RIGOROUS DIMENSION ESTIMATES FOR CANTOR SETS ARISING IN ZAREMBA THEORY

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Abstract. We address the question of the accuracy of bounds used in the study of Zaremba's conjecture. Specifically, we establish rigorous estimates on the Hausdorff dimension of certain Cantor sets which arise in the analysis of Zaremba's conjecture in [5, 18, 19, 23].

1. Introduction

Given any rational number $\frac{p}{q} \in (0,1)$ a simple application of Euclid's algorithm shows there exist coefficients $a_1, \cdots, a_n \in \mathbb{N}$ such that

$$\frac{p}{q} = [a_1, \cdots, a_n] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_n}}}}$$

(see e.g. [15, Thm. 161]). Given a finite subset $A \subset \mathbb{N}$, however, a natural question is to enquire as to what restriction is imposed on the denominators of such rational numbers in the case where $a_1, \cdots, a_n \in A$, in other words to study the corresponding denominator set

$$Q_A = \left\{ q \in \mathbb{N} : \exists p \in \mathbb{N}, a_1, \cdots, a_n \in A \text{ such that } \frac{p}{q} = [a_1, \cdots, a_n] \right\}.$$

More specifically, Zaremba [28] conjectured that when $A = \{1,2,3,4,5\}$, all natural numbers occur as denominators q for suitable choices of $a_1, \cdots, a_n \in A$, i.e. that $Q_{\{1,2,3,4,5\}} = \mathbb{N}$.

The choice of numbers up to 5 in this conjecture is natural, since the corresponding result fails for the smaller set $A = \{1,2,3,4\}$, where for example it is known that the numbers 6, 54 and 150 do not lie in the denominator set $Q_{\{1,2,3,4\}}$ (see [23, p. 193]).

The original motivation of Zaremba to study this problem was related to numerical integration and the use of the method of "good lattice points". Although Zaremba's conjecture remains open, there is various numerical evidence supporting it (see e.g. the discussion in [23, §2]); indeed in the article [4], the authors cite work of Borosh showing that all the denominators $q \leq 10^4$ occur in $Q_{\{1,2,3,4,5\}}$, and quote Knuth as having established the same result in the range $10^4 \leq q \leq 3.2 \times 10^6$.

In a significant recent paper, Bourgain & Kontorovich [5] showed that for the larger set $A = \{1,2,\cdots,50\}$, the corresponding denominator set Q_A has *density one* as a subset of $\mathbb{N}$, in other words

$$\lim_{N \to +\infty} \frac{Q_A \cap \{1,\ldots,N\}}{N} = 1.$$

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There is known to be a close connection between this kind of problem and the *Hausdorff dimension* of certain related sets. For a finite subset $A \subset \mathbb{N}$, let E_A denote the set of all $x \in (0,1)$ such that the digits $a_1(x), a_2(x), \ldots$ in the (infinite) continued fraction expansion

$$x = [a_1(x), a_2(x), a_3(x), \ldots] = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}}$$

all belong to A. Sets of the form E_A are said to be of *bounded type* (see e.g. [23, 26]); in particular they are Cantor sets, and study of their Hausdorff dimension has attracted significant attention (see e.g. [7, 8, 9, 10, 13, 14, 16, 17, 20, 21, 22]).

In the context of the Zaremba conjecture, the following result of Huang [18, 19] illustrates the connection with bounds on the Hausdorff dimension of bounded type sets E_A:

Theorem 1 (Huang). *For the set $A = \{1, 2, 3, 4, 5\}$, the corresponding denominator set Q_A has density one in $\mathbb{N}$ provided $\dim(E_A) > \frac{5}{6}$.*

In particular, Huang's theorem represents an improvement on the above result of Bourgain-Kontorovich (in that the set $\{1, 2, \ldots, 50\}$ is replaced by the smaller set $\{1, 2, 3, 4, 5\}$), provided the lower bound $\dim(E_A) > \frac{5}{6}$ does indeed hold. In fact Huang [18, 19] cites as evidence of this bound a paper of the first author [20], where the techniques of [21] were used to give a non-rigorous indication that $\dim(E_A) \approx 0.8368 > 0.8333\ldots = \frac{5}{6}$. Although the method introduced in [21] yielded high quality empirical approximations, it is only in our more recent paper [22] that effective techniques have been introduced for converting these heuristics into a rigorous proof of the quality of a specific computation. In view of the conditional nature of Huang's Theorem 1, and the recent availability of techniques potentially capable of rendering rigorous the heuristic estimate of $\dim(E_A)$ in [20], in this paper we employ the technology of [22] in order to rigorously prove the following:

Theorem 2. *If $A = \{1, 2, 3, 4, 5\}$ then $\dim(E_A) > \frac{5}{6}$.*

Indeed in §5 we give a rigorous proof, stated as Theorem 8, of a significantly more accurate estimate on $\dim(E_{\{1,2,3,4,5\}})$. Combining Theorems 1 and 2 we deduce the following unconditional version of Huang's Theorem.

Corollary 1 (after Huang). *For the set $A = \{1, 2, 3, 4, 5\}$, the corresponding denominator set Q_A has density one in $\mathbb{N}$.*

A stronger conjecture due to Hensley [17, Conj. 3, p. 16] was that provided $\dim(E_A) > \frac{1}{2}$ then every sufficiently large natural number occurs as a corresponding denominator, i.e. that Q_A contains all sufficiently large natural numbers. However, Bourgain & Kontorovich [5] indicated that $A = \{2, 4, 6, 8, 10\}$ provides a counterexample to this conjecture, noting that in this case Q_A does not contain any natural numbers which are equal to 3 (mod 4), and that moreover $\dim(E_A) \approx 0.517 > 1/2$ (see [5, p. 139]). Their approximation to $\dim(E_{\{2,4,6,8,10\}})$, using an implementation of the algorithm of [21], is a heuristic one, in the spirit of the empirical computations in [20, 21] rather than the rigorously validated version of [22]. In view of the importance of this Bourgain-Kontorovich counterexample to Hensley's conjecture, it is of interest to rigorously establish the lower bound on the dimension of $E_{\{2,4,6,8,10\}}$ (which we present in §4 as Theorem 7):

Theorem 3. *If* $A = \{2, 4, 6, 8, 10\}$ *then* $\dim(E_A) > \frac{1}{2}$.

In particular, this confirms the assertion of [5, p. 139], yielding:

Corollary 2 (after Bourgain-Kontorovich)**.** *The set* $A = \{2, 4, 6, 8, 10\}$ *provides a counterexample to the Hensley conjecture.*

Finally, we recall that Bourgain & Kontorovich proved [5, Thm. 1.26] the existence of $h \in \mathbb{N}$ such that there are infinitely many prime numbers d which have a primitive root b (mod d) with the property that the partial quotients of the rational b/d are bounded by h, and they indicated that h could be chosen to equal 51. The following sharpening of this result due to Huang [19, Cor. 1.1.12] is reliant on a lower bound for the Hausdorff dimension of the Cantor set $E_{\{1,2,3,4,5,6\}}$:

Theorem 4 (Huang)**.** *If* $\dim(E_{\{1,2,3,4,5,6\}}) > \frac{19}{22}$ *then there are infinitely many prime numbers* d *which have a primitive root* b (mod d) *such that the partial quotients of* b/d *are* ≤ 7.

Although Huang indicates that $\dim(E_{\{1,2,3,4,5,6\}}) > \frac{19}{22}$ is true, citing the empirical approximation $\dim(E_{\{1,2,3,4,5,6\}}) \approx 0.8676 > 0.86363636\ldots = \frac{19}{22}$ of [20], there was no rigorous proof of this result. Once again, therefore, there is considerable interest in rendering Theorem 4 an unconditional result by providing a rigorous validation of the Hausdorff dimension bound. This we do in our third main result:

Theorem 5. *If* $A = \{1, 2, 3, 4, 5, 6\}$ *then* $\dim(E_A) > \frac{19}{22}$.

In fact we give a rigorous proof of a significantly more accurate estimate on $\dim(E_{\{1,2,3,4,5,6\}})$ as Theorem 9 in §6. A corollary of Theorem 5 is the unconditional analogue of Huang's Theorem 4:

Corollary 3 (after Huang)**.** *There are infinitely many prime numbers* d *which have a primitive root* b (mod d) *such that the partial quotients of* b/d *are* ≤ 7.

The organisation of this article is as follows. After some preliminaries in §2 on Hausdorff dimension and the thermodynamic underpinnings of our computational approach, in §3 we describe (after [22]) the way in which these computations can be converted into rigorous effective bounds. In §4 we prove that the Hausdorff dimension of $E_{\{2,4,6,8,10\}}$ is greater than 1/2, in §5 we establish a rigorous bound on $\dim(E_{\{1,2,3,4,5\}})$ which in particular shows this dimension to be larger than 5/6, and in §6 we rigorously approximate the dimension of $E_{\{1,2,3,4,5,6\}}$, which in particular implies it is larger than 19/22.

2. Preliminaries

In this section we collect a number of results (see also [21, 22]) which underpin our algorithm for approximating Hausdorff dimension.

We begin by recalling some results for continued fractions. For a non-empty finite subset $A \subset \mathbb{N}$, let E_A denote the set of all $x \in (0, 1)$ such that the digits $a_1(x), a_2(x), \ldots$ in the continued fraction expansion

$$x = [a_1(x), a_2(x), a_3(x), \ldots] = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}}$$

all lie in A. Equivalently, if

$$T_n(x) := (n+x)^{-1}$$

then E_A is the smallest non-empty closed set satisfying the self-similarity condition

$$E_A = \cup_{n\in A} T_n(A)\,.$$

The *Gauss map*

$$T(x) = \frac{1}{x} \pmod 1$$

is such that $T \circ T_n$ is the identity map for each n, and all of the sets E_A satisfy $T(E_A) = E_A$.

Each of the sets $E_A \subset [0,1]$ is a Cantor set of zero Lebesgue measure, and a natural way to describe their size is via Hausdorff dimension.

Definition 1. For a general set $E \subset \mathbb{R}$, if we define

$$H^\delta_\varepsilon(E) := \inf\left\{ \sum_i \mathrm{diam}(U_i)^\delta : \mathcal{U} = \{U_i\} \text{ is an open cover of } E \text{ such that each } \mathrm{diam}(U_i) \le \varepsilon \right\},$$

and $H^\delta(E) := \lim_{\varepsilon\to 0} H^\delta_\varepsilon(E)$, then the *Hausdorff dimension* of E, denoted $\dim(E)$, is defined to be the infimum of the set $\inf\{\delta : H^\delta(E) = 0\}$.

For the sets E_A, their Hausdorff dimension coincides with their *box dimension* (see e.g. [12]).

For a general continuous function $f : E_A \to \mathbb{R}$, its *pressure* $P(f)$ is defined to be

$$P(f) = \lim_{n\to+\infty} \frac{1}{n} \log\left(\sum_{\substack{T^n x = x \\ x\in E_A}} e^{f(x)+f(Tx)+\ldots+f(T^{n-1}x)} \right),$$

and making the particular choice $f = -s\log|T'|$ leads to an important characterisation of the Hausdorff dimension of E_A (see [3, 6, 12, 24]):

Lemma 1. *The function $\mathbb{R}\to\mathbb{R}$ defined by $s \mapsto P(-s\log|T'|)$ is strictly decreasing, and its unique zero is precisely the Hausdorff dimension* $\dim(E_A)$.

For $s \in \mathbb{R}$, and finite $A \subset \mathbb{N}$, define the *transfer operator* $\mathcal{L}_{A,s}$ by

$$\mathcal{L}_{A,s} f(x) = \sum_{n\in A} \frac{f(T_n x)}{(n+x)^{2s}}\,.$$

This operator is known to leave invariant a number of natural function spaces, notably the Hilbert Hardy spaces considered below, or for example the Banach space of Lipschitz functions on $[0,1]$. On these spaces the value $e^{P(-s\log|T'|)}$ is an eigenvalue of strictly largest modulus, and is a simple eigenvalue. Consequently, Lemma 1 implies that the Hausdorff dimension of E_A is the unique value $s\in\mathbb{R}$ such that $\mathcal{L}_{A,s}$ has spectral radius equal to 1.

When acting on suitable Hilbert Hardy spaces, the *trace* $\mathrm{tr}(\mathcal{L}^n_{A,s})$ of each n-th power $\mathcal{L}^n_{A,s}$ is given (see [21, 25]) by

$$\mathrm{tr}(\mathcal{L}^n_{A,s}) = \sum_{\underline{i}\in A^n} \frac{|T'_{\underline{i}}(z_{\underline{i}})|^s}{1 - T'_{\underline{i}}(z_{\underline{i}})} = \sum_{\underline{i}\in A^n} \frac{\prod_{j=0}^{n-1} T^j(z_{\underline{i}})^{2s}}{1-(-1)^n \prod_{j=0}^{n-1} T^j(z_{\underline{i}})^2}\,, \qquad (1)$$

where the point $z_{\underline{i}}$ is the unique fixed point in $(0,1)$ of the n-fold composition $T_{\underline{i}} = T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}$ (and hence a period-n point for the Gauss map T), and in particular is a quadratic irrational. The function defined on the complex disc $|z| < e^{-P(-s\log|T'|)}$ (i.e. the disc of convergence of $\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(\mathcal{L}_{A,s}^n)$) by

$$\Delta(z,s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(\mathcal{L}_{A,s}^n) \right) \tag{2}$$

extends by analytic continuation to an entire function of $\mathbb{C}$, called the *determinant* of $\mathcal{L}_{A,s}$.

When acting on suitable Hilbert Hardy spaces, the eigenvalues of $\mathcal{L}_{A,s}$ are precisely the reciprocals of the zeros of its determinant. In particular, the zero of the function $\Delta(s,\cdot)$ with smallest modulus is $e^{-P(-s\log|T'|)}$, therefore the Hausdorff dimension of E_A is precisely the value of s such that 1 is the zero of minimum modulus of $\Delta(s,\cdot)$.

In fact, when $\mathcal{L}_{A,s}$ acts on such a space of holomorphic functions, its approximation numbers decay at an exponential rate (see [22, Cor. 2]), so that $\mathcal{L}_{A,s}$ belongs to an exponential class (cf. [1, 2]) and is in particular a trace class operator, from which the existence and above properties of trace and determinant follow (see [27]).

This allows us (cf. [21, 22]) to write $\Delta(z,s)$ as the series $\Delta(z,s) = 1 + \sum_{n=1}^{\infty} \delta_n(s) z^n$, and then set $z = 1$ to define the *dimension determinant* $\mathfrak{D}$ by

$$\mathfrak{D}(s) := \Delta(1,s) = 1 + \sum_{n=1}^{\infty} \delta_n(s)\,,$$

a holomorphic function which is known to be entire (see [21, 25]). Solutions s of

$$0 = 1 + \sum_{n=1}^{\infty} \delta_n(s) = \mathfrak{D}(s) \tag{3}$$

are such that the value 1 is an eigenvalue for the operator $\mathcal{L}_{A,s}$, and in particular the largest real zero of $\mathfrak{D}$ is precisely the dimension $\dim(E_A)$ (cf. Proposition 1), being the value of s such that 1 is the leading eigenvalue (i.e. of maximum modulus) for the operator $\mathcal{L}_{A,s}$.

The coefficients $\delta_n(s)$ are computable (to arbitrary precision, for a given s) in terms of those periodic points of $T|_{E_A}$ whose period is $\leq n$, using the formula (1). Therefore, for any given $N \in \mathbb{N}$, we may define $\mathfrak{D}_N$ by

$$\mathfrak{D}_N(s) := 1 + \sum_{n=1}^{N} \delta_n(s)\,, \tag{4}$$

so that a solution s_N to the equation $\mathfrak{D}_N(s) = 0$ will be an approximate solution to (3), and the smaller $\sum_{n=N+1}^{\infty} \delta_n(s)$ is the better this approximation will be. In what follows, we use rigorous upper bounds on (the absolute value of) $\sum_{n=N+1}^{\infty} \delta_n(s)$ to yield rigorous estimates on $|s_N - \dim(E_A)|$.

3. Bounding dimension determinant coefficients

We now begin the serious task of converting these theoretical estimates into practical bounds that can be used to complete the proofs of the results stated in the introduction. The key point is that we can employ a number of technical innovations introduced in [22] in order to make estimates both effective and rigorous.

Let $A \subset \mathbb{N}$ be finite. An open disc $D \subset \mathbb{C}$ is said to be *admissible (for A)* if $\cup_{i\in A} T_i(D) \subset D$.

For an admissible disc D of radius ϱ, centred at c, let D' be the smallest disc, concentric with D, such that $\cup_{i\in A} T_i(D) \subset D'$, and let ϱ' denote the radius of D'. The associated *contraction ratio* $\theta = \theta_{A,D}$ is then defined as

$$\theta = \theta_{A,D} := \frac{\varrho'}{\varrho}\,.$$

Introducing the notation

$$E_n(\theta) := \frac{\theta^{n(n+1)/2}}{\prod_{i=1}^n (1-\theta^i)}\,, \tag{5}$$

we note the super-exponential decay $E_n(\theta) = O(\theta^{\frac{n^2}{2}})$ as $n \to \infty$.

Definition 2. The *Hilbert Hardy space* $H^2(D)$ consists of those functions f which are holomorphic on D such that $\|f\|^2 := \sup_{r<\varrho} \int_0^1 |f^*(c+re^{2\pi i t})|^2\, dt < \infty$, with inner product given by

$$(f,g) = \int_0^1 f^*(c+\varrho e^{2\pi i t})\overline{g^*(c+\varrho e^{2\pi i t})}\, dt\,,$$

where f^* and g^* denote the respective non-tangential limit functions of f and g.

The monomials

$$m_k(z) = \varrho^{-k}(z-c)^k \tag{6}$$

constitute an orthonormal basis of $H^2(D)$.

Admissibility of D ensures that for $s \in \mathbb{R}$, the transfer operator $\mathcal{L}_{A,s}$ preserves $H^2(D)$. In particular,

$$\mathcal{L}_{A,s}(m_k)(z) = \sum_{j\in A} \frac{(T_j(z)-c)^k}{\varrho^k (z+j)^{2s}}\,,$$

and we may use numerical integration to explicitly compute (to arbitrary precision) the norm $\|\mathcal{L}_{A,s}(m_k)\|$ as

$$\|\mathcal{L}_{A,s}(m_k)\|^2 = \int_0^1 \left| \sum_{j\in A} \frac{(T_j(\gamma(t))-c)^k}{\varrho^k(\gamma(t)+j)^{2s}} \right|^2 dt\,, \tag{7}$$

where $\gamma(t) = c + \varrho e^{2\pi i t}$.

For $j \in A$ the functions

$$w_{j,s}(z) = \frac{1}{(z+j)^{2s}}$$

are holomorphic on the admissible disc D, and we use their *uniform* norms

$$\|w_{j,s}\|_\infty = \sup_{z\in D} |w_{j,s}(z)|\,,$$

together with the contraction ratio θ, to define the constant

$$K_s = K_{s,A,D} := \frac{\sum_{j\in A} \|w_{j,s}\|_\infty}{\theta\sqrt{1-\theta^2}}\,. \tag{8}$$

For $s \in \mathbb{R}$ and $n, Q, M, N \in \mathbb{N}$ with $n \leq Q \leq M \leq N$, if we introduce the quantities

$$\alpha_{n,N,+}(s) := \left(\sum_{k=n-1}^{N} \|\mathcal{L}_{A,s}(m_k)\|^2 + \left(\sum_{j \in A} \|w_{j,s}\|_\infty \right)^2 \frac{\theta^{2(N+1)}}{1-\theta^2} \right)^{1/2}, \quad (9)$$

$$\beta^{M,-}_{l,N,+}(s) := \sum_{i_1 < \ldots < i_l \leq M} \prod_{j=1}^{l} \alpha_{i_j,N,+}(s)\,, \quad (10)$$

$$J_{Q,N,s} := K_s \left(1 + \theta^{2(N+2-Q)}\right)^{1/2}, \quad (11)$$

$$\beta^{M,+}_{n,N,+}(s) := \beta^{M,-}_{n,N,+}(s) + \sum_{l=0}^{n-1} J^{n-l}_{Q,N,s}\, \beta^{M,-}_{l,N,+}(s)\, \theta^{M(n-l)} E_{n-l}(\theta)\,. \quad (12)$$

then the following bound was established in [22]:

Theorem 6. *Let $A \subset \mathbb{N}$ be finite, and D an admissible disc, with contraction ratio $\theta = \theta_{A,D}$. If $s \in \mathbb{R}$, and $Q, M, N \in \mathbb{N}$ with $n \leq Q \leq M \leq N$, then the dimension determinant coefficients $\delta_n(s)$ satisfy*

$$|\delta_n(s)| \leq \min\left(K_s^n E_n(\theta)\ ,\ \beta^{M,+}_{n,N,+}(s) \right).$$

Remark 1. Theorem 6 was proved in [22] using the theory of approximation numbers in Hilbert space. The inequality $|\delta_n(s)| \leq K_s^n E_n(\theta)$ from Theorem 6 is referred to as the *Euler bound*, acknowledging Euler's work [11] on the identity $E_n(\theta) = \sum_{i_1<\ldots<i_n} \theta^{i_1+\ldots+i_n}$. The term $K_s^n E_n(\theta)$ has a simple closed form, and is $O(\gamma^{n^2})$ as $n \to \infty$ for any $\gamma \in (\theta^{1/2}, 1)$, though the constant $K_s = K_{s,A,D}$ may be large enough (if A is large) to render the tail estimate $|\sum_{n>Q} \delta_n(s)| \leq \sum_{n>Q} K_s^n E_n(\theta)$ insufficiently sharp if Q is chosen to be small. The terms $\beta^{M,+}_{n,N,+}(s)$, referred to as *upper computed Taylor bounds* in [22], have the virtue of being readily computable to arbitrary precision, but are not available in closed form; their utility, therefore, is in bounding $|\delta_n(s)|$ for $n \leq Q$, where Q is chosen so that the tail estimate derived from the Euler bound is sufficiently sharp. In practice M and N will be chosen so that $\beta^{M,+}_{n,N,+}(s)$ agrees with $\beta^{M,-}_{n,N,+}(s)$ (which is given by a notably simpler formula) to very high precision (e.g. several hundred decimal places), i.e. the more complicated term $\sum_{l=0}^{n-1} J^{n-l}_{Q,N,s}\, \beta^{M,-}_{l,N,+}(s)\, \theta^{M(n-l)} E_{n-l}(\theta)$ in (12) effectively plays no computational role; similarly, $\alpha_{n,N,+}(s)$ will in practice agree with $(\sum_{k=n-1}^{N} \|\mathcal{L}_{A,s}(m_k)\|^2)^{1/2}$ to very high precision, so that the term $(\sum_{j\in A} \|w_{j,s}\|_\infty)^2 \frac{\theta^{2(N+1)}}{1-\theta^2}$ in (9) effectively plays no computational role.

4. The Hausdorff dimension of $E_{\{2,4,6,8,10\}}$ is greater than $1/2$

Motivated by the work of Bourgain & Kontorovich [5] described in §1, specifically [5, p. 139] (see also [23, Lem. 2.20]), our aim in this section will be to provide a rigorous proof of the fact that the Hausdorff dimension of $E_{\{2,4,6,8,10\}}$ is greater than $1/2$, a result which heretofore has enjoyed a folklore status, based on convincing but non-rigorous numerical work.

Our approach is motivated by the following observation:

Proposition 1. *For any finite alphabet A, if $s_0 \in \mathbb{R}$ is such that the corresponding dimension determinant $\mathfrak{D} = \mathfrak{D}_A$ satisfies $\mathfrak{D}(s_0) < 0$, then $\dim(E_A) > s_0$.*

Proof. The method is to show firstly that $\mathfrak{D}$ cannot have real zeros that are larger than $\dim(E_A)$, so that $\mathfrak{D}$, being a continuous function, does not change sign on the interval $(\dim(E_A), \infty)$, and secondly that the derivative $\mathfrak{D}'(s)$ is strictly positive at its zero $s = \dim(E_A)$. This then implies that $\mathfrak{D}$ is strictly positive on $(\dim(E_A), \infty)$, or in other words the desired result that if $\mathfrak{D}(s_0) < 0$ then necessarily $\dim(E_A) > s_0$.

To show that if $s > \dim(E_A)$ then $\mathfrak{D}(s) \neq 0$, recall that $s \mapsto p(s) = P(-s \log |T'|)$ is strictly decreasing on $\mathbb{R}$, with $s = \dim(E_A)$ its unique zero. Therefore $s \mapsto z_1(s) = e^{-p(s)}$, which is the zero of minimum modulus of $\Delta(\cdot, s)$, is a strictly increasing function. In particular, $z_1(\dim(E_A)) = 1$, so if $s > \dim(E_A)$ then $z_1(s) > 1$; thus all zeros of $\Delta(\cdot, s)$ must have modulus strictly larger than 1. Therefore in particular the equation $\Delta(1, s) = 0$ has no solutions for $s > \dim(E_A)$, i.e. the equation $\mathfrak{D}(s) = 0$ has no solutions for $s > \dim(E_A)$, i.e. $\mathfrak{D}$ has no zeros that are strictly larger than $\dim(E_A)$.

To complete the proof it remains to show that the derivative $\mathfrak{D}'(s)$ is strictly positive at $s = \dim(E_A)$. To see this we use the infinite product

$$\Delta(z, s) = \prod_{r=1}^{\infty} (1 - z\lambda_r(s)),$$

where $\lambda_r(s)$ are the eigenvalues of $\mathcal{L}_{A,s}$, listed according to algebraic multiplicity, and ordered so that their absolute values are non-increasing, with in particular $\lambda_1(s) > |\lambda_r(s)|$ for all $r \geq 2$ (since the leading eigenvalue $\lambda_1(s)$ is simple).

If $\Gamma(s) := \prod_{r=2}^{\infty} (1 - \lambda_r(s))$ then $\mathfrak{D}(s) = (1 - \lambda_1(s))\Gamma(s)$, so

$$\mathfrak{D}'(s) = -\lambda_1'(s)\Gamma(s) + (1 - \lambda_1(s))\Gamma'(s)\,,$$

and since $\lambda_1(\dim(E_A)) = 1$ then

$$\mathfrak{D}'(\dim(E_A)) = -\lambda_1'(\dim(E_A))\Gamma(\dim(E_A)).$$

But $s \mapsto \lambda_1(s) = e^{p(s)}$ is strictly decreasing, so $-\lambda_1'(\dim(E_A)) > 0$, and therefore it remains to show that $\Gamma(\dim(E_A)) > 0$. For this, note that if $s \in \mathbb{R}$ (in particular if $s = \dim(E_A)$), the coefficients in the power series expansion of $\Delta(z, s)$ are all real, by (1). Therefore non-real zeros of Δ arise as conjugate pairs, both with the same multiplicity. Multiplying out those factors in the product representation of Γ corresponding to conjugate pairs, we see that $\Gamma(\dim(E_A))$ is an infinite product of strictly positive terms (since $|\lambda_r(\dim(E_A))| < 1$ for each $r \geq 2$). The sequence of terms converges to 1, since $|\lambda_r(\dim(E_A))| \to 0$, therefore the infinite product converges to a strictly positive value. That is, $\Gamma(\dim(E_A)) > 0$, as required. □

Having established Proposition 1, our strategy for proving that $\dim(E_{\{2,4,6,8,10\}}) > 1/2$ will be to show that $\mathfrak{D}(1/2) < 0$ for the corresponding dimension determinant $\mathfrak{D} = \mathfrak{D}_{\{2,4,6,8,10\}}$.

In view of the central role of the value $s = 1/2$ in this section, we shall write

$$\mathcal{L}_A := \mathcal{L}_{A,1/2}\,,$$

and

$$\delta_n := \delta_n(1/2)\,,$$

so that

$$\mathfrak{D}(1/2) = 1 + \sum_{n=1}^{\infty} \delta_n \,.$$

It will turn out to be sufficient to work with Gauss map orbits of periods 1, 2 and 3, and in Lemmas 2, 3, 4 below we record exact formulae for the corresponding traces of the operator $\mathcal{L}_A$.

Lemma 2. *If $A = \{2,4,6,8,10\}$ then $u_1 = \mathrm{tr}(\mathcal{L}_A)$ is given by the exact formula*

$$\begin{aligned} u_1 &= \frac{\sqrt{2}-1}{1+\left(\sqrt{2}-1\right)^2} + \frac{\sqrt{5}-2}{1+\left(\sqrt{5}-2\right)^2} + \frac{\sqrt{10}-3}{1+\left(\sqrt{10}-3\right)^2} + \frac{\sqrt{17}-4}{1+\left(\sqrt{17}-4\right)^2} + \frac{\sqrt{26}-5}{1+\left(\sqrt{26}-5\right)^2} \\ &= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{5}} + \frac{1}{2\sqrt{10}} + \frac{1}{2\sqrt{17}} + \frac{1}{2\sqrt{26}} \,. \end{aligned}$$

Proof. From (1), $u_1 = \sum_{n\in A} z_n/(1+z_n^2)$, where

$$z_n = [n,n,n,\ldots] = \sqrt{k_n^2+1} - k_n \,,$$

for $k_n = n/2$, and the result follows. □

Lemma 3. *If $A = \{2,4,6,8,10\}$ then $u_2 = \frac{1}{2}\mathrm{tr}(\mathcal{L}_A^2)$ is given by the exact formula*

$$\begin{aligned} u_2 = \frac{1}{2}\Bigg(&\frac{3-2\sqrt{2}}{1-\left(3-2\sqrt{2}\right)^2} + \frac{19-6\sqrt{10}}{1-\left(19-6\sqrt{10}\right)^2} + \frac{33-8\sqrt{17}}{1-\left(33-8\sqrt{17}\right)^2} + \frac{51-10\sqrt{26}}{1-\left(51-10\sqrt{26}\right)^2} \\ &+ \frac{3\left(9-4\sqrt{5}\right)}{1-\left(9-4\sqrt{5}\right)^2} + \frac{2\left(17-12\sqrt{2}\right)}{1-\left(17-12\sqrt{2}\right)^2} + \frac{2\left(7-4\sqrt{3}\right)}{1-\left(7-4\sqrt{3}\right)^2} + \frac{2\left(5-2\sqrt{6}\right)}{1-\left(5-2\sqrt{6}\right)^2} \\ &+ \frac{19-6\sqrt{10}}{1-\left(19-6\sqrt{10}\right)^2} + \frac{2\left(31-8\sqrt{15}\right)}{1-\left(31-8\sqrt{15}\right)^2} + \frac{2\left(11-2\sqrt{30}\right)}{1-\left(11-2\sqrt{30}\right)^2} + \frac{2\left(25-4\sqrt{39}\right)}{1-\left(25-4\sqrt{39}\right)^2} \\ &+ \frac{2\left(13-2\sqrt{42}\right)}{1-\left(13-2\sqrt{42}\right)^2} + \frac{2\left(41-4\sqrt{105}\right)}{1-\left(41-4\sqrt{105}\right)^2} + \frac{2\left(21-2\sqrt{110}\right)}{1-\left(21-2\sqrt{110}\right)^2} \Bigg) . \end{aligned}$$

Proof. From (1),

$$u_2 = \frac{1}{2} \sum_{(m,n)\in A^2} \frac{z_{m,n}}{1-z_{m,n}^2} \,,$$

where it can be shown that

$$z_{m,n} = k_{m,n} - \sqrt{k_{m,n}^2 - 1} \,,$$

for

$$k_{m,n} = 1 + \frac{mn}{2} \,.$$

Note that $z_{4,4} = z_{2,8} = z_{8,2} = 9 - 4\sqrt{5}$, contributing the term with coefficient 3 in the above expression for u_2. Otherwise the four remaining fixed points contribute the terms with coefficient 1, and the 9 remaining period-2 orbits contribute the terms with coefficient 2 (since $z_{m,n} = z_{n,m}$). □

Lemma 4. *If $A = \{2,4,6,8,10\}$ then $u_3 = \frac{1}{3}\mathrm{tr}(\mathcal{L}_A^3)$ is given by the exact formula*

$$\begin{aligned} u_3 \;=\; & \frac{1}{3}\Bigg(\frac{5\sqrt{2}-7}{1+\left(5\sqrt{2}-7\right)^2} + \frac{17\sqrt{5}-38}{1+\left(17\sqrt{5}-38\right)^2} + \frac{37\sqrt{10}-117}{1+\left(37\sqrt{10}-117\right)^2} + \frac{65\sqrt{17}-268}{1+\left(65\sqrt{17}-268\right)^2} \\ & + \frac{101\sqrt{26}-515}{1+\left(101\sqrt{26}-515\right)^2} + \frac{3\left(\sqrt{145}-12\right)}{1+\left(\sqrt{145}-12\right)^2} + \frac{3\left(\sqrt{290}-17\right)}{1+\left(\sqrt{290}-17\right)^2} + \frac{3\left(\sqrt{442}-21\right)}{1+\left(\sqrt{442}-21\right)^2} \\ & + \frac{3\left(\sqrt{485}-22\right)}{1+\left(\sqrt{485}-22\right)^2} + \frac{3\left(\sqrt{730}-27\right)}{1+\left(\sqrt{730}-27\right)^2} + \frac{6\left(\sqrt{901}-30\right)}{1+\left(\sqrt{901}-30\right)^2} + \frac{6\left(\sqrt{1522}-39\right)}{1+\left(\sqrt{1522}-39\right)^2} \\ & + \frac{6\left(\sqrt{2305}-48\right)}{1+\left(\sqrt{2305}-48\right)^2} + \frac{3\left(\sqrt{3026}-55\right)}{1+\left(\sqrt{3026}-55\right)^2} + \frac{6\left(\sqrt{3137}-56\right)}{1+\left(\sqrt{3137}-56\right)^2} + \frac{6\left(\sqrt{4762}-69\right)}{1+\left(\sqrt{4762}-69\right)^2} \\ & + \frac{3\left(\sqrt{5185}-72\right)}{1+\left(\sqrt{5185}-72\right)^2} + \frac{3\left(\sqrt{5330}-73\right)}{1+\left(\sqrt{5330}-73\right)^2} + \frac{3\left(\sqrt{6401}-80\right)}{1+\left(\sqrt{6401}-80\right)^2} + \frac{3\left(\sqrt{7922}-89\right)}{1+\left(\sqrt{7922}-89\right)^2} \\ & + \frac{6\left(\sqrt{8101}-90\right)}{1+\left(\sqrt{8101}-90\right)^2} + \frac{6\left(\sqrt{11026}-105\right)}{1+\left(\sqrt{11026}-105\right)^2} + \frac{3\left(\sqrt{12322}-111\right)}{1+\left(\sqrt{12322}-111\right)^2} + \frac{6\left(\sqrt{16901}-130\right)}{1+\left(\sqrt{16901}-130\right)^2} \\ & + \frac{3\left(\sqrt{19045}-138\right)}{1+\left(\sqrt{19045}-138\right)^2} + \frac{3\left(\sqrt{23717}-154\right)}{1+\left(\sqrt{23717}-154\right)^2} + \frac{6\left(\sqrt{29242}-171\right)}{1+\left(\sqrt{29242}-171\right)^2} + \frac{3\left(\sqrt{36482}-191\right)}{1+\left(\sqrt{36482}-191\right)^2} \\ & + \frac{3\left(\sqrt{41210}-203\right)}{1+\left(\sqrt{41210}-203\right)^2} + \frac{3\left(\sqrt{44945}-212\right)}{1+\left(\sqrt{44945}-212\right)^2} + \frac{6\left(\sqrt{63505}-252\right)}{1+\left(\sqrt{63505}-252\right)^2} + \frac{3\left(\sqrt{97970}-313\right)}{1+\left(\sqrt{97970}-313\right)^2} \\ & + \frac{3\left(\sqrt{110890}-333\right)}{1+\left(\sqrt{110890}-333\right)^2} + \frac{3\left(\sqrt{171397}-414\right)}{1+\left(\sqrt{171397}-414\right)^2} + \frac{3\left(5\sqrt{74}-43\right)}{1+\left(5\sqrt{74}-43\right)^2} \Bigg). \end{aligned}$$

Proof. From (1),

$$u_3 = \frac{1}{3} \sum_{(l,m,n)\in A^3} \frac{z_{l,m,n}}{1+z_{l,m,n}^2}\,, \tag{13}$$

where it can be shown that

$$z_{l,m,n} = \sqrt{k^2+1} - k\,,$$

for

$$k = k_{l,m,n} = \frac{1}{2}\left(lmn + l + m + n\right)\,. \tag{14}$$

The 35 terms in the above expression for u_3 correspond to the 35 distinct values of $z_{l,m,n}$ as (l,m,n) ranges over A^3. Of the 125 terms in (13), five correspond to fixed points, and the remaining 120 correspond to points of least period 3. Of the 40 period-3 orbits, half of them are such that l, m, and n are distinct elements of A; in such cases the distinct orbits coded by (l,m,n) and (l,n,m) satisfy $z_{l,m,n} = z_{l,n,m}$ (note that (14) is symmetric in l, m, n), thus contributing 6 identical terms in the sum (13), i.e. the terms with coefficient 6 in the above expression for u_3. The other 20 period-3 orbits, for which precisely two of l, m, n are equal, contribute 3 identical terms in the sum (13), i.e. the terms with coefficient 3 in the above expression for u_3. Thus u_3 is naturally written as a sum of $35 = 5 + 10 + 20$ terms. □

Using the exact formulae of Lemmas 2, 3, and 4 we are now able to evaluate the order-3 approximation $\mathfrak{D}_3(1/2)$ to $\mathfrak{D}(1/2)$:

Lemma 5. *For $E = E_{\{2,4,6,8,10\}}$, the order-3 approximation $\mathfrak{D}_3(1/2)$ satisfies*

$$\mathfrak{D}_3(1/2) = 1 + \delta_1 + \delta_2 + \delta_3 < -\frac{1}{20}\,. \tag{15}$$

Proof. Using the definitions of the δ_i, and Lemmas 2, 3, and 4, we bound[1]

$$\delta_1 = -u_1 < -954/1000\,,$$
$$\delta_2 = \frac{1}{2}u_1^2 - u_2 < -102/1000\,,$$
$$\delta_3 = u_1 u_2 - u_3 - \frac{1}{6}u_1^3 < 2/1000\,,$$

therefore

$$\mathfrak{D}_3(1/2) = 1 + \delta_1 + \delta_2 + \delta_3 < -54/1000 < -1/20\,.$$

□

Lemma 6. *The error term for the approximation of $\mathfrak{D}(1/2)$ by $\mathfrak{D}_3(1/2)$ is bounded by*

$$|\mathfrak{D}(1/2) - \mathfrak{D}_3(1/2)| < \frac{1}{20}$$

Proof. Let $D \subset \mathbb{C}$ be the disc of radius $\varrho = 3/2$ centred at $c = 1/2$. For the alphabet $A = \{2,4,6,8,10\}$ this disc has contraction ratio $\theta = 1/3$ (the point $-1 \in \partial D$ satisfies $T_2(-1) = 1$, which has distance $1/2 = \theta\varrho$ from the centre of D, see Figure 1).

For each $n \in A = \{2,4,6,8,10\}$ the function $w_n(z) = 1/(z+n)$ has maximum modulus on $\overline{D}$ when $z = c - \varrho = -1$, in other words

$$\|w_n\|_\infty = \frac{1}{n-1}\,, \tag{16}$$

and therefore

$$\sum_{n\in A} \|w_n\|_\infty = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} = \frac{563}{315}\,,$$

so

$$K = \frac{\sum_{n\in A}\|w_n\|_\infty}{\theta\sqrt{1-\theta^2}} = \frac{563}{70\sqrt{2}} < 6\,.$$

Now

$$|\delta_n| \le K^n E_n(\theta) < 6^n E_n(1/3) = \frac{6^n 3^{-n(n+1)/2}}{\prod_{i=1}^n (1-3^{-i})} =: F_n\,,$$

from which we readily derive[2] the required bound

$$|\mathfrak{D}(1/2) - \mathfrak{D}_3(1/2)| \le \sum_{n=4}^\infty |\delta_n| < \sum_{n=4}^\infty F_n < 1/20\,.$$

□

[1] These bounds are conveniently checked by numerically evaluating the explicit formulae for u_1, u_2, u_3 given in Lemmas 2, 3, and 4, using either a pocket calculator or a package such as Mathematica. One finds that $u_1 = 0.95459995\ldots$, $u_2 = 0.55800098\ldots$, and $u_3 = 0.38595811\ldots$.

[2] Note that $F_4 = 81/2080 \approx 0.0389$, $F_5 = 243/251680 \approx 0.000965512$, $F_6 = 729/91611520 \approx 0.0000079$, etc., and in fact $\sum_{n=4}^\infty F_n = 0.039915\ldots$.

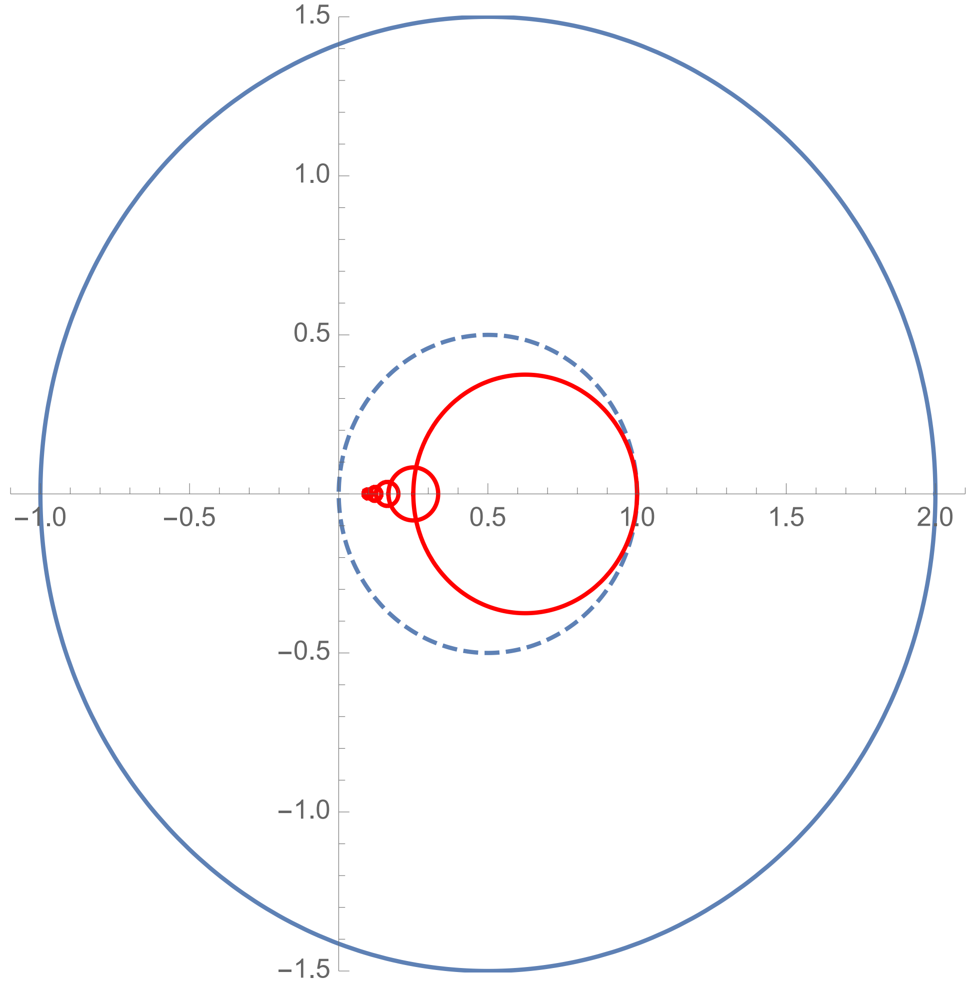

FIGURE 1. Inner disc D' (dashed) contains images $T_2(D)$, $T_4(D)$, $T_6(D)$, $T_8(D)$, $T_{10}(D)$ of the outer disc D (centre $1/2$, radius $3/2$), in the proof that $\dim(E_{\{2,4,6,8,10\}}) > 1/2$

Remark 2. The specific disc D used in the proof of Lemma 6 ensures that error bounds are both reasonably sharp and take a conveniently simple form. Its contraction ratio $1/3$ is fairly close to optimal, though in fact the minimum possible contraction ratio is slightly smaller than $3/10$, and if we were wishing to establish very high accuracy rigorous bounds on $\dim(E_{\{2,4,6,8,10\}})$ then it would be preferable to work with a disc whose contraction ratio is (close to) optimal. Note that our choice of D here is not available in the case of alphabets A containing the number 1, since the point -1 on the boundary of D is then a pole for the function $1/(z+1)$ which arises in defining the associated transfer operator.

We can now prove the desired result:

Theorem 7. *The Hausdorff dimension of* $E_{\{2,4,6,8,10\}}$ *is strictly larger than* $1/2$.

n	s_n
1	0.48423601174084654015914428125801664463082136184352
2	0.51785646889922347669500264756892828759037033720127
3	0.51735835552554373712759333961028665424316762904677
4	0.51735703035327082175724494790903719578904071340121
5	0.51735703093697422452618598486769311779169231777479
6	0.51735703093701730520259909968044128779914246471704
7	0.51735703093701730466662960310305782115301520544050
8	0.51735703093701730466662847483603679980173115413977
9	0.51735703093701730466662847483643973376603352818029
10	0.51735703093701730466662847483643973379049172430329

TABLE 1. Approximations s_n to $\dim(E_{\{2,4,6,8,10\}})$

Proof. For $A = \{2, 4, 6, 8, 10\}$, Lemmas 5 and 6 together give $\mathfrak{D}(1/2) < 0$ where $\mathfrak{D} = \mathfrak{D}_A$. Proposition 1 then implies that $\dim(E_A) > 1/2$. $\square$

To end this section we provide (see Table 1) a sequence of approximations to the dimension of $E_{\{2,4,6,8,10\}}$, indicating that

$$\dim(E_{\{2,4,6,8,10\}}) = 0.517357030937017304666628474836439733 7\ldots.$$

With extra work, the majority of these empirically observed decimal digits could be rigorously justified using the techniques involving computed bounds (along the lines of §5 and §6), though in this section our preference was to establish, in a rather explicit way not relying on computer assistance, the more conservative lower bound $\dim(E_{\{2,4,6,8,10\}}) > 1/2$ which is of specific number-theoretic interest (see [5, p. 139] and [23, Lem. 2.20]).

5. The Hausdorff dimension of $E_{\{1,2,3,4,5\}}$

Here we consider the set $E_{\{1,2,3,4,5\}}$, corresponding to the choice $A = \{1, 2, 3, 4, 5\}$. The approximation s_N to $\dim(E_{\{1,2,3,4,5\}})$, based on periodic points of period up to N, is the zero (in the interval $(0, 1)$) of the function $\mathfrak{D}_N$ defined by (4); these approximations are tabulated in Table 2 for $1 \le n \le 8$. We note that the 7th and 8th approximations to $\dim(E_{\{1,2,3,4,5\}})$ share the first 13 decimal digits[3] 0.8368294436812.

It turns out that we can *rigorously* justify 8 of these decimal digits. Define

$$s^- = 0.83682944$$

and

$$s^+ = 0.83682945 = s^- + 10^{-8}.$$

We then claim:

Theorem 8. *The Hausdorff dimension* $\dim(E_{\{1,2,3,4,5\}})$ *lies in the interval* (s^-, s^+).

[3]Note that Hensley [17, p. 16] gives the ten decimal digit approximation 0.8368294437, where the first 9 digits are correct, and the final digit is rounded up.

n	s_n
1	0.705879459442766674905124438813
2	0.848104427201487198901594372491
3	0.837214988477016376170810547613
4	0.836824477038318042493697933421
5	0.836829420428362177143803729319
6	0.836829443722239849891499678185
7	0.836829443681235947667216097180
8	0.836829443681208815677961682649

TABLE 2. Approximations s_n to $\dim(E_{\{1,2,3,4,5\}})$

Proof. Since E_A is a subset of $\mathbb{R}$, its Hausdorff dimension is smaller than 1, and by Proposition 1 we know that $\dim(E_A)$ is the largest real zero of $\mathfrak{D}$. Our strategy is to firstly show that $\mathfrak{D}(s^-) < 0 < \mathfrak{D}(s^+)$, so that the continuous function $\mathfrak{D}$ has a zero in (s^-, s^+), and secondly show that $\mathfrak{D}$ is strictly increasing on $(s^+, 1)$, from which it follows that $\mathfrak{D}$ has no real zeros larger than s^+, hence that $\dim(E_A)$ must lie between s^- and s^+.

Let $D \subset \mathbb{C}$ be the open disc centred at c, of radius ϱ, where c is the largest real root of the polynomial

$$5c^7 + 60c^6 + 243c^5 + 309c^4 - 225c^3 - 459c^2 + 225c - 21\,,$$

so that

$$c \approx 0.87125926704398872810485343206695409630164248025156401329070629881 5\,,$$

and

$$\varrho = -2 + \sqrt{c^2 + 6c + 8 - 3/c}\,, \tag{17}$$

so that

$$\varrho \approx 1.2470534929824824598483785751791096246979111741665500043001273 5\,.$$

The relation (17) ensures that $T_1(c-\varrho)$ and $T_5(c+\varrho)$ are equidistant from c, and this common distance is denoted by $\varrho' = T_1(c-\varrho) - c = c - T_5(c+\varrho)$, so that

$$\varrho' \approx 0.7307765383817149373582105355817758624954070500891639699965633 49\,.$$

The specific choice of c is to ensure that the contraction ratio $\theta = \varrho'/\varrho$ is minimised, taking the value

$$\theta = \frac{\varrho'}{\varrho} \approx 0.5860025592278103347716108078872601737057117182784609220519 57\,.$$

Having computed the points of period up to $P = 8$ we can form the functions $s \mapsto \delta_n(s)$ for $1 \le n \le 8$, and evaluate these at $s = s^-$ to give

$$\mathfrak{D}_8(s^-) = 1 + \sum_{n=1}^{8} \delta_n(s^-) = (-7.23265042091732132359\ldots) \times 10^{-9} < -7 \times 10^{-9} < 0\,, \tag{18}$$

and at $s = s^+$ to give

$$\mathfrak{D}_8(s^+) = 1 + \sum_{n=1}^{8} \delta_n(s^+) = (1.24148369391570553114\ldots) \times 10^{-8} > 10^{-8} > 0\,. \qquad (19)$$

We now aim to show that the approximation $\mathfrak{D}_8$ is close enough to $\mathfrak{D}$ for (18) and (19) to imply, respectively, the negativity of $\mathfrak{D}(s^-)$ and the positivity of $\mathfrak{D}(s^+)$. In other words, we seek to bound the tail $\sum_{n=9}^{\infty} \delta_n(s)$, and this will be achieved by bounding the individual Taylor coefficients $\delta_n(s)$, for $n \geq 9 = P+1$. It will turn out that for $n \geq 13$ the cruder Euler bound on $\delta_n(s)$ is sufficient, while for $P+1 = 9 \leq n \leq 12 = Q$ we will use the upper computed Taylor bound (cf. Remark 1) $\beta^{M,+}_{n,N,+}(s)$ for suitable $M, N \in \mathbb{N}$.

Henceforth let $Q = 12$, $M = 150$, $N = 200$, and consider the case $s = s^-$. We first evaluate the $H^2(D)$ norms of the monomial images $\mathcal{L}_{A,s}(m_k)$ for $0 \leq k \leq N = 200$, as

$$\begin{aligned}
\|\mathcal{L}_{A,s}(m_0)\| &= 1.18094153698482882249447608084779380079799521014296\ldots \\
\|\mathcal{L}_{A,s}(m_1)\| &= 0.50373481635455365839901987777081994881907010494221\ldots \\
\|\mathcal{L}_{A,s}(m_2)\| &= 0.25538908510961660244036590250705094646855677581007\ldots \\
&\vdots \\
\|\mathcal{L}_{A,s}(m_{200})\| &= (9.2211490601699406685842370009793893017\ldots) \times 10^{-48}.
\end{aligned}$$

Using these norms $\|\mathcal{L}_{A,s}(m_k)\|$ we then evaluate, for $1 \leq n \leq M = 150$, the terms $\alpha_{n,N,+}(s) = \alpha_{n,200,+}(s)$ defined (cf. (9)) by

$$\alpha_{n,N,+}(s) = \left(\sum_{k=n-1}^{N} \|\mathcal{L}_{A,s}(m_k)\|^2 + \left(\sum_{i=1}^{5} \|w_{i,s}\|_\infty \right)^2 \frac{\theta^{2(N+1)}}{1-\theta^2} \right)^{1/2}$$

to be

$$\begin{aligned}
\alpha_{1,200,+}(s) &= 1.31924766289256695924356827596610055341618618514631\ldots \\
\alpha_{2,200,+}(s) &= 0.58804037469497804159060266597641325581551232133109\ldots \\
\alpha_{3,200,+}(s) &= 0.30338542658416252872670480452558662518433118485741\ldots \\
&\vdots \\
\alpha_{150,200,+}(s) &= (8.4073197947570136649265418048602686584245204793167\ldots) \times 10^{-36}.
\end{aligned}$$

The terms $\alpha_{n,200,+}(s)$ are then used to form the upper computed Taylor bounds $\beta^{M,+}_{n,N,+}(s) = \beta^{M,-}_{n,N,+}(s) + \sum_{l=0}^{n-1} J^{n-l}_{Q,N,s}\, \beta^{M,-}_{l,N,+}(s)\, \theta^{M(n-l)} E_{n-l}(\theta)$, where

$$\beta^{M,-}_{n,N,+}(s) = \beta^{150,-}_{n,200,+}(s) = \sum_{i_1<\ldots<i_n\leq 150} \prod_{j=1}^{n} \alpha_{i_j,200,+}(s)\,,$$

which for $9 \leq n \leq 12 = Q$ are[4]

$$\beta^{M,+}_{9,N,+}(s) = (3.8691484792014233501009508862660178562663259339993\ldots) \times 10^{-9}$$

[4]Although not needed in this proof, we record here that the values of $\beta^{M,+}_{n,N,+}(s)$ for $1 \leq n \leq 8$ are $\beta^{M,+}_{1,N,+}(s) \approx 2.58$, $\beta^{M,+}_{2,N,+}(s) \approx 2.22$, $\beta^{M,+}_{3,N,+}(s) \approx 0.84$, $\beta^{M,+}_{4,N,+}(s) \approx 0.16$, $\beta^{M,+}_{5,N,+}(s) \approx 0.015$, $\beta^{M,+}_{6,N,+}(s) \approx 0.00085$, $\beta^{M,+}_{7,N,+}(s) \approx 0.000025$, $\beta^{M,+}_{8,N,+}(s) \approx 4.15 \times 10^{-7}$.

$$\beta^{M,+}_{10,N,+}(s) = (2.041028155630093895625799528930764710962712003414\ldots) \times 10^{-11}$$
$$\beta^{M,+}_{11,N,+}(s) = (6.130924622613936837872004195147235402486502450229\ldots) \times 10^{-14}$$
$$\beta^{M,+}_{12,N,+}(s) = (1.052236362635027746065630357473084205235777881109\ldots) \times 10^{-16}$$

so in particular

$$\sum_{n=9}^{12} |\delta_n(s)| \le \sum_{n=9}^{12} \beta^{M,+}_{n,N,+}(s) < 3.9 \times 10^{-9}. \tag{20}$$

It remains to derive the Euler bounds on the Taylor coefficients $\delta_n(s)$ for $n \ge 13$. For $s > 0$ and $i \in \{1, 2, 3, 4, 5\}$, the function $w_{i,s}(z) = 1/(z+i)^{2s}$ has maximum modulus on D when $z = c - \varrho$, so

$$\|w_{i,s}\|_\infty = 1/(i + c - \varrho)^{2s}\,. \tag{21}$$

A computation using (21) gives

$$\|w_{1,s}\|_\infty \le 2.20065253120324840404447910422664240546255334143101505817715 5\,,$$
$$\|w_{2,s}\|_\infty \le 0.444077465889954989420982559661627815714819270961004072921669\,,$$
$$\|w_{3,s}\|_\infty \le 0.198948407046876624291927334956495986322487588119823603200126\,,$$
$$\|w_{4,s}\|_\infty \le 0.115896001097710230802023825180791690553618611817392771206340\,,$$
$$\|w_{5,s}\|_\infty \le 0.077082300149426430401659913390892369783863063355289787925134\,,$$

thus

$$\sum_{i=1}^{5} \|w_{i,s}\|_\infty \le 3.036656705387216678961072737416450267837341875684525293430\,,$$

and therefore

$$K_s = \frac{\sum_{i=1}^{5} \|w_{i,s}\|_\infty}{\theta\sqrt{1-\theta^2}} \le 6.39507165244054791777743776407948610 7\,.$$

Now $|\delta_n(s)| \le K_s^n E_n(\theta)$, and we readily compute that

$$K_s^{13} E_{13}(\theta) < (1.400110201142029734380103146354603164131262801 65\ldots) \times 10^{-10}\,,$$
$$K_s^{14} E_{14}(\theta) < (5.044817236971637679074225231056832130389446340 54\ldots) \times 10^{-13}\,,$$

and the super-exponential decay of the terms $K_s^n E_n(\theta)$ means we easily bound

$$\left|\sum_{n=13}^{\infty} \delta_n(s)\right| \le \sum_{n=13}^{\infty} K_s^n E_n(\theta) < 1.5 \times 10^{-10}\,. \tag{22}$$

Combining (22) with (20) gives, for $s = s^-$,

$$\left|\sum_{n=9}^{\infty} \delta_n(s)\right| < 4 \times 10^{-9}\,. \tag{23}$$

Combining (23) with (18) then gives

$$\mathfrak{D}(s^-) = 1 + \sum_{n=1}^{\infty} \delta_n(s^-) < -3 \times 10^{-9} < 0\,. \tag{24}$$

We now show that $\mathfrak{D}(s^+)$ is positive. In view of (19), for this it is sufficient to show that $|\sum_{n=9}^{\infty} \delta_n(s)| < 10^{-8}$ for $s = s^+$. In fact the stronger inequality (23) (which we have proved for $s = s^-$) can also be established for $s = s^+$, using the same general method as for $s = s^-$, since the intermediate computed values for the norms $\|\mathcal{L}_{A,s}(m_k)\|$, the terms $\alpha_{n,N,+}(s)$, the computed Taylor bounds $\beta_{n,N,+}^{M,+}(s)$, and the Euler bounds $K_s^n E_n(\theta)$, are sufficiently close to those for $s = s^- = s^+ - 10^{-8}$. Combining (19) with inequality (23) for $s = s^+$ gives the required positivity

$$\mathfrak{D}(s^+) = 1 + \sum_{n=1}^{\infty} \delta_n(s^+) > 0\,. \tag{25}$$

Since $\mathfrak{D}$ is continuous, (24) and (25) imply that it has a zero in (s^-, s^+), and in particular that $\dim(E_A)$, as the largest zero of $\mathfrak{D}$ (by Proposition 1), is larger than s^-. To prove that $\dim(E_A) < s^+$ it now suffices to show that $\mathfrak{D}$ is strictly increasing on $(s^+, 1)$, and hence has no zeros in this interval. For this we use that the function $\mathfrak{D}_8(s) = 1 + \sum_{n=1}^{8} \delta_n(s)$ is available to us in closed form, together with an estimate on the derivative of the remainder function

$$\mathfrak{R}_8(s) := \mathfrak{D}(s) - \mathfrak{D}_8(s) = \sum_{n=9}^{\infty} \delta_n(s)\,.$$

In particular, $\mathfrak{D}_8$ can be shown to be both strictly increasing and strictly concave on the interval $(s^+, 1)$ (cf. Figure 2, showing the restriction of $\mathfrak{D}_8$ to $[0, 1]$), with

$$\mathfrak{D}_8'(s) > \mathfrak{D}_8'(1) = 1.3546901785\ldots > \frac{13}{10} \quad \text{for all } s \in (s^+, 1)\,. \tag{26}$$

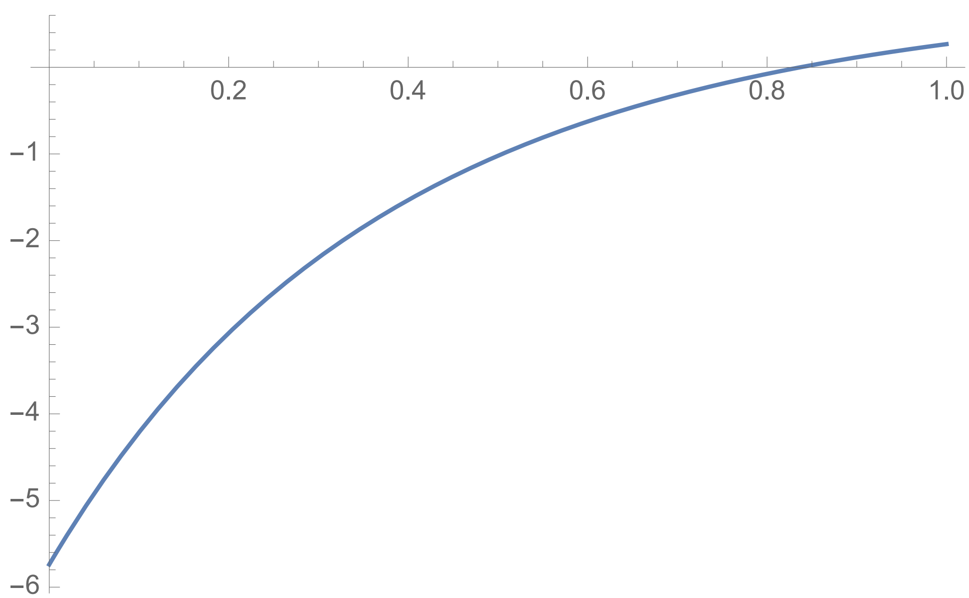

FIGURE 2. Order-8 approximation $\mathfrak{D}_8$ to the dimension determinant for $E_{\{1,2,3,4,5\}}$.

Let U denote the ε-neighbourhood in $\mathbb{C}$ of the real interval $(s^+, 1)$, where for concreteness we choose $\varepsilon = 1/10$. We shall bound the modulus of $\mathfrak{R}_8$ on U via Euler bounds on the coefficients $\delta_n(s)$ for $n > 8$, $s \in U$, and then use Cauchy's integral formula to derive a bound on $\mathfrak{R}_8'(s)$ for $s \in (s^+, 1)$. Recall (see Theorem 6) that $|\delta_n(s)| \le K_s^n E_n(\theta)$, where $E_n(\theta) = \theta^{n(n+1)/2} \prod_{i=1}^n (1-\theta^i)^{-1}$ is independent of s, and

$$K_s = \frac{\sum_{i=1}^5 \|w_{i,s}\|_\infty}{\theta\sqrt{1-\theta^2}} = \frac{\sum_{i=1}^5 (i+c-\varrho)^{-2s}}{\theta\sqrt{1-\theta^2}} .$$

It is readily shown that

$$\sup_{s\in U} |K_s| = K_{1+\varepsilon} = K_{11/10} = 7.11229430658606518348\ldots ,$$

so that

$$\sup_{s\in U} |\delta_9(s)| \le K_{11/10}^9 E_9(\theta) = 0.01024367095233740092\ldots ,$$

$$\sup_{s\in U} |\delta_{10}(s)| \le K_{11/10}^{10} E_{10}(\theta) = 0.00034957413642133622\ldots ,$$

$$\sup_{s\in U} |\delta_{11}(s)| \le K_{11/10}^{11} E_{11}(\theta) = 0.00000697687020201114\ldots ,$$

$$\sup_{s\in U} |\delta_{12}(s)| \le K_{11/10}^{12} E_{12}(\theta) = 0.00000008150368808892\ldots ,$$

and we can therefore bound

$$\sup_{s\in U} |\mathfrak{R}_8(s)| \le \sum_{n=9}^\infty \sup_{s\in U} |\delta_n(s)| < \frac{11}{1000} . \tag{27}$$

If C_s denotes the positively oriented circular contour of radius $\varepsilon = 1/10$, centred at s, then Cauchy's integral formula gives the derivative formula $\mathfrak{R}_8'(s) = \frac{1}{2\pi i}\oint_{C_s} \frac{\mathfrak{R}_8(t)}{(t-s)^2}\,dt$, and U is the union over $s \in (s^+, 1)$ of the open discs bounded by the C_s, so (27) yields

$$|\mathfrak{R}_8'(s)| \le \frac{1}{\varepsilon^2} \sup_{t\in C_s} |\mathfrak{R}_8(t)| \le 100 \sup_{t\in U} |\mathfrak{R}_8(t)| < \frac{11}{10} \quad \text{for all } s \in (s^+, 1) .$$

In particular,

$$\mathfrak{R}_8'(s) > -\frac{11}{10} \quad \text{for all } s \in (s^+, 1) , \tag{28}$$

so combining (26) and (28) gives

$$\mathfrak{D}'(s) = \mathfrak{D}_8'(s) + \mathfrak{R}_8'(s) > 0 \quad \text{for all } s \in (s^+, 1) ,$$

so indeed $\mathfrak{D}$ is strictly increasing on $(s^+, 1)$, as required. □

Remark 3. The analysis of Bourgain-Kontorovich and Huang also applies to more general finite subsets $A \subset \mathbb{N}$, see [18]. More precisely, if $\dim(E_A) > \frac{5}{6}$ then the corresponding subset $Q_A \subset \mathbb{N}$ has density one. Therefore, it is natural to consider other non-sequential finite subsets A for which we can rigorously show $\dim(E_A) > \frac{5}{6}$.

Using the same method as that used in this section, one can show rigorously that the dimensions of the sets E_A associated to the choices $A = \{1, 2, 3, 5, 6, 8\}$ or $A = \{1, 2, 3, 4, 6, 18\}$, for example, are greater than $\frac{5}{6}$.

n	s_n
1	0.742538972647559226233764770933
2	0.878373312250454078800953132613
3	0.867983314494266000322362181011
4	0.867614537223019698282665367406
5	0.867619151810964612388367917252
6	0.867619173277394444047871397558
7	0.867619173240135215103752503105
8	0.867619173240110928010919906321

TABLE 3. Approximations s_n to $\dim(E_{\{1,2,3,4,5,6\}})$

6. The Hausdorff dimension of $E_{\{1,2,3,4,5,6\}}$

Here we consider the set $E_{\{1,2,3,4,5,6\}}$, corresponding to the choice $A = \{1,2,3,4,5,6\}$. The approximation s_N to $\dim(E_{\{1,2,3,4,5,6\}})$, based on periodic points of period up to N, is the zero (in the interval $(0,1)$) of the function $\mathfrak{D}_N$ defined by (4); these approximations are tabulated in Table 3 for $1 \le n \le 8$. We note that the 7th and 8th approximations to $\dim(E_{\{1,2,3,4,5,6\}})$ share the first 13 decimal digits 0.8676191732401.

It turns out that we can *rigorously* justify 7 of these decimal digits. Define

$$s^- = 0.8676191$$

and

$$s^+ = 0.8676192 = s^- + 10^{-7}\,.$$

We then claim:

Theorem 9. *The Hausdorff dimension* $\dim(E_{\{1,2,3,4,5,6\}})$ *lies in the interval* (s^-, s^+).

Proof. The strategy of proof is similar to that used for Theorem 8, firstly showing that $\mathfrak{D}(s^-) < 0 < \mathfrak{D}(s^+)$ so that $\mathfrak{D}$ has a zero in (s^-, s^+), and secondly arguing that this is the largest zero of $\mathfrak{D}$, hence must be $\dim(E_A)$.

Let $D \subset \mathbb{C}$ be the open disc centred at c, of radius ϱ, where this time

$$c \approx 0.888786621704996501948480357049568065602437524401186717911139539201$$

is chosen as the largest real root of the polynomial

$$384c^7 + 5376c^6 + 25872c^5 + 42560c^4 - 16660c^3 - 67228c^2 + 26803c - 2744\,,$$

and

$$\varrho = -\frac{5}{2} + \frac{1}{2}\sqrt{4c^2 + 28c + 45 - 14/c} \approx 1.284639341742533191143484074021163452454469\,.$$

It follows that

$$\varrho' = T_1(c-\varrho) - c = c - T_6(c+\varrho) \approx 0.76643890552427727077005511585427320750401107808160\,,$$

and therefore

$$\theta = \frac{\varrho'}{\varrho} \approx 0.596617961648792936828996037574102515963872474543358842308573\,.$$

The points of period up to $P = 8$ determine the functions $s \mapsto \delta_n(s)$ for $1 \leq n \leq 8$, which when evaluated at $s = s^-$ and $s = s^+$ give

$$\mathfrak{D}_8(s^-) = 1 + \sum_{n=1}^{8} \delta_n(s^-) = (-1.49837375936920427086\ldots) \times 10^{-7} < -10^{-7} < 0\,, \quad (29)$$

$$\mathfrak{D}_8(s^+) = 1 + \sum_{n=1}^{8} \delta_n(s^+) = (5.47463816560924051357\ldots) \times 10^{-8} > 5 \times 10^{-8} > 0\,. \quad (30)$$

We now claim that $\mathfrak{D}_8$ is close enough to $\mathfrak{D}$ for the inequalities (29) and (30) to imply that $\mathfrak{D}(s^-) < 0 < \mathfrak{D}(s^+)$, and will establish this by bounding $\delta_n(s)$, for $n \geq 9 = P + 1$. As previously, for $n \geq 13$ the Euler bound on $\delta_n(s)$ turns out to be sufficient, while for $9 \leq n \leq 12 =: Q$ we use upper computed Taylor bounds $\beta^{M,+}_{n,N,+}(s)$, where once again we set $M := 150$, $N := 200$. To introduce some variety in the part of the proof presented in full, and in recognition of the fact that in the present case $\mathfrak{D}_8(s^+)$ is closer to zero than $\mathfrak{D}_8(s^-)$ is, we here consider the case $s = s^+$.

The norms $\|\mathcal{L}_{A,s}(m_k)\|$ are computed via numerical integration, and then used to form the terms $\alpha_{n,200,+}(s)$, which are then used to form the upper computed Taylor bounds which for $9 \leq n \leq 12$ take the values

$$\beta^{M,+}_{9,N,+}(s) = (1.262194624669540668569841998650141041089448447560 1\ldots) \times 10^{-8}$$
$$\beta^{M,+}_{10,N,+}(s) = (8.314966430413518627081622024687687663503710477628\ldots) \times 10^{-11}$$
$$\beta^{M,+}_{11,N,+}(s) = (3.176610018228192242136810148998407171840692198466\ldots) \times 10^{-13}$$
$$\beta^{M,+}_{12,N,+}(s) = (7.061524069747938884792482724386219757269895805839\ldots) \times 10^{-16}\,,$$

from which

$$\sum_{n=9}^{12} |\delta_n(s)| \leq \sum_{n=9}^{12} \beta^{M,+}_{n,N,+}(s) < 1.3 \times 10^{-8}. \quad (31)$$

To compute the Euler bounds on $\delta_n(s)$ for $n \geq 13$ we note, as previously, that $\|w_{i,s}\|_\infty = 1/(i + c - \varrho)^{2s}$, whence $\|w_{1,s}\|_\infty = 2.39\ldots$, $\|w_{2,s}\|_\infty = 0.44\ldots$, $\|w_{3,s}\|_\infty = 0.18\ldots$, $\|w_{4,s}\|_\infty = 0.10\ldots$, $\|w_{5,s}\|_\infty = 0.07\ldots$, $\|w_{6,s}\|_\infty = 0.05\ldots$, and

$$\sum_{i=1}^{6} \|w_{i,s}\|_\infty \leq 3.2569770652183742209338412506577 7\,,$$

therefore

$$K_s = \frac{\sum_{i=1}^{6} \|w_{i,s}\|_\infty}{\theta\sqrt{1-\theta^2}} \leq 6.8023596969991813862882005017255101914 55\,.$$

It follows that[5]

$$K_s^{13} E_{13}(\theta) < (1.751608670306048305544710571625984526147775\ldots) \times 10^{-9}\,,$$
$$K_s^{14} E_{14}(\theta) < (8.632960731433444691012413481027827464799512\ldots) \times 10^{-12}\,,$$

[5]Note that $K_s^{12} E_{12}(\theta) = (2.119\ldots) \times 10^{-7}$, which is slightly too large for our purposes, thus justifying the choice of $Q = 12$ as the largest index for which the computed Taylor bound, rather than the Euler bound, is used.

and

$$\left|\sum_{n=13}^{\infty}\delta_n(s)\right| \le \sum_{n=13}^{\infty} K_s^n E_n(\theta) < 2\times 10^{-9}\,, \tag{32}$$

so (31), (32) together give, for $s = s^+$,

$$\left|\sum_{n=9}^{\infty}\delta_n(s)\right| < 1.5\times 10^{-8}\,, \tag{33}$$

and combining (33) with (30) gives

$$\mathfrak{D}(s^+) = 1 + \sum_{n=1}^{\infty}\delta_n(s^+) > 3.5\times 10^{-8} > 0\,. \tag{34}$$

It remains to show that $\mathfrak{D}(s^-)$ is negative. In view of (29), for this it is sufficient to show that $|\sum_{n=9}^{\infty}\delta_n(s)| < 10^{-7}$ for $s = s^-$. In fact the stronger inequality (33) (which we have proved for $s = s^+$) can also be established for $s = s^-$, using the same general method as for $s = s^+$, since the intermediate computed values for $\|\mathcal{L}_{A,s}(m_k)\|$, $\alpha_{n,N,+}(s)$, $\beta_{n,N,+}^{M,+}(s)$, and $K_s^n E_n(\theta)$, are sufficiently close to those for $s = s^+ = s^- + 10^{-8}$. Combining (29) with inequality (33) for $s = s^-$ gives the required negativity

$$\mathfrak{D}(s^-) = 1 + \sum_{n=1}^{\infty}\delta_n(s^-) < 0\,. \tag{35}$$

Since $\mathfrak{D}$ is continuous, (34) and (35) imply that it has a zero in (s^-, s^+), and in particular that $\dim(E_A)$, as the largest zero of $\mathfrak{D}$ (by Proposition 1), is larger than s^-.

To prove that $\dim(E_A) < s^+$ it now suffices to show that $\mathfrak{D}$ has no zeros in $(s^+, 1)$. For this, it is technically convenient to deviate slightly from the approach used in the proof of Theorem 8, by firstly establishing that $\mathfrak{D}$ is strictly increasing on $(s^+, 9/10)$ (hence has no zeros in this interval, since $\mathfrak{D}(s^+) > 0$), and then showing directly that $\mathfrak{D}$ is strictly positive on $[9/10, 1]$.

The function $\mathfrak{D}_8$ can be shown to be both strictly increasing and strictly concave on the interval $(s^+, 9/10)$, with

$$\mathfrak{D}_8'(s) > \mathfrak{D}_8'(9/10) = 1.8898838248\ldots > \frac{3}{2} \quad \text{for all } s \in (s^+, 9/10)\,. \tag{36}$$

Define $\mathfrak{R}_8(s) := \mathfrak{D}(s) - \mathfrak{D}_8(s) = \sum_{n=9}^{\infty}\delta_n(s)$, and let U denote the ε-neighbourhood in $\mathbb{C}$ of the interval $(s^+, 9/10)$, where $\varepsilon = 1/5$. Now $|\delta_n(s)| \le K_s^n E_n(\theta)$ (by Theorem 6), where

$$K_s = \frac{\sum_{i=1}^{6}\|w_{i,s}\|_\infty}{\theta\sqrt{1-\theta^2}} = \frac{\sum_{i=1}^{6}(i+c-\varrho)^{-2s}}{\theta\sqrt{1-\theta^2}}\,,$$

and it is readily shown that

$$\sup_{s\in U}|K_s| = K_{9/10+\varepsilon} = K_{11/10} = 7.5658074521937180033 5\ldots\,,$$

so that

$$\sup_{s\in U}|\delta_9(s)| \le K_{11/10}^9 E_9(\theta) = 0.04376686701280541755\ldots\,,$$

$$\sup_{s\in U}|\delta_{10}(s)| \le K_{11/10}^{10}E_{10}(\theta) = 0.00190306136091161412\ldots\,,$$

$$\sup_{s\in U}|\delta_{11}(s)| \le K_{11/10}^{11}E_{11}(\theta) = 0.00004925505297772416\ldots\,,$$

$$\sup_{s\in U}|\delta_{12}(s)| \le K_{11/10}^{12}E_{12}(\theta) = 0.00000075953226513723\ldots\,,$$

and we can therefore bound

$$\sup_{s\in U}|\mathfrak{R}_8(s)| \le \sum_{n=9}^{\infty}\sup_{s\in U}|\delta_n(s)| < \frac{1}{20}\,. \tag{37}$$

If C_s denotes the positively oriented circular contour of radius $\varepsilon = 1/5$, centred at s, then Cauchy's integral formula together with (37) yields

$$|\mathfrak{R}_8'(s)| \le \frac{1}{\varepsilon^2}\sup_{t\in C_s}|\mathfrak{R}_8(t)| \le 25\sup_{t\in U}|\mathfrak{R}_8(t)| < \frac{5}{4} \quad \text{for all } s\in(s^+,9/10)\,,$$

and in particular,

$$\mathfrak{R}_8'(s) > -\frac{5}{4} \quad \text{for all } s\in(s^+,9/10)\,, \tag{38}$$

so combining (36) and (38) gives

$$\mathfrak{D}'(s) = \mathfrak{D}_8'(s) + \mathfrak{R}_8'(s) > 0 \quad \text{for all } s\in(s^+,9/10)\,,$$

so indeed $\mathfrak{D}$ is strictly increasing on $(s^+,9/10)$, as claimed.

It remains to show that $\mathfrak{D}$ has no zeros in the interval $[9/10,1]$. Since $\mathfrak{D}_8$ is increasing on this interval,

$$\mathfrak{D}_8(s) \ge \mathfrak{D}_8(9/10) = 0.06368315529812853238\ldots > \frac{1}{20} \quad \text{for all } s\in[9/10,1]\,. \tag{39}$$

Now $s\mapsto K_s^n$ is increasing on $[9/10,1]$, so if $s\in[9/10,1]$ then

$$|\delta_n(s)| \le K_s^n E_n(\theta) \le K_1^n E_n(\theta) = \frac{K_1^n\theta^{n(n+1)/2}}{\prod_{i=1}^n(1-\theta^i)} < AK_1^n\theta^{n(n+1)/2}$$

where $A := \prod_{i=1}^n(1-\theta^i)^{-1} = 6.780731869\ldots$, therefore

$$\sum_{n=9}^{\infty}|\delta_n(s)| < A\sum_{n=9}^{\infty}K_1^n\theta^{n(n+1)/2} < AK_1^9\theta^{45}\sum_{i=0}^{\infty}(K_1\theta^{10})^i = \frac{AK_1^9\theta^{45}}{1-K_1\theta^{10}} = 0.02845\ldots < \frac{3}{100}\,,$$

and hence

$$\sup_{s\in[9/10,1]}|\mathfrak{R}_8(s)| \le \sup_{s\in[9/10,1]}\sum_{n=9}^{\infty}|\delta_n(s)| < \frac{3}{100}\,. \tag{40}$$

From (39) and (40) it follows that

$$\mathfrak{D}(s) = \mathfrak{D}_8(s) + \mathfrak{R}_8(s) > \frac{1}{20} - \frac{3}{100} > 0 \quad \text{for all } s\in[9/10,1]\,,$$

so indeed $\mathfrak{D}$ has no zeros in the interval $[9/10,1]$, and the proof is complete. □

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