Metric Dimension: from Graphs to Oriented Graphs

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Abstract

The metric dimension $\mathrm{MD}(G)$ of an undirected graph G is the cardinality of a smallest set of vertices that allows, through their distances to all vertices, to distinguish any two vertices of G. Many aspects of this notion have been investigated since its introduction in the 70's, including its generalization to digraphs.

In this work, we study, for particular graph families, the maximum metric dimension over all strongly-connected orientations, by exhibiting lower and upper bounds on this value. We first exhibit general bounds for graphs with bounded maximum degree. In particular, we prove that, in the case of subcubic n-node graphs, all strongly-connected orientations asymptotically have metric dimension at most $\frac{n}{2}$, and that there are such orientations having metric dimension $\frac{2n}{5}$. We then consider strongly-connected orientations of grids. For a torus with n rows and m columns, we show that the maximum value of the metric dimension of a strongly-connected Eulerian orientation is asymptotically $\frac{nm}{2}$ (the equality holding when n, m are even, which is best possible). For a grid with n rows and m columns, we prove that all strongly-connected orientations asymptotically have metric dimension at most $\frac{2nm}{3}$, and that there are such orientations having metric dimension $\frac{nm}{2}$.

Keywords: Resolving sets, Metric dimension, Strongly-connected orientations.

1 Introduction

1.1 Resolving sets and metric dimension in undirected graphs

The *distance* $\mathrm{dist}_G(u, v)$ (or simply $\mathrm{dist}(u, v)$ when no ambiguity is possible) between two vertices u, v of an undirected graph G is the length of a shortest path from u to v. A *resolving set* R of G is a subset of vertices that permits to distinguish all vertices of G according to their distances to the vertices of R. In other words, R is resolving if and only if, for every two distinct vertices u, v of G, there exists $w \in R$ such that $\mathrm{dist}_G(w, u) \neq \mathrm{dist}_G(w, v)$. The *metric dimension* $\mathrm{MD}(G)$ of G is the minimum cardinality of a resolving set of G. Since $V(G) \setminus \{v\}$ is a resolving set of G for every $v \in V(G)$, this parameter $\mathrm{MD}(G)$ is defined for every undirected graph G.

The notions of resolving sets and metric dimension have been widely studied since their introduction in the 70's by Harary and Melter [8], and Slater [14], notably because they can be used to model many real-life problems. Many related aspects have been investigated to date,

*This work has been partially supported by ANR project MultiMod and ANR program "Investments for the Future" under reference ANR-11-LABX-0031-01. Due to lack of space, several proofs are only sketched. Full proofs can be found at `https://hal.inria.fr/hal-01938290`.

including algorithmic and complexity aspects, and bounds on the metric dimension of particular graph families. Our main focus in this paper being the metric dimension of oriented graphs, we refer the interested reader to surveys (e.g. [1, 2]) for more details about investigations in the undirected context.

1.2 Resolving sets and metric dimension in digraphs

A natural way of generalizing graph theoretical problems is to consider their directed counterparts. In the context of the metric dimension of graphs, this was first considered by Chartrand, Rains, and Zhang in [3], before receiving further consideration in several works (see [5, 6, 9, 11, 12]). It is worthwhile recalling that, in digraphs, distances have behaviours that differ from those in undirected graphs. Notably, an important point that should be addressed is that, in the context of general digraphs D, we might have $\mathrm{dist}(u,v) \neq \mathrm{dist}(v,u)$ for any two vertices u, v, where $\mathrm{dist}(u,v)$ here refers to the length of a shortest directed path from u to v. A digraph D is *strongly-connected* (or *strong* for short) if, for every $u, v \in V(D)$, there is a directed path from u to v, and conversely one from v to u. Hence, if D is not strong, then there are vertices $u, v \in V(D)$ such that no directed paths from u to v exist. In such a case, we set $\mathrm{dist}(u,v) = +\infty$.

These peculiar aspects of distances in digraphs must be taken into account when defining directed notions of resolving sets and metric dimension. Throughout this work, the notions of resolving sets and metric dimension in digraphs are with respect to the following definitions. Let R be a subset of vertices of a digraph D. Two vertices u, v of D are said to be *distinguished*, denoted by $u \nsim_R v$, if there exists $w \in R$ such that $\mathrm{dist}(w,u) \neq \mathrm{dist}(w,v)$. Otherwise, u and v are *undistinguished* by R, which is denoted by $u \sim_R v$. In particular, if $\mathrm{dist}(w,u)$ is finite and $\mathrm{dist}(w,v)$ is not for some $w \in R$, then $u \nsim_R v$. A set $R \subseteq V(D)$ is called *resolving* if all pairs of vertices of D are distinguished by R. The *metric dimension* $\mathrm{MD}(D)$ of D is then the smallest size of a resolving set. Note that $\mathrm{MD}(D)$ is defined for every digraph D; in particular, we have $\mathrm{MD}(D) < |V(D)|$ since $R = V(D) \setminus \{v\}$ is a resolving set for any $v \in V(D)$ (as having any vertex in a resolving set makes it distinguished from all other vertices).

Our definitions of directed resolving sets and metric dimension actually differ from those originally introduced by Chartrand, Rains, and Zhang. On the one hand, in their definition of resolving sets, they consider the distances from each of the vertices not in R to the vertices in R in order to distinguish the vertices of D. In our definition, the distances from each of the vertices in R to the vertices not in R are considered. Note that both definitions are equivalent on that point, as, given a digraph D, if we reverse the direction of all arcs, resulting in a digraph $\widetilde{D}$, then any shortest path from u to v in D becomes a shortest path from v to u in $\widetilde{D}$.

On the other hand, their definition of resolving sets requires that the distances from each pair of distinct vertices to the vertices in R which distinguish them be defined, while our definition (with distances from vertices in R to the other vertices) allows for undefined distances ($+\infty$) to be used as well. Contrary to our definition, this implies that their definition of metric dimension is not defined for all digraphs. As far as we know, the characterization of digraphs that admit a metric dimension (following their definition) is still an open problem [3].

Although our definitions and those of Chartrand, Rains, and Zhang are different, it is worthwhile mentioning that most of our investigations in this paper also apply to their context, as we mainly focus on strong digraphs, in which case our definitions and theirs are equivalent (up to reversing all arcs).

To date, the investigations on the metric dimension of digraphs have thus been with respect to the definitions originally introduced by Chartrand, Rains, and Zhang. As a first step, they notably gave in [3], a characterization of digraphs with metric dimension 1. Complexity aspects were considered in [12], where it was proved that determining the metric dimension of a strong digraph is NP-complete. Bounds on the metric dimension of various digraph families were later exhibited (Cayley digraphs [5], line digraphs [6], tournaments [9], digraphs with cyclic covering [11], De Bruijn and Kautz digraphs [12], etc.).

1.3 From undirected graphs to oriented graphs

To avoid any confusion, let us recall that an *orientation* D of an undirected graph G is obtained when every edge uv of G is oriented either from u to v (resulting in the arc (u,v)) or conversely (resulting in the arc (v,u)). An *oriented graph* D is a directed graph that is an orientation of a simple graph. Note that when G is simple, D cannot have two vertices u,v such that (u,v) and (v,u) are arcs. Such symmetric arcs are allowed in digraphs, which is the main difference between oriented graphs and digraphs. Throughout this paper, when simply referring to a *graph*, we mean an undirected graph.

In [4], Chartrand, Rains, and Zhang considered the following way of linking resolving sets of undirected graphs and digraphs. They considered, for a given graph G, the worst orientations of G for the metric dimension, i.e., orientations of G with maximum metric dimension. Looking at our definition of resolving sets and metric dimension, this is a legitimate question as it has to be pointed out that, for a graph, the metric dimension might or might not be preserved when orienting its edges. An interesting example (reported e.g., in [3, 9]) is the case of a graph G with a Hamiltonian path: while $\mathrm{MD}(G)$ can be arbitrarily large in general (consider e.g., any complete graph), there is an orientation D of G verifying $\mathrm{MD}(D) = 1$ (just orient all edges of a Hamiltonian path from the first vertex towards the last vertex, and all remaining edges in the opposite direction). Conversely, there exist orientations D of G for which $\mathrm{MD}(D)$ can be much larger than $\mathrm{MD}(G)$. As an example, let us consider any path P with $2n+1$ vertices $v_0, ..., v_{2n}$. Clearly, $\mathrm{MD}(P) = 1$; however, the orientation D of P obtained by making every vertex v_{2k+1} ($k = 0, ..., n-1$) become a source (i.e., orienting its incident edges away) verifies $\mathrm{MD}(D) = n$. As shown in this paper, this phenomenon occurs for strong orientations as well.

In [4], the authors proved that, for every positive integer k, there exist infinitely many graphs for which the metric dimension of any of its strongly-connected orientations is exactly k. They have also proved that there is no constant k such that the metric dimension of any tournament is at most k.

1.4 Our results

Motivated by these observations, we investigate, throughout this work, the parameter WOMD defined as follows. For any connected graph G, let $\mathrm{WOMD}(G)$ denote the maximum value of $\mathrm{MD}(D)$ over all strong orientations D of G. Let us extend this definition to graph families as follows. For any family $\mathcal{G}$ of 2-edge-connected graphs[1], let $\mathrm{WOMD}(\mathcal{G}) = \max_{G \in \mathcal{G}} \frac{\mathrm{WOMD}(G)}{|V(G)|}$. Section 2 first introduces tools and results that will be used in the next sections. In Section 3, bounds on $\mathrm{WOMD}(\mathcal{G}_\Delta)$ are proved, where $\mathcal{G}_\Delta$ refers to the family of 2-edge-connected graphs with maximum degree Δ. In particular, we prove that we asymptotically have $\frac{2}{5} \leq \mathrm{WOMD}(\mathcal{G}_3) \leq \frac{1}{2}$. In Section 4, we then consider the families of grids and tori. For the family $\mathcal{T}$ of tori, we prove that we asymptotically have $\mathrm{WEOMD}(\mathcal{T}) = \frac{1}{2}$, where the parameter $\mathrm{WEOMD}(\mathcal{T})$ is defined similarly to $\mathrm{WOMD}(\mathcal{T})$ except that only strong Eulerian orientations of tori (i.e., all vertices have in-degree and out-degree 2) are considered. For the family $\mathcal{G}$ of grids, we then prove that asymptotically $\frac{1}{2} \leq \mathrm{WOMD}(\mathcal{G}) \leq \frac{2}{3}$. Remaining open questions and problems are gathered in Section 5.

Terminology and notation Let D be a digraph. For a vertex v of D, we denote by $d^-_D(v)$ (resp. $d^+_D(v)$) the *in-degree* (resp. *out-degree*) of v which is the number of in-coming

[1] The edge-connectivity requirement, here and further, is to guarantee the good definition of $\mathrm{WOMD}(G)$ for every $G \in \mathcal{G}_\Delta$, as it is a well-known fact that a graph has strong orientations if and only if it is 2-edge-connected (see [13]).

(resp. out-going) arcs incident to v. For every arc (v, u) (resp. (u, v)) in-coming to (resp. out-going from) u, we call u an *out-neighbour* (resp. *in-neighbour*) of v. The set of all in-neighbours (resp. out-neighbours) of v is denoted by $N_D^-(v)$ (resp. $N_D^+(v)$). The subscripts in the previous notations might be dropped when no ambiguity is possible. We denote by $\Delta^+(D)$ (resp., $\Delta^-(D)$) the maximum out-degree (resp., maximum in-degree) of a vertex in D. Note that, in an oriented graph D, the in-degree (resp. out-degree) of a vertex corresponds to the cardinality of its in-neighbourhood (resp. out-neighbourhood).

2 Tools and preliminary results

We start off by pointing out the following property of resolving sets in digraphs having vertices with the same in-neighbourhood. This result will be one of our main tools for building digraphs with large metric dimension.

Lemma 1. *Let D be a digraph and $S \subseteq V(D)$ be a subset of $|S| \geq 2$ vertices such that, for every $u, v \in S$, we have $N^-(u) = N^-(v)$. Then, any resolving set of D contains at least $|S| - 1$ vertices of S.*

Proof. If two vertices $u, v \in S$ do not belong to a resolving set R, then $\text{dist}(w, u) = \text{dist}(w, v)$ for every $w \in R$, contradicting that R is a resolving set. $\square$

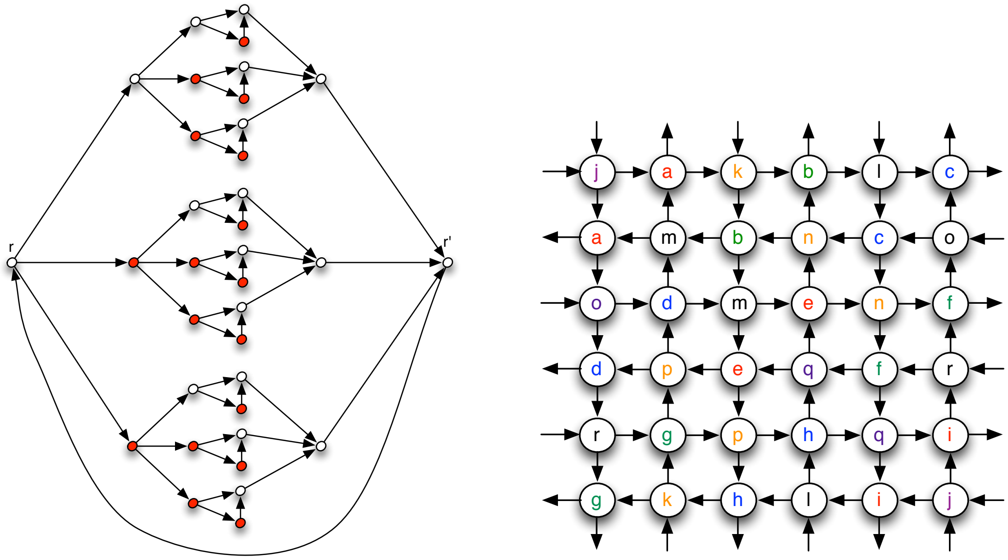
We now introduce a technique that will be used in the next sections for exhibiting upper bounds on the metric dimension of strong digraphs with maximum out-degree at least 2. The technique is based on a connection between the resolving sets of a such digraph and the vertex covers of a particular graph associated to it. A *vertex cover* of a graph G is a subset $S \subseteq V(G)$ of vertices such that, for every edge uv of G, at least one of u and v belongs to S. To any digraph D we associate an *auxiliary (undirected) graph* D_{aux} constructed as follows: the vertices of D_{aux} are those of D; for every two distinct vertices u, v of D such that $N_D^-(u) \cap N_D^-(v) \neq \emptyset$, let us add the edge uv to D_{aux}. In other words, D_{aux} is the simple undirected graph depicting the pairs of distinct vertices of D sharing an in-neighbour. By construction, note that, in D_{aux}, every two distinct vertices are joined by at most one edge.

It turns out that, for strong digraphs D with maximum out-degree at least 2, a vertex cover of D_{aux} is resolving in D.

Lemma 2. *Let D be a strong digraph with $\Delta^+(D) \geq 2$. Any vertex cover of D_{aux} is a resolving set of D.*

Proof. Towards a contradiction, assume the claim is false, i.e., there exists a set $S \subseteq V(D)$ which is a vertex cover of D_{aux} but not a resolving set of D. Since $\Delta^+(D) \geq 2$, there are edges in D_{aux} and thus $S \neq \emptyset$. Let v_1, v_2 be two vertices that cannot be distinguished through their distances from S; in other words, for every $w \in S$ (note that $w \neq v_1, v_2$), we have $\text{dist}_D(w, v_1) = \text{dist}_D(w, v_2)$, and that distance is finite since D is strong. Now consider such a vertex $w \in S$ at minimum distance from v_1 and v_2. In D, any shortest path P_1 from w to v_1 has the same length as any shortest path P_2 from w to v_2.

Because $v_1 \neq v_2$ and P_1, P_2 are shortest paths, note that all vertices of P_1 and P_2 cannot be the same; let thus x_1 denote the first vertex of P_1 that does not belong to P_2, and, similarly, let thus x_2 denote the first vertex of P_2 that does not belong to P_1. In other words, the first vertices of P_1 and P_2 coincide up to some vertex x, but the next vertices x_1 (in P_1) and x_2 (in P_2) are different. So, D_{aux} contains the edge x_1x_2, and at least one of x_1, x_2 belongs to S. Furthermore, x_1 and x_2 are closer to v_1, v_2 than w is; this is a contradiction to the original choice of w. $\square$



(a) (b)

Figure 1: (a) The oriented graph $D_{3,3,2}$. The set of red vertices is an example of an optimal resolving set. (b) A strong orientation D of the $6 * 6$ torus $T_{6,6}$ verifying $\mathrm{MD}(D) = |V(T_{6,6})|/2$. Every two vertices marked with a same letter have the same in-neighbourhood; thus, every resolving set must contain at least one of them.

Lemma 2 shows that a resolving set of any strong digraph (with maximum out-degree at least 2) can be obtained by considering every vertex and choosing at least all of its out-neighbours but one. The proof suggests that this is because this is a way to distinguish all shortest paths from a vertex to other ones.

Corollary 3. *For every strong digraph D with $\Delta^+(D) \geq 2$, the metric dimension $\mathrm{MD}(D)$ of D is at most the size of a minimum vertex cover of D_{aux}.*

Unfortunately, determining the minimum size of a vertex cover of a given graph is an NP-complete problem in general [7]. However, in the context of Corollary 3, we are mostly interested in having reasonable upper bounds on the size of a minimum vertex cover of D_{aux}. Such upper bounds can be exhibited when D has particular additional properties, as will be shown in the next sections.

3 Strong oriented graphs with bounded maximum degree

By the *maximum degree* $\Delta(D)$ of a given oriented graph D, we mean the maximum degree of its underlying undirected graph (i.e., the maximum value of $d^-(v) + d^+(v)$ over the vertices v of D). In this section, we investigate the maximum value that $\mathrm{MD}(D)$ can take among all strong orientations D of a graph with given maximum degree. Since a strong oriented graph D with $\Delta(D) = 2$ is a directed cycle, in which case $\mathrm{MD}(D)$ is trivially 1, we focus on cases where $\Delta(D) \geq 3$.

All our lower bounds in this section are obtained through the following constructions. For any $k \in \mathbb{N}$ and $\Delta \geq 2$, we denote by $T_{\Delta,k}$ the rooted Δ-ary complete tree with depth k. More precisely, $T_{\Delta,k}$ is a rooted tree such that every non-leaf vertex has Δ children and all leaves are at distance k from the root. Note that $|V(T_{\Delta,k})| = \frac{\Delta^{k+1}-1}{\Delta-1}$ and $T_{\Delta,k}$ has Δ^k leaves and maximum degree $\Delta+1$. For any $k \in \mathbb{N}$ and $\Delta, i \geq 2$, let $D_{\Delta,k,i}$ be the oriented graph defined as follows (see Figure 1a for an illustration). Start with T being a copy of $T_{\Delta,k-1}$ with all edges oriented from the root to the leaves. Let $v_1^{k-1}, \cdots, v_{\Delta^{k-1}}^{k-1}$ be the leaves of T and let r be its root. For every $1 \leq j \leq \Delta^{k-1}$, add i out-neighbours $u_1^j, \cdots, u_i^j$ to v_j^{k-1}. Then, for $1 \leq j \leq \Delta^{k-1}$ and $1 \leq \ell < i$, add the arc (u_ℓ^j, u_i^j). Then, add a copy T' of $T_{\Delta,k-2}$ where all edges are oriented from the leaves to the root. Let $v'_1, \cdots, v'_{\Delta^{k-2}}$ be the leaves of T' and let r' be its root. For every $1 \leq j \leq \Delta^{k-2}$ and for every $1 \leq \ell \leq \Delta$, add the arc $(u_i^{\Delta(j-1)+\ell}, v'_j)$. Finally, add the arc (r', r); note that this ensures that $D_{\Delta,k,i}$ is strong.

Theorem 4. *For every $k \in \mathbb{N}$ and $\Delta, i \geq 2$, $D_{\Delta,k,i}$ is a strong oriented graph with maximum degree $\Delta+1$,*

$$|V(D_{\Delta,k,i})| = \frac{\Delta^k - 1}{\Delta - 1} + i\Delta^{k-1} + \frac{\Delta^{k-1}-1}{\Delta-1} \qquad \textit{and} \qquad \mathrm{MD}(D_{\Delta,k,i}) \geq \Delta^{k-1} - 1 + \Delta^{k-1}\max\{1, i-2\}.$$

Proof. We only need to prove the last statement. For every $1 \leq \ell \leq \Delta^{k-1}$, let $v_1^\ell, \cdots, v_{\Delta^\ell}^\ell$ denote the vertices of $D_{\Delta,k,i}$ at distance ℓ from $r = v_1^0$. Note that, for every $0 \leq \ell \leq k-2$ and $1 \leq j \leq \Delta^\ell$, the vertices $v_{\Delta(j-1)+1}^{\ell+1}, \cdots, v_{\Delta(j-1)+\Delta}^{\ell+1}$ have the same in-neighbourhood $\{v_j^\ell\}$. By Lemma 1, every resolving set of $D_{\Delta,k,i}$ thus has to include at least $\Delta - 1$ of these vertices. For every $1 \leq j \leq \Delta^{k-1}$, the vertices $v_{i(j-1)+1}^k, \cdots, v_{i(j-1)+i-1}^k$ have the same in-neighbourhood $\{v_j^{k-1}\}$. Again by Lemma 1, every resolving set of $D_{\Delta,k,i}$ must thus include at least $i-2$ of these vertices. Moreover, it can be checked that, when $i = 2$, every resolving set of $D_{\Delta,k,i}$ must include at least one of $v_{2(j-1)+1}^k, v_{2(j-1)+2}^k$. Figure 1a shows an example of a resolving set of $D_{3,3,2}$.

Hence, any resolving set R of $D_{\Delta,k,i}$ verifies the following inequation and the result follows:

$$|R| \geq \left(\sum_{\ell=0}^{k-2} \Delta^\ell(\Delta-1)\right) + \Delta^{k-1}\max\{1, i-2\}.$$

□

In the rest of this section, we exhibit upper bounds on $\mathrm{MD}(D)$ for oriented graphs D with bounded maximum degree, some of which are close to lower bounds that can be established using Theorem 4.

We begin with an upper bound for strong subcubic (i.e., with maximum degree 3) oriented graphs D.

Lemma 5. *For every strong subcubic n-node oriented graph D, we have* $\mathrm{MD}(D) \leq \frac{n}{2}$.

Sketch of proof. In D, there are only 3 types of vertices v, namely: either $d^-(v) = d^+(v) = 1$; or $d^-(v) = 1$ and $d^+(v) = 2$; or $d^-(v) = 2$ and $d^+(v) = 1$. Only the vertices v verifying $d^+(v) = 2$ create edges in D_{aux} and there are at most $\frac{1}{2}n$ of these vertices v since $\sum_{v \in V(D)} d_D^-(v) = \sum_{v \in V(D)} d_D^+(v)$. Thus, D_{aux} contains at most $\frac{1}{2}n$ edges and thus, admits a vertex cover S with size at most $\frac{1}{2}n$. The result follows Corollary 3. □

In the next result, we exhibit a general upper bound on $\mathrm{MD}(D)$ for every strong digraph D with given maximum in-degree and maximum out-degree (at least 2). Recall that a *proper vertex-colouring* of an undirected graph is a partition of the vertices into stable sets.

Theorem 6. *For every strong n-node digraph D with max. in-degree Δ^- and max. out-degree $\Delta^+ \geq 2$,*

$$\mathrm{MD}(D) \leq \frac{\Delta^-(\Delta^+-1)}{\Delta^-(\Delta^+-1)+1} n.$$

Proof. The maximum degree of a vertex v of D_{aux} is $\Delta^-(\Delta^+ - 1)$: this is because v has at most Δ^- in-neighbours in D, each of which, if it has an out-neighbour different from v, might yield a new edge incident to v in D_{aux}. So each of these at most Δ^- in-neighbours of v in D might create, in D_{aux}, up to $\Delta^+ - 1$ edges incident to v. Hence, the maximum degree of D_{aux} is $\Delta^-(\Delta^+ - 1)$. From greedy colouring arguments, it thus follows that D_{aux} admits a proper vertex-colouring using at most $\Delta^-(\Delta^+ - 1) + 1$ colours.

The claim now follows from Lemma 2 by just noting that, for any graph with a given proper vertex-colouring, a vertex cover can be obtained by taking all colour classes but one. In particular, since a proper k-vertex-colouring of an n-node graph always has a colour class with size at least $\frac{1}{k}n$, we deduce the claim by considering, as a vertex cover of D_{aux}, all colour classes but a biggest one of any proper $(\Delta^-(\Delta^+ - 1) + 1)$-vertex-colouring. $\square$

Theorems 4 and 6 and Lemma 5 yield the following:

Corollary 7. *Let $\mathcal{G}_\Delta$ be the family of 2-edge-connected graphs with maximum degree Δ. Then, for any $\epsilon > 0$,*

$$\frac{2}{5} - \epsilon \leq \text{WOMD}(\mathcal{G}_3) \leq \frac{1}{2}, \qquad \text{and} \qquad \frac{1}{2} - \epsilon \leq \text{WOMD}(\mathcal{G}_4) \leq \frac{6}{7}, \qquad \text{and} \qquad \lim_{\Delta \to \infty} \text{WOMD}(\mathcal{G}_\Delta) = 1;$$

4 Strong orientations of grids and tori

By a *grid* $G_{n,m}$, we refer to the Cartesian product $P_n \Box P_m$ of two paths P_n, P_m. A *torus* $T_{n,m}$ is the Cartesian product $C_n \Box C_m$ of two cycles C_n, C_m. In the undirected context, it is easy to see that $\text{MD}(G_{n,m}) = 2$ while $\text{MD}(T_{n,m}) = 3$ (see e.g., [10]); however, things get a bit more tricky in the directed context.

Grids and tori have maximum degree 4; thus, bounds on the maximum metric dimension of a strong oriented grid or torus can be derived from our results in Section 3. In this section, we improve these bounds through dedicated proofs and arguments. We first consider strong Eulerian oriented tori (all vertices have in-degree and out-degree 2), for which we exhibit the maximum value of the metric dimension. We then consider strong oriented grids, for which we provide improved bounds.

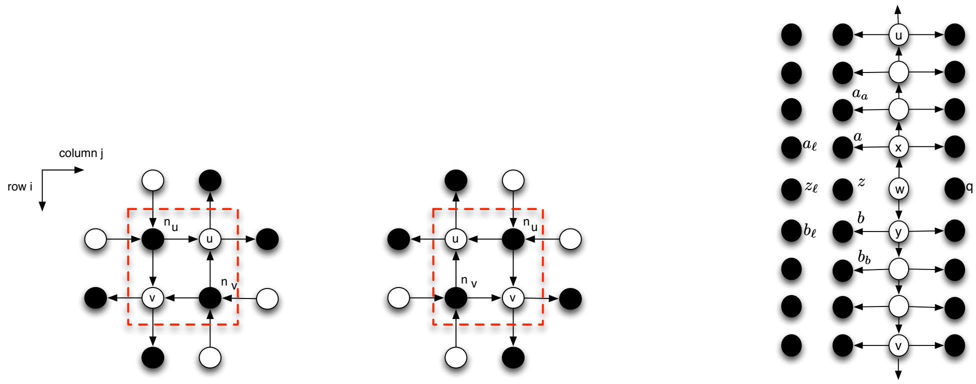
4.1 Strong Eulerian orientations of tori

Let $0 < n \leq m$ be two integers, and let $T_{n,m}$ be the torus on nm vertices. That is, $V(T_{n,m}) = \{(i,j) \mid 0 \leq i < n, 0 \leq j < m\}$, and $(i,j),(k,\ell) \in E(T_{n,m})$ if and only if $|i-k| \in \{1, n-1\}$ and $j = \ell$, or $|j - \ell| \in \{1, m-1\}$ and $i = k$. By convention, the vertex $(0,0)$ is regarded as the topmost, leftmost vertex of the torus. That is, $\{(0,j) \in V(T_{n,m}) \mid 0 \leq j < m\}$ is the topmost (or first) row, and $\{(i,0) \in V(T_{n,m}) \mid 0 \leq i < n\}$ is the leftmost (or first) column.

As a main result in this section, we determine the maximum metric dimension of a strong Eulerian oriented torus. More precisely, we study the following slight modifications of the parameter WOMD. For a connected graph G, we denote by $\text{WEOMD}(G)$ the maximum value of $\text{MD}(D)$ over all strong Eulerian orientations D of G. For a family $\mathcal{G}$ of 2-edge-connected graphs, we set $\text{WEOMD}(\mathcal{G}) = \max_{G \in \mathcal{G}} \frac{\text{WEOMD}(G)}{|V(G)|}$.

Theorem 8. *For the family $\mathcal{T}$ of tori, we have* $\text{WEOMD}(\mathcal{T}) = \frac{1}{2}$.

Sketch of proof. Let us consider the case of a torus $T_{n,m}$ when n and m are even. We first show that there exists a strong Eulerian orientation D of $T_{n,m}$ with $\text{MD}(D) \geq \frac{nm}{2}$. Indeed, let us orient $T_{n,m}$ such that odd (resp., even) columns are oriented bottom to top (resp., top to bottom) and odd (resp., even) rows are oriented right to left (resp., left to right). See Figure 1b



(a) (b)

Figure 2: (a) The two cases of "bad squares" in the torus. Black vertices are the ones in the initial set R. (b) Configuration with two undistinguished vertices u and v in the grid. Black vertices are those in R and white ones are in $V(G_{n,m}) \setminus R$. The vertex w is the LCV of u and v.

for an illustration. The lower bound follows from Lemma 1 since vertices can be partitioned into pairs of vertices having a common in-neighbourhood.

For the upper bound, the proof is constructive and provides a resolving set of size at most $\frac{nm}{2}$. The algorithm starts with the set $R = \{(i,j) \in V(D) \mid i+j \text{ even}\}$ (note that it is a minimum vertex cover and a stable set of size $\frac{nm}{2}$) and iteratively performs local modifications (swaps one vertex in R with one of its neighbours not in R) without changing the size of R until R becomes a resolving set R^*. Precisely, if R is not a resolving set (otherwise, we are done) then at least two vertices are not distinguishable by their distances to the vertices in R. Let u and v be two such vertices. We prove that such two vertices belong to a so-called *bad square* as depicted in Figure 2a (there are two cases). We then prove that all bad-squares are pairwise vertex-disjoint. Finally, we prove (by a case analysis) that the vertex set R^* obtained from R by exchanging vertices u and n_v (as defined in Figure 2a) for every bad square is a resolving set. □

4.2 Strong oriented grids

In this section, we consider the maximum metric dimension of a strong oriented grid. For every such grid, we deal with its vertices using the same terminology introduced in Section 4.1 for tori (i.e., the vertices of the topmost row have first coordinate 0, and the vertices of the leftmost column have second coordinate 0). Our main result to be proved in this section is the following.

Theorem 9. *Let $\mathcal{G}$ be the family of grids. For any $\epsilon > 0$, we have*

$$\frac{1}{2} - \epsilon \leq \mathrm{WOMD}(\mathcal{G}) \leq \frac{2}{3} + \epsilon.$$

Sketch of proof. Let us consider the case of a grid $G_{n,m}$ when n and m are even. We first show that there exists a strong orientation D of $G_{n,m}$ with $\mathrm{MD}(D) \geq \frac{nm-(n+m)}{2}$. Indeed, let us orient $G_{n,m}$ similarly to $T_{n,m}$ in Theorem 8 (and Figure 1b). The lower bound follows from Lemma 1 since vertices can be partitioned into *bad pairs* of vertices having a common in-neighbourhood (except for the vertices on the borders (first/last row/column) in which case only half of the vertices are included in the bad pairs).

For the upper bound, let us assume that $m \equiv 0 \mod 3$. The proof is constructive and provides a resolving set of size at most $\frac{2nm}{3} + \epsilon$. The algorithm starts with the set $R = \{V(G_{n,m}) \setminus (i, 3j-1) | 0 \leq i \leq n-1, 1 \leq j \leq m/3\}$ (i.e., R contains the first 2 out of every 3 columns from left to right in the grid) and iteratively performs local modifications (swaps one vertex in R with one of its neighbours not in R) without changing the size of R until R becomes a resolving set R^*. Precisely, if R is not a resolving set (otherwise, we are done) then at least two vertices are not distinguishable by their distances to the vertices in R. Let u and v be two such vertices. We prove that, for any such two vertices u and v, they belong to the same column C (not including any vertex in R) and there exists a unique vertex $w \in C$ (called the *Last Common Vertex* (*LCV*) of u and v) at the same distance from u and v (see Figure 2b, where superscripts ^{w} have been omitted.). In that case, let z^w be the vertex on the left of w, x^w be the vertex above w, a^w be the vertex on the left of x^w and above z^w, and b^w be the vertex below z^w. We show that, for every *LCV* w, the vertices $\{w, z^w, a^w, b^w\}$ and the vertices around them are pairwise vertex-disjoint. Finally, we prove (by a tedious case analysis) that the vertex set R^* is a resolving set where R^* is obtained from R by exchanging, for every *LCV* w, vertices z^w and x^w (if (a^w, z^w) or (b^w, z^w) is an arc) or exchanging vertices a^w and x^w otherwise. $\square$

5 Conclusion

In this work, we have investigated, for a few families of graphs, the worst strong orientations in terms of metric dimension. In particular settings, such as when considering strong Eulerian orientations of tori, we managed to identify the worst possible orientations (Theorem 8). For other families (graphs with bounded maximum degree and grids), we have exhibited both lower and upper bounds on WOMD that are more or less distant apart. As further work on this topic, it would be interesting to lower the gap between our lower and upper bounds, or consider strong orientations of other graph families.

In particular, two appealing directions could be to improve Corollary 7 (for max. degree 3) and Theorem 9. For graphs with maximum degree 3, we do wonder whether there are strong orientations for which the metric dimension is more than $\frac{2}{5}$ of the vertices. It is also legitimate to ask whether our upper bound ($\frac{1}{2}$ of the vertices), which was obtained from the simple technique described in Corollary 3, can be lowered further.

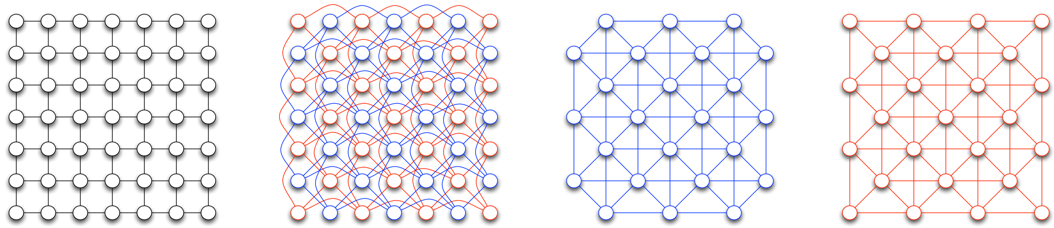
In Theorem 9, we proved that any strong orientation of a grid asymptotically has metric dimension at most $\frac{2}{3}$ of the vertices. Towards improving this upper bound, one could consider applying Corollary 3, for instance as follows. For a given oriented grid D, let A^* be the graph obtained as follows (where we deal with the vertices of D using the same terminology as in Section 4):

- $V(A^*) = V(D)$.
- We add, in A^*, an edge between two vertices (i, j) and (i', j') if they are joined by a path of length exactly 2 in the grid underlying D. That is, the edge is added whenever (i', j') is of the form $(i-1, j-1)$, $(i-2, j)$, $(i-1, j+1)$, $(i, j+2)$, $(i+1, j+1)$, $(i+2, j)$, $(i+1, j-1)$, or $(i, j-2)$.

Note that A^* has two connected components C_1, C_2 being basically obtained by glueing K_4's along edges. See Figure 3 for an illustration.

It can be noticed that for any oriented grid D, its auxiliary graph D_{aux} is a subgraph of A^*. From Corollary 3, any upper bound on the size of a minimum vertex cover of A^* is thus also an upper bound on $\text{MD}(D)$ (assuming D is strong, in which case it necessarily verifies $\Delta^+(D) \geq 2$). Unfortunately, we have observed that any minimum vertex cover of A^* covers $\frac{3}{4}$ of the vertices, which is not better than our upper bound in Theorem 9.

There is still hope, however, to improve our upper bound using the vertex cover method. Indeed, under the assumption that D is a strong oriented graph, actually D_{aux} can be far from



(a) $G_{9,9}$ (b) A^* (c) C_1 (d) C_2

Figure 3: The grid $G_{9,9}$ and the associated graph A^*.

having all the edges that A^* has. For instance, it can easily be proved that, in D_{aux}, it is not possible that a vertex (i, j) is adjacent to all four vertices $(i-2, j)$, $(i, j+2)$, $(i+2, j)$, $(i, j-2)$ (if they exist). Using a computer, we were actually able to check on small grids that, for all strong orientations D, the minimum vertex cover of D_{aux} has size at most $\frac{1}{2}$ of the vertices. This leads us to raising the following two questions related to our upper bound in Theorem 9:

Problem 10. *For any strong orientation D of a grid $G_{n,m}$, do the minimum vertex covers of D_{aux} have size at most $\frac{nm}{2}$?*

Problem 11. *For any strong orientation D of a grid $G_{n,m}$, do we have $\mathrm{MD}(D) \leq \frac{nm}{2}$?*

Note that if the upper bound in Problem 11 held, then it would be close to the lower bound in Th. 9.

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