditable, dependable and confirmable (Guba, 1981; Shenton, 2004).

Course Design

At Canterbury Christ Church University (CCCU), SKE mathematics courses start with an online induction, followed by an initial computer-based multiple-choice assessment. An individual action plan is then negotiated with a tutor via email to focus subsequent learning on individual's development needs. Students participate in weekly online tutorials and work through self-directed online resources accessed through the University's Virtual Learning Environment (VLE), evidenced by a developing e-portfolio. At the end of the course, a final test measures a student's progression in mathematics. Success criteria for the course relate to engagement with the self-study materials, an increase in audit score, and a satisfactory e-portfolio submission. Course lengths range from eight to twenty weeks in duration and we tutor participants with mathematics degrees who require a refresher, and those without mathematics A-level within the same cohort. Applicants are pre-trainees on university-led or employment-led ITE courses, training to teach age ranges 7-14, 11-16, 11-18 or 14-19 and have a range of previous experiences of online learning.

The online mathematics resources are structured according to topics that correspond to the needs and requirements of the mathematics national curriculum and are modelled on how children learn mathematics in the classroom. In order to promote autonomy, each unit (approximately 8 hours' work) can be studied in sequence or standalone, giving students the ultimate flexibility in creating their own path in response to their initial mathematics skills audit. As well as having a wide range of on demand sessions to select from in order to design their own pathway (there are more than 50 sessions available), the sessions themselves were designed by an experienced team of mathematics educators following a social-constructivist model of learning mathematics. For example, in the session entitled "From Paper Folding to Angle", students explore and develop their understanding of angle rules through investigating the properties of A4 paper.

It is relatively easy to ensure that on-demand materials provide flexibility and autonomy. Doing so for live tutorials is more problematic, and a number of models have been explored in order to meet this need. The current delivery model aims to mitigate both of these challenges and consists of a 20-week rolling cycle of Key Stage Three and GCSE up to Grade Four, a 16-week rolling cycle of Key Stage Three and Foundation GCSE, a 12-week rolling cycle of GCSE only topics and an 8-week rolling cycle of Higher GCSE and introduction to A-level. Students enrol on an 8, 12, 16 or 20-week course according to their development needs. There are four tutorials a week, one for each of the rolling cycles. The rolling cycles are designed so that a student can join in at any stage, thus the students at each live tutorial will be at different stages of the course. Students do not have to commit to any one of the four rolling cycles - they are free to swap from week to week, or attend more than one tutorial a week. For example, an engineering graduate may choose to skip the mechanics session taking place that week and attend the foundation GCSE proof tutorial instead.

The course design therefore offers a combination of flexible learning, through both access to and the pedagogical design of on demand resources, and supported learner autonomy, through the structuring of live tutorials, which lead to both the development of mathematical knowledge and understanding and the confidence of learners.

Analysis

Our analysis considers how the CCCU SKE mathematics course provides both flexibility and supported autonomy using Brookfield's four lenses as its framework (Brookfield, 2017). Firstly, by considering student learning, we critically reflect upon the lens of student eyes and personal experience in which to ascertain the perceived learning benefits and limitations of SKE mathematics provision from the viewpoint of the learner. Secondly, by considering tutor pedagogy, we reflect upon the lens of colleague (tutor) perceptions and theory to highlight the benefits and limitations of the course from the viewpoint of teaching strategies.

Student Learning

The current course design is intended to allow students to enhance their understanding though flexible engagement at a pace, time and location that is convenient to their wider professional commitments and priorities. In this subsection, we consider the on-demand sessions and live tutorials through the lens of the student and their personal experience, considering three main areas: how students manage the design of their own pathway through the on-demand materials, how students perceive the social-constructivist nature of the on-demand materials, and how they use the live tutorials.

Many students are initially overwhelmed by the quantity of on-demand materials available to them. One adaptation that has been made to the course design in response to this is to provide direction towards sessions which will address the needs identified within the audit. In their feedback students will be told, for example, that if they answered question 22 incorrectly, in which they had to solve a system of simultaneous equations, then they should complete the on demand session 16.2, solving simultaneous equations. Students are also provided with a gap analysis in the form of a spreadsheet in which they RAG-rate their confidence against each session title, and use this to prioritise sessions. Some students use this to make a strategic plan, others report that it feels like empty bureaucracy and take a more ad hoc approach to selecting sessions. There is some evidence that a strategic pathway based on audit feedback and gap analysis leads to better outcomes as illustrated in Table 1.

	Pathway through on demand sessions (session numbers in order)	Initial audit result	Final audit result	Overall grade
Student 1	15, 1, 2, 3, 4 (ad hoc)	52	61	Satisfactory
Student 2	1, 2, 3, 4, 5, 8, 15, 11, 12, 6, 16, 17, 20, 23, 19, 21, 25, 26, 31, 33, 34, 36, 24, 43, 44, 46, 51, 58, 57, 56, 55, 54, 53, 52, 50, 49, 59, 70, 41, 42, 29, 18, 14, 13, 7, 6, 17, 16, 23, 46, 38, 37, 45, 47, 48 (moderate structure)	32	49	Good
Student 3	2, 4, 6, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 29, 36, 1, 3, 5, 9, 24, 25, 28 (strong structure)	41	101	Excellent

Table 2. Comparison of student pathway choices to audit scores and overall grade

Many students find the investigative nature of the on-demand sessions to be problematic. The social-constructivist principles which informed the design of these sessions work well in a classroom where learners can interact with their peers and more knowledgeable others. The second and third stated aims of the SKE programme are to develop mathematical thinking, and to place mathematical knowledge within meaningful contexts, and so it is vital that students perceive mathematics as a discursive, social discipline, but this can be hard to achieve when learners are isolated both geographically and in time. Attempts to address this have included the provision of solutions (which include notes on methods and alternative approaches) and the availability of the tutor to discuss sessions via email. Additionally, tutors are sensitive to this in the planning and delivery of live tutorials, when the essential discursive nature of mathematics and its learning can be addressed.

In their final reflections, many students comment on how the live tutorials were the most useful part of the course to them, for example:

> "The weekly tutorials were very informative and highlighted areas that I needed to revise further, this for me was the most practical part of the course."
>
> "The questions we solved … were pivotal for learning progression."
>
> " I found the online live lessons to be helpful and has given me some confidence in what I am doing,"

The model of rolling cycles differentiated at four levels across four separate tutorials each week was intended to enable students to select the live tutorial most appropriate to them. Many students attended all four tutorials every week, which meant that they encountered the same materials up to four times, but delivered at different speeds. Students explained that they were happy to be overwhelmed by the materials in early sessions, knowing that they would revisit it and grow in confidence. One said that the first time round she felt like an outsider observing others doing the maths, the next time she was a consumer of the mathematics, before finally moving into the roles of expert and leader. As the tutorials were on a rolling programme with new students joining every four weeks, this created a supportive learning environment in which not only the tutor was able to act as provocateur and more knowledgeable other, but students were able to do so too. Issues of poor student engagement due to lack of confidence in an unfamiliar learning environment is reduced as new cohorts join groups who have already established learning habits and the new social norms of the online classroom.

Tutor pedagogy

From the perspective of a theoretical lens, students' access to a range of online maths materials and resources follows the principles of connectivism by providing a diverse and open space in which to autonomously develop their understanding. Given that students have both the flexibility and autonomy to develop their own understanding via engagement with these materials, the responsibility of the tutor becomes less about knowledge transference and more about provocation – the tutor challenges students to think more deeply about their understanding which, in turn, induces a more adaptable and contextual approach to the knowledge they have acquired. Tutor and student interaction during tutorials provided opportunities to both challenge students' understanding and provide contextual and individual guidance to enhance understanding of mathematics topics.

Tutors were able to act as a provocateur in the on demand sessions. In one session students were guided through the steps to fold a sheet of A4 paper to create equilateral triangles and then use these to construct tetrahedra and octahedra, but were then later challenged to use this activity to prove the ratio of the lengths of the sides of the paper. In an introduction to calculus, students are supported in understanding both the fundamentals and applications of differentiation through film clips of a car chase.

Online tutorials typically begin with a series of challenges to problematize students' understanding of topics studied via engagement with online resources. The provision of mathematical problems allows both the student and tutor to confirm the current level of understanding and identify potential gaps or issues that can then be addressed. After potential gaps in understanding have been identified, the tutor is then able to recognise errors and provide guidance that is bespoke to student's individual context and experience – it is in this sense that, from a social constructivist point of view, both tutor and peers can act as More Knowledgeable Others who can challenge and question students within the context of their own understanding. In one particular tutorial that was videoed for self and peer observation purpose, students were invited to use their existing knowledge to suggest which mathematical object best exemplifies key mathematical terminology, such as "expression" or "inequality". Drawing mainly on their knowledge of the English language, students suggest pairings and are prompted by the tutor to explain their thinking. The tutor is particularly interested to hear the thinking behind incorrect pairings. As this example demonstrates, by identifying the symptom of errors and the reasoning behind them, the tutor is able to provide a solution and explanation that connects with the student's own context. From the lens of tutor, it would therefore appear that students' confidence and understanding of mathematics is increased by combining independently accessed online resources with challenging and contextual tutor interaction.

Conclusion

Our reflections through the lens of theory, designer, tutor and student has found that by combining online learning materials with the support of a 'More Knowledgeable Other', students effectively increase their knowledge and understanding of mathematics. The increasing accessibility and flexibility of online learning resources changes the role of the tutor from that of didactic pedagogue, to that of the provocateur who challenges the understanding of the student in which to advance their knowledge (Osberg & Biesta, 2008). A combination of flexibility in learning with supported learner autonomy leads to both the development of learners' understanding and confidence.

The importance of differentiation is highlighted as a key issue in presenting and delivering materials. Students use diagnostic assessment to autonomously develop an individualised learning programme. This learning journey is both informed by this action plan but can then be altered as the course unfolds. These differentiated asynchronous course resources have been found to promote independent active engagement by participants in their mathematics, evidenced by their asking their own questions and constructing their own understanding of the content.

Whilst evidence supports autonomous online learning as leading to an effective comprehension of relevant mathematical knowledge, by itself it lacks the opportunity to enrich, adapt and negotiate understanding within the context of challenging and practical applications. Our reflections suggest that students benefit from the social interaction during online tutorials which enhances and extends their knowledge through a variety of challenging problems and questions to support and extend their developing conceptual understanding.

The multiple needs of the learners and the large choice in course length currently means that an 'ideal' tutorial structure is difficult to achieve; several models have been used in order to tailor the real-time tutorials to the individual needs of students. Our current 'rolling structure' model has proved most able to fulfil the very different needs of students whilst maintaining the flexibility and autonomy identified as being so important to online learners.

Whereas digital learning, epitomised in the theory of connectivism, allows students to flexibly engage with learning at a pace, time and location suitable to their individual needs, a reflection on the experiences of students and tutors concludes that students' deeper and enhanced understanding of mathematics benefits from the complementary use of a social-constructivist model of learning.

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Attuning to the mathematics of difference: Haptic constructions of number

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CAPTeaM develops and trials activities that Challenge Ableist Perspectives on the Teaching of Mathematics. The project involves teachers and researchers from the UK and Brazil in reflecting upon the practices that enable or disable the participation of disabled learners in mathematics. In this paper, we focus on two themes that emerged from data analyses generated in the first phase of the study: deconstructing the notion of the normal mathematics student/classroom and attuning mathematics teaching strategies to student diversity. Here, we address these themes through exemplifying participants' haptic constructions of number in the context of a multiplication task in terms of four strategies they devise: "counting fingers"; "tracing the sum"; "negotiating signs to indicate place value"; "decomposing".

Keywords: Teacher Education; inclusion; embodiment; ableism.

Inclusive mathematics in an ableist landscape

Educational systems throughout the world continue to be profoundly structured around the construct of the "normal student", a socially constructed student, not a living, flesh and blood person. This construct can be employed to imply that there exists some kind of universal trajectory by which mathematical knowledge can be expected to be learnt, deviation from which is evidence of abnormality and, often, deficiency. Organising the teaching of mathematics according to imposed norms can obscure or even disallow variations in learning associated with different sensory, physical, linguistic, social and cultural experiences and identities – and contributes to a culture in which disability tends to be considered a lamentable condition, a disadvantage that must be overcome (Nardi, Healy, Biza, & Fernandes, 2018). It also results in educational practices developed with students in mind who do not actually exist, rather than for students who will be subjected to these practices.

Our study CAPTeaM (Challenging Ableist Perspectives on the Teaching of Mathematics), aims to challenge beliefs, processes and practices related to mathematics teaching which produce "a particular kind of self and body (the corporeal standard) that is projected as the perfect, species-typical and therefore essential and fully human" (Campbell, 2001, p. 44) and which contribute to the exclusion of disabled learners (e.g., Nardi et al. 2018). CAPTeaM involves inviting practising and future teachers to engage with tasks that encourage them to reflect upon the challenges of attuning mathematics teaching strategies to student diversity and to avoid privileging the notion of a normal student. To this end, we have collected data in Brazil and the UK as participants interact with two different types of tasks.

In the first (Type I), teachers are presented with classroom episodes which show the mathematical activities of disabled students. They are invited to consider how they might enable the engagement of disabled learners within inclusive learning communities. In the second (Type II), small groups of teachers solve a mathematical problem while at least one of them is temporarily and artificially deprived of access to a sensory field or familiar channel of communication.

In this paper, we focus on Type II data and analyses. We begin by outlining the theoretical basis for the task design, which involved linking ideas from the historical-cultural perspective of Vygotsky with aspects of embodied cognition. We then evidence the participants' discursive practices, especially in relation to deconstructing the notion of the normal mathematics student/classroom and attuning mathematics teaching strategies to student diversity. Here, we exemplify said attunement through illustrating participants' haptic constructions of number in the context of a task that invited them to communicate about multiplying a three digit number by a two digit number.

The theoretical underpinnings of CAPTeaM

A major concern expressed by Vygotsky (1997) in his seminal work with disabled learners in the 1920s and 1930s was that the dominant quantitative approaches of his time reduced the question of development to performance on measures that imply deficit not potential. For him, children whose learning is shaped by a disability can be expected to develop differently from their non-disabled peers, but this does not imply lesser development. In a nutshell, Vygotsky's position can be put as follows: if a disabled child achieves the same level of development as a child without a disability, then the child with a disability achieves this in another way, by another course, by other means. For the teacher, he argues, it is particularly important to know the uniqueness of the course along which to lead the child and thus to transform the barriers associated with an impediment into possibilities for development.

Our interpretation of this position (Nardi et al. 2018) is that learning can be defined as participating in, and appropriating (or making one's own), discourses associated with the knowledge discipline we know as mathematics. The process of making something one´s own is shaped by the tools used to act with it – and this includes tools of the body as well as material and semiotic artefacts. Part of understanding the mathematical discourses of learners (with or without disabilities) involves considering how and when the substitution of one (semiotic, material or bodily) tool by another engenders alternative mathematical discourses, which in turn empower the participation of those who have difficulties in interacting with conventional forms. Treating tools of the body as knowledge mediators is consistent with embodied approaches to cognition, which posit that perceptual-motor activities represent a constituent part of our thought processes (Gallese & Lakoff, 2005) and that feeling is part of knowing mathematics (Healy & Fernandes, 2014). Moreover, since that construction and use of all mediational tools have both social and individual dimensions, cognition is as much an interpersonal process as an intrapersonal one.

In teaching, the interpersonal side of cognition is particularly cogent, as it occurs in the context of contact with actions, emotions and senses of others. Indeed, Gallese (2010) has suggested that, when we come into contact with others, our implicit awareness of our bodily similarities result in the activation of the same neural resources when we perceive the actions, emotions and sensations of others as when we experience or execute them ourselves. We accept this suggestion with some caution: not all human bodies are similar and restricting empathy in this way could be used to reinforce exactly the idea of "normal" development that we are trying to avoid. For us, teaching mathematics involves engaging in discourses in ways explicitly aimed at involving learners in sharing the feelings of the teacher about aspects of mathematics, in a process during which the teacher also endeavours to feel the mathematics of the student. This involves a reciprocity of intentions: the teacher attempts to communicate so that her intentions come to inhabit the bodies of her learners, while simultaneously allowing their intention to inhabit hers (Healy & Fernandes, 2014). Given that not all bodies feel things in the same way, this necessarily requires the legitimisation of different ways of expressing and doing mathematics so that difference as well as similarity can be felt as one's own.

This brings us back to Vygotsky and the idea that, as teachers, we need to seek the mediational means that make most sense to the learners we teach and not to expect that the same means will necessarily be appropriable by all – or, even, that the impossibility of using certain tools necessarily impedes mathematics learning. In short, the mediational means that we make available (or not) in learning situations should be attuned to the learners involved.

The aims and methods of CAPTeaM

To explore the role of using different tools of the body in mathematical activities in ways which engage us in recognising and challenging ableism and in developing pedagogies that empower rather than disable learners, we use situation-specific tasks (Biza, Nardi, & Zachariades, 2007). These are research-informed tasks which invite teachers to consider mathematics teaching situations grounded on seminal learning and teaching issues and likely to occur in actual practice (ibid.). Situation-specific tasks can contribute towards generating nuanced accounts of teachers' pedagogical and mathematical discourses as well as facilitate teacher reflection and discursive shifts with respect to how teachers work towards enhancing learners' (disabled or not) opportunities to participate in mathematical activity (Biza, Nardi, & Zachariades, 2018). CAPTeaM involves engaging practising and future teachers with two types of situation-specific tasks, Type I and II, briefly described in the introduction.

Here we focus on Type II data and analyses. Type II tasks are designed with the aim of provoking reflections about how access to mediational means differently shapes mathematical activity. Participants work in groups of three. One member (A) acts as an observer and films the interaction of the other two members. The second member (B) has a learner role and is asked to solve a mathematical problem whilst, temporarily and artificially, deprived of use of a particular sensory field and/or communicational mode (e.g., seeing). The third member (C) has a teacher role, communicating the problem and intervening as judged necessary, but without access to another sensory field or communicational mode (e.g., speaking). In this paper, we focus on one of the Type II tasks (Figure 1).

For the task we consider in the rest of this paper, in each trio (A, B, C), the problem involved multiplying a three-digit number by a two-digit number, e.g., 347x26, although numbers given varied across trios. Then, all convened for plenary discussion of the strategies that had emerged in the small groups. Small-group activity, as well as plenary discussions, were video-recorded. We wish to stress that the aim of the task was not that the participants would attempt to role play the part of someone with a disability. Rather, we would argue that the temporary suspension of a mediation tool that someone is accustomed to use can serve to heighten awareness of alternative possibilities for communicating and expressing mathematics and to encourage participants to consciously attune their interactions according to the particular needs of the other (be they teacher or learner in this task).We chose to constrain the activity of both teacher and learner in the Type II task to highlight the reciprocity of these roles.

Artificially restricting mathematical interactions

For this activity, we will split in groups of three.

One member of the group (A) is the observer.

A second group member (B) will temporarily lose access to the visual field (by shutting their eyes or being blindfolded).

The third member (C) can see but cannot speak.

C will be given a piece of paper with the rest of the instructions.

Instructions to C: Your task is to ask (without speaking) B to multiply 347 by 26 and to indicate whether or not the answer suggested by B is correct.

B should not have access to these instructions.

Once the task is complete, A, B and C have a short discussion about how the restrictions influenced their strategies.

Figure 1. The *Artificially restricting mathematical interactions* Task (Type II).

Data was collected in Brazil and the UK from 91 pre- and in-service teachers (70 from Brazil and 21 from the UK). Bar a small number of in-service mathematics teachers (none with SEND coordinator responsibilities), participants in the UK were students on a Secondary Mathematics PGCE programme. Participants in Brazil included four practicing teachers with some Special Education responsibilities, ten teachers who were also undertaking a two-year Masters in Mathematics Education course, 38 undergraduate students on a four-year course in Mathematics Education (future mathematics teachers) and 18 undergraduate students studying on a four course in Pedagogy (to become generalist primary teachers).

Participants completed four tasks (three of Type I and one of Type II) in three-hour sessions. Data consists of written responses to the tasks (for Type I only) and audio / video recordings of small-group and plenary discussions of the responses. Data collection was carried out once ethical approval by the Research Ethics Committees in both the UK and Brazil institutions had been granted. Analysis of the data aimed to identify participants' perspectives on teaching mathematics to people with different disabilities. The following five themes emerged (see more details in Nardi et al. 2018, p. 154): valuing and attuning; classroom management; experience and confidence; institutional possibilities and constraints; and, resignification.

As we scrutinised the data on each of the above themes, the need started to emerge for robust, factual accounts of the participants' strategies for coping with the tasks. For example, we started asking questions such as what types of bodily involvement do we observe in the participants' interaction? or what communicational channels do the participants deploy during their interaction? In relation to the task in Figure 1, these transformed into questions such as: How do participants communicate about number? How is place value dealt with? Are some numbers more difficult than others? How do participants negotiate ways of communicating Yes/No (Right/Wrong)?

How do participants express, and overcome (if so), any difficulties they experience in this communication? In this paper, we share data excerpts which illustrate answers to these questions and showcase the resourceful ways in which the participants coped with the challenges posed by the task in Figure 1.

Data: Haptic constructions of number and place data

In addressing the aforementioned questions, a suite of strategies emerged that showcase how the participants invented novel ways of doing mathematics, particularly with regard to how they express number when team members B and C cannot see and speak respectively. In doing so, resorting to the communicational channels afforded by the sense of touch – thereafter haptic constructions – became a pivotal characteristic of what the participants chose to do[1]. We exemplify four of these strategies, S1-S4.

S1. Counting fingers. Participants indicate each digit in order, starting with hundreds, then tens and then units, by counting or raising the corresponding number of fingers. Communicating about each digit was easy but sharing the understanding that the three digits were meant as the components of a three-digit number was not. We identified four ways in which the participants coped with this challenge, less or more successfully. Each emerged after the three digits were identified through finger-counting: (1.1) Creating a sign intended to suggest joining the numbers into one. This was generally unsuccessful as it was interpreted by the blindfolded team member as a sign, for example, to add the numbers (Figure 2). (1.2) Continuing directly to indicate the multiplication sign, in a variety of ways (crossing two index fingers or arms, tracing a cross on hand or arm). Usually this had to be repeated a number of times before 3 4 7 became 347 and, even when this was understood, the number tended to be uttered as “three four seven” rather than “three hundred and forty-seven”. (1.3) Guided writing of number using the blindfolded team member’s hand and a pen or pencil. Finally (1.4), using objects, usually pens, instead of fingers. This was typically quickly abandoned. We return to this in S3 where objects were also used to communicate place value.

Figure 2: Treat the 3 numbers as one (S1.1).

Figure 3. Tracing the written symbol for number on inside of arm (S2).

S2. Tracing the sum. Participants communicate the number as a whole and without explicit attention to place value through tracing on centre of hand, back or arm (Figure 3). We identified three ways in which the participants did so: (2.1) Tracing the written symbol for number on centre of hand, digits signed one after the other on the same location, without indication of the position of each in the whole number and then moving on to tracing the multiplication sign. (2.2) Tracing the complete number on hand, back or arm, with position felt – that is, for 347, the index finger is moved to the

[1] We note that all participants in the role of C chose touch over sound to interact with their partner B.

right as the participant draws hundreds, tens and units – and then moving on to tracing the multiplication sign. (2.3) Guiding hand to write number on paper.

S3. Negotiating signs to indicate place value. Here the digits and the place value are communicated together with two different embodied notations which are used simultaneously in four different ways: (3.1) Placing B's hand in three locations after counting the numbers on the hand (Figure 4). (3.2) Counting fingers on arm, moving the position to different locations on the arm to indicate place value. (3.3) Using objects (screwed up paper balls) and placing on different locations on table. (3.4) Using objects to represent hundreds, tens and units. In 3.4 examples of objects used include pens or the vertical bars on a metal frame door.

Figure 4: Placing hand in different places on the table to indicate place value (S3.1).

Figure 5: Communicating the 300 part of 347 (S4).

S4. Decomposing. This strategy involved breaking down the number according to place value. For example, 347 was communicated as 300 plus 40 plus 7 through finger indication of 3 followed by two 0s, followed by 4 and 0, followed by 7. Figure 5 shows the zeros expressed through forming a circumference with index finger and thumb.

Of the four main strategies, the first two (S1 and S2) occurred more frequently than strategies S3 and S4 in which more attention was given to explicitly representing place value. Generally speaking, these latter strategies emerged in cases in which an S1 strategy was initially employed but those in role B (learner) had difficulty in understanding that a number with more than one digit was involved. Some of the participants in role C (teacher) showed an initial reluctance to change their strategy, choosing to repeat the same pattern of actions in a slightly slower form or by tapping the hand or securing their partners fingers more firmly. We might liken this to repeating commands more slowly, with particular emphasis on certain words. This accentuating sometimes made things clearer, but was more frequently unhelpful. Because the learners were permitted to speak, some chose to explain their needs very clearly. As they asked questions or provided information about their difficulties in interpreting the specific intentions behind the haptic constructions ("*do you want me to add*", "*it could be a 6 or a zero"*), their teachers were motivated to modify – or attune - their strategies accordingly. In some cases, learners explicitly told teachers how to proceed. This was invariably associated with the development of an efficient and effective task resolution. It was also common for the learners to suggest signs for "yes" and "no" (as in "*tap my arm twice for yes and once for no*").

In a small number of cases, the learners seemed reluctant to question or even provide their teachers with details of any interpretation problems. Perhaps they didn´t see this as part of the role of being a learner. Where the teachers were flexible about changing their strategies, this reluctance was not necessarily an impediment to success, and sometimes led to a more vocal participation from the learner as the task progressed.

Least successful were interactions in which the teacher repeated the same strategy and the student only communicated their lack of understanding.

Reflections on alternative mathematical expressions

In our analyses, we consider if and how engaging in this multiplication task motivated the participants to reflect upon how mathematical objects and operations might be expressed in ways that would make sense given the restrictions imposed on both team members C (in the role of a teacher who cannot speak) and B (in the role of a learner who cannot see). We stress that, despite the fact that most of those assigned the teaching role expressed concerns, even desperation, that their task initially seemed an impossible one, in most cases this did not turn out to be the case. Shared signs which enabled successful outcomes generally emerged fairly rapidly. In relation to the strategies S1-S4, devised in the absence of access to spoken or visibly written symbols, objects and gestures were combined in different ways that were gradually attuned to the resources available to the learners. On the way, effective ways of substituting temporarily disabled channels were invented.

This process of attunement drew heavily on what the teachers knew about the learners' previous mathematical experiences, and all of the strategies that were employed appeared to be directed at enabling the learners to re-enact previously experienced mathematical practices – albeit by activating expressive forms not commonly associated with multiplying numbers. Place value was not being introduced to the learners, it was being triggered through haptic means. The different haptic realisations of number allowed, eventually, the learner to feel the intentions of the teacher. This however was not always immediate, as it required some time for the teacher to accustom to simultaneously inhabiting a body which could not speak (their own) and a body that could not see (the learner's).

As described above, the attempts of both teachers and learners to appropriate each other's intentions were facilitated when the participants in the role of learner also assumed some of the responsibility for communication. It was also common for those in the role of learner to assign to the sighted teacher the task of remembering numbers that the blind learner could write down but could not see, here the other becomes a substitute tool. In the group discussions, these were issues that the participants highlighted as they reflected on how the temporarily imposed restrictions opened windows on pedagogical strategies that might be employed with disabled students. The idea of giving the student a role in guiding the teacher was one approach suggested:

> I was thinking before [about teaching a blind student], I would be lost, I wouldn´t know what to do to teach someone who is blind. But you have to listen to the person. It was her who showed me the way, in this case blind, she gave me the way. "Do it like this, do it like this". She gave me a way of communicating with her.

By requiring diverse forms of bodily involvement, Type II tasks provide opportunities for participants to consider the many and varied ways in which a mathematical problem can be approached. They permit a moving in and out of their long-established mathematical and pedagogical comfort zones, and a growing appreciation of difference as well as the enactment of agency shifts that these moves may imply. These are attributes of mathematics teaching that are pertinent at large and by no means exclusive to the teaching of disabled learners.

We see CAPTeaM tasks as inviting us to attune our teaching strategies in ways that harness the different potentials of different students, and to not associate mathematical ability with fluency with mediational means that are not available to all.

Acknowledgements

CAPTeaM is funded by British Academy International Partnership and Mobility grant PM160190.

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Collaborative task design with student partners in a STEM foundation mathematics course: visual support for the multiplication of matrices

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This paper concerns part of a collaborative project involving ex-students from a university Foundation Studies Programme working with teacher-researchers and analytic assistants in designing computer-based tasks for Foundation level tutorial sessions. This paper focuses on the design and use of a GeoGebra file to assist with students becoming proficient with matrix multiplication. The visual support of highlighting particular rows and columns of two matrices which are to be multiplied together proved helpful for students to develop success with carrying out matrix multiplication but also had a negative effect as well. A particular issue arose concerning the lack of variation in the size of matrices within the first set of questions. This led to difficulties with later questions involving matrices of different sizes.

Keywords: technology; matrices; collaborative; imagery.

Introduction and background

GeoGebra (https://www.geogebra.org/) is an open-source piece of software which has versatility to be used for many topics within mathematics. Examples of its use can be found in modelling (Hidiroğlu & Bukova Güzel, 2013) and problem situations which relate to functions (Ofra & Tabach, 2013). In our case, the GeoGebra program is used differently. Firstly, we use it in the context of matrix multiplication. Secondly, we created a series of questions, with sliders and buttons which did not require the learner to know or learn any of the functionality of GeoGebra itself (more details about the file created are given below).

Relatively little research has been carried out on the learning of arithmetic with matrices. Chang (2011) offered lesson notes for working with students on a linear algebra course and used questioning alongside the transformation of the image of a face to support a physical meaning for matrix multiplication. She offered two ways to define matrix multiplication. Larson (2010) explored how students think about the multiplication of a matrix with a vector including seeing it as either a matrix acting on a vector or a vector acting on a matrix. Hannah, Stewart and Thomas (2014) analysed students' views as their university teacher experimented with different orders of presentation on topics, including matrices and solving matrix equations of the form A**x**=**b** (where A was a matrix and **x** and **b** were vectors). Students favoured the use of pictures and examples over being given definitions. Imagery is an important aspect of what technology can bring to a topic and this includes the possibility of dynamic images rather than just static versions. Taylor, Pountney and Malabar (2007) looked at topics at undergraduate level, including matrix multiplication, and compared animated images with static versions of these. With matrix multiplication, numbers from the first matrix were multiplied by certain numbers in the second matrix. Their animation involved these numbers coming together through copies of numbers from the first matrix moving to be alongside the appropriate numbers in the second matrix. Findings suggested this aided students to see which numbers from the first matrix should be multiplied by which numbers from the second. In addition, students considered that the animated version speeded up their understanding of what was involved compared to the static versions.

This paper seeks to add to the limited literature of using visual imagery to assist students' learning of matrix multiplication. It is also part of a wider project (2016-18 The Catalyst Project) to (a) incorporate computer-based tasks into a university Foundation Studies course and (b) involve ex-students from Foundation Studies in the design and implementation of computer tasks for that course. As such this was one part of a highly collaborative project. It involved recruiting ex-students from the previous year's course, who responded to an initial announcement and who were selected following an interview. These we call Student Partners. We also involved current PhD students as Analytic Assistants, who helped with the data collection and analysis of that data. Lastly, there were three teacher-researchers who were members of staff, one of whom taught the Foundation course. Teachers can sometimes fail to see tasks from the students' perspective (Choy, 2016) and Johnson, Coles and Clarke (2017) suggest that both teachers and students should be part of the task design process. This was a crucial aspect of our research project. The overall aim was to investigate how the co-development of computer-based tasks with Student Partners could enhance the students' learning on the course. In particular, this paper focuses on matrix multiplication as this was one aspect of the course the Student Partners identified as being problematic. The particular research question related to this paper was to find out to what extent interactive visual supports enhance students' ability to multiply matrices.

Methodology

During the academic year 2016-2017, three tutorial sessions on matrices took place in a computer laboratory, each lasting 50 minutes. Every student on the Foundation Studies course attended one of these tutorials. Across the three tutorials there were 18 computers set-up with screen-capture software. This captured the screen and mouse movements along with conversations between the students. Two students did not complete any of the matrix multiplication tasks; hence we collected data for this paper from 16 screens. In addition we had, for the purpose of analysis, (i) audio data and photographs from the task design meetings carried out with the Student Partners (SPs), analytic assistants and teacher-researchers, (ii) reflections from SPs, (iii) feedback questionnaires from 13 Foundation Students commenting on the computer tasks and one interview with a Foundation student who attended the matrices tutorial.

Initially, a real-time analysis of each screen-capture was carried out where the video was played in real time and a factual summary of what happened was noted along with timings. Rich conversations, related to the task or reflecting upon the visual imagery, were identified for later transcription. Following transcription, the videos were viewed in greater detail at specific points which were either (a) key moments when mistakes were made; (b) significant movements of the sliders controlling the visual support; (c) where there was change from correct to incorrect entries or vice versa; or (d) rich conversations. A grounded theory (Strauss & Corbin, 1990) approach was taken where codes were developed related to (a) – (d) above. A factual list for each computer screen was created showing whether successive answers entered were right or wrong and whether the visual support was positioned correctly each time. A focus was then taken on the developing success, or otherwise, of the students across their tutorial time on the matrix multiplication questions and, in particular, the role of the visual support offered from the design of the GeoGebra file.

The design of the matrix tasks

Four design meetings took place between the teacher-researchers, SPs and the analytical assistants to design tasks based upon the use of Autograph and GeoGebra. The purpose of the design meetings was to identify topic areas which the SPs felt would benefit from additional support. Once topics were identified the SPs were involved with contributing to the design of computer tasks, offering visual support for the Foundation Students on the course.

Prior to the first design meeting the SPs were asked to revisit the two topics of complex numbers and matrices and identify what they recalled having difficulty with. One topic identified was matrix multiplication as a topic where it was easy to make mistakes. One SP (SPs will be labelled, SP1, SP2, etc.), SP1, reflected prior to the first meeting that remembering the rules of matrix multiplication was an area of difficulty. At the design meeting SP2 commented that it was confusing as to whether you go "down and along or along and along". After some discussion about it was decided that GeoGebra might be an appropriate tool to produce a useful file for matrix multiplication. The ideas for two GeoGebra files were developed and collectively discussed in the first design meeting and draft versions of the files brought to the second design meeting. One concerned the multiplication of matrices and is the focus of this paper.

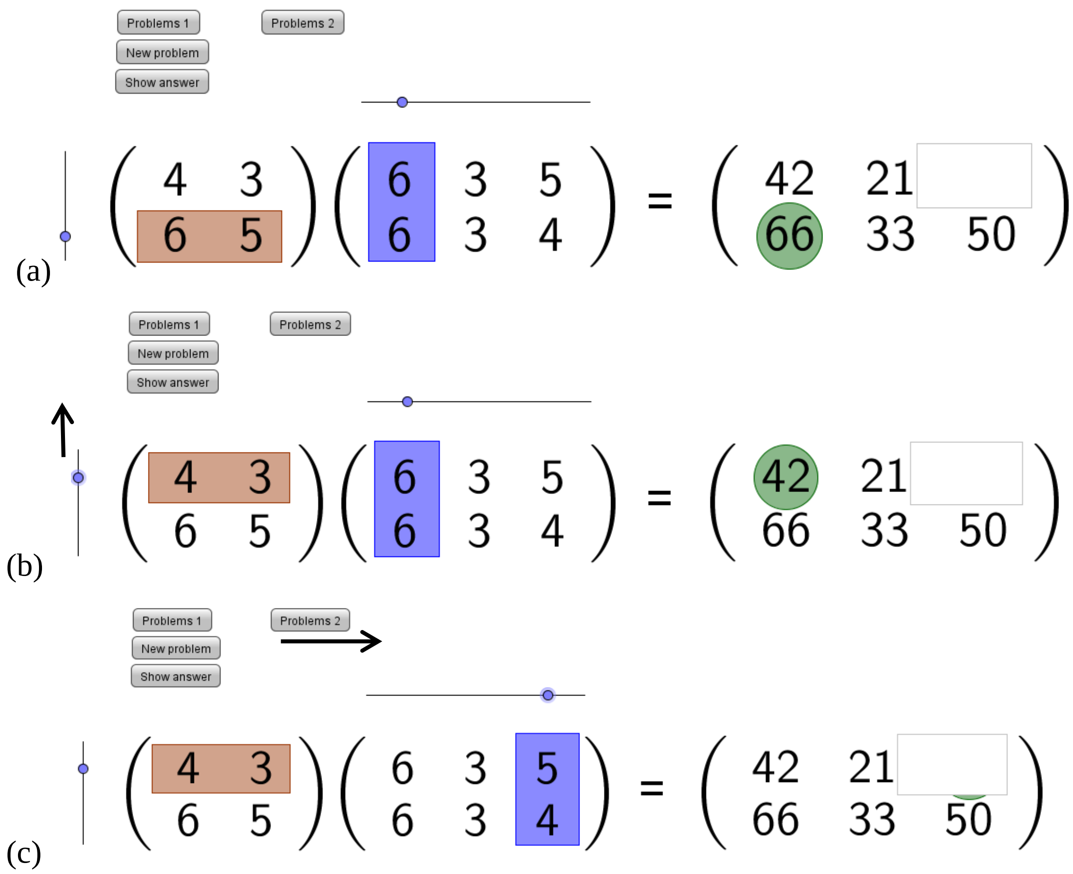

Figure 1 (a-c): Movement of sliders to correct position behind the empty box in the answer matrix.

The GeoGebra file showed two matrices, with randomly generated positive numbers, which were to be multiplied together. All but one value of the answer matrix was shown. The task was to enter in the remaining number. There were shaded rectangles offering visual support and a circle in the answer matrix indicated where the resultant number appeared for the given coloured rectangles. There were sliders alongside the matrices to move the coloured rectangles, with the circle moving automatically as a consequence (see Figure 1(a-c)).

At the first design meeting SP3 was positive about the initial idea and suggested the highlighting of related elements. SP3 felt you could “see in front of your eyes which one is multiplied with which”. SP2 felt it would help visualise the process which they had originally identified as something which could be confusing. In particular they mentioned it helped “forge the link between the… cause and effect”. SP3, however, pointed out that this computer task encourage robotic thinking and it was acknowledged that this task would not be about developing understanding but about assisting with the process.

At the second design meeting a re-worked version of the file was shown where randomly generated positive integers from one to seven appeared in the matrices which were to be multiplied. This version incorporated two new aspects. The first was a button which, when pressed, showed the full answer matrix. This allowed the learner to compare the number they entered with the correct number in the answer matrix. The second aspect was a ‘Problems 2’ (P2) set of questions in addition to the original P1 (see Figure 1) set of questions. Here larger randomly chosen matrices, involving negative as well as zero and positive integers, were involved without the visual support of sliders or coloured rectangles. This showed the partially completed answer matrix with two missing numbers, which were to be entered. As with P1 the complete answer matrix could be revealed as well, and an infinite series of problems of a similar nature produced.

Analysis

Prior to the tutorial sessions, the students on the Foundation course had received a lecture and course notes which included how to multiply matrices. However, seven of the 16 screen-captures revealed students needing help from one of their fellow students in order to even make a start with multiplying matrices. Table 1 below gives an overview of the eventual success of the students on the 16 screen-captures, irrespective of initial difficulties.

Consistent success with P1 & P2	11	Inconsistent success	1
Success with P1 but not with P2	1	Consistently incorrect	1
Success with P1. Did not attempt P2	1	Too little data to comment	1

Table 1: Eventual success of the students, irrespective of initial difficulties

When errors were made most students took time to look at the correct answer matrix and their own, incorrect, entries. With the P1 set of questions, with the exception of one student, every time a student entered an incorrect number this was followed by getting the next task correct. The one exception was a student who did not use the sliders to move the coloured rectangles into their correct place on three consecutive occasions; thus not using the visual support. It was only when the student used the sliders, so that the coloured rectangles were in the appropriate position, that they obtained correct answers. The remainder of the questions on P1 were then answered correctly.

Power of the visual

The GeoGebra file was designed to offer a strong visual cue to help students recall the process of multiplying matrices. However, on some occasions the students did not use the sliders to move the highlighted rectangles to the correct row and column for the particular missing number in the answer matrix. Instead, they incorrectly carried out the calculation based on the incorrect row and column already highlighted. Thus the visual lure of the highlighted rectangles was stronger than the logic of considering if the row and column were appropriate.

Generally, however, the imagery appeared to offer useful support for many of the students. There were examples of students spending time when the highlighted boxes were positioned so that the circle in the answer matrix was over a number which could be seen (for example see Figure 1(a)). In this way they could see an answer and check how it could be arrived at from the highlighted row and column. Mouse movements were seen ‘pointing’ to the relevant numbers which had to be multiplied. Students then moved the sliders so that the highlighted boxes were in the correct position and then entered a correct answer. At times when mistakes were made, it triggered a discussion with fellow students who were working on a nearby computer, particularly when students struggled with the P2 problems, which no longer provided the visual support of the coloured rectangles. Students were heard helping others by saying “Imagine how it was before [with the highlighted boxes]”, “I kind of imagine… the boxes on the screen”. Many students started off moving the highlighted rectangles to the correct position before answering, thus making use of the visual support. After a while they no longer bothered moving the rectangles as they could answer correctly without that support.

Transition from P1 to P2

The transition from P1 questions (which involved a 2 by 2 matrix multiplied by a 2 by 3 matrix with visual support) to P2 questions (which involved a 3 by 3 matrix multiplied by a 3 by 2 matrix including negative numbers and no visual support) was difficult for some students. Students made mistakes which appeared to relate to difficulties with calculations with negative numbers. However, an interesting finding came from the way in which some students transferred the imagery from the P1 questions to the larger matrices in P2. Many students had difficulty with the first P2 question even after they had been successful with the P1 questions. Some students gave a verbal outburst on seeing the first P2 question on the screen, such as “Oh my lord”. Five students switched back to look again at P1 before returning to P2. Three of these appeared then to transfer literally what they did with the P1 questions; they started off with the larger matrices in P2 by multiplying just the first two pairs of numbers as they had done with P1, rather than the three pairs needed for the larger matrices.

Feedback from Foundation Students

Following the tutorial sessions, we gathered feedback about the matrices tasks from the Student Partners, written feedback from 11 of the Foundation Students who engaged with the tasks in the tutorial and an interview with one Foundation Student. The comments from the Foundation Students were all very positive. One said “I have to admit I missed the lectures on that [matrices] and hadn't done it before I came to the tutorial. And I can now do it. So I think that just proves that it was helpful… it does give you a visual understanding of how it all works." Several others also mentioned the visual support. For example one commented: "The boxes indicating which columns and rows were good to get me started. After a while I stopped using these." The screen-capture analysis showed that there was a clear issue about the transition from P1 to P2 questions. Although one student commented that they liked the fact that the visual hints were no longer visible for P2 questions, another student suggested that having "an option to add or remove the sliders would be good because then students of varying abilities can use the program". We intend to meet again with the Student Partners to reflect upon this very useful recommendation and all the tasks and to consider changes for their use in future years.

Discussion

The GeoGebra file allowed students the freedom to move sliders in order to see how different rows and columns were associated with particular positions within the answer matrix. There was evidence that some students made good use of these sliders and positioned them so that they could work on how a visible number in the answer matrix might have been obtained. At other times the act of moving the sliders, so that the circle in the answer matrix was in the position of the missing number, meant that the students' attention was taken to the relevant row and column to be multiplied. There was evidence that students were successful when moving the sliders to the correct position when previously they had entered incorrect answers without having moved the sliders.

The GeoGebra file offered instant feedback by showing the correct answer matrix underneath the matrix which had a missing number for the students to enter. Thus, students could immediately see whether they were correct or not. Most students took time, after entering an incorrect number, to see where they had gone wrong. This led to success with the next question. This finding aligns with Lozano (2017) who talked about future actions being shaped due to immediate feedback from the computer. Yildiz and Baltaci (2016) also commented upon students correcting errors due to feedback from GeoGebra and reported that making mistakes created discussion between the students. This was also a feature from our study as on some occasions students consulted another student when they were making mistakes.

Success with P1 questions (which had visual support) did not always translate into success with the P2 questions where the matrices were larger and there was no visual support. Even though students were getting success with P1 questions, the question arises as to what sense of generality was carried forward when meeting the first P2 question. As teachers we might feel that the P1 questions offered examples of the generality of multiplying any two matrices together. However, students do not necessarily know this generality and have to make sense of what they are seeing (Caglayan, 2014), and try to construct rules from the particular examples they meet (Mason, 1996). The fact that quite a few students did not find the transition straightforward, with some multiplying only two pairs of numbers together despite the matrices being larger, indicated there may not be sufficient variation within the P1 questions for students to develop appropriate generality.

Conclusions and ways forward

There is evidence that the visual support offered helped 11 of the 16 groups of students to gain consistent success with multiplying matrices with both the P1 and P2 problems. Only one student was consistently incorrect with their answers. The feedback from the Foundation Students indicates that they found the visual support helpful. In particular, the act of moving the sliders to the correct position resulted in correct answers being entered. However, there were clear difficulties when moving from questions where visual support is provided to questions where this is no longer the case and the matrices are a different size. It appears that there is not enough variation in the size of matrices with the P1 questions. All P1 questions involve multiplying a 2 by 2 matrix with a 2 by 3 matrix. The size of the matrix does not change. This was mainly done due to technical difficulties with producing a file where the size of the matrix can vary. This raises the issue of technical considerations which are a practical reality when designing computer tasks, and the educational variation which is desired for the task to be effective. We feel that the result of this study has shown that although the current file did enhance the students' ability to multiply matrices, more variation within the P1 questions will make the tasks more effective. The GeoGebra file offers a new kind of visual support for students learning to multiply matrices; our research adds to the limited literature in this area.

Seven of the 16 screen captures showed students having no idea of how to even start to multiply matrices. The fact that the Foundation Students had a lecture already and notes that clearly showed how to multiply matrices means that these workshop tasks were invaluable. As such the Student Partners were accurate in identifying matrix multiplication as an important area where additional support could be useful. As teacher-researchers we have found the presence of the student voice in the whole process invaluable - from task initiation, design and reflection on the process. This will feed into the re-designing of the tasks for future use whilst we continue to use the Student Partners in re-working the computer tasks.

Acknowledgements

This project is supported by HEFCE (Higher Education Funding Council for England) Catalyst Fund: Innovations in learning and teaching, and addressing barriers to student success A: Small-scale, 'experimental' innovation in learning and teaching (Project code K20).

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Increasing post-16 mathematics participation in England: the early implementation and impact of Core Maths

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Core Maths, a relatively new and distinct post-16 qualification, has been developed to address a key UK government policy imperative – that of increasing post-compulsory mathematics participation in England from its low comparative position internationally. In the light of recent policy developments to increase uptake in post-compulsory maths, we discuss emerging findings from a large-scale three-year mixed-methods project on Core Maths, funded by The Nuffield Foundation. In particular, we use national data to investigate the wide range of other qualifications that Core Maths students are taking, but find little emergent evidence of any early impact on attainment in these courses. We also present interview data from teachers and senior leaders demonstrating how Core Maths is being implemented in a wide variety of ways in schools and colleges.

Keywords: Post-16; Core Maths.

Introduction

This paper outlines early findings from a three-year longitudinal mixed-methods project funded by The Nuffield Foundation. We described the aims and background to the study more fully in a previous paper (Homer et al., 2017). In the current paper, we consider whether there is any early evidence of enhanced attainment in other subjects studied by students taking Core Maths, and outline some of the emerging qualitative findings from interviews with stakeholders (teachers, curriculum managers, and senior leaders) in a sample of 13 schools and colleges in England.

Post-16 mathematics education policy developments in England

Post-16 participation in mathematics in England (i.e. once compulsory study ends) is low, compared with our main international economic competitors (Hodgen et al., 2010). The UK government is committed to meeting an aspiration voiced in the recent review of post-16 mathematics (Smith, 2017) that in ten years' time all students will be studying some mathematics post-16 (Department for Education, 2018; HM Treasury, 2017). This can only be achieved through offering students an appropriate set of mathematical pathways. Core Maths is a new and distinct alternative to Advanced Level (A-level) Mathematics, the long-established academic mathematics pathway post-16. It is offered in various guises by the different awarding bodies in England (Homer et al., 2017), and was first taught in 2014 and first examined in 2016. It is designed primarily to support the mathematics in students' main programme of study, or at work and in everyday life (Core Maths Support Programme, 2016). The course is intended to be studied over two years, alongside A-levels or other Level 3 (i.e. advanced) qualifications, but with only half the number of hours devoted to it than a full A-level entails. Its focus is on applying already-learned mathematical knowledge and concepts in authentic contexts, and on developing confidence, competence and fluency (Department for Education, 2015); only 20% of the qualification is intended to be new content. This makes it suitable for any students who pass their General Certificate of Secondary Education (GCSE) in Mathematics with at least a Grade 4 at the transition point to advanced study beyond the age of 16, making it a crucial addition to the portfolio of post-16 mathematics qualifications available in England. Core Maths has a high profile role to play if the government's education policy, and indeed wider economic policy (Lingard, 2011), aims are to be met.

Methodology

National data

Core Maths is intended to support other subjects that have elements of mathematical demand (Glaister, 2015). One major strand of our project, the analysis of data from the National Pupil Database, enables us to estimate the impact of studying Core Maths on students' attainment in other post-16 curriculum subjects. For the first cohort of Core Maths students (examined in 2016), we identified the five most popular subjects also being studied by these students. For each of these, we then carried out a modelling approach to compare attainment in these subjects between students who had and who had not studied Core Maths. We controlled for a range of potential fixed factors (gender, measures of socio-economic status, attainment at 16, ethnicity, and institution type) in a multi-level (clustering in school/college) random intercept only variance components model with the outcome variable A-level (or equivalent points) adjusted for qualification 'size'. Students entered for any other Level 3 mathematics qualifications were removed from the analysis. The key outcome of this modelling is an estimate of the Core Maths 'effect' on student attainment in each post-16 subject.

Interview data

Another key strand of the study seeks to answer a research question regarding what institutions are doing to maximise the success of Core Maths, and what barriers and challenges they are facing. This qualitative strand explores the views and experiences of staff and students within 13 English schools and colleges where Core Maths is currently being offered. Over 40 centres were initially identified, either through contact with Maths Hubs (regional maths education support networks in England) or directly via institutions' websites, as potential case studies. These were gradually approached to take part, bearing in mind a desire to ensure representation of the different types of post-16 setting which exist in England, until enough, and a reasonable spread of, institutions expressed an interest in participating.

The first round of fieldwork interviews took place in September/October 2017, to harness views at the start of the academic year, with follow-up visits taking place later in the project. Semi-structured interviews were conducted with teachers, students, and senior leaders responsible for institutional curriculum policy, focusing partly on relevant issues identified from the literature, but also allowing participants to talk freely about their experiences of and perspectives on Core Maths. Interviews with 15 Core Maths teachers, 12 Heads of Maths, and 11 senior leaders (defined as a Head of Faculty, Vice Principal, Headteacher or Principal) were transcribed and coded, and thematic analysis was carried out using inductive and deductive approaches.

Emerging findings

Core Maths and other subjects – national data

For the first cohort of students, Core Maths is combined with a very wide range of other subjects and qualification types. There is no predominant link with any other particular subject, albeit Table 1 shows that the majority of other courses taken by Core Maths students lean towards the scientific/quantitative as opposed to the arts/humanities.

Level 3 subject in 2016	**Qualification type**	**N**	**% within Core Maths cohort**
Engineering Studies	Advanced level vocational qualifications (BTEC Diplomas)	239	8.7
Applied Sciences		233	8.5
Computer Appreciation		213	7.8
Psychology	Advanced level academic qualifications (A-levels)	207	7.6
Biology		161	5.9

Table 1. The most popular subject/qualifications awarded to Core Maths students in 2016

Only a small percentage of the first Core Maths cohort took even the most popular subjects, and Table 1 also shows that Core Maths is taken alongside both academic and vocational courses.

Table 2 shows the total number of students awarded each of these five qualifications in 2016, and this is the sample size for each of the statistical models when estimating the impact of doing Core Maths on attainment (there was missing data for some co-variates which explains the lower Core Maths numbers compared to Table 1).

Post-16 subject examined in 2016	**Total number of students awarded each subject**	**Number of students also awarded Core Maths**	**Core Maths students as percentage of total in each subject**
Engineering Studies (BTEC)	7,655	206	2.69
Applied Sciences (BTEC)	15,019	196	1.31
Computer Appreciation (BTEC)	20,209	178	0.88
Psychology (A-level)	42,236	196	0.46
Biology (A-level)	21,660	148	0.68

Table 2. Sample sizes in comparative analyses

For the three BTEC subjects in Table 2, there is a positive but non-significant effect of doing Core Maths which is on average approximately 12% of an A-level grade (or equivalent). For the two A-level subjects, the Core Maths ‘effect’ is small but negative but again non-significant (13% of a grade worse for Core Maths compared to non-Core Maths students). To an extent, these non-significant findings are a result of the actual sample sizes within the Core Maths group being quite small (Table 2) so estimates have relatively large standard errors. It could also be the case that the actual effect on outcomes is hard to detect, since it is likely to be quite small – compare with, for example, Gill’s (2017) work on the Extended Project Qualification, which found that the impact of doing that qualification was of the order of one A-level grade higher for a student taking four A-levels.

Results presented here should be treated with considerable caution. There was some missing prior attainment and demographic data, and the possibility of confounding variables that were not included in the analysis.

Perspectives on Core Maths in schools and colleges – interview data

This necessarily brief overview of the findings emerging most strongly from the qualitative data presents some of the themes which resonate (or not) with previous research or with stated policy intentions.

The need for more mathematics post-16

Participants echo the need acknowledged in the literature (British Academy, 2015; Department for Business, Energy and Industrial Strategy, 2017; Glaister, 2017; HM Treasury, 2017; Hodgen, Marks, & Pepper, 2013) for more students to be studying mathematics in some form post-16 for a range of reasons: to support other subjects, to assist with progression to and success in higher education or employment, and in order to become numerate citizens. Teachers and managers describe the benefits of Core Maths, as seen in this 11-18 school headteacher's comment that "it provided our students with an opportunity to continue maths, and maintain that subject within their profile, for the future, which we felt was a really strong thing to do".

Senior leaders express as much support for Core Maths as do mathematics teachers, provided that class sizes are sustainable and student outcomes are deemed satisfactory in the context of the institution.

Awareness of Core Maths

Participants describe strategies used to promote awareness of Core Maths among their colleagues. They report support from, for example, Psychology, Business Studies, and Science (particularly Biology) staff, but believe the potential benefits to students of taking Core Maths alongside such subjects need to be communicated more widely:

> "I think this year they're probably more aware than they ever have been, because of [Core Maths teacher] going into the morning sessions to sell it and going into the classrooms and things like that […] I think it's kind of gaining a bit more popularity and people are a bit more aware of it now, but I wouldn't be confident enough to say that everyone would know." FE College Curriculum Leader.

Little, if any, evidence is reported of awareness of Core Maths amongst students or their parents unless it is specifically mentioned to them by the post-16 institution hoping to recruit or retain those students:

> "…we have to explain what it is, because people don't know… the word is not out there massively. Everybody knows what A-levels are. Most people I think would know what BTECs are…it's not an awful lot of people who would be able to tell you what Core Maths is... until they get to the point where they're actually making their options." University Technical College Head of Maths.

Core Maths therefore seems to have relatively little currency as yet. Despite endorsements from universities on both the Core Maths Support Programme website (STEM Learning, 2017) and their own websites, there is a notable preference from some universities/HE courses for a particular GCSE Maths grade (e.g. some courses specify GCSE grade B, now a grade 6, in their admissions criteria), and will not take Core Maths in its place, despite Core Maths demonstrating progression beyond Level 2 (GCSE) and into Level 3. This is leading some centres to support a resit of GCSE Mathematics to improve a student's grade post-16, in preference to taking Core Maths:

> "…while students have picked it thinking it was, going to be a requirement, when they've looked at, as they've started to look at university requirements, they're more likely to say we want an A or B in GCSE...so some are thinking well would

> it have been better to resit the GCSE?… So I, I'm not sure there is a great awareness at university." Sixth Form College Vice Principal.

Positioning in the post-16 curriculum

Core Maths is also seen to have another problem related to awareness: it does not bear the more familiar title of 'A-level' or 'BTEC', but is a 'Level 3 Certificate'. As an AS-sized qualification (half the teaching time of an A-level, and only 40% of the value of an A-level in its contribution to university admission), it is an anomaly at a time when, our data suggests, the two-year linear model for three full A-levels (or equivalents) is becoming the norm, and the AS a thing of the past. This leaves centres struggling to work out how to integrate Core Maths into option blocks and timetabling:

> "…when we did four courses […] they were all ASes, and part of our problem was selling Core Maths as it wasn't an AS. And there was no second year studying it, which is the big problem with Core Maths in terms of selling it." Sixth Form College Head of Maths.

Current post-16 funding supports 600 guided learning hours (GLH) per year, which allows for three two-year A-level courses or the equivalent (180 GLH annually each), and 60 GLH for tutorial time, careers work and enrichment. Core Maths, at 180 GLH in total, is designed to be offered in addition to those three full courses over two years, and does not fit neatly within the funding formula. Managers justify the extra cost in terms of benefit. As one Head of Department explains, "[providing Core Maths is] bonkers from a funding point of view, but it's the right thing to do for the learner's progression […] we do balance the books, but there's the humane element of it as well".

Core Maths was designed to support students over the typical two-year post-16 study period (Department for Education, 2013), at 90 GLH per year. Whilst some institutions do run a two-year course, others run Core Maths over one year, which suits some institutions where it is not uncommon for students to leave after one year. It also frees students to focus on their main study programme in the second year:

> "we do it in Year 12 […] It seems to work better that way, so that they've got it out of the way, ready to go into Year 13." Studio School Assistant Principal.

On the other hand, a two-year course can better suit an institution which sees Core Maths as supporting other subjects, and where it fits with their timetabling if Core Maths has fewer teaching hours per week than an A-level/BTEC subject:

> "...as we were moving two or three years ago from modular A-levels to linear A-levels, we began to wonder how we might use a Core Maths qualification integrated into a larger programme of study. And so we sold Core Maths to them, that it would support their subject but also give them a freestanding qualification." Sixth Form College Vice Principal.

Core Maths can also be set up as an enrichment, which any student can opt into but which is additional to the (usually three) main subjects a student is taking:

> "I think it is quite a hard sell, 'cause you're asking the students to do something extra than what they actually need, to go to university. And even though it benefits them, I think they might think well I've got enough on my plate already, with three A-levels." 11-18 School Head of Maths.

> "Core Maths doesn't sit in the normal option blocks. It's as part of our additional enrichment and tutorial programme that we do so the students would study their three subjects, and Core Maths." 11-18 School Headteacher.

In some centres, Core Maths is offered at enrolment just as other subjects are, such as A-level History, with the same number of taught periods. However, Core Maths has no second year into which students can progress. Students will then typically be directed towards an Extended Project Qualification (EPQ) in their second year:

> "…some of them would be doing something equivalent to two, Core Maths, and then they'd build up the extra UCAS points to make it equivalent to three with something like the EPQ as well." Studio School Head of Maths.

Progression is a common concern for centres attempting to position Core Maths within their mathematics provision. There are instances from our case study centres of students not studying Core Maths directly after GCSE in the first year post-16, but moving into Core Maths either from a GCSE retake in the first year post-16, or from a year studying AS Maths, where the student is not progressing into the second year of A-level. These possibilities have been seen to work well.

The most significant negative comment from centres not offering Core Maths is precisely the difficulty of incorporating it into the institution's curriculum offer. There is a particular sense of mismatch in institutions where the now-defunct AS/A-level *Use of Mathematics* qualification (see Noyes & Adkins, 2017; Noyes, Wake, & Drake, 2011) has previously been taken successfully by students. Respondents regard the removal of *Use of Mathematics*, and its replacement by something half its size, as an incomprehensible move on the part of the government:

> "…if they haven't done *Use of Maths* before, they think, yeah that's [Core Maths is] not a bad idea, but if you have done *Use of Maths*, you're just thinking, it's such an appalling substitute, for what was, and the students really liked it, you know, then the kids are committed…" Sixth Form College Head of Maths.

Issues of student 'choice'

The take-up of Core Maths by students is relatively low, even in institutions where support for Core Maths seems robust (Homer et al., 2017). Allowing students to choose Core Maths voluntarily is perhaps a fair approach to recruitment, but can be a risky strategy where student numbers are under scrutiny. Tying participation in Core Maths to particular study programmes seems to result in a bigger cohort, as more students are directed onto the course to support their studies in, for example, Applied Science (BTEC), or Psychology (A-level). This means some students find themselves obliged to take Core Maths, perhaps initially with some resentment, having thought they had given up mathematics after passing their GCSE. As part of our research, we are monitoring and will be reporting on the developing mathematical dispositions of Core Maths students.

There remains some concern in institutions about the long-term prospects of Core Maths, particularly bearing in mind the fate of *Use of Mathematics*. The future of Core Maths within an institution can depend on student numbers, and also on results, whether that be the outcomes of Core Maths itself, or the outcomes for students in other subjects, which participation in Core Maths is designed to support (Glaister, 2015; Homer et al., 2017; Smith, 2017):

> "…and as I say my massive concern is they'll drop Core Maths 'cause as well we're gonna become an academy [i.e. funded centrally, not locally] in February…so I don't know what that's gonna entail, in terms of, they might just say, right, you can forget Core Maths, because you've got a small number, you know, I really don't know what's gonna happen." 11-18 School Head of Maths.

Related reforms, CPD, and teacher supply

Amongst the staff interviewed, there is a feeling of weariness with reform (see Golding, 2017). With the arrival of Core Maths, there were the concurrent pressures of adapting to the new Mathematics GCSE and A-level, so the amount of time and energy available for thinking about the delivery of Core Maths, and training for staff, has been variable, and in some cases minimal or non-existent. Our data show that choice of awarding body and specification has often been made on the simple basis of availability of resources, or even familiarity with the layout of the exam paper, and less often to a thorough comparison of available specifications. Engagement with local maths hubs, other teacher networks, or the Core Maths Support Programme prior to its demise, is also variable: some teachers in the study find this kind of networking and support invaluable, whereas others have developed their Core Maths provision independently.

There is a national concern over mathematics teacher shortages (Smith, 2017). However, in our case study centres, specialist mathematics teachers are delivering Core Maths, and are often enthusiastic, energetic and motivated about the new course:

> "…I think [joint Head of Maths] was quite keen to take it himself at one point, because he quite liked the sound of the set-up of the lessons and this idea of, well, here's a real world problem, what maths can we throw at it? And that's quite, there's something quite freeing about that." Studio School Head of Maths.

Frequently, Core Maths is deliberately allocated to teachers who formerly taught *Use of Mathematics*, or who came into teaching from other careers.

Concluding remarks

It could be argued that our case study centres represent a biased sub-section of school/college maths departments, since they value Core Maths enough to be running it in its early years, and agreed to take part in our research. It is also possible that findings from the first cohort of national data could differ from those of later cohorts, the first consisting of mainly enthusiastic 'Early Adopters' (Advisory Committee on Mathematics Education, 2014). Implementation of a new, innovative qualification is likely, in practice, to take time to mature. Hence, we will analyse national data from later cohorts to compare quantitative findings with those presented here. Future analysis will also focus on the longitudinal aspects of the study, monitoring any change in patterns of uptake and attitudes of students and other stakeholders. We will attempt to link the qualitative and quantitative aspects of the research as we gather more data, surveying a wider range of stakeholders, and bringing different theoretical perspectives to the analysis. Finally, our ongoing exploration of why institutions are choosing *not* to offer Core Maths will give us deeper insight into the challenges faced by this new qualification.

The data presented here generally indicate support for the wider policy imperative of ensuring more students study mathematics post-16. The two main challenges for centres are the logistics of positioning Core Maths within the curriculum framework and funding conditions now characterising the post-16 sector, and whether to target certain students or allow students to opt in. These questions are inextricably linked, and are themes that merit further investigation over the remainder of the project.

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Talk in Mathematics: teachers collaboratively working on developing students' mathematical language use in lessons

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Talk in Mathematics is a collaborative project with two school mathematics departments investigating ways of enabling students to develop their mathematical talk during lessons. During the project the mathematics teachers videoed their own teaching and chose clips to share in the regular group meetings. In this paper we outline three aspects of classroom interaction that received particular attention during the project: mathematical explanations; silence and wait time; and mathematical language. We examine the scope and depth of professional learning afforded by the approaches we took in the project and explore the wider implications these have for teaching and learning of mathematics.

Keywords: classroom interaction; explanations; wait time; vocabulary; teacher development; discipline of noticing.

Introduction

Opportunities for students to talk in mathematics classrooms have an important role to play in developing students' thinking and ability to communicate mathematically (Prediger & Erath, 2014). Research into promoting student talk in classrooms has largely focused on questions or prompts that teachers can use to encourage and support students (Franke et al., 2009) or on the social norms that need to be established to enable fruitful discussions to occur (Mercer & Sams, 2006). In this paper we describe how two mathematics departments used a video club to work on developing their students' mathematical talk in their lessons. Over the two years of the project the teachers focused on three aspects of students' talk: explanations, pausing, and their use of mathematical vocabulary.

In this paper we address the question of the scope and depth of professional learning afforded by a video club that is driven by the concerns and priorities of the teachers themselves. Reflecting the responsive nature of the project, the structure of this paper differs from that usually found in academic papers. Whilst we include a discussion of the research methodology and the way that we worked with the teachers, there is no specific literature review section. Instead the literature is introduced within each section of the paper as and when it is relevant, such as in the analysis of the teachers' conversations during the video club and the discussion of findings arising from the examination of the teachers' lessons.

Research Methodology

The findings reported in this paper come from a larger project titled *Talk in Mathematics* (TiM). Two groups of mathematics teachers worked with two researchers over a period of two years. The participants involved in the project changed over the two years as teachers moved school and other colleagues joined in. The data discussed in this paper is drawn from the teachers in the first school group (Kevin, Imogen, and Julia), and the second school group (Freya, Charlie, Emma, Anna, and Laura). There were other teachers at some of the meetings, but as they did not attend regularly or share clips of their own teaching in the meetings they have not been included in the analysis shared in this paper.

The groups met regularly over the course of the project. Throughout the two years the teachers videoed themselves teaching or, within a school group, videoed each other. The teachers chose which class to video, when to video and what to video. All videos of entire lessons were shared with the researchers. Short clips from some of these videos were then chosen by the teachers to share for discussion during meetings. Some teachers only videoed one class whilst others shared videos of more than one class. This resulted in a collection of lessons covering the full age and attainment range, including mixed attainment year 7 classes, low attaining year 8 classes and A level further mathematics lessons.

The whole class interactions from these videos were transcribed using Jefferson transcription (Jefferson, 2004) and subsequently analysed by the researchers. In addition, all meetings were audio recorded and transcribed verbatim. The analysis of the data occurred on two levels: the classroom interactions within the videos were analysed using conversation analysis (Sidnell & Stivers, 2012), and when asked for, this analysis was shared with the teachers. The meetings were analysed using thematic analysis. The initial codes for this thematic analysis were based on the foci identified by the teachers themselves as they introduced the video clips of their teaching within the broader topics of pausing, vocabulary and explanations. Further codes emerged inductively. This coding has allowed us a qualitative analysis of the scope of teachers' professional learning afforded by the project.

Teachers working collaboratively in a video club

Video-based professional development opportunities have been shown to have a positive impact on both teacher and student learning (van Es & Sherin, 2010). Most video clubs have focused on shifting teachers' attention from their own practice to their students' thinking. The teachers in this study broadened this focus to consider *how* students construct what they want to say, *how* they make use of opportunities to talk, and what resources they draw upon.

In each meeting one or two of the teachers would share a short clip from their teaching. This teacher would also set the focus of the discussion that followed. The meetings were based on the Discipline of Noticing (Mason, 2002), which combines reflective practice and action research. Through this approach the teachers shaped their own professional development opportunities through their choice of focus and consideration of associated actions. Mason (2012) asserts that through noticing aspects of our own practice, we become sensitised to noticing this aspect in the future, which in turn gives us opportunities to act differently moving forwards. The initial discussion of the video clip was led by the teacher sharing the video and offering *accounts of* the interactions in the clip before *accounting for* (Coles, 2013; Jaworski, 1990) what was observed. Both the teachers and the researchers in the club would ask questions and prompt each other during the discussions, drawing on their experiences across the meetings and their own teaching. These conversations supported systematic reflection and noticing of different aspects of practice with the dual goals of recognising choices and possibilities for acting differently in the future (Mason, 2002, p. 70).

One aim of the TiM project was for research findings about ways of supporting and developing students' talk in mathematics to be addressed during the meetings in order to enable the teachers to use these in their own practice. In contrast to other studies, this was primarily done by asking questions of the teachers about their own practice, as observable in their videos and in the discussions, in order to explore ways of acting differently or to consider what might constitute effective practice. In this way results and recommendations from research were not explicitly shared but the teachers were supported to draw upon their own expertise and knowledge of their classroom and their practice, in order to express these for themselves. Thus, the teachers constructed ways of acting differently that generally matched recommendations made in the literature, though the teachers did not use the technical language that researchers do. For example, the teachers explored aspects of using wait time (Ingram & Elliott, 2016), revoicing (Herbel-Eisenmann, Drake, & Cirillo, 2009; O'Connor & Michaels, 1993), scaffolding (Moschkovich, 2015b), the relationships and differences between semantic and lexical aspects of word use, and grammatical structures that are specifically mathematical (Pimm, 1987; Schleppegrell, 2014), but not using any of these specific words or making any reference to the research. To emphasise this point further, we would highlight that in the first meeting of the project where the teachers chose to focus on their use of pauses the researchers introduced the phrase *wait time* (a break between turns, as opposed to a pause that is a break within a turn), but this term was not taken up by the teachers who continued to talk about *pausing*.

Findings

The preliminary findings are grouped by the three foci that the teachers chose to work on over the course of the two years: explanations, pausing, and vocabulary. The analysis is ongoing and in this paper we focus on those findings that were presented at the BCME conference and that prompted questions from the audience. In particular, we offer evidence for the scope of professional learning afforded by a video club that is driven by the concerns and priorities of the teachers themselves and we indicate more tentative observations about professional growth (Clarke & Hollingsworth, 2002), that we express here as deep learning, in respect of explanations, pausing and vocabulary. Connections are made to existing literature where they resonate with the ideas that the teachers themselves focused on. This reflects the projects' principle of reference to literature arising from what the teachers identified as issues or strategies, rather than the literature directing what the teachers discussed or enacted in their teaching.

Explanations

Our analysis of the scope for professional learning afforded by a video club highlighted two key emergent areas associated with explanations that. The first area that teachers discussed was the nature of prompts and tasks that led students to giving explanations. The second was what the teachers counted as an explanation and more specifically a mathematical explanation. The second area is more fundamental that the first, and emerged from a shift away from the initial focus on practice to a focus on beliefs.

There was particular depth to the first focus on the nature of prompts and task through consideration of examples other than those that explicitly asked students for an explanation. As researchers we examined the transcripts of lessons to identify scenarios where students gave these sorts of explanations and these are given in more detail in Ingram, Andrews, and Pitt (2018). These scenarios included where, without the teacher's invitation, a student refuted another student's response to a teacher question; in order to justify their refutation, the student had spontaneously offered this explanation. We describe these types of student explanations that are not explicitly asked for by the teacher as *naturally occurring*. Extracts illustrating naturally occurring student explanations were shared with the teachers. Additionally, the teachers focused on using both pauses (discussed below) and tasks that would generate a debate between students so as to make naturally occurring student explanations more likely. One of the teachers subsequently focused on using tasks where misconceptions and multiple answers were likely to arise. Examples of her use of such tasks was evident in clips that she shared later in the project, but the focus of the discussions then was on the students' use of reasoning and mathematical vocabulary rather than the task choice (for details of this discussion see Ingram & Andrews, 2018).

The second focus on what the teachers counted as an explanation (and more specifically a mathematical explanation) occurred in two ways. Firstly, the teachers examined which student explanations they accepted in the clips they shared. Secondly, they imagined varying the content of these explanations in order to test the boundaries of what they might or might not accept. For example, explanations given by students included "because a hundred and fifty take away sixty is ninety", "because you can't", "because when it's in minus, numbers you take away get like a higher number". The teachers either attended to the reasoning within a student utterance or the language used as their criteria for whether something was acceptable as an explanation, with some explanations being accepted as mathematical by some teachers and not others. The question was raised as to whether a student referring to mathematical objects or processes using 'thingy' rather than the mathematical words was giving a mathematical explanation. While there was evidence from the meetings therefore that the scope of professional learning opportunities extended to beliefs about what constituted a mathematical explanation, further analysis of lessons is needed to identify whether these nuanced professed beliefs coincide with enacted practice.

Pausing

Pausing was the first area of focus identified by one of the schools and was the focus of the first three meetings with that school. We have already discussed how in the project pausing became a catch-all term for a range of strategies than involved the use of silences. Different teachers focused on different aspects of pausing during their interactions, but the scope of professional learning opportunities afforded by the video club included the areas of wait time, when it was appropriate to pause, and the issues around establishing pausing as a classroom norm. All of the teachers in this particular school group focused on the pauses they left after a student had given a response to a question, which Rowe (1986) named Wait Time II. We have reported elsewhere on this particular focus (Andrews, Ingram, & Pitt, 2016), including the importance the teacher placed on allowing students the opportunity to provide a fuller, or in another way revised, response. In the current paper we look at themes that the teachers returned to across the project. These include instances when the focus had shifted away from pausing, which might suggest more sustained changes to teachers' practice.

Previous research has detailed the difficulty teachers have in changing the time they wait after asking a question or after a student answers a question in a sustained way (Black, Harrison, Lee, Marshall, & Wiliam, 2003; Rowe, 1986), citing the uncomfortableness of pausing as a possible reason. The teachers in this study also experienced the uncomfortableness of pausing: "I was trying desperately not to look like it was uncomfortable so the kids didn't notice" (Charlie, meeting 2) and "I felt excruciating embarrassed and so did they" (Anna, meeting 3). Anna's uncomfortableness seemed to be empathetic to the uncomfortableness of the students whom she saw as being placed "in the spotlight" by her use of pausing.

Different teachers found different ways of managing the uncomfortableness of pausing, establishing more refined strategies to promote fuller student responses than utilising Wait Time II alone. Charlie and Freya both talked about the nuances of *when* they were pausing (Ingram & Elliott, 2016) and focused on the types of question and forms of student response that would make leaving an additional pause appropriate. They talked about leaving pauses when they knew the students had an answer but just needed more time to articulate it. From the video data, this typically featured as part of a longer sequence of interaction that featured a strategy to enact Ingram and Elliott's (2016) Wait Time I(i) (time between teacher question and student response) as well as Wait Time II. In some cases, this Wait Time I strategy was to pose a question and allow students time to work on this individually on paper before taking in responses, with the students' writing providing the warrant for knowing that, given time, the student could articulate a response. Anna, on the other hand, did not talk about how the nature of the question or the students' responses might affect whether she paused or not. She engaged in what she called *displacement activities*, such as writing on the whiteboard or walking round the classroom, during the pauses she found uncomfortable. This was a strategy to allow the students the time a pause offers whilst minimising the uncomfortableness she felt leaving these pauses and relieving the pressure she perceived as being placed on students. These additional strategies of Anna, Charlie and Freya, although varying in their subtlety in terms of subject pedagogy, indicate a depth of professional learning beyond enacting pausing alone.

Vocabulary

Vocabulary was a theme that the teachers in both groups returned to over several meetings and that interacted with the other foci. A particular example of this was whether an explanation needed to include mathematical vocabulary and this was indicative of the depth of professional learning the video club approach afforded. A further indication of deeper learning was critical engagement with whole-school initiatives. Both the schools had a whole school focus on students' literacy and there were policies in place about sharing key words and explicitly supporting students to learn the technical vocabulary associated with specific curriculum areas. Sharing video clips of mathematics lessons brought these whole-school approaches into sharper focus.

Three key areas associated with vocabulary that emerged through the project indicated the scope for professional learning afforded by a video club. Firstly, there was a focus on assessing and developing the meaning students associated with particular words. Secondly, teachers discussed offering opportunities for the students to use technical vocabulary in meaningful ways (in the ways suggested by Moschkovich, 2015). Finally, consideration was given to the strategies that are effective in helping students to learn mathematics specific vocabulary, including making the distinction between learning nouns (such as expression and equation) and learning process words (such as simplify or solve).

Within the first area of interest, one teacher from each school, Kevin and Freya, separately and independently shared a clip where a student was having difficulty with the meaning of the word equation. In Kevin's clip a student asks "what's the difference between an expression and an equation?" and in Freya's clip the students are categorising statements written on the board into equations, expressions and identities and a student explains that "it can be an equation 'cause you add them all up and that's an equation" and follows with "'cause an equation's with numbers and sums in it" and "an expression is with algebra". The sharing of these clips both resulted in an exploration of the choices between teachers giving definitions and students working from these, and giving students experiences where they can develop their own meanings and lead towards a definition consistent with one that would generally be accepted in the mathematics community. These discussions arose even though in both clips shared the teacher ended the discussions by defining equation and then moving on to a new task or word. Freya in particular wanted to explore ways of acting differently, but faced a dilemma as in her clip the student had developed alternative meanings for equation and expression that would not generally be accepted and correcting this was the only action that came to mind in the moment.

Emma shared a clip where the students were debating, giving reasons for their answers. Over time they shifted to using the mathematical word *multiple* within their reasons after prompting from the teacher, but were unable to do the same with the word *factor*. The intention here was that the technical vocabulary was introduced at a time when it could be used meaningfully by the students to support their reasoning. This clip was brought to the meeting by Emma in order to unpick the differences between the two scenarios where in the first case her support to use the mathematical language of multiple was successful, and in the second case this was not successful (described in more detail in Ingram & Andrews, 2018).

The focus on learning mathematical nouns as opposed to verbs was not addressed in a sustained way in the meetings. The teachers involved in the project did not share any clips with an explicit focus on the difficulties of students understanding grammatical aspects of mathematical language. However, the issue was raised by several of the teachers in several of the meetings. Freya in particular was concerned that her students have difficulty understanding phrases such as 'show that', 'find' and 'solve' and that these difficulties underlie obstacles they experience decoding examination questions more than the actual mathematics involved. As with the themes of explanations and pausing, professional learning here was deeper than offering opportunities in lessons for students to experience technical language alone and included introducing carefully constructed situations in which it was more likely that students would choose to the technical language themselves.

Conclusion

In this paper we have presented preliminary analysis from a two-year collaborative project where two groups of mathematics teachers worked on improving their students' talk within mathematics lessons. Our analysis to date of the meetings and the videoed lesson indicates the scope of professional learning afforded by a video club that is driven by the concerns and priorities of the teachers themselves. The breadth of ideas explored within several key issues related to student talk in the current project in encouraging. Yet the analysis presented in this paper is ongoing and any conclusions we might draw about the professional growth with respect to the ideas raised at this stage are cautious. We have evidence of connections between the discussions held in the meetings and the practice observable in the videos given to the researchers when focusing on teachers' use of pausing, but not for the other two issues identified in this paper of explanations and vocabulary. Whether the ideas that come to mind for teachers in meetings also come to mind in the midst of a lesson remains uncertain. Furthermore, not all the specific issues of focus in the meetings beyond these three have been examined. The analysis so far has been driven by those aspects of talk on which the teachers within the project particularly focused. Conversation analysis in particular is an approach which also enables us to examine more implicit aspects of practice that we have not yet fully made use of.

The focus on vocabulary is particularly interesting in terms of the opportunities for deeper learning through critical engagement with whole-school initiatives. Whilst the teachers and the researchers talk about a need to situate vocabulary learning within wider mathematical practices, the school-initiated professional development opportunities the teachers were offered and the school policies within which they were working emphasised distinct approaches that specifically focused just on learning vocabulary. Wessel and Erath (2018) suggest that these two approaches of learning and using vocabulary need combining with explicit teaching strategies embedded within broader meaningful discursive practices.

A challenge of working in this way with a group of experienced teachers is the need to be responsive to what the teachers notice and attend to as they examine their own practice (Coles, 2014). Whilst most video clubs are established with a view of changing practice, it is not always the case that certain aspects of practice need or are wanted to be changed. The teachers in this project noticed aspects of their practice that they wanted to change but also aspects that they felt were effective at accomplishing what they were aiming to achieve. Both these situations give rise to a growing awareness of ways of acting differently.

The early findings suggest that the way of working with these teachers may support sustained change in practice, which is not dependent upon the presence of the researchers and theoretical ideas they can offer. Rather it uses a supportive community or change environment (Clarke & Hollingsworth, 2002) in which to examine your own practice in detail and to consider ways of acting differently is key. This enables teachers to apply strategies that resonate with those established through academic research effectively within the context of their own classroom, with the motivation and focus for change coming from the teachers themselves. This raises a challenge of how to effectively use an approach to professional learning which draws extensively upon teachers' existing expertise to enable sustained change in areas of practice identified by researchers. For deep professional learning, from the perspective of Clarke and Hollingsworth (2002), is shaped by teachers' goals, beliefs, knowledge and practice alongside external influences such as research findings and school priorities, as well as the context within which the teachers are working. The approach adopted in this project brought to the surface teachers' beliefs and goals, and supported teachers' professional growth through systematic reflection between these beliefs, goals and their practice alongside enactment of new practices prompted by their school's priorities.

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What is teaching with variation and is it relevant to teaching and learning mathematics in England?

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This paper considers the literature to understand a teaching approach used by mathematics teachers in Shanghai, known as teaching with variation or *bianshi*. What the bianshi framework involves is explored; also what impact bianshi has had on learning mathematics in Shanghai and how relevant this approach might be to primary classrooms in England. Bianshi involves generalising from examples using *conceptual* and *procedural* variation for concept development. There is some evidence to suggest that bianshi has a positive impact on mathematics learning in parts of China and is complementary to *constructivist* principles and thus potentially transferable to classrooms in England.

Keywords: procedural variation; conceptual variation; Shanghai; bianshi; primary; generalisation

Introduction

In recent years, the Department for Education for England (DfE), through the channels of the National Centre for Excellence in the Teaching of Mathematics (NCETM) and the Maths Hubs network, have promoted mathematics teaching practices observed in Shanghai, China (Department for Education, 2016). ‘Expert teachers’ from Shanghai have provided show-case lessons for primary teachers from England in both China and in English primary schools. One feature of these lessons, of interest to the author, is the attention the expert teachers pay to the deliberate variation of problems and examples as their lessons unfold. This is called bianshi jiaoxue (“teaching with variation”). The use of ‘variation’ is just one part of the reform agenda in England however, this paper will argue that particular attention to this might indicate how well the desired reform can be achieved.

This paper considers the theoretical framework for bianshi in order to understand the implications for its use in the afore-mentioned English context and contributes to the author’s doctoral research that seeks to answer the following research question:

> *What changes and what stays the same when primary teachers incorporate pedagogical practices associated with promoting learning from variation?*

The paper attempts to address the following questions:

1. How is ‘teaching with variation’ evident in Shanghai, China?
2. What evidence suggests that ‘teaching with variation’ has a positive impact on teaching and learning mathematics?

3. What evidence suggests that 'teaching with variation' is a practice that could be transferable to primary classrooms in England?

Teaching with Variation in Shanghai – The Bianshi Framework

Worldwide interest in the mathematics classroom practices in East Asian countries has arisen as a result of international student achievement tests such as TIMSS 2011 (Mullis, Martin, Foy, & Aurora, 2012) and PISA (Organisation for Economic Co-operation and Development, 2014), which have revealed the repeated success of Chinese pupils in the mathematics element of these tests. Numerous studies continue to examine Chinese mathematics instruction in an attempt to draw out the features of teaching and learning mathematics that might contribute to this 'superior' performance (L. Gu, Yang, & He, 2015) and the presence of bianshi has been noted (Clarke, Keitel, & Yoshinori, 2006). Bianshi is based on the Chinese maxim "*only by comparing can one distinguish*" (F. Gu, Huang, & Gu, 2017). It involves the architect of learning (teacher or textbook author) devising opportunities for learners to distinguish variant and invariant properties of a mathematical object (concept or procedure) in order to gain a deeper understanding of a mathematical concept or process. A longitudinal study conducted in the region of Qingpu in the 1980s and 1990s in Shanghai observed two types of variation used by Chinese teachers and defined them as *conceptual* and *procedural* variation (L. Gu, Huang, & Marton, 2004). Such "indigenous" approaches are strongly evident in task design in China (Sun, 2013).

Conceptual variation

Conceptual variation is concerned with experiencing a concept from multiple perspectives which contributes to deeper understanding of a concept. The teacher provides examples of a concept by offering deliberately varied contexts or representations. E.g. to understand the meaning of 'three' as a quantity, one should experience threes of different objects, sounds, movements, in different arrangements as well as how it relates to 2 and 4. These combined experiences all contribute to a deeper understanding of its mathematical structure. Examples such as these are handled carefully and frequently by Chinese teachers (Cai & Nie, 2007). In a show-case Year 4 lesson in England in 2017, an expert teacher from Shanghai compared three-quarters with one-quarter using a number line, a bar model and an area model and used them to orchestrate discussions to lead the pupils to draw out the generalisation "to compare two fractions they must be of the same whole".(See fig. 1).

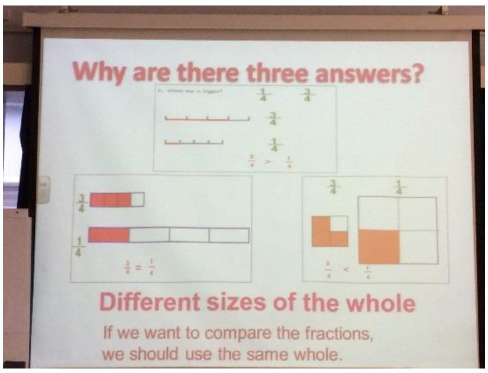

Fig 1. Representations for comparing two fractions each of a different whole.

Procedural variation

Procedural variation is used by teachers to promote conceptual understanding of a mathematical object through the use of 'problem sets' (Sun, 2011). In her analysis of Chinese textbooks, Sun (2011) categorised procedural variation 'problem sets' in the following way:

I. Varying one condition in a problem to become aware of the variant and invariant relationships. (one problem multiple changes, OPMC)
II. Varying the approach to solve a single problem. (one problem multiple solutions, (OPMS)
III. Varying the problems that use a single approach to solve a variety of problems. (multiple problems one solution, MPOS)

The use of conceptual and procedural variation offers the learner a 'space of relations' from which learners can abstract generalisations that contribute to the building of a comprehensive structure of a mathematical object (Sun, 2011). Sun only focuses her attention on the presence of problems that are chosen with procedural variation. She does not describe how teachers design and promote learning by using varied problems. It is the author's view that how these problems unfold is of central importance to the bianshi framework. L. Gu et al. (2004) provide further constructs within the bianshi framework in relation to procedural variation.

Pudian, anchoring point and potential distance

In procedural variation, the order in which a teacher chooses to introduce a set of problems is called *pudian* (L. Gu et al., 2004). This sequence offers pupils experience of steps that carefully unfold the intended concept. The first step that is chosen is familiar to the pupils and defined as the *anchoring point of knowledge* (L. Gu et al., 2004). Teachers consider how to bridge the gap between the *anchoring point* and the intended new learning which is defined as the *potential distance* (L. Gu et al., 2004). Teachers must understand how to vary the steps of the pudian to create 'proper learning distances' (Ding, Jones, Mei, & Sikko, 2016).

The impact of bianshi on teaching and learning mathematics in Shanghai

Bianshi has sparked interest with several researchers from western cultures as a possible explanation of why Chinese learners achieve so much better than their international counterparts in comparative tests (Clarke et al., 2006). It is not argued that bianshi is solely responsible for the high performance in these international comparison tests, however a number of studies have evaluated the effectiveness of variation in Chinese classrooms, concluding positive effects on pupil learning. L. Gu et al. (2004); Bao et al (2003) (cited in Shao, Fan, Huang, Ding, and Li (2013)) and F. Gu et al. (2017). Other studies are reported in Mandarin making the evidence difficult to assess for non-Mandarin-speakers.

The findings from a large longitudinal study conducted in the poor Shanghai district of Qingpu (L. Gu et al. (2004); (F. Gu et al., 2017)) shed further light. As part of educational reform in China in the late 1970s, when standards of achievement in mathematics were generally poor in this district, Lingyuan Gu and colleagues began exploring the use of variation in classrooms with a small number of experimental schools. The number of experimental schools increased and by 1986 the pass rate for entrance to the junior high schools had risen to 85% from 16% in 1979 (L. Gu et al., 2015). As a result, the reformed teaching approaches were extended to the whole of China. Other features of Chinese mathematics classrooms such as coherence (Wang & Murphy, 2004); teacher dominated lessons (Mok, 2006) and profound teacher knowledge (Ma, 1999) might also contribute to their success but in each case these features can be referred back to the presence of deliberate variation.

Some research on the use of deliberate variation in secondary mathematics classrooms in England has shown that promoting learning from variation "is a powerful design strategy for producing exercises that encourage learners to engage with mathematical structure, to generalize and to conceptualize, even when doing apparently mundane questions…" but "…knowing more about its impact on learning is going to take more experimentation and longer immersion" (Watson & Mason, 2006, p. 108).

Bianshi and constructivism

Learning theories related to constructivism are widely and implicitly used as the principles of teaching and learning mathematics in the US, UK and mainland Europe. Over the last 50 years the works of Piaget, Bruner and Vygostsky have influenced mathematics educators keen to apply research findings to improve the teaching and learning of mathematics. Constructivism is based on new learning forming as a result of personal experiences that build on prior knowledge: learners construct knowledge on the basis of links with previous knowledge. However, the principles for teaching and learning mathematics in China have been exposed to influences over millennia; none more so than those derived from their Confucian heritage (Shao et al., 2013). Since the turn of the new millennium, mathematics curriculum reform in China has begun to incorporate classroom practices heavily influenced by the US National Council of Teachers of Mathematics (NCTM) standards, which promotes pupils' learning using constructivist principles (Clements & Battista, 1990). However, bianshi has remained a feature despite this reform. So, if it has been possible to map constructivist principles to practices that make use of bianshi in China, then one must consider if it is possible to map bianshi to practices that make use of constructivist principles in western cultures.

The term *scaffolding* is associated with Wood, Bruner, and Ross (1976) and involves the teacher providing suitable support for learning. The teachers' deliberate removal of scaffolding is defined as *fading* by van de Pol, Volman, and Beishuizen (2010). Scaffolding is akin to pudian (Ding et al., 2016) however F. Gu et al. (2017) argue that pudian devotes greater attention to the hierarchy of the steps i.e. to build relational understanding (Skemp, 1976) between each varied example. An untrained eye may perceive this as rote learning (or instrumental learning (Skemp, 1976). This has been referred to as "the paradox of the Chinese learner" (Huang & Leung, 2004) because such approaches, leading to success in international tests appear contradictory to constructivist theories of learning.

Similarities can be drawn between bianshi and the work of Dienes (1971) who suggested that for pupils to abstract and generalise mathematics they need to experience *perceptual* and *mathematical variability* both of which are akin to conceptual variation. Perceptual variability involves experiencing a mathematical structure in different observable situations to perceive its structural properties. Mathematical variability involves experiencing essential features of a mathematical concept being varied so that a generality of the concept can be achieved. Gattegno's (1971) four 'powers of mind' - the powers of *extraction*, *transformation*, *abstraction* and the power of *stressing and ignoring* – describe a processes that learners experience to make sense of mathematical examples presented to them. Such experiences are implicit in the design of problems using conceptual and procedural variation. The author argues that bianshi provides practical applications of the learning frameworks offered by Dienes (1971) and Gattegno (1971).

In Chinese text-books, new learning is stimulated from a familiar situation or context which are then used to develop the abstract concepts (Sun, Teresa B, & Loudes E, 2013). Using realistic starting points is also a feature of Realistic Mathematics Education (RME) (Freudenthal, 1973). In both bianshi and RME, the mathematiztion of realistic examples into abstract mathematical concepts is promoted. Both approaches also make use of pupils' own solution strategies through discussion led by the teacher (Schleppenbach, Perry, Miller, Sims, & Fang, 2007). Pupils' varied solutions provide a teaching tool for collective discussion and comparison in the *Theory of Didactical Situations* (TDS) (Brousseau & Balacheff, 1997). I argue that the use of pupil-generated solutions from authentic mathematical situations as featured in both the RME and TDS theoretical frameworks and developed within the constructivist paradigm, are what Sun (2011) describes as type-II procedural variation.

So what is it about bianshi that separates it from western practices? A number of studies have sought to explore what differences exist between Chinese and US Grade-6 pupils (Cai, 2000, 2004). These have shown that Chinese pupils generally outperform their US counterparts and that an indicator of this success is Chinese pupils' ability to represent word problems using algebraic generalisations (Cai, 2000). Interviews with a sample of teachers from the US and China, revealed that Chinese teachers had a clearly articulated teaching goal to lead learners to generalisations when solving problems but the US teachers were satisfied with any representation of the problem solution (Cai, 2004). This suggests that the abstraction to a generalisation, a feature of bianshi, may explain one key difference between practices.

Conclusion

In this paper I have sought to answer three afore-mentioned questions. Bianshi is an approach developed in Shanghai from Chinese "indigenous practice" (Sun, 2011) where teachers use pudian - sequences of problems - designed with *procedural variation* for pupils to build relational understanding (Skemp, 1976) of a mathematical process or concept. Chinese teachers also use *conceptual variation* to representation concepts in multiple ways and in varied contexts. In both cases the variations experienced are used to lead pupils to draw generalisations of the intended concepts; a goal strongly expressed by Chinese teachers (Cai & Hwang, 2002). Bianshi can have a positive impact on learning (L. Gu et al., 2015) and could be relevant to teaching mathematics in England because it is complementary to constructivist principles implicit in mathematics classrooms there. Little is written about how Chinese teachers come to use bianshi effectively and as such may present problems for the intended pedagogical reform in England. I conjecture that to teach with variation effectively, teachers in England will not only have to understand the features of the bianshi framework but also appreciate how pudian is carefully designed to lead pupils to draw generalisations of the intended mathematical concept being taught. The author's doctoral research will examine a sample of primary teachers' classroom practices in England as they learn to use the bianshi framework to promote learning from vaiationin mathematics. This will contribute to the knowledge of how bianshi becomes an accomplished classroom practice in cultures where variation is not an indigenous practice.

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Teachers' use of resources for mathematics teaching: The case of teaching trigonometry

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This paper draws on the *documentational approach* and *knowledge quartet* to analyse a trigonometry Year 13 lesson of a secondary mathematics teacher who uses a range of paper based and electronic resources. Data were collected during one lesson observation and a follow-up interview with the teacher. Analysis identifies the resources and *schemes of use* of these resources: *aims* of the teaching activity, *rules of actions*, *operational invariants* and *inferences* in relation to the trigonometry lesson but also in relation to Year 13 teaching, especially towards student preparation for the exams. It also explores this teacher's work in the class by using the different dimensions of the knowledge quartet: *foundation*, *transformation*, *connection* and *contingency*. The findings explore teacher's use of resources and the potencies of using the knowledge quartet in tandem with the documentational approach.

Keywords: documentational work; resources; scheme; knowledge quartet.

Introduction

Mathematics teachers, throughout their work, attempt to attune the different factors in their working environment (Gueudet & Trouche, 2009). The integration of technology resources along with other materials adds to the complexity of this attuning (Clark-Wilson & Noss, 2015). The study presented here is part of the PhD research of the first author that explores secondary mathematics teachers' ways of balancing the different factors that influence their work, especially when they use a range of resources including mathematics-education software. We are interested in how teachers plan their teaching and use the available resources in relation to their teaching aims and their views about mathematics and teaching of mathematics (Gueudet & Trouche, 2009). Also, we are interested in how teachers materialise and manage their lessons (Jaworski, 1994; Rowland, Huckstep, & Thwaites, 2005) in the light of the available resources. To this aim we draw on two theoretical lenses: the *documentational approach* (Gueudet & Trouche, 2009) to address our former interest and the *knowledge quartet* (Rowland et al., 2005) in relation to the latter. Specifically, we interview teachers, observe their teaching over lessons where mathematics-education software is used amongst other resources, and do follow-up interviews in which we invite their views on what happens during observations. Here, we present preliminary analysis of a lesson observation and a follow-up interview from one participant. With this analysis, we aim, first, to investigate the characteristics of this teacher's work with resources and, second, to identify the potencies of using the aforementioned theoretical perspectives together in such investigation.

Documentational approach

The documentational approach examines teachers' interactions with resources. A *resource* is something that interferes in a teacher's work or activity, whether it is an artefact, a teaching material, or even an interaction with a student or colleague (Gueudet & Trouche, 2009). According to Adler (2000) a "resource" can also be "the verb re-source, to source again or differently" (p.207). As a result of their interactions with resources towards their teaching aims, teachers develop personal *schemes of use* where a scheme is a set of procedures carried out on a specific set of resources across different situations (Gueudet, 2017). A scheme of use consists of the *aim* of the teaching activity (e.g. to teach trigonometry); *rules of action*, which represent the teacher's actions (e.g. solving past-exam questions on trigonometry); *operational invariants*, which are the explanations adopted by a teacher to justify her stable actions in a range of similar situations (e.g. it is useful to use Autograph to show graphical presentations); and, *inferences* (e.g. one activity on Autograph went well, so it is to be used in the future). A *document* is an "association of resources and the scheme of use of these resources" (Gueudet, 2017, p. 201). In this paper, we aim to investigate the characteristics of a teacher's document also by investigating how his scheme of use is activated and applied in actual teaching. To this purpose, we draw on the knowledge quartet proposed by Rowland and colleagues (Rowland et al., 2005).

Knowledge quartet

The *knowledge quartet* (Rowland et al., 2005) is a tool for analysing and reflecting on teachers' knowledge and beliefs with the aim of developing mathematics teaching. It "is a framework for the observation, analysis and development of mathematics teaching, with a focus on the teacher's mathematical content knowledge" (Thwaites, Jared, & Rowland, 2011, p. 227). It is defined by its four dimensions: *foundation*, *transformation*, *connection* and *contingency* (Rowland et al., 2005). The first represents teachers' knowledge and beliefs: their knowledge about mathematics as well as about the teaching and learning of mathematics, and their beliefs about mathematics and its teaching and learning. 'Transformation' "concerns the ways that teachers make what they know accessible to learners, and focuses in particular on their choice and use of representations and examples" (Thwaites et al., 2011, p. 227). 'Connection' focuses on teacher's choices in terms of the plan of lesson, the sequence of activities, connecting ideas and concepts and doing so in a coherent way. 'Contingency' considers how teachers act in response to "unanticipated and unplanned events", this involves "responses to unexpected pupil contributions, and […] notable 'in-flight' teacher insights" (Thwaites et al., 2011, p. 227). Table 1 below shows the constituent codes of each of these dimensions as listed in Rowland et al. (2005), and Thwaites et al. (2011). This paper is our first attempt to use both the documentational approach and knowledge quartet to investigate the characteristics of a teacher's work. The first lens is for investigating his document, and the second affords a focused look at the details of his work in the class based on the four dimensions of the quartet and their components in Table 1. Our use of the knowledge quartet together with the documentational approach, aims to analyse "the situations in which" a scheme of use "is activated and applied" (Rowland, Thwaites, & Jared, 2015, p. 75 with our addition), the *scheme of use quartet* will reflect the details of these situations in relation to the used resources.

Foundation	Transformation	Connection	Contingency
• Adheres to textbook • Awareness of purpose • Concentration on procedures • Identifying errors • Overt subject knowledge • Theoretical underpinning • Use of terminology	• Choice of examples • Choice of representation • Demonstration • Use of instructional materials	• Anticipation of complexity • Decisions about sequencing • Making connections between procedures • Making connections between concepts • Recognition of conceptual appropriateness	• Deviation from agenda • Responding to children's ideas • Use of opportunities • Responding to the (un)availability of tools and resources

Table 3: The knowledge quartet dimensions and their constituent codes

Methodology

This paper reports preliminary outcomes from a PhD project conducted in England that looks at upper secondary mathematics teachers' use of resources, especially in teaching design and implementation that employs mathematics-education software. The study is based on an interpretative research methodology (Stake, 2010) and analyses qualitative data from classroom observations and interviews with teachers. Here, we discuss one video-recorded lesson observation and the audio-recorded follow-up interview of one participant, George, with 15 years of teaching experience mostly in upper secondary education. The interview was conducted by the first author after the first analysis of the observation where the teacher's main steps and choices were identified. In the interview, George was invited to reflect on these choices and on the flow of the lesson. The lesson was chosen as a characteristic of a series of 11 lessons from George to demonstrate the process and the preliminary findings of an analysis that employed both the documentational approach and the knowledge quartet. The documentational analysis was conducted first and identified the used resources as well as the schemes of use (aims, rules of actions, operational invariants and inferences) in the context of the observed lesson and summarised them in a documentational work table (part of it in Table 2), similar to the one used by Gueudet (2017) in the analysis of university teachers' work. Whereas, the analysis using the knowledge quartet employed the four dimensions of the quartet to explore the details of specific situations in which the scheme of use is developed and applied.

Data summary

The lesson observation on which we report here was 50-minutes long. George was teaching a mixed gender group of Year 13 students (17-18 years old) preparing for their A-level examination (a high school leaving qualification in the UK). In this lesson, he was teaching the trigonometry formulae in Figure 1, using a textbook by Wiseman and Searle (2005). He started by showing the students the graph of a function (Figure 2) on Autograph (a mathematics-education graph software), on the interactive white board. He then asked them to estimate the graph's equation. The students gave a variation of estimates, and then agreed that $y=3.6 \sin (x+35)$ seemed the closest. At that point, the teacher revealed what the equation of the graph was on Autograph: "Autograph says it is $y=3 \sin x +2 \cos x$". Then, he continued: "the computer must be right, but we also know that this is sine translated and stretched, so the two must be equal". Based on that argument, George concluded that $R \sin (x+ \theta) = 3 \sin x +2 \cos x$. A student suggested the use of compound angles rule, so George wrote that:

$$R \sin (x + \theta) = R \sin x \cos \theta + R \sin \theta \cos x$$
$$= 3 \sin x + 2 \cos x$$
$$R \cos \theta = 3 \quad \text{and} \quad R \sin \theta = 2$$
$$\cos \theta = 3/R \quad \text{and} \quad \sin \theta = 2/R$$

$$a \sin x \pm b \cos x = R \sin(x \pm a)$$
$$a \cos x \pm b \sin x = R \cos(x \mp a)$$
$$R = \sqrt{a^2 + b^2}$$
$$R \cos \alpha = a \quad \text{and} \quad R \sin \alpha = b$$
$$\cos \alpha = \frac{a}{R}\ , \ \sin \alpha = \frac{b}{R}\ , \ \tan \alpha = \frac{b}{a}$$

Figure 1: Trigonometry formulae

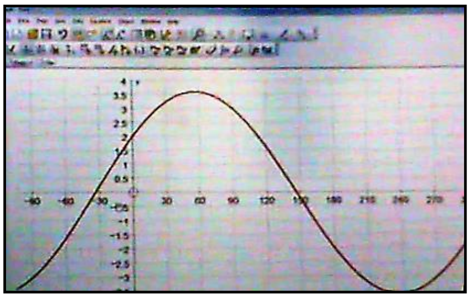

Figure 2: Function graph on Autograph

A student suggested Pythagoras, but George did not react and continued to find tan θ: $\sin \theta / \cos \theta = \tan \theta = 2/3$. Using their calculators, the students told the teacher that $\theta = 33.7°$. George was now about to rearrange the equation to find R, the same student who mentioned Pythagoras before said "Pythagoras", so the teacher commented "Oh yes, Pythagoras even better". He then drew the right triangle in Figure 3 and proceeded: $R = \sqrt{(3^2 + 2^2)}$.

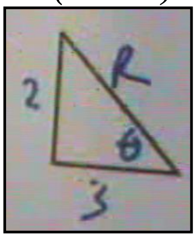

Figure 3: "Pythagoras"

It is given that $3\cos\theta - 2\sin\theta = R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$.

(a) Find the value of R. *(1 mark)*

(b) Show that $\alpha \approx 33.7°$. *(2 marks)*

(c) Hence write down the maximum value of $3\cos\theta - 2\sin\theta$ and find a positive value of θ at which this maximum value occurs. *(3 marks)*

Figure 4: Past-exam questions from the school website

At this point, George finished his demonstration of the formulae in Figure 1, and started solving an exercise from the school website, which was a past-exam question (Figure 4). He chose a problem that had $R \cos (x + \theta)$, unlike the one before which was about $R \sin (x + \theta)$. George proposed a question to his students about how they would know if the tangent was 3/2 or 2/3. He commented:

> The long way is to work it out, to crunch these through and see what you get. The short way, now look at this… if it starts with a cos, they always ask you to do it in terms of cos here and if it starts with sin, its sin… Now let's double check.

George started checking if that was the case by looking at past-exam questions, which all followed that rule when a student asked whether that was always the case. To answer the question, George wondered what if the question had "*3 sin x +4 cos x*" and "*R cos (θ + x)*", a student suggested to swap around the terms in "*3 sin x +4 cos x*", and George agreed: "Exactly, just swap them around" and gave an example from the textbook to confirm (Wiseman & Searle, 2005, question 1, p.198). After that, a student asked about what would happen if they were asked about tangent. George explained that as "*tan α = sin α / cos α*, so I guess in this case it's 3/4 instead of 4/3. Yea, but if we were to swap them around… Ah if it matches up then the short version is the second number over the first number, ya…". George then showed the question in Figure 4 which he described as a "classic" one. He explained that the three marks were given for the last part of the question because it asked for the maximum value of the whole function and the angle. He also said that it would have been one mark if the question was about the maximum value of the function only without asking about the angle.

The teacher then said he was going to look at his notes to check if he had forgotten something, showed the formulae in Figure 1 again and set the homework from the textbook (Wiseman & Searle, 2005, p.198, exercise 9E questions 1, 2 and 4). Then, he put a question on the board for the students to solve. The students started working independently, in pairs or in groups, as they wished. After some time, George solved on the board one of the questions the students worked on. He re-explained how to find R, he went back to the interactive whiteboard and to what he had written before, and told the students that instead of using Pythagoras to find R they could square $R\cos\theta=3$ and $R\sin\theta=2$ (i.e. $R^2\cos^2\theta=9$ and $R^2\sin^2\theta=4$). As $\cos^2\theta+\sin^2\theta=1$, he explained that adding the previous two equations will give $R^2(\sin^2\theta+\cos^2\theta)=13$. A student inquired if instead he could substitute the values of sine and cosine, George advised him that it is better not to, and that by using whole numbers he would avoid the "tiny roundings" his calculator would do. George said that as $\sin^2\theta+\cos^2\theta=1$ then $R^2=13$ and so $R=\sqrt{13}$. The students now went back to solving textbook questions, and George was going around the tables answering students' questions until the end of the lesson.

In the post-observations interview, George commented on his choice of resources for this lesson and on his aim from using Autograph. Quotations from the interview are included in the analysis, where appropriate.

Data Analysis

Using the documentational approach

Preliminary analysis of the data, from one lesson observation and a follow up interview with George, draws an overview of his documentational work towards the introduction of the trigonometry formulae in Figure 1 (i.e. his teaching aims, rules of actions, operational invariants and inferences). Georges uses a range of resources, paper based and electronic, including interactive whiteboard, whiteboard, curriculum, textbook, past exam papers, teaching experience, homework sheet, students' previous knowledge, calculators, notebooks, Autograph, formulae sheet (hard and soft copies). Besides the schemes of use that are related to this specific lesson, we have also identified more general schemes mainly towards students' preparation for the exams. So, George had a specific aim: to introduce and use the formulae in Figure 1, and a general aim: to prepare students for the exams. We also identified his specific and general rules of actions (numbered A1-A10) and operational invariants (numbered O1-O5) in Table 2, and we will refer to these in the analysis and discussion using their numbers in the table.

George's rules of action were based on solving a range of examples for the students (A5 & A8); asking for their contributions (A1 & A9); giving them the time to practice independently and ask questions (A10); making connections with what they already learned (A4); showing some exam-style questions (A6); setting up homework (A7); and using graphs and Autograph to demonstrate concepts (A1-A2). The operational invariants were identified in his teaching approach (through the observation) and his reflection in the interview. For example, George said during the interview that he had the algebra and the trigonometry work typed up (A3 & O2), so he can go back to it if a mistake was made during the lesson:

> If we make a mistake in the algebra then they can get a bit confused […] And, so it's important for me to try and check and I'm checking all the way through is that correct, is that correct, is that correct so if I forget a step then it kind of they get a bit misled. But, it usually goes ok. If it does go wrong then we are usually able to go back and go ok look I know what I'm expecting so was it this step, this step, this step, this step? And, at, it's at that point that I can then go back I've got the

> trigonometry typed up. […] if I need to as backup, when I can show them it's supposed to be like this so they understand it that way.

In terms of the inferences, George mentioned that, for this lesson, the activity on Autograph worked well and he would use it in the future: "as long as we need to teach that, I'll probably carry on that way because it seems to work fairly well".

Rules of action	Operational invariants
Specific: A1. Use Autograph to show the graphs of a trigonometric function on the main whiteboard, where the teacher does the work and students contribute when appropriate A2. The trigonometric function was prepared before the lesson on Autograph A3. Use typed trigonometry formulae *General:* A4. Connect the new ideas to students' previous knowledge A5. Choose exercises from the textbook, these exercises are at variety of difficulty levels A6. Use past exam questions to show styles of questions and mark schemes and offer some practice and preparation for exam A7. Set up homework with due date in one week A8. Find patterns and similarities/ differences between exercises A9. Answer students' questions A10. Students are given the time to solve questions and practice independently during class time	*Specific:* O1. It is easier to use Autograph to connect trigonometric ideas in this case O2. It helps to have the trigonometry typed up so at any point he can go back to that, if needed, and show the students the correct steps *General:* O3. It is useful to connect new ideas to previous knowledge O4. Choose exercises from the textbook that are at variety of difficulty levels, to show students how to answer questions at each of these levels O5. The use of past-exam questions helps students practice and experience exam-style questions, and is something requested by students

Table 4: George's documentational table regarding one trigonometry lesson

Using the knowledge quartet

The knowledge quartet: foundation, transformation, connection and contingency and their constituent codes in Table 1, is our lens to look at George's work in the class. In relation to the foundation dimension, we noticed George's mathematical and pedagogical knowledge unfolding throughout the lesson and the interview. We noticed his *awareness of purpose* in his explicit aims of teaching the trigonometry formulae as well as preparing students for the exam. And, we noticed moments of him *concentrating on procedures* for example when he was finding *R*, he moved away from using Pythagoras (which was suggested by a student) in his initial demonstration of the formulae and followed the method he had in mind when he was solving exercises. Also, in his suggestion of mnemonic rules in the identification of the right formula. Additionally, George adhered to exams' requirement, that is why he was solving past-exam questions (A6 & O5) and he expressed that this is important to prepare students for the exams: "Once you are ready you just practise past papers because they are the best way to get you the most experience of exam style questions". Also, George followed the school policy in relation to homework assignments (A7), which meant he gave one homework assignment every week and each assignment was due in a week time (e.g. for this lesson the homework was from the textbook and was due in a week time). Based on this data, we have observed that "exam requirements" and "school policy" are part of teacher's foundation that affect his choices in mathematics teaching. Concerning the transformation dimension, George chose a translated and stretched sine function on Autograph and used that to demonstrate how trigonometry formulae could be related (A1, A2 & A4, O1 & O3). *Instructional materials* included Autograph and the textbook, which was the main source for the questions suggested in the class and for homework. In relation to the connection dimension, the examples and representations were used to connect concepts and procedures as George commented: "So, I was using it to make them think one thing and then force them to see it in a different way and then make them make the connection between". Also, through his choice of textbook exercises and past-exam questions (A5 & A6, O4 & O5), George attempted to connect the new concepts and procedures to the textbook and to the exam requirements. From this observation, we see also the "connection between resources" and the mathematical meaning they bring as an important characteristic of George's teaching actions.

We see foundation (overt subject knowledge), transformation (use of instructional materials: non-use of calculator in this case) and connection (anticipation of complexity); all evident in George's response to the student who asked him if he could use his calculator to substitute the values of $\sin\theta$ and $\cos\theta$ in the equation $R^2(\sin^2\theta+\cos^2\theta)=13$, instead of using $\sin^2\theta+\cos^2\theta=1$, to find R. George did not recommend that method as he wanted his students to "avoid the tiny roundings" their calculators would do if they followed that way. He explained that by using a whole number ($\sin^2\theta+\cos^2\theta=1$) the answer would be more accurate.

In terms of the contingency dimension, one case is when the same student recommended the use of Pythagoras to find $\tan\theta$ and then to find R. George did not react to the student's suggestion. There is no evidence whether he heard the student or not. We note that when the same student repeated his suggestion, George followed his recommendation. But later in the lesson (in solving past-exam questions) he returned to the method he had initially planned for (squaring the equations $R\cos\theta = a$ and $R\sin\theta = b$ and adding them) to find R, which is connected to *Pythagoras* implicitly.

Discussion and Conclusion

Preliminary analysis of the data, from one lesson observation and a follow up interview with George, draws an overview of his documentational work towards the introduction of some trigonometry formulae (Figure 1), as well as preparing students for the exams. A more detailed look into his actions in class was reached by using the knowledge quartet. Regarding the knowledge quartet, with having George's documentational work in mind, the data analysis proposed tentatively that "exam requirements" and "school policy" are aspects that can be considered in his foundation. In addition, "connection between resources" and the mathematical meaning these resources bring can be considered in George's choices in planning the lesson and the way he connects ideas. This is a tentative suggestion that we investigate further in the analysis of George's and other teachers' series of sessions.

Our findings demonstrate the potencies of the documentational approach together with the knowledge quartet in offering insights into George's work and capturing the dynamic nature of his teaching. For example, George's choice of the example and its demonstration in Autograph during the lesson, in combination with his reflection in the interview suggested how he aimed to connect ideas:

> So, the aim of Autograph at that point, I was just using Autograph just to show them the graph. I deliberately wanted them to think that it's, I wanted them to tell me it's a sine graph that has been transformed. So, I wanted them to tell me it was like whatever it was 3.5 sin (x + 35), so I wanted that answer and then I wanted to show them. So, I was then using Autograph to show them that it was something else, 3 whatever it was sin + 2 cos or whatever it was. So, to deliberately make them think hang on a second, we are right but we can't be right, and so to force that idea that these two things must be the same and we must be able to work things out between them.

This was reflected in George's scheme of use (see Table 2) and related to the knowledge quartet dimensions: his mathematical foundation, connection of mathematical ideas and use of examples and representations towards making these mathematical ideas accessible to the students. In this first attempt of combined analysis, we argue that while the documentational approach offers a good overview of teachers' work in relation to the used resources, the knowledge quartet is potent in the investigation of how teacher scheme of use "is activated and applied" in teaching. In future work, we aim to investigate further the use of the documentational approach in tandem with the knowledge quartet.

Acknowledgements

This research is funded by a studentship from the Faculty of Social Sciences, UEA.

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NRICH and collaborative problem-solving: An investigation into teachers' use of NRICH teaching materials

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This paper reports on a pilot study undertaken as part of the NRICH's ongoing Habits of Mind project. The project aims to support teachers to nurture the skills that young mathematicians will need in their future studies or careers. Here I report the results of a pilot study exploring how teachers adapted existing NRICH collaborative problem-solving materials for their own settings. This report draws upon case studies conducted in ten primary schools. Although the participating schools adapted the resources for their own local needs, the wider findings reveal a number of key messages suitable both for teachers and resource designers for maximising the potential of collaborative problem-solving activities in the mathematics classroom.

Keywords: Problem-solving; collaborative; curriculum; teacher; classroom.

Introduction

In November 2017, collaborative problem-solving (CPS) attracted worldwide media attention following the publication of the first set of international CPS rankings as part of the Programme for International Student Assessment project (OECD, 2017). It is likely that the future employment market will require employees to be able to problem-solve more than ever before. Such problem solving is also likely to be collaborative and therefore, it is essential that teachers understand how to support young learners develop their CPS skills. Nevertheless, CPS has a complex set of requirements which we need to understand in order to support classroom teachers to plan and deliver effective CPS lessons.

This pilot study, led by the NRICH team based at the University of Cambridge, was a response to calls for the development of curriculum-aligned CPS materials to support teachers better prepare their pupils for their future working environments (Luckin, Baines, Cukurova, Holmes & Mann, 2017).

Literature

Luckin et al. (2017, p. 9) defined CPS as "the process of a number of persons working together as equals to solve a problem". CPS makes considerable demands on both adults and pupils since it requires problem-solving skills as well as the ability to work with others. Focusing on the literature relating to teachers and CPS, it was clear that there were several key issues to address in order to develop CPS in the classroom. Some teachers admitted avoiding CPS due to their concerns regarding possible classroom disruption (Cohen, 1994) whilst others reportedly lacked either the training or the confidence required to deliver CPS sessions in their classrooms (Kutnick, Blatchford,

& Baines, 2005). The literature also revealed concerns regarding the mathematical subject knowledge of many primary school teachers (Williams, 2008). Formal assessments played a role in the development, or rather lack of development, of CPS since the mathematics SATs (Standard Assessment Tests) for primary-aged pupils focused exclusively on individual, rather than collaborative, problem-solving skills. The literature addressing the learners' perspective regarding CPS includes a recent Finnish study wherein adult learners reported that some of the participating adults actively disliked working collaboratively and others found it a stressful experience (Järvenoja & Järvelä, 2013). Taking into account the above issues regarding teacher concerns, teacher subject knowledge and the willingness of the participants to partake in CPS sessions, it was evident that much more work was required in order to fully embed CPS in classrooms. Nevertheless, the literature also revealed that interest in CPS had tripled since the 1980s (Books.google.com, 2017).

In order to develop CPS skills in the classroom, Luckin et al. (2017) argued that the three key areas to be addressed were task design, leadership support and teaching style. This paper focuses on task design. Luckin et al. (2017) reported that effective CPS task design required positive interdependence, promotive interaction, individual accountability, interpersonal and group skills and group processing (Table 1).

Table 1. The five essential features for effective CPS. Adapted from *Solved! Making the case for collaborative problem-solving.* (Luckin et al., 2017, p. 34).

CPS Feature	Definition
Positive interdependence	This means that the task cannot be completed by one person alone. Group members must synchronise their efforts.
Promotive interaction	Members are willing to support each other to complete the task.
Individual accountability	Students must undertake their share of the work and feel responsible for the group's success.
Interpersonal and group skills	It cannot be assumed that students naturally have (or will use) high-level collaboration skills. Hence students may need support in developing such skills.
Group processing	Members reflect on the quality of their working relationship and seek to improve it through personal and joint effort.

Although CPS is not assessed in SATs, the literature consistently revealed calls for schools to develop mathematical skills beyond the basic curriculum, including Cuoco, Goldenberg and Mark (1996) and Kilpatrick, Swafford and Findell (2001). Kilpatrick et al. (2001) shared their vision of mathematical fluency which required five inter-connected aspects. They categorised those five aspects as procedural fluency, conceptual understanding, strategic competence, adaptive reasoning and a productive disposition (Kilpatrick et al., 2001, p. 116). CPS arguably covers at least two of those categories by requiring the strategic competence to recognise the need to work together to solve the problem as well as the productive disposition to want to work together to reach a solution. Kilpatrick et al. (2001) reported the findings of an earlier project comparing the impact of a CPS-led curriculum with a more traditional approach towards mathematics teaching (Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998). The project focused on developing proportional reasoning with Year 8 students (n = 187). Following the intervention, the CPS group achieved 53% correct answers in an end-of-study evaluation compared with 28% correct answers from the traditionally schooled group. The findings revealed the potential impact of CPS on later academic achievement. Nevertheless, Kilpatrick et al. (2001) stressed that teachers needed to "select, organise and present tasks that are well suited to both collaborative work and to the curriculum" (pp 348-349). Moreover, Cuoco et al. (1996) noted that whilst the future roles of learners were unknown, the skills that they would require could be taught in the mathematics classroom; they urged practitioners to ensure that "part of students' experience should be in a classroom culture in which they work in collaboration with each other and in which they feel free to ask questions of each other and to comment on each other's work" (pp 4-5). The list of skills that they believed students should acquire - including pattern sniffing, experimenting, tinkering and guessing – were gathered together under the title 'habits of mind.' Drawing upon those habits of mind (Cuoco et al., 1996) and the five aspects of fluency (Kilpatrick et al., 2001) the NRICH team identified a set of their existing primary teaching resources which were collated together under the heading of 'being collaborative'. Following the publication of the Luckin et al. (2017) report, the five essential features of CPS offered NRICH an opportunity to revisit those 'being collaborative' resources in order to maximise their potential to develop CPS in the primary classroom. The NRICH team worked alongside ten Cambridgeshire schools and Cambridgeshire Local Authority's Primary Mathematics Team.

Methodology

This pilot study adopted a multiple case study approach involving ten primary schools. Yin (2008) suggested that case studies were appropriate for in-depth research where the context played a key part in developing an understanding of the situation. This study addressed the following five research questions derived from the five aspects of CPS task design suggested by Luckin et al. (2017):

- RQ1: To what extent can NRICH activities promote positive interdependence, requiring more than one person to reach a solution and requiring group members to synchronise their efforts?
- RQ2: How can teachers support promotive interaction, enabling their pupils to support each other to complete a NRICH task?
- RQ3: How can teachers ensure individual accountability using NRICH tasks; pupils undertaking their share of the work and feeling responsible for the group's success?
- RQ4: How can teachers plan the NRICH task to develop interpersonal and group skills?
- RQ5: How can teachers ensure that their pupils reflect on the quality of their working relationship and seek to improve it through personal and joint effort when using NRICH tasks?

The project involved three distinct phases. First, the teachers attended a professional development session exploring CPS which included a summary of the findings of the Luckin et al. (2017) report as well the opportunity to explore existing NRICH 'being collaborative' resources. Second, the teachers were encouraged to take time to reflect on that session in order to adapt the NRICH resources for CPS sessions in their own classrooms. Third, an NRICH team member visiting each of the ten participating schools after the CPS lessons had taken place. During the visits, the teachers were interviewed about their experiences adapting existing NRICH resources for CPS and focus groups of pupils were interviewed about the activities and their attitudes towards CPS. Both the interviews (n = 16) and 16 focus group discussions (n = 128) were recorded on a handheld audio recorder. The use of a video recorder, which might have captured useful non-verbal data, was rejected due to issues with school policies regarding the use of video cameras. The audio recordings were transcribed prior to the analysis stage. Throughout the project, the research approach respected the ethical requirements of British Educational Research Association (BERA, 2011). In particular, no real names were stored electronically and neither the names of the pupils or their schools were stated in the final report. The resulting transcripts were analysed using a framework analysis approach (Ritchie & Spencer, 1994).

Findings

The findings reported in this section are organised by research question. Although the schools explored five different existing NRICH activities during this study, this paper focuses on the findings for the activity 5 Steps to 50. In this activity, pairs of pupils rolled a die to generate a two-digit number, such as 23. They needed to either add or subtract in steps of one, ten or a hundred to reach their target of fifty … but they were only allowed five steps to reach the target (Figure 1).

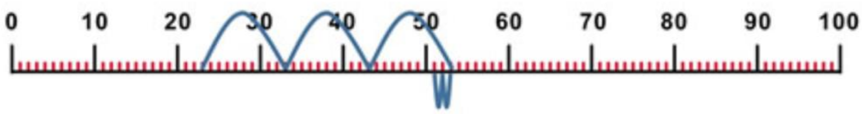

Figure 1. Attempting to reach 50 using in five steps from 23 to 50.

Promoting positive interdependence

Not all of the pupils valued working together to reach a solution; 43 of them felt that they should be left alone to work on their calculations, some complaining that paired work led to increased noise levels or frequent interruptions to their work. "Everyone tries to interrupt me when I'm trying to think of the answer," complained one of the pupils (School 1, Pupil A). In contrast, 18 of the pupils reported that they preferred paired work because there were fewer issues than trying to be heard within a larger group. However, the activity 5 Steps to 50 was designed to be played in pairs and most of the teachers set it up that way, allowing the children to choose their own partners although there were three cases where pupils refused to work in a pair. Seventy-three pupils, though, reported that they enjoyed the paired aspect of the activity. For example, one boy noted about his paired work, "When I was stuck, he knew the answer. Then when he was stuck, I knew the answer. I didn't think it would work out, because we can be a bit silly sometimes, but it did" (School 2, Pupil A).

In one school, the teacher identified a high attaining pupil (School 3, Pupil A) who struggled to match his working pace with others, so the teacher chose a suitable partner for him and that arrangement seemed to work well. In the follow-up interview, the high attaining pupil admitted preferring working individually but acknowledged that working with a partner might be helpful if either of them got stuck with their work. Six of the pupils complained that their partners ignored their efforts and three others claimed that they simply told their partners what to do, not appearing to work collaboratively at all. However, over half of the pupils were positive about paired work, and nine of them claimed that they found it easier to stay on track when working in pairs.

Supporting promotive interaction

The teacher interviews revealed two different ways in which the teachers supported promotive interaction during the NRICH task – modelling and lowering the mathematical requirements of the activity. Twelve of the teachers modelled two roles for each pair - rolling the dice and recording the answers - which the pupils felt made it fair. Three of those teachers also reported that it was far better to carefully choose numbers which allowed their pupils time to understand the activity rather than relying on the numbers randomly generated by rolling a die. A teacher from School 4 described a successful lesson where she started off with the whole class completing the example on the NRICH website, then splitting up in pairs but still working with the class for two more examples, "They were there with their number lines and we were doing it step-by-step together," before working with just their partners. Taking into account concerns regarding balancing the mathematical expectations of a task alongside the collaboration required, six of the teachers reported that using a KS1 activity worked extremely well for a CPS lesson, and in three of the KS2 classes the task continued into break time and beyond. Each of the teachers who completed the activity stressed that it complemented other number work in their current mathematics lessons - the CPS aspect of the activity was seen as an added advantage of choosing the task.

Ensuring individual accountability

At the initial training session, several of the teachers raised concerns that some of their pupils did not always make effective contributions during group work sessions. In contrast, others reported that certain individuals tended to dominate groups, to the detriment of the other pupils. As part of this project, all of the teachers were asked to consider ways to encourage individual accountability in their lesson plans. The most successful approach appeared to involve feedback; the ten teachers who insisted that each pupil should be prepared to feedback their findings to their class stated that their approach ensured that most of their pupils were motivated to support their partner.

Although their approach seemed to be effective, ensuring individual accountability required the pupils to be willing to listen to one another too. However, the focus group interviews revealed that around a third of the interviewees actively disliked paired work for this activity. Their most common criticism was that they believed they were cheating when working with another pupil during a lesson, especially if that lesson involved number skills. Those comments may reflect the current assessment system for primary schools which focuses on individual, rather than group, assessments for mathematics. When questioned further, some of those pupils felt that number work was far better suited for individuals and would have preferred to work alone, but acknowledged that CPS might be useful for other areas of the mathematics curriculum such as shape or graph work. Those pupils were not actively avoiding work, in most cases they were keen to make progress, but they strongly disliked paired work. One boy (School 2, Pupil B) noted that he preferred working individually because "You can get on with your own work without somebody asking you loads of questions when you just want to get on." Although three pupils refused to work in pairs during the activity, others worked mostly alone but shared their findings before deciding on a joint solution. When prompted, all of the pupils who disliked paired work admitted that a partner might have been helpful if they got stuck.

Planning to develop interpersonal and group skills

Both teaching time and resources were key aspects of developing interpersonal and group skills. The importance of devoting class time to developing CPS was a frequent comment during the teacher interviews. However, in order to fit in both the mathematical aims of their lessons and the CPS aspects too, over half of the teachers reported that their lessons overran. Nevertheless, all of those teachers felt that the extra time was worthwhile, and the pupil interviews also revealed that the pupils usually understood the importance of learning to work together. The classroom environment was also important for developing their interpersonal and group skills. Twelve of the teachers set aside extra time to prepare their classrooms for the activity, organising a range of potentially useful resources such as number lines, Numicon and bead strings on desk tops, with dice and wipe boards also easily accessible in most classes. One of the teachers described the benefits of using a range of resources for supporting pupils who became over-reliant on concrete resources, “This has bridged that gap beautifully. I thought it would work well and it did. Plus there were so many different ways to do it” (School 6, Teacher 1). In another class, a higher attaining pupil very effectively represented which numbers he could, and could not, reach when taking five steps to 50 on a 100 square (School 7, Pupil A). His approach was quickly adopted by other members of the class, highlighting the potential of the activity for sharing ideas and working flexibly (Figure 2).

Ensuring that pupils reflect on the quality of their working relationships

The focus groups unexpectedly revealed a potential approach for continuing to develop CPS skills, and six of the teachers predicted following it up after the end of the project; during the focus group interviews, most of the teachers reported that the question asking their pupils to rate their CPS skills and justify their score offered valuable insights into their pupils’ thinking. The majority of teachers also reported that some of their pupils over-estimated their CPS skills whilst others did not recognise how well they already worked with their other pupils. One boy boasted that “I just told him what to do and he did it” (School 8, Pupil A). Another boy, who struggled to engage with the task admitted, “I don’t really work well with other people that my teacher puts me with, I want to work with my friends” (School 1, Pupil B). One of the girls recognised the challenges in group work, “Working with partners is quite tricky and I really don’t know how you work in partners” (School 9, Pupil A). However, all of the pupils did manage to rate their CPS out of five, and most of them could provide reasons to justify their scores and suggest areas for improvement.

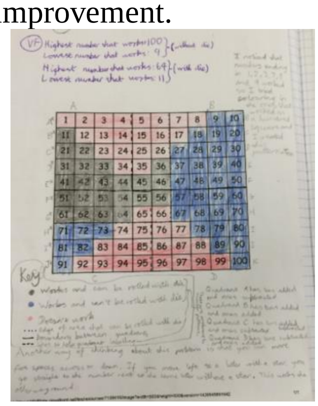

Figure 2. Example of a colour-coded solution using a hundred square. Click here for link to colour version of this work.

Discussion and conclusions

In the introduction, it was noted that this pilot study was motivated by the publication of the first set of international CPS rankings (OECD, 2017). It is essential that teachers understand how to support young learners develop their CPS skills. This study identified several key strategies for further consideration when developing CPS activities for the mathematics classroom, as well as some concerning evidence regarding some of the pupils' attitudes towards CPS. Most, but not all, of the pupils in this study enjoyed CPS. However, some of the pupils actively disliked it, reflecting the findings by Järvenoja and Järvelä (2013) for Finnish adults which were reported in the literature review. Comments from some of the pupils which indicated that CPS was possibly cheating were deeply concerning. Since our schools assess pupils individually, those comments might reflect the assessments experienced by primary-aged pupils which focus on individual workbooks dominated by calculation work. However, the growing call for CPS skills cannot be solely addressed with a series of CPS-focused lessons unless pupils have opportunities to explore reasons for working together as well as ways to improve their CPS skills. This study revealed that such an approach demands valuable class time. It also highlighted the importance of planning CPS activities which offered a range of specific roles for the pupils and modelling those roles with the class before setting the pupils off to work in pairs. Suitable roles for pupils might include recording answers and taking turns using the equipment. Another effective strategy was ensuring that all of the pupils understood that they might be expected to share their work with the whole class later in the lesson. The focus group approach encouraging pupils to rate their own CPS skills was welcomed by class teachers who acknowledged that such activities require time but deemed the time spent worthwhile. In order to nurture CPS in the classroom, this pilot study has revealed the importance of ensuring teachers have sufficient time to develop those skills.

Acknowledgements

NRICH would like to thank Nesta for funding this pilot study and Cambridgeshire Local Authority's Primary Mathematics Team for their support.

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Mathematics for the reformed science A-levels: Implications for science teaching

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Post-16 mathematics remains high on the political agenda in England with attempts to increase the mathematical engagement, confidence and competence of young people being supported by various qualification reforms. This includes *adding* new qualifications under the banner of Core Maths and *embedding* mathematics as mandated percentages in the assessment of science A-levels. Achieving the full aspirations of the *adding* policy will require substantial increases in the number of teachers of mathematics. Successfully delivering the *embedding* policy will require science teachers well-equipped to teach the increased mathematical demands of the reformed science A-levels. This paper explores some of the challenges associated with this *embedding* strategy by drawing on our quantitative analysis of reformed science A-levels, new evidence from chemistry Examiners' Reports and insights from the literature. We discuss curriculum alignment, the need for dialogue between science and mathematics teachers within schools and colleges, as well as implications for teacher professional development.

Keywords: Mathematics; A-level science; teachers; embedding

Introduction

In recent years, mathematics has featured prominently in the UK Government's education policy agenda. Sir Adrian Smith's review of post-16 mathematics (Smith, 2017) highlighted the low uptake of post-16 mathematics in England and the UK more generally (see also Hodgen, Pepper, Sturman & Ruddock, 2010; ACME, 2011; Royal Society, 2008), and drew attention to regional disparities in progression to post-16 mathematics qualifications. The report discussed the shortfall of mathematics teachers in England, identifying this as constraining factor in achieving long-term national goals for the growth towards universal participation in post-16 mathematics. The economic need for a mathematically well-qualified workforce and the expanding need for employees with so called Science, Technology, Engineering and Mathematics (STEM) skills, was a message reinforced further in the UK Government's subsequent Industrial Strategy (Department for Business, Energy & Industrial Strategy, 2017) and has been articulated widely both in the UK (Royal Society, 2014; House of Lords, 2012; Roberts, 2002) and internationally (e.g. in Europe (Gago, 2004), the USA (National Academies, 2007) and Australia (The Australian Industry Group, 2015)).

As a subject, the range and scope of applications of mathematics is diverse and multi-faceted. It underpins much of modern technology and finds widespread application within higher education across disciplines in the natural sciences, engineering, computing, the social sciences and humanities, both within undergraduate and postgraduate study. Yet, evidence shows that there are substantial weaknesses in the levels of awareness and understanding of the prominent role mathematics plays within disciplines in higher education (Hodgen, McAlinden & Tomei, 2014). The reasons behind this observation are complex, and include the failure of universities to adequately signal the mathematical requirements of their degree programmes through their entrance requirements (McAlinden & Noyes, 2018; Hodgen et al., 2014) as well as deep-rooted negative cultural attitudes to mathematics as a subject (Smith, 2017).

In England, recent qualification reforms have had a strong focus on the mathematical needs for higher education study within disciplines. These reforms have been phased in over several years with some of the new qualifications still awaiting their first formal assessment. Elsewhere, as part of an historical case study of England, we have set out the drivers and policy levers that have been instrumental in bringing about these qualification reforms (McAlinden & Noyes, 2018). Mathematics for post-16 study is now being developed in two ways: (i) an *adding* policy seeks to increase uptake of post-16 mathematical study, in part through the introduction of new Core Maths qualifications; (ii) an *embedding* policy mandates mathematical assessment requirements within other disciplines (McAlinden & Noyes, 2017).

In this paper we build on our earlier analysis of the mathematics within reformed science A-levels (McAlinden & Noyes, 2017), and present a preliminary analysis of the information regarding mathematics that can be gleaned from Examiner Reports of the first live assessment of the reformed A-level Chemistry. We analyse the messages within these reports pertinent to achieving the aspirations of the *embedding* policy. Then we proceed to consider the implications for science teachers of implementing both the *adding* and *embedding* strategies, with particular reference to the opportunities and challenges within school and college settings.

Qualifications in England and their reforms

In England the study of mathematics is compulsory for the first five years of secondary education, at which point young people take their General Certificate of Secondary Education (GCSE) qualifications at age 16. If they achieve sufficiently good results in their GCSEs young people can progress to further academic study, which, for the majority, takes the form of 3 or 4 subjects at advanced level (A-level). The A-level qualifications are high stakes national qualifications, taught over two years and administered by a small number of independent awarding organisations. The curriculum is set by the Government's Department for Education, with Ofqual having regulatory authority for implementation in line with statutory requirements.

In 2016/17 the result reporting system for GCSE Mathematics in England changed from alphabetic gradings (A-G and U) to numeric gradings (9-1) (Ofqual, 2015a). The achievement of a 'good' pass in GCSE, equivalent to a grade C or a grade 4 in the new system, is identified as the attainment of Level 2 in mathematics. Level 3 qualifications include A-levels, the Advanced Subsidiary (AS) qualifications (approximately equivalent to half of an A-level) and, in the case of mathematics, the recently introduced Core Maths qualifications. The latter provide a post-16 mathematics route for young people who have passed GCSE Mathematics but are not continuing on to AS/A-level Mathematics.

The reformed A-level Physics, Chemistry and Biology now contain statutory minimum percentages for the assessment of disciplinary-relevant mathematical content at Level 2 or above (Department for Education, 2014). These qualifications, along with the new GCSE Mathematics were assessed for the first time in the summer of 2017.

First teaching of the new A-level Mathematics was deferred until the following September to facilitate more coherent progression through the mathematics qualifications. However, given the role of GCSE Mathematics as an implicit prerequisite for A-level science study, there is a less obvious misalignment between the timeframes for the introduction of these qualifications. We return to this issue later.

Embedding mathematics in reformed A-level science assessments

Information available prior to the first assessment point

The reference point for the current study is our earlier analysis of the sample assessment materials (SAMs) for the reformed Biology, Chemistry and Physics A-levels, across three awarding organisations (McAlinden & Noyes, 2017). This work was carried out before the first live assessments of these qualifications. These SAMs will have been a key resource used by teachers in developing curriculum and preparing students for the qualifications, having been previously subjected to scrutiny by the qualifications regulator, Ofqual, to ensure that they gave an accurate indicator of the assessment of the qualifications. Building on the approach of Noyes, Drake, Wake and Murphy (2010) in the Evaluating Mathematics Pathways Project, we undertook a quantitative analysis of the mathematics within the SAMs and investigated a range of areas including: (i) the mark allocations for mathematical work; (ii) the nature of the assessed mathematical content (e.g. numerical, graphical, algebraic etc); (iii) the level of mathematics (whether at GCSE or above); (iv) the mathematical processing skills required (e.g. representing, procedural analysis, reasoning, interpreting etc); (v) the practical or theoretical nature of the tasks in which mathematics arose; (vi) the mathematical complexity; and (vii) the extent of the mathematical embedding within the science subject.

The mathematics in A-level Chemistry SAMs

The results of our earlier analysis of the A-level Chemistry SAMs found that the marks for mathematical content in the SAMs met the 20% statutory requirements (Department for Education, 2014). Based on our findings we developed the following synoptic mathematical portrait of the mathematics within the SAMs of the reformed A-level Chemistry.

> The mathematics within the qualifications is deeply embedded and so is not easily accessible without knowledge of chemistry. The mathematical work is predominantly procedural with most marks coming from questions requiring decisions to be made. The majority of the mathematics requires only standard level GCSE Mathematics although the complexity of calculations is greater than what would be expected at GCSE. It is predominantly numerical, with smaller amounts of algebra and graphical work also being required. (McAlinden & Noyes, 2017, p.11)

We also developed similar mathematical portraits for A-level Biology and Physics.

Of necessity, the chemistry mathematical portrait is based purely on the SAMs published in advance of first live assessment of the qualifications and not on the actual student learning or the achievement of the learning and assessment objectives. An in-depth understanding of the latter will have had to await a detailed evaluation of the mathematical performance of the first student cohort taking the reformed qualifications. In the absence of such, useful insights can be gleaned from the Examiners' Reports from the various awarding organisations.

Review of Examiners' Reports from first actual assessments

The Examiners' Reports on all of the relevant Chemistry A-levels are not in the public domain yet. However, we obtained the reports for the full suite of examination papers from one awarding organisation, which, in line with our earlier work, we have chosen not to identify. We have analysed these reports by searching for information about the assessed mathematical content. Our key observations are summarised below.

Observation (1): Level-2 nature of the mathematics
The synoptic comments within the reports mention the requirement for greater assessment of mathematics at Level 2 within the qualification. The reports identify that the less successful candidates struggled with the calculations, and lost marks on how they used significant figures. This characteristic was identified in the reports as being prominent in achieving the 20% Level 2 mathematical requirement.

Observation (2): Practising calculations within questions
The Examiners' Reports also identified that candidates needed more practice with the new style of questions and particularly the calculations within them.

Observation (3): Interpretation of solutions within subject context
Another weakness that was identified in candidates' work was the submission of mathematical answers which were clearly impossible from a chemistry perspective.

Observation (4): Question structure
The reports also pointed to the wider use of less structured/scaffolded calculations and that those candidates who were most successful were able to carry out such calculations.

Observation (5): Tackling unfamiliar problems
The inclusion of unfamiliar problems within examination papers, (i.e. of a type not previously seen by candidates), was also highlighted within the reports.

Commentary on findings

Our mathematical portrait for A-level Chemistry (McAlinden & Noyes, 2017) has identified the heavy reliance on GCSE Mathematics content, with only small amounts of post-GCSE material. The latter is an area that was not discussed in the Examiners' Reports. Observation (1) draws attention to the importance placed within the mark schemes on the correct use of significant figures as a factor in achieving the Level 2 mathematical assessment requirements within the qualification. This observation points to a need to ensure that a skewed and disproportionate emphasis is not placed on one particular area of Level 2 mathematical content (e.g. significant figures) at the expense of coverage of other more challenging areas. This is a characteristic that should be kept under review and given due consideration in the setting of future examination papers and their accompanying marking schemes. In this context we note that this point is particularly pertinent to the way in which marks are awarded for partially correct solutions.

Observation (2) relates to the revised question styles and the way in which calculations arise within questions. Our subject portrait for the mathematics within A-level Chemistry has identified a high level of mathematical embedding and as such the ability to access the calculations can, in many cases, be reliant on a grasp of the underlying chemistry. The change of question style is also one area that is likely to have posed challenges for teachers who will have had to adapt their teaching approaches to the new specifications and its mathematical requirements, with fewer sources of the new style of questions.

While mathematical embedding features substantially within A-level Chemistry, observation (3) points to the detachment of the mathematical calculation from the chemistry in question by some candidates. This behaviour can be symptomatic of a decontextualisation of the outcome of a calculation from the underpinning chemistry, and/or a failure to interpret the answer in a meaningful way.

Collectively, observations (3), (4) and (5) can all be linked to the general characteristics of (mathematical) problem solving. The unfamiliarity of questions, the use of unstructured questions and the interpretation of solutions are all characteristics which could be expected to arise within mathematical problem solving (ACME, 2016). This is particularly relevant, given the greater emphasis on problem solving within the reformed GCSE and A-level Mathematics qualifications (Ofqual, 2015b, 2015c). In this context it is worth noting that the student cohort about which the Examiners' Reports were written, will not have taken this reformed GCSE Mathematics qualification, which was also assessed for the first time in 2017.

Discussion

Achieving the long-term aspirations of the two-pronged *adding* and *embedding* policies poses many challenges, not least of which is the need for a highly skilled teaching workforce able to deliver new mathematics qualifications and reformed curricula in other disciplines, each including revised mathematical requirements. Current numbers of mathematics teachers in England are insufficient to meet the needs of the 'maths for all to 18' agenda and there is a recognition that teachers from other quantitative disciplines, with appropriate professional development, will have to be recruited to assist with the teaching of Core Maths (Smith, 2017). Less obvious, but perhaps equally pertinent, is the need for renewed, targeted professional development for teachers in other disciplines, such as chemistry, in which mathematical requirements have increased but have actually been playing a well-established role for many years. Such diversification of the training needs of those involved in the teaching of mathematics in the classroom, in whatever form it may take, represents a shift in the overall mathematics education landscape. Of necessity, this is likely to be accompanied by a broadening of the pool of educators involved in its delivery and a greater emphasis on peer learning between teachers across discipline boundaries within school and college settings.

The sharing of sound mathematical knowledge and pedagogy across disciplines, while highly desirable, is non-trivial and the challenges associated with conducting informative conversations in this domain should not be underestimated. In particular, the Association for Science Education (ASE) has drawn attention to differences in the terminologies used by teachers of sciences and mathematics when referring to mathematical concepts and ideas (ASE, 2016a). For example, a reference to a 'line' in the sciences can be taken to mean a straight line or a curve, while in mathematics these two entities are considered distinct and different (p.2). The acquisition of an awareness of these differences has great potential in enabling teachers to facilitate young people in making more effective connections between their different subjects of study.

In secondary school education in England, mathematics and the sciences are traditionally taught separately as distinct, standalone subjects. Consequently, young people will either need to have met mathematical concepts and techniques before they arise in science classrooms, or the teaching of these topics will have to take place within the sciences. From the ages of 11-16 young people in England will be working towards the compulsory GCSE Mathematics qualification. As such, opportunities do exist for curriculum alignment within schools to ensure that the mathematics is taught first within mathematics lessons before it is required within science classes. There is also scope for mathematics and science teachers to work together in planning curriculum delivery in order to assist young people in making connections across the boundaries between their mathematics and science subjects. Examples of such collaborative practice have been identified in recent ASE (2016b) work.

The scenario at A-level is somewhat different. The successful achievement of the aims of the *embedding* policy are inextricably linked to progress towards the *adding* policy. At present there is still no statutory requirement that young people embarking on science A-levels will be studying for a parallel Level 3 mathematics qualification, although there are substantial benefits from so doing (McAlinden & Noyes, 2017). In particular, A-level science classes are very likely to contain some young people studying Level 3 mathematics, along with others who are not. (This is particularly relevant for chemistry and biology, but perhaps less so for physics.) For the latter group of young people, the role of the teacher of Level 3 mathematical content will, of necessity, default to the science teacher. If such teaching is to go beyond purely procedural approaches, the science teacher will also need to have a sound understanding of the mathematics in question, as well as the requisite pedagogic knowledge to teach it effectively. The extent to which science teachers will have had opportunities to acquire and develop this expertise is open to question.

While recognising the importance of the context of the English qualification system in our discussion, it is also constructive to consider if relevant insights can be acquired from experiences in other international contexts. More specifically, some of the likely challenges for teachers that accompany implementation of the *embedding* policy are not dissimilar from those observed in studies of interdisciplinary curricula and teaching across mathematics and science in the USA. For example, in a study in middle schools Burghardt, Lauckhardt, Kennedy, Hech and McHugh (2015) reported a "significant increase in mathematical content scores" for young people who experienced a "mathematics-infused science" curriculum, in which mathematics was taught within science as well as in mathematics (p. 204). However, the authors did note that variability both in the implementation of the mathematics-infusion and in teacher effectiveness, were limitations of their study. They also postulated that some science teachers may have been better placed to reinforce mathematics within a science context, rather than to introduce the mathematical content to young people for the first time. The latter point resonates with the work of Weinberg and Sample McMeeking (2017) who investigated the barriers and enablers to integrated science and mathematics teaching in high schools, again in the USA. They concluded that one aspect that contributed to a lack of success of interdisciplinary teaching approaches was what they referred to as "interdisciplinary pedagogical content knowledge". They identified that the teachers in their study "… expressed some level of discomfort in knowing how to teach interdisciplinary content" (p. 211). With the increased mathematical emphasis within A-levels across subjects this is an area likely to become increasingly important in the future.

Our discussion of the opportunities for dialogue between science and mathematics teachers would be incomplete without some mention of the challenges presented by the timeframes for qualification reform implementation. Specifically, the simultaneous introduction of the reformed GCSE Mathematics and the reformed science A-levels will have complicated such teacher conversations. Queries from science teachers about the mathematical backgrounds of young people on the reformed science A-levels will have required mathematics teachers to respond with reference to the pre-reformed GCSE Mathematics, rather than the curriculum they were in the process of teaching. Furthermore, the greater use of unfamiliar problems identified by the A-level Chemistry Examiners' Reports, could perhaps have been better supported if young people had a background of the reformed GCSE Mathematics, with its stronger emphasis on problem solving (Ofqual, 2015b). Such complications to cross-disciplinary dialogue are neither constructive nor desirable and are symptomatic of a lack of coherence in the overall qualifications reform process.

Conclusion

Our analysis of the A-level science SAMs has demonstrated clear benefits to young people in continuing with post-16 mathematics study in terms of their preparation for A-level sciences (McAlinden & Noyes, 2017). Indeed, such is also the case for many other A-level subjects (e.g. geography, economics, psychology). Constructive conversations between mathematics teachers and science teachers (and conversations between mathematics teachers and those in other quantitative subjects) about mathematical curricula and pedagogy can contribute much towards enhancing the effectiveness of delivery of the *embedding* policy. The consequent increase in awareness of curriculum interdependencies within schools and colleges also has great potential to foster better signalling from teachers to young people regarding the usefulness and value of taking post-16 mathematics qualifications alongside A-levels in the sciences and other quantitative subjects. Such small steps should be encouraged and strongly reinforced by powerful messages from policy influencers, employers and higher education about the long-term value of post-16 mathematics study (McAlinden & Noyes, 2018; Smith, 2017).

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Group Flow When Engaged with Mathematics

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In this paper I discuss the phenomena of group flow and its relationship with the mathematics classroom. Flow is a linguistic term describing an intrinsically rewarding state of mind; being 'in the zone', and totally absorbed in a situation to the exclusion of all else. The term was initially articulated by the Hungarian psychologist Mihály Csíkszentmihályi in the seminal work 'Beyond Boredom and Anxiety'. This article considers a group of 40 students over 6 lessons and by means of question responses and video analysis the approach to mathematics group work is studied through the lens of group flow. I will argue that during the mathematics lessons studied the group manifest a shared peak experience and that the shared experience of group flow provides a 'joyful' element to the mathematics classroom.

Keywords: Flow; social flow; group flow; individual flow; solitary flow; mathematics classroom; mathematics; experience sampling method.

Introduction

Social interaction within mathematics classrooms and verbalising mathematical thinking has gained in popularity with teachers in recent years (Armstrong, 2008; Walker, 2010; Tatsis & Koleza, 2008). The notion of group flow adds another dimension and motivation towards Vygotskian notions of social learning and away from individual centred learning, including a focus on social and socio-mathematical norms. This article is part of a larger body of research apropos of flow (both solitary and group) and its relationship with the mathematics classroom, (Morrison, 2017). It interconnects existing research of both solitary and group flow with the mathematics classroom.

Described initially as a solitary experience, much of the existing research on flow has concentrated on this aspect of 'individuality', in many contexts including education. Only in more recent years the term group or social flow has been used distinct from solitary or individual flow (e.g., Sawyer, 2007; Nakamura & Csíkszentmihályi, 2009; Armstrong, 2008). Walker (2010, p.3) suggests 'basic research on the conditions and forms of social flow is limited'.

The original solitary flow experience has 9 multi-faceted pre conditions that engender its occurrence. These form flow theory (Csíkszentmihályi, 1975) and are shown in table 1. The conditions have been loosely classified as contributors and indicators, contributors being 'inputs', and indicators the 'outputs'. Although there is crossover between the two, by categorising in this way a teacher will gain an insight into what is within their influence to *contribute* to flow and what is less tangible to impact teaching practice that nevertheless *indicates* flow is occurring.

Contributors (Outputs)	Indicators (Inputs)
Skill vs Challenge	Autotelic (enjoyment for its own self, not necessarily for an outcome)
Clear objectives	Choice/ control of the task
Clear and immediate feedback loop	Total immersion
	No worry of failure
	Time distortion (usually feeling shorter than reality)
	Loss of awareness

Table 1: Solitary Flow pre-conditions

Group flow

Considerable research exists to suggest learners act differently in social situations than in solitary examples (e.g., Lee, 2006; Pimm, 1989). Group flow is a social, collective and peak experience in which, as the name suggests, occurs when learners become immersed together in an optimal state. Group flow, has the earlier described prerequisites of solitary flow, yet its social nature necessitates other conditions. Sawyer (2007) suggests ten pre-conditions for group flow: the group's goal; close listening; complete concentration; being in control; blending ego; equal participation; familiarity; communications; moving it forward; and the potential for failure.

In addition, Walker (2010) proposes group or social flow can be broadly split into two categories, namely those that are interactive, and those that are co-active. Both have different effects on the learner's flow experience within the mathematics classroom.

Co-active group flow

Occurs when mathematical activities that require co-operation but not co-involvement, for example a teacher led question and answer session. The learner conceivably has a flow experience, but the possible passive learning within the experience signifies that it may or may not be a shared flow experience. The experience becomes shared through individual involvement.

Interactive group flow

Occurs in activities and tasks which require co-involvement and require co-operation such as a group problem solving activity, a team quiz or partner matching dominoes game. When group participants rely on each other's mathematical skills, it necessitates social interaction and flow becomes a shared experience. Here contribution is pivotal, and every participant will at some point play a role in creating the experience, creating what Kotler and Wheal (2017, p.14) labels 'dynamic subordination'. The experience becomes shared by the group involvement.

The promising benefits of flow

It is likely that the optimal state of group flow is highly desirable for learners in the secondary mathematics classroom. There is ample investigation to suggest continued exposure to flow assists in providing happiness and joyfulness (e.g., Seligman, Ernst, Gillham, Reivich, & Linkins, 2009; Csíkszentmihályi, 1990). By prioritising flow in the mathematics classroom, mathematics becomes more joyous to learners (Williams, 2006). For example, a study by Nakamura, (Csíkszentmihályi & Csíkszentmihályi, 1992), illustrates those who did mathematics and learned to enjoy the discipline, advanced their skills, whereas those who did not enjoy the subject made little or no progress. One aim of this research is to give a structure and explanation to the intrinsic happiness and creativity that arrives from this state of mind when carrying out mathematical tasks within the classroom.

Upon emerging from a state of flow, a learner is in a state of elation (Csíkszentmihályi, 1975). Repeated exposure to 'flow' has been shown in previous studies to cause resilience and subsequent heightened self-esteem in young people (Seligman, 2011). In particular group flow seems to engender a heightened creativity (Sawyer, 2007), possibly useful in the mathematics classroom. Throughout mathematical enquiry, flow occurs 'when a learner identifies a mathematical complexity, spontaneously decides to explore it, and subsequently develops new conceptual knowledge' (Williams, 2006, p.394).

In sum, the overall purpose of this research is to investigate the phenomena of 'group flow'; a multi-dimensional model occurring in the mathematics classroom, construct an understanding of its appearance and pay attention to its perceived benefits.

Method

The described longitudinal study took place over an 18-month period. It took an approach of enquiry based practitioner research. Conceivably typical of this approach, the phenomena of group flow (as distinct from solitary flow) became apparent after analysis of the first cycle of research (Nixon, 1981). It was focused entirely within an inner city, mixed gender, secondary mathematics classroom setting. Qualitative data sets from 6 lessons (n=40) were collected using instruments including video and audio recordings, stimulated video recall transcripts, questionnaires and both learner journals and researcher field diaries.

Making sense through a phenomenological framework

An activity based in social settings required a distinct and clear methodology. Emotional frames of mind often are associated with phenomenology, along with "perception, time-consciousness, self-consciousness, awareness of the body and consciousness of others." (Smith, 2014, p.1). Establishing a phenomenological framework enabled the research to focus entirely on the phenomena itself i.e. when group flow occurred. As an example, the conditions of solitary and social flow (described earlier) were used solely as identifiers for occurrences of the group flow state. 'Normal' mathematics lessons took place, and flow was measured. The study examined and observed group flow, collecting data about group flow and therefore the research did not require a control group.

Measurement

Measurement through questionnaire responses is inherent with complexities such as recollection, leading questions and other bias. The multi-faceted and subjective nature of experiences and the 'ephemeral' nature of the flow experience (Hektner, Schimidt, & Csíkszentmihályi, 2007) complicates matters further. A rejection of the notion of objectivity is unavoidable, the learner 'attaches meaning because each individual has an experience of flow' (Morrison, 2017, p.1). In addition, respondents succumbing to social pressure and social desirability and the consequent influence on results is particularly inherent within the secondary mathematics classroom.

The Experience Sampling Method (ESM) is used to mitigate these considerations. ESM is an established data collection method and was used extensively in systematic phenomenological studies developed at the University of Chicago (Hektner et al., 2007) and consists of participants having pagers that randomly sample everyday experiences. The research discussed here used a classroom timer to randomly assign the times participants fill out the questionnaires in the classroom. Initially and necessarily to enable accurate and robust responses, learners are engaged in a dialogue as to the meaning of flow, and the differences between group and solitary flow. Participants are asked to self-assess their experiences of solitary flow and group flow. The questions for this systematic phenomenology were adapted from the Flow Short Scale (FSS) and are illustrated in table 2. FSS is a series of psychometrically valid questions, established by Jackson and Marsh (1996) which measure all nine pre-conditions for the flow state. It was originally used in situations where recording of flow was not possible immediately, for example playing music or athletics.

Methods of Analysis

The extracted data is analysed using a technique of examining visual evidence; a method called multi slice imagining. This innovative technique was developed by Konecki (2011), whereby a rich descriptive narrative of the visual evidence (video in this case) is then combined with visual grounded theory and situational analysis. A visual narrative grammar is considered including the context of the creation of the data and the structure of the sequence of the images. In this study the images and sections of video as visual evidence are chosen, with the focus and context including the lesson, students and environment as criteria. A narrative is then written and intertwined with group flow as a lens. This is in turn coded using visual grounded theory methods and combined and triangulated using the other data points including journals and questionnaire responses. Using visual grounded theory is still a relatively new concept (Mey & Dietrich, 2016), and with multi slice imagining this enables the videos from the six lessons to be utilised systematically, triangulated and combined with the comments in the questionnaire responses and the journal entries. As Strauss (1987) suggested there is discussion amongst grounded theory advocates as to its inductive or abductive nature (abductive in the sense of using existing knowledge to inform and reconstruct considering the newly discovered ideas). Within this study an abductive GTM approach is useful.

Results and Discussion

Although analysis is still ongoing, this method has produced an indicative list of flow indicators such as:

1. Head bent down, not sitting upright instead moving their eyes head closer to the work.
2. Generally moving closer to the work
3. Looking at the work/task/problem/worksheet; either alone or with another.
4. Pointing at the work or indicating at it in some way.

These assisted in validating the responses shown in tables 2, 3 & 4.

How does flow start?	'yes' response
The activity itself	84%
Concentrating	74%
The mathematical challenges	51%
My inside motivation	51%
My positive mood	56%
The classroom atmosphere	49%
Talking with my friends	62%
Talking with my teacher	24%
My mathematical skills	49%
Ignoring distractions	47%

Table 2. The emergence of Flow during 6 mathematics lessons (n=98).

Why would students report that flow was more likely when they talked with their friends? Communicating mathematically with peers may produce more challenging tasks (Lee, 2006). When talking about mathematics the skillset is often shared. Challenge vs skill is a central tenet of flow theory. Possibly this confirms the popularity of learning from peers in a social setting, although whether group flow is more enjoyable than solitary flow, would be a matter for a further discussion.

Why would 'talking with my teacher' engender flow far less than 'talking with my friends'? Perhaps a teacher inhibits group flow by presenting a barrier of authority; one source of the very knowledge that the learner seeks is the barrier to that knowledge. Often in the classroom, the teacher is the dominant player. The researcher was the participants' mathematics teacher and therefore often is involved in the group flow within the lesson. Conceivably co-active group flow occurred when the teacher is involved in activities, and interactive group flow occurred when learners are communicating with peers. The comments in table 3 confirm this. Certainly, more discussion is needed here, and the ongoing analysis of the study is accepted as a limitation of this article.

The excerpts from the journals in table 4 show many comments relating to talking and involving other individuals confirming the existence of group flow. The highly scored 'positive mood' in table 2 suggests when group flow occurred the lessons were a more enjoyable experience for the learner.

By establishing a culture of group flow, the teacher becomes tasked with creating a positive atmosphere in which talking, and specifically talking about mathematics is an objective in itself. Learners are able to articulate more readily about mathematics, pupil voice is expressed, listened to and heard and students are also writing about their experiences of mathematics. The information presented here adds another layer to the existing research showing verbalising, talking and communicating about mathematics leads to better mathematicians (e.g., Pimm, 1989; Lee, 2006; Cobb, Yackel, & Wood, 1992). As Sawyer (2007, p.43) points out, 'conversation leads to flow, and flow leads to creativity'.

Tell us about a time you were in flow?
Participant: like when we having like a normal discussion not like too loud but we just talking about the math and something that like excites us and like everyone wants to like put their hand up and speak that puts us in flow and then when you talking about something and then like you curious about it we like want to talk like to our partners next to each other like we talk about like it and add more like information on it Interviewer: Ok Participant: So that gets us more in to flow and when we talk about it a little bit more and then listens to you which makes it more exciting. Participant: I think the best place for me to concentrate is like to sit to a person that you know it's like when I use to sit next to [?] I use to be like I have never talked to him in my life and then it was just really like difficult and I sit next to [?] I can ask her anything and its gets easier

Table 3. Post lesson video stimulated interviews excerpt. Transcript is verbatim.

I'm in flow because…
We talked
Ignoring distractions
Sit next to friend
Friend explained a new method
(The teacher) telling stories
(The teacher) long explanations!
I explained things to #2 I felt like I could understand them a bit more
Quieter than the classroom
Have people around me
With the class
Talking blends into background and I can't really hear it
When *(the teacher)* was talking about the flow books :)
(The teacher)'s Birmingham metaphor
Silence in the room helped my 'zone out'
When I start writing I feel in flow
Engaged with the work because it was a challenge
I'm enjoying talking about maths because it helps me get into flow
When *(the teacher)* is talking, I'm not looking at the board. I'll be colouring in or reading. I find it easier to understand the work because when I don't focus on what he is writing on the board my hearing becomes stronger.
We were in silence as there were no distractions and it was easier to concentrate
Right level of skill, and a little bit easy (as we had done some work on this before)
When someone brings up an idea(mathematical) and when you think about what they say and discuss it
(The teacher) talking about how you treat inequalities like equations except when you divide by a negative number
During the xy talk
Briefly meditating before a task
Work was challenging
Work was new for me
Comfortable

Table 4. Excerpts from participants flow journals. All comments are verbatim, apart from italics parenthesis.

A major benefit of entering group flow, particularly here in regard to mathematics, is the suggested increase in creativity during this heightened emotional condition. Sawyer (2007, p.43) suggests 'people who participated in group flow were the highest performers'. The creativity and associated flow that emerges from jazz musicians playing and improvising 'jam sessions' (Wrigley & Emmerson, 2011);

perhaps is an adequate metaphor as to what was happening within the classroom during the most optimum moments. Students take responsibility and deliver their input into the task and a number of data points showed this to be the case.

From observing a pattern in certain activities where flow is present, the question arises as to whether it is possible to 'hack' flow; i.e. bring about the optimal state of flow deliberately and intentionally. This is a possible future research opportunity. Some researchers have proposed that starting flow deliberately is not possible, (Csíkszentmihályi, 1990; Jackson & Marsh, 1996), nevertheless improving the pre-conditions for flow can increase its likelihood (Wrigley & Emmerson, 2011).

This sample is not necessarily representative of the population at large, nevertheless this research seeks to clarify and demonstrate a practical way into making mathematics enjoyable in the classroom, and is a feasible vector for teacher development.

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Empathy in Interactions in a Grade 8 Mathematics Classroom in Chile

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This paper presents an analysis of the interactions through conversations between a teacher and their students in a Chilean grade 8 classroom (13-14 years old) when they are doing mathematics in their usual way. Enactivist theory and Gallagher's neuroscientific idea of mirror neurons relate empathy to biological roots. This study considers empathy in the interaction between participants associated with the mathematical interpretation from a teacher to their students, showing ways in which mathematics understanding emerged.

Keywords: Interaction; empathy; enactivist; actions.

Introduction

> ". . . Affect or emotion is the origin of what we do every day, in our doing and interaction with the world . . ." (Varela, 2000, p. 247, translated from Spanish).

Students' emotions regarding mathematics have been widely researched from different educational approaches. A review of this situation is presented by Zan, Brown, Evans and Hannula (2006).

Hannula (2012) drew a theoretical framework related to the affect of mathematics, including how it is related to cognitive, motivational and emotional aspects of affect.

Other authors considered emotions related to students' interest in a mathematical task, which could be manifested through their engagement and affected by the teacher's role in that engagement (Nyman, 2017). For example, a study with German students (aged 12–13 years) about interest in mathematics showed that interest in mathematics could be considered as a predictor for mathematics achievement (Heinze, Reiss, & Rudolph, 2005, p. 212).

Reid and Drodge (2000) argued that, "emotions play a positive and central role in mathematics" (p. 249), when students and teachers are doing mathematics in the classroom, describing this activity as *emotional orientations* or shared preferences (Reid & Drodge, 2000 p. 249). Another author who discussed positive emotions suggested that Chilean grade 8 students "have very positive views of mathematics: three-quarters report liking or enjoying this subject, and more than half would like a job involving mathematics in the future" (Ramirez, 2005, p.109). From the point of view of the emotions, mathematics is perceived by these students as enjoyable.

Despite the large number of studies on emotion in mathematics and, taking into account that "emotion is the origin of what we do every day, in our doing and interaction with the world" (Varela, 2000, p. 247, translated from Spanish), there is little focus in the literature reviewed on how emotion is manifested through empathy in the interactions between teachers and their students in a mathematics classroom.

Lord-Kambitsch (2014) says the meaning of empathy is ambiguous, because it "differs according to specific context" (p. 4). In the everyday actions of human beings, including those interactions that may happen in a mathematics classroom, empathy could be characterised with a "special status that is distinct from every type of social cognition" (Gallager, 2012, p. 355). To explore the interaction of how empathy arises between teachers and their students within a mathematics classroom, I have adopted enactivist theory.

Enactivist Position

Enactivism is a cognitive theory (Reid, 2014, p. 159), where "cognition is not the representation of a pregiven world by a pregiven mind but rather the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs" (Varela, Thompson, & Rosch, 1993, p. 9), which is strongly implicated in the interaction of each person with the environment. For example, any action or interaction between each person (including objects) and their surroundings bring forth a particular mathematical world for each one, arising through these interactions (based on Maturana & Varela, 1992, p. 26).

From this perspective, a person is structurally determined because she/he operates showing coherent actions in the environment that make sense for her/him, thereby presenting a unique person in its behaviour. In this context of coherent actions within an environment, each person generates his/her actions and it is possible to observe the autonomy of each person through the decisions that she/he makes within a set of interactions that entail options (based on Varela, 1994). However, although each person generates their actions, it is important to say that these can be triggered by the environment in which they act, because each person is structurally coupled to the environment. This means teacher and students are part of the classroom and the classroom is part of them (based on Coles & Brown, 2016), living in that moment and ready to act.

The environment is the set of interactions (surroundings) between each person structurally coupling with others. Therefore, each student, teacher and their experiences are part of an environment in which interactions are occurring (based on Ramirez, 2017). For example, a person can receive a stimulus, such as a mathematical question, and he/she can engage (or not), by answering that question. Whether they answer that question or not is determined by their structural determinism.

As a consequence of this recursive process of interactions in a mathematics classroom, where each participant (teacher-students) shows changes in their interaction, mathematics knowledge can be seen as unique, in its historicity of interaction and the context when this took place (based on Depraz, Varela, & Vermersch, 2003, p.156).

Finally, we can distinguish shifts of actions observed when a teacher and student are interacting, specifying what he/she brings forth with the distinction making (based on Maturana, 2000) and thus, noting patterns of interactions when the learning happens.

Emotion expressed by empathy

Cooper (2004) considered empathy in teaching and learning, she was interested in the role of facial expressions, language and tone of voice into functional empathy within a group, which can include discipline and relations amongst the group; She described and profound empathy as the development of positive emotions and interactions, such as acting and taking responsibility; and adaptive and integrated concepts and moral aspects

(p. 15). From my enactivist point of view, the description of empathy given by Cooper (2004) suggests actions performed by students or teacher when their interaction in the classroom took place.

Other researchers have considered the importance of empathy to be "a characteristic of a teacher that enables adequate communication between the participants of the educational process" (Stojiljkovic, Djigic, & Zlatkvic, 2012, p. 961), centred this characterisation, from an enactivist point of view on, how the teacher's actions may, or may not, trigger an adequate communication with their students.

Gallagher (2012) noted that empathy can (by default) be driven by the neuroscience of mirror neurons, which "provide a mechanism by which we can understand the actions of others by mapping the actions of the other people onto our own motor system" (Kaplan & Lacoboni, 2006, p. 175).

In this study, and inspired by the mirror neurons idea mentioned by Gallager (2012), I will consider empathy in the interaction between teacher and their students when "there is recognition of the other's experience as belonging to the other, without losing the distinction between self and other" (Thompson, 2001, p. 6) that is, I project myself in the other, but I am still being me, i.e., when certain interactions happen between a teacher and their students, the teacher can project an action led by empathy from the mathematical action of their student.

To specify this recognition of how empathy can be observed through interaction between teacher and their students, in my analysis I will considerer
the third complementary stage of empathy, identified by Depraz and Cosmelli (2003) that is: "an interpretative understanding of yourself as being alien to me" (p.172) which "involves expression (verbal or not) and interpretation, which lead, to the possibility of understanding (and misunderstanding, of course): it is a cognitive step" (Depraz & Cosmelli, 2003, p. 173).

The aim of this paper is to explore interactions through conversations of how empathy arises between a teacher and their students in a grade 8 classroom (ages 13-14 years old) when they are doing mathematics in their usual way.

Study Context

This study is part of an ongoing doctoral research project that involves characterising interactions in the mathematics classroom between teacher and students and amongst students themselves.

Within a 2.5-month period, I collected the data, which comprised eight 90-minute mathematics classroom observations, five audio-recorded lessons and three video-recorded lessons, in which the students and their teacher were working in their usual way to solve word problems and on exponents and powers, square roots and percentages. Furthermore, in two of the video-recorded lessons, they were working on a mathematical modelling task that was new to them because the country's national curriculum had recently integrated this learning goal (Bases Curriculares, 2012; 2013).

The Grade 8 classroom comprised of a mathematics teacher with 10 years of teaching experience, and 23 students. The small number of students allowed me to stay close to the details in the interaction between students and their teacher when my observations took place.

In addition, I carried out in the same period four interviews with the teacher and two interviews with a group of students comprising four or five students respectively. The method of choosing the participants in the interview was according to the frequency of interactions that I observed in the classroom.

In this paper, I report on my recent analysis related to empathy in the interactions based on observations, audio-video recordings and an interview with the teacher.

Analysis

In the recursive actions performed by students and teachers, when there is a shift in their actions, a distinction is noted by the observer. For this paper, I am analysing the transcription using the third stage of empathy proposed by Depraz and Cosmelli (2003), as I mentioned previously, which is intrinsically related to a person's interactions in his/her mathematics world.

The following text presents a conversation that was observed in a lesson between the teacher and a student. In this lesson, the class was solving operations associated with calculating the square root of a number, which sometimes involves solving an equation. However, I recognise that within the time that I observed, the students had not been working on any particular strategy to solve such an equation.

This episode of interaction has been chosen because it provides an account of patterns I also observed with other students in this classroom.

C*n* refers to the number of the contributions in this excerpt from the transcript (text translated from Spanish). Numbers in parentheses indicate the time when the interaction happened between the participants. The symbol // means more than a one-second pause and ~ indicates faster speaking. T: means teacher and S: means student. Words in square brackets have been added by the researcher in order to clarify the situation.

Square Root Chain

The following transcript begins with a question from a student, which is directed at the teacher, regarding solving the equation $\sqrt{16 \cdot 18} = 4x$. The point in the equation has been left, because in the Chilean context the use of the symbol " · " means multiplication, i.e., $16 \cdot 81$ means 16 x 81.

> C1: (17.22_17.25): T: Let's see; what have you done? [deleted for coherence] // What did you get?
> C2: (17.26): S: The square root of 81 [the student wrote in his/her exercise book $\sqrt{9}$].
> C3: (17.28_17.31): T: That is nine, not the square root of nine // because you said the number multiplied by itself gives [81].
> C4: (17.41_17.43): T: Not [that], because the square root [of nine] is three.
> C5: (17.43_17.51): T: Are you understanding? This is the process. It's not necessary to follow operating in square roots. What is the square root of 81?
> C6: (17.52): S: Okay, so . . .
> C7: (17.53_17.54): T: What is the square root of 81?
> C8: (17.55): S: Nine.
> C9: (17.56): T: Done it; it's not the square root [of nine].
> C10: (17.58_17.59): S: And then, for example, aha! Here, I can't follow because it is wrong.
> C11: (17.59_18.01): T: Mmm
> C12: (18.02_18.04): S: Then there, // could be it?' [referring to $\sqrt{25 \cdot x} = 35$.
> C13: (18.06_18.07): T: What is the square root of 25?
> C14: (18.07_ 18.10): S: Ah, no; I could have //. I got this $\sqrt{81} = \sqrt{9}$; $\sqrt{625} = \sqrt{25}$; $\sqrt{36} = 6$

C15: (18.12_18.23): T: Of course, because what you are doing is a square root chain. This means that, what you get, you calculate the square root again. Is it asking which number, when multiplied by itself, gives 625?
C16: (18.23_18.25): S: Five.
C17: (18.25_18.27): T: Twenty-five, not the square root of twenty-five.
C18: (18.28_18.33): S: No, not yet because I had written the number with the square root.//
C19: (18.43_18.48): T: It was the [square] root of 625; you got 25.
[Interruption: Another student asks the teacher if he can go look for something that he needs.]
C20: (19.03): T: [Square] root of 625 is . . .
C21: (19.05): S: Twenty-five.
C22: (19.07_19.19): T: So, it is equal to 25. That is asking again, and you're not writing the answer to your question, ~ what is that number that multiplied by itself gives 625 ~ [this number is 25].
C23: (19.21_19.28): S: Ah // I've multiplied twice; for example, here [referring to what she had written before C14]
C24: (19.30_19.32): T: That is alright, // delete the square and ask yourself again.
C25: (19.32_19.35) S: Oh! // done it!
C26: (19.36): T: What is the number that multiplied by itself gives 36?
C27: (19.37_19.38): S: So, the same here.
C29: (19.40): T: Exactly//
C30: (19.42): S: OK, alright.

In the transcript, although the structural coupling is happening all the time because the participants are interacting with others in this mathematics classroom, I observed in particular a structural coupling from the teacher and this student in the dialogue "multiplied by itself gives . . ." in C3, C15 and C22, which triggers an action mathematically, that can be evidenced later in C8, C16, C23 in the answer provided by the student. Similarly, the same happened when the student replies to the question about what the square root of 625 was (in C21), given that previously the teacher told the student the answer to the square root of 625 (as shown in C19).

In addition, I observed that when a change was triggered by a chain of events, within the historicity of interaction, the action increased. In C10, C14 and C18, the student manifested his/her understanding of the situation of what she/he was doing saying "it is wrong" but being more specific about his/her mistake, "I had written the number with the square root", which means she/he has written square root nine as the result. Later, I noted other distinctions between what happened before those actions related to the mistake of the square root and C23. This distinction, and the shift in the action, was triggered by the teacher's intervention in C22. The student finds a way to make sense, as shown by the contribution C23, "I've multiplied twice", but I must recognise that the student receives other stimuli by the teacher before, for example, comments such as "square root chain" in contribution C15, including, "What is the square root of . . ." in C5, C7 and C13.

However, after contribution C22, the student is now showing what she/he has noted as evidence (C23, C27) or more specifically he/she manages to make sense as shown in C23.

In addition, in contribution C1, when the teacher says "Let's see; what have you done?", this shows empathy, which leads to a possible interpretation of what the student had completed. The teacher starts the interaction from the action of the student, showing empathy that could be related to stage three mentioned previously by Depraz and Cosmelli (2003). Later, the teacher makes explicit his/her interpretation of what the student had done in contribution C15 with the phrases, "What you are doing is a square root chain"; "you calculate the square root again"; and in C19, "you got 25". The teacher interprets what the student is doing, which is “leading with the possibility of understanding” (Depraz & Cosmelli, 2003) by mentioning what was done previously in their mathematical interaction. A consequence of this kind of interaction is recognition (or noting) of what the student accomplished mathematically, based on the teacher’s interpretation.

This kind of empathy, which is manifested through the interpretation, was evident in other episodes in other lessons. See the transcript below from my field-note observation regarding the definition of recurring decimals in fraction conversions. The *italic* words mean what I have noted in their transcript related to empathy.

> Teacher: It is not [pure] mathematics language; it is colloquial [language]. But, *it is for your understanding.*

The phrase, “it is for your understanding”, shows empathy based on the interpretation of the teacher regarding the students (based on Depraz & Cosmelli, 2003). A definition written in colloquial language could generate understanding.

In addition, in the sixth interview with the teacher, one of the aspects addressed was his/her mathematics teaching history

> *I know which questions* I have to ask to trigger the content in some way. I’m encouraging my students [referring to students from 8 grade] so that *they can connect and do* with the knowledge that was there, which I assembled and re-appropriated…(Teacher)

The sentences, “I know which questions I have to ask . . . the content” and “I’m encouraging . . . connect and doing”, can be associated with the teacher and his/her view of doing mathematics in his/her classroom and his/her interpretation and understanding of what happens in the classroom. This shows that he/she is part of the classroom (structurally coupled) but also reaffirms empathy based on the interpretation mentioned above.

Discussion

When emotion in the interaction is characterised as empathy, considering “an interpretative understanding of yourself as being an alien to me” (Depraz & Cosmelli, 2003, p. 173) I noted empathy as a recurrent coupling in the action performed by the teacher to their student and also a recognition of what the *other* and *they* do mathematically.

Through the lens of empathy allows the researcher to observe the possible understanding of the teacher, according to the student’s understanding (or misunderstanding), as mentioned previously, of being here (as a teacher) but being there (interpreting what the other does or could do), as evidenced in the transcript about “square root chain”. It also makes explicit what the teacher and the student are noting in that particular moment when mathematics emerges in their actions.

The concepts of *being here* and *being there* can also possibly be observed in other teacher interventions, such as the definition of recurring numbers.

Due to the characteristics of this study, based on interaction observed through the teacher and one student in their conversation, one of the limitations is the lack of attention to physical interaction, for example moving a hand to express something mathematically.

In future research, it would be interesting to explore the other complementary stages of empathy such as those associated with the physicality of empathy (Depraz & Cosmelli, 2003). It is especially important when considering that a recent study by

Kardas and O'Brien (2018) suggested that observing what others are doing (without doing the same actions observed) allowed the observer to believe that he/she could perform the skill as well, creating an illusion of learning. How can empathy, expressed physically, be enacted in mathematics in order for students to make sense of what they are doing?

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Exploring the role of mindset in shaping student perceptions of inquiry based instruction in mathematics

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In this paper I present findings of an exploratory multiple case study from two UK secondary schools. Two teachers and their bottom set year 9 classes took part in a single lesson intervention in 2017. The intervention was designed according to the principles of inquiry based instruction (IBI), in which students explored a novel problem before receiving formal instruction. Through class questionnaires and follow-up interviews, I explore the perceptions of four students with mathematics difficulties (MD) towards IBI and whether mindset (growth versus fixed) moderates these perceptions in these students. Four themes emerge from the analysis: mathematics disaffection, inquiry as a form of neglect, inquiry as a form of empowerment, and teacher influence. Students with fixed mindsets expressed views of inquiry as a form of neglect more than those with growth mindsets, whereas students with growth mindsets expressed stronger views of inquiry as a form of empowerment.

Key Words: Inquiry based instruction; mindset; mathematics difficulties

Introduction

Reform efforts to improve the teaching of mathematics have been ongoing for some time (Cobb & Jackson, 2011). Central to these efforts is a migration away from traditional instruction, such as tell-and-practice, towards teaching that places greater emphasis on student inquiry. In general terms, such inquiry based instruction (IBI) in mathematics describes a pedagogic approach in which the teacher provides the students with problems in a domain of mathematics, however, offers limited guidance in favour of student exploration. Students are expected to explore the problem space with the aim of discovering knowledge. The exact amount of guidance given during these inquiry exercises is highly variable, and in many ways, this leads to the lack of clear delimitations between inquiry and non-inquiry approaches. Pure discovery approaches, in which students receive zero instruction, have been shown to be ineffective (Bruder & Prescott, 2013; Hmelo-Silver, Duncan, & Chinn, 2007; Kirschner, Sweller, Kirschner, & R, 2018). The key to effective IBI in mathematics is to determine the optimal mix of teacher led guidance and student led exploration.

The exact cognitive and non-cognitive factors that make IBI effective are unclear. Subjecting students to the cognitively demanding process of exploring the problem space during IBI would seem to increase their cognitive load, thereby reducing their capacity to create long term memory (Kirschner et al. 2018; Sweller, 2016). Therefore, IBI would seem to be incompatible with cognitive load theory (CLT). Various explanations are put forward as to why certain types of IBI are effective despite a potential conflict with CLT. Metacognitive mechanisms are one such area of explanation. By allowing students to explore the problems it is proposed they become conscious of their knowledge gaps and that these impasses facilitate the assimilation of new "missing pieces" (Schwartz & Martin, 2004). Studies also suggest inquiry based tasks activate deeper awareness of the learning processes and prepare students for subsequent direct instruction (Kapur, 2011; Schwartz, Chase, Oppezzo, & Chin, 2011).

Despite the popularity of IBI, its effectiveness for students with mathematics difficulties (MD) has been mixed and teachers have demonstrated a reluctance to use inquiry techniques with this population of students. Approximately 45 percent of teachers believe that the sort of higher order thinking needed for IBI is not appropriate for low achieving students (Zohar, Degani, & Vaaknin, 2001). A yearlong study by Woodward and Baxter (1997) found students with learning disabilities, and similar low attaining peers, made marginal gains when given IBI, whereas those that followed a traditional curriculum made dramatic gains. Similar results were found in a meta-analysis of 58 intervention studies by Kroesbergen and van Luit (2003). On the surface, this would all seem to support the view that IBI should not be used for students with MD, however relatively little research has looked into how these students' mindsets and attitudes may play a role.

Mindset

Why should IBI be effective for some students and not others? One area that has not received much attention is the extent non-cognitive factors, such as mindset, influence the effectiveness of inquiry based approaches. It is known that non-cognitive factors such as behaviours, perseverance, learning strategies, and mindset can influence a student's overall performance. The impact of mindset is particularly interesting given its recent popularity (Boaler, 2013). According to Dweck (2006), mindset can be categorised in three ways: 'growth mindset' (40 percent of the population), 'fixed mindset' (40 percent of the population), and 'mixed mindset' (20 percent of the population). Individuals with growth mindset believe that intelligence is not fixed but malleable. They view learning as a process governed by effort, as opposed to some ingrained ability. By framing learning within this context individuals are able to perform at a higher level. Alternatively, individuals with a fixed mindset believe intelligence cannot be altered and ability or 'smartness' is something you either have or do not have. These individuals tend to focus on performance and set objectives around demonstrating strong ability in the areas in which they believe they are superior. As such, they avoid challenges that might compromise this view (Dweck, 2006; Yorke & Knight, 2004). Students with growth mindsets, however, see challenges as learning opportunities and implement flexible learning goals. Thus, students with growth mindsets typically respond positively to failure and see it as an opportunity for increased learning and effort. Given that IBI often requires students to persist in their exploration of a problem despite possible failure, is it possible that such techniques are unsuitable for students with fixed mindsets?

Mathematics difficulties

The idea that a student can be deficient in a domain of knowledge is well understood and evidenced by the widespread acceptance of reading difficulties such as dyslexia. The application of this notion to mathematics has emerged over the last two decades. Terms such as mathematics disability, mathematics difficulties, and dyscalculia are used, somewhat interchangeably, to describe poor mathematics performance. It is unclear as to the origin of mathematics specific domain deficiency, or indeed if these many terms are different constructs with similar phenotypes. Cognitive and other non-cognitive causes are proposed, both with supporting evidence (Geary, 2004; Shalev, Auerbach, Manor, & Gross-Tsur, 2000). In general, the prominent thinking is of 'mathematics learning disabilities' as being a distinct cognitive construct from 'mathematics difficulties', with MD having its origins in non-cognitive factors. It is this view of MD that I will adopt within this study.

A common method for identifying students with MD is the use of students' standardised test scores (Murphy, Mazzocco, Hanich, & Early, 2007). The 10th to 25th percentile emerges as common cut-off criteria, often supplemented with additional criteria, such as excluding confounding diagnosed disabilities (e.g., dyslexia).

Mathematics difficulties and mindset in IBI effectiveness

In the UK and US, the preponderance of ability grouping in mathematics means students with MD are typically grouped in 'low ability' classrooms. This 'ability grouping' has been shown to propagate fixed mindset and the stereotype that abilities are somehow genetic and fixed (Plomin, Kovas, & Haworth, 2007). Teachers often adopt fixed mindsets in how they instruct these ability groupings, despite believing they are adopting growth or mixed mindsets (Marks, 2013). The prevalence of fixed mindset tendencies is greater in students with low prior academic achievements (Snipes & Tran, 2017). Before concluding that IBI is ineffective for students with MD researchers need a greater understanding of how a student's mindset could moderate its effectiveness.

A more comprehensive exploration of this issue is the subject of a larger ongoing study. However, described herein, are the results of an exploratory multiple case study that sought to investigate two questions. Firstly, how do students with MD perceive IBI? And secondly, does mindset moderate the effectiveness of IBI for students with MD?

Methods

This study used a multiple case study approach with cross case analysis of the data as described by Yin (2013). Each case is a bottom set mathematics class within a UK secondary school. Case 1, Esterwick College[1] is a larger than average secondary school based in Cambridgeshire. Case 2, Brighthedge College is a smaller than average sized secondary school. In both cases the classrooms selected were lower set mathematics classes. Selecting the lower set provided the most likely group of students with MD, as both schools set their students based upon standardised test scores. Students diagnosed with confounding disabilities were excluded from data collection.

Prior to the intervention all students within each case (16 at Esterwick and 16 at Brighthedge) were asked to participate in two questionnaires. The first questionnaire was the Attitudes Toward Mathematics Inventory (ATMI; Tapia & Marsh II, 2004), which measures students' attitudes toward mathematics on four subscales; (1) enjoyment of mathematics; (2) motivation towards mathematics; (3) self-confidence in mathematics; and (4) value of mathematics. The second questionnaire administered was the Implicit Theories of Intelligence Scale for Children (ITIS; Hong, Chiu, & Dweck, 1995). This three-question instrument, validated through a number of studies, is designed to determine a student's mindset (either fixed, growth, or mixed). For the

[1] All names of schools and students have been changed for anonymity.

purposes of this study it is assumed that a student's mindset did not change over the course of the IBI or follow up interviews.

The interventions themselves consisted of a single IBI lesson focused on a novel problem in an area of mathematics. In both cases the lesson plans were built in collaboration with the lead teacher and the degree to which the lesson conformed to inquiry was gauged using the Electronic Quality of Inquiry Protocol (EQUIP; Marshall, Smart, & Horton, 2010). EQUIP is a series of four rubrics used to assess a lesson's degree of inquiry on a scale of one to four, being pre-inquiry (level 1), developing inquiry (level 2), proficient inquiry (level 3), and exemplary inquiry (level 4). In Esterwick the teacher and students had little prior experience working with IBI. By contrast, the Brighthedge teacher described herself as "comfortable" with IBI, having previously been a science teacher. However, her students within the selected class had previously received little IBI as it was felt inappropriate for this group. Prior to the intervention the teachers in both cases conducted a practice IBI lesson using the rubric and received feedback from myself.

The IBI lessons to be included in the study were conducted several weeks after the above practice lessons. The Esterwick lesson covered the topic of estimating the area of a circle, in which students received several circles on grid paper and were asked to estimate their area using any method. The Brighthedge lesson covered the topic of linear relationships, in which students read a story and then enacted the story by adding marbles to a beaker of water and recording their observations. Both lessons were followed by a whole class discussion and met the level of proficient inquiry (level 3). Each IBI lesson was video recorded and detailed field notes were made. Particular attention was paid to those students who would later be called for interviews. The principle purpose of these observations was to provide subject and context specific data to aid the follow up interviews.

Two students from each class were chosen for interviewing, making a total of four students interviewed across both cases. Selection was purposively conducted to ensure that within each case one interviewee held a growth mindset and the other held a fixed mindset as determined by the ITIS. Interviews took place in a quiet room separately from their regularly scheduled mathematics lesson. In the case of Esterwick College the interviews were conducted in the presence of the teacher's aide. However, in the case of Brighthedge College the interviews were conducted in private. Interviews followed a semi-structured format, meaning the discussions were conversational in style but with important pre-determined questions to be addressed. The sessions were audio recorded to allow accurate transcription and data analysis.

Data were analysed separately since the IBI lesson and teacher were different for each case. Interviews were initially transcribed, and in an iterative process, each transcript was interpreted line by line (within the context) and coded (Merriam, 2009). These were then reduced through numerous rounds of coding. During this data exploration phase, possible themes and explanations were noted to determine how each student perceived the IBI. Subsequently, the interview data were analysed across the cases to identify and refine those themes as well as any sub-themes. Finally, the students' mindsets were considered, and any differences in theme expression between these mindsets were noted.

Results

The interview analysis indicated participants' perceptions of IBI fell into four themes: (1) mathematics disaffection; (2) inquiry as a form of neglect; (3) inquiry as a form of empowerment; and (4) teacher influence.

Mathematics disaffection

When questioned about their feelings towards mathematics, all four students expressed views that suggested disaffection with mathematics. This was expressed in several ways. The idea that mathematics is boring was recurrent.

> It's been alright. It's a bit boring. – Fred, Brighthedge

This was often paired with notions that mathematics is something that is only done because it is required. This might be best described as 'quiet disaffection' in which students routinely comply with class expectations but their engagement is limited to 'resigned acceptance' (Nardi & Steward, 2003).

> When you really just don't understand it or you don't want to do it … it's like, 'Ugh, do I really have to do this? I don't get it.' It's like you can't be bothered type-thing. It makes it annoying. – Gloria, Esterwick

Inquiry as a form of neglect

At several points during the interview I asked the students to reflect on the inquiry lesson. To some extent, all four students expressed negative views towards the lack of teacher support. Many felt that it was the responsibility of the teacher to explain the concepts before they tried it on their own.

> If they're not going to help you, then there's no point you doing it cause … it's sort of a bit like they don't want to show you how to achieve it. So, there's not really much point in doing it. Like, we're in school to learn stuff and if teachers don't show you how to do it or help you then there's not much point in being there. It's a bit of a waste of everyone's time. – Fred, Brighthedge

In a similar vein, all four students said teachers should make themselves available to answer questions.

> If the teacher doesn't make you understand it then it's on me to ask the teacher for another way of understanding, because that's what they're there for. – Gloria, Esterwick

Inquiry as a form of empowerment

Somewhat contrary to the previous theme, all four students expressed views that suggested inquiry mathematics was a form of empowerment. Students were mirroring some of the concepts that major proponents of IBI put forward such as the idea that inquiry learning engages students to "think more", forces them to "use their brain," and results in greater understanding.

> It kind of made your brain work a bit more so therefore you would understand a bit more because instead of just listening you actually have to guess it, you have to figure it out. Instead of listening to what you're being told. Which I think made it just better and easier. Well not easier, but it made it more understandable. – Gemma, Brighthedge

Teacher influence

The final theme to emerge was around the role of the teacher. Sub-themes to emerge included the idea that teachers drive a substantial portion of the enjoyment of a lesson and strong classroom management is a crucial factor in how well students perform.

> There's certain ways teachers teach. I think they're all different. Some of them are similar, but I think it's the way that the teacher wants to teach and then you got to adapt the way they teach you. And then you just sort of compromise with each other to see what you like and what they like and how they can teach you better. – Faye, Esterwick

> She just made it a lot more fun. Like sometimes she would put it in a song, and this kind of sounds cheesy, but she would put some bits in a song that we would laugh to and then you'd remember it. – Gloria, Esterwick

Theme strength between mindset groups

Whilst the four themes were present in every interview, it is interesting to consider how the relative strength of the themes differed between students of different mindsets. The themes of mathematics disaffection and teacher influence were expressed to similar extents across both fixed and growth mindset participants. However, participants presenting a fixed mindset expressed views of inquiry as a form of neglect much more strongly than those with a growth mindset. This was evidenced by numerous factors, such as a greater proportion of the interview time or the number of times the interviewee returned to this theme. Furthermore, within this theme, students with fixed mindsets expressed views towards a marked lack of persistence.

> If something's really hard, and I just can't do it then I'll ask once/twice about it, and then if I can't do it then I'll just end up talking, and then I'll never do it. …Like if it didn't work the first time … in my head I then say to myself I can't do it. And then I just lose interest and focus and everything. – Fred, Brighthedge

Conversely, participants presenting a growth mindset expressed ideas of inquiry as a form of empowerment more than participants with a fixed mindset. Furthermore, these growth mindset participants demonstrated a sub-theme which indicated the link between persistence and outcome.

> I was sort of determined to find out the answer, like 'Oh it can't be that hard'. If that one doesn't work, like with puzzles or something, if one piece doesn't fit into another piece then you always try to look for that one piece that will fit. – Gloria, Esterwick

ATMI Survey Results

The above analysis of the interview data explores differences in the perceptions of IBI based upon the students' mindsets. However, it's possible that observed differences may also arise from attitudinal differences between the students. It is therefore important to investigate whether mindset is distinct from other relevant measures, such as attitude towards mathematics. Looking at the results of the ATMI it was observed that in both cases there was no significant correlation between mindset and any of the four subscales of the ATMI. Further, there was no significant difference between the mean attitude scores across the three mindset groups. Taken together, this suggests mindset is a distinct construct from these attitudinal subscales.

Discussion

The absence of any significant difference in attitudes toward mathematics between the different mindset groups supports previous studies which have suggested that mindset is a separate construct distinct from those measured in the ATMI, namely enjoyment, motivation, self-confidence, and value perception.

Clearly the small sample size of this study limits the ability to make any generalisations, however the emergence of the four themes provides an interesting insight into how students with MD perceive inquiry based instruction in mathematics. These themes underscore the persistence of disaffection within this population group as well as the important role of the teacher in creating engagement.

The observation that the two students with growth mindsets were much more likely to see inquiry approaches as empowering is particularly interesting. This observation, combined with the propensity for teachers to shy away from IBI for students with MD, suggests an interesting area for further study. Might teachers be missing out on an important opportunity to address disaffection with their students with MD by using more inquiry based practices? Furthermore, studies have demonstrated that mindset is not fixed and may be amenable to intervention. By providing a lesson targeted at teaching students to have a growth mindset there is evidence that students with fixed mindsets can improve their performance. This suggests the possibility that IBI might be made more effective for students with both MD and fixed mindsets following a mindset intervention.

Acknowledgement

I am grateful to the Cambridge Trust for funding this study.

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Primary pre-service teachers: reasoning and generalisation

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Generalising tasks, in the context of mathematical reasoning, have featured in our work with primary pre-service teachers (PSTs). We used two particular problems - 'matchstick squares' and 'flower beds' - to explore the generalisation approaches taken by PSTs. In this paper, we analyse the ways in which one of them, Terry, uses recursive or functional approaches to generalisation, and how he attends to looking for a relationship and seeing sameness and difference between figures in a sequence. We consider what motivates shifts in attention, the significance of the PST's prior experience and of PST-collaboration in our teaching sessions. We conclude with a discussion about the significance of this activity in the PST's preparation for teaching, with reference to Mason's (2010) notions of pro-spection and retro-spection.

Keywords: generalisation; reasoning; pre-service primary mathematics teacher education.

Introduction

The current National Curriculum (Department for Education, DfE, 2013) for children at primary schools in England now includes reasoning as an explicit aim of its programme of study for primary mathematics. This has renewed the place of reasoning in the debate about teaching and learning of children in primary school. For example, national testing for children aged 7 and 11 now includes written papers on mathematical reasoning (DfE, 2017).

However, the term 'mathematical reasoning' covers many different thinking processes and strategies, and DfE exemplification focuses on reasoning associated with answering closed questions (DfE 2016). This sort of reasoning does not necessarily match the aim of the National Curriculum, which focuses on conjecturing and generalisation.

The authors of this paper are members of a larger group of primary mathematics educators, each with a commitment to research in mathematics education. The group has met about twice a year, for 10 years. As primary mathematics teacher educators in five universities, we have found that we promote mathematical reasoning in similar ways in our programmes. We have a shared belief in the value of reasoning associated with pattern, algebra and generalisation, and find that we use very similar activities in our sessions. In order to enrich our work as tutors on pre-service teacher education programmes, we wanted to investigate how student teachers respond to university-based training sessions which aim to prepare them to teach reasoning, and to explore the approaches to generalisation that student teachers adopt themselves when engaging with such activities.

Generalisation

Within the broader context of mathematical reasoning, a common context for generalising, sometimes referred to as 'growing patterns,' is a sequence of geometric figures constructed from, for example, matchsticks, squares or dots. Learners' attempts to generalise such a pattern can involve "manipulating the figure itself to make counting easier; finding a local rule (recursion) which reflects one way to build the next term from previous ones; (and) spotting a pattern which leads to a direct formula" (Mason, 1996, pp. 75-76). One important theme of the research on pattern generalising is this distinction between finding a local, *recursive* relationship and a direct, *functional* relationship. Research points to learners' preferences towards finding a local rule of recursion between figures in a sequence, and the relative difficulty of finding a functional relationship (MacGregor & Stacey, 1993; Stacey & MacGregor, 2001).

For example, in the 'growing pattern' of matchstick-squares shown below (Figure 1):

Figure 1. Matchstick squares

a recursive response would observe that each one has 3 more matches than the previous one. So 4, 7, 10, …. The number of matches in, say, the 10^{th} configuration could be found by extending the number sequence …13, 16, etc. A functional insight would observe that when there are n squares, the number of matches can be expressed as $3n+1$. In this way I can find how many matchsticks there would be if there were 10 squares, without having to list the previous 9.

Ferrara and Sinclair (2016) argue that while noticing a recursive relationship requires an understanding of horizontal 'mobilities,' identifying a functional rule requires an understanding of vertical 'mobilities,' i.e. understanding the relationship between the independent and dependent variable.

Wider literature also identifies the significance of visualisation in pattern generalisation. Wilkie and Clarke (2016) explored the different ways in which individual students *see* a pattern, by inviting them to use colour to show how they saw elements of the geometric shape. They found that the subsequent generalisations reflected the ways in which students initially perceived the pattern. Seeing the structure of a figure as the result of 'growth' from previous figures led to a recursive rule, while other ways of seeing led to a functional rule. Different ways of seeing and counting elements in a pattern can lead to different, equivalent generalisations.

Bills and Rowland (1999) contrast two ways of arriving at a functional generalisation, which they call 'empirical' and 'structural'. The fundamental distinction is between knowing *that* and knowing *why*. In the case of the squares growing pattern (Figure 1), an *empirical* approach would reason: I have the numbers 4, 7, 10, 13, … and I observe that these can all be expressed as $3n+1$. It's just a fact, and it works, though I don't know why. A *structural* insight might perceive some general structure in the situation – for example, that in every case, there is a row of C-shapes, each with 3 matchsticks, and one to complete the last square. So there are $3n+1$ matchsticks in the nth configuration.

For a striking numerical example of the distinction, consider the sequence 1, 1+3, 1+3+5, etc. A functional generalisation – that the terms are all perfect squares, and the nth term equals n^2 – follows fairly readily. In the first instance this might well be an empirical generalisation – I don't (yet) know *why* these sums are squares. The generalisation becomes structural if, for example, we envisage a 3x3 square array of dots (Figure 2), with 1 dot bottom left, 3 dots adjacent to the first one, (building a 2x2 square array), then 5 dots above and to the right of that (2x2) square, completing the 3x3 square array. The first n odd numbers are then seen as a set of dots from which an n x n array is constructed.

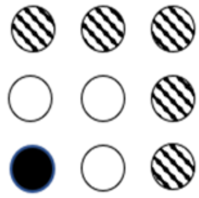

Figure 2. Square array

In summary, seeing the structure of a geometric figure supports what Bills and Rowland (1999) refer to as 'structural' generalisation. This is in contrast to 'empirical' generalisation which, in the context of a geometric sequence, describes a consistent relationship identified between quantifiable elements, such as the figure number and number of matchsticks. The resulting (empirical) generalisation is then "divorced from the structure of the pattern" (Küchemann, 2010, p.233). Küchemann (2010) makes a compelling case for focussing on structure *within* a single figure in a sequence rather than presenting learners with a systematic sequence of elements. Such analysis of the structure of a *generic example* fosters "seeing a generality through the particular" (Mason, 1996, p.65). (The above account of the 3x3 square of dots (with Figure 2) was intended to be generic in connection with 1+3+ … (2n-1)). Beyond working with a generic example, teachers have an array of pedagogic choices which may shape pattern perception and visualisation. These include the use of concrete materials, drawings, diagrams and technological environments (Wilkie & Clarke, 2016).

While, in the literature, relatively little attention has been paid to teacher knowledge in relation to generalising and functional thinking, there is evidence that this is an area of difficulty for primary teachers and primary pre-service teachers (Wilkie, 2016; Goulding *et al*., 2002). Wilkie's research highlighted "the importance of teachers developing their own ability to generalise patterns and to learn to understand the process by which students develop functional thinking through recursive and explicit generalisation" (p.270). Our own study explores these important ideas, as pre-service teachers work on tasks which challenge them to reason, yet are 'sufficiently close' to primary mathematics.

The Study

This paper presents the approach that one student teacher - we call him Terry - took to tackling a problem involving reasoning and generalisation. Terry was on a one year graduate primary teacher education course, specialising in mathematics. The session that Terry reflects on below was designed to enable students to explore growing patterns, whilst working together with peers to explore possible alternative approaches. Students were presented with the Flowerbed pattern (original source unknown) where square slabs are placed around the border of a square flowerbed - see Figure 3 below. They were asked to generalise about the number of paving slabs required around each square bed. Students were given some time initially to consider the problem, then they worked together, sharing their approaches. Shortly after the taught session, Terry was interviewed about his approach to the problem. We used Wilkie's notions of 'recursive' and 'functional' thinking to analyse his responses.

Terry's response and our analysis

Terry had a degree in Theatre Studies and had studied mathematics at A level. In the interview, he said that he had been confident with the subject in the first year of his A level study but had found the second year "quite a lot more challenging". Terry was enjoying teaching mathematics and had found the experience of applying his mathematics knowledge in his teaching practice rewarding. The specialist course had changed his view of the subject by introducing him to mathematics pedagogy.

> Terry: I think my view of mathematics was quite narrow until coming onto the course and just seeing how everything can be broken down and made so much more accessible, even … even things like fractions which is like this feared term in primary schools.

Terry recounted his approach to the Flowerbeds problem with reference to his notes from the session. During the interview, and while he was explaining his train of thought, Terry made additional notes on a printed illustration of the pattern that was provided by the interviewer (one of the authors). He explained that his initial approach was to focus on the number of squares that formed the centre of the shape for each case. He wrote the corresponding numbers (1, 4, 9) under each case and then counted the number of white squares that surrounded the dark-shaded centre of the shape in each case (8, 12, 16) (Figure 1).

> Terry: I started off by noting down, we had case 1, case 2, case 3, and I noted down how many squares were in the centre of the flowerbed … Yeah, so I was drawn to that, so we had 1, 4 and 9. And then I calculated …

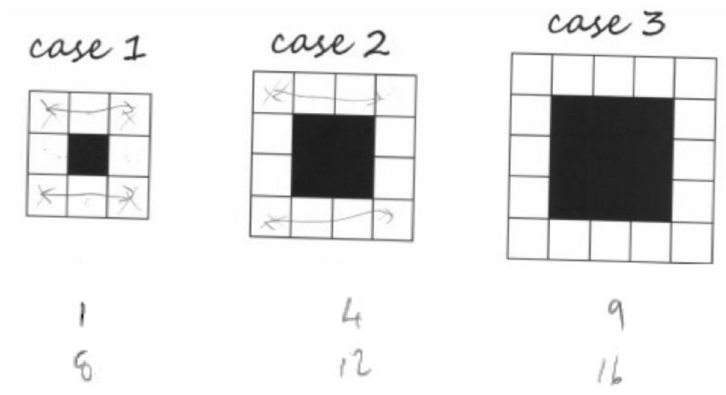

Figure 3. Terry's jottings while explaining his initial approach to the pattern.

Terry continued his explanation referring to his own notes from the session.

> Terry: And then I started off by trying to figure out some kind of pattern or link or connection between those numbers, and I wasn't really getting anywhere to be honest. And then I … I thought back to a previous university session, when we did something similar to generalising, where we found something that stays the same each time.
>
> Interviewer: OK.
>
> Terry: So this is obviously where I'd gone to in the middle, originally that is different each time …so I thought what is the same each time.…And it ended up being the four corners. …Were the same each time, there's always going to be four corners, so that's where I ended up going down this route.
>
> Interviewer: OK, and following that, after you saw the four corners as staying the same, what did you do next? Where did you go after that?

> Terry: So for this one I would have, N would be 1 (referring to case 1), so I'd have … four lots I think of N, and then I would be adding on … oh no hang on … this is 4 here, that's always … I've just confused myself.

Terry's initial approach was to count the squares of each case in the sequence with the view of identifying a functional relationship (determined empirically) between two quantifiable aspects of each case; the number of squares that constitute the central part of each case and the number of white squares that surround the central part (Figure 3).

The difficulty that he encountered in identifying a link between these numbers prompted a move to a recursive approach whereby he looked for what remained the same and what changed in each item of the sequence. This was supported by his recollection of a similar activity and strategy that he had learned in a previous university session. Terry found it difficult to conclude his explanation. The interviewer prompted a bit more.

> Interviewer: Right, so you have the four corners as a constant feature.
>
> Terry: Yes.
>
> Interviewer: And then what happens? Are you looking at the squares between the corners now?
>
> Terry: Yeah, so then there's, we've got … four here and then obviously one, two, so it's two lots …
>
> Interviewer: So you're still looking at the middle part or not anymore?
>
> Terry: I, yes, to base off this one.
>
> Interviewer: OK.
>
> Terry: So you've got the, I guess we call that, maybe that can be called N, so it's 4N….Plus 4 …
>
> Interviewer: … N is the centre one with four around it?
>
> Terry: Yes, so there's four lots of N around it.

In the above extract, Terry goes back to focusing his attention on a single case of the sequence (case 1) seeking to identify a general rule with attention to the *structure* of the shape. He associates N with the central black square. He refers to 4N as representing the four adjacent white squares and to "Plus 4" as representing the four constant corners. When moving his attention to case 2, he becomes confused and returns to recursive reasoning.

> Terry: And then … plus four, this one, but then I'm, I've not accounted for this one, have I? Or have I? No, I haven't.

Here, "plus four, this one" refers to Terry's observation that the sides of the square in case 2 (excluding the four corners) are formed out of eight, in total, white squares that are adjacent to the centre (i.e. four more than the squares that constitute the sides in case 1). However, at that point Terry realises that he has not accounted for how the central square has grown moving from case 1 to case 2 and remains puzzled.

At this point, Terry recalls his collaboration with one of his peers during the session, and describes an alternative approach that they took when seeking the general (functional) rule for the sequence.

> Terry: Yeah, well we had ways of looking at it, I mean I think, that was one way of seeing it. The other way I saw was I'd looked at this as like a 1, 2, 3 (draws a line across the three white squares in the first and third row of case 1).

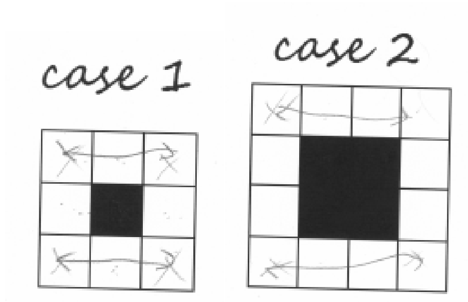

Figure 4. Terry's jottings on the printed pattern

> Terry: And then there was the middle ones and these, (referring to the central square of case 1 and the squares on either side of it) and then the same with this one (case 2), the top … (draws a line across the top and bottom rows of case 2, Figure 4).
>
> Interviewer: And you are still looking at the middle part, the dark part, yeah?
>
> Terry: Yes, so this one (goes back to case 1) I guess would be N and then there's, so there's two lots of N isn't there, and then on the top there's plus two, so two lots of N plus 2.
>
> Interviewer: Where are the two lots of N? What is the two lots of N? The four squares in the middle of case 2?
>
> Terry: Ehm … so 2, it's case 2 and then we've got on the top 1 and 2, 3, 4, so N plus 2 …Two lots of N plus 2.

With the assistance of one of his peers, the 'structure' perceived by Terry has now changed. Focusing on case 1, Terry associates N with 1 and explains that the number of squares in the top and bottom row is represented by N+2 so the top and bottom row are "two lots of N plus 2". He provides the same explanation for case 2 (Figure 4) noting the relationship between N and the number of squares that form the top and bottom row but without accounting the central, dark square and the adjacent white squares. Although he did not complete the formula here, he had generalised about all sections of the pattern separately by that point.

Towards the end of the session, the interviewer asked Terry to indicate one thing that he had learned from this session and would apply when he teaches mathematics.

> Terry: Giving children plenty of opportunity to discuss, I think that's quite important, and just to encourage people to discuss in the classroom because I know …
>
> Interviewer: Why do you think it's important?
>
> Terry: Because that's what helped me in terms of when I heard …
>
> Terry: … anything like that, that often was like a hook into allowing me to access the problem in which, without that I wouldn't have been able to. If it was just silent, I would have been sat there in my own space, staring at the one way I could identify it, trying to see it in some other way, but probably struggling and failing miserably. But being able to hear other people discuss it, allowed me like access into the problem a little bit more.

In his response, Terry highlights, on the basis of this experience, the value of opportunities for classroom discussion that encourage learners to see patterns in different ways, and to allow all learners to access tasks that might have been too challenging for them to tackle on their own.

Conclusion

Terry's account of different approaches to the exploration for a general rule indicated shifts of reasoning and attention to recursive as well as functional relationships (Ferrara & Sinclair, 2016). In this case, shifts of reasoning appeared to be prompted by difficulty in completing a particular line of exploration, which steered Terry to draw from his prior experience with similar activities, and also, by his observation of alternative approaches that others had adopted, in a setting that encouraged peer collaboration. Through the reported shifts between functional and recursive thinking, Terry appeared to maintain, largely, his focus and attention to the structural elements of the sequence (Küchemann, 2010).

Although Terry explicitly referred to "other ways of looking at it [the pattern]", we cannot know whether he was aware of his move between different kinds of mathematical reasoning. A question that is raised for us, as primary mathematics teacher educators, is whether this matters, and whether it would require greater and explicit emphasis as part of our sessions. Terry considered the opportunity to see the structure in different ways, in discussion with his peers, to be the key learning from this experience, and that that would influence his own teaching in the classroom. This highlights the value of including such activities in mathematics teacher-preparation sessions, offering pre-service teachers the opportunity to experience generalisation explorations for themselves, and to identify aspects of practice that would be important in their own classrooms.

Next Steps

In the next phase of this research, we are investigating how best to prepare our pre-service primary teachers to introduce and support children in school to work with generalising activities. The mathematics specialist PSTs in one of our universities have already worked on pattern generalisation tasks with a group of children in school, and discussed that experience at a follow-up session with their university tutor (one of the authors). As a theoretical framework for analysing their feedback, we are working with Mason's (2010) dictum that "in order to learn from experience it is necessary to do more than engage in activity" (p. 23). Mason (2010) suggests that teachers can do the following – for themselves and each other – to engage with pro-spection (anticipation) and retro-spection (reflection) on teaching: (i) work on mathematics for themselves to "sensitise themselves to the struggles that pupils experience" (p. 43), and (ii) collaborate in their enquiries – "to direct each other's attention to salient features so that finer distinctions can be made" (p. 43). This pro-spective and retro-spective activity relate both to their own learning (about generalisation) and their own teaching. Analysis of data from the university-based follow-up session is ongoing, against a framework derived from these insights from Mason (2010).

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Inclusion and disability in the primary mathematics classroom: Examples of teaching staff discourses on the participation of visually impaired pupils

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Research on inclusion and disability is underdeveloped in mathematics education. This two-phase doctoral study investigates inclusion and disability in the discourses of teaching staff and pupils in British mainstream primary mathematics classrooms with visually impaired (VI) pupils, first in an exploratory phase and then in an experimental phase. The study endorses the following tenets: that inclusion can be achieved when pupils' academic and social needs are considered and met in lessons; and, that disability is socially constructed. Teaching staff and pupils of four classrooms are taking part in this ongoing study. Data is collected through classroom observations and interviews with teaching staff and pupils. One of the preliminary findings of the first phase concerns teachers' and teaching assistants' frequently different practices of inclusion/exclusion and of enabling/disabling of VI pupils. In this paper, we report a Year 3 (Y3) classroom episode which illustrates said differences.

Keywords: inclusion; disability; discourse; VI pupils

Introduction

"Inclusion" and "disability" are conceptualised in several and different ways in educational research (Nardi, Healy, Biza, & Fernandes, 2018). One proliferating difference in meanings attributed to "inclusion" is between the proponents of the special education model and those of disability studies in education (Slee, 2011). The former consider inclusion as a reconstruction of special education, situated in mainstream settings, while the latter consider inclusion as an educational model of social justice, eliminating any forms of discrimination produced by the special education model. "Disability" as well is endorsed differently in the two prevalent models of disability discourse: the medical model and the social model (LoBianco & Sheppard-Jones, 2007). The medical model considers disability as a medical condition attributed to the individual's impairment, while the social model considers disability as socially constructed.

The study we report from in this paper is an ongoing two-phase doctoral study funded by the University of East Anglia. Phase 1 is exploratory and investigates how disability and inclusion are constructed in the discourses of teaching staff and pupils in mainstream primary mathematics classrooms with VI pupils in England. Phase 2 is experimental, considers issues on inclusion and disability identified in Phase 1 and involves collaboratively designed mathematics lessons that aim to be fully inclusive and minimise disability in the mathematics classroom. We use the terms "inclusion"/ "exclusion" to denote when the VI pupils are invited, or not, to participate in a lesson activity on an equal basis with the rest of their peers, albeit not necessarily with the same tools (sensory, material, semiotic) (Vygotskii, 1978). We use the terms "enabling"/ "disabling" when the teaching staff consider, or not, the VI pupils' perceptual needs in a lesson activity.

In what follows, we outline key developments in the relevant research literature and conclude with an outline of the study's significance and research questions. We then discuss the theoretical underpinnings of the study and introduce the study's methodology and context of data collection. Preliminary findings from the first phase of the study are then presented, with reference to a particular episode extracted from a Y3 classroom. We close with implications that our conclusions from this episode have for our ongoing analyses and the second phase of the study.

Literature review and theoretical underpinnings

A limited number of studies have been conducted in the area of inclusion of VI pupils in mathematics classrooms. Amongst the foci in the literature are the following: VI pupils' forms of accessing, expressing mathematics and development of inclusive teaching strategies (e.g. Fernandes & Healy, 2013); VI pupils' experiences in mainstream mathematics classrooms (e.g. Bayram, Corlu, Aydın, Ortaçtepe, & Alapala, 2015); and, design of inclusive mathematics teaching and learning materials (e.g. Leuders, 2016).

While the existing literature has certainly been informative and has started to prepare the grounds for the creation of more inclusive mathematics classrooms, research studies that design, trial and evaluate inclusive mathematics lessons in the classroom are sparse. It is in response to this sparsity that this study was conceived.

The study addresses two research questions. How are inclusion and disability constructed in the discourses of teaching staff and pupils in mathematics classrooms? How do collaboratively designed mathematics lessons impact upon teaching staff's and pupils' discourses on inclusion and disability?

The study's theoretical framework is sociocultural and endorses theoretical tools from Vygotskian sociocultural theory of learning (Vygotskii, 1978); Sfard's discursive perspective, known as the theory of commognition (Sfard, 2007); the social model of disability (Oliver, 2009); and, the theory of embodied cognition (Gallese & Lakoff, 2005).

Drawing upon Vygotskii's (1978) sociocultural theory of learning, we see mathematical learning as a social and cultural process which involves the use of a variety of sensory tools (e.g. hands, ears, eyes) in mathematical meaning making and expression. Our consideration of bodily tools as indicators of mathematical meaning making and expression makes us infuse our sociocultural framework with elements from the neuroscientific theory of embodied cognition (Gallese & Lakoff, 2005). Apart from speech, we consider voice, gestures and facial expressions as vital factors for meaning making and expressing mathematics.

Drawing upon Sfard's (2007) discursive perspective, we discern teaching staff's and pupils' discursive activity – word use, visual mediators, endorsed narratives and routines – and particularly the elements of their activity that concern inclusion and disability, as evident in their speech as well as through their bodies, such as voice, gestures and facial expressions.

Drawing upon the social model of disability (Oliver, 2009), we consider disability as socially constructed and arising for people with impairments when environmental and attitudinal factors prevent their participation in activities on an equal basis with others (United Nations, 2006). In this respect, we endorse the tenet that disability would be significantly mitigated if disabling barriers were removed. We consider inclusive education as an appropriate form of education through which disability can be drastically minimised in the mathematics classroom.

Methodology and context

The study is qualitative in both its phases. Its methodology has ethnographic characteristics (Bryman, 2016), as the data is collected in the naturalistic environment of mathematics classrooms with the aim of investigating the discourses of teaching staff and pupils on inclusion and disability in depth.

Ethical approval for the study has been granted by the School of Education Research Ethics Committee. Participants' anonymity, confidentiality and right to withdraw from the study have been guaranteed to the participants, who have all provided consent for participation in the study (including parental consent for the participating children).

Data has been collected in four mainstream primary mathematics classrooms in England. Criteria for the selection of the classrooms were the presence of VI pupils in them and willingness of the teaching staff and pupils to participate. There is one VI pupil in three of the classes and two in the fourth. Most of the participating VI pupils have severe visual impairment and none of them is blind in both their eyes. Two pupils have congenital visual impairment while three have adventitious visual impairment.[1]

Every class has at least one teaching assistant but the teaching assistant's role differs from class to class. While two of the classes have a teaching assistant supporting the VI pupils almost exclusively, in the other two the teaching assistants support pupils who need help at particular instances and their role does not focus on supporting the VI pupils specifically. We now present an account of the study's first phase of data collection, as this is the phase conducted so far.

We collected data through observations of 26 mathematics lessons (30 hours in total); individual interviews with 5 class teachers (five interviews, 2 hours in total); individual interviews with 4 teaching assistants (four interviews, 2 hours in total); focussed-group interviews with 27 pupils (ten interviews, 1.5 hours in total); and, one ten-minute individual interview with one pupil. During observations, written notes were kept in all lessons. 18 lessons were audio-recorded and 12 lessons out of them were video-recorded, too. All teaching staff and pupil interviews were audio-recorded, except three, during which written notes were kept due to the interviewees' preference.

Each method of data collection used in the study serves distinct purposes. The main focus of the observations is to report classroom evidence showing inclusion, exclusion, enabling and disabling of VI pupils. Such evidence is reported in the discursive activity of teaching staff and pupils in the mathematics classroom. The main focus of the teaching staff interviews is to gather their perspectives on inclusion and disability. Finally, the main focus of the pupil interviews is to gather the pupils' experiences of learning mathematics in the particular classrooms.

[1] "Congenital" and "adventitious" have to do with the age of onset of visual impairment. Congenitally VI are the individuals who have been born with visual impairment while adventitiously VI are the individuals whose visual impairment has appeared later in their life.

We have coded the names of classrooms[2] and of teaching staff[3] and we have used pseudonyms for the names of pupils.

Phase 1 data collection was completed in March 2018 and data analysis of the Phase 1 data is starting now. There are currently two focal points: teaching staff and pupil discourses related to inclusion/exclusion of VI pupils in mathematics classrooms; teaching staff and pupil discourses related to enabling/disabling of VI pupils in mathematics classrooms.

With regard to the first focal point (inclusion/exclusion), we are currently scrutinising the data for evidence of the following: discourses related to academic inclusion/exclusion of VI pupils; discourses related to social inclusion/exclusion of VI pupils.

Here we present a sample of this first scrutiny of our data focusing on one episode which illustrates variation in inclusion and disability discourses: first within the teacher herself and then between the teacher and the teaching assistant. In this episode, the focus is primarily on academic inclusion/exclusion of VI pupils.

A Y3 episode

The following episode was extracted from a lesson on addition and subtraction as inverse operations in Week 2 of the observations during Phase 1. It comes from S1Y3. We first present a factual account – and then a preliminary analysis – of the episode. We conclude with a discussion in which we zoom out of the particular episode and into our analysis of the whole Phase 1 dataset.

As contextual information about S1Y3, we note the following: Fred has severe visual impairment in both his eyes. Ian is VI in one eye and sighted in the other one. TA1a works with them individually, sits in between the two pupils and supports them both perceptually (namely, facilitating their sensory access to materials and resources that may be impeded due to their visual impairment) and substantively (namely, communicating with them about the mathematical content of the lessons). TA1b is the general teaching assistant of the class.

A factual account of the episode

In order to check that 216 is the sum of 176 and 40, the teacher writes the subtraction 216-176 on the interactive whiteboard using the column method. She asks the class what she should write each time in order to find the difference. The class finds the units' digit correctly and the teacher writes the digit in the units' place on the interactive whiteboard. Fred and Ian have access to the interactive whiteboard through an iPad and a computer, respectively. They sit at the front, with TA1a sitting in between them. Some sighted pupils sit on the carpet and others on their tables.

The class struggles with "1 take away 7" (the tens' column) and the teacher asks three sighted pupils to stand up on the carpet facing the rest of the class. She gives a place value hat to each of the three pupils to put on – one hat labelled "H" (for Hundreds), one hat labelled "T" (for Tens) and one hat labelled "O" (for Ones).

[2] The name of each classroom consists of two main parts collated with each other: SNumber YNumber. "S" signifies "School" and "Y" signifies "Year group".

[3] We use "T" for "Teacher" and "TA" for "Teaching Assistant". The names are followed by a number, which signifies the school in which each of the staff teaches. In cases where there is more than one teacher or teaching assistant in a class, the number is followed by a small letter.

The teacher creates 216 with concrete base 10 blocks, giving 2 blocks of Hundred to the 'Hundred pupil', 1 block of Ten to the 'Ten pupil' and 6 Ones to the 'One pupil'. She subtracts 176 gradually: she first removes the 6 Ones from the 'One pupil', ending up with 0 Ones; she then exchanges 1 Hundred of the 'Hundred pupil' with 10 Tens, which she then brings and gives to the 'Ten pupil'. Before completing the subtraction with the blocks, she returns to the incomplete column subtraction on the interactive whiteboard and explains what she has done with the Tens and the Hundreds, drawing on her previous actions with the concrete base 10 blocks. She then returns to the concrete blocks to complete the rest of the subtraction steps, which she subsequently follows on the interactive whiteboard.

TA1a asks Fred to use his iPad and zoom in with his camera so that he can see the teacher's actions. Ian's computer does not have such a function.

TA1a helps Ian follow the teacher's actions, drawing each of the teacher's subtraction steps on a whiteboard, placed in front of her and next to Ian, in the following way (Figure 1):

Figure 1: How TA1a illustrated the subtraction 216-176 for Ian[4].

A preliminary analytical account of the episode

When the teacher works on the subtraction on the interactive whiteboard

We see evidence of inclusion and enabling of Fred and Ian in this part of the lesson. The teacher includes both Fred and Ian through providing them with assistive technology – an iPad and a computer, respectively – connected to her computer. This connection allows the VI pupils to be part of the lesson, as it helps them access the teacher's work on the interactive whiteboard independently and at the same time with the rest of the class. The only difference in the VI pupils' case is that the teacher's work is mediated through a different tool – an iPad and a computer – and not the interactive whiteboard. Furthermore, through the provision of assistive technology connected to her computer without any technical problems, the teacher enables the VI pupils to access her work on the interactive whiteboard, without missing any of this work. Therefore, in this part of the lesson, the teacher both includes and enables the VI pupils. The inclusion and enabling are achieved with the same practice – the provision of assistive technology connected to her computer without any technical problems arising.

When the teacher works on the subtraction with concrete base 10 blocks

We see evidence of exclusion and disabling of Fred and Ian in this part of the lesson. We see exclusion in the use of at least one practice which, albeit concrete, is not considerate of the VI pupils' sensory needs and divides the class into two groups of

[4] The shape of the pictorial representations in each column is similar to that of the concrete base 10 blocks, used by the teacher.

pupils: the sighted pupils, who can access this practice, and the VI pupils, who cannot. The practice is that of exchanging of 1 of the Hundreds with 10 Tens, which are then added to the pre-existing 1 Ten and allow the subtraction in the tens' column to be carried out. The teacher's practice also disables the VI pupils because it is not designed to be accessible to them (at this distance, they cannot see what the teacher does). Therefore, in this part of the lesson, the VI pupils are excluded and disabled by the teacher through non-access to her concrete demonstrations.

When TA1a asks Fred to use his iPad to access the teacher's work with the blocks

TA1a's practice aims at including Fred and enabling him to access the teacher's work with the blocks. Indeed, with the zooming in function of his iPad's camera, Fred is invited to participate alongside those who have access to the teacher's work and he is enabled to access it, too. Therefore, in this action of TA1a, we see evidence of Fred's inclusion and enabling.

When TA1a works with Ian on the whiteboard

We also see evidence of Ian's inclusion and enabling through TA1a's work with Ian towards accessing the teacher's work with the blocks, albeit with a different mediational means to Fred's. While the iPad allows Fred to independently access the teacher's work directly and at the same time as it occurs, the lack of zooming in function of Ian's computer camera makes TA1a be the mediational means, with the help of a whiteboard too, for Ian. Ian is allowed to access the teacher's work indirectly, through TA1a, and with some delay compared to the rest of the class. The delay is attributed to TA1a, who is the mediator between the teacher and Ian, looking at each of the steps that the teacher follows and then adapting these steps to Ian's needs using a whiteboard.

Brief discussion of the episode

This episode is selected to evidence teachers' and teaching assistants' different practices of inclusion/exclusion and of enabling/disabling. Teachers frequently implement practices addressed only to the sighted community of learners and, as a result, they exclude the VI learners from the particular parts of the lesson. They also often rely on teaching assistants for the inclusion of VI pupils. The following excerpt from T1a's Phase 1 interview evidences this reliance on TA1a for the inclusion of the VI pupils in her mathematics lessons, particularly in those occasions when she carries out a demonstration at the front of the class: "[I]f you're modelling something at the […] front of the class and you can't really see that to access it, so it's making sure you've then got someone else in the class that can model what you are doing, do exactly what you are doing".

In this episode, the intervention of the teaching assistant was vital for the inclusion and enabling of the two VI pupils, who would have been excluded and disabled if they had had to follow the teacher's practice through their eyes and without using the additional mediational means. The teaching assistant's sitting in between the two VI pupils allowed her to readily realise that the pupils had no access to the teacher's work and to promptly intervene.

Despite its inclusion and enabling intention, the teaching assistant's intervention did not result in the inclusion of Fred at all times during the teacher's work with the blocks. Frequently, Fred appeared to disengage, by not focusing his iPad's camera on the teacher and by focusing it instead on other things irrelevant to the lesson that captured his attention. While we do not elaborate this issue of Fred's engagement further here – the focus of the episode in this paper is on the teaching staff's actions – we stress its importance and we note that our subsequent analyses will focus very intently on said elaboration.

Fred's responses in this episode exemplify another potent focal point in our emerging analyses: the cases where VI pupils choose to disengage, to self-exclude from the lesson despite being offered an alternative that would allow them to be included. We also note as of potential interest in our developing analyses that the teacher uses a concrete, tactile practice with the sighted community of learners while the teaching assistant invites the VI learners to use their limited vision rather than their touch to access this practice. In Fred's case, his access to the teacher's practice is achieved with the zooming in function of his iPad's camera while in Ian's case, such access is achieved with the teaching assistant's transformation of the teacher's concrete practice into a pictorial, visual one. At face value, the teacher's work on a tactile practice with the sighted pupils – and the teaching assistant's invitation of the VI pupils to access this tactile practice through their limited vision – may look paradoxical. We discern here though the possibility that what TA1a does resonates with a broader set of institutional and teaching staff's perspectives and practices which prioritise vision as a prevalent sense for learning and working in mathematics. Our ongoing analyses explore this further.

We now conclude with implications that our conclusions from this episode have for our ongoing analyses and for the second phase of the study.

Concluding remarks, also towards Phase 2 of the study

The conclusions from the above episode have several implications for the second phase of the study. One of the implications concerns the teaching staff's role in the VI pupils' inclusion. We are currently designing lessons in a way that brings the teacher into a position of sole responsibility for the inclusion of the VI pupils – and in line with the analogous responsibility she has for the rest of the class. While ensuring the VI pupils' inclusion by the teachers, in close collaboration with the class teachers, we engineer the lessons so that the teacher can ask the teaching assistants to support pupils who need help at particular instances (we noted this need in at least half of the lessons during Phase 1). Another implication of the analysis we discussed briefly in this paper concerns paying attention, to the greatest extent possible, to implementing inclusive teaching practices across the whole class, rather than differentiating practices for sighted and VI pupils – was the frequent occurrence in Phase 1, including the episode we discussed in this paper. With the teacher being the only one responsible for the inclusion of the VI pupils and with designing practices which are common to the whole class, we argue that we can achieve better inclusion of VI pupils in the mathematics classroom. We see better inclusion as being achieved when VI pupils feel included in the lesson: they do not self-exclude and are happy to be part of the lesson.

Another implication of the above episode, which we are currently considering in Phase 2, concerns the institutional and teaching staff's perspectives on vision as the prevalent sense for learning and working in mathematics. Rather than aiming to always take advantage of the limited vision of the VI pupils - and thus use typically visual ways to teach mathematics - we are designing lessons with the participating teachers that invite the whole class to experience mathematics also through non-visual ways. In our current collaboration with teaching staff on said design, we also explore their perspectives on the feasibility of this invitation and we examine potential benefits that this invitation may bring to the class.

Acknowledgements

We would like to thank wholeheartedly: the regional educational authorities who have supported the study and have facilitated access to classrooms; schools' head teachers and Special Educational Needs and Disabilities Coordinators (SENDCOs) for their support and encouragement; and, the University of East Anglia for funding the study through a Social Sciences Faculty Doctoral Studentship. Most importantly, we would like to thank teaching staff and pupils, who have embraced participation in the study.

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Dotty triangles: two different approaches to analysing young children's responses to a pattern replication activity

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This article presents results from an exploratory study into developing pattern awareness with children aged 3 to 5, following the work of Mulligan and Mitchelmore (2009) on Awareness of Mathematical Pattern and Structure (AMPS). When the children copied a 6-dot triangular pattern, we similarly found diverse responses, which we analysed using the AMPS levels and then Biggs and Collis' SOLO taxonomy. The latter approach revealed that children responded to up to 5 elements in the pattern. This approach allowed us to identify positively the beginning stages of structural understanding, when children recognised 1 or 2 elements of the pattern. It also emphasised the challenge that the apparently simple task of copying an image can present to young children.

Keywords: patterns; early years mathematics; SOLO taxonomy

Introduction

As humans we are particularly prone to search for regularity and patterns in our environment: for example, in music we find pleasure in listening to notes arranged in a predictable manner and with a regular rhythm, whereas we tend to dislike random sounds (Orton, 1999). But it is in mathematics where pattern comes to the fore, with mathematics referred to as the 'science of patterns' because it involves the search for, construction and communication of patterns and regularity (Smith, 2003).

Young children's pattern awareness has recently been linked to general mathematical competence and to be predictive of later achievement. Mulligan and Mitchelmore (2009) identified that young children had different levels of Awareness of Mathematical Pattern and Structure (AMPS) which was consistent across pattern types (repeating, spatial, growing) and modes (spatial regularity, colour, shape, number). Children with higher levels of AMPS tended to also perform better in other measures of mathematics (Mulligan & Mitchelmore, 2009). Rittle-Johnson, Fyfe, Hofer and Farran (2017) identified patterning at age 5 (assessed in terms of repeating patterns) as predictive of mathematics achievement at age 11. Furthermore, Papic, Mulligan and Mitchelmore (2011) found that the AMPS levels are not immutable: with focused teaching, pre-school children's AMPS scores could be improved, with positive effects on their mathematics, particularly with regard to number and pre-algebraic thinking. Recent studies have also shown that teaching pattern awareness can have particular benefits for the mathematics of low achieving or disadvantaged children (Papic et al., 2011; Kidd et al., 2014). However, Kidd et al. (2014) also conclude that the mechanisms whereby pattern instruction helps maths performance are currently unknown. Rittle-Johnson et al. (2017), while recommending a greater focus on patterning in pre-school and the early primary grades, also argue that 'patterning knowledge requires more attention in theories of mathematical development' (p. 12), and that more reliable and appropriate assessments are needed.

We consequently became interested in developing ways of teaching pattern awareness in order to enhance young children's mathematics. We also felt that pattern as a mathematical topic would be likely to appeal to the various talents and interests of both early years teachers and children. Early years teachers in England are required to teach pattern: it is included within the spatial mathematics goal for five-year-olds (DfE, 2017). However, there is no clear progression in learning, as it is subsequently included in the national curriculum first within Numbers and then within Geometry (DfE, 2013). It therefore seemed a relevant and potentially fruitful area for collaborative enquiry with teachers.

Our study originated from an Australian research programme which developed a pattern awareness assessment (PASA: Mulligan, Mitchelmore & Stephanou, 2015) and teaching programmes (PASMAP: Mulligan & Mitchelmore, in press; Papic, et al., 2011). We used and adapted several tasks from the PASA assessment and then used this information to develop an intervention. In this article we focus on one task from the assessment which involved copying and extending a 6-dot triangular pattern. We found that children gave a surprising variety of responses that did not fit easily with the AMPS levels. We therefore reanalysed the children’s responses with the SOLO taxonomy (Biggs & Collis, 1982). Hence, the research question we examined in this study was: What affordances do the two frameworks—AMPS and the SOLO taxonomy—offer when analysing young children’s response to copying a 6-dot triangle?

Literature review and theoretical framework

Pattern may be defined in many ways, but mathematical patterns must involve some kind of regularity (Orton,1999). Papic et al. (2011) regard pattern as including 'any replicable regularity', which may include ‘simple repetition’ (p. 238) or 'consistent relations' between elements (p. 240). The Erikson Early Math Collaborative (2018) define pattern as a sequence with a rule.

Developing pattern awareness is considered important because it develops mathematical thinking: recognising pattern structure involves the analysis and simplification of complex information, focusing on mathematical relationships while ignoring other features (Rittle-Johnson et al., 2013). Even spatial single object patterns, such as a triangle, give an opportunity for abstracting and generalising: ‘the aim is to find consistent relations within specific categories of geometrical shape’ (Papic et al., 2011, p. 240). This type of pattern includes arrangements of dots on a dice, recognition of which develops important subitising skills (Sarama & Clements, 2009). According to Mitchelmore and Mulligan (2009), AMPS also includes a motivational tendency to seek and analyse patterns.

According to the AMPS framework, children with low levels of pattern awareness may recognise features of a pattern but not the way they are organised, whereas those with high levels will recognise and generalise the pattern structure to other contexts. The definition of the AMPS levels was partly derived from Biggs and Collis’ (1982) generic SOLO taxonomy (Mulligan & Mitchelmore, 2009), which analyses the quality of children’s learning on a particular task. The SOLO taxonomy proposes five possible progressive levels of responses to a task: when the child does not give an appropriate response to the task, their response is at the *pre-structural* level. At the *uni-structural* level the child only focuses on one aspect of the task, whereas at the *multi-structural* level the child focuses on several relevant aspects of the task but treats them as if they are independent. At the *relational* level, the child has integrated all the aspects of the task into a coherent whole and at the *extended abstract* level the child can generalise the knowledge to a new topic. Other research has shown that 4- to 6-year-old children transition from focusing on only one aspect of a task to coordinating their attention on two dimensions (Case & Okamoto, 1996), or from a *uni-structural* understanding of tasks to a *multi-structural* understanding of tasks. This implies that children's pattern awareness may relate to more generalised measures of learning quality: we therefore decided to use both measures to analyse children's responses, to see what insights this gave us as to their interpretation of the pattern.

Methods

Participants and Setting

All of the participants came from four schools in an inner-borough in London, UK. All of these schools had pupils from a wide variety of minority ethnic groups, with a higher proportion than average who spoke English as an additional language (EAL). There were 26 children aged between 36 and 62 months at the beginning of this study, with fourteen of them in reception and twelve in nursery. Fourteen were girls and twelve were boys. Fourteen of the children were identified by their teachers as high achieving in mathematics and twelve as low achieving. There were 15 children aged between 43 and 69 months during the post-assessment period of this study, with five children in reception and ten in nursery. Nine were girls and six were boys. Ten had been identified by their teacher as high-achieving in mathematics and five as low-achieving. Twelve children were unavailable for reassessment for a variety of reasons. One child with special educational needs only participated in the post-assessment because of communication difficulties at the beginning of the year.

Procedures

Either the teachers or the researchers conducted pre- and post-assessments (derived from Mulligan et al., 2015); these included copying and extending an ABC pattern, creating an AB border pattern, copying and extending a triangular pattern, and subitising eight dots. While our main study included an intervention (for intervention activities, see Gifford, 2017) to help the children improve their understanding of pattern, this article will focus on how the children copied the triangular pattern (see Figure 1).

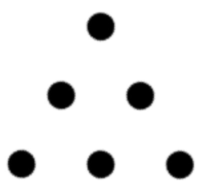


Figure 1: Triangular pattern that the children were asked to copy

In the original PASA assessment (Mulligan et al., 2015) the children were asked to draw the image of 6 dots that they had only seen for 2 seconds, and then asked to extend the pattern. As we were working with younger children we decided to follow the protocol in Papic, Mulligan and Mitchelmore (2011), so asked the children to copy the pattern in front of them.

Data Analysis

The data comprised the children's drawings and notes of what they said as they were asked to copy the triangular pattern. Responses were initially assigned levels using Mulligan et al.'s (2015) criteria. As we reviewed the children's varied responses to the problem, we noticed that these did not fit easily with Mulligan et al.'s (2015) levels and so we reanalysed the data using the SOLO taxonomy (Biggs & Collis, 1982). We found the new levels more closely reflected what these young children were producing; we then chose exemplars to illustrate these.

Results

Mulligan et al.'s (2015) scale

We found that few children could accurately copy the 6-dot pattern (see Table 1). Only 19% of the children could do this at pre-assessment and 33% at post-assessment.

Table 1: Pre- and post-assessment levels in comparison with Mulligan et al.'s (2015) levels

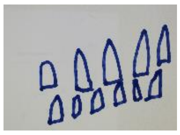
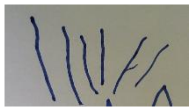
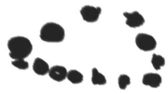
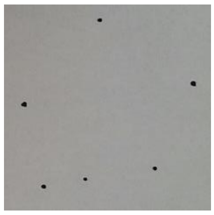
Levels	**Pre-assessment % of 26 children (actual numbers in brackets)**	**Post-assessment % of 15 children (actual numbers in brackets)**
1 Pre-structural: Does not copy the given pattern	23 (6)	7 (1)
2 Emergent: Draws a triangular group of dots not arranged in rows	50 (13)	53 (8)
3 Partial: Draws a triangular group of dots not correctly arranged in rows	8 (2)	7 (1)
4 Structural: Draws a correct copy but an incorrect extension	19 (5)	33 (5)
5 Advanced: Draws and extends the pattern correctly	0 (0)	0 (0)

We found that, using Mulligan et al.'s (2015) classification scale, most children were classified as either pre-structural or emergent at both assessment points (19 children or 73% at pre-assessment and (9 children or 60% at post-assessment: see Table 1). Despite these similar classifications we saw patterns of finer gradations within the children's responses that we thought would give us more information. For example, using Mulligan et al.'s (2015) scale both the child who just scribbled and the child who drew six lines were classified as pre-structural (see Table 2), even though the latter had clearly responded to the image.

SOLO taxonomy

We subsequently reanalysed the data by looking at the number of pattern elements the children represented and using the SOLO taxonomy (Biggs & Collis, 1982). There were five possible elements to the pattern: the shape of the dots, the numerosity of six, the triangular shape, equal spacing and rows (see Table 2). We numbered the pre-structural level as 0, for children who either made no response to the prompt, scribbled, or wrote something unrelated to the prompt (e.g. writing the numerals 1-8). Children at the uni-structural level (1) represented only one element of the pattern, either the dots, the number, or the triangle shape. Children at the multi-structural level (2) focused on two elements of the pattern, which were either dots and the number, dots and the shape, or rows and the dots. Children who focused on three elements of the pattern—either dots, spacing and shape; dots, number and rows; or dots, rows and shape—were assigned level 3, which we later decided was transitional. At the relational level (4) children represented at least four elements and correctly replicated the image. However, we noticed they had produced this in different ways. Some children put dots along the sides of the triangle, some placed the dots in rows, and one placed dots at the corners and then put dots at the mid-points of the sides. We noted that the six-dot arrangement could be seen as six dots forming the sides of a triangular space, rather than as three rows, of one, two and three dots.

Table 2: Elements of the pattern represented by the children

Revised Levels	Descriptor	Examples	
0 Pre-structural	The child makes no response, scribbles, or writes something unrelated to the prompt.		
1 Uni-structural	The child focuses on one element of the pattern, either the dots, the number, or the triangle shape.	 **Triangles**	 **Six** lines
2 Multi-structural	The child focuses on two elements of the pattern, either dots and number, dots and shape, or rows and dots.	 **Dots** in a **triangle**	
3 Transitional	The child focuses on three elements of the pattern: either dots, spacing and shape; dots, number and rows; or dots, rows and shape.		**6 dots** drawn in **rows**
4 Relational	The child produces at least four elements of the pattern, either placing the dots as sides of the triangle or in rows.	 	**6 dots** drawn as sides of the **triangle** with roughly **equal spacing.** **6-dot triangular** pattern built up **row by row**.

Re-analysis of our data allowed us to discriminate children's responses in greater detail and more positively, especially at the lower levels. Rather than 23% being assessed as pre-structural and 50% as emergent, the children's responses were distributed across three levels (see Table 3), identifying their attention to one or two structural features of the pattern.

In both scales there is an interesting dip at level 3, suggesting that there is not a gradual progression in the number of elements that children notice. This is likely to be because level 3 is a transitional state, which occurs when children are beginning to focus on the relationships between the elements but have not yet coordinated all the relationships needed to see the pattern as a whole (Biggs and Collis, 1982). Siegler (2006) regards this as a fleeting but vital state for learning, as it occurs only while cognitive change is happening.

Table 3: Percentage of children in each category

Percentage of children numbers in brackets	0 Pre-structural	1 Uni-structural	2 Multi-structural	3 Transitional	4 Relational
Pre-assessment % (26)	12 (3)	27 (7)	35 (9)	12 (3)	15 (4)
Post-assessment % (15)	0 (0)	7 (1)	47 (7)	20 (3)	27 (4)

Discussion

While Mulligan et al.'s (2015) classification scale was a useful starting point for analysing children's responses to the 6-dot triangular pattern, it was not sufficient. There was a need for a finer-grained scale, particularly at the lower end of the scale where the AMPS scale grouped children who were beginning to show appreciation of the structure of the pattern with those who demonstrated no understanding of structure.

We suggest that the 6-dot triangular pattern was difficult for the young children to copy because they see it as composed of five separate elements: dots, number, triangular shape, equal spacing and rows. When there are multiple features to focus on, young children have to decide where and how to focus their attention. Some children can only focus on and represent one element at a time, some can integrate 2 or 3 elements but only a few can relate all the elements to produce a whole that resembles the original. This supports Papic and Mulligan's (2005) finding that some children can only see one element in a pattern whereas others can spot multi-modal patterns. One interesting finding was that, whereas we had expected children to interpret the image as a growing pattern, they saw a different, but equally valid structure, interpreting it as a spatial single object, or an empty triangle.

Using the SOLO taxonomy (Biggs & Collis, 1982) rather than Mulligan et al.'s (2015) classification scale we can see more developmental growth because it allows for more discrimination at the lower levels. It is plain that this taxonomy is a potentially useful way of measuring young children's quality of learning: There was a good spread of responses across the first four levels of the taxonomy with a dip in the transitional level suggesting a cognitive change in learning to relate the various elements together. While none of our children attained the top extended abstract level, there was sufficient discrimination between the levels to comment on the quality of the children's understanding.

The fact that the SOLO taxonomy (Biggs & Collis, 1982) was a good model in this instance implies that children's developing pattern awareness may reflect a more generic development in the number of elements they can pay attention to at once and in their appreciation of complex images. It highlights young children's difficulties in focusing on an image as a whole, requiring them to synthesise multiple elements. However, pattern contexts provide an appropriate level of complexity to discriminate levels of learning among young children and can give us insights into their development and the way they interpret such images.

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Hearing the whistle: how children can be supported to be active and influential participants in mathematics lessons through effective use of assigning competence and pre-teaching

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In this paper, we report on an action research project focused on disadvantaged children, funded by Devon County Council and two maths hubs. The project explored the use of pre-teaching (intervention in advance of mathematics lessons) and assigning competence (intervention during mathematics lessons, focused on raising status) to support children to be active and influential participants in mathematics lessons, successfully accessing the mathematics. Thirty-nine primary teachers from seventeen schools across Devon participated in the project, supported by five maths advisers. Questionnaires, interviews, observations, collaborative research lessons, case studies and end of key stage data were used to provide evidence of the impact of pre-teaching and assigning competence on the participation and the influence of focus children in mathematics lessons and on their outcomes. The interventions had a positive impact on the focus children; the main finding was that the impact relied on the interventions being undertaken by the class teacher.

Keywords: pre-teaching; assigning competence; disadvantage; intervention

Introduction

> "The start of maths lessons used to feel like being in a race where everyone else heard the whistle and started to run but I didn't." (Y6 child)

The aims of the English national curriculum include the following statement:

> "The expectation is that the majority of pupils will move through the programmes of study at broadly the same pace" (Department of Education, 2013, p. 99)

This links to one of the key aims of teaching for mastery as set out by the National Centre for Excellence in the Teaching of Mathematics (NCETM), which is to ensure that **all** children have a deep understanding of mathematics (Askew et al., 2015). Since children come into school with vastly different experiences, one of the challenges is in how to provide each child with the necessary support for them to understand deeply.

> "Our basic task in education is to find strategies which will take individual differences into consideration but which will do so in such a way as to promote the fullest development of the individual… given time, enough, all students can conceivably attain mastery of a learning task." (Bloom, 1968, p.3)

The phrase 'given time, enough' contains a challenge: the challenge of providing additional time for children who need it, in a form that will maximise impact whilst minimising disruption to the rest of their learning. Within NCETM literature related to teaching for mastery in mathematics in England, the practice in Shanghai has proved influential and 'rapid intervention' is suggested.

> "If a pupil fails to grasp a concept or procedure, this is identified quickly and early intervention ensures the pupil is ready to move forward with the whole class in the next lesson." (NCETM, 2016, p.1)

An alternative way to provide additional time, which has additional benefits, is to intervene in advance of mathematics lessons: pre-teaching.

Pre-teaching and assigning competence

During 2015/6 the authors were involved in an action research project focused on teaching for mastery in mixed-age classes. Pre-teaching emerged as a successful strategy for supporting struggling learners to access mathematics in lessons (Trundley et al., 2016).

Research on pre-teaching includes Carnine (1980) who found that pre-teaching component skills for multiplication to at-risk six-year olds increased performance. In addition, Lalley and Miller (2006) compared the impact of pre-teaching and re-teaching eight-year olds and found that whilst both had a (similar) impact on performance, pre-teaching alone had an impact on children's confidence and motivation. A personal account of pre-teaching from a second-grade teacher echoes this finding:

> "Remediation is often a terrible way to help kids catch up. Pre-teaching is more effective and more fun…For the same 20-minute investment of time, we can change the way a child sees himself as a reader, thinker, or mathematician. For children accustomed to struggle, those moments can be transformative… The feeling of confidence can linger long after the class has moved on to the next concept." (Minkel, 2015, para 11 and 26)

The 2015/6 project (Trundley et al., 2016) also explored complex instruction, an instructional approach that aims to achieve equitable classrooms (Cohen et al., 1999), and in particular the idea of assigning competence in order to raise the status of certain children in mathematics lessons. Cohen et al. (1999) found that assigning competence to low-status children increased their participation without a negative impact on the contributions of high-status children.

Assigning competence is not about praising a contribution simply because it has been made; it has to be of value to the whole group:

> "… if student feedback is to address status issues, it must be public, intellectual, specific and relevant to the group task. The public dimension is important, as other students learn that the student offered the idea; the intellectual dimension ensures that the feedback is an aspect of mathematical work; and the specific dimension means that students know exactly what the teacher is praising." (Boaler, 2016, p.134)

The interest in pre-teaching and assigning competence, generated by the 2015/6 project, and the potential for using these approaches to support children who struggle to be involved in maths lessons led to this research project being set up to explore the question: How can pre-teaching and assigning competence be used to effectively support children to access age-appropriate mathematics and be active and influential participants in maths lessons?

In our research we were interested in exploring areas which had not been the focus of previous research: different structures, and content for pre-teaching sessions and how to use pre-teaching in conjunction with assigning competence in order to change children's participation and their influence in maths lessons.

Structure of the action research project

Thirty-nine primary teachers from seventeen schools across Devon participated in the project which ran from September 2016 to July 2017. The teachers involved taught from Y1 to Y6 with the majority of teachers working with Y3 and Y4.

Each school had either a pair or a trio of teachers working together as learning partners. These pairs/trios were grouped to form five geographical clusters; each cluster was supported by a mathematics adviser. The project was part of a focus on closing the gap, a priority for Devon, and each teacher was asked to identify three vulnerable or disadvantaged focus children for the project.

The mathematics advisers supported the teachers with collecting data at the start of the project. Data collected throughout the project included:

- Interviews – the focus children were interviewed pre- and post-project; the interviews were filmed. Selected classmates of the focus children were interviewed at the end of the project.
- Questionnaires – teachers completed questionnaires pre- and post-project and were filmed at the end of the project talking about the impact.
- Observation – the participation of focus children was observed pre-project both in a whole class maths lesson and in their trio exploring a maths question.
- Journals – teachers reflected in journals on the participation and influence of focus children in pre-teaching sessions, in everyday maths lessons and in research lessons. They also collected informal observations from other adults including parents.
- Cluster meeting discussions – these were taped and minutes taken.
- Collaborative research lessons – plans, video and notes from follow up discussion were all used to capture information from the live research lessons.
- Case studies – in advance of the final meeting, teachers gathered end of project data and prepared a case study on one of their focus children.

A **launch meeting** explored the key principles and research underpinning the project and included reading Cohen et al. (1999) and Minkel (2015). Teachers were asked to provide regular pre-teach sessions for their focus children and to explore assigning competence in class maths lessons. Whilst the focus for the project and the research question were established by the maths advisers, the teachers made most of the decisions throughout the project, including the selection of focus children, the structures for pre-teaching sessions, the content of pre-teaching sessions and how to assign competence in lessons.

The **cluster meetings** provided an opportunity for teachers from different schools to meet, to share their experiences of pre-teaching and assigning competence, to explore ideas from related research and to plan their next steps. They also provided some of the planning time for the collaborative lesson research cycles.

The teachers were all involved in four cycles of **collaborative lesson research** (CLR); three in their schools and one joint one which took place in one of the schools. The model for the CLR cycles was based closely on the Japanese model (see Takahashi & McDougal, 2016). For the three cycles of CLR that took place in each school, the teachers had at least two planning sessions to explore the mathematics, to create the detailed research proposal for the research pre-teach session and to identify opportunities for assigning competence in the following research maths lesson. The focus for the observation of the live sessions was on the impact of the joint decisions made on the focus children; this was discussed after the research sessions. The focus children were also asked to reflect on the lesson and how the pre-teaching had helped them. This informed the post-lesson discussion.

The final cycle of CLR brought all the teachers together with the five mathematics advisers. The focus for the research session was decided by the trio of teachers from the host school. The research proposal was drafted by the mathematics advisers with the teacher who would be teaching the lesson. The draft proposal was then shared at the cluster meetings allowing all the teachers to contribute, in particular to anticipate responses and suggest how to deal with these. The live session, in the school hall, was observed by all of the project teachers and maths advisers.

At the end of the project there was a meeting to collect data and share findings; each teacher completed a case study report on one of their focus children.

Project findings

> "We have had the privilege of witnessing teachers change children's lives through this project. Children who had no belief in themselves as learners in mathematics now believe in themselves, and are actively involved in their own learning and in the learning of others." (Trundley et al., 2017, p.3)

Both teachers and children involved in this project reported that the combination of the two strategies (pre-teaching and assigning competence) had a positive impact on levels of participation and on the ability to be influential in lessons. For many of the children it not only allowed them to access age-appropriate mathematics, it also had a positive impact on their attainment in tests.

> "The impact has been extraordinary. The three children who have made the most progress this year have been the three focus children who have experienced more pre-teaching than others." (Y6 teacher)

For the seventeen schools involved in the project, there was an average increase in end of KS2 results of 10.5 percentage points in 2017 compared with 2016. End of key stage data was available for only a small number of individual children as most of the children involved were not in Y2 or Y6. Except in one school, at the beginning of the year, the children selected were children who were not on track to achieve expected at the end of the year. Of the 18 children in Y2, 14 achieved expected (78%) and of the 15 children in Y6, 13 achieved expected (87%).

> "The three children began the year with little confidence in maths and very little confidence in themselves. Their first assessment in December showed them scoring 33%, 28% and 34% respectively…The summer SATs showed the children being incredibly confident …The children's attitude in their own ability had completely changed and they felt that they could answer the questions and had a very reasonable chance of passing. There were smiles on their faces the whole week... Their scaled scores were 100, 106 and 107" (Y6 teacher)

There were five key findings; the first of these was by far the most important.

Finding 1: Pre-teach sessions must be run by the class teacher

The research showed that the pre-teaching benefited the children in the maths lesson because it was run by the class teacher. This was evident in a number of ways. Firstly, the children valued the time because it was with their class teacher.

> "It's more than just maths. It's the process of building engagement and self-confidence." (Y2 teacher)

The pre-teach sessions provided the children and class teacher with a shared experience which gave them a shared understanding and common references which they took into the whole class lesson. This meant the children knew that the teacher knew they knew the mathematics going into the lesson because they had been in the pre-teach session together. There was also a sense of the children wanting to work hard in the lesson because they had been given the extra small group time with the teacher. This additional time with the class teacher was seen as a privilege by the children, it was something that was envied by others in the class.

This was in stark contrast to the attitude towards remedial interventions run by someone other than the class teacher. An issue that was raised early on was the status of interventions in schools and how this has a knock-on effect in terms of the status of the children attending the interventions. Some of the older children, initially, were unhappy at having been identified for the group but this soon changed when they realised it gave them time with their class teacher and that it helped raise their status in the maths lessons.

> "The main challenge has been totally rethinking the way that I see intervention working in my class." (Y6 teacher)

It also provided teachers with an opportunity to reflect on the planned lesson in advance, as the pre-teaching often revealed things which prompted adjustments to make the lesson more effective; in effect it was a pre-teach for the teachers as well as for the children, all of whom were then more focused at the start of the lesson.

Finding 2: Pre-teaching and assigning competence maximise learning in lessons

By having class teachers provide the 'additional time' and putting it before the learning in a maths lesson, rather than after it, children are able to maximise learning in the lesson. This is because it makes the lesson a meaningful experience for the children, rather than experiencing the lesson as something they don't understand, leaving them to feel they have failed. Teachers are also better placed to support the learning in the lesson.

Increased participation and access to the mathematics lesson allowed the focus children to become more influential. Across the duration of the project, assigning competence became easier because the children were contributing more and were focused on the mathematics rather than on their emotions.

Finding 3: Pre-teaching and assigning competence have a positive impact on children's confidence in themselves as mathematical thinkers

An increase in confidence was the most common observation made by the participating teachers about their focus children. Confidence is difficult to measure but it is possible to look at the effects of an increase in confidence. The main effect was increased participation in the class lesson, and an understanding of how to participate as a learner in different situations, indicated by various changes in behaviour including:

- Engagement from the start of the lesson

> "Having a sneaky preview of the lesson gives me a head start and if I didn't have that I wouldn't be ready." (Y3 child)

- Offering contributions and being ready to respond

> "Before I just sat there doing nothing and was unsure. Now I'm like PICK ME because I feel more confident." (Y3 child)

- Asking different questions and seeking out a challenge

> "The children were excited to choose challenging numbers. Children said: I am going to go for the trickiest." (Y4 teacher)

- Accessing resources independently

> "Maths group helps me because I know what is coming up. I feel more confident to have a go. If I forget I can use the working wall or the resources we used." (Y4 child)

- Explaining thinking

> "She won't let me go until I fully understand. She would go through it with me and at the end she would ask me questions just to make sure I get it." (Y6 partner of focus child)

- Supporting others

> "It wasn't just in his head. He actually showed me how to work it out. He showed me more than once which helped me remember it all." (Y3 partner of focus child)

- Active participation in conversations

> "He used to agree with everything I said but now we have discussions about the maths." (Y3 partner of focus child)

- Changes in behaviour and attitude outside of the classroom

> "She is a different child. She talks about what she has been doing in maths at home. She is keen to practise skills at home." (Parent of Y1 focus child)

Finding 4: Pre-teaching can have different structures and focus on different things

The project did not find that a particular structure worked best but there were key elements that had an impact on the success of the pre-teach sessions:

- Pre-teaching must provide children with access to the mathematics in the maths lesson, allowing them to actively participate. It is not about being able to replicate in the lesson the maths from the pre-teach session nor is it about teaching the whole lesson in the pre-teach session. It is about preparing the children to be able to engage in the struggle of the mathematics in the lesson by removing additional barriers.
- Identifying **one** thing that will allow the children to access the mathematics in the lesson. This could include:
    - Introducing new mathematics, images, resources or contexts
    - Rehearsing prior learning
    - Rehearsing language
    - Allowing confusion to happen
    - Exploring misconceptions
- Timing – most teachers found that having the session on the same day as the maths lesson worked best. Some teachers liked to run the session immediately before the lesson whilst others liked a gap as it allowed them time to reflect on how they might want to adjust the lesson in light of the pre-teach session.
- Frequency – teachers varied in terms of how frequently they ran pre-teach sessions but at least once a week worked best, with most teachers running two or more sessions a week. The impact on the focus children in this project relied on their sustained involvement in the pre-teach sessions for the full year.
- Length – there is no set length for a pre-teach session, the important thing is clarity about the purpose of the session and taking the time needed. Fifteen

minute sessions were often needed when the focus was on introducing something new, but sometimes a few minutes immediately before a lesson prepared the children for participation, for example through rehearsing language.

Finding 5: Assigning competence is a powerful tool but can be more challenging for teachers to use effectively.

> "Assigning competence is about drawing attention to a child's thinking that everyone can learn from, rather than drawing attention to a child by getting them to perform. It is about being explicit about the child's contribution that is of intellectual value." (Trundley et al., 2017, p. 25)

Observations from teachers about how to make this work include: be subtle; comment on the thinking or the idea not the child; use simple phrases to draw attention to valuable thinking; anticipate and monitor responses; support other children to publicly state how they have been helped by a class member; subvert hierarchies that exist in the classroom; and attend to classroom culture and school culture.

Conclusion and discussion

Asked what a lesson would look like without a pre-teach, a Y6 focus child replied:

> "A better question would be what would my year look like without it? I would still think I couldn't do maths! What would secondary school look like without it? SCARY!"

This study shows that pre-teaching and assigning competence are two tools which have the potential to increase the participation and influence of low-status children in maths lessons, when used by class teachers, and are particularly effective when used in combination. However, this is a small study; there is a need for further research on a larger scale with a more focused exploration of the impact of frequency and timing for different age groups. Much of the data presented is qualitative and future research would benefit from collecting quantitative data on all the focus children and on a control group if possible.

Throughout the project it was essential to keep the teachers focused on the aim, which was to help the children to be active and influential in maths lessons. Issues arose when teachers thought they should try to cover the whole lesson in the pre-teach session, attempting to prevent struggle in the lesson, and were focused on 'doing'. A pre-teach session is an opportunity to consider the additional barriers that the focus children might have in accessing the struggle in a lesson, preparing them to 'hear the whistle', rather than preventing them from struggling in a lesson.

The biggest barrier for teachers was time. In some schools, teachers struggled to find the time for the sessions, especially when they were not supported by senior leaders. There is a need to explore how senior leaders might support teachers across whole schools with providing pre-teaching sessions.

It needs to be recognised that the year-long project was structured to provide teachers with a sustained professional development experience. The two elements of the project identified by the teachers as being most influential on their own practice, were the collaborative lesson research cycles and the support of a maths adviser; it is not possible to know how influential these were in terms of the impact on learners. Further projects would need to examine whether or not the professional development model was significant.

Acknowledgement

The project would not have been possible without funding and we are grateful for the funding support received from Devon County Council, Jurassic Maths Hub and Cornwall and West Devon (CODE) Maths Hub.

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Challenging the fear: a framework for addressing anxiety in adults learning mathematics

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This paper extends the initial findings of research into undergraduate perceptions related to learning mathematics, where a positive change in attitude was observed in a group of 75 first year education students on the completion of their first mathematics education unit. Analysis of the early stages of research identified a range of factors which may have supported this change, including the role of the teacher and the teaching strategies used, personal perceptions related to learning mathematics and the role of discussion. Focus group discussions were held to probe more deeply into these findings and the role of the teacher and the teaching strategies were identified as the top influencing factors in developing students' confidence in learning mathematics. Triangulated results were analysed against Knowles' six assumptions for adult learning (Knowles, Holton III, & Swanson, 2005) and a framework for supporting the teaching of mathematics education in adults was constructed.

Keywords: attitudes; perceptions; confidence; understanding; teaching

Introduction

Having identified a trend in student anxiety related to learning mathematics when embarking on undergraduate education studies, the early stages of my doctoral research focussed on exploring the views of a cohort of 75 first year undergraduate students who were attending a part-time BA (Hons) Applied Education Studies degree (Wicks, 2014). The purpose was to identify their perceptions related to learning mathematics and any factors that they felt affected them whilst learning. The first mathematics education unit they attended focussed on the development of personal subject knowledge in mathematics and introduced theory related to the learning and teaching of mathematics. This gave rise to the opportunity to explore perceptions about how they felt about learning mathematics and factors that might affect them. Students were asked to complete questionnaires before and after this unit, which explored perceptions related to confidence, understanding and any factors affecting their views of learning mathematics.

68 students completed the pre-teaching questionnaire and 64 the post-teaching questionnaire. Analysis of the questionnaires indicated an increase in the use of positive language related to the learning of mathematics (2:1 positive to negative, compared to 1: 2 in the pre-teaching questionnaire) and an increase in comparative perceptions related to understanding and confidence, whereby 55/64 (86%) identified themselves with a higher, or much higher, level of understanding and 53/64 (83%) identified

themselves with a higher, or much higher, level of confidence. The initial findings from the research also identified a range of factors that had affected the students in learning mathematics during the first unit and these were: matters relating to the teacher – either the teaching strategies used or personal characteristics - being the top rated influence (43/59, 73%), with personal perceptions being the second highest influence (14/59, 24%). Setting arrangements and the role of discussion were the third highest influences identified in the pre and post-teaching questionnaires respectively.

In order to probe more deeply into the findings of the questionnaires, focus group discussions were held with a group of ten students, with the aim of identifying any strategies that they perceived might support them in learning mathematics. This paper focuses on presenting these results and triangulating the findings with the questionnaires and the literature base addressing adult learning.

Perceptions related to learning and teaching mathematics

Concerns regarding adults' perceptions related to learning mathematics have been previously established, with Richardson and Suinn (1972) suggesting that mathematics anxiety can interfere with the process of manipulating numbers and solving problems. Others have expressed similar concerns, identifying that anxiety can limit the ability of those affected in completing mathematical activities (Ashcraft & Krause, 2007; Boaler, 2009; Evans, 2002; Tobias, 1993). Further exploration of such anxieties provides a concern that where teachers are anxious about learning mathematics themselves, this may have an effect on their teaching of the subject, particularly at primary (elementary) level (Ball, 1990; Relich, 1996). In particular, there are those who suggest that those who are anxious about mathematics themselves may pass this on to the pupils they teach (Bekdemir, 2010; Brady & Bowd, 2005; Haylock, 2010).

It is possible to identify a range of influences that affect how people feel about learning mathematics; however there is a consistency in those who suggest that the teacher is the key influencing factor (Bekdemir, 2010; Finlayson, 2014; Hodgen & Askew, 2006). Further exploration of this area suggests that the teacher has the power to create either a positive or negative view of mathematics through either encouragement or humiliation (Bibby, 1999, 2002) and that the influence of the mathematics teacher can affect learners' levels of anxiety about the subject (Ward-Penny, 2009; Ashcraft and Moore, 2009). Whilst the role of the teacher is identified as a consistent influence on how others feel about learning mathematics, other potential influences in this area include the fear of failure in front of others (DCSF, 2008; Welder & Champion, 2011), personal perceptions related to the subject of mathematics (Buxton, 1981; Dweck, 2007), and the potential effect of confusion between relational and instrumental understanding of mathematics (Tall, 2013); however, although these are acknowledged, it is not possible to explore these in depth within the constraints of this paper and the focus is maintained on unpicking the role of the teacher in further depth. .

With the potential that anxiety in learning mathematics may be passed on to students, further consideration needs to be given to how to support adults who are anxious about learning mathematics – particularly those working with children in schools. Coben (2006) suggests that there is limited consideration of how to support adults in this area, but there are those who have considered how adults might learn and could potentially provide a structure for this. In particular, Knowles, Holton III and Swanson (2005) built on the work of Lindeman (1926), offering an andragogical model for adult learning. His six assumptions are based on the idea that adults need to be prepared to learn, rather than just focussed on content, and that they need to be supported in making connections between one aspect of learning and another. This is consistent with the connectionist approach advocated by Klinger (2011) to support adults in leaning mathematics. In order to establish potential strategies to support adults learning mathematics, Knowles et al's (2005) six assumptions for adult learning will form part of the theoretical base for analysis of data discussed within this paper.

Methodology

The aim of the research was to identify strategies to support adults learning mathematics on an undergraduate education degree. Creswell and Piano Clark (2009) advocate the use of a range of appropriate methods to explore a research problem, and hence pre-course audits, pre and post-teaching questionnaires and focus group discussions were used to track students through their first year of study. Survey research was used to explore the students' perceptions before and after their first mathematics education unit, the results of which have already been summarised and shared in previous research and within the introduction.

In order to probe more deeply into the results of the questionnaires, and to identify what factors might have affected any changes in their perceptions, focus group discussions were held with ten students, organised in three groups. Activities were designed to encourage the students to talk with each other, rather than focus on the facilitator. The activities explored students' perceptions about learning mathematics, factors which had influenced them during their mathematics unit and discussions related to the role of the teacher and teaching.

Thematic coding of the pre and post-teaching questionnaires was used to identify factors that might potentially affect how students felt about learning mathematics and arose, as advised by Robson (2011), from interaction with the data. Newby (2010) suggests that such themes (or templates) can be used to examine views in more depth. A template coding approach, advocated by King (2004), was utilised to support a constructivist position, building together a key themes though a range of interpretations of different aspects of the data. Hence template themes were identified by comparing previous literature findings and the two questionnaires. These were coded and the focus group responses were then analysed against the constructed templates.

The sample of students chosen for the full research study was 75 first year undergraduate students enrolled on a part-time BA Applied Education Studies degree in 2011/12. Of these students, all were offered the opportunity to complete the pre-and post teaching questionnaires identified in the introduction. The final phase of the research was to identify students for whom there had been a potential change in perceptions and invite them to be a part of the focus group discussions. Since all of the questionnaires were completed anonymously, students were identified who had made the greatest rates of progress from their initial mathematics test audit completed prior to their first mathematics unit (June 2011) and the mathematics tests associated with the first unit (March 2012). They were then invited to be a part of the focus group discussions. The rationale behind this identification was rooted in the links made within literature identifying a positive correlation between confidence and performance (Ashcraft & Krause, 2007; Buxton, 1981) and within a previously conducted pilot study (Wicks, 2011). Ten students took part in the activities and discussions.

Consideration was given to the reliability and validity of the study, and in particular my role as practitioner-researcher. In order to minimise bias, as advised by Cohen, Manion and Morrison (2007), the process of triangulation was achieved by using two or more methods to examine the research aims. The student audits and pre-teaching questionnaires were designed to explore the students' past perceptions and experiences related to learning mathematics; the post-teaching questionnaires were designed to identify any changes in perceptions, factors affecting these changes and strategies that might support learning mathematics. By using more than one method for each research aim, I aimed to corroborate or question my findings by comparing the different elements of the data (Denscombe, 2010). Although I cannot discount the fact that some participants could have responded in order to please their teacher (BERA, 20110, steps were taken to minimise these. Students were advised in writing that participation was voluntary and that involvement in the research would not affect their studies in any way. To support this, all questionnaires were completed anonymously, where it was not possible to identify any student. For the focus groups, an impartial facilitator would have been preferable, but as this was not possible, the discussions were held once students had completed all assessments related to the mathematics unit, so that they could be sure that there would be no additional implications.

Results

Activity 1 was used to identify whether or not the sample group was representative of those identifying themselves more positively about mathematics than before they started their degree course. Students were asked to identify words they associated with how they felt about mathematics from a group of ten words taken from the pre- and post-questionnaires (strong, weak, fear, interest, easy, confident, unconfident, struggle, enjoy, difficult). They identified these words in the ratio 9 to 22 negative to positive, similar to the findings of the post-teaching questionnaire, indicating that the sample was representative of the whole group in relation to their perceptions of mathematics.

Activity 2 involved students ranking and discussing a range of potential influences identified from the literature base and were those included in the student questionnaires (attendance at sessions, teaching, other students, tests and exams, online materials, discussion boards and blogs, websites, outside influences, drop in sessions, in class discussion). Table 1 identifies the top three influences claimed for each group.

	Focus Group 1	**Focus Group 2**	**Focus Group 3**
1	Teacher Teaching	Attendance at sessions	Teacher Teaching
2	In-class discussion Other students Attendance at sessions	Teacher Teaching	Online materials Websites
3	Online materials Outside influences	Online materials	In-class discussion Other students Attendance at sessions

Table 1: Summary of the top influences in learning mathematics as adults

Activity 3 focussed on unpicking the only theme that had been identified by all students as an influencing factor in the post-teaching questionnaire, and that related to 'teaching' (73% of students). This included specific reference to techniques used within the classroom, such as clear explanations and modelling techniques, along with teacher characteristics such approachability and personal support. In order to explore these two strands further, atudents had flip chart paper on which 'The Teacher' and 'Teaching' were listed as the top. They were asked to identify what characteristics they would put under each heading and how they affected their feelings about learning mathematics.

The discussions within the focus group identified that the characteristics of the teacher helped to make them feel comfortable, specifically nurturing type characteristics and being approachable. Alongside this, the strategies used by the teacher to support learning were also identified as important, in particular in being able to break things down into achievable steps. Table 2 summarises the characteristics identified by the three focus groups.

Teacher characteristics	Teaching characteristics
• Good subject knowledge • Nurturing characteristics • Clarity in use of language and explanation	• Clear demonstrations to break down methods • Range of teaching strategies • Suitable pace (not too fast or slow) • Time for practice • Opportunities to ask questions

Table 2: Summary of characteristics related to teaching

Activity 4 allowed the students to discuss the second- and third- rated influencing factors identified in the post-teaching questionnaire, related to discussion with others and personal perceptions. Students were firstly asked to explore what effect discussion and working with others had on their learning of mathematics and this was followed up with a discussion about how their own personal view about mathematics had affected them.

Students acknowledged that working with others could be supportive in developing understanding, but only if peers were like minded. Discussions related to personal perceptions acknowledged that there were a range of positive and negative perceptions related to learning mathematics, and some surprise that students knew more than they originally thought!

Discussion and implications

The aim of the research was to identify strategies that may support adults learning mathematics on an undergraduate education degree. The results of the post teaching questionnaires and focus group discussions has supported this process, in particular in identifying three top influences related to learning mathematics and a consistency with others exploring this area: firstly the teacher and teaching, similar to the findings of Hodgen and Askew (2006) and Finlayson (2014); secondly, the role of personal perceptions, as identified by Dweck (2007) and Tall (2013); and finally the role of discussion (Vygotsky, 1978; Wittgenstein, 1978). However, as matters related to the teacher and teaching were identified as having the highest positive influence on how students felt about learning mathematics, this forms the main focus of this discussion.

Template coding analysis of the audit, pre and post-teaching questionnaires and the relevant literature base supported the identification of five themes relating to: the teaching; the teacher; personal perceptions; the role of others and setting arrangements. Aiming to identify strategies that might support adults in learning mathematics, the focus group discussions were analysed against these original templates, and then refined with additional detail. All strands of data relating to the teacher identified the need to have good subject knowledge and nurturing type characteristics. The strands relating to teaching revealed similarities to Knowles et al's (2005) six assumptions for adult learning and as a result seven teaching strategies have been constructed for consideration when working with adults learning mathematics. With the suggestion by Klinger (2011) that adults needed to be supported to make connections in mathematics, the analysis identifies that Knowles' framework could potentially be extended to focus specifically on mathematics. Table 3 identifies the seven proposed teaching strategies alongside these assumptions. There is no hierarchical order.

Teaching Strategy (TS)	Student Comments	Assumption
TS1: Clear modeling and explanation of strategies TS2: Break down mathematics to show how each area is developed	Students valued clear explanation to support understanding of why and how they were exploring specific areas of mathematics	The need to know Motivation
TS3: Make connections between different aspects of mathematics.	Students identified the need to build on understanding and make connections to other aspects of learning.	The role of experience
TS4: Have a pace appropriate to the level of students	Students identified the need to have a pace appropriate to their level of understanding	Readiness to learn
TS5: Allow time for questioning and discussion TS6: Provide time for practice within sessions	Students identified the need to be able to ask questions and for the teacher to respond to individual needs. Time to practice concepts also allowed for individual support.	Readiness to learn Orientation to learning
TS7: Provide (online) practice materials outside of teaching sessions	Students identified a need for personal practice outside of teaching sessions	The learner's self-concept.

Table 3: Identification of seven proposed teaching strategies to support adults learning mathematics compared with Knowles' six assumptions for adult learning

The findings from the study suggest that there had been a change in perceptions related to learning mathematics by the end of the students' first mathematics unit on their undergraduate education degree. A higher proportion of students identified a positive disposition towards mathematics than at the start of the course, including higher levels of perceived confidence and understanding. Analysis suggests seven underlying teaching strategies could be considered to support adults learning mathematics; however, it is also acknowledged that other factors may also play a part, in particular the role of others and personal perceptions and that this requires further research.

The results of the study and the application of these proposed teaching strategies now need to be explored through further research and across a wider range of experiences involving adults learning mathematics. Plans are in place to extend this research with colleagues to allow for a wider range of independent discussions with students, and further exploration of additional theoretical frameworks related to adults learning and teaching mathematics.

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Pre-service teachers' perceptions of theory – the case of compressed knowledge in mathematics

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This paper aimed to examine how pre-service teachers perceive the theory of compressed knowledge in mathematics; hoping to bring new knowledge to the issue of how to link theory and practice in initial teacher education. Twelve pre-service teachers attended two focus groups (one prior to a teaching placement and one after it). Participants were selected using the following break characteristics: self-assessed reflectivity and receptiveness to theory; self-reported judgement of whether the idea of compressed knowledge was useful. The data showed that both the training provider and the accessibility and relevance of the theory are influential in how much pre-service teachers value theory. In addition, compressed knowledge theory was interpreted in two distinct ways: as a 'tool' or as a 'source of enlightenment'. A model of how interpretation of theory may evolve is proposed including the need for a 'state of perplexity' during practice to move a theory into knowledge for future action.

Keywords: Pre-service teacher; theory; compressed knowledge

Introduction

What is compressed knowledge in mathematics?

In 2000, Ball and Bass proposed that teachers often hold their mathematics content knowledge in a desirably compressed form, allowing them to be fluent and competent. Ironically, however, knowledge held in this final polished form is inadequate for teaching as it makes it difficult for teachers to discern how learners are thinking and requires them to unpack their content knowledge to ensure that it is visible and accessible to their learners (Ball & Bass, 2000). Thomas (1997), following analysis and amalgamation of the previous twenty-five years of theory definition, concluded that theory in education should be defined as an idea or a possible explanation that comes from a culmination of intellectual endeavour. Ball and Bass' (2000) ideas of compressed knowledge are exactly that; a possible explanation coming from a culmination of intellectual endeavour. For the purposes of this paper, therefore, I shall classify compressed knowledge in mathematics as theory, and will refer to it as CKT.

Theory in initial teacher education

The divide between theory and practice for pre-service teachers (PSTs) has pervaded the literature for decades (Dillon, 2017). I decided to use CKT as a platform to explore this issue due to a number of similar experiences I had with PSTs. During seminars, PSTs seemed to gain great insight and comfort from the idea that their knowledge may be compressed when they felt frustrated at their inability to clearly explain mathematical ideas to children or to their peers during micro-teaching. I was intrigued. What is it about CKT that appears to make it more useful and immediately accessible than other 'more standard' learning theories that are commonly seen not to impact on practice? The following research question therefore emerged: How do Primary PSTs perceive CKT? -do they value it? -how do they interpret it?

Research background

PSTs' perceptions of theory

Conclusions regarding PSTs' perceptions of the validity of theory are hugely disparate even when using large samples. Tang et al. (2016) found that, whilst a proportion of their PSTs valued theoretical input and could articulate the impact it had on their practice, a significant proportion stated that theory was not valid information. Hobson (2003) also found that approximately 20% of respondents to their questionnaire stated that theory was of no value to their development as a teacher. In his interviews with PSTs, White (2005) found that some PSTs interpreted theory as something that had be used or applied as it was, whilst others felt that you could pick specific aspects of a theory and could mix theories together. To further understand PSTs' interpretations of theory, links can be made to the suggestion that use of theory falls into two categories: it becomes knowledge for action or it becomes knowledge for understanding (Eraut, 2003). In a study of PSTs in four settings in Northern England, Hobson (2003) was able to categorise the PSTs into two main interpretations of theory. He found that the majority of PSTs interpreted theory as wider knowledge that may or may not influence their practice; perhaps suggesting theory becoming knowledge for action. Contrastingly, a small minority interpreted it as a vehicle to aid their understanding of practice; perhaps suggesting theory becoming knowledge for understanding.

PSTs' perceptions of CKT

Although there is no literature about PSTs and CKT, case study analysis of practising teachers shows that decompressing their knowledge supports them to plan series of lessons (e.g., Provost, 2013) and helps them to successfully select and sequence examples (e.g., Burke, 2013).

Factors which may affect PSTs' reported perceptions of theory

Differences in reports of PSTs' opinions on the validity of theory in may reflect a genuine continuum but may alternatively be due to other factors. For example, both Smith and Hodson (2010) and Williams and Soares (2002) acknowledge the potential influence of lecturers when reporting that their participants all seemed to value theory to an extent. In conjunction, in a study which deliberately collected data from nine PSTs *before* any formal training had begun, Holt-Reynolds (1992) found that the most common response to theory was downright rejection. In addition, in-service teachers have been found to find theory inaccessible due to the language used (Zeuli & Tiezzi, 1993), whilst Dye (1999) found that PSTs who saw little value in theory claimed that it was full of 'elitist jargon'. Similarly, PSTs and teachers are reported to state that theory is meaningless if it is not relevant to experience (Davis, 1999).

Methodology

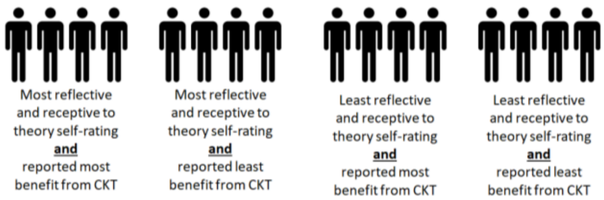
Use of focus groups

Of particular concern in this study is the need to bridge the gap between myself and my students. Focus groups were therefore chosen as they allow insight into the perceptions of participants using their own language (Litosseliti, 2003), and are thought to be an excellent method to help bridge experiential gaps between researcher and participants (Morgan, 1998). Just as the social aspect of focus groups acts as a major benefit, it also leaves it open to criticism due to the multitude of social influences (Gibbs, 1997). To reduce these effects, I held the focus groups in a neutral, non-teaching area of the campus; elevating it above a normal social interaction or seminar situation. I also explicitly highlighted the ongoing relationships between myself and the participants, assuring them that my role in the group was as a researcher. To further reduce any effects of prior relationships with researchers, I chose an assistant who was from another university and was therefore unknown to the participants. I guided the discussion to reduce the sequential and peacock effect and, as advised by Albrecht et al. (1993), I requested that my assistant note down any voice tones or non-verbal subtleties which may indicate social influence.

Participants, questions and a follow up focus group

I chose to sample second-year undergraduate PSTs because I hoped to capture their perceptions post their first year (when they are predominantly likely to find theory abstract (Davis, 1999)), but pre their final year (when they are most likely to find theory insightful (Furlong et al., 2000)). CKT was introduced during a seminar with the second year PSTs as a possible explanation for why teachers may struggle to explain concepts to pupils. Drawing on Thomas' (1997) definition of theory, I chose to use the words idea, explanation and theory interchangeably throughout the seminar.

To enhance the validity of my data as representative of the population, I chose to survey all my 2nd year PSTs and use the following 'break characteristics' (Knodel, 1993) to select four groups of four PSTs:



I used self-assessed reflection and receptiveness to theory in general as a break characteristic based on the continuing theme in education literature that 'reflection' is a process that is essential for PSTs to be receptive to new ideas (Clarke & Hollingsworth, 2002), e.g., theory. My second break characteristic came from a question in the survey in which I asked them whether they had found CKT useful to their development as a teacher of mathematics. Again, I selected the extremes of this characteristic: those in the survey who not only replied 'yes' to this question, but also gave an informed example of how it had been helpful, versus those who said 'no' with an exclamation mark. In addition, as advised by Knodel (1993), I used the survey results to ensure that the following characteristics were equally spread across the groups: age; gender; mathematics attainment; teaching grade attainment; beliefs and anxiety about mathematics; participant's definition of subject knowledge.

In order to 'hear the voice' of participants, Litosseliti (2003) suggests a 'funnel design' whereby questions begin on broad, general topics before progressively becoming more focussed beneath the research question. As predicted by Litosseliti (2003), the research evolved and I decided to do an additional focus group with each group *after* they had been on SBT (school-based-training). This was due to the pre-SBT focus group revealing a trend of the PSTs suggesting ways in which they could use CKT, but wanting to experience a teaching practice (with their new knowledge of CKT). The final transcribed data therefore consisted of four pre-SBT focus groups of 45 minutes, four debriefing discussions between myself and my assistant immediately after each pre-SBT focus group (30 minutes each) and four post-SBT focus groups of 20 minutes.

Analysis of the data

I utilised the principles of inter-rater reliability by asking my assistant to conduct a second, independent analysis of the data separately from myself (Gillham, 2000). We began by separately devising an initial coding of the data that encompassed ideas of co-terminals, nesting and overlapping if and when they became necessary. The data was repeatedly recoded until we were happy with the assigned basic codes. In order to verify the broad themes, consistency was sought between myself and my assistant. We both independently analysed the entirety of the pre-SBT focus group transcriptions, including the 'settling in' initial questions and the 'top of the funnel' questions. This allowed us to check for parity in our analysis of these sections to further affirm the validity of our analysis of the parts of the focus group that were directly related to the research question. The reliability of my results is affirmed by the pervasive similarities between statements and viewpoints about theory given by the PSTs in my study, and those of the PSTs reported within a number of previous studies (e.g., Allen & Wright, 2014). Due to this striking correspondence between PSTs' perceptions of theory in general, it can be extrapolated that (despite no existing data to compare it to) my data regarding PSTs' perceptions of CKT is also reliable. Direct quotes from the focus groups use the pseudonyms PST1, PST2 etc.

Results and discussion

Did the PSTs value CKT?

Literature suggests that in an average population of PSTs there will be a distinct, albeit minor, proportion of PSTs who resolutely do not perceive theory as a valid source of information (e.g., Tang et al., 2016). My PSTs were deliberately chosen to try to capture this end of a possible spectrum as 25% of them were at the extreme in self-assessing as 'non-reflective' alongside their original declaration that CKT was not a valid source of information. And yet, at the end of the research, <u>**all** the PSTs stated that CKT was valid and useful</u>. What could have caused this?

The way CKT was introduced to the PSTs

CKT was introduced as part of a seminar. The subsequent focus groups created an opportunity for the participants to engage in deeper practical reasoning alongside their personal goals. Once they began SBT the PSTs had a chance to try out ideas from CKT and reflect on their impact. Additionally, by meeting my PSTs again after their SBT, they were encouraged to reflect further on the impact of their use of CKT during their practice. Inadvertently, the design of my research has utilised a range of aspects of teacher development that are recommended: including drawing on teachers' prior experience and beliefs (Forgasz & Leder, 2008); creating a community of practice (Wenger, 1998) where critical analysis is encouraged (Jaworski, 2006); allowing enactment of an idea (Allen & Wright, 2014) and facilitating reflection following enactment of an idea (Clark & Hollingsworth, 2002). Indeed, Smith and Hodson (2010) attributed a large part of the reason that their PSTs felt positive towards theory to the opportunities given to discuss and rework theory based on prior experience.

Accessibility and relevance of CKT

A strong trend in the literature is the idea that teachers and pre-service teachers find theory inaccessible, potentially due to the language used (e.g., Zeuli & Tiezzi, 1993). However, inaccessibility seemed less evident in the following comment which neatly captures a further feature of CKT, that the language (compressed; decompress; unpack; deconstruct) induces visual interpretations, which Cunningham and Stewart (2003) argue make learning theory easier to understand and therefore access: "It's just like having a big box - long multiplication...when you unpack it you find all the little bits that you have to understand to be able to do it" (PST1).

Relevance of theory is a key predictor for receptiveness by teachers (e.g., Furlong et al., 2000). A strong umbrella code identified by both analysts was 'PSTs applying CKT to previous experience'; nested beneath this code was a range of sub-codes indicating the relevance of CKT to a breadth of experiences across the PSTs. Interestingly, a further code arose from across all the focus groups both before and after placement; that of my PSTs demonstrating positive language and facial expression when talking about CKT whilst predominantly showing negative language and facial expression when talking about 'theory'. Furthermore, no instances were found of PSTs using the word theory alongside CKT; whilst a weak code evident across all the groups was that of PSTs stating explicitly that they did not perceive CKT as a theory.

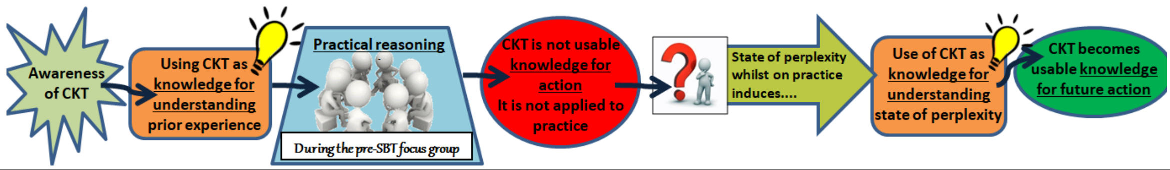
How did the PSTs interpret CKT?

A significant overarching theme in the codes linked to the PSTs' interpretations of CKT was their interpretation of it either as something to be used to support their mathematics teaching practice, "I think it will hugely affect my planning..... how you're going to break it down into steps for the child." (PST2); or as a 'source of enlightenment', "I had a lightbulb moment....I always thought that because I couldn't explain it, that I was like, thick. But it wasn't... I just couldn't break it down." (PST3).

I am immediately drawn to the possible similarities between this theme and 'knowledge for action' versus 'knowledge for understanding' (as defined by Eraut, 2003). Being able to articulate a theory as knowledge for action or knowledge for understanding should allow my PSTs to use that theory during their upcoming SBT and my PSTs emphatically claimed that they would apply CKT to practice from the beginning of SBT. Interestingly though, none of them did. Drawing on literature on belief change, Dewey (1933) claimed that teacher development can only occur in response to a 'state of perplexity'; a theme that has remained in teacher development (e.g., Korthagen, 2005). During the post-SBT focus groups, my PSTs reported noticing a 'point of need' when their planning was unsuccessful. This 'point of need' seemed to induce the use of

CKT as knowledge for understanding, enabling them to analyse their 'state of perplexity'. None of my PSTs used CKT to inform their planning and teaching until they reached this 'state of perplexity'. Subsequently, the PSTs seemed to perceive CKT as knowledge for action, using it to inform their teaching decisions from that moment forward. Despite probing, none of the PSTs spoke of needing more than one 'state of perplexity' to cause a resultant impact of CKT on that aspect of their practice. I propose, therefore, that my PSTs went through the following process in their perception of CKT (fig 1):

Figure 1: CKT becoming knowledge for future action



The idea of transitioning through knowledge for understanding before theory can be used as knowledge for action reflects findings from Furlong et al.'s (2000) survey of nearly six hundred newly qualified teachers where a large proportion described a 'click' moment when they perceived how theory and practice linked together.

Conclusions

In common with the literature, the data indicated that there are two key influences on how much PSTs value theory: the training provider and the accessibility and relevance of the theory. As a training provider, the structure of the research allowed my PSTs to discuss CKT through practical reasoning, then try it out during school experience, and finally return to the community in which CKT was initially discussed and reflect on the way they had used it. Drawing on previous literature on belief change, this structure is highly likely to have affected how my PSTs perceived CKT.

In addition, coding of the data revealed that my PSTs found CKT to be relevant to their experience, accessible in its language and categorised differently to other theories that they had encountered. I was initially struck by how my PSTs' interpretations of CKT seemed subtly different to my interpretation of CKT. However, through the process of the research I developed a strong appreciation of the power of CKT being interpreted in a range of ways by PSTs, allowing them to take ownership of the theory and utilise it in the best way for themselves. In contrast to Thomas (1997), who argued that if teacher educators sanction multiple meanings for theory we are in danger of theory losing its usefulness in education, I suggest that a future point of discussion should be around whether it matters how PSTs interpret theory as long as they find it valuable for their practice; perhaps linking to ideas of personal theories and private theories (e.g., Korthagen & Lagerwerf, 1996). This may also of course re-open a discussion around what 'theory' in education actually is. Perhaps, under a different definition, CKT should not be considered a 'theory'.

The data also revealed that CKT was being interpreted in two distinct ways: as a 'tool' or as a 'source of enlightenment'. This distinction mirrors ideas from Eraut (2003) of 'knowledge for action' as opposed to 'knowledge for understanding'. I propose that these themes could be used to more deeply analyse how perceptions of theory may evolve over time, potentially requiring that PSTs experience a 'state of perplexity' that is resolved through theory as 'knowledge for understanding' before knowledge of that theory can become 'knowledge for (future) action'.

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