Star-Shaped Metrics for Mechanical Metamaterial Design

Addendum: The Link Between the Regular Voronoi Diagram and the Growth-model Voronoi Diagram

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This is an addendum to the paper of the same title published in ACM Transactions on Graphics, special issue for SIGGRAPH 2019, by the same authors. The text below assumes that the reader has the main article at hand.

Recall that when using different polygonal distances at each site, the Voronoi diagram of a cell $V_{\mathcal{S}}(p)$ might have several connected components, only one of which, $Con(p)$ contains the site p. We seek a diagram $V'_{\mathcal{S}}$, as close as possible to $V_{\mathcal{S}}$, in which $Con(p)$ is the only component of $V'_{\mathcal{S}}(p)$, that is, $Con(p) = V'_{\mathcal{S}}(p)$.

In the rest of this section, a *component* is always a connected component of a cell of some site.

An intuitive idea to remove a connected component V of $V_{\mathcal{S}}(p)$ is to identify the set Γ' of the sites of the components immediately surrounding V and then replacing V by the Voronoi diagram of Γ' restricted to V. V then acts as a "cutout" letting us see a different Voronoi diagram through.

In the context of the discrete version of the algorithm (Algorithm 1 in the main text), we prove (Proposition 5) that this is the correct intuition for reasoning about the growth-model Voronoi diagram.

Given a site $p \in \Gamma$,

- $V_{\mathcal{S}}(p)$ is its real Voronoi cell, a subset of the plane $\mathbb{R}^2$,
- $V^g_{\mathcal{S}}(p)$ is its Growth-model Voronoi cell,
- $DV_{\mathcal{S}}(p)$ is its discrete Voronoi cell, a subset of pixels,
- $DV^g_{\mathcal{S}}(p)$ is its discrete Growth-model Voronoi cell,
- $DCon(p)$ is the connected component of $DV_{\mathcal{S}}(p)$ that contains p (note that $DCon(p)$ may be empty).

Proposition 1. *The discrete Growth-model Voronoi cell $DV^g_{\mathcal{S}}(p)$ computed by the algorithm is either empty or 4-connected.*

Proof. Follows from the algorithm. The cell can be empty if several sites live in the same pixel, in which case the one closest to the pixel center takes the cake. It can also be empty if another site, far away in the grid, uses a particularly aggressive distance. □

Proposition 2. $DCon(p) \subset DV^g_{\mathcal{S}}(p)$.

Proof. Let c be a pixel in $DCon(p)$. So there is a 4-connected path from p to c. And for each pixel along this path, p is the closest site, because $DCon(p) \subset DV_{\mathcal{S}}(p)$. So all the pixels along this path are assigned to $DV^g_{\mathcal{S}}(p)$ by the algorithm. □

For the purpose of building intuition, we prove the following

Proposition 3. *If a connected component V of $DV_{\mathcal{S}}(p)$, not containing p, is completely surrounded by $DCon(q)$ ($p \neq q$) then V disappears in the growth model and makes way for $DV^g_{\mathcal{S}}(q)$. In our notation: $V \subset DV^g_{\mathcal{S}}(q)$.*

Proof. The external boundary ∂V of V is in $DCon(q) \subset DV^g_{\mathcal{S}}(q)$ (by Proposition 2). ∂V shields its interior from the other sites, which can not cross ∂V in the growth model, so that $DV^g_{\mathcal{S}}(q)$ is free to fill the area. □

Now, we describe an algorithm which starts with the discrete Voronoi diagram and attempts to patch it until all Voronoi cells are connected. What we obtain at each iteration is a mosaic of pieces of Voronoi diagrams, so it is not a Voronoi diagram, so we call each such mosaic a "labeling." A labeling ℓ assigns a site to each pixel: $\forall x \in \mathbb{Z}^2, \ell(x) \in \Gamma$. The first labeling, ℓ^0, is the discrete Voronoi diagram: $\ell^0(p) = DV_{\mathcal{S}}(p)$, $DCon^0(p) = DCon(p)$.

In the labeling ℓ^i, $i \geq 0$, the *sea* is the union of all the components that contain their respective site:

$$\text{sea}^i = \bigcup_{p \in \Gamma} DCon^i(p).$$

An *island* is a connected component of the complement of the sea. An island may contain several components assigned to different sites, and/or be unbounded.

The *neighborhood of a pixel* x is the set $N(x)$ of 4 pixels immediately adjacent to x: up, down, left and right.

The *external boundary* ∂I of an island I is

$$\partial I = \left(\bigcup_{x \in I} N(x) \right) \setminus I.$$

Proposition 4. *The external boundary of an island I contains only pixels in the sea. For each pixel x in ∂I assigned to site p, there exists a 4-connected path from x to p along which all pixels are labeled p (by definition of the sea).*

So, the proposed algorithm iteratively kills all the islands (Definition 1), in any order. Let $j \geq 0$ be the index of the first labeling without island (assuming that such a j exists). We claim that this j-th labeling is equal to the Growth-model Voronoi diagram: $\ell^j = DV^g_{\mathcal{S}}$, thereby establishing an intuitive link between $DV_{\mathcal{S}}$ and $DV^g_{\mathcal{S}}$.

The proof is by recurrence. Proposition 2 is the initialization of the recurrence. It says that

$$\forall p \in \Gamma, DCon^0(p) \subset DV^g_{\mathcal{S}}(p).$$

When we kill an island, the sea components surrounding the island grow a little bit and we show that Proposition 2 is still satisfied. When we kill an island, smaller islands may remain, but we show that eventually, all islands disappear except in some specific circumstances.

Definition 1. *(Killing an island) Let I be an island in labeling ℓ^i. We kill I in order to obtain the next labeling ℓ^{i+1}. To do so, we gather the set of conqueror sites $K = \{\ell^i(x) \mid x \in \partial I\}$. We then replace the labeling of the pixels in I by the labeling given by the discrete Voronoi diagram of the conquerors: DV_K. Note that $DCon^i(p) \subset DCon^{i+1}(p)$.*

Proposition 5 (recurrence). *For all labelings $\ell^i, i \geq 0$, for all sites p, $DCon^i(p) \subset DV^g_{\mathcal{S}}(p)$. This implies that $\ell^i = DV^g_{\mathcal{S}}$ when both are restricted to sea^i.*

Proof. By recurrence over i. This is true for $i = 0$. Assume this is true for a given $i \geq 0$. Let I be the island killed in order to obtain ℓ^{i+1}. Let K be the conquerors. Let p be a site in Γ.

In I, $DV^g_{\mathcal{S}}$ assigns only labels from K. Indeed if another label $q \notin K$ appears in I, that is, $DV^g_{\mathcal{S}}(q) \cap I \neq \emptyset$, then this pixel labeled q is not path-connected to the site q, because of the recurrence hypothesis that $DCon^i(p) \subset DV^g_{\mathcal{S}}(p)$: the external boundary ∂I of I has ℓ^i-labels in K ($\ell^i(\partial I) \subset K$), and $DV^g_{\mathcal{S}}$ coincides with ℓ^i over sea^i $= \bigcup_{p \in \Gamma} DCon^i(p)$. This contradicts Proposition 1.

Let x be a pixel in $DCon^{i+1}(p)$. If $x \in DCon^i(p)$ then $x \in DV^g_{\mathcal{S}}(p)$ and we are done. Otherwise, x is in I and there is a 4-connected path $\mathcal{P}$ inside $I \cap DCon^{i+1}(p)$, from x to a pixel on the external boundary ∂I of I, and from there, a path to p fully labeled with p in both $DCon^i(p)$ (Proposition 4) and $DV^g_{\mathcal{S}}(p)$ (by our recurrence hypothesis).

For each pixel along the path $\mathcal{P}$, p is the closest site (among the site in K) and the growth-model algorithm propagates only labels from K inside I. So p is guaranteed to win all the pixels along $\mathcal{P}$. In other words, all the pixels along path $\mathcal{P}$ are assigned to $DV^g_{\mathcal{S}}(p)$ by the growth-model algorithm. We conclude that $DCon^{i+1}(p) \subset DV^g_{\mathcal{S}}(p)$ and therefore sea$^{i+1} \subset DV^g_{\mathcal{S}}$. □

From Proposition 5 we can conclude that if we can kill all the islands, then the resulting labeling is equal to $DV^g_{\mathcal{S}}$. Therefore, this island-killing algorithm let us intuitively see what the growth-model algorithm is doing. It nicely corresponds to creating a mosaic of regular Voronoi diagrams so that all Voronoi cells become connected.

It might happen, however, that the process of killing an island results in exactly the same island. This may happen when the labels inside the island are the same as the set of conquerors. In this case, the algorithm described here does not terminate and strays away from modeling the behavior of the Growth-model Voronoi diagram. A finer grained island killing may recover the correct behavior, but the island-killing algorithm as presented above is sufficient to give us a good understanding of how $DV^g_{\mathcal{S}}$ can be obtained from $DV_{\mathcal{S}}$ without having to follow the very unintuitive front-propagation of Algorithm 1.