Dimension Métrique des Graphes Orientés

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La dimension métrique $\mathrm{MD}(G)$ d'un graphe non-dirigé G est le nombre minimum de sommets qui permettent, via leurs distances à tous les sommets, de distinguer les sommets de G les uns des autres. Cette notion a été beaucoup étudiée depuis sa conception dans les années 70 car elle permet notamment de modéliser la localisation d'une cible par ses distances à un réseau de capteurs dans un graphe. Nous considérons ici sa généralisation aux digraphes. Nous étudions, pour certaines classes de graphes, la dimension métrique maximum parmi toutes les orientations fortement connexes en donnant des bornes sur cette valeur. Notamment, nous étudions ce paramètre dans les graphes de degré maximum borné, les grilles et les tores. Pour ces derniers, nous trouvons la valeur exacte asymptotiquement.

Mots-clefs : Graphes, Dimension Métrique, Ensembles Resolvants, Orientations fortement connexes

1 Introduction

The *distance* $\mathrm{dist}(u,v)$ between two vertices u,v of a (di)graph $D=(V,E)$ is the length of a shortest (directed) path from u to v. A digraph D is *strong* if, for every $u,v \in V(D)$, there is a directed path from u to v, and one from v to u. Let $R \subseteq V$. Two vertices u,v of D are said to be *distinguished* by R if there exists $w \in R$ such that $\mathrm{dist}(w,u) \neq \mathrm{dist}(w,v)$. Otherwise, u and v are *undistinguished* by R. A set $R \subseteq V(D)$ is called *resolving* if all pairs of vertices of D are distinguished by R. The *metric dimension* $\mathrm{MD}(D)$ of D is then the smallest size of a resolving set. Note that, $\mathrm{MD}(D) < |V(D)|$ since, for any $v \in V(D)$, $R = V(D) \setminus \{v\}$ is a resolving set (as any vertex in a resolving set is distinguished from all other vertices).

In the undirected context, these notions have been widely studied since their introduction in the 70's in [8, 14], notably because they can be used to model many real-life problems (e.g., see the surveys [1, 3]). Typically, for example, placing sensors at every vertex of a resolving set would allow to locate targets in ad-hoc networks. The goal is then to minimize the number of sensors ensuring that any target will be uniquely located (e.g., [2]). Similarly, this problem can be seen as a robot required to geolocate itself in an environment modelled by a graph, via distance sensors, with the goal being to minimize the number of sensors that need to be probed by the robot to always be geolocatable [9]. As most networks are directed, the metric dimension was first generalized to digraphs in [4], where notably, a characterization of digraphs with metric dimension 1 was given. In [13], it was proved that determining the metric dimension of a strong digraph is NP-complete. Bounds on the metric dimension of various digraph families were later exhibited (Cayley digraphs [6], line digraphs [7], tournaments [10], digraphs with cyclic covering [12], De Bruijn and Kautz digraphs [13], etc.).

From undirected graphs to oriented graphs. Recall that an *orientation* D of an undirected simple graph G is obtained by replacing each edge uv by exactly one of the arcs (u,v) or (v,u). An *oriented graph* D is a directed graph that is an orientation of a simple graph. Throughout this paper, when simply referring to a *graph*, we mean an undirected graph.

Note that, for a graph, the metric dimension might or might not be preserved when orienting its edges. An interesting example [4, 10] is the case of a graph G with a Hamiltonian path : while $\mathrm{MD}(G)$ can be arbitrarily large in general (consider e.g., any complete graph), there is an orientation D of G verifying $\mathrm{MD}(D)=1$ (just orient all edges of a Hamiltonian path from the first vertex towards the last vertex, and all remaining edges in the opposite direction). Conversely, there exist orientations D of G for which $\mathrm{MD}(D)$ can be much

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larger than $\mathrm{MD}(G)$, which is the case for grids and tori as shown in Section 4. In [5], the authors initiated the study of the orientations of G with maximum metric dimension giving only basic results.

Outline and Our results. Motivated by these observations, we investigate the parameter WOMD (Worst Orientation for the Metric Dimension) defined as follows. For any connected graph G, let $\mathrm{WOMD}(G)$ be the supremum of $\mathrm{MD}(D)$ over all strong orientations D of G, and for any family $\mathcal{G}$ of 2-edge-connected graphs [‡], let $\mathrm{WOMD}(\mathcal{G}) = \sup\limits_{G\in\mathcal{G}} \frac{\mathrm{WOMD}(G)}{|V(G)|}$.

Section 2 introduces tools and results that will be used in the next sections. In Section 3, bounds on $\mathrm{WOMD}(\mathcal{G}_\Delta)$ are proved, where $\mathcal{G}_\Delta$ refers to the family of 2-edge-connected graphs with maximum degree Δ. In Section 4, we consider the families of grids and tori. For the family $\mathcal{T}$ of tori, we prove that we asymptotically have $\mathrm{WEOMD}(\mathcal{T}) = \frac{1}{2}$, where the parameter $\mathrm{WEOMD}(\mathcal{T})$ is defined similarly to $\mathrm{WOMD}(\mathcal{T})$ except that only strong Eulerian orientations of tori (i.e., all vertices have in-degree and out-degree 2) are considered. For the family $\mathcal{G}$ of grids, we then prove that asymptotically $\frac{1}{2} \leq \mathrm{WOMD}(\mathcal{G}) \leq \frac{2}{3}$.

Terminology and notation. Let D be a digraph. For a vertex v of D, we denote by $d^-_D(v)$ (resp. $d^+_D(v)$) the *in-degree* (resp. *out-degree*) of v. The set of all in-neighbours (resp. out-neighbours) of v is denoted by $N^-_D(v)$ (resp. $N^+_D(v)$). Let $\Delta^+(D)$ (resp. $\Delta^-(D)$) be the max. out-degree (resp. in-degree) of a vertex in D.

2 Tools and preliminary results

First, we point out the following property of resolving sets in digraphs having vertices with the same in-neighbourhood. This result will be one of our main tools for building digraphs with large metric dimension.

Lemma 1. *Let D be a digraph and $S \subseteq V(D)$ be a subset of at least two vertices such that, for every $u, v \in S$, we have $N^-(u) = N^-(v)$. Then, any resolving set of D contains at least $|S| - 1$ vertices of S.*

We now show a technique based on a connection between the resolving sets of a strong digraph D, with $\Delta^+(D) \geq 2$, and the vertex covers of a particular graph associated to it. A *vertex cover* of a graph G is a subset $S \subseteq V(G)$ of vertices such that, for every edge uv of G, at least one of u and v belongs to S. To any digraph D we associate an *auxiliary (undirected) graph* D_{aux} constructed as follows : the vertices of D_{aux} are those of D ; for every two distinct vertices u, v of D such that $N^-_D(u) \cap N^-_D(v) \neq \emptyset$, add the edge uv to D_{aux}. So, D_{aux} is the simple undirected graph depicting the pairs of distinct vertices of D sharing an in-neighbour.

Lemma 2. *Let D be a strong digraph with $\Delta^+(D) \geq 2$. For any vertex cover $S \subseteq V(D_{\mathrm{aux}})$ of D_{aux}, S is a resolving set of D and hence,* $\mathrm{MD}(D) \leq |S|$.

Proof. Towards a contradiction, assume there exists a set $S \subseteq V(D)$ which is a vertex cover of D_{aux} but not a resolving set of D. Since $\Delta^+(D) \geq 2$, there are edges in D_{aux} and thus $S \neq \emptyset$. Let v_1, v_2 be two vertices that cannot be distinguished by S, i.e., for every $w \in S$ (note that $w \neq v_1, v_2$), we have $dist(w, v_1) = dist(w, v_2)$ in D, and that distance is finite since D is strong. Now consider such a vertex $w \in S$ at minimum distance from v_1 and v_2. In D, any shortest path P_1 from w to v_1 has the same length as any shortest path P_2 from w to v_2. Because $v_1 \neq v_2$ and P_1, P_2 are shortest paths, note that all vertices of P_1 and P_2 cannot be the same ; let thus x_1 (x_2, resp.) denote the first vertex of P_1 (P_2, resp.) that does not belong to P_2 (P_1, resp.) So, D_{aux} contains the edge x_1x_2, and at least one of x_1, x_2 belongs to S. Furthermore, x_1 and x_2 are closer to v_1, v_2 than w is ; this is a contradiction to the original choice of w. $\square$

3 Strong oriented graphs with bounded maximum degree

The *maximum degree* $\Delta(D)$ of a given oriented graph D, is the maximum degree of its underlying undirected graph. In this section, we investigate the maximum value that $\mathrm{MD}(D)$ can take among all strong orientations D of a graph with given maximum degree. Since a strong oriented graph D with $\Delta(D) = 2$ is a directed cycle, in which case $\mathrm{MD}(D)$ is trivially 1, we focus on cases where $\Delta(D) \geq 3$.

Theorem 1. *Let $\mathcal{G}_\Delta$ be the family of 2-edge-connected graphs with maximum degree Δ. Then, $\frac{2}{5} \leq \mathrm{WOMD}(\mathcal{G}_3) \leq \frac{1}{2}$, $\frac{1}{2} \leq \mathrm{WOMD}(\mathcal{G}_4) \leq \frac{6}{7}$, and $\lim\limits_{\Delta\to\infty} \mathrm{WOMD}(\mathcal{G}_\Delta) = 1$.*

‡. Ensures $\mathrm{WOMD}(G)$ is defined for every $G \in \mathcal{G}_\Delta$, since a graph has strong orientations if and only if it is 2-edge-connected.

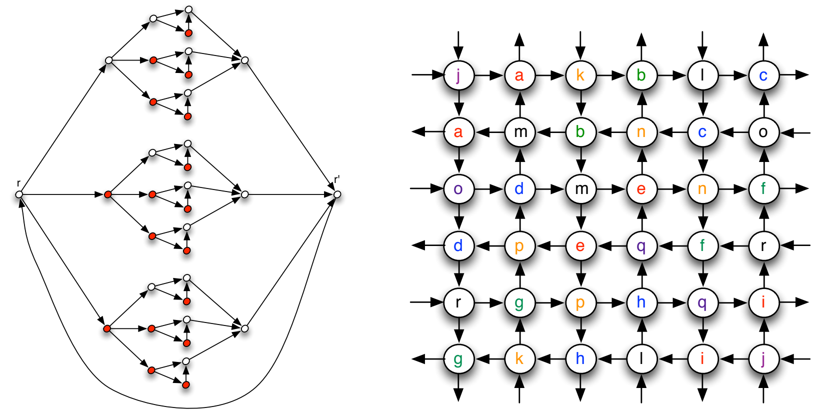

FIGURE 1: (Left) The oriented graph $D_{3,3}$. Red vertices are an example of an optimal resolving set. (Right) A strong orientation D of the $6*6$ torus $T_{6,6}$ where $\mathrm{MD}(D) = |V(T_{6,6})|/2$. Every two vertices with the same letter have the same in-neighbourhood ; thus, every resolving set must contain at least one of them.

Sketch of proof. To prove the lower bounds, we use Lemma 1 applied to the digraph $D_{\Delta,k}$ obtained (roughly) from one Δ-ary complete tree of depth k glued (via leaves) to a Δ-ary complete tree of depth $k-2$ in reversed orientation (see Fig. 1 (left) for $\Delta = k = 3$).

By definition, $\mathrm{WOMD}(\mathcal{G}_\Delta) \leq 1$ for any Δ. Lemma 2 is used to prove the upper bounds. In the case $\Delta = 3$, only the vertices v verifying $d^+(v) = 2$ create edges in D_{aux} and there are at most $\frac{n}{2}$ of these vertices v since $\sum_{v \in V(D)} d^-_D(v) = \sum_{v \in V(D)} d^+_D(v)$. Thus, D_{aux} contains at most $\frac{n}{2}$ edges and so, admits a vertex cover of size at most $\frac{n}{2}$. For general $\Delta \geq 3$, we prove that, for any n-node digraph D in $\mathcal{G}_\Delta$ with $\Delta^-, \Delta^+ \geq 2$, D_{aux} has max. degree at most $\Delta^-(\Delta^+ - 1)$, and so admits a proper colouring with at most $\Delta^-(\Delta^+ - 1) + 1$ colours and therefore, has a vertex cover (and so $\mathrm{MD}(D)$) of size at most $\frac{\Delta^-(\Delta^+-1)}{\Delta^-(\Delta^+-1)+1} n$, implying the upper bound for $\Delta = 4$. $\diamond$

4 Strong orientations of grids and tori

A *grid* $G_{n,m}$, is the Cartesian product $P_n \square P_m$ of two paths P_n, P_m. A *torus* $T_{n,m}$ is the Cartesian product $C_n \square C_m$ of two cycles C_n, C_m. We denote the vertices of both these graphs by their coordinates, i.e., for $0 \leq i < n$, $0 \leq j < m$, the vertex (i, j) has abscissa i and ordinate j. In the undirected case, $\mathrm{MD}(G_{n,m}) = 2$ and $\mathrm{MD}(T_{n,m}) = 3$ (see [11]). We determine the maximum metric dimension of a strong Eulerian oriented torus and a strong oriented grid.

Theorem 2. *For the family $\mathcal{T}$ of tori, we have* $\mathrm{WEOMD}(\mathcal{T}) = \frac{1}{2}$.

Sketch of proof. Let n,m be even. Orient $T_{n,m}$ such that only alternating (entire) columns and rows are oriented in the same direction (see Fig. 1 (right)). The lower bound follows from Lemma 1 since the vertices can be partitioned into pairs of vertices having a common in-neighbourhood.

For the upper bound, we design an algorithm that starts with the set $R = \{(i, j) \in V(D) \mid i + j \text{ even}\}$ and iteratively performs local modifications (swaps one vertex in R with one of its neighbours not in R) without changing the size of R until R becomes a resolving set R^*. Precisely, if R is not a resolving set (otherwise, we are done), then at least two vertices, say u and v, are not distinguishable by R. We prove that u and v belong to a so-called *bad square* as depicted in Fig. 2 (left) (there are two cases). We then prove that all bad-squares are pairwise vertex-disjoint. Finally, we prove (by a case analysis) that the vertex set R^* obtained from R by exchanging vertices u and n_v (as defined in Fig.. 2 (left)) for every bad square is a resolving set. $\diamond$

Theorem 3. *Let $\mathcal{G}$ be the family of grids. Then,* $\frac{1}{2} \leq \mathrm{WOMD}(\mathcal{G}) \leq \frac{2}{3}$.

Sketch of proof. The lower bound follows by orienting $G_{n,m}$ similarly to $T_{n,m}$ as in Th. 2 (and Fig. 1 (right)). For the upper bound, let us assume that $m \equiv 0 \mod 3$. We design an algorithm that starts with the set

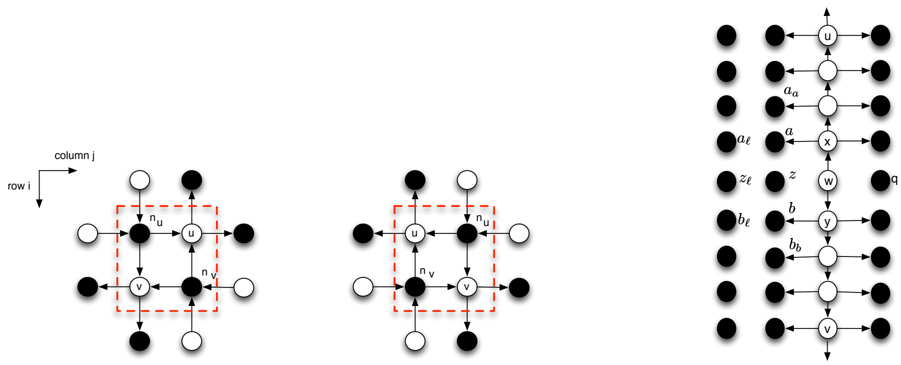

FIGURE 2: (Left) The two cases of "bad squares" in the torus. Black vertices are the ones in the initial set R. (Right) Configuration with two undistinguished vertices u and v in the grid. Black vertices are those in R and white ones are in $V(G_{n,m}) \setminus R$. The vertex w is the *LCV* of u and v.

$R = \{V(G_{n,m}) \setminus (i,3j-1) | 0 \leq i \leq n-1, 1 \leq j \leq m/3\}$ (i.e., R contains the first 2 out of every 3 columns from left to right in the grid) and iteratively performs local modifications (swaps one vertex in R with one of its neighbours not in R) without changing the size of R until R becomes a resolving set R^*. Precisely, if R is not a resolving set (otherwise, we are done), then at least two vertices, say u and v, are not distinguishable by R. We prove that, for any such two vertices u and v, they belong to the same column C (not including any vertex in R) and there exists a unique vertex $w \in C$ (called the *Last Common Vertex* (*LCV*) of u and v) at the same distance from u and v (see Fig. 2 (right), where superscripts ^{w} have been omitted). We show that, for every *LCV* w, the vertices $\{w, z^w, a^w, b^w\}$ (as defined in Fig. 2 (right)) and the vertices around them are pairwise vertex-disjoint. Finally, we prove (by a case analysis) that the vertex set R^* is a resolving set where R^* is obtained from R by exchanging, for every *LCV* w, vertices z^w and x^w (if (a^w, z^w) or (b^w, z^w) is an arc) or exchanging vertices a^w and x^w otherwise. $\diamond$

Conclusion. As further work on this topic, it would be interesting to lower the gap between our lower and upper bounds, or consider strong orientations of other graph families.

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