FAST ESCAPING POINTS OF ENTIRE FUNCTIONS: A NEW REGULARITY CONDITION

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Abstract. Let f be a transcendental entire function. The fast escaping set, $A(f)$, plays a key role in transcendental dynamics. The quite fast escaping set, $Q(f)$, defined by an apparently weaker condition is equal to $A(f)$ under certain conditions. Here we introduce $Q_2(f)$ defined by what appears to be an even weaker condition. Using a new regularity condition we show that functions of finite order and positive lower order satisfy $Q_2(f) = A(f)$. We also show that the finite composition of such functions satisfies $Q_2(f) = A(f)$. Finally, we construct a function for which $Q_2(f) \neq Q(f) = A(f)$.

1. Introduction

Let f be a transcendental entire function and denote by $f^n, n = 0, 1, ...,$ the nth iterate of f. An introduction to the theory of iteration of transcendental entire and meromorphic functions can be found in [1].

The set
$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$$
is called the *escaping set* and was first studied for a general transcendental entire function by Eremenko in [6], where he conjectured that all the components of $I(f)$ are unbounded. Although much progress has been made towards the conjecture, it still remains an open problem.

Results on Eremenko's conjecture for a general transcendental entire function have been obtained by Rippon and Stallard in [10], [12] by considering a subset of the escaping set known as the *fast escaping set*, $A(f)$, and showing that all the components of $A(f)$ are unbounded. This set was introduced by Bergweiler and Hinkkanen in [3]. We will use the definition given in [10] according to which
$$A(f) = \{z : \text{ there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\},$$
where
$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|, \quad \text{for} \quad r > 0,$$
and $R > 0$ is large enough to ensure that $M(r) > r$ for $r \geq R$.

The set $A(f)$ now plays a key role in complex dynamics (see [10]) and so it is useful to be able to identify points that are fast escaping. In [10, Theorem 2.7], it is shown that points which eventually escape faster than the iterates of any

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function of the form $r \mapsto \varepsilon M(r), r > 0$, where $\varepsilon \in (0,1)$, are actually fast escaping.

It is natural to ask whether μ_ε can be replaced in this result by a smaller function. In this context, Rippon and Stallard introduced the *quite fast escaping set* $Q(f)$ in [13]. Let $\mu_\varepsilon(r) = M(r)^\varepsilon$, where $r > 0$ and $\varepsilon \in (0,1)$. The quite fast escaping set is defined as follows:

$$Q(f) = \{z : \exists\, \varepsilon \in (0,1) \text{ and } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \mu_\varepsilon^n(R), \text{ for } n \in \mathbb{N}\},$$

where $R > 0$ is such that $\mu_\varepsilon(r) > r$ for $r \geq R$. The function μ_ε that is used in the definition of $Q(f)$ is smaller than the function defined by $r \mapsto \varepsilon M(r)$.

The set $Q(f)$ arises naturally in complex dynamics and so it is of interest to establish when $Q(f)$ is equal to $A(f)$. Although Rippon and Stallard were the first to define $Q(f)$, points that belong to $Q(f)$ were used earlier in results concerning the Hausdorff measure and Hausdorff dimension of the escaping set and the Julia set of some classes of functions (see [4] and [8]). Rippon and Stallard showed that $Q(f) = A(f)$ for many classes of functions, but they also constructed examples where $Q(f) \neq A(f)$. One well studied class of functions for which $Q(f) = A(f)$ is the Eremenko-Lyubich class $\mathcal{B}$ which consists of the functions for which the set of singularities of the inverse function, f^{-1}, is bounded (see [7]).

The following family of functions $\mu_{m,\varepsilon}$ is a natural generalisation of the function μ_ε defined by $\mu_\varepsilon(r) = M(r)^\varepsilon$:

$$\log^m \mu_{m,\varepsilon}(r) = \varepsilon \log^m M(r), \quad m \in \mathbb{N}, \quad \varepsilon \in (0,1),$$

whenever $\mu_{m,\varepsilon}(r)$ is defined. Note that the function μ_ε used in the definition of $Q(f)$ is equal to $\mu_{1,\varepsilon}$.

In this paper we focus on the case $m = 2$, that is, we consider

$$\mu_{2,\varepsilon}(r) = \exp((\log M(r))^\varepsilon), \quad 0 < \varepsilon < 1, \tag{1.1}$$

and we set

$$Q_2(f) = \{z : \exists\, \varepsilon \in (0,1) \text{ and } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \mu_{2,\varepsilon}^n(R), \text{ for } n \in \mathbb{N}\}, \tag{1.2}$$

where $R > 0$ is such that $\mu_{2,\varepsilon}(r) > r$ for $r \geq R$. Note that $Q_2(f)$ is independent of R.

For $0 < \varepsilon < 1$ we have $\mu_{2,\varepsilon}(r) < \mu_\varepsilon(r)$, for sufficiently large r, so

$$A(f) \subset Q(f) \subset Q_2(f).$$

Unlike the functions μ_ε that were introduced in earlier papers, for $\mu_{2,\varepsilon}$ we do not know a priori that, for any given transcendental entire function, $\mu_{2,\varepsilon}(r) > r$ for r large enough. This means that, for some slowly growing functions f, there exist points in $Q_2(f)$ that are not even escaping. However, $Q_2(f) \subset I(f)$ for a large class of functions. We seek to identify functions for which $Q_2(f) = A(f)$.

Recall that the order $\rho(f)$ and lower order $\lambda(f)$ of f are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}, \quad \lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

We prove the following result:

Theorem 1.1. *Let $f = f_1 \circ f_2 \circ \cdots \circ f_j$ be a finite composition of transcendental entire functions, where f_1 has finite order and positive lower order. Then $Q_2(f) = A(f)$.*

An immediate consequence of Theorem 1.1 is the following:

Corollary 1.2. *We have $Q_2(f) = A(f)$ whenever $f = f_1 \circ f_2 \circ \cdots \circ f_j$ is a finite composition of transcendental entire functions and f_1 satisfies one of the following:*

(a) *there exist $A, B, C, r_0 > 1$ such that*

$$A \log M(r, f_1) \le \log M(Cr, f_1) \le B \log M(r, f_1), \quad \textit{for } r \ge r_0;$$

(b) *$f_1 \in \mathcal{B}$ and is of finite order.*

In fact, functions of type (a) were studied by Bergweiler and Karpińska in [2] where it was shown that they are of finite order and positive lower order. All functions in class $\mathcal{B}$ have lower order not less than 1/2 (see [9, Lemma 3.5]) and finite compositions of such functions of finite order were considered by Rottenfusser, Rückert, Rempe and Schleicher in [14].

The proof of Theorem 1.1 is in three steps. We first introduce a new regularity condition as follows:

A transcendental entire function f is *strongly log-regular* if, for any $\varepsilon \in (0, 1)$, there exist $R > 0$ and $k > 1$ such that

$$\log M(r^k) \ge (k \log M(r))^{1/\varepsilon}, \quad \text{for } r > R. \tag{1.3}$$

Using (1.1) we see that for each $\varepsilon \in (0, 1)$, (1.3) is equivalent to

$$\mu_{2,\varepsilon}(r^k) \ge M(r)^k, \quad \text{for } r > R, \tag{1.4}$$

which implies that

$$\mu_{1,\varepsilon}(r^k) \ge M(r)^k, \quad \text{for large } r, \tag{1.5}$$

or equivalently, there exist $R_0 > 0, k, d > 1$ such that

$$M(r^k) \ge M(r)^{kd}, \quad \text{for } r > R_0. \tag{1.6}$$

The latter condition is equivalent to the condition called *log-regularity* that was used in [13] as a sufficient condition for $Q(f) = A(f)$. The name strong log-regularity arises from the fact that strong log-regularity implies log-regularity.

It seems natural to generalise (1.2) and (1.4) for any $m \in \mathbb{N}$ as follows:
Let

$$Q_m(f) = \{z : \exists\, \varepsilon \in (0, 1), \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \ge \mu^n_{m,\varepsilon}(R), \text{ for } n \in \mathbb{N}\},$$

where $R > 0$ is such that $\mu_{m,\varepsilon}(r) > r$ for $r \geq R$. If $\mu_{m,\varepsilon}$ satisfies the generalised (1.4), that is, for any $\varepsilon \in (0,1)$ and any $m \in \mathbb{N}$, there exist $R > 0$ and $k > 1$ such that

$$\mu_{m,\varepsilon}(r^k) \geq M(r)^k, \text{ for } r > R, \tag{1.7}$$

then we can show that $Q_m(f) = A(f)$. However, the larger m is, the more difficult it is for f to satisfy (1.7). For example, it is not hard to check that, for $m = 3$, $f(z) = e^z$ does not satisfy (1.7). In forthcoming work we give alternative, but more complicated, regularity conditions which, for any $m \geq 2$, guarantee that $Q_m(f) = A(f)$ for a wide range of functions f.

In Section 2, we give the first two steps of the proof of Theorem 1.1. In the first step, we show that strong log-regularity is a sufficient condition for $Q_2(f) = A(f)$. In the second step, we prove that any transcendental entire function of finite order and positive lower order is strongly log-regular.

The last step of the proof is given in Section 3 where we show that strong log-regularity is preserved under finite composition of transcendental entire functions where the first function of the composition is strongly log-regular.

Finally, in the last section, we construct two functions. The first is an example of a strongly log-regular function with zero lower order and positive, finite order and the second is a function for which $Q_2(f) \neq A(f)$ whereas $Q(f) = A(f)$.

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2. Sufficient conditions for $Q_2(f) = A(f)$

In this section we give the first two steps of the proof of Theorem 1.1. We first prove the following result about strongly log-regular functions:

Theorem 2.1. *Let f be a transcendental entire function which is strongly log-regular. Then $Q_2(f) = A(f)$.*

Proof. Clearly $A(f) \subset Q_2(f)$, as noted earlier. Suppose now that $z \in Q_2(f)$. Then (1.2) implies that there exist $\varepsilon \in (0,1)$ and $\ell \in \mathbb{N}$ such that

$$|f^{n+\ell}(z)| \geq \mu^n_{2,\varepsilon}(R), \text{ for } n \in \mathbb{N}, \tag{2.1}$$

where $R > 0$ is such that $\mu_{2,\varepsilon}(r) > r$ for $r \geq R$. As f is strongly log-regular it satisfies (1.4) and so there exist $R_0 > R$ and $k > 1$ such that, for $r > R_0$,

$$\mu_{2,\varepsilon}(r^k) \geq M(r)^k, \text{ for } r > R_0. \tag{2.2}$$

By applying (2.2) twice we obtain

$$\mu^2_{2,\varepsilon}(r^k) = \mu_{2,\varepsilon}(\mu_{2,\varepsilon}(r^k)) \geq \mu_{2,\varepsilon}((M(r))^k) \geq (M(M(r)))^k, \text{ for } r > R_0$$

since $\mu_{2,\varepsilon}(r^k) > R$. By applying (2.2) repeatedly in this way we obtain that

$$\mu^n_{2,\varepsilon}(r^k) \geq (M^n(r))^k \geq M^n(r), \text{ for } r > R_0. \tag{2.3}$$

But $M^n(r) \to \infty$ as $n \to \infty$ for $r \geq R$ and so there exists $n_0 \in \mathbb{N}$ such that $M^{n_0}(R) \geq R^k$ and hence, (2.1) and (2.3) imply that

$$|f^{n+n_0+\ell}(z)| \geq \mu_{2,\varepsilon}^{n+n_0}(R) \geq M^{n+n_0}(R^{1/k}) \geq M^n(R),$$

and the result follows. □

We now show that all functions of finite order and positive lower order are strongly log-regular.

Theorem 2.2. *Let f be a transcendental entire function of finite order and positive lower order. Then f is strongly log-regular and hence $Q_2(f) = A(f)$.*

Proof. Let f be a transcendental entire function of finite order and positive lower order. Then there exist $0 < q < p$ such that

$$e^{r^q} \leq M(r) \leq e^{r^p}, \text{ for sufficiently large } r, \tag{2.4}$$

or equivalently

$$r^q \leq \log M(r) \leq r^p.$$

So, for each $\varepsilon \in (0, 1)$ and sufficiently large r,

$$(\log M(r^k))^\varepsilon \geq (r^{qk})^\varepsilon = r^{\varepsilon qk}. \tag{2.5}$$

It follows from (2.4) that, for $k > p/(q\varepsilon)$, there exists $R > 0$ such that, for $r > R$,

$$r^{\varepsilon qk} \geq kr^p \geq k \log M(r), \tag{2.6}$$

so (1.3) is satisfied, by (2.5) and (2.6). □

3. Composition and strong log-regularity

In this section we complete the proof of Theorem 1.1 by showing that the finite composition of transcendental entire functions, where the first function of the composition is strongly log-regular, is a strongly log-regular function.

Theorem 3.1. *Let $f_1, f_2, ..., f_j$ be transcendental entire functions and suppose f_1 is strongly log-regular. Then $g = f_1 \circ f_2 \circ ... \circ f_j$ is strongly log-regular.*

Theorem 3.1 implies that if f is strongly log-regular then the n-th iterate f^n is strongly log-regular as well.

In order to prove the theorem we need the following lemma of Rippon and Stallard [11, Lemma 2.2].

Lemma 3.2. *Let f be a transcendental entire function. Then there exists $R_0 > 0$ such that, for all $r \geq R_0$ and all $c > 1$,*

$$\log M(r^c) \geq c \log M(r). \tag{3.1}$$

We also need the following lemma of Sixsmith [15, Lemma 2.4].

Lemma 3.3. *Suppose that f is a non-constant entire function and g is a transcendental entire function. Then, given $\nu > 1$, there exist $R_1, R_2 > 0$ such that*

$$M(\nu r, f\circ g) \geq M(M(r,g),f) \geq M(r, f\circ g), \quad \text{for } r \geq R_1 \tag{3.2}$$

and

$$M(\nu r, g\circ f) \geq M(M(r,f),g) \geq M(r, g\circ f), \quad \text{for } r \geq R_2. \tag{3.3}$$

Proof of Theorem 3.1. It is sufficient to prove the result for $j = 2$. Let f_1 be strongly log-regular, that is, for any $\varepsilon \in (0,1)$ there exist $R > 0$ and $k > 1$ such that

$$(\log M(r^k, f_1))^{\varepsilon} \geq k \log M(r, f_1), \quad \text{for } r \geq R, \tag{3.4}$$

and let f_2 be any transcendental entire function.

Given $\varepsilon' \in (0,1)$ we take $\varepsilon = \frac{2}{3}\varepsilon'$. Then there exist $R > 0$ and $k > 1$ such that (3.4) holds with this ε. Now take $\nu = k^{1/2}$ and put $k' = \nu k = k^{3/2}$. Note that $\varepsilon' = \frac{3}{2}\varepsilon = \varepsilon(1 + \log \nu/\log k)$. Then we apply Lemma 3.3 with $f = f_2$ and $g = f_1$, where R_2 is the constant in (3.3) and R_0 is the constant in (3.1) for $f = f_2$. So, for $r \geq \max\{e, R_0, R_2\}$, we have

$$\begin{aligned} M(r^{k'}, f_1\circ f_2) &\geq M(\nu r^k, f_1 \circ f_2) \\ &\geq M(M(r^k, f_2), f_1), \qquad \text{by (3.3)} \\ &\geq M(M(r, f_2)^k, f_1), \qquad \text{by (3.1)}. \end{aligned}$$

Hence, for $r \geq R' = \max\{e, R, R_0, R_2\}$,

$$\begin{aligned} (\log M(r^{k'}, f_1\circ f_2))^{\varepsilon} &\geq (\log M(M(r, f_2)^k, f_1))^{\varepsilon} \\ &\geq k \log M(M(r, f_2), f_1), \quad \text{by (3.4)} \\ &\geq k \log M(r, f_1 \circ f_2). \end{aligned}$$

Hence, for $r \geq R'$,

$$\begin{aligned} (\log M(r^{k'}, f_1\circ f_2))^{\varepsilon'} &= (\log M(r^{k'}, f_1\circ f_2))^{(3/2)\varepsilon} \\ &\geq (k \log M(r, f_1\circ f_2))^{3/2} \\ &= k'(\log M(r, f_1\circ f_2))^{3/2} \\ &\geq k' \log M(r, f_1\circ f_2), \end{aligned}$$

as required. So $f_1 \circ f_2$ is strongly log-regular. □

4. Examples

In this section we construct two examples of functions with specific properties.

Example 4.1. There exists a transcendental entire function of zero lower order and positive, finite order which is strongly log-regular.

Example 4.2. There exists a transcendental entire function f which is log-regular such that $Q_2(f) \neq A(f)$. Hence, $Q(f) = A(f)$ but $Q_2(f) \neq Q(f)$.

In order to construct these functions we use the following lemma (see [5]).

Lemma 4.1. *Let ϕ be a convex increasing function on $\mathbb{R}$ such that $\phi(t) \neq O(t)$ as $t \to \infty$. Then there exists a transcendental entire function f such that*

$$\log M(e^t, f) \sim \phi(t) \quad as \quad t \to \infty.$$

We showed in Section 2 that all transcendental entire functions of finite order and positive lower order are strongly log-regular. However, a strongly log-regular function of finite order does not need to have positive lower order. Indeed, Example 4.1 gives a function of zero lower order and positive, finite order which is strongly log-regular.

Proof of Example 4.1. We first take a fixed value of ε, say $\tilde{\varepsilon} \in (0,1)$, and a fixed value of k, say $\tilde{k}$, such that $\tilde{k} > 2/\tilde{\varepsilon} \geq \frac{2\log(\tilde{k}+1)}{\log \tilde{k}}$ and construct a convex increasing function ϕ on $\mathbb{R}$ such that:

a) $\displaystyle\liminf_{t\to\infty} \frac{\log \phi(t)}{t} = 0$;
b) $\displaystyle 1 \leq \limsup_{t\to\infty} \frac{\log \phi(t)}{t} \leq \tilde{k}$;
c) there exists $T > 0$ such that, for $t > T$,

$$\phi(\tilde{k}t) \geq (\tilde{k}\phi(t))^{1/\tilde{\varepsilon}}. \tag{4.1}$$

Once this is done, we show that this function ϕ satisfies (4.1) for *any* $\varepsilon \in (0,1)$ with a suitable $k > 1$.

Take $\tilde{d} = 1/\tilde{\varepsilon}$. Then $\tilde{k} > \tilde{d} > 1$. Take $a_0 = 1$, and choose t_0 so large that

$$\frac{\log \tilde{k}}{t_0} < \frac{1}{2} \tag{4.2}$$

and

$$\tilde{k}^{\tilde{d}} e^{\tilde{d}t} \leq e^{\tilde{k}t}, \quad \text{for} \quad t \geq t_0. \tag{4.3}$$

Then we set $t_n = \tilde{k}^n t_0$, for $n \in \mathbb{N}$, and define

$$a_n = e^{t_n}, \quad n = N_1, N_2, \ldots, N_m, \ldots, \tag{4.4}$$

where (N_m) is an increasing sequence, to be chosen shortly, and

$$a_n = (\tilde{k}a_{n-1})^{\tilde{d}}, \quad \text{elsewhere.} \tag{4.5}$$

We will show that, for each $m \in \mathbb{N}$, we can choose N_m so that

$$\frac{\log a_{N_m - 1}}{t_{N_m - 1}} < \frac{1}{2^m} \tag{4.6}$$

and

$$e^{t_{N_m}} \geq (\tilde{k}a_{N_m-1})^{\tilde{d}}. \tag{4.7}$$

Then we let ϕ be the real function that is linear on each of the intervals $[t_n, t_{n+1}]$ with $\phi(t_n) = a_n$, for $n \in \mathbb{N}$.

Suppose there is no $N_1 \in \mathbb{N}$ which satisfies (4.6) with $m = 1$. Then $a_n = (\tilde{k} a_{n-1})^{\tilde{d}}$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} \frac{\log a_n}{t_n} &= \frac{\log(\tilde{k} a_{n-1})^{\tilde{d}}}{t_n} \\ &= \frac{\tilde{d}}{\tilde{k}} \left(\frac{\log \tilde{k}}{t_{n-1}} + \frac{\log a_{n-1}}{t_{n-1}} \right). \end{aligned} \tag{4.8}$$

Now let $x_n = \frac{\log a_n}{t_n}$, $c = \tilde{d}/\tilde{k} < 1$ and $\varepsilon_{n-1} = \frac{\log \tilde{k}}{t_{n-1}}$. We have that

$$x_n = c(\varepsilon_{n-1} + x_{n-1}), \quad \text{for all } \ n \in \mathbb{N},$$

so

$$\begin{aligned} \limsup_{n\to\infty} x_n &\leq c \limsup_{n\to\infty} \varepsilon_{n-1} + c \limsup_{n\to\infty} x_{n-1} \\ &= c \limsup_{n\to\infty} x_{n-1}, \end{aligned}$$

as $\varepsilon_{n-1} \to 0$ as $n \to \infty$ and $c < 1$. Hence

$$\limsup_{n\to\infty} \frac{\log a_n}{t_n} = 0$$

and so we obtain a contradiction. Therefore, (4.6) is true for some $N_1 \in \mathbb{N}$.

Suppose now that (4.6) is true for $N_1, N_2, ..., N_m \in \mathbb{N}$ but it fails to be true for all $n > N_m$. Following the above argument, we again obtain a contradiction and so there exists $N_{m+1} \in \mathbb{N}$ which satisfies (4.6).

It follows from (4.2),(4.6) and the fact that $\tilde{k} > 2\tilde{d} > 1$ that

$$\frac{\log(\tilde{k} a_{N_m-1})^{\tilde{d}}}{t_{N_m}} = \frac{\tilde{d}}{\tilde{k}} \left(\frac{\log \tilde{k}}{t_{N_m-1}} + \frac{\log a_{N_m-1}}{t_{N_m-1}} \right) \leq \frac{\log a_{N_m-1}}{t_{N_m-1}} < \frac{1}{2^m} < 1,$$

and so $e^{t_{N_m}} > (\tilde{k} a_{N_m-1})^{\tilde{d}}$, which means that (4.6) implies (4.7).

In order to prove a) we note that it follows from (4.6) that

$$\frac{\log \phi(t_{N_m-1})}{t_{N_m-1}} = \frac{\log a_{N_m-1}}{t_{N_m-1}} < \frac{1}{2^m}, \quad \text{for } \ m \in \mathbb{N},$$

and so

$$\liminf_{t\to\infty} \frac{\log \phi(t)}{t} \leq \liminf_{m\to\infty} \frac{1}{2^m} = 0.$$

We also note that it follows from (4.4) that

$$\frac{\log \phi(t_{N_m})}{t_{N_m}} = \frac{\log a_{N_m}}{t_{N_m}} = 1, \quad \text{for } \ m \in \mathbb{N},$$

and so, in order to prove b), it remains to show that

$$\limsup_{t\to\infty} \frac{\log \phi(t)}{t} \leq \tilde{k}.$$

It suffices to show that $\phi(t) \le e^{\tilde{k}t}$ for large values of t. We will first show that $\phi(t_n) \le e^{t_n}$, for n large enough.

Suppose that $\phi(t_n) \le e^{t_n}$ for some n. Then either

$$\phi(t_{n+1}) = e^{t_{n+1}}$$

or

$$\phi(t_{n+1}) = (\tilde{k}\phi(t_n))^{\tilde{d}} \le (\tilde{k}e^{t_n})^{\tilde{d}} \le e^{\tilde{k}t_n} = e^{t_{n+1}}, \tag{4.9}$$

by (4.3). In either case, we deduce that $\phi(t_n) \le e^{t_n}$ implies that $\phi(t_{n+1}) \le e^{t_{n+1}}$. Since $\phi(t_{N_m}) = e^{t_{N_m}}$, for all $m \in \mathbb{N}$, we conclude that $\phi(t_n) \le e^{t_n}$, for $t_n \ge t_{N_1}$. Now take any $t \in [t_n, t_{n+1}], n \ge N_1$. Then

$$\phi(t) \le \phi(t_{n+1}) \le e^{t_{n+1}} = e^{\tilde{k}t_n} \le e^{\tilde{k}t},$$

and the result follows.

We now show that (4.1) is true for $t \ge t_0$. In order to do so, we consider the functions $g(t) = \phi(\tilde{k}t)$ and $h(t) = (\tilde{k}\phi(t))^{\tilde{d}}$. For each $n \ge 0$, g is a linear, increasing function on $[t_n, t_{n+1}] = [t_n, \tilde{k}t_n]$ and h is convex on $[t_n, t_{n+1}]$. We will find the values of the two functions g and h at the endpoints of each interval and we will use the fact that the graph of a convex function which has the same or smaller values at the endpoints than a linear function is always below the graph of the linear function. Thus, to show that $h(t) = (\tilde{k}\phi(t))^{\tilde{d}} \le \phi(\tilde{k}t) = g(t)$ for all $t \ge t_0$, it is sufficient to show that

$$(\tilde{k}\phi(t_n))^{\tilde{d}} \le \phi(t_{n+1}), \quad \text{for} \quad n \ge 0.$$

This is evidently true if (4.5) holds and if (4.4) holds it is true by (4.7).

Finally, we need to show that ϕ is convex. It suffices to show that the sequence of gradients $g_n = \frac{a_n - a_{n-1}}{t_n - t_{n-1}}$, $n \in \mathbb{N}$, of the line segments in the graph of ϕ is increasing, or equivalently that, for $n \in \mathbb{N}$,

$$\frac{a_{n+1} - a_n}{t_{n+1} - t_n} \ge \frac{a_n - a_{n-1}}{t_n - t_{n-1}}. \tag{4.10}$$

Since $t_n = \tilde{k}^n t_0$, for $n \in \mathbb{N}$, we need to show that

$$a_{n+1} \ge (\tilde{k} + 1)a_n - \tilde{k}a_{n-1}.$$

But

$$a_{n+1} + \tilde{k}a_{n-1} \ge (\tilde{k} + 1)a_n$$

since, by (4.7) and the fact that $\tilde{d} = \frac{1}{\tilde{\varepsilon}} \ge \frac{\log(\tilde{k}+1)}{\log \tilde{k}}$,

$$a_{n+1} \ge (\tilde{k}a_n)^{\tilde{d}} \ge \tilde{k}^{\tilde{d}}a_n \ge (\tilde{k} + 1)a_n, \quad \text{for} \quad n \in \mathbb{N},$$

and the result follows.

We have constructed a function ϕ such that (4.1) holds for $\tilde{\varepsilon}$ which is a specific value of $\varepsilon \in (0, 1)$. In fact for any other $\varepsilon \in (0, \tilde{\varepsilon})$ we can find a large enough $k > 1$ such that (4.1) holds for the same function ϕ. Indeed, suppose first that

(4.1) holds for $\varepsilon = \tilde{\varepsilon}$ and set $\tilde{d} = 1/\tilde{\varepsilon}$, as before.

Now take $\varepsilon \in (0, \tilde{\varepsilon})$ and suppose that $1/\varepsilon = d = \tilde{d}^p$, for some $n \le p < n+1, n \in \mathbb{N}$. It follows from (4.1) that, for $t \ge t_0$,

$$\begin{aligned} \phi(\tilde{k}^{2n+2}t) &\ge (\tilde{k}\phi(\tilde{k}^{2n+1}t))^{\tilde{d}} = \tilde{k}^{\tilde{d}}\phi(\tilde{k}^{2n+1}t)^{\tilde{d}} \\ &\ge \tilde{k}^{\tilde{d}}\tilde{k}^{\tilde{d}^2}\phi(\tilde{k}^{2n}t)^{\tilde{d}^2} \\ &\ge \tilde{k}^{\tilde{d}+\tilde{d}^2+\cdots+\tilde{d}^{2n+2}}\phi(t)^{\tilde{d}^{2n+2}}. \end{aligned} \tag{4.11}$$

We now show that

$$\tilde{k}^{\tilde{d}+\tilde{d}^2+\cdots+\tilde{d}^{2n+2}}\phi(t)^{\tilde{d}^{2n+2}} \ge (\tilde{k}^{2n+2}\phi(t))^{\tilde{d}^p}. \tag{4.12}$$

As $p < n+1$, it suffices to show that

$$\tilde{d} + \tilde{d}^2 + \cdots + \tilde{d}^{2n+2} \ge (2n+2)\tilde{d}^{n+1}. \tag{4.13}$$

We will prove (4.13) using the inequality of arithmetic and geometric means, which implies that

$$\begin{aligned} \frac{\tilde{d} + \tilde{d}^2 + \cdots + \tilde{d}^{2n+2}}{2n+2} &\ge \sqrt[2n+2]{\tilde{d}\tilde{d}^2\cdots\tilde{d}^{2n+2}} \\ &= \sqrt[2n+2]{\tilde{d}^{(2n+2)(2n+3)/2}} \\ &= \tilde{d}^{(2n+3)/2} = \tilde{d}^{n+3/2} > \tilde{d}^{n+1}, \end{aligned}$$

as required. Combining (4.11) and (4.12) gives

$$\phi(kt) \ge (k\phi(t))^{\varepsilon}, \text{ for } t \ge t_0,$$

where $k = \tilde{k}^{2n+2}$. Thus (4.1) holds for any $\varepsilon \in (0, \tilde{\varepsilon})$ and hence for any $\varepsilon \in (0, 1)$.

Now we can apply Lemma 4.1 to ϕ to give a transcendental entire function f such that

$$\log M(e^t, f) = \phi(t)(1 + \delta(t)), \tag{4.14}$$

where $\delta(t) \to 0$ as $t \to \infty$. Then

$$\frac{\log\log M(e^t, f)}{t} = \frac{\log\phi(t)}{t} + \frac{O(\delta(t))}{t}$$

and so $\lambda(f) = 0$ and $1 \le \rho(f) \le \tilde{k}$, by properties (a) and (b) respectively.

It remains to show that f satisfies (1.3). We know that for any $\varepsilon \in (0, 1)$ there exists $k > d = 1/\varepsilon$ such that

$$\phi(kt) \ge (k\phi(t))^d, \text{ for } t \ge t_0. \tag{4.15}$$

Let $0 < \varepsilon' < \varepsilon$ and set $d' = 1/\varepsilon'$. Then, by (4.14) and (4.15),

$$\log M(e^{kt}, f) \ge \frac{1 + \delta(kt)}{(1 + \delta(t))^{d'}}(k \log M(e^t, f))^{d'}, \text{ for } t \ge t_0.$$

Since

$$\frac{1 + \delta(kt)}{(1 + \delta(t))^{d'}} \to 1 \text{ as } t \to \infty,$$

we deduce that

$$\log M(e^{kt}, f) \geq (k \log M(e^t, f))^{1/\varepsilon},$$

for large t and so f satisfies (1.3) for sufficiently large r with $r = e^t$. □

Throughout the paper, we are interested in sufficient conditions for $Q_2(f) = A(f)$. However, these two sets are not always equal. We now construct a function for which $Q_2(f)$ is not equal to $A(f)$.

Proof of Example 4.2. We construct a transcendental entire function f which is log-regular and hence, by [13, Theorem 4.1], $Q(f) = A(f)$, but for which $Q_2(f) \neq A(f)$. Obviously, this function cannot be strongly log-regular. In order to construct such a function we will again use the result of Clunie and Kövari (see Lemma 4.1). The idea is to find a real, increasing, convex function ϕ such that:

- there exist $k > 1$ and $d > 1$ such that

$$\phi(kt) \geq kd\phi(t), \quad \text{for large } t, \tag{4.16}$$

  and

- if f is produced from ϕ using Lemma 4.1 then the iterates of the function $\mu_{2,\varepsilon}(r)$, for $\varepsilon \in (1/2, 1)$, grow much more slowly than the iterates of $M(r) = M(r, f)$.

Let $\phi(t) = t^2, t > 0$. Then ϕ is increasing and convex. Let $k > 1$ and $1 < d < k$. Then

$$\phi(kt) = k^2t^2 > kdt^2 = kd\phi(t)$$

and so (4.16) is satisfied.

Now we apply Lemma 4.1 to ϕ to give a transcendental entire function f such that

$$\log M(e^t, f) = \phi(t)(1 + \delta(t)) = t^2(1 + \delta(t)), \tag{4.17}$$

where $\delta(t) \to 0$ as $t \to \infty$.

Now (4.16) implies that

$$\log M(e^{kt}, f) \geq kd\frac{1 + \delta(kt)}{1 + \delta(t)} \log M(e^t, f),$$

where

$$\frac{1 + \delta(kt)}{1 + \delta(t)} \to 1 \text{ as } t \to \infty.$$

Hence, there exists $1 < d' < d$ such that

$$\log M(e^{kt}, f) \geq kd' \log M(e^t, f),$$

for large t, and so, by (1.6), f is log-regular which implies that $Q(f) = A(f)$.

Now we show that, for $\varepsilon \in (1/2,1)$, the iterates of $\mu_{2,\varepsilon}(r)$ grow more slowly than the iterates of $M(r)$.

By (4.17), we have

$$M(r) = \exp((\log r)^2(1+\nu(r))), \tag{4.18}$$

where $\nu(r) = \delta(\log r) \to 0$, as $r \to \infty$, and

$$\mu_{2,\varepsilon}(r) = \exp((\log M(r))^{\varepsilon}) = \exp(((\log r)^2(1+\nu(r)))^{\varepsilon}) = \exp((\log r)^{2\varepsilon}(1+\nu(r))^{\varepsilon}). \tag{4.19}$$

Now fix $\varepsilon \in (1/2,1)$. It then follows from (4.19) that there exists $R > 0$ such that we have $\mu_{2,\varepsilon}(r) > r$, for $r \geq R$ and so $\mu^n_{2,\varepsilon}(R) > R$, for $n \in \mathbb{N}$.

The idea is to show that, for any $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$ with $n > N$, we have

$$\mu^{m+n}_{2,\varepsilon}(R_0) < M^n(R_0), \tag{4.20}$$

for some $R_0 \geq R$. We then show that this implies that $Q_2(f) \neq A(f)$.

Since $\varepsilon \in (1/2,1)$, it follows from (4.18) and (4.19) that there exist $R_1, R_2 > 0$ and $c, \tilde{c} \in \mathbb{R}$ such that

$$1 < 2\varepsilon < \tilde{c} < c < 2, \tag{4.21}$$

$$M(r) \geq \exp((\log r)^c), \quad \text{for } r > R_1, \tag{4.22}$$

and

$$\mu_{2,\varepsilon}(r) \leq \exp((\log r)^{\tilde{c}}), \quad \text{for } r > R_2. \tag{4.23}$$

Hence, by (4.22) and (4.23), we obtain, for $n \in \mathbb{N}$,

$$M^n(r) \geq \exp((\log r)^{c^n}), \quad \text{for } r > R_1,$$

and

$$\mu^n_{2,\varepsilon}(r) \leq \exp((\log r)^{\tilde{c}^n}), \quad \text{for } r > R_2.$$

By (4.21), we can easily see that, for any $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$ with $n > N$, we have

$$\tilde{c}^{n+m} < c^n,$$

and hence

$$\mu^{n+m}_{2,\varepsilon}(r) \leq \exp((\log r)^{\tilde{c}^{n+m}}) < \exp((\log r)^{c^n}) \leq M^n(r), \quad \text{for } r > R_0,$$

where $R_0 = \max\{R, R_1, R_2\}$. Therefore, (4.20) is satisfied for $R_0 = \max\{R, R_1, R_2\}$. We will now show that (4.20) implies that $Q_2(f) \setminus A(f)$ is non-empty. For this purpose we will use the following theorem of Rippon and Stallard (see [13, Theorem 3.1]).

Theorem 4.2. *Let f be a transcendental entire function. There exists $R = R(f) > 0$ with the property that whenever (a_n) is a positive sequence such that*

$$a_n \geq R \ \text{ and } \ a_{n+1} \leq M(a_n), \ \text{ for } n \in \mathbb{N},$$

there exists a point $\zeta \in \mathbb{C}$ and a sequence (n_j) with $n_j \to \infty$ such that

$$|f^n(\zeta)| \geq a_n, \ \text{ for } n \in \mathbb{N}, \ \text{ but } |f^{n_j}(\zeta)| \leq M^2(a_{n_j}), \ \text{ for } j \in \mathbb{N}.$$

Now, by Theorem 4.2, with $a_n = \mu^n_{2,\varepsilon}(R), n \in \mathbb{N}$, there exists a point ζ and a sequence (n_j) with $(n_j) \to \infty$ as $j \to \infty$, such that, for our function f,

$$|f^n(\zeta)| \geq \mu^n_{2,\varepsilon}(R), \quad \text{for } n \in \mathbb{N}, \tag{4.24}$$

and

$$|f^{n_j}(\zeta)| \leq M^2(\mu^{n_j}_{2,\varepsilon}(R)), \quad \text{for } j \in \mathbb{N}. \tag{4.25}$$

It follows from (4.24) that $\zeta \in Q_2(f)$. Also, (4.20) and (4.25) together imply that, for each $m \in \mathbb{N}$ and $n_j > m$, we have

$$\begin{aligned} |f^{(n_j-m+2)+m-2}(\zeta)| &= |f^{n_j}(\zeta)| \\ &\leq M^2(\mu^{n_j}_{2,\varepsilon}(R)) \\ &< M^2(M^{n_j-m}(R)) \\ &= M^{n_j-m+2}(R). \end{aligned}$$

Hence, $\zeta \notin A(f)$, so $Q_2(f) \neq A(f)$, as required. □

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