CENTRE for ECONOMIC PERFORMANCE

CEP Discussion Paper No 840

November 2007

Robustly Optimal Monetary Policy

Kevin D. Sheedy

THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

E·S·R·C ECONOMIC & SOCIAL RESEARCH COUNCIL

Abstract
This paper analyses optimal monetary policy in response to shocks using a model that avoids making specific assumptions about the stickiness of prices, and thus the nature of the Phillips curve. Nonetheless, certain robust features of the optimal monetary policy commitment are found. The optimal policy rule is a flexible inflation target which is adhered to in the short run without any accommodation of structural inflation persistence, that is, inflation which it is costly to eliminate. The target is also made more stringent when it has been missed in the past. With discretion on the other hand, the target is loosened to accommodate fully any structural inflation persistence, and any past deviations from the inflation target are ignored. These results apply to a wide range of price stickiness models because the market failure which the policymaker should aim to mitigate arises from imperfect competition, not from price stickiness itself.

Keywords: Inflation persistence, optimal monetary policy, rules versus discretion, stabilization bias, inflation targeting
JEL Classification: E5

This paper was produced as part of the Centre's Macro Programme. The Centre for Economic Performance is financed by the Economic and Social Research Council.

Acknowledgements
I thank Petra Geraats for helpful discussions and Michael Woodford for his comments on an earlier draft of this paper, entitled 'Resistance to Inflation Persistence: Optimal Monetary Policy Commitment'. This research has also benefited from comments received at the 2007 Money, Macro and Finance research conference and from workshop participants at the University of Cambridge. I thank the ESRC for financial support received during the writing of this paper.

Kevin D. Sheedy is an Associate of the Macro Programme at the Centre for Economic Performance. He is also a lecturer in the Department of Economics, London School of Economics. Email: k.d.sheedy@lse.ac.uk

Published by
Centre for Economic Performance
London School of Economics and Political Science
Houghton Street
London WC2A 2AE

ISBN 978-0-85328-215-0

1 Introduction

There is much debate among both economists and central bankers about the appropriate balance between rules and discretion in monetary policy. A central question in this debate is whether it is possible to find a rule that performs well in a broad range of circumstances, given the pervasive uncertainty which surrounds the monetary policy transmission mechanism and the shocks hitting the economy. This paper contributes to the debate by studying the nature of the optimal monetary policy commitment under a wide range of assumptions about the price-setting behaviour of firms, and thus a large set of alternative Phillips curves consistent with different degrees of inflation persistence.

In spite of the breadth of models covered, the optimal policy commitment is reducible to some simple principles. First, discretion needs to be constrained by resisting the temptation to accommodate inflation persistence. A policymaker acting with discretion would have an incentive to make any disinflation sub-optimally slow if inflation displays any "intrinsic" persistence: that is, persistence implied directly by the price-setting behaviour of firms. Second, if inflation exceeds its target value then the target in the future must be tightened by an equivalent amount. This ensures that the price level is stabilized in the long run. A policymaker acting with discretion would have no incentive to pay any heed to past failures to meet the inflation target. By credibly resisting the urge to return inflation slowly to target, and by committing to correct past failures to control inflation, a better response of inflation and aggregate output to the shocks hitting the economy is achieved.

Intrinsic inflation persistence is said to occur when firms' past price-setting decisions, as manifested in changes in inflation and relative prices, affect the current position of the short-run Phillips curve, and thus the range of feasible inflation-output gap combinations. Another way of saying this is that intrinsic inflation persistence occurs if past inflation rates and relative prices appear as state variables in the Phillips curve. In the widely used Calvo (1983) price-setting model, the resulting New Keynesian Phillips curve exhibits no intrinsic inflation persistence because it implies inflation is a purely forward-looking variable. However, this paper considers a far wider range of price-setting models in which the Phillips curve generally contains a backward-looking component and where intrinsic inflation persistence is ubiquitous.

Calvo pricing is based on the assumption that every firm in the economy has the same probability of changing price at all times. But rather than confine attention to models with this highly restrictive assumption, this paper allows for a completely general specification of time-dependent pricing, and one that also permits heterogeneity in price stickiness between industries. In other words, the probability of a firm choosing a new price can be any function of the time elapsed since a price change last occurred, and the industry the firm is based in. This includes the alternative time-dependent pricing models of Taylor (1980), Wolman (1999), and many others as well. But irrespective of the particular assumptions made about price stickiness, it turns out that it is possible to construct a targeting rule that is optimal within this class of models. Importantly, this means that it is not necessary to know which model within the class is true in order to find the appropriate optimal targeting rule.

To understand why it is possible to find an optimal policy targeting rule that is valid for all these different models of price stickiness, it is first necessary to identify the market failures that monetary policy has a role in mitigating. There are two of these and both stem from firms being imperfectly competitive price setters. The first results from firms' failure to take account of the effects of their pricing decisions on aggregate demand, creating an aggregate demand externality of the type discussed by Mankiw (1985). The second is that the market power of firms can lead to relative-price distortions and thus an inefficient allocation of resources in the economy.

A further source of relative-price distortions is the presence of price stickiness. Because price-adjustment times are not coordinated in general, the timing of firms' price changes affects relative prices, even when there is no change in demand or costs to warrant a relative-price adjustment. But if firms are maximizing profits when they select new prices then it turns out that such optimizing behaviour is very helpful to the policymaker. It is shown that for the policymaker to eliminate this type of relative-price distortion it is only necessary to intervene in the economy to the extent that firms having fully flexible prices would want to set the prices that support the Pareto-efficient allocation of resources. As these prices can be known without reference to the particular model of price stickiness, the policymaker's problem is accordingly simplified.

This convenient feature of firms' behaviour is found because there is some common ground between the policymaker's aim of minimizing price distortions and firms' goal of maximizing profits. A firm's profits are influenced to some extent by the gap between the price of its product and the prices of similar substitutable products sold by its competitors. If the gap becomes too wide then profits will suffer, but price distortions will clearly be increased too because an excessive spread of prices for similar products is inefficient. If part of the task of maximizing profits is to ensure that a firm's price gap relative to its rivals is not too wide, then it follows that profit maximization may actually help reduce price distortions in some situations. Thus profit-maximizing behaviour is a double-edged sword: it leads firms with market power to create some price distortions, but it also induces them to minimize other distortions that result from price stickiness.

The policymaker has one policy instrument (the nominal interest rate) with which to address the two market failures identified above, namely the aggregate demand externality and the relative-price distortions generated by imperfect competition and sticky prices. As it is not simultaneously possible to resolve both market failures, a compromise between these two objectives leads to an optimal monetary policy based around a flexible inflation target, that is, a target for a weighted average of the inflation rate and the output gap (the percentage deviation of aggregate output from its efficient level). It is shown that the relative need for correction of the two market failures can be judged without reference to the stickiness of prices, so there is just one optimal weighting of inflation and the output gap that applies to all models of price stickiness. The weights attached to the two objectives depend only on the extent of imperfect competition.

The differences between the optimal targeting rule and the optimal conduct of policy with discretion are manifested in the level the flexible inflation target is set at. With discretion, the target is revised frequently to accommodate fully any "intrinsic" inflation, that is, inflation which depends on state variables appearing in the Phillips curve such as past inflation rates and relative prices. But if a commitment is made at a certain time to set monetary policy according to a rule thereafter, the optimal targeting rule mandates a flexible inflation target that is instead lowered in line with the cumulative overshoot of inflation from its target value since the commitment came into force. The policymaker is not allowed to slacken the target temporarily even if inflation has picked up in the meantime and has resulting in a higher current level of "intrinsic" inflation. Indeed, the past failure to control inflation actually calls for a tighter target in the present. However, any intrinsic inflation which is already present when the commitment comes into force should be fully accommodated, making the optimal targeting rule time inconsistent. If the policymaker is required to use a rule that is time consistent in its treatment of intrinsic inflation, then the optimal time-consistent rule adopts the stance of resisting the temptation to accommodate all intrinsic inflation, irrespective of when it took root.

Such differences can be explained by showing that there is a difference between how intrinsic inflation should be seen ex ante and ex post. A policymaker contemplating the possibility of intrinsic inflation arising in the future sees it as bad for price distortions. However, a policymaker faced with a situation in which intrinsic inflation has already emerged because of a past failure to achieve price stability sees accommodation of it as the best course to take in minimizing price distortions. As the policymaker with discretion is free to revise any plans made in the past, the ex post view of intrinsic inflation drives monetary policy. When a commitment is made from a particular date onwards, the ex ante view predominates, even though an ex post view of intrinsic inflation existing at the time of the commitment is also taken. Time-consistent optimal policy requires that the ex ante view of intrinsic inflation be adopted at all times.

This paper builds upon the analysis of optimal monetary policy presented in Woodford's (2003) *Interest and Prices*, drawing on many of the techniques used in that work such as the use of a utility-based loss function to evaluate the conduct of monetary policy, and the notion of optimality from the timeless perspective when studying time-consistent policy rules. The major contribution of this paper is to address the optimal monetary policy problem when there is intrinsic inflation persistence present. The baseline model used by Woodford incorporates a Calvo pricing assumption and thus implies no intrinsic inflation persistence. While this leads to significant differences in the results where the analysis of discretionary policy is concerned, there is at least one timelessly optimal targeting rule presented by Woodford which is found to apply just as well to all the models of price stickiness considered in this paper.

There have been a number of modifications suggested to the baseline model of Woodford in order to generate intrinsic inflation persistence. One of the most widespread of these is the idea of Galí and Gertler (1999) that a certain fraction of firms might use a rule of thumb when setting prices. The optimal monetary policy implications of adding this feature to Woodford's model have been explored by Steinsson (2003). But there is a crucial difference between the intrinsic inflation persistence created by the rule of thumb and the intrinsic inflation persistence found in this paper. The former relaxes the assumption that firms maximize profits when setting prices, but maintains Calvo's assumption about the likelihood of price adjustment. This paper pursues the opposite strategy whereby the assumption of profit maximization is maintained but the Calvo pricing assumption is relaxed. Because the use of the rule of thumb means that firms are not maximizing profits, an additional market failure is created in addition to those highlighted earlier, and monetary policy acquires a role it would not otherwise have possessed. This leads to differences between the optimal policy implications found in this paper and those presented by Steinsson. Similar differences are also found when intrinsic inflation persistence is introduced using the assumption that firms use an indexation rule linked to past inflation (Christiano, Eichenbaum and Evans, 2005). The implications for optimal monetary policy in this case are analysed by Giannoni and Woodford (2005), and the conclusions differ from those of this paper because the use of the indexation rule means that firms are not maximizing profits when they change price.

The plan of this paper is as follows. Section 2 sets out the assumptions of the model, incorporating the full range of time-dependent pricing models. Then section 3 studies the optimal behaviour of firms and the policymaker and derives the equilibrium of the model, taking log-linear approximations as necessary. Section 4 examines the resulting Phillips curve and the extent of intrinsic inflation persistence. Section 5 derives the optimal policy implications in the cases of discretion and commitment from both a specific initial date and the timeless perspective. Some examples of these results applied to particular price-setting models are given in section 6, along with a discussion of how the results relate to other findings in the optimal monetary policy literature. Finally, some conclusions are drawn in section 7.

2 The model

2.1 Households

The economy contains a continuum of households. Each household consumes a basket of goods and devotes some fraction of its time to work. Each household specializes in providing one differentiated labour input, though each labour input is supplied by a large number of households. Households are indexed by a pair $(\imath, \jmath)$ denoting the $\jmath$-th household supplying the $\imath$-th labour input, with $\imath, \jmath \in \Omega$ where Ω is the unit interval. All households have the same set of preferences defined over consumption and leisure. In what follows, let $c_t(\imath, \jmath)$ denote household $(\imath, \jmath)$'s consumption of the basket of goods at time t, and $h_t(\imath, \jmath)$ denote the number of type-$\imath$ labour hours that household $(\imath, \jmath)$ supplies. The lifetime utility function of household $(\imath, \jmath)$ is

$$\mathcal{U}_t(\imath, \jmath) \equiv \max_{\substack{c_\tau(\imath,\jmath) \\ h_\tau(\imath,\jmath)}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[u(c_\tau(\imath, \jmath)) - v(h_\tau(\imath, \jmath)) \right] \tag{2.1.1}$$

where $u(\cdot)$ is a strictly increasing and concave function, $v(\cdot)$ is a strictly increasing and convex function, and $0 < \beta < 1$ is the household's subjective discount factor.

Households can purchase a basket of consumption goods at money price P_t. Hours of type-$\imath$ labour are remunerated by money wage $W_t(\imath)$. Because many households supply each type of labour input, households are price takers in both labour and goods markets. All households are assumed to start with equal financial wealth. Asset markets for state-contingent securities are complete. It is assumed these asset markets are open to households before they know what type of labour they will supply, and that each household is equally likely to be assigned to supply each specialized labour input. Since asset markets are complete, and initial financial wealth and expected lifetime labour income are equal ex ante before households know their type, and the utility function (2.1.1) is additively separable between consumption and leisure, households will choose to have their consumption fully insured against idiosyncratic shocks. There is thus a level of consumption $C_t = c_t(\imath, \jmath)$ common to all households at time t. The number of hours of labour supplied may differ across households, but only because there is potentially a dispersion of real wages for different types of labour input. All households supplying type-$\imath$ labour receive the same wage, and thus supply a common number of hours $H_t(\imath) = h_t(\imath, \jmath)$ each. Let $w_t(\imath) \equiv W_t(\imath)/P_t$ denote the real wage for labour of type $\imath$ at time t. The first-order condition for maximizing (2.1.1) with respect to the labour supply $H_t(\imath)$ subject to a given real wage $w_t(\imath)$ is

$$\frac{v_h(H_t(\imath))}{u_c(C_t)} = w_t(\imath) \tag{2.1.2}$$

where $u_c(\cdot)$ is the marginal utility of consumption and $v_h(\cdot)$ is the marginal utility of leisure.

The prices of the complete set of assets available to households are reflected in the asset-pricing kernel $\mathfrak{M}_{\tau|t}$. This gives the price of a basket of consumption goods in one particular state of the world at time τ relative to the conditional probability of that state occurring, the price being expressed in terms of period-t consumption baskets. Maximizing utility (2.1.1) with respect to consumption intertemporally and across different states of the world leads to the following Euler equation:

$$\frac{\beta^{\tau-t} u_c(C_\tau)}{u_c(C_t)} = \mathfrak{M}_{\tau|t} \tag{2.1.3}$$

2.2 Industries and differentiated goods

Households' consumption basket is made up of a range of differentiated goods. Each good is produced by a single firm using just one of the differentiated labour inputs, though each labour input is used by a large number of firms. Goods are indexed by a pair $(\imath, \jmath)$ denoting the $\jmath$-th good produced using the $\imath$-th labour input, with $\imath, \jmath \in \Omega$. The same system is used to index firms because production of each differentiated good is monopolized by a single firm.

While each firm's product is in some way unique, firms producing goods with similar characteristics are grouped together into industries. Each firm belongs to one and only one industry. There are $n \geq 1$ industries in total, with industry i having size $0 < \omega_i \leq 1$. An industry's size is measured by the fraction of the economy's firms that are based in it, so industry weights ω_i must sum to one. The unit interval Ω

is partitioned into industries Ω_i as follows:

$$\bigcup_{i=1}^{n} \Omega_i = \Omega \equiv [0,1) \quad , \quad \Omega_i \equiv \left[\sum_{j=1}^{i-1} \omega_j \, , \, \sum_{j=1}^{i} \omega_j \right) \tag{2.2.1}$$

Firm $(\imath, \jmath)$ is said to belong to industry i if $\imath \in \Omega_i$. It is assumed that all firms using the same differentiated labour input belong to the same industry, and each industry i employs a range of inputs $\imath \in \Omega_i$.

Let $C_t(\imath, \jmath)$ denote all households' total consumption of the good produced by firm $(\imath, \jmath)$. Households are willing to substitute between the products of firms within industries. That willingness is captured by the elasticity of substitution within industry i, denoted by $\varepsilon_{it} > 1$, which may vary exogenously across industries and over time. The elasticity represents the degree of competitiveness within industry i. Households' consumption basket C_{it} of the products of industry i and the corresponding price index P_{it} are

$$C_{it} \equiv \left(\frac{1}{\omega_i} \int_{\Omega_i} \int_{\Omega} C_t(\imath, \jmath)^{\frac{\varepsilon_{it}-1}{\varepsilon_{it}}} d\jmath d\imath \right)^{\frac{\varepsilon_{it}}{\varepsilon_{it}-1}} \quad , \quad P_{it} \equiv \left(\frac{1}{\omega_i} \int_{\Omega_i} \int_{\Omega} P_t(\imath, \jmath)^{1-\varepsilon_{it}} d\jmath d\imath \right)^{\frac{1}{1-\varepsilon_{it}}} \tag{2.2.2}$$

where $P_t(\imath, \jmath)$ is the money price charged by firm $(\imath, \jmath)$ for the good it produces at time t. If households minimize the expenditure required to achieve a given level of consumption C_{it} of industry i's products according to the basket in (2.2.2), then they allocate their spending between the firms within that industry as follows:

$$C_t(\imath, \jmath) = \left(\frac{P_t(\imath, \jmath)}{P_{it}} \right)^{-\varepsilon_{it}} C_{it} \tag{2.2.3}$$

This is the demand function faced by firm $(\imath, \jmath)$ in industry i, with the price elasticity of demand being the elasticity of substitution ε_{it}.

Just as households are prepared to substitute between goods produced within an industry, they are also willing to substitute between the products of different industries. The overall consumption basket and corresponding price index are

$$C_t \equiv \left(\sum_{i=1}^{n} \omega_i C_{it}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad , \quad P_t = \left(\sum_{i=1}^{n} \omega_i P_{it}^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}} \tag{2.2.4}$$

where $\varepsilon > 1$ is the elasticity of substitution between the products of different industries. The size of ε relative to the ε_{it} is not restricted. Again, minimizing the expenditure necessary to achieve a particular level of the overall consumption basket C_t in (2.2.4) leads to the following industry demand curve that determines the allocation of spending across industries:

$$C_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\varepsilon} C_t \tag{2.2.5}$$

The industry demand curve has price elasticity ε.

2.3 Production, costs and profits of firms

If firm $(\imath, \jmath)$ is based in industry i then its production function for output $Y_t(\imath, \jmath)$ is

$$Y_t(\imath, \jmath) = A_{it} H_t(\imath, \jmath)^{\eta_{yh}} \tag{2.3.1}$$

where A_{it} is the common exogenous level of technology in industry i, and $H_t(\imath, \jmath)$ denotes the number of hours of type-$\imath$ labour employed by firm $(\imath, \jmath)$. Technology may vary across industries and over time. The parameter $0 < \eta_{yh} \leq 1$ is the elasticity of output with respect to the employment of labour hours.

As each labour input is used by a large number of firms, every firm is a price taker in the labour market. So a firm paying real wage $w_t(\imath)$ in an industry with technology level A_{it} subject to production function (2.3.1) faces the following real total and marginal costs of producing output y_t at time t:

$$\mathcal{C}(y_t; A_{it}, w_t(\imath)) = w_t(\imath)\left(\frac{y_t}{A_{it}}\right)^{\frac{1}{\eta_{yh}}} \quad , \quad \mathcal{C}_Y(y_t; A_{it}, w_t(\imath)) = \frac{w_t(\imath)}{\eta_{yh}} \frac{y_t^{\frac{1-\eta_{yh}}{\eta_{yh}}}}{A_{it}^{\frac{1}{\eta_{yh}}}} \tag{2.3.2}$$

Firms are the sole producers of their individual products, so they have the power to set prices. Prices are posted in money terms, and $P_t(\imath, \jmath)$ denotes the money price used by firm $(\imath, \jmath)$ at time t. The resulting level of demand $Y_t(\imath, \jmath)$ is found by combining the demand curves in (2.2.3) and (2.2.5):

$$Y_t(\imath, \jmath) = \left(\frac{P_t(\imath, \jmath)}{P_{it}}\right)^{-\varepsilon_{it}} \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Y_t \tag{2.3.3}$$

Demand depends on the firm's own price $P_t(\imath, \jmath)$, the industry price level P_{it}, the aggregate price level P_t, and the level of aggregate demand Y_t. The price indices P_{it} and P_t are defined in (2.2.2) and (2.2.4). An individual firm takes all industry and aggregate variables as given, so its price elasticity of demand is ε_{it}.

For a given choice of price $P_t(\imath, \jmath)$, the resulting level of real profits can be obtained from demand function (2.3.3) and cost function (2.3.2). It is assumed also that all firms receive a proportional wage-bill subsidy of $0 \leq \mathfrak{s} < 1$ from the government. Suppose a firm in industry i using labour input $\imath$ at time t charges a price $p_{it}(\imath)$ in money terms, with implied relative price $\varrho_{it}(\imath) \equiv p_{it}(\imath)/P_t$ compared to the economy-wide price index. Industry i itself has an overall relative price of $\varrho_{it} \equiv P_{it}/P_t$ at time t compared to all other industries in the economy. The real wage of the specialized labour input $\imath$ is currently $w_t(\imath)$. The real profits of this firm are denoted by the function $\digamma(\varrho_{it}(\imath); \varrho_{it}, Y_t, A_{it}, w_t(\imath), \varepsilon_{it})$, where Y_t is aggregate output, A_{it} is the current level of technology in industry i, and ε_{it} is the elasticity of demand in industry i:

$$\digamma(\varrho_{it}(\imath); \varrho_{it}, Y_t, A_{it}, w_t(\imath), \varepsilon_{it}) \equiv \varrho_{it}(\imath)^{1-\varepsilon_{it}} \varrho_{it}^{\varepsilon_{it}-\varepsilon} Y_t - (1-\mathfrak{s})\mathcal{C}\left(\varrho_{it}(\imath)^{-\varepsilon_{it}} \varrho_{it}^{\varepsilon_{it}-\varepsilon} Y_t; A_{it}, w_t(\imath)\right) \tag{2.3.4}$$

2.4 Price setting

Not all firms choose a new money price at every point in time. For each firm, there is a probability of changing price which is a function of the industry it is based in and the time that has elapsed since it last changed its money price. This represents a model of time-dependent pricing augmented to allow for heterogeneity in price stickiness between industries.

Let $\mathcal{A}_t$ denote the set of firms that change price at time t, and $\mathcal{D}_t(\imath, \jmath)$ the duration of price stickiness for firm $(\imath, \jmath)$ at time t. The probability of a firm in industry i choosing a new price now given that its price was last changed j periods ago is denoted by α_{ij}:

$$\alpha_{ij} \equiv \mathbb{P}\left(\mathcal{A}_t \,\middle|\, \Omega_i \,,\, \mathcal{D}_{t-1} = j-1\right) \tag{2.4.1}$$

The probabilities of price adjustment are collected into sequences $\{\alpha_{ij}\}_{j=1}^{\infty}$, referred to as hazard functions for price changes. Apart from being well defined probabilities, there are two weak restrictions imposed on the probabilities $\{\alpha_{ij}\}_{j=1}^{\infty}$ in (2.4.1). The first is that some prices in each industry are sticky for at least one period; the second is that there exists a lower bound $\underline{\alpha}_i > 0$ on the probabilities of price adjustment:

$$\alpha_{i1} < 1 \quad , \quad \alpha_{ij} \geq \underline{\alpha}_i \tag{2.4.2}$$

The restrictions in (2.4.2) are needed to rule out the case where all prices are fully flexible and thus monetary policy has no real effects, and also the case where all prices are permanently sticky and no meaningful equilibrium exists. These restrictions do not actually have much substantive economic content because the results of this paper will apply to economies arbitrarily close to full price flexibility, or arbitrarily close to the case where prices can be completely sticky for a period of time. The assumptions are therefore broad enough to accommodate almost any pricing hazard function, including the well-known special cases of Calvo (1983) price setting ($\alpha_{ij} = \alpha_i$) and Taylor (1980) contracts ($\alpha_{ij} = \underline{\alpha}_i$ for j less the length of the contract in industry i and $\alpha_{ij} = 1$ otherwise, then letting $\underline{\alpha}_i$ tend to zero). No restrictions are placed on how much the hazard function in one industry can differ from that in another.

By accepting the trivial restrictions in (2.4.2), there must exist unique stationary distributions of the duration of price stickiness for each industry.[1] Let θ_{ij} denote the proportion of firms in industry i using a price set j periods ago:

$$\theta_{ij} \equiv \mathbb{P}\left(\mathcal{D}_t = j \,\middle|\, \Omega_i\right) \tag{2.4.3}$$

Each sequence $\{\theta_{ij}\}_{j=0}^{\infty}$ represents a probability distribution so $0 \leq \theta_{ij} \leq 1$ and $\sum_{j=0}^{\infty} \theta_{ij} = 1$ for all i. From (2.4.1) and (2.4.3), the price durations in an industry are linked to that industry's pricing hazard function by:

$$\theta_{i,j+1} = (1 - \alpha_{i,j+1})\theta_{ij} \tag{2.4.4}$$

This is demonstrated by noting that the proportion of firms using a price set $j+1$ periods ago can be obtained by multiplying the proportion of firms using a price set j periods ago by the probability that these firms will not choose a new price. Using (2.4.4) and the assumptions in (2.4.2), the sequence $\{\theta_{ij}\}_{j=0}^{\infty}$ must also satisfy:

$$\theta_{i,j+1} \leq \theta_{ij} \quad , \quad \theta_{i1} > 0 \quad , \quad \theta_{ij} \leq (1 - \underline{\alpha}_i)^j \theta_{i0} \tag{2.4.5}$$

These mean, respectively for each industry, that there are always more firms using a newer price than an older price; that there are at least some firms using a price set in the past; and that the distribution of price durations must eventually decay no slower than at a geometric rate.

The hazard functions $\{\alpha_{ij}\}_{j=1}^{\infty}$ are not parameterized directly. Instead, the recursive parameterization developed in Sheedy (2007*b*) is exploited here to simplify the subsequent analysis. This involves generating industry i's hazard function $\{\alpha_{ij}\}_{j=1}^{\infty}$ using the recursion:

$$\alpha_{ij} = \alpha_i + \sum_{k=1}^{\min\{j-1, m_i\}} \varphi_{ik} \left(\prod_{\ell=j-k}^{j-1} (1 - \alpha_{i\ell}) \right)^{-1} \tag{2.4.6}$$

It is shown by Sheedy (2007*b*) that the parameter α_i controls the level of the hazard function, and the sequence of parameters $\{\varphi_{ij}\}_{j=1}^{m_i}$ controls its slope. The number m_i is the order of the recursion used for industry i. Proposition 2 in Sheedy (2007*b*) shows that by making m_i sufficiently large, any hazard function satisfying the assumptions in (2.4.2) can be represented in this way.

It is convenient to collect all $m_i + 1$ hazard-function parameters together by defining $\phi_{i0} \equiv 1 - \alpha_i$ and $\phi_{i,j+1} \equiv -\varphi_{ij}$ for $j = 1, \ldots, m_i$. It is proved in Proposition 4 of Sheedy (2007*b*) that (2.4.6) implies the age distribution of prices $\{\theta_{ij}\}_{j=0}^{\infty}$ for industry i must be generated by a linear recursion of order $m_i + 1$ using parameters $\{\phi_{ij}\}_{j=1}^{m_i+1}$:

$$\theta_{ij} = \sum_{k=1}^{\min\{j, m_i+1\}} \phi_{ik}\theta_{i,j-k} \quad , \quad \theta_{i0} = 1 - \sum_{k=1}^{m_i+1} \phi_{ik} \tag{2.4.7}$$

In what follows, let $m \equiv \max\{m_1, \ldots, m_n\}$ denote greatest order of recursion needed to approximate the

[1] This statement is proved in Proposition 4 of Sheedy (2007*b*).

pricing hazard function for any industry in the economy.

A final assumption concerns the coordination (or independence) of price changes. It is assumed that price adjustment times cannot be coordinated at the level of the industry or the national economy. But for simplicity, it is supposed that all firms using the same specialized labour input (a small subset of one industry) perfectly coordinate the timing of their price changes.

2.5 Market clearing

Equilibrium in labour and goods markets requires that

$$C_t(\imath,\jmath) = Y_t(\imath,\jmath) \quad , \quad C_{it} = Y_{it} \quad , \quad C_t = Y_t \quad , \quad \int_\Omega H_t(\imath,\jmath)d\jmath = \int_\Omega h_t(\imath,\jmath)d\jmath \tag{2.5.1}$$

for all goods $(\imath,\jmath)$, for all industries i, and for all types of labour input $\imath$. The terms Y_{it} and Y_t denote the baskets of industry i output and economy-wide output respectively, defined in the same way as the consumption baskets C_{it} and C_t in (2.2.2) and (2.2.4):

$$Y_{it} \equiv \left(\frac{1}{\omega_i}\int_{\Omega_i}\int_\Omega Y_t(\imath,\jmath)^{\frac{\varepsilon_{it}-1}{\varepsilon_{it}}} d\jmath d\imath\right)^{\frac{\varepsilon_{it}}{\varepsilon_{it}-1}} \quad , \quad Y_t \equiv \left(\sum_{i=1}^n \omega_i Y_{it}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{2.5.2}$$

2.6 Policymaker

Optimal policy from the perspective of a benevolent policymaker is defined with respect to a utilitarian social welfare function. This objective function is denoted by $\mathcal{U}_t$ and is obtained by integrating lifetime utility (2.1.1) over the set of all households:

$$\mathcal{U}_t \equiv \int_\Omega\int_\Omega \mathcal{U}_t(\imath,\jmath)d\jmath d\imath \tag{2.6.1}$$

This paper abstracts from the active use of fiscal policy and concentrates on how monetary policy can be used to maximize objective (2.6.1). Fiscal policy simply comprises a sequence of lump-sum taxes and transfers, and a proportional wage-bill subsidy paid to all firms. The wage-bill subsidy rate $\mathfrak{s}$ is set to ensure the economy operates at an efficient level of aggregate economic activity on average. The government budget does not always have to be in balance, and any deficits are financed by issuing one-period risk-free nominal bonds. But fiscal policy is Ricardian and lump-sum taxes are assumed to be adjusted to ensure government solvency in all circumstances.

Monetary policy is implemented by setting the gross interest rate $\mathcal{I}_t$ on one-period risk-free nominal bonds. For there to be no arbitrage opportunity in the presence of a range of other securities, the nominal interest rate $\mathcal{I}_t$ must be related to the asset-pricing kernel $\mathfrak{M}_{t+1|t}$ and economy-wide price index P_t as follows:

$$\mathbb{E}_t\left[\mathfrak{M}_{t+1|t}\frac{P_t}{P_{t+1}}\right] = \mathcal{I}_t^{-1} \tag{2.6.2}$$

By substituting the first-order condition (2.1.3) for the optimal allocation of consumption spending and the goods market clearing condition (2.5.1) into (2.6.2), and then taking expectations, the intertemporal Euler equation is obtained:

$$\beta\mathcal{I}_t\mathbb{E}_t\left[\frac{u_c(Y_{t+1})}{u_c(Y_t)}\frac{P_t}{P_{t+1}}\right] = 1 \tag{2.6.3}$$

This establishes a channel through which monetary policy can affect aggregate demand. All that remains to be specified is whether the policymaker maximizing (2.6.1) acts with discretion or is able to make binding commitments. Both cases are considered in the following sections of the paper.

3 Optimal behaviour of firms and the policymaker

The aim of this section is to identify the variables in the economy which are relevant to firms' and the policymaker's pursuit of their objectives, and to derive metrics that allow their success in meeting those goals to be assessed. This analysis is important because it will illustrate the extent to which these objectives are in conflict.

The optimal behaviour of both firms and the policymaker is studied for the economy described in section 2 by deriving equations characterizing the behaviour of the agents. As an exact solution to these equations cannot be found, recourse is had to the technique of log linearization. A steady state is first identified around which a log-linear approximation can be made. To simplify the analysis, a steady state is chosen in which prices are fully adjusted, inflation is zero, and where there is symmetry across all industries. Further details of this steady state are provided in appendix A. In what follows, a bar over a variable denotes its steady-state value, and sans serif letters denote the log deviation (approximate percentage difference) of the corresponding roman letter from its steady-state value. When a variable is indeterminate in the steady state (for instance, any money price), the sans serif letter denotes just the logarithm of that variable. So for example, $\bar{Y}$ is steady-state aggregate output, and $\mathsf{Y}_t \equiv \log Y_t - \log \bar{Y}$ is the log-deviation of aggregate output Y_t from its steady-state level; whereas $\mathsf{P}_t \equiv \log P_t$ is just the log of the aggregate price level P_t. Second- and higher-order terms in the log-deviations are suppressed unless otherwise specified in all following equations.

3.1 Policymaker

The first step is to establish a benchmark by which to judge the success or failure of any monetary policy. This is done by characterizing a Pareto-efficient allocation of resources in the economy, hypothetically supposing the policymaker could set aside the price mechanism and act as a social planner. Because there are potentially many Pareto-efficient allocations, uniqueness is obtained restricting attention to allocations in which all households have the same level of consumption, mirroring the equal consumption of the market allocation found in Section 2.1 when initial financial wealth and expected lifetime income is equal and asset markets are complete.

This Pareto-efficient allocation is found by maximizing total utility (2.6.1) (involving the industry-specific and overall consumption baskets C_{it} and C_t, defined respectively in (2.2.2) and (2.2.4)) subject to the production function (2.3.1) and all households having the same level of consumption. As there are no technological differences between goods in the same industry and because households' preferences for leisure do not depend on the type of labour input they supply, all firms in the same industry produce the same level of output in the Pareto-efficient allocation. The values of variables in this efficient allocation are marked by asterisks, with $Y_t^* = C_t^*$ being the efficient level of aggregate output and consumption, Y_{it}^* being the efficient level of output of firms in industry i, and H_{it}^* the corresponding efficient level of labour usage by firms in that industry. Shadow relative prices and real wages for each industry are denoted by ϱ_{it}^* and w_{it}^* respectively. These are the prices that would support the efficient allocation of resources under the market mechanism.

Using the equations for the production function (2.3.1), the marginal cost function (2.3.2), the industry demand curve (2.2.5), the optimal labour supply condition (2.1.2), the output basket (2.5.2), and goods and labour market equilibrium conditions (2.5.1), the following system of equations characterizes the Pareto-efficient allocation of resources:

$$Y_{it}^* = A_{it} H_{it}^{*\eta_{yh}} \tag{3.1.1a}$$

$$Y_{it}^* = \varrho_{it}^{*-\varepsilon} Y_t^* \tag{3.1.1b}$$

$$\frac{v_h(H_{it}^*)}{u_c(Y_t^*)} = w_{it}^* \tag{3.1.1c}$$

$$Y_t^* = \left(\sum_{i=1}^{n} \omega_i Y_{it}^{*\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{3.1.1d}$$

$$\varrho_{it}^* = \mathcal{C}_Y \left(Y_{it}^*; A_{it}, w_{it}^* \right) \tag{3.1.1e}$$

A key equation that ensures efficiency is (3.1.1e), equating prices to marginal costs. This is violated under conditions of imperfect competition. As has already been mentioned, the system of equations in (3.1.1) is difficult to solve, so the next result gives a log-linear approximation to the solution.

Lemma 1 *A log-linear approximation to the solution of system of equations (3.1.1) for the Pareto-efficient level of aggregate output Y_t^* and shadow relative prices ϱ_{it}^* is given by*

$$\mathsf{Y}_t^* = \eta_y \mathsf{A}_t \quad , \quad \rho_{it}^* = -\eta_\rho (\mathsf{A}_{it} - \mathsf{A}_t) \tag{3.1.2}$$

where Y_t^* is the log-deviation of efficient output Y_t^*, $\rho_{it}^* \equiv \log \varrho_{it}^*$ is the log of the industry i shadow relative price, and $\mathsf{A}_t \equiv \sum_{i=1}^{n} \omega_i \mathsf{A}_{it}$ is the weighted average of the log-deviation of technology across all industries. Both parameters η_y and η_ρ are strictly positive.

Proof. See appendix B.1. ∎

The efficient level of aggregate output simply depends on the average level of technology in the economy at any given point in time. And if one industry experiences a productivity advantage over another owing to better technology, then this is reflected in that industry having a lower Pareto-efficient shadow relative price.

The solution outlined above is of course not attainable in practice. But monetary policy can still be appraised by looking at how far the actual level of output and actual relative prices are from their efficient levels derived in Lemma 1. To this end, the economy-wide output gap $\mathcal{Y}_t \equiv Y_t / Y_t^*$ is defined. Similarly, if $\varrho_{it} \equiv P_{it}/P_t$ is the actual industry i relative price, the relative price gap $\varrho_{it}/\varrho_{it}^*$ is important in seeing the extent to which the efficient allocation of resources between industries is perturbed. Misallocation of resources can also take place within industries, and the extent of this problem can be seen from the size of the cross-sectional variance of (log) prices within an industry, denoted by $\sigma_{it}^2 \equiv \mathbb{V}_{\Omega_i} [\log P_t(\imath, \jmath)]$ for industry i at time t. The following result formalizes these claims by deriving a second-order approximation of the policymaker's objective function.

Lemma 2 *A second-order approximation of the negative of the social welfare function $-\mathcal{U}_t$, defined in (2.6.1), is given by the following quadratic loss function $\mathfrak{U}_t$,*

$$\mathfrak{U}_t = \frac{1}{2} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\frac{\eta_x}{\varepsilon} \mathsf{y}_\tau^2 + \sum_{i=1}^{n} \omega_i \left((\rho_{i\tau} - \rho_{i\tau}^*)^2 + \sigma_{i\tau}^2 \right) \right] \tag{3.1.3}$$

where third- and higher-order terms have been suppressed, along with terms that are independent of monetary policy. The loss function depends on the (log) output gap $\mathsf{y}_t \equiv \mathsf{Y}_t - \mathsf{Y}_t^*$, the (log) relative price gap $\rho_{it} - \rho_{it}^*$, where $\rho_{it} \equiv \log \varrho_{it}$, and the cross-sectional variance σ_{it}^2 of (log) prices within industries. Both Y_t^* and ρ_{it}^* are defined in Lemma 1 and the parameter η_x is strictly positive.

Proof. See appendix B.2. ∎

Therefore, it is the job of monetary policy to close the output and relative price gaps and minimize the cross-sectional dispersion of prices within industries as much as possible.

The policymaker affects these variables by varying nominal interest rates, which affect the level of aggregate demand. This works through the consumption Euler equation (2.6.3), which can be log linearized as follows,

$$\mathsf{y}_t = \mathbb{E}_t \mathsf{y}_{t+1} - \eta_{ic}^{-1}\{\mathsf{i}_t - \mathbb{E}_t \pi_{t+1} - \mathfrak{R}_t^*\} \tag{3.1.4}$$

where $\pi_{t+1} \equiv \mathsf{P}_{t+1} - \mathsf{P}_t$ denotes inflation in the aggregate price level between period t and $t+1$, $\mathsf{i}_t \equiv \log \mathcal{I}_t - \log \bar{\mathcal{I}}$ is the log-deviation of the (gross) nominal interest rate from its steady-state value, and $\mathfrak{R}_t^* \equiv \mathbb{E}_t \mathsf{Y}_{t+1}^* - \mathsf{Y}_t^*$ is the natural real interest rate, an exogenous variable depending on the expected change in the efficient level of aggregate output Y_t^*. The details of this log linearization are contained in appendix A. The strictly positive parameter η_{ic}^{-1} is the elasticity of intertemporal substitution. When prices are not completely flexible, equation (3.1.4) indicates how nominal interest rates can affect the output gap y_t. Changes in the output gap then have a knock-on effect on the other variables with which the policymaker is concerned.

3.2 Firms

This section now considers how firms should set prices in order to maximize profits, with the aim of discovering how this affects the relationship between the output gap, relative prices and inflation. It is also important to know whether the actions of firms are likely to assist or to hinder the policymaker in pursuing the objectives set out in section 3.1.

3.2.1 Profit-maximizing price setting with flexible prices

It is useful to begin with the hypothetical case of perfectly flexible prices. Suppose that some firms using a particular labour input $\imath$ and based in a particular industry i have the opportunity to change their prices without constraint now and in the future. If these firms maximize profits then they must clearly choose a relative price $\hat{\varrho}_{it}(\imath)$ for which the first derivative of the profit function (2.3.4) is zero:

$$\digamma_{\varrho}(\hat{\varrho}_{it}(\imath); \varrho_{it}, Y_t, A_{it}, w_t(\imath), \varepsilon_{it}) = 0 \tag{3.2.1}$$

An expression for this derivative can be obtained from equation (2.3.4),

$$\digamma_{\varrho}(\varrho; \varrho_{it}, Y_t, A_{it}, w_t(\imath), \varepsilon_{it}) = (1-\varepsilon_{it})\varrho^{-\varepsilon_{it}-1}\varrho_{it}^{\varepsilon_{it}-\varepsilon} Y_t \left\{ \varrho - \frac{(1-\mathfrak{s})\varepsilon_{it}}{(\varepsilon_{it}-1)} \mathcal{C}_Y \left(\varrho^{-\varepsilon_{it}} \varrho_{it}^{\varepsilon_{it}-\varepsilon} Y_t; A_{it}, w_t(\imath) \right) \right\} \tag{3.2.2}$$

where $1/(\varepsilon_{it}-1)$ denotes these firms' desired (net) markup of price on marginal cost. The elasticity of demand ε_{it} is greater than unity so the markup is always well defined.

If firms in industry i using labour input type $\imath$ set profit-maximizing relative price $\hat{\varrho}_{it}(\imath)$ then these firms will face a common level of demand $\hat{Y}_{it}(\imath)$ and need to employ labour hours $\hat{H}_{it}(\imath)$. This level of employment then affects the real wage $\hat{w}_t(\imath)$ faced by the firms with flexible prices using labour input $\imath$. The equations necessary to find the values of all these variables are the production function (2.3.1), the demand function (2.3.3), and the labour supply function (2.1.2). It turns out that any group of firms in industry i with this pricing freedom would like to set the same relative price $\hat{\varrho}_{it}$ irrespective of the particular type of labour input it uses. This is because the production and cost functions are the same for all labour input types used by a particular industry. The index $\imath$ is thus dropped from the relative price $\hat{\varrho}_{it}$, output $\hat{Y}_{it}$ and employment

$\hat{H}_{it}$ of all firms using labour input $\imath$ and having flexible prices. The system of equations determining these variables is:

$$\hat{Y}_{it} = A_{it}\hat{H}_{it}^{\eta_{yh}} \tag{3.2.3a}$$

$$\hat{Y}_{it} = \hat{\varrho}_{it}^{-\varepsilon_{it}} \varrho_{it}^{\varepsilon_{it}-\varepsilon} Y_t \tag{3.2.3b}$$

$$\frac{v_h(\hat{H}_{it})}{u_c(Y_t)} = \hat{w}_{it} \tag{3.2.3c}$$

$$\hat{\varrho}_{it} = \left(\frac{\varepsilon_{it}}{\varepsilon_{it}-1}\right)(1-\mathfrak{s})\mathcal{C}_Y(\hat{Y}_{it}; A_{it}, \hat{w}_{it}) \tag{3.2.3d}$$

Just like the equations in (3.1.1), the system (3.2.3) is difficult to solve analytically. The next result presents a log-linearization of the solution.

Lemma 3 *A log-linear approximation to the profit-maximizing flexible relative price $\hat{\varrho}_{it}$ defined by the system of equations* (3.2.3) *is given by*

$$\hat{\rho}_{it} = \eta_x \mathsf{y}_t + \rho^*_{it} + \eta_\epsilon \epsilon_{it} \tag{3.2.4}$$

where $\hat{\rho}_{it} \equiv \log \hat{\varrho}_{it}$, y_t is the output gap, ρ^*_{it} is the efficient relative price defined in Lemma 1, and $\epsilon_{it} \equiv \log(\varepsilon_{it}/(\varepsilon_{it}-1)) - \log(\varepsilon/(\varepsilon-1))$ is the log-deviation of the optimal (gross) markup. The coefficients η_x and η_ϵ are both strictly positive.

Proof. See appendix B.3. ■

The profit-maximizing flexible relative price depends positively on the output gap y_t because higher output leads to higher costs as a result of diminishing returns to labour and higher real wages. It moves one-for-one with the efficient relative price ρ^*_{it} because this reflects changes in technology that affect costs at the industry level. Finally, variations in industry-specific competitiveness as measured by changes in desired markups are captured by the shocks ϵ_{it}. A positive shock means that firms are pushing for higher relative prices.

3.2.2 Profit-maximizing, forward-looking price setting with sticky prices

Attention now returns to the original problem of the finding the profit-maximizing price when prices are sticky. This task turns out to be closely related to the profit-maximization problem with flexible prices treated in section 3.2.1. But when prices are expected to be sticky it is no longer sufficient to consider the effects of a new price on profits in the current period; future periods' profits need to be taken into account for as long as a price could potentially remain in use.

Suppose a firm gets an opportunity to choose a new money price at a particular point in time. The optimal price the firm chooses is referred to as a reset price. It shown below that there is one common profit-maximizing reset price for each industry at each point in time, denoted by R_{it} for industry i at time t. As the reset price is likely to be sticky in the future, the choice of R_{it} potentially affects firms' profits at all times from period t onwards. But the effect on future profits needs to be discounted for two reasons. First, because the profits occur in the future the income stream is discounted by financial markets when calculating a present value. Second, while reset price R_{it} is definitely used in period t, in each period afterwards there is a probability that a new reset price will be chosen before these profits are actually realized. Let ς_{ij} denote the survival function for prices set by firms in industry i, where ς_{ij} is the probability that a price set by one of the firms in this industry at time t is still in use at time $t+j$. From repeated application of equation (2.4.4) it follows that the survival probabilities are proportional to the distribution of the duration of price stickiness:

$$\varsigma_{ij} = \prod_{k=1}^{j}(1-\alpha_{ik}) = \frac{\theta_{ij}}{\theta_{i0}} \tag{3.2.5}$$

Now consider a firm in industry i using type-$\imath$ labour. If at time τ this firm is still using a reset price from period t then its current level of real profits is given by $\digamma(R_{it}/P_\tau; \varrho_{i\tau}, Y_\tau, A_{i\tau}, w_\tau(\imath), \varepsilon_{i\tau})$, where the profit function is defined in (2.3.4). As will be demonstrated below, the equilibrium real wage $w_t(\imath)$ depends only on the industry the firm is in and the time period in which the currently used reset price was chosen. Its equilibrium value at time τ is thus denoted by $w_{i,\tau|t}$. The objective function $\mathcal{F}_{it}$ that firms maximize when choosing reset price R_{it} is obtained by multiplying the level of profits by the asset-pricing kernel $\mathfrak{M}_{\tau|t}$ and the survival probability $\theta_{i,\tau-t}/\theta_{i0}$ from (3.2.5), and then summing over all time periods from t onwards:

$$\mathcal{F}_{it} \equiv \max_{R_{it}} \sum_{\tau=t}^{\infty}(\theta_{i,\tau-t}/\theta_{i,0})\mathbb{E}_t[\mathfrak{M}_{\tau|t}\digamma(R_{it}/P_\tau; \varrho_{i\tau}, Y_\tau, A_{i\tau}, w_{i,\tau|t}, \varepsilon_{i\tau})] \tag{3.2.6}$$

The first-order condition for maximizing (3.2.6) is obtained by differentiating (3.2.6) with respect to the reset price R_{it},

$$\sum_{\tau=t}^{\infty}(\theta_{i,\tau-t}/\theta_{i,0})\mathbb{E}_t[\mathfrak{M}_{\tau|t}P_\tau^{-1}\digamma_\varrho(R_{it}/P_\tau; \varrho_{i\tau}, Y_\tau, A_{i\tau}, w_{i,\tau|t}, \varepsilon_{i\tau})] = 0 \tag{3.2.7}$$

where the first derivative of profit function (2.3.4) is given in (3.2.2). The equilibrium real wage $w_{i,\tau|t}$ is obtained as in section 3.2.1 by combining the production function (2.3.1), the demand curve (2.3.3), and the labour supply function (2.1.2). Let $Y_{i,\tau|t}$ and $H_{i,\tau|t}$ be the common levels of output and employment for firms in industry i at time τ using the reset price R_{it} set in period t. By using these equations, and by substituting in the expression for the derivative of the profit function from (3.2.2) into (3.2.7), the following system of equations is obtained:

$$Y_{i,\tau|t} = A_{i\tau}H_{i,\tau|t}^{\eta_{yh}} \tag{3.2.8a}$$

$$Y_{i,\tau|t} = \left(\frac{R_{it}}{P_{i\tau}}\right)^{-\varepsilon_{i\tau}}\left(\frac{P_{i\tau}}{P_\tau}\right)^{-\varepsilon} Y_\tau \tag{3.2.8b}$$

$$\frac{\upsilon_h(H_{i,\tau|t})}{u_c(Y_\tau)} = w_{i,\tau|t} \tag{3.2.8c}$$

$$\sum_{\tau=t}^{\infty}(\theta_{i,\tau-t}/\theta_{i,0})\mathbb{E}_t\left[\mathfrak{M}_{\tau|t}\left(\frac{R_{it}}{P_\tau}\right)^{-\varepsilon_{i\tau}}\varrho_{i\tau}^{\varepsilon_{i\tau}-\varepsilon}Y_\tau\left\{\frac{R_{it}}{P_\tau} - \left(\frac{\varepsilon_{i\tau}}{\varepsilon_{i\tau}-1}\right)(1-\mathfrak{s})\mathcal{C}_Y(Y_{i,\tau|t}; A_{i\tau}, w_{i,\tau|t})\right\}\right] = 0 \tag{3.2.8d}$$

The close connection between the solution of this system and the corresponding system of equations (3.2.3) for firms with flexible prices can be seen by log-linearizing the solution to (3.2.8).

Lemma 4 *A log-linear approximation of the solution to the system of equations (3.2.8) defining the profit-maximizing reset price is given by*

$$\mathsf{R}_{it} = \sum_{\tau=t}^{\infty}\vartheta_{i,\tau-t}\mathbb{E}_t[\mathsf{P}_\tau + \hat{\rho}_{i\tau}] \quad , \quad \vartheta_{ij} \equiv \frac{\beta^j\theta_{ij}}{\sum_{k=0}^{\infty}\beta^k\theta_{ik}} \tag{3.2.9}$$

where $\mathsf{R}_{it} \equiv \log R_{it}$ *is the log reset price,* $\mathsf{P}_t \equiv \log P_t$ *is the log price level, and* $\hat{\rho}_{it}$ *is the profit-maximizing flexible relative price given in Lemma 3. The sequence of weights* $\{\vartheta_{ij}\}_{j=0}^{\infty}$ *sums to one, is non-increasing, and each weight is non-negative.*

Proof. See appendix B.4. ■

Equation (3.2.9) states that the profit-maximizing reset price R_{it} is a weighted average of the sequence of current and expected future profit-maximizing flexible prices, that is, $\mathsf{P}_\tau + \hat{\rho}_{i\tau}$, using weights based on the discount factor β and survival probabilities $\varsigma_{ij} = \theta_{ij}/\theta_{i0}$.

Once the choice of reset price for each firm in each industry has been determined, the industry price levels and aggregate price level themselves can be calculated. As all firms in the same industry changing price at the same time find it optimal to choose the same reset price, it follows from equation (2.2.2) and the age distribution of prices in (2.4.3) that the price level for industry i can be written as a function of past and present reset prices:

$$P_{it} = \left(\sum_{j=0}^{\infty} \theta_{ij} R_{i,t-j}^{1-\varepsilon_{it}}\right)^{\frac{1}{1-\varepsilon_{it}}} \tag{3.2.10}$$

This equation, and the aggregate price level equation (2.2.4), can be log-linearized as follows:

$$\mathsf{P}_t = \sum_{i=1}^{n} \omega_i \mathsf{P}_{it} \quad , \quad \mathsf{P}_{it} = \sum_{j=0}^{\infty} \theta_{ij} \mathsf{R}_{i,t-j} \tag{3.2.11}$$

The aggregate price level P_t is thus a weighted average of industry price levels P_{it}, which in turn are weighted averages of current and past reset prices R_{it}. The weights are respectively the sizes of each industry, and the age distribution of prices within industries.

Just as Lemma 2 derived a loss function for the policymaker, it is possible to obtain an analogous loss function for firms (that is, a second order approximation of the negative of the profit function). Later on, this will be used to assess the extent to which the objectives of firms and the policymaker conflict.

Lemma 5 *The objective function $\mathcal{F}_{it}$ of firms in industry i changing price at time t is defined in equation (3.2.6). The following loss function $\mathfrak{F}_{it}$ is a second-order approximation of $-\mathcal{F}_{it}$,*

$$\mathfrak{F}_{it} \equiv \frac{1}{2} \sum_{\tau=t}^{\infty} \vartheta_{i,\tau-t} \mathbb{E}_t[\{\mathsf{R}_{it} - \mathsf{P}_\tau - \hat{\rho}_{i\tau}\}^2] \tag{3.2.12}$$

where third- and higher-order terms have been suppressed, along with terms that are independent of these firms' pricing decisions. The profit-maximizing flexible relative price $\hat{\rho}_{it}$ is defined in Lemma 3, and the sequence of weights $\{\vartheta_{ij}\}_{j=0}^{\infty}$ in equation (3.2.9).

Proof. See appendix B.5. ■

Firms' losses stem from their reset prices differing from the prices they would choose in the absence of constraints on price adjustment. When prices are sticky it is impossible to find a reset price that can match all of the optimal flexible prices in every time period simultaneously. But it is clear that the profit-maximizing reset price given in Lemma 4 minimizes loss function (3.2.12) to the greatest possible extent.

4 The Phillips curve and intrinsic inflation persistence

Section 3 has identified how firms would choose prices to maximize profits in a world in which those prices are likely to remain sticky. The next step is to aggregate this optimizing behaviour into a Phillips curve relating the (gross) inflation rate $\mathit{\Pi}_t \equiv P_t/P_{t-1}$ and relative prices $\varrho_{it} \equiv P_{it}/P_t$ to real economic activity, as measured by the output gap $\mathcal{Y}_t \equiv Y_t/Y_t^*$, and the exogenous shocks. A crucial feature of the Phillips curve is the extent to which it exhibits intrinsic inflation persistence.

4.1 Phillips curve

As there is a potentially large number of industries, the aggregated system of pricing equations is most conveniently represented using vector and matrix notation. In what follows, boldface letters are used to denote the $n \times 1$ vector of the corresponding industry-specific variables. Hence the vector of industry-specific price levels is written as $\mathsf{\mathbf{P}}_t \equiv (\mathsf{P}_{1t} , \ldots , \mathsf{P}_{nt})'$. Similarly, $\boldsymbol{\omega} \equiv (\omega_1 , \ldots , \omega_n)'$ is the vector of industry sizes. Using this notation, the aggregate price level P_t in (3.2.11) can be expressed as $\mathsf{P}_t = \boldsymbol{\omega}'\mathsf{\mathbf{P}}_t$.

The (gross) rate of price inflation specific to industry i is defined as $\mathit{\Pi}_{it} \equiv P_{it}/P_{i,t-1}$, and the log-deviation of this inflation rate from its steady-state value is denoted by π_{it}. The vector of industry-specific inflation rates is $\boldsymbol{\pi}_t$, and consequently economy-wide inflation is given by $\pi_t \equiv \boldsymbol{\omega}'\boldsymbol{\pi}_t$. The vectors $\boldsymbol{\rho}^*_t$ and $\hat{\boldsymbol{\rho}}_t$ refer to (log deviations of) the relative prices that would be desired respectively by the policymaker and profit-maximizing firms in a world where prices are completely flexible. These are the relative prices characterized in Lemmas 1 and 3. Equation (3.2.4) shows that $\hat{\boldsymbol{\rho}}_t$ and $\boldsymbol{\rho}^*_t$ are related according to

$$\hat{\boldsymbol{\rho}}_t = \eta_x \boldsymbol{\iota} \mathsf{y}_t + \boldsymbol{\rho}^*_t + \eta_\epsilon \boldsymbol{\epsilon}_t \tag{4.1.1}$$

where y_t is the (log) aggregate output gap, $\boldsymbol{\iota}$ is an $n \times 1$ vector of 1s, and $\boldsymbol{\epsilon}_t$ is a vector of industry-specific competitiveness shocks.

Lemma 4 describes how a profit-maximizing firm would behave once it decides to change its price given the likelihood of this price being sticky in the future, as indicated by the hazard function in (2.4.1). This behaviour is aggregated to obtain a system of equations describing the determination of the inflation rate in each industry.

Proposition 1 *Using the log-linearizations of the equations describing the profit-maximizing behaviour of firms in Lemmas 3 and 4, the log-linearizations of the price indices in (3.2.11), and the recursive parameterization in (2.4.7), the following set of pricing equations is obtained. There exist a sequence of $n \times 1$ vector coefficients $\{\boldsymbol{\gamma}_j\}_{j=1}^m$, sequences of $n \times n$ matrices $\{\boldsymbol{\Lambda}_j\}_{j=1}^{m+1}$ and $\{\boldsymbol{\Xi}_j\}_{j=0}^{\infty}$, and an $n \times n$ invertible matrix $\boldsymbol{\aleph}$ such that the vector of industry-specific inflation rates $\boldsymbol{\pi}_t$ is given by*

$$\boldsymbol{\pi}_t = \sum_{j=1}^{m} \boldsymbol{\gamma}_j \pi_{t-j} - \boldsymbol{\rho}_{t-1} + \sum_{j=1}^{m+1} \boldsymbol{\Lambda}_j \boldsymbol{\rho}_{t-j} + \boldsymbol{\aleph} \sum_{j=0}^{\infty} \boldsymbol{\Xi}_j \mathbb{E}_t[\eta_x \boldsymbol{\omega} \mathsf{y}_{t+j} + \boldsymbol{\Omega}\boldsymbol{\rho}^*_{t+j} + \eta_\epsilon \boldsymbol{\Omega}\boldsymbol{\epsilon}_{t+j}] \tag{4.1.2}$$

where $\pi_t \equiv \boldsymbol{\omega}'\boldsymbol{\pi}_t$ is the economy-wide inflation rate, $\boldsymbol{\rho}_t$ is the relative price vector, y_t is the economy-wide output gap, $\boldsymbol{\rho}^*_t$ is the vector of efficient relative prices, $\boldsymbol{\epsilon}_t$ is the vector of competitiveness shocks, and m is the order of the hazard-function recursion. The diagonal matrix $\boldsymbol{\Omega}$ contains the industry sizes ω_i along the principal diagonal. The matrix $\boldsymbol{\Xi}_0$ is equal to the $n \times n$ identity matrix $\mathbf{I}$, and $\boldsymbol{\iota}$ is an $n \times 1$ vector of 1s.

Proof. See appendix C.1. ■

Equation (4.1.2) is stated in terms of the vector of industry-specific inflation rates $\boldsymbol{\pi}_t$. The Phillips curve for the whole economy is obtained by averaging the equations in (4.1.2) over industries using the industry sizes as weights. The following Phillips curve is obtained

$$\pi_t = \sum_{j=1}^{m} \gamma_j \pi_{t-j} + \sum_{j=1}^{m+1} \boldsymbol{\lambda}'_j \boldsymbol{\rho}_{t-j} + \boldsymbol{\omega}'\boldsymbol{\aleph} \sum_{j=0}^{\infty} \boldsymbol{\Xi}_j \mathbb{E}_t[\eta_x \boldsymbol{\omega} \mathsf{y}_{t+j} + \boldsymbol{\Omega}\boldsymbol{\rho}^*_{t+j} + \eta_\epsilon \boldsymbol{\Omega}\boldsymbol{\epsilon}_{t+j}] \tag{4.1.3}$$

where $\gamma_j \equiv \boldsymbol{\omega}'\boldsymbol{\gamma}_j$ and $\boldsymbol{\lambda}'_j \equiv \boldsymbol{\omega}'\boldsymbol{\Lambda}_j$ have been defined, and the fact that the weighted average of the relative price vector $\boldsymbol{\rho}_t$ is always zero has been used.

Both equations (4.1.2) and (4.1.3) effect a decomposition of the determinants of inflation into forward- and backward-looking components. The content of Proposition 1 is in showing which variables enter into each component and with what coefficients. The backward-looking part depends on m lags of economy-wide inflation π_t and $m+1$ lags of the relative price vector $\boldsymbol{\rho}_t$. The number m is the order of the recursive representation (2.4.7) needed to capture all the details of the hazard functions (2.4.1). This could be any non-negative number: it is not possible to set an upper bound on it that applies to all models of price stickiness. The forward-looking component contains all the current and expected future values of the output gap y_t, efficient relative prices $\boldsymbol{\rho}^*_t$, and competitiveness shocks $\boldsymbol{\epsilon}_t$. These are the variables affecting the profit-maximizing flexible relative prices $\hat{\boldsymbol{\rho}}_t$ given in equation (4.1.1).

The equations in (4.1.2) constitute a set of constraints on a policymaker caring about both inflation and the output gap. But it is important to recognize a fundamental difference in respect of the forward- and backward-looking components of the constraints. As of time t, the backward-looking component cannot be affected by any current policy actions because it is predetermined. And while it is true that in a rational expectations equilibrium, the expectations in the forward-looking may depend on the past because the variables turn out to be serially correlated, there is no necessity for them to do so. They are thus amenable to being influenced by monetary policy, though the extent of this influence will depend on the ability of the policymaker to enter into binding commitments to undertake actions in the future. Therefore it is the backward-looking component that is used to formulate a precise definition of intrinsic inflation persistence, which together with the exogenous shocks forms the only part of (4.1.2) that must ultimately be taken as given by a policymaker with the ability to steer expectations.

4.2 Intrinsic inflation persistence

The history of inflation and relative prices that enters the backward-looking component of (4.1.2) is denoted by $\mathcal{H}_t$,

$$\mathcal{H}_t \equiv \{ \pi_{t-1} \, , \, \ldots \, , \, \pi_{t-m} \, , \, \boldsymbol{\rho}_{t-1} \, , \, \ldots \, , \, \boldsymbol{\rho}_{t-m-1} \, , \, \mathsf{P}_{t-1} \, \} \tag{4.2.1}$$

where the set $\mathcal{H}_t$ has been augmented by the past general price level P_{t-1}, which is needed if equation (4.1.2) is to be used to obtain the current price level as well as inflation rates and relative prices. The expression for the backward-looking component of (4.1.2) is taken as the definition of the level of intrinsic inflation prevailing at time t given history $\mathcal{H}_t$ in (4.2.1). The current level of intrinsic inflation in each industry is denoted by an $n \times 1$ vector $\boldsymbol{\text{ш}}_t$:

$$\boldsymbol{\text{ш}}_t \equiv \sum_{j=1}^{m} \boldsymbol{\gamma}_j \pi_{t-j} - \boldsymbol{\rho}_{t-1} + \sum_{j=1}^{m+1} \boldsymbol{\Lambda}_j \boldsymbol{\rho}_{t-j} \tag{4.2.2}$$

For economy-wide inflation π_t the relevant level of intrinsic inflation is obtained by taking a weighted average of $\boldsymbol{\text{ш}}_t$ across industries, which is denoted by $\text{ш}_t \equiv \boldsymbol{\omega}'\boldsymbol{\text{ш}}_t$.

The trade-off the policymaker faces is made apparent by separating the forward-looking component of (4.1.2) into a component that depends on the output gap and a component that depends on various exogenous shocks that affect inflation. The influence of aggregate demand on inflation $\boldsymbol{\pi}_t$ is denoted by $\mathbf{y}_t$, which is defined as follows:

$$\mathbf{y}_t \equiv \eta_x \boldsymbol{\aleph} \sum_{j=0}^{\infty} \boldsymbol{\Xi}_j \boldsymbol{\omega} \mathbb{E}_t \mathsf{y}_{t+j} \tag{4.2.3}$$

The remaining influences on inflation come from shocks to industry-specific technology and competitiveness. These are referred to collectively as "cost-push" shocks. The technology shocks are "efficient" cost-push shocks because they create a need for price adjustment if an efficient allocation of resources is to be attained. Lemma 1 has shown that efficient relative prices $\boldsymbol{\rho}^*_t$ are a function only of the exogenous technology shocks. On the other hand, the competitiveness shocks $\boldsymbol{\epsilon}_t$ are "inefficient" cost-push shocks because they relate to pressure for price adjustment that would perturb the allocation of resources further away from what is efficient. The two types of cost-push shock are denoted by $\boldsymbol{\wp}_t$ and $\boldsymbol{\epsilon}_t$ respectively, and expressions for these are taken from (4.1.2):

$$\boldsymbol{\wp}_t \equiv \boldsymbol{\aleph} \sum_{j=0}^{\infty} \boldsymbol{\Xi}_j \boldsymbol{\Omega} \mathbb{E}_t \boldsymbol{\rho}^*_{t+j} \quad , \quad \boldsymbol{\epsilon}_t \equiv \eta_\epsilon \boldsymbol{\aleph} \sum_{j=0}^{\infty} \boldsymbol{\Xi}_j \boldsymbol{\Omega} \mathbb{E}_t \boldsymbol{\epsilon}_{t+j} \tag{4.2.4}$$

By using definitions (4.2.2), (4.2.3) and (4.2.4), the pricing equations in (4.1.2) can be stated succinctly as follows:

$$\boldsymbol{\pi}_t - \mathbf{y}_t = \boldsymbol{\text{ш}}_t + (\boldsymbol{\wp}_t + \boldsymbol{\epsilon}_t) \tag{4.2.5}$$

When averaged over all industries this becomes $\pi_t - \text{y}_t = \text{ш}_t + (\wp_t + \epsilon_t)$, where ш_t, y_t, $\wp_t$ and ϵ_t denote the weighted averages of the vectors in (4.2.2), (4.2.3) and (4.2.4).

Equation (4.2.5) demonstrates that the current level of intrinsic inflation ш_t should properly be seen as a constraint on what monetary policy can currently achieve. The left-hand side of (4.2.5) comprises the variables that are the objectives of the policymaker. For a given value of the right-hand side of (4.2.5), inflation π_t can only be lowered at the expense of aggregate demand y_t. Hence, at time t, for a given history $\mathcal{H}_t$ and in the absence of current and expected future cost-push shocks, $\pi_t < \text{ш}_t$ if and only if $\text{y}_t < 0$. That is, without favourable cost-push shocks, the policymaker can only achieve an actual inflation rate below the current level of intrinsic inflation by having a negative level of aggregate demand y_t, which from (4.2.3) is a weighted average of output gaps from period t onwards. Thus there is a real cost of achieving an inflation rate different from the current level of intrinsic inflation.

This brings to the fore an important feature of intrinsic inflation persistence which can easily be misconstrued. Intrinsic inflation does *not* refer to a level of inflation that is inevitable; the policymaker still has a choice and can trade off inflation for real activity in the short run to some extent. Instead, an increase in the level of intrinsic inflation constitutes a deterioration in the range of feasible inflation and output gap combinations, reducing the level of aggregate economic activity consistent with each given rate of inflation. Or equivalently, intrinsic inflation is the level of inflation which would result from a given history if the output gap were zero both now and in the future, and there were no further cost-push shocks. This follows immediately from (4.2.5) on setting y_t, $\wp_t$ and ϵ_t to zero. Hence the concept of intrinsic inflation is closely related to the real costs of disinflation, but this should not hide the fact that the policymaker still has a choice to make.

In studying how past inflation affects the range of feasible inflation and output gap combinations available today and in the future, it is useful to define the time-path of intrinsic inflation. The time-path of intrinsic inflation is the series of inflation rates following on from a given history and supposing that all intrinsic inflation is fully accommodated ($\boldsymbol{\pi}_t = \boldsymbol{\text{ш}}_t$) now and in the indefinite future, so that any cost-push shocks are absorbed by output gap fluctuations. Formally, starting from history $\mathcal{H}_t$ in (4.2.1), let $\tilde{\boldsymbol{\pi}}_\tau(\mathcal{H}_t)$ be the inflation vector that is realized in period τ if $\boldsymbol{\pi}_\tau = \boldsymbol{\text{ш}}_\tau$ in all periods $\tau \geq t$. Using the expression for $\boldsymbol{\text{ш}}_t$ in (4.2.2), it is shown in Lemma 8 how this path can be calculated using a system of difference equation for the inflation vector $\tilde{\boldsymbol{\pi}}_\tau(\mathcal{H}_t)$ and relative price vector $\tilde{\boldsymbol{\rho}}_\tau(\mathcal{H}_t)$.[2] The time-path of intrinsic inflation $\tilde{\boldsymbol{\pi}}_\tau(\mathcal{H}_t)$ is the solution of

$$\tilde{\boldsymbol{\pi}}_\tau(\mathcal{H}_t) \equiv \sum_{j=1}^{m} \gamma_j \tilde{\pi}_{\tau-j}(\mathcal{H}_t) - \tilde{\boldsymbol{\rho}}_{\tau-1}(\mathcal{H}_t) + \sum_{j=1}^{m+1} \boldsymbol{\Lambda}_j \tilde{\boldsymbol{\rho}}_{\tau-j}(\mathcal{H}_t) \tag{4.2.6a}$$

$$\tilde{\boldsymbol{\rho}}_\tau(\mathcal{H}_t) \equiv \tilde{\boldsymbol{\pi}}_\tau(\mathcal{H}_t) - \boldsymbol{\iota}\tilde{\pi}_\tau(\mathcal{H}_t) + \tilde{\boldsymbol{\rho}}_{\tau-1}(\mathcal{H}_t) \tag{4.2.6b}$$

where the convention is adopted that $\tilde{\pi}_\tau(\mathcal{H}_t) = \pi_\tau$ if $\tau < t$, and similarly for $\tilde{\boldsymbol{\rho}}_\tau(\mathcal{H}_t)$. The implied time-path

[2] Details are provided in appendix B.8.

of the price vector $\mathbf{P}_\tau$ is calculated once the paths of inflation and relative prices from (4.2.6) are known using the formula $\tilde{\mathbf{P}}_\tau(\mathcal{H}_t) \equiv \boldsymbol{\iota}\mathsf{P}_{t-1} + \boldsymbol{\rho}_{t-1} + \sum_{j=0}^{\tau-t} \tilde{\boldsymbol{\pi}}_{t+j}(\mathcal{H}_t)$. From these definitions and (4.2.5) it is apparent that, starting from history $\mathcal{H}_t$ and in the absence of further cost-push shocks from period t onwards, the output gap is zero in all current and future periods if actual inflation is allowed to follow the time-path of intrinsic inflation, that is $\pi_\tau = \tilde{\pi}_\tau(\mathcal{H}_t)$, in all periods from t onwards.

It is shown in Lemma 8 that all models of price stickiness considered here have the property that $\lim_{\tau\to\infty} \tilde{\pi}_\tau(\mathcal{H}_t) = 0$ starting from any history $\mathcal{H}_t$. This means that any currently prevailing intrinsic inflation must eventually decay, though the precise rate of decay is sensitive to the shape of the hazard function for price changes and the extent of heterogeneity. It is not possible to make any other general statements about the pattern of decay; for example, it need not even be monotonic. But the fact that the path of intrinsic inflation has limit zero means that there is always some speed of disinflation which is sustainable without sacrificing real output. If the policymaker tries to achieve a disinflation more rapidly than this then aggregate output will be at a suboptimal level. So the class of price stickiness models studied here does not predict a cost of disinflation per se, but rather a cost of insufficiently slow disinflation.

Both $\boldsymbol{\text{ш}}_t$ and $\tilde{\pi}_\tau(\mathcal{H}_t)$ in (4.2.2) and (4.2.6) depend on the particular history $\mathcal{H}_t$ of price setting in the economy up to period t. But it is useful to have some measure of how much intrinsic inflation persistence results from the shocks that the economy is typically subject to. The impulse response function of intrinsic inflation is a concise way of summarizing this information. It is calculated using the time-path of intrinsic inflation supposing a history in which price stability has been achieved prior to period t, but where a one-off cost-push shock has occurred with equal magnitude in all industries at time t. The size of this shock is normalized so that the initial effect on aggregate inflation is 1%. Then the impulse reponse function $\text{п}(j)$ gives the level of intrinsic inflation $\tilde{\pi}_{t+j}(\mathcal{H}_t)$ in period $t+j$. So when for example $0 < \text{п}(j+1) < \text{п}(j)$ for $j = 0, 1, 2, \ldots$, the inflation initially resulting from the cost-push shock must be reduced gradually and monotonically if a recession is to be avoided.

The expression in (4.2.2) for the current level of intrinsic inflation shows that it generally depends on past relative prices, as well as past aggregate inflation rates. But there is one important special case in which information about relative prices can be ignored when examining intrinsic inflation at the aggregate level, and in which aggregate inflation is independent of the efficient cost-push shocks. This is where all firms in all industries share the same hazard function for price adjustment.

Proposition 2 *If the pricing hazard functions in (2.4.1) satisfy $\alpha_{ij} = \alpha_j$ for all industries $i = 1 \ldots, n$ then economy-wide intrinsic inflation $\text{ш}_t \equiv \boldsymbol{\omega}'\boldsymbol{\text{ш}}_t$ and the efficient cost-push shock $\wp_t \equiv \boldsymbol{\omega}'\boldsymbol{\wp}_t$ defined in (4.2.2) and (4.2.4) are given by:*

$$\text{ш}_t = \sum_{j=1}^{m} \gamma_j \pi_{t-j} \quad , \quad \wp_t = 0 \tag{4.2.7}$$

Thus only m lags of the economy-wide inflation rate π_t are needed to determine the current level of intrinsic inflation.

Proof. See appendix C.2. ■

This section has demonstrated how intrinsic inflation persistence affects the current short-run menu of inflation and output gap combinations available to the policymaker. But it turns out that this concept is also very useful in understanding the objectives of the policymaker.

4.3 The objective function of the policymaker

An expression for the policymaker's loss function has already been derived in Lemma 2. But that result stated the losses in terms of the cross-sectional variances σ_{it}^2 of prices within each industry. This is incomplete because these variances can in fact be obtained from the history of inflation rates and relative prices once the hazard functions are specified. This section shows that there is a very close connection between the cross-sectional variances, the profit functions of firms, and the concept of intrinsic inflation introduced in section 4.2.

Written using vector and matrix notation, the expression for the policymaker's loss function in equation (3.1.3) is

$$\mathfrak{U}_t \equiv \frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\frac{\eta_x}{\varepsilon}\mathsf{y}_\tau^2 + (\boldsymbol{\rho}_\tau - \boldsymbol{\rho}_\tau^*)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}_\tau^*) + \boldsymbol{\sigma}_\tau'\boldsymbol{\Omega}\boldsymbol{\sigma}_\tau\right] \tag{4.3.1}$$

where $\boldsymbol{\rho}_t$ is the vector of relative prices, $\boldsymbol{\rho}_t^*$ is the vector of efficient relative prices given in (3.1.2), and $\boldsymbol{\sigma}_t$ is a vector of cross-sectional standard deviations σ_{it} of prices within industries. The diagonal matrix $\boldsymbol{\Omega}$ contains the industry sizes ω_i along the principal diagonal.

There is in fact considerable overlap between equation (4.3.1) and the loss functions of firms (that is, the negative of the profit function) characterized in Lemma 5. Equation (3.2.12) gives the definition of loss function $\mathfrak{F}_{it}$ relevant to firms in industry i choosing a reset price at time t. The prices chosen by such firms impinge on the losses of the policymaker, and in turn, the policymaker's decisions impinge on the profits of firms. Current and future monetary policy decisions also have an impact on the realized losses of those firms which are still using prices set in the past. To account for this, denote by $\mathfrak{F}_{i,t|T}$ the expected current and future losses from period t onwards of a firm in industry i that set its current price at time T. An expression for $\mathfrak{F}_{i,t|T}$ is obtained by discarding the first $t-T$ terms of the sum in (3.2.12):

$$\mathfrak{F}_{it} = \sum_{\tau=t}^{\infty}\vartheta_{i,\tau-t}\mathbb{E}_t\left[(\mathsf{R}_{it} - \mathsf{P}_\tau - \hat{\rho}_{i\tau})^2\right] \quad , \quad \mathfrak{F}_{i,t|T} = \frac{1}{2}\sum_{\tau=t}^{\infty}\vartheta_{i,\tau-T}\mathbb{E}_t[(\mathsf{R}_{iT} - \mathsf{P}_\tau - \hat{\rho}_{i\tau})^2] \tag{4.3.2}$$

where the sequence of weights $\{\vartheta_{ij}\}_{j=0}^{\infty}$ is defined in (3.2.9). The next result formally states the relationship between firms' and the policymaker's loss functions, and the connection with the concept of intrinsic inflation introduced in section 4.2.

Proposition 3 *Given the profit-maximizing behaviour of firms outlined in Lemmas 3 and 4, the benevolent policymaker's loss function $\mathfrak{U}_t$ in (4.3.1) can be partially stated in terms of firms' loss functions $\mathfrak{F}_{it}$ and $\mathfrak{F}_{i,t|T}$ defined in (4.3.2) as follows*

$$\begin{aligned}\mathfrak{U}_t = {}& \frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\frac{\eta_x}{\varepsilon}\mathsf{y}_\tau^2 + (\boldsymbol{\rho}_t - \boldsymbol{\rho}_t^*)'\boldsymbol{\Omega}(\boldsymbol{\rho}_t - \boldsymbol{\rho}_t^*) - (\boldsymbol{\rho}_\tau - \hat{\boldsymbol{\rho}}_\tau)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau - \hat{\boldsymbol{\rho}}_\tau)\right] \\ & + \sum_{i=1}^{n}\frac{\omega_i\theta_{i0}}{\vartheta_{i0}}\mathbb{E}_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathfrak{F}_{i\tau} + \sum_{\tau\to-\infty}^{t-1}\beta^{\tau-t}\mathfrak{F}_{i,t|\tau}\right]\end{aligned} \tag{4.3.3}$$

where $\boldsymbol{\rho}_t^*$ and $\hat{\boldsymbol{\rho}}_t$ are the relative prices desired by policymakers and firms respectively in the absence of any impediment to immediate price adjustment. The weights θ_{i0} and ϑ_{i0} are defined in equations (2.4.3) and (3.2.9).

A consequence of (4.3.3) is that the derivative of $\mathfrak{U}_t$ with respect to the price vector $\mathbf{P}_\tau$ depends only on the gap between the relative prices $\hat{\boldsymbol{\rho}}_\tau$ and $\boldsymbol{\rho}_\tau^*$ that are desired in the absence of price stickiness, and the derivative with respect to reset price vector $\mathbf{R}_\tau$ is zero:

$$\frac{\partial\mathfrak{U}_t}{\partial\mathbf{P}_\tau} = \beta^{\tau-t}\boldsymbol{\Omega}(\hat{\boldsymbol{\rho}}_\tau - \boldsymbol{\rho}_\tau^*) \quad , \quad \frac{\partial\mathfrak{U}_t}{\partial\mathbf{R}_\tau} = \mathbf{0} \tag{4.3.4}$$

Using these results, the loss function $\mathfrak{U}_t$ can also be stated in terms of the deviation of actual inflation $\boldsymbol{\pi}_t$

from intrinsic inflation $\mathbf{\text{ш}}_t$ *and the efficient cost-push shock* $\wp_t$ *defined in* (4.2.2) *and* (4.2.4) *respectively:*

$$\mathfrak{U}_t = \frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\frac{\eta_x}{\varepsilon}\mathrm{y}_\tau^2 + (\boldsymbol{\pi}_\tau - \mathbf{\text{ш}}_\tau - \wp_\tau)'\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_\tau - \mathbf{\text{ш}}_\tau - \wp_\tau)\right] \tag{4.3.5}$$

The $n \times n$ *matrix* $\boldsymbol{\aleph}$ *is symmetric and positive definite.*

Proof. See appendix C.3. ∎

The first part of the result in equations (4.3.3) and (4.3.4) indicates the extent of the overlap between the interests of firms and those of the benevolent policymaker. There are two sources of conflict between these parties where the respective loss functions do not coincide. The first concerns the level of aggregate output. Individual firms set their prices without taking account of the effect this might have on aggregate demand, and thus on the welfare of other agents in the economy. This is the so-called aggregate demand externality identified by Mankiw (1985). The second conflict results from the differences between the relative prices that firms and the policymaker would like to see in the hypothetical case where prices are perfectly flexible, namely $\hat{\boldsymbol{\rho}}_t$ and $\boldsymbol{\rho}_t^*$. Notice that no knowledge of the hazard functions for price adjustment is required when calculating the extent of this conflict. The terms in the loss function (4.3.1) that do require knowledge of the hazard functions are also found in the loss functions of firms. Therefore, the need for a benevolent policymaker to correct price distortions is indicated only by the disagreement between the policymaker and firms conditional on the absence of price stickiness.

The second part of Proposition 3 demonstrates the difference between the policymaker's attitude to intrinsic inflation ex ante and ex post. The vector of inflation rates $\boldsymbol{\pi}_t$ only enters the loss function (4.3.5) as a deviation from the current level of intrinsic inflation $\mathbf{\text{ш}}_t$ and efficient cost-push shocks $\wp_t$. So while allowing inflation to fluctuate does increase price distortions when intrinsic inflation is zero; once intrinsic inflation is created by past fluctuations in inflation, price distortions are minimized by fully accommodating the intrinsic inflation.

There is a further connection between price distortions, profit-maximization and intrinsic inflation. It turns out that the complete accommodation of existing intrinsic inflation requires that both firms and the policymaker should fully agree on the prices they would like to choose if thought themselves able to make price adjustments in every time period. If this agreement can be reached using monetary policy then the accommodation of intrinsic inflation is fully consistent with both the policymaker minimizing price distortions and firms maximizing profits. This result is stated formally below, along with a description of the determinants of inflation if this policy of accommodating intrinsic inflation is pursued to its fullest extent.

Proposition 4 *When firms act to maximize profits as described in Lemmas 3 and 4, the following three statements are equivalent:*

(i) *The following terms in the loss function* $\mathfrak{U}_t$ *from* (4.3.1) *corresponding to relative-price distortions are minimized from period* t_0 *onwards:*

$$\mathfrak{P}_{t_0} \equiv \frac{1}{2}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbb{E}_{t_0}\left[(\boldsymbol{\rho}_t - \boldsymbol{\rho}_t^*)'\boldsymbol{\Omega}(\boldsymbol{\rho}_t - \boldsymbol{t}_t^*) + \boldsymbol{\sigma}_t'\boldsymbol{\Omega}\boldsymbol{\sigma}_t\right] \tag{4.3.6}$$

(ii) *Existing intrinsic inflation and current and future efficient cost-push shocks are perfectly accommodated, that is,* $\boldsymbol{\pi}_t = \mathbf{\text{ш}}_t + \wp_t$ *for all* $t \geq t_0$*;*

(iii) *The interests of the policymaker and firms would be aligned were no impediments to full price flexibility perceived, that is,* $\hat{\boldsymbol{\rho}}_t = \boldsymbol{\rho}_t^*$ *in all periods* $t \geq t_0$.

Proof. See appendix C.4. ∎

Hence if monetary policy were solely directed towards the complete accommodation of intrinsic inflation and efficient cost-push shocks from some date onwards, then price distortions would be eliminated to the greatest possible extent, and inflation would be determined by the intrinsic inflation that exists prior to date t_0 and the cost-push shocks that occur from period t_0 onwards.

To see the extent to which the presence of efficient cost-push shocks creates a case for time-varying inflation when prices are sticky, let $\pi^{**}_{t|t_0}$ be the weighted average of the vector $\boldsymbol{\pi}^{**}_{t|t_0}$, which defined to be the solution of the following system of difference equations,

$$\boldsymbol{\pi}^{**}_{t|t_0} \equiv \sum_{j=1}^{m} \gamma_j \pi^{**}_{t-j|t_0} - \boldsymbol{\rho}^{**}_{t-1|t_0} + \sum_{j=1}^{m+1} \boldsymbol{\Lambda}_j \boldsymbol{\rho}^{**}_{t-j|t_0} + \boldsymbol{\wp}_t \tag{4.3.7a}$$

$$\boldsymbol{\rho}^{**}_{t|t_0} \equiv \boldsymbol{\pi}^{**}_{t|t_0} - \boldsymbol{\iota}\pi^{**}_{t|t_0} + \boldsymbol{\rho}^{**}_{t-1|t_0} \tag{4.3.7b}$$

where $\boldsymbol{\wp}_t$ is defined in (4.2.4) and the convention is adopted that $\boldsymbol{\pi}^{**}_{t|t_0} = \mathbf{0}$ and $\boldsymbol{\rho}^{**}_{t|t_0} = \mathbf{0}$ if $t < t_0$. The level of inflation that minimizes relative-price distortions when prices are sticky is characterized by the following result.

Proposition 5 *Let $\pi^*_{t|t_0}$ be the inflation rate that minimizes the loss from relative-price distortions from period t_0 onwards, as measured by $\mathfrak{P}_{t_0}$ in (4.3.6).*

(i) The rate of inflation needed to minimize price distortions is equal to

$$\pi^*_{t|t_0} = \tilde{\pi}_t(\mathcal{H}_{t_0}) + \pi^{**}_{t|t_0} \tag{4.3.8}$$

where $\tilde{\pi}_t(\mathcal{H}_{t_0})$ is the time-path of intrinsic inflation defined in (4.2.6), and $\pi^{**}_{t|t_0}$ is the solution of (4.3.7), which depends only on the exogenous cost-push shocks $\wp_t$ from period t_0 onwards.

(ii) As the initial period t_0 recedes into the past ($t_0 \to -\infty$), the inflation rate $\pi^*_{t|t_0}$ in (4.3.8) tends to a stochastic process π^*_t that is solely a function of the history of exogenous cost-push shocks $\wp_t$:

$$\pi^*_t \equiv \lim_{t_0 \to -\infty} \pi^*_{t|t_0} \tag{4.3.9}$$

(iii) If the pricing hazard functions in (2.4.1) satisfy $\alpha_{ij} = \alpha_j$ for all industries $i = 1 \ldots, n$ then

$$\pi^*_{t|t_0} = \tilde{\pi}_t(\mathcal{H}_{t_0}) \quad , \quad \pi^*_t = 0 \tag{4.3.10}$$

so the rate of inflation that minimizes price-distortions from t_0 onwards depends only on the time-path of intrinsic inflation starting from history $\mathcal{H}_{t_0}$. If inflation is chosen to minimize price distortions in all periods ($t_0 \to -\infty$) then the optimal inflation rate is zero at all times.

Proof. See appendix C.5. ∎

It is noteworthy that even when no intrinsic inflation persistence is present, the inflation rate that minimizes price distortions may vary over time when the speed of price adjustment differs across industries. This time-variation results from efficient cost-push shocks, which occur when there are fluctuations in the Pareto-efficient relative prices $\boldsymbol{\rho}^*_t$. But when all industries have the same hazard function for price changes, the only source of time-variation in the inflation rate minimizing price distortions is intrinsic inflation that was already present before this goal was adopted.

A policy directed exclusively towards accommodating intrinsic inflation persistence and efficient cost-push shocks is not optimal in general because it focuses too much attention on minimizing price distortions, ignoring the market failure that results from the aggregate demand externality. The trade-off between mitigating these two market failures means that a strict inflation target is not optimal. The optimal trade-off is characterized in the next section.

5 Optimal monetary policy

Now the objectives and the constraints of the policymaker have been clarified, the question of optimal monetary policy can be addressed more precisely. It should be clear from both the constraints and objective functions identified in section 4 (especially equations (4.2.5) and (4.3.5)) that intrinsic inflation will play a key role. Furthermore, Proposition 4 shows that there is an equivalence between minimizing price distortions, accommodating existing intrinsic inflation, and aligning the interests of firms and policymakers concerning the relative prices they would choose in the absence of price stickiness. Because of this latter result, the conflict between firms and the policymaker over price distortions in a world with price stickiness is reducible to conflict in a world without price stickiness. This allows certain general principles of optimal policy to be derived that apply to the whole range of time-dependent pricing models, and ensures that the targeting rules which implement optimal monetary policy are relatively simple in all cases.

The result that minimizing price distortions from a given point in time onwards is equivalent to accommodating intrinsic inflation *which already exists at that point* provides the key to understanding the differences between optimal policy with discretion and an optimal commitment to a policy rule. The case of rules is further subdivided into commitments that are optimal from the point of view of the policymaker at a particular point in time (referred to as t_0–commitment), and timeless-perspective commitment where the policy rule is constrained to be time consistent, forbidding any special treatment of the initial date on which the commitment comes into force. The differences between t_0–commitment and timeless-perspective commitment also turn out to be explicable in terms of intrinsic inflation persistence. The three cases of discretion, t_0–commitment, and timeless-perspective commitment are analysed separately below.

5.1 Discretion

When the policymaker acts with discretion it means that current monetary policy can be set without reference to any commitments made in the past. The policymaker decides optimal policy by computing the likely paths of the target variables for each choice of interest rates, and thus evaluates the expected loss from each available option.

The interest rate decision is not the only factor that affects the expected paths of the target variables; there are also state variables which are beyond the control of the policymaker. These can be classified as either exogenous or predetermined. The set of relevant predetermined endogenous variables is given in the history $\mathcal{H}_t$ defined in (4.2.1) which is used to calculate the current level of intrinsic inflation. This set contains all the predetermined variables that appear in equation (4.1.2). Because this history affects the range of inflation and output gap pairs that are available to the policymaker, it will also affect the combination that is optimally chosen. Thus if the public has rational expectations they will make their beliefs about the policymaker's future actions depend on the endogenous variables influenced by policy today that will be part of the relevant history in the future. So even though a policymaker with discretion cannot directly control future expectations, the indirect effect of changing current endogenous variables on these future histories must be taken into account. Equilibria in which the public's expectations depend only on endogenous variables which are actually needed to compute the likely future paths of the target variables are referred to as Markovian equilibria. The following result shows that there is an intermediate target for monetary policy which implements the discretionary Markovian equilibrium.

Theorem 1 *Suppose the policymaker maximizes (2.6.1) subject to the behaviour of profit-maximizing firms as characterized in Lemma 4. Suppose also the policymaker acts with discretion. Then the discretionary Markovian equilibrium is implemented by setting monetary policy to pursue the following intermediate target*

$$\pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = \text{ш}_t + \wp_t \tag{5.1.1}$$

in all time periods. This flexible inflation target is adjusted in the short run to accommodate fully any intrinsic inflation ш_t *and any efficient cost-push shocks* $\wp_t$*, defined in (4.2.2) and (4.2.4) respectively.*

Proof. See appendix D.1. ■

This flexible inflation target has a number of notable features. First, the weights attached to the inflation and output gap variables are independent of the pattern of price adjustment and depend solely on the extent of imperfect competition in the economy, as measured by the average price elasticity of demand ε. Since this elasticity must be greater than unity in a monopolistically competitive world, percentage deviations of output from potential count for less than deviations of inflation from its target value. In more competitive economies the weight attached to the output gap is lower, and as the limit of perfect competition is reached this weight tends to zero. The two objectives only receive equal weight in the limiting case of the lowest amount of price competition consistent with the existence of an equilibrium. But in general, the inflation target is not strict, and some attention must be paid to the output gap.

However, the most striking feature of the intermediate target which implements the discretionary equilibrium is its complete accommodation of intrinsic inflation persistence. As was discussed in section 4.2, intrinsic inflation refers to inflation which would occur in the absence of cost-push shocks and where the output gap is expected to remain zero, taking the history of pricing decisions as given. A consequence of this definition is that reducing actual inflation below the current level of intrinsic inflation will lead to a widening of the output gap, unless there happens by coincidence to be a favourable cost-push shock at the same time. But Theorem 1 shows that a policymaker with discretion would never want to carry out such a disinflation. Once intrinsic inflation is present, the policymaker sees no benefit in tackling it because to do so would actually increase price distortions as well as lead to a deterioration in the output gap. And because the inflation target is not strict, the policymaker will not refrain from allowing inflation to deviate from target if an excessively large output gap would result. This allows intrinsic inflation to become established, which is then accommodated by the policymaker in the future.

A final feature of (5.1.1) is that the flexible inflation target is also adjusted to accommodate fully the current and future efficient cost-push shocks in $\wp_t$. With adverse cost-push shocks, the policymaker will allow inflation to rise, but favourable shocks lead to an opportunistic disinflation.

5.2 Commitment made at a specific initial date

The results of section 5.1 highlight a serious flaw in having too much discretion in the setting of monetary policy. On the one hand, intrinsic inflation is bad for price distortions ex ante, but once present, it should be tolerated to minimize price distortions ex post. A policymaker with discretion will not tackle intrinsic inflation persistence because once intrinsic inflation has arisen there is simply no incentive to do so. This section explores the benefits of making a binding commitment to a policy rule that does not allow intrinsic inflation persistence to be accommodated.

The policymaker now considers not just the current interest rate decision but the full range of state-contingent paths for interest rates from some initial period t_0 onwards. Supposing that the public believes that the policymaker is indeed bound by these commitments, it possible to calculate state-contingent paths for the target variables, evaluate the expected loss, and find the optimal path to commit to. The following result shows that there is a targeting rule which implements this optimal state-contingent commitment.

Theorem 2 *Suppose the policymaker maximizes (2.6.1) and sets policy subject to the behaviour of profit-maximizing firms as characterized in Lemma 4. Suppose also that the policymaker makes a commitment to conduct monetary policy according to a state-contingent rule in all time periods from t_0 onwards. The optimal state-contingent path for the target variables is achieved by committing monetary policy to the following intermediate target,*

$$\pi_t + \frac{1}{\varepsilon}\mathrm{y}_t = \pi^*_{t|t_0} - \sum_{\tau=t_0}^{t-1}(\pi_\tau - \pi^*_{\tau|t_0}) \tag{5.2.1}$$

where the optimal inflation target $\pi^*_{t|t_0}$ is equal to $\tilde{\pi}_t(\mathcal{H}_{t_0}) + \pi^{**}_{t|t_0}$ as is shown in equation (4.3.8) of Proposition 5. The first component $\tilde{\pi}_t(\mathcal{H}_{t_0})$ is the time-path of intrinsic inflation from t_0 onwards defined in (4.2.6), and the second component $\pi^{**}_{t|t_0}$ in (4.3.7) and depends only on the sequence of efficient cost-push shocks $\{\wp_t\}$ from t_0 onwards.

Proof. See appendix D.2. ■

Just as with the intermediate target (5.1.1) for discretion, the t_0-commitment is to a flexible inflation target based on a weighted average of inflation and the output gap. The weights placed on these two target variables are independent of the hazard functions and are the same as in the case of discretion. And also as in the case of discretion, the level of the flexible inflation target is history dependent. However, the nature of this history dependence is very different.

The first point to note is that any new intrinsic inflation which gets into the economy from period t_0 onwards is not accommodated by the rule, even though there is a temptation to do so. However, any intrinsic inflation which already exists at the point when the commitment is made is still accommodated completely. So unlike the flexible inflation target in the case of discretion, the level of the target is not revised based on the history $\mathcal{H}_t$, but can only depend in a predetermined way on the history $\mathcal{H}_{t_0}$ from the time when the commitment came into force.

While the level of the flexible inflation target is not influenced by the relevant state variables in $\mathcal{H}_t$, it has additional history dependence arising from some irrelevant state variables in the form of all the past misses of the inflation target that have occurred since it was first adopted. These target deviations are both possible and likely because the inflation target is flexible and puts some weight on output gap fluctuations. But equation (4.1.2) shows that they need not be relevant for determining the set of feasible inflation and output gap combinations open to the policymaker. Commitment to a rule involving irrelevant state variables is used as a means of steering private-sector expectations in order to open up a better set of possibilities for the target variables. This form of optimal history dependence has been emphasized by Woodford (2000). The precise nature of the commitment in the present case is to revise down the current level of the flexible inflation target in response to every failure to hit the target in the past. The revision is made one-for-one with past target deviations, so basically a 1% cumulative overshoot entails a 1% lower target in the current period. Notice that this policy is effectively stabilizing the price level in the long run. This supports the conclusion found by Woodford for the special case of the Calvo pricing.

In summary, the optimal t_0–commitment is to a rule in which only intrinsic inflation that has arisen prior to t_0 is accommodated, and in which all failures to hit the inflation target from period t_0 onwards are eventually corrected in future periods. But both of these features clearly make the policy rule time inconsistent, which is an unattractive feature that risks damaging its credibility.

5.3 Commitment from the timeless perspective

An optimal commitment made at a specific date has been seen to prescribe a rule which treats intrinsic inflation differently depending on whether it has already taken root prior to the adoption of the rule, or has arisen afterwards. This creates a time-inconsistency problem because if new intrinsic inflation occurs after the commitment, a reconsideration of the rule at a later date will propose that the new intrinsic inflation be accommodated, contrary to the stiffer response called for by the original rule. The revised rule will also want to wipe the slate clean in terms of past misses of the inflation target.

One resolution of the time-inconsistency problem of optimal monetary policy commitments has been suggested by Woodford (2003). A commitment made from the *timeless perspective* permits no special treatment of the initial period in which the rule is adopted. This works by making a binding commitment to minimize the loss function from some point in time onwards, but subject to a finite number of constraints on the values of the target variables during a transitional period. And crucially, these constraints must themselves be of the same time-consistent form as is optimal for the future paths of the target variables. Thus a policy rule that is optimal from the timeless perspective is one in which if the policymaker were initially forced to ensure that the target variables move in accordance with the rule, it would be optimal to adopt the same rule (and the same evolution of the target variables) in all future periods.

One feature of this equilibrium concept is that there are frequently multiple timelessly optimal paths for the target variables. However, Woodford shows that these timelessly optimal equilibria differ from each other only by a deterministic component that eventually approaches zero as time passes. Rather than attempt to characterize all possible timelessly optimal rules, the following result focuses on one particular class that emerges as the natural generalization of Theorem 2 for t_0–commitment.

Theorem 3 *Suppose the policymaker aims to maximize* (2.6.1) *and sets policy subject to the behaviour of profit-maximizing firms as characterized in Lemma 4. If the policymaker is constrained for the first m periods by the necessity of inflation evolving in accordance with a function of the history of inflation and relative prices $\mathcal{H}_t$ and the current exogenous state vector then the resulting timelessly optimal equilibrium can be implemented by the following targeting rule:*

$$\pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = \pi_t^* - (\mathsf{P}_{t-1} - \mathsf{P}_{t-1}^*) \tag{5.3.1}$$

The optimal price-level path $\{\mathsf{P}_t^*\}$ is composed of the sum of $\bar{\mathsf{P}}$, an arbitrarily chosen long-run target, and P_t^{**}, which is related to the inflation target π_t^* defined in equation (4.3.9) *of Proposition 5 according to $\mathsf{P}_t^{**} - \mathsf{P}_{t-1}^{**} = \pi_t^*$ and depends only on the exogenous efficient cost-push shocks $\{\wp_t\}$.*

Proof. See appendix D.3. ■

This targeting rule can also be characterized as a flexible inflation target, just like the intermediate targets for discretion and t_0–commitment in (5.1.1) and (5.2.1). As before, the weights attached to the target variables are unaffected by the shape of the hazard functions, and are the same as those found to be optimal under discretion and t_0–commitment. However, it is apparent from the right-hand side of (5.3.1) that the history dependence of the level of the flexible inflation target is different from those earlier results.

The most striking difference is the complete absence of any intrinsic inflation term. This means that the policymaker should resist the temptation to accommodate intrinsic inflation at all times, no matter when it arose. Furthermore, the targeting rule has the property that any failure of inflation to be on target in the past leads to an adjustment to the level of the inflation target in the present, which effectively stabilizes the price level in the long run. As discussed in section 5.2, the rule optimal for a t_0–commitment looks at inflation target deviations only from t_0 onwards. But since the timeless-perspective policy cannot make reference to a specific start date it achieves the same result by looking at the past deviation of the price level from some long-run target path. Over time, this gap will grow wider whenever inflation misses its target since the evolution of the long-run target path $\{\mathsf{P}_t^*\}$ is completely exogenous.

The timeless-perspective targeting rule (5.3.1) is also the simplest of the three intermediate targets (5.1.1), (5.2.1) and (5.3.1). The only component of rule that requires knowledge of the pattern of price adjustment is the exogenous series $\{\pi_t^*\}$ for the long-run level of the inflation target. As is shown in Proposition 5, time variation in π_t^* arises only because of differences in the speed of price adjustment between industries coupled with the presence of different technology shocks hitting each industry. If either of these features is absent, a constant and zero long-run inflation target is optimal.

6 Discussion

6.1 Some examples of specific price setting models

A number of specific price-setting models are now considered to illustrate the similarities and differences between the optimal policy prescriptions of section 5 in the cases of discretion and commitment, and to compare these with existing work on optimal monetary policy. The models cover a range of assumptions about the shape of the hazard function for price adjustment, and the variation in hazard functions across industries. The extent to which the models exhibit intrinsic inflation persistence differs accordingly as well.

Example 1 (Identical flat hazard functions for each industry) This case occurs where there is one single probability of price adjustment that applies to all firms at all times, that is, $\alpha_{ij} = \alpha$. This is of course the well-known Calvo (1983) model of price setting, leading to the New Keynesian Phillips curve:[3]

$$\pi_t = \aleph \sum_{j=0}^{\infty} \beta^j \mathbb{E}_t[\eta_x \mathsf{y}_{t+j} + \eta_\epsilon \epsilon_{t+j}]$$

The Phillips curve given above is a special case of equation (4.1.3). The key point to note about this model is its prediction of the complete absence of any intrinsic inflation persistence. Inflation is an exclusively forward-looking variable, depending only on current and expected future output gaps and cost-push shocks. Therefore, the impulse response function $\mathrm{\pi}(j)$ of intrinsic inflation, defined in section 4.2, is equal to 0 for all $j \geq 1$.

The welfare implications of this Calvo pricing model are derived by Woodford (2003). He shows that the utility-based loss function (4.3.5) is given by the following expression,

$$\mathfrak{U}_t = \frac{1}{2} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\frac{\eta_x}{\varepsilon} \mathsf{y}_\tau^2 + \frac{1}{\aleph} \pi_\tau^2 \right]$$

Note that this has the same form as the "ad hoc" quadratic loss functions widely used in the optimal policy literature, though the weights on the two target variables are pinned down precisely by the parameters of the model. All that matters for households' welfare is the deviation of aggregate output from its efficient level and the deviation of inflation from its long-run target value of zero.

Woodford also derives the intermediate targets (5.1.1), (5.2.1) and (5.3.1) for this model, corresponding

[3] This is just the usual expression for the New Keynesian Phillips curve, $\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \aleph(\eta_x \mathsf{y}_t + \eta_\epsilon \epsilon_t)$, solved forward.

to the cases of discretion, t_0–commitment and timeless perspective commitment respectively,

$$\begin{aligned} \textbf{Discretion}: &\qquad \pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = 0 \\ t_0\textbf{–commitment}: &\qquad \pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = -\sum_{\tau=t_0}^{t-1}\pi_\tau \\ \textbf{Timeless perspective}: &\qquad \pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = \bar{\mathsf{P}} - \mathsf{P}_{t-1} \end{aligned}$$

where the form of these expressions and the notation have been modified slightly to make them comparable with the results of this paper in Theorems 1–3. The discretionary target is a classic flexible inflation target with no time variation or history dependence in the level of the target. This is because there is no intrinsic inflation to accommodate, and no efficient cost-push shocks because all industries have the same speed of price adjustment. The t_0–commitment modifies the flexible inflation target by adjusting the current level of the target in line with past inflation target deviations. Since the long-run inflation target is zero, the cumulative overshoot of the target is obtained by summing actual inflation rates from t_0 up to the period before the current one. There is no intrinsic inflation prevailing at time t_0 to accommodate and no efficient cost-push shock. Similarly, the only modification to the flexible inflation target for the timeless-perspective commitment is that it is necessary to take account of the past price-level deviation from a constant long-run target.

Example 2 (Identical upward-sloping hazard functions for each industry) This case maintains the homogeneity of the pricing hazard function across industries as in Example 1, but supposes that newer prices are stickier than older prices, resulting in an upwards-sloping hazard function. It is shown in Sheedy (2007*b*) that there exists a first-order recursive representation (2.4.6) in which the hazard function is upwards sloping everywhere, that is, $\alpha_j < \alpha_{j+1}$. Since $m = 1$, there is one lag of inflation in the Phillips curve (4.1.3):

$$\pi_t = \gamma\pi_{t-1} + \aleph\sum_{j=0}^{\infty}\beta^j\left(\frac{1-\gamma^{j+1}}{1-\gamma}\right)\mathbb{E}_t[\eta_x\mathsf{y}_{t+j} + \eta_\epsilon\epsilon_{t+j}]$$

By appealing to Theorem 1 of Sheedy (2007*b*), the upward-sloping hazard function implies $\gamma > 0$. Therefore, this price setting model exhibits positive intrinsic inflation persistence, as can be seen either from this positive coefficient on lagged inflation or the impulse response function $\pi(j)$ of intrinsic inflation, which is positive everywhere, that is $\pi(j) > 0$ for all $j \geq 1$.

The presence of intrinsic inflation persistence means that disinflation cannot be achieved costlessly even once the factors giving rise to it have dissipated. But the presence of intrinsic inflation also changes the form of the utility-based loss function (4.3.5). This is modified by including the deviation of inflation from the current level of intrinsic inflation, rather than just the rate of inflation itself:

$$\mathfrak{U}_t = \frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\frac{\eta_x}{\varepsilon}\mathsf{y}_\tau^2 + \frac{1}{\aleph}(\pi_\tau - \gamma\pi_{\tau-1})^2\right]$$

As a consequence, some of the optimal policy implications are altered relative to the case of a flat hazard function. The intermediate targets in equations (5.1.1), (5.2.1), and (5.3.1) now become:

$$\textbf{Discretion}: \quad \pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = \gamma\pi_{t-1}$$

$$t_0\textbf{–commitment}: \quad \pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = \gamma^{t-t_0+1}\pi_{t_0-1} - \sum_{\tau=t_0}^{t-1}(\pi_\tau - \gamma^{\tau-t_0+1}\pi_{t_0-1})$$

$$\textbf{Timeless perspective}: \quad \pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = \bar{\mathsf{P}} - \mathsf{P}_{t-1}$$

Notice that both discretion and t_0–commitment accommodate some intrinsic inflation. For discretion, it is all intrinsic inflation present at time t; for t_0–commitment, it is only the intrinsic inflation that prevailed at t_0, the date when the commitment came into force. As the timeless perspective commitment ignores all intrinsic inflation, its targeting rule has the same form as that in Example 1 for a flat hazard function. There are no efficient cost-push shocks to accommodate in any of these cases because all industries share the same hazard function.

Example 3 (Different flat hazard function for each industry) Here the second key assumption of Example 1 is relaxed, while the first one is maintained. The hazard function is now allowed to vary across industries, but unlike the case considered in Example 2, these hazard functions are all required to be flat. Formally, the price-adjustment probabilities in (2.4.1) are such that $\alpha_{ij} = \alpha_i$ for all j, and $\alpha_i \neq \alpha_j$ if $i \neq j$. The precise distribution of the price-adjustment probabilities across the n industries is left unspecified. To aid comparison with Examples 1 and 2 it is assumed that all industries have identical technology ($\mathsf{A}_{it} = \mathsf{A}_t$) so $\boldsymbol{\rho}_t^* = \mathbf{0}$. The Phillips curve (4.1.3) in this case is given by:

$$\pi_t = \boldsymbol{\omega}'\boldsymbol{\Lambda}\boldsymbol{\rho}_{t-1} + \boldsymbol{\omega}'\boldsymbol{\aleph}\sum_{j=0}^{\infty}\beta^j\boldsymbol{\Lambda}^j\mathbb{E}_t[\eta_x\boldsymbol{\omega}\mathsf{y}_{t+j} + \eta_\epsilon\boldsymbol{\Omega}\boldsymbol{\epsilon}_{t+j}]$$

Intrinsic inflation persistence here comes from the one lag of the relative price vector $\boldsymbol{\rho}_t$. There are no lags of economy-wide inflation. This does not mean, however, that intrinsic inflation persistence is irrelevant when considering economy-wide shocks. When the speed of price adjustment varies across industries then a given economy-wide shock leads to a range of price responses in different industries and hence affects relative prices. In the next period, this perturbation of the past relative price vector puts limits on how quickly inflation can return to its long-run level without aggregate output deviating from its efficient level.

The properties of this model have been studied extensively in Sheedy (2007*a*). In particular, it has been shown that the intrinsic inflation persistence presence in this model is of a somewhat peculiar form. The impulse response function of intrinsic inflation satisfies $\pi(j) < 0$ for all $j \geq 1$, that is, there is negative intrinsic persistence. In other words, there is a tendency for inflation to overshoot its average level after a shock. However, in spite of this unusual behaviour of intrinsic inflation, the analysis developed in section 5 still applies. At no stage was it necessary to impose restrictions on the shape of the impulse response function of intrinsic inflation.

The utility-based loss function can easily be obtained from equation (4.3.5),

$$\mathfrak{U}_t = \frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\frac{\eta_x}{\varepsilon}\mathsf{y}_\tau^2 + (\boldsymbol{\pi}_t - (\boldsymbol{\Lambda}-\mathbf{I})\boldsymbol{\rho}_{t-1})'\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_t - (\boldsymbol{\Lambda}-\mathbf{I})\boldsymbol{\rho}_{t-1})\right]$$

and the optimal intermediate targets (5.1.1), (5.2.1), and (5.3.1) for the cases of discretion, t_0–commitment, and timeless–perspective commitment respectively are:

$$\textbf{Discretion}: \quad \pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = \boldsymbol{\omega}'\boldsymbol{\Lambda}\boldsymbol{\rho}_{t-1}$$

$$t_0\textbf{–commitment}: \quad \pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = \boldsymbol{\omega}'\boldsymbol{\Lambda}^{t-t_0-1}\boldsymbol{\rho}_{t_0-1} - \sum_{\tau=t_0}^{t-1}(\pi_\tau - \boldsymbol{\omega}'\boldsymbol{\Lambda}^{\tau-t_0-1}\boldsymbol{\rho}_{t_0-1})$$

$$\textbf{Timeless perspective}: \quad \pi_t + \frac{1}{\varepsilon}\mathsf{y}_t = \bar{\mathsf{P}} - \mathsf{P}_{t-1}$$

As before, all intrinsic inflation prevailing at time t is fully accommodated with discretion. When a t_0–commitment is made, only existing intrinsic inflation as of period t_0 is accommodated. And the targeting rule used for the timeless–perspective commitment ignores intrinsic inflation completely, and has exactly the same form as in Examples 1 and 2.

6.2 Relation to other analyses of optimal monetary policy

In recent years the problem of optimal monetary policy has attracted an increasing amount of research.[4] This section will make a brief attempt to point out the key differences in assumptions and conclusions this paper has in comparison with some of that work.

6.2.1 Inflation bias and stabilization bias

It is first necessary to mention a classic issue in monetary policy analysis that has been neglected in this paper. Since the work of Kydland and Prescott (1977) and Barro and Gordon (1983), the rules versus discretion debate in monetary policy analysis has often been conducted around the issue of the inflation bias. An inflation bias occurs when a policymaker with discretion tries to engineer a surprise inflation to raise aggregate output to its efficient level. This course of action is thought desirable because the level of output that would otherwise prevail (even after prices or expectations have fully adjusted) is below the efficient level. An inflation bias results because the public rationally foresees this policy and raises its expectations of inflation, thus robbing the policymaker of the ability to use a surprise change in monetary policy to achieve this end, but nonetheless leading to a ratcheting up of actual inflation. A desirable feature of a monetary policy rule is thus to commit to a lower average level of inflation than would prevail with discretion. The problem is absent from the model in this paper because the wage-bill subsidy assumed here eliminates the gap between the natural and efficient levels of aggregate output on average.

In some versions of the model of the inflation bias, such as that of Barro and Gordon, once a commitment to a lower level of inflation is successfully made, all the drawbacks of discretionary policy are eliminated. The problem is thus relatively simple to resolve in principle, though possibly difficult in practice. However, it has been frequently pointed out that eliminating the inflation bias is a necessary but not sufficient condition for a policy rule to be optimal in a wider class of Phillips curve models. Not only does the average level of inflation differ for an optimal commitment relative to discretion, but so too must the response to shocks. This issue is referred to as the stabilization bias of discretionary monetary policy, and has been analysed by Clarida *et al.* (1999); Jonsson (1997); Svensson (1997) and Woodford (2003) among others. It arises because a monetary policy rule is able to steer the public's expectations about the future after a shock has occurred in a way that would not be credible if the policymaker had the discretion to change course later. The characterization of the optimal commitment highlighted in this paper refers to the features of a policy rule that ensure the stabilization bias of discretionary policymaking is removed. Removing the inflation

[4] For a flavour of this see the papers of Clarida, Galí and Gertler (1999); Khan, King and Wolman (2003); Kollmann (2004); Schmitt-Grohé and Uribe (2004*a*) as well as Woodford (2003).

bias by means of a wage-bill subsidy allows attention to be focused on the issue of the stabilization bias, which has so far received less attention.

6.2.2 The objectives of firms and the policymaker

In relation to other work concerned with the appropriate response of monetary policy to shocks, most of the major differences with this paper can be reduced to the assumptions made about the objectives of the policymaker and of firms, and the extent to which these agents rationally pursue those objectives. Traditionally, optimal monetary policy has been defined with respect to a quadratic loss function including, at least, inflation and output (or unemployment) deviations from some desirable levels.[5] That these variables do correspond to the implicit or explicit goals of many central banks and governments is not to be denied, but for a normative analysis of policy, it should not be assumed that a loss function taking this form is automatically congruent with the welfare of households in the economy. Since the pioneering work of Rotemberg and Woodford (1998) and Woodford (2003), many of the technical challenges in deriving utility-based loss functions have been overcome for some of the models typically used in monetary policy analysis. And in at least one case it is found that the utility-based loss function has the same form as the usual "ad hoc" quadratic loss function (albeit with the weights precisely pinned down by the parameters of the model). However, it can be seen from Proposition 3 that this is not a general feature of all time-dependent pricing models. While the output gap appears as normal in equation (4.3.5), the inflation rate must be replaced by its deviation from the current level of intrinsic inflation and efficient cost-push shocks. Thus whenever intrinsic inflation persistence is present, the "ad hoc" loss function will not coincide with the utility-based loss function.

Analogous to differences in the extent to which the policymaker maximizes the welfare of households, there is also a range of assumptions in the literature about the extent to which firms maximize profits when they set prices. In this paper it is supposed that firms are always forward looking and rational in seeking to maximize profits whenever they set a new price. The fact that firms are forward-looking profit maximizers does not prevent the Phillips curve having a "backward-looking" component. Intrinsic inflation persistence is introduced by relaxing the assumption of Calvo pricing, namely that the pricing hazard function is flat (and the same for all industries). But it should be stressed that the most popular way of introducing such persistence into the standard New Keynesian model actually takes the opposite approach, maintaining the Calvo pricing assumption but relaxing the requirement of profit maximization (at least for some group of firms).

Galí and Gertler (1999) posit a "rule of thumb" that is used by a certain fraction of firms to set prices, with the remaining firms continuing to be forward-looking profit maximizers. The rule of thumb is backward looking and does not maximize profits in general. The optimal policy implications of this idea are analysed by Steinsson (2003). As in the work by Woodford, Steinsson is careful to base the policymaker's loss function on the level of household utility implied by the model. But because of the rule of thumb, some firms are not maximizing profits when they set their prices, and thus the analysis of the extent of market failure and the extent of disagreement between firms and the policymaker in Propositions 3 and 4 breaks down. The rule of thumb actually creates an additional market failure that would not be present in an economy with imperfect competition and price stickiness alone. This leads to differences in policy implications compared with this paper, even though the extent of intrinsic inflation persistence may be similar. One difference that is apparent is the optimal commitment in Steinsson's work does not fully correct for past failures to hit the inflation target.

[5]Several papers making this assumption are found in Taylor (1999).

6.2.3 Heterogeneity in price adjustment and the optimal weighting of the price index

The case where some firms' prices are stickier than others on average has been considered in work by Aoki (2001) and Benigno (2004). One of the findings of those studies is that the overall inflation rate that is relevant for the welfare of households should be weighted in favour of those prices which are stickier. This prescription essentially argues for the use of a core inflation measure to guide monetary policy. In this paper, price stickiness can vary between industries, but the aggregate inflation rate appearing in the intermediate targets is weighted solely on the basis of the relative sizes of different industries.

It is possible to reconcile this with the findings of Aoki and Benigno by noting that this paper considers a case in which differences in price stickiness do not create a need for the weights to be adjusted. This case depends on assuming that the inter-industry elasticity of substitution is the same as the steady-state within-industry elasticities of substitution, though when the economy is not in the steady state these elasticities can of course differ. This is done purely in the interests of keeping the analysis as simple as possible, but it does show that *both* heterogeneity in price-adjustment probabilities and differences in elasticities of substitution are needed to obtain the core inflation result.

6.2.4 Targeting rules and instrument rules

The policy implications of this paper take the form of targeting rules for monetary policy rather than instrument rules. A targeting rule is a requirement for the policymaker to adjust the monetary policy instrument until some given criterion is satisfied, whereas an instrument rule is a specific mapping from observable economic variables to the monetary policy instrument. This distinction is discussed further in Svensson (1999). Targeting rules are used here because it is generally not possible to find one instrument rule which is optimal for a broad range of models of price stickiness. Different parameterizations of the hazard functions for price adjustment will lead to different instrument rules being optimal, as would differences in many other factors such as the combination of exogenous shocks hitting the economy. Another advantage of targeting rules here is their relative simplicity even when fully optimal. An optimal instrument rule designed to apply to just one price-setting model will often need a very complicated expression. The complexity of optimal instrument rules has led some to consider optimal simple instrument rules, where the instrument rule is constrained to depend linearly on a restricted subset of variables. An example of work in this area is Schmitt-Grohé and Uribe (2004*b*). On the other hand, this paper has shown that there may be simple optimal targeting rules that apply to a wide range of models, even when these models potentially feature a huge number of free parameters.

It should be stressed however that this paper has focused on the implications of price stickiness for optimal monetary policy and has excluded many other potentially relevant factors. Papers such as Khan *et al.* (2003); Kollmann (2004); Schmitt-Grohé and Uribe (2004*a*) also consider other features such as transaction frictions, wage-stickiness, capital goods and investment in addition to price stickiness. But while adding any of these features to the model may change the optimal targeting rule, it is unlikely to remove completely the differences between the optimal targeting rule and the optimal discretionary policy highlighted here. Thus by focusing on one particular aspect of the optimal monetary policy problem, a precise analytical characterization of the differences between commitment and discretion when prices are sticky is obtained, which also helps to provide some intuitive understanding of why these differences are found. It avoids the problems of understanding and interpreting results that arise from the "black-box" of a numerical solution of a calibrated model with many different features.

7 Conclusions

This paper has analysed optimal monetary policy in a model of time-dependent pricing that imposes no restrictions on how the probability of price adjustment depends on the number of periods a price has remained fixed for. In addition, there is no restriction on the amount of heterogeneity in these price-adjustment probabilities between industries. Taking this widest possible range of time-dependent pricing models, it has been shown that certain principles of optimal monetary policy apply to all of them. In the three cases of complete discretion over policy, commitment to a targeting rule from a specific initial date onwards, and commitment from the timeless perspective, it is possible to obtain an analytical solution for an intermediate target that implements the best policy. This intermediate target always takes the form of a flexible inflation target. Interest rates should be adjusted to ensure that a weighted average of the inflation rate and the output gap reaches a target value. The weights attached to these two variables depend only on the average amount of price competition in the economy and are otherwise the same for all the models of price adjustment considered.

The differences between monetary policy with discretion, commitment from some initial date, and timeless-perspective commitment are apparent in the generally time-varying level the flexible inflation target is set at. In general, the current inflation target should be history dependent in all three cases, but the nature of this history dependence changes radically as the scope for discretionary action varies. It is always optimal for a policymaker with complete discretion to accommodate fully any intrinsic inflation persistence by revising up the inflation target one-for-one with any rise in current intrinsic inflation. In other words, if current inflation has a tendency to be too high as a result of high inflation in the past, and if the only way to avoid this is a tightening of monetary policy which would have a deleterious effect on the output gap, then the policymaker should passively accept this higher level of inflation.

But once binding commitments are allowed, the policymaker finds it optimal to resist accommodating intrinsic inflation persistence. An optimal targeting rule that comes into force after some initial date will stop the policymaker accommodating any intrinsic inflation that arises subsequent to the initial date. However, at the same time it should also accommodate intrinsic inflation which is present when the rule came into force. The asymmetric treatment of intrinsic inflation depending on whether it arises before or after the commitment is made makes the optimal commitment time-inconsistent. If the choice of rule is constrained to be time-consistent by considering the optimal commitment from the timeless perspective, then it is found that no intrinsic inflation should be accommodated, irrespective of when it arose.

A second difference between discretion and commitment is in the response to deviations of inflation from the level of the flexible inflation target in the past. With discretion, it is optimal to pay no attention to these past deviations. On the other hand, if a binding commitment is feasible, the policymaker finds it optimal to revise the current level of the flexible inflation target in response to past deviations of inflation from the target. The policymaker takes the cumulative overshoot of inflation relative to target since the rule came into force and subtracts this from the target that would otherwise be used. But like the response to intrinsic inflation, this privileges the initial date on which the rule is introduced and thus is not time consistent. If time consistency is enforced, the policymaker chooses a long-run price level target and uses the past deviation of prices from this target to adjust the current level of the target. This has the same effect of punishing excessively high inflation in the past with a tighter inflation target today.

The broad applicability of these results comes from a careful analysis of the market failures to which the economy is subject and extent to which these can be mitigated by policy intervention. The paper thus approches the optimal monetary policy problem from the perspective of public economics. When firms are profit maximizers, the scope for intervening to correct distortions can be judged by asking whether firms and the policymaker would currently be in agreement were all prices in the economy perceived to be fully flexible. This is true even though price stickiness can itself be a cause of relative-price distortions. The finding allows simple targeting rules to be drawn up which are optimal for a very wide range of models of price stickiness.

References

Aoki, K. (2001), "Optimal monetary policy responses to relative price changes", *Journal of Monetary Economics*, **48**:55–80. 31

Barro, R. J. and Gordon, D. B. (1983), "A positive theory of monetary policy in a natural rate model", *Journal of Political Economy*, **91(4)**:589–610. 29

Benigno, P. (2004), "Optimal monetary policy in a currency area", *Journal of International Economics*, **63(2)**:293–320. 31

Calvo, G. A. (1983), "Staggered prices in a utility-maximizing framework", *Journal of Monetary Economics*, **12**:383–398. 1, 7, 26

Christiano, L. J., Eichenbaum, M. J. and Evans, C. L. (2005), "Nominal rigidities and the dynamic effects of a shock to monetary policy", *Journal of Political Economy*, **113**:1–45. 3

Clarida, R., Galí, J. and Gertler, M. (1999), "The science of monetary policy: A New Keynesian perspective", *Journal of Economic Literature*, **37**:1661–1707. 29

Galí, J. and Gertler, M. (1999), "Inflation dynamics: A structural econometric analysis", *Journal of Monetary Economics*, **44**:195–222. 3, 30

Giannoni, M. P. and Woodford, M. (2005), "Optimal inflation targeting rules", in B. S. Bernanke and M. Woodford (eds.), "The Inflation Targeting Debate", University of Chicago Press, Chicago. 3

Jonsson, G. (1997), "Monetary politics and unemployment persistence", *Journal of Monetary Economics*, **39**:303–325. 29

Khan, A., King, R. G. and Wolman, A. (2003), "Optimal monetary policy", *Review of Economic Studies*, **70(4)**:825–860. 29, 31

Kollmann, R. (2004), "Welfare maximizing operational monetary and tax policy rules", *Discussion paper 4782*, Centre for Economic Policy Research. 29, 31

Kydland, F. and Prescott, E. (1977), "Rules rather than discretion: The inconsistency of optimal plans", *Journal of Political Economy*, **85(3)**:473–492. 29

Mankiw, N. G. (1985), "Small menu costs and large business cycles: A macroeconomic model of monopoly", *Quarterly Journal of Economics*, **100(2)**:529–538. 1, 20

Rotemberg, J. J. and Woodford, M. (1998), "An optimization-based econometric framework for the evaluation of monetary policy", *Technical working paper 233*, National Bureau of Economic Research. 30

Schmitt-Grohé, S. and Uribe, M. (2004*a*), "Optimal fiscal and monetary policy under sticky prices", *Journal of Economic Theory*, **114**:198–230. 29, 31

——— (2004*b*), "Optimal simple and implementable monetary and fiscal rules", *Working paper 10253*, National Bureau of Economic Research. 31

Sheedy, K. D. (2007*a*), "Inflation persistence when price stickiness differs between industries", *Working paper*, London School of Economics. 28

——— (2007*b*), "Intrinsic inflation persistence", *Working paper*, London School of Economics. 7, 27, 54

Steinsson, J. (2003), "Optimal monetary policy in an economy with inflation persistence", *Journal of Monetary Economics*, **50**:1425–1456. 3, 30

Svensson, L. E. O. (1997), "Optimal inflation targets, 'conservative' central banks, and linear inflation contracts", *American Economic Review*, **87**:98–114. 29

——— (1999), "Inflation targeting as a monetary policy rule", *Journal of Monetary Economics*, **43**:607–654. 31

Taylor, J. B. (1980), "Aggregate dynamics and staggered contracts", *Journal of Political Economy*, **88**:1–24. 1, 7

——— (1999), *Monetary Policy Rules*, Chicago University Press. 30

Wolman, A. L. (1999), "Sticky prices, marginal cost, and the behavior of inflation", *Federal Reserve Bank of Richmond Economic Quarterly*, **85(4)**:29–48. 1

Woodford, M. (2000), "Pitfalls of forward-looking monetary policy", *American Economic Review*, **90(2)**:100–104. 24

——— (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, New Jersey. 3, 25, 26, 29, 30

Youla, D. C. (1961), "On the factorization of rational matrices", *IEEE Transactions on Information Theory*, **7(3)**:172–189. 50

A Steady state and log linearizations

This section gives more details about the steady state around which the equations of section 3 are log-linearized. The steady state features flexible (or fully adjusted) prices, zero inflation, symmetry across industries, and is also Pareto efficient. In what follows, a bar above a variable denotes its steady-state value.

The steady-state level of technology common to all industries is some exogenous value $\bar{A} > 0$, and the common steady-state elasticity of substitution $\varepsilon > 1$ is also given. Starting from this point, steady-state output $\bar{Y}$, consumption $\bar{C}$, real wage $\bar{w}$, hours worked $\bar{H}$, and real marginal cost $\bar{\mathcal{C}}_Y$ can be calculated. Any variables that refer to money prices have no steady-state value, but the (gross) inflation rate has steady-state value $\bar{\mathit{\Pi}} = 1$, and because all industries are treated symmetrically and prices are fully adjusted, the steady-state relative price is $\bar{\varrho} = 1$.

The production function (2.3.1), the first-order condition for optimal labour supply (2.1.2), the expression for real marginal cost (2.3.2), and the goods market clearing condition (2.5.1) respectively imply that:

$$\bar{Y} = \bar{A}\bar{H}^{\eta_{yh}} \quad , \quad \frac{v_h(\bar{H})}{u_c(\bar{C})} = \bar{w} \quad , \quad \bar{\mathcal{C}}_Y = \frac{\bar{w}\bar{Y}^{\frac{1-\eta_{yh}}{\eta_{yh}}}}{\eta_{yh}\bar{A}^{\frac{1}{\eta_{yh}}}} \quad , \quad \bar{C} = \bar{Y} \tag{A.0.1}$$

Equation (2.1.3) determines the steady-state value of the asset-pricing kernel $\bar{\mathfrak{M}}_{\tau|t}$ as a function only of the gap between τ and t and households' subjective discount factor β:

$$\bar{\mathfrak{M}}_{\tau|t} = \beta^{\tau-t} \tag{A.0.2}$$

Using the above, equation (2.6.2), and the steady-state (gross) inflation rate $\bar{\mathit{\Pi}} = 1$, the steady-state gross nominal interest rate is $\bar{\mathcal{I}} = 1/\beta$. This steady-state interest rate is strictly positive as required.

As all prices are flexible or fully adjusted, the derivative of the profit function (2.3.4) with respect to its own relative price must be zero in the steady state. From equation (3.2.2), and using (A.0.1), $F_{\varrho}(\bar{\varrho}; \bar{\varrho}, \bar{Y}, \bar{A}, \bar{w}, \varepsilon) = 0$ implies

$$\frac{\left(\frac{\varepsilon}{\varepsilon-1}\right)(1-\mathfrak{s})\bar{Y}^{\frac{1-\eta_{yh}}{\eta_{yh}}} v_h(\bar{H})}{\eta_{yh}\bar{A}^{\frac{1}{\eta_{yh}}} u_c(\bar{Y})} = 1 \tag{A.0.3}$$

where $\mathfrak{s}$ is the government's wage-bill subsidy.

There are no steady-state values for reset prices because these are expressed in terms of money. But the steady-state value of a reset price relative to other prices, denoted $\bar{r}$, does exist. Using the assumption $\bar{\mathit{\Pi}} = 1$ together with $\bar{\varrho} = 1$ and equations (2.2.4) and (3.2.10), it is clear that $\bar{r} = 1$.

The additional requirement of Pareto efficiency means that the steady-state ratio of the marginal utility of leisure to the marginal utility of consumption is equal to the marginal product of labour. These can be obtained from utility function (2.1.1) and production function (2.3.1), leading to the following efficiency condition:

$$\frac{v_h(\bar{H})}{u_c(\bar{Y})} = \frac{\eta_{yh}\bar{A}^{\frac{1}{\eta_{yh}}}}{\bar{Y}^{\frac{1-\eta_{yh}}{\eta_{yh}}}} \tag{A.0.4}$$

By comparing (A.0.3) and (A.0.4), the twin requirements of prices being fully adjusted and the steady-state being Pareto efficient necessitate a wage-bill subsidy $\mathfrak{s}$ that satisfies $\left(\frac{\varepsilon}{\varepsilon-1}\right)(1-\mathfrak{s}) = 1$. It is clear that setting $\mathfrak{s} = \varepsilon^{-1}$ ensures both requirements are met. As $\varepsilon > 1$, the resulting subsidy satisfies the inequality $0 < \mathfrak{s} < 1$, so is always well defined. This fully characterizes the steady state.

Having established the steady state, now the log-linearizations of the equations of the model are derived. In what follows, unless otherwise stated, log-deviations of variables from their steady-state values are denoted using sans serif letters. When the variable in question has no steady-state value (for instance, any money price) the sans serif letter denotes just the logarithm of the variable. So for example, $\mathsf{A}_{it} \equiv \log A_{it} - \log \bar{A}$ is the log-deviation of technology in industry i from its steady-state value. Then ϵ_{it} is defined to be the log-deviation of the (gross) desired markup from its steady-state value, that is, $\epsilon_{it} \equiv \log\left(\frac{\varepsilon_{it}}{\varepsilon_{it}-1}\right) - \log\left(\frac{\varepsilon}{\varepsilon-1}\right)$. The variables A_{it} and ϵ_{it} are the only exogenous shocks introduce into the model, and so in principle, all the log-deviations of the endogenous variables can be expressed in terms of them. If each industry-specific shock is collected into $n \times 1$ vectors $\mathbf{A}_t$ and $\boldsymbol{\epsilon}_t$, then the $2n \times 1$ vector $\boldsymbol{\upsilon}_t \equiv (\ \mathbf{A}'_t\ ,\ \boldsymbol{\epsilon}'_t\ )'$ contains all the exogenous shocks. Where it is necessary to indicate it, terms of order k in $\boldsymbol{\upsilon}_t$ are denoted by $\mathcal{O}(||\boldsymbol{\upsilon}_t||^k)$. In the log-linearizations that follow, second- and higher-order terms are suppressed.

The log-linearized version of the first-order condition (2.1.2) for the optimal supply of labour of type $\imath$ is

$$\eta_{wh}\mathsf{H}_t(\imath) + \eta_{ic}\mathsf{C}_t = \mathsf{w}_t(\imath) \quad , \quad \eta_{ic} \equiv -\frac{\bar{C}u_{cc}(\bar{C})}{u_c(\bar{C})} > 0 \quad , \quad \eta_{wh} \equiv \frac{\bar{H}v_{hh}(\bar{H})}{v_h(\bar{H})} > 0 \tag{A.0.5}$$

where the coefficients η_{ic} and η_{wh} are strictly positive because the utility function $u(\cdot)$ is strictly increasing and concave, and the function $v(\cdot)$ is strictly increasing and convex. And the log-linearization of first-order condition (2.1.3) is

$$\mathsf{m}_{\tau|t} = \eta_{ic}(\mathsf{C}_\tau - \mathsf{C}_t) \tag{A.0.6}$$

where the positive coefficient η_{ic} is defined in (A.0.5) above. Log-linear approximations of the industry price level and aggregate price level indices defined in (2.2.2) and (2.2.4) are given by:

$$\mathsf{P}_{it} = \frac{1}{\omega_i}\int_{\Omega_i}\int_{\Omega}\mathsf{P}_t(\imath,\jmath)d\jmath d\imath \quad , \quad \mathsf{P}_t = \sum_{i=1}^{n}\omega_i\mathsf{P}_{it} \tag{A.0.7}$$

The production function (2.3.1) for a firm $(\imath,\jmath)$ in industry i is

$$\mathsf{Y}_t(\imath,\jmath) = \mathsf{A}_{it} + \eta_{yh}\mathsf{H}_t(\imath,\jmath) \tag{A.0.8}$$

when log-linearized. The log-linear approximation of real marginal cost (2.3.2) for firm $(\imath,\jmath)$ in industry i with production function (A.0.8) is denoted by $\mathsf{x}_t(\imath,\jmath)$:

$$\mathsf{x}_t(\imath,\jmath) = \mathsf{w}_t(\imath) + \frac{1-\eta_{yh}}{\eta_{yh}}\mathsf{Y}_t(\imath,\jmath) - \frac{1}{\eta_{yh}}\mathsf{A}_{it} \tag{A.0.9}$$

This firm also faces the following log-linearized demand function (2.3.3) for its product:

$$\mathsf{Y}_t(\imath,\jmath) = -\varepsilon(\mathsf{P}_t(\imath,\jmath) - \mathsf{P}_t) + \mathsf{Y}_t \tag{A.0.10}$$

Finally, the goods market clearing conditions in (2.5.1) simply become:

$$\mathsf{C}_t(\imath,\jmath) = \mathsf{Y}_t(\imath,\jmath) \quad , \quad \mathsf{C}_{it} = \mathsf{Y}_{it} \quad , \quad \mathsf{C}_t = \mathsf{Y}_t \quad , \quad \int_{\Omega}\mathsf{H}_t(\imath,\jmath)d\jmath = \mathsf{H}_t(\imath) \tag{A.0.11}$$

This provides all the basic log linearizations necessary to obtain the results in section 3.

B Proofs of lemmas

B.1 Proof of Lemma 1

The equations characterizing the Pareto-efficient allocation of resources (the one that gives all households equal consumption) are given as a block in (3.1.1). The log-linearizations of the production function (3.1.1a), demand curve (3.1.1b), and labour supply curve (3.1.1c) are shown in equations (A.0.8), (A.0.10) and (A.0.5) respectively,

$$\mathsf{Y}^*_{it} = \mathsf{A}_{it} + \eta_{yh}\mathsf{H}^*_{it} \tag{B.1.1a}$$

$$\mathsf{Y}^*_{it} = -\varepsilon\rho^*_{it} + \mathsf{Y}^*_t \tag{B.1.1b}$$

$$\eta_{wh}\mathsf{H}^*_{it} + \eta_{ic}\mathsf{Y}^*_t = \mathsf{w}^*_{it} \tag{B.1.1c}$$

where the resource constraint in (A.0.11) has also been used. The steady state around which these approximations are made is described in appendix A. The output aggregator (3.1.1d) can be log-linearized as follows:

$$\mathsf{Y}^*_t = \sum_{i=1}^{n}\omega_i\mathsf{Y}^*_{it} \tag{B.1.1d}$$

Finally, the real marginal cost and relative price condition (3.1.1e) and be log-linearized using the result from equation (A.0.9):

$$\rho^*_{it} = \mathsf{w}^*_{it} + \left(\frac{1-\eta_{yh}}{\eta_{yh}}\right)\mathsf{Y}^*_{it} - \frac{1}{\eta_{yh}}\mathsf{A}_{it} \tag{B.1.1e}$$

By solving linear equations (B.1.1a)–(B.1.1d) the solutions $\mathsf{Y}^*_t = \eta_y\mathsf{A}_t$ and $\rho^*_{it} = -\eta_\rho(\mathsf{A}_{it} - \mathsf{A}_t)$ are obtained, where $\mathsf{A}_t \equiv \sum_{i=1}^{n} \omega_i \mathsf{A}_{it}$ denotes the weighted average of A_{it} across industries, and η_y and η_ρ are the following positive coefficients:

$$\eta_y \equiv \frac{1+\eta_{wh}}{1-\eta_{yh}+\eta_{wh}+\eta_{yh}\eta_{ic}} \quad , \quad \eta_\rho \equiv \frac{1+\eta_{wh}}{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon} \tag{B.1.2}$$

This proves the claim of the lemma.

B.2 Proof of Lemma 2

The aim is to obtain a second-order approximation to the average lifetime utility of all households from period t onwards, denoted by $\mathcal{U}_t$ in equation (2.6.1). This welfare function is broken down first into utility accruing to a single household $(\imath, \jmath)$ at a point in time t, denoted by $U_t(\imath, \jmath)$. This is defined with reference to the household utility function (2.1.1):

$$U_t(\imath, \jmath) \equiv u(c_t(\imath, \jmath)) - v(h_t(\imath, \jmath)) \tag{B.2.1}$$

It has been argued in section 2.1 that the existence of complete asset markets leads to full consumption insurance, so $c_t(\imath, \jmath) = C_t$ for all $(\imath, \jmath)$. As a result, from (2.1.2), all households supplying the same labour input $\imath$ (and hence receiving the same wage) choose to supply the same number of hours, $h_t(\imath, \jmath) = H_t(\imath)$ for all $\jmath$. A second-order accurate approximation of the utility function in (B.2.1) in terms of the log-deviations of consumption C_t and hours $\mathsf{H}_t(\imath)$ is

$$U_t(\imath, \jmath) = \bar{U} + \bar{C} u_c \left(\mathsf{C}_t - \frac{(\eta_{ic}-1)}{2} \mathsf{C}_t^2 \right) - \bar{H} v_h \left(\mathsf{H}_t(\imath) + \frac{(1+\eta_{wh})}{2} \mathsf{H}_t^2(\imath) \right) + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.2}$$

where $\bar{U}$ is the steady-state value of utility function (B.2.1) and $\mathcal{O}(||\boldsymbol{v}_t||^3)$ refers to terms that are third-order or higher in the exogenous shocks $\boldsymbol{v}_t$ as described in appendix A. The positive parameters η_{ic} and η_{wh} are defined in equation (A.0.5), and the constants u_c and v_h denote partial derivatives of the functions $u(\cdot)$ and $v(\cdot)$ evaluated at steady-state consumption $\bar{C}$ and hours $\bar{H}$ as given in appendix A.

The next step is find an approximation of the average utility U_t accruing to all households in one time period t, which is defined as:

$$U_t \equiv \int_{\Omega} \int_{\Omega} U_t(\imath, \jmath) d\jmath d\imath \tag{B.2.3}$$

The goods market equilibrium condition (2.5.1) implies that $\bar{C} = \bar{Y}$, and in log-linear terms from (A.0.11), $\mathsf{C}_t = \mathsf{Y}_t$. The efficiency condition (A.0.4) for the steady state and the production function in (A.0.1) mean that $\bar{H} = \eta_{yh} \bar{Y}$. Hence, averaging (B.2.2) over all households $(\imath, \jmath)$ yields the following second-order approximation of (B.2.3),

$$U_t = \bar{U} + \bar{Y} u_c \left(\mathsf{Y}_t - \frac{(\eta_{ic}-1)}{2} \mathsf{Y}_t^2 - \eta_{yh} \sum_{i=1}^{n} \int_{\Omega_i} \left\{ \mathsf{H}_t(\imath) + \frac{(1+\eta_{wh})}{2} \mathsf{H}_t^2(\imath) \right\} d\imath \right) + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.4}$$

where the integration over household hours worked has been broken down into blocks of labour types associated with different industries. Substituting in the expression for the log-linearized production function (A.0.8) and the labour-market equilibrium conditions in (A.0.11) into (B.2.4):

$$\begin{aligned} U_t = \bar{U} + \bar{Y} u_c \Bigg(& \mathsf{Y}_t - \frac{(\eta_{ic}-1)}{2} \mathsf{Y}_t^2 \\ & - \sum_{i=1}^{n} \int_{\Omega_i} \int_{\Omega} \left\{ \mathsf{Y}_t(\imath, \jmath) - \mathsf{A}_{it} + \frac{(1+\eta_{wh})}{2\eta_{yh}} (\mathsf{Y}_t(\imath, \jmath) - \mathsf{A}_{it})^2 \right\} d\jmath d\imath \Bigg) + \mathcal{O}(||\boldsymbol{v}_t||^3) \end{aligned} \tag{B.2.5}$$

A second-order approximation of the output aggregator Y_t across industries in (2.5.2) is given by

$$\mathsf{Y}_t = \mathbb{E}_I[\mathsf{Y}_{it}] + \frac{1}{2}\left(\frac{\varepsilon-1}{\varepsilon}\right)\mathbb{V}_I[\mathsf{Y}_{it}] + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.6}$$

where $\mathbb{E}_I[\cdot]$ denotes the cross-sectional weighted mean across the n industries, and $\mathbb{V}_I[\cdot]$ the inter-industry cross-sectional variance. Similarly, the within-industry aggregator Y_{it} in (2.5.2) can be approximated as follows

$$\mathsf{Y}_{it} = \mathbb{E}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)] + \frac{1}{2}\left(\frac{\varepsilon_{it}-1}{\varepsilon_{it}}\right)\mathbb{V}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)] + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.7}$$

where $\mathbb{E}_{\Omega_i}[\cdot]$ and $\mathbb{V}_{\Omega_i}[\cdot]$ denote the cross-sectional mean and variance within industry i. The variation over time and across industries of the elasticity of substitution ε_{it} can be removed by noting $\varepsilon_{it} = \varepsilon + \mathcal{O}(||\boldsymbol{v}_t||)$. It follows that $\varepsilon_{it}/(\varepsilon_{it}-1) = \varepsilon/(\varepsilon-1) + \mathcal{O}(||\boldsymbol{v}_t||)$ and so (B.2.7) becomes:

$$\mathsf{Y}_{it} = \mathbb{E}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)] + \frac{1}{2}\left(\frac{\varepsilon-1}{\varepsilon}\right)\mathbb{V}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)] + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.8}$$

The second-order approximations of the across- and within-industry aggregators (B.2.6) and (B.2.8) can be combined to produce:

$$\mathsf{Y}_t = \mathbb{E}_I\mathbb{E}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)] + \frac{1}{2}\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\mathbb{V}_I[\mathsf{Y}_{it}] + \mathbb{E}_I\mathbb{V}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)]\right) + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.9}$$

The consolidated aggregator (B.2.9) can be substituted into the second-order approximation of U_t in (B.2.5) to give

$$\begin{aligned} U_t = \bar{U} + \bar{Y}u_c\Bigg(& \mathbb{E}_I\mathbb{E}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)] + \frac{1}{2}\frac{(\varepsilon-1)}{\varepsilon}\left(\mathbb{V}_I[\mathsf{Y}_{it}] + \mathbb{E}_I\mathbb{V}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)]\right) - \frac{(\eta_{ic}-1)}{2}\mathsf{Y}_t^2 \\ & - \mathbb{E}_I\mathbb{E}_{\Omega_i}\left[(\mathsf{Y}_t(\imath,\jmath) - \mathsf{A}_{it}) + \frac{(1+\eta_{wh})}{2\eta_{yh}}(\mathsf{Y}_t(\imath) - \mathsf{A}_{it})^2\right]\Bigg) + \mathcal{O}(||\boldsymbol{v}_t||^3) \end{aligned} \tag{B.2.10}$$

where the definitions of the cross-sectional expectation operators $\mathbb{E}_I[\cdot]$ and $\mathbb{E}_{\Omega_i}[\cdot]$ have been used. By simplifying (B.2.10) and introducing the notation "t.i.p." for terms that are independent of monetary policy, the following expression for U_t is obtained:

$$\begin{aligned} U_t = -\frac{\bar{Y}u_c}{2}\Bigg(& (\eta_{ic}-1)\mathsf{Y}_t^2 + \frac{(1+\eta_{wh})}{\eta_{yh}}\mathbb{E}_I[\mathbb{V}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)] + (\mathsf{Y}_{it} - \mathsf{A}_{it})^2] \\ & + \frac{(1-\varepsilon)}{\varepsilon}\left(\mathbb{V}_I[\mathsf{Y}_{it}] + \mathbb{E}_I\mathbb{V}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)]\right)\Bigg) + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \end{aligned} \tag{B.2.11}$$

The definition of the cross-sectional variance operator $\mathbb{V}_I[\cdot]$ implies that $\mathbb{E}_I[(\cdot)^2] = \mathbb{V}_I[\cdot] + \mathbb{E}_I[\cdot]^2$, so equation (B.2.11) can be written as:

$$\begin{aligned} U_t = -\frac{\bar{Y}u_c}{2}\Bigg(& (\eta_{ic}-1)\mathsf{Y}_t^2 + \frac{(1+\eta_{wh})}{\eta_{yh}}(\mathsf{Y}_t - \mathsf{A}_t)^2 + \frac{(1-\varepsilon)}{\varepsilon}\mathbb{V}_I[\mathsf{Y}_{it}] + \frac{(1+\eta_{wh})}{\eta_{yh}}\mathbb{V}_I[\mathsf{Y}_{it} - \mathsf{A}_{it}] \\ & + \left(\frac{(1-\varepsilon)}{\varepsilon} + \frac{(1+\eta_{wh})}{\eta_{yh}}\right)\mathbb{E}_I\mathbb{V}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)]\Bigg) + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \end{aligned} \tag{B.2.12}$$

The expression above is now written in terms of the Pareto-efficient levels of aggregate output Y_t^* and industry-specific output Y_{it}^*. These can be obtained from the log-linearizations in (B.1.1b) and (B.1.2).

Noting that both are terms which are independent of monetary policy, equation (B.2.12) can be restated as:

$$U_t = -\frac{\bar{Y}u_c}{2}\left(\frac{(1-\eta_{yh}+\eta_{wh}+\eta_{yh}\eta_{ic})}{\eta_{yh}}(\mathsf{Y}_t-\mathsf{Y}_t^*)^2 + \frac{(\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon)}{\eta_{yh}\varepsilon}\mathbb{V}_I[\mathsf{Y}_{it}-\mathsf{Y}_{it}^*] \right.$$
$$\left. + \frac{(\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon)}{\eta_{yh}\varepsilon}\mathbb{E}_I\mathbb{V}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)]\right) + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.13}$$

The demand curve for the products of industry i in (2.2.5) can be log-linearized as $\mathsf{Y}_{it} = -\varepsilon\rho_{it}$, where the goods-market equilibrium conditions in (A.0.11) have also been used. Combining this with (B.1.1b), the cross-sectional variance of the deviation of industry-specific outputs from their efficient levels is given by

$$\mathbb{V}_I[\mathsf{Y}_{it}-\mathsf{Y}_{it}^*] = \mathbb{E}_I[\{(\mathsf{Y}_{it}-\mathsf{Y}_{it}^*)-(\mathsf{Y}_t-\mathsf{Y}_t^*)\}^2] + \mathcal{O}(||\boldsymbol{v}_t||^3) = \varepsilon^2\mathbb{E}_I[(\rho_{it}-\rho_{it}^*)^2] + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.14}$$

where the efficient relative price ρ_{it}^* is defined in (B.1.2). Likewise, the log-linearization of the individual firm demand function in (A.0.10) shows that:

$$\mathbb{V}_{\Omega_i}[\mathsf{Y}_t(\imath,\jmath)] = \varepsilon^2\mathbb{V}_{\Omega_i}[\mathsf{P}_t(\imath,\jmath)] + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.15}$$

Putting equations (B.2.13), (B.2.14) and (B.2.15) together with the definition of the output gap, $\mathsf{y}_t \equiv \mathsf{Y}_t - \mathsf{Y}_t^*$, yields the following second-order approximation of U_t,

$$U_t = -\frac{\varepsilon(\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon)}{\eta_{yh}}\frac{\bar{Y}u_c}{2}\left\{\frac{\eta_x}{\varepsilon}\mathsf{y}_t^2 + \mathbb{E}_I[(\rho_{it}-\rho_{it}^*)^2] + \mathbb{E}_I\mathbb{V}_{\Omega_i}[\mathsf{P}_t(\imath,\jmath)]\right\} + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.16}$$

where the coefficient η_x is defined as $\eta_x \equiv (1-\eta_{yh}+\eta_{wh}+\eta_{yh}\eta_{ic})/(\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon)$. By denoting the cross-sectional variance of prices in industry i by $\sigma_{it}^2 \equiv \mathbb{V}_{\Omega_i}[\mathsf{P}_t(\imath,\jmath)]$ and defining the coefficient $\eta_\epsilon \equiv \eta_{yh}/(\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon)$, equation (B.2.16) can be expressed as:

$$U_t = -\frac{\varepsilon\bar{Y}u_c}{2\eta_{yh}\eta_\epsilon}\left[\frac{\eta_x}{\varepsilon}\mathsf{y}_t^2 + \sum_{i=1}^n \omega_i(\{\rho_{it}-\rho_{it}^*\}^2 + \sigma_{it}^2)\right] + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.17}$$

Finally, lifetime utility averaged over all households $\mathcal{U}_t$ in (2.6.1) can be obtained from U_t given definitions (B.2.1), (B.2.3) and (2.1.1):

$$\mathcal{U}_t = \sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t U_\tau \tag{B.2.18}$$

By summing (B.2.17) over time, a second-order accurate approximation of average lifetime utility $\mathcal{U}_t$ is obtained:

$$\mathcal{U}_t = -\frac{\varepsilon\bar{Y}u_c}{2\eta_{yh}\eta_\epsilon}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\frac{\eta_x}{\varepsilon}\mathsf{y}_\tau^2 + \sum_{i=1}^n \omega_i(\{\rho_{i\tau}-\rho_{i\tau}^*\}^2 + \sigma_{i\tau}^2)\right] + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.2.19}$$

This proves the result stated in the lemma.

B.3 Proof of Lemma 3

The system of equations defining the profit-maximizing flexible relative price is given as the block (3.2.3). The log-linearizations of the production function (3.2.3a), the demand function (3.2.3b), and the labour supply function (3.2.3c) can be taken from (A.0.8), (A.0.10), and (A.0.5) respectively:

$$\hat{\mathsf{Y}}_{it} = \mathsf{A}_{it} + \eta_{yh}\hat{\mathsf{H}}_{it} \tag{B.3.1a}$$

$$\hat{\mathsf{Y}}_{it} = -\varepsilon\hat{\rho}_{it} + \mathsf{Y}_t \tag{B.3.1b}$$

$$\eta_{wh}\hat{\mathsf{H}}_{it} + \eta_{ic}\mathsf{Y}_t = \hat{\mathsf{w}}_{it} \tag{B.3.1c}$$

Equation (3.2.3d) describing the profit-maximizing markup of price on marginal cost becomes

$$\hat{\rho}_{it} = \hat{\mathsf{w}}_{it} + \frac{1-\eta_{yh}}{\eta_{yh}}\hat{\mathsf{Y}}_{it} - \frac{1}{\eta_{yh}}\mathsf{A}_{it} \tag{B.3.1d}$$

where (A.0.9) has been used. By solving linear equations (B.3.1a)–(B.3.1d) the following solution is obtained:

$$\hat{\rho}_{it} = \left(\frac{1-\eta_{yh}+\eta_{wh}+\eta_{yh}\eta_{ic}}{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon}\right)\mathsf{Y}_t - \left(\frac{1+\eta_{wh}}{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon}\right)\mathsf{A}_{it} + \left(\frac{\eta_{yh}}{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon}\right)\epsilon_{it} \tag{B.3.2}$$

The equation above can be restated in terms of the output gap $\mathsf{y}_t \equiv \mathsf{Y}_t - \mathsf{Y}_t^*$ and efficient relative price ρ_{it}^*, as defined in Lemma 1, yielding $\hat{\rho}_{it} = \eta_x \mathsf{y}_t + \rho_{it}^* + \eta_\epsilon \epsilon_{it}$ with η_x and η_ϵ defined by:

$$\eta_x \equiv \frac{1-\eta_{yh}+\eta_{wh}+\eta_{yh}\eta_{ic}}{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon} \quad , \quad \eta_\epsilon \equiv \frac{\eta_{yh}}{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon} \tag{B.3.3}$$

Therefore the result stated in the lemma is proved.

B.4 Proof of Lemma 4

The first step in proving the result is to log-linearize the equations (3.2.8a)–(3.2.8c). This can be done by using the log-linearizations of the production function, demand function, and labour supply equation in (A.0.8), (A.0.10) and (A.0.5):

$$\mathsf{Y}_{i,\tau|t} = \mathsf{A}_{i\tau} + \eta_{yh}\mathsf{H}_{i,\tau|t} \tag{B.4.1a}$$

$$\mathsf{Y}_{i,\tau|t} = -\varepsilon(\mathsf{R}_{it} - \mathsf{P}_\tau) + \mathsf{Y}_\tau \tag{B.4.1b}$$

$$\eta_{wh}\mathsf{H}_{i,\tau|t} + \eta_{ic}\mathsf{Y}_\tau = \mathsf{w}_{i,\tau|t} \tag{B.4.1c}$$

The first-order condition for the profit-maximizing reset price (3.2.8d) can be log-linearized to produce

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t}\theta_{i,\tau-t}\mathbb{E}_t\left[\mathsf{R}_{it} - \mathsf{P}_\tau - \left(\mathsf{w}_{i,\tau|t} + \frac{1-\eta_{yh}}{\eta_{yh}}\hat{\mathsf{Y}}_{i,\tau|t} - \frac{1}{\eta_{yh}}\mathsf{A}_{i\tau}\right) - \epsilon_{i\tau}\right] = 0 \tag{B.4.1d}$$

where the fact that $\mathfrak{M}_{\tau|t} = \beta^{\tau-t} + \mathcal{O}(||\boldsymbol{\upsilon}_t||)$ and the definition $\epsilon_{it} \equiv \log(\varepsilon_{it}/(\varepsilon_{it}-1)) - \log(\varepsilon/(\varepsilon-1))$ have been used. The term in parentheses in (B.4.1d) is the real marginal cost at time τ of a firm in industry i using a reset price chosen in period t. Using (B.4.1a)–(B.4.1c), the following expression for this real marginal cost is obtained:

$$\begin{aligned}\mathsf{w}_{i,\tau|t} + \frac{1-\eta_{yh}}{\eta_{yh}}\hat{\mathsf{Y}}_{i,\tau|t} - \frac{1}{\eta_{yh}}\mathsf{A}_{i\tau} = {} & -\frac{\varepsilon}{\eta_{yh}}(1-\eta_{yh}+\eta_{wh})(\mathsf{R}_{it} - \mathsf{P}_\tau) \\ & + \left(\frac{1-\eta_{yh}+\eta_{wh}+\eta_{yh}\eta_{ic}}{\eta_{yh}}\right)\mathsf{Y}_\tau - \left(\frac{1+\eta_{wh}}{\eta_{yh}}\right)\mathsf{A}_{i\tau}\end{aligned} \tag{B.4.2}$$

This equation can be restated in terms of the output gap $\mathsf{y}_t \equiv \mathsf{Y}_t - \mathsf{Y}_t^*$ using the definitions in (B.1.2):

$$\begin{aligned}\mathsf{w}_{i,\tau|t} + \frac{1-\eta_{yh}}{\eta_{yh}}\hat{\mathsf{Y}}_{i,\tau|t} - \frac{1}{\eta_{yh}}\mathsf{A}_{i\tau} = &-\frac{\varepsilon}{\eta_{yh}}(1-\eta_{yh}+\eta_{wh})(\mathsf{R}_{it}-\mathsf{P}_\tau) \\ &+ \left(\frac{1-\eta_{yh}+\eta_{wh}+\eta_{yh}\eta_{ic}}{\eta_{yh}}\right)\mathsf{y}_\tau + \left(\frac{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon}{\eta_{yh}}\right)\rho_{i\tau}^*\end{aligned} \tag{B.4.3}$$

By substituting (B.4.3) back into (B.4.1d) and rearranging terms, the following expression for the profit-maximizing reset price is obtained:

$$\begin{aligned}\mathsf{R}_{it} = \sum_{\tau=t}^{\infty}\left(\frac{\beta^{\tau-t}\theta_{i,\tau-t}}{\sum_{j=0}^{\infty}\beta^j\theta_{ij}}\right)\mathbb{E}_t\Bigg[\mathsf{P}_\tau + &\left(\frac{1-\eta_{yh}+\eta_{wh}+\eta_{yh}\eta_{ic}}{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon}\right)\mathsf{y}_\tau \\ &+ \rho_{i\tau}^* + \left(\frac{\eta_{yh}}{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon}\right)\epsilon_{i\tau}\Bigg]\end{aligned} \tag{B.4.4}$$

Comparing this to the result in equation (B.3.3), the reset price can be written as a weighted average of current and expected future profit-maximizing flexible prices:

$$\mathsf{R}_{it} = \sum_{\tau=t}^{\infty}\left(\frac{\beta^{\tau-t}\theta_{i,\tau-t}}{\sum_{j=0}^{\infty}\beta^j\theta_{ij}}\right)\mathbb{E}_t[\mathsf{P}_\tau + \hat{\rho}_{i\tau}] \tag{B.4.5}$$

By noting the definition of sequence $\{\vartheta_{ij}\}_{j=0}^{\infty}$ in (3.2.9), the result stated in the lemma is proved.

B.5 Proof of Lemma 5

The first step in establishing the result is to obtain a second-order approximation of the profit function for a single time period. Denote the profits made in period t by firm $(\imath,\jmath)$ in industry i with relative price $\varrho_t(\imath,\jmath)$ by $F_t(\imath,\jmath) \equiv \digamma(\varrho_t(\imath,\jmath);\varrho_{it},Y_t,A_{it},w_t(\imath),\varepsilon_{it})$. The profit function in (2.3.4) implies this can be written as

$$F_t(\imath,\jmath) = \varrho_t(\imath,\jmath)Y_t(\imath,\jmath) - (1-\mathfrak{s})\mathcal{C}(Y_t(\imath,\jmath);A_{it},w_t(\imath)) \tag{B.5.1}$$

where output $Y_t(\imath,\jmath)$ is obtained from the demand function (2.3.3).

A second-order accurate approximation of the real revenue component of (B.5.1) around a symmetric steady state ($\bar{\varrho}(\imath,\jmath)=1$) is given by:

$$\varrho_t(\imath,\jmath)Y_t(\imath,\jmath) = \bar{Y} + \bar{Y}\left(\rho_t(\imath,\jmath) + \mathsf{Y}_t(\imath,\jmath) + \frac{1}{2}(\rho_t(\imath,\jmath)+\mathsf{Y}_t(\imath,\jmath))^2\right) + \mathcal{O}(||\boldsymbol{\upsilon}_t||^3) \tag{B.5.2}$$

where $\mathcal{O}(||\boldsymbol{\upsilon}_t||^3)$ refers to third-order and higher terms in the exogenous disturbances $\boldsymbol{\upsilon}_t$. By taking logs of the demand function (2.3.3), the following exact expression is obtained:

$$\mathsf{Y}_t(\imath,\jmath) = -\varepsilon_{it}\rho_t(\imath,\jmath) - (\varepsilon-\varepsilon_{it})\rho_{it} + \mathsf{Y}_t \tag{B.5.3}$$

Note this expression includes the price elasticity of demand ε_{it}, which varies over time and across industries. The log-deviation of the (gross) markup is denoted by ϵ_{it}, so ε_{it} and ϵ_{it} are related by $\epsilon_{it} \equiv \log(\varepsilon_{it}/(\varepsilon_{it}-1)) - \log(\varepsilon/(\varepsilon-1))$. It follows that an approximate linear relationship between ε_{it} and ϵ_{it} is given by:

$$\varepsilon_{it} = \varepsilon - \varepsilon(\varepsilon-1)\epsilon_{it} + \mathcal{O}(||\boldsymbol{\upsilon}_t||^2) \tag{B.5.4}$$

Hence, a second-order accurate expression for demand $\mathsf{Y}_t(\imath,\jmath)$ can be obtained by combining (B.5.4) with

(B.5.3):

$$\mathsf{Y}_t(\imath,\jmath) = -\varepsilon\rho_t(\imath,\jmath) + \mathsf{Y}_t + \varepsilon(\varepsilon-1)(\rho_t(\imath,\jmath)-\rho_{it})\epsilon_{it} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.5.5}$$

By substituting (B.5.5) into (B.5.2) a second-order accurate expression for real revenue is obtained,

$$\varrho_t(\imath,\jmath)Y_t(\imath,\jmath) = \bar{Y}\left((1-\varepsilon)\rho_t(\imath,\jmath) + \frac{1}{2}((1-\varepsilon)\rho_t(\imath,\jmath)+\mathsf{Y}_t)^2 - \varepsilon(1-\varepsilon)\rho_t(\imath,\jmath)\epsilon_{it}\right) + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.5.6}$$

where "t.i.d." denotes terms that are independent of one individual firm's pricing decision such as ρ_{it} and Y_t.

The second component of the profit function (B.5.1) is total real cost. The total cost function is given in (2.3.2) and has the following second-order approximation,

$$\mathcal{C}(Y_t(\imath,\jmath);A_{it},w_t(\imath)) = \eta_{yh}\bar{Y}\left(\frac{1}{\eta_{yh}}\mathsf{Y}_t(\imath,\jmath) + \frac{1}{2}\left(\mathsf{w}_t(\imath) + \frac{1}{\eta_{yh}}(\mathsf{Y}_t(\imath,\jmath)-\mathsf{A}_{it})\right)^2\right) + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.5.7}$$

where the steady-state labour supply condition in (A.0.1) and efficiency condition (A.0.4) have been used to deduce that $\bar{w}(\bar{Y}/\bar{A})^{1/\eta_{yh}} = \eta_{yh}\bar{Y}$. By combining (B.5.7) with the approximation of the demand function in (B.5.5) the following is obtained:

$$\begin{aligned}\mathcal{C}(Y_t(\imath,\jmath);A_{it},w_t(\imath)) = \bar{Y}\Bigg(&-\varepsilon\rho_t(\imath,\jmath) + \frac{\eta_{yh}}{2}\left(\frac{1}{\eta_{yh}}(-\varepsilon\rho_t(\imath,\jmath)+\mathsf{Y}_t-\mathsf{A}_{it}) + \mathsf{w}_t(\imath)\right)^2 \\ &-\varepsilon(1-\varepsilon)\rho_t(\imath,\jmath)\epsilon_{it}\Bigg) + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3)\end{aligned} \tag{B.5.8}$$

A second-order accurate approximation of profits $F_t(\imath,\jmath)$ in (B.5.1) is obtained by combining the approximations of real revenue (B.5.6) and total real cost (B.5.8), and noting that the efficiency condition for the steady state described in appendix A implies that $1-\mathfrak{s} = (\varepsilon-1)/\varepsilon$:

$$\begin{aligned}F_t(\imath,\jmath) = \frac{\bar{Y}}{2}\Bigg(&((1-\varepsilon)\rho_t(\imath,\jmath)+\mathsf{Y}_t)^2 + \frac{\eta_{yh}(1-\varepsilon)}{\varepsilon}\left(\frac{1}{\eta_{yh}}(-\varepsilon\rho_t(\imath,\jmath)-\mathsf{A}_{it}+\mathsf{Y}_t) + \mathsf{w}_t(\imath)\right)^2 \\ &-2(1-\varepsilon)\rho_t(\imath,\jmath)\epsilon_{it}\Bigg) + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3)\end{aligned} \tag{B.5.9}$$

By expanding the brackets and collecting terms, the expression for profits in (B.5.9) can be simplified as follows:

$$\begin{aligned}F_t(\imath,\jmath) = \frac{(1-\varepsilon)\bar{Y}}{2}\rho_t(\imath,\jmath)\Bigg(&\left(1-\varepsilon+\frac{\varepsilon}{\eta_{yh}}\right)\rho_t(\imath,\jmath) - 2\left(\frac{1}{\eta_{yh}}-1\right)\mathsf{Y}_t \\ &+2\frac{1}{\eta_{yh}}\mathsf{A}_{it} - 2\mathsf{w}_t(\imath) - 2\epsilon_{it}\Bigg) + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3)\end{aligned} \tag{B.5.10}$$

Noting that all of Y_t, A_{it}, $\mathsf{w}_t(\imath)$ and ϵ_{it} are independent of the pricing decision of any single firm, equation

(B.5.10) can be rewritten as:

$$F_t(\imath,\jmath) = -\frac{\eta_{yh}(\varepsilon-1)\bar{Y}}{2(\eta_{yh}+(1-\eta_{yh})\varepsilon)}\Bigg(\left(\frac{\eta_{yh}+(1-\eta_{yh})\varepsilon}{\eta_{yh}}\right)\rho_t(\imath,\jmath) - \left(\frac{1}{\eta_{yh}}-1\right)\mathsf{Y}_t + \frac{1}{\eta_{yh}}\mathsf{A}_{it} - \mathsf{w}_t(\imath) - \epsilon_{it}\Bigg)^2 + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.5.11}$$

It is argued in section 2.4 that all firms using the same labour input $\imath$ will choose the same price, denoted by $\mathsf{P}_t(\imath)$ with corresponding relative price $\rho_t(\imath)$. This affects the determination of the real wage $\mathsf{w}_t(\imath)$, which is found by solving equations $\mathsf{w}_t(\imath) = \eta_{wh}\mathsf{H}_t(\imath) + \eta_{ic}\mathsf{Y}_t$, $\mathsf{Y}_t(\imath) = \mathsf{A}_{it} + \eta_{yh}\mathsf{H}_t(\imath)$ and $\mathsf{Y}_t(\imath) = -\varepsilon\rho_t(\imath) + \mathsf{Y}_t$ obtained from log-linearizations (A.0.5), (A.0.8) and (A.0.10):

$$\mathsf{w}_t(\imath) = -\frac{\varepsilon\eta_{wh}}{\eta_{yh}}\rho_t(\imath) - \frac{\eta_{wh}}{\eta_{yh}}\mathsf{A}_{it} + \left(\frac{\eta_{wh}}{\eta_{yh}} + \eta_{ic}\right)\mathsf{Y}_t + \mathcal{O}(||\boldsymbol{v}_t||^2) \tag{B.5.12}$$

Substituting the expression for the real wage in (B.5.12) into (B.5.11) produces a second-order approximation of the profits $F_t(\imath)$ made by a firm setting relative price $\rho_t(\imath)$:

$$F_t(\imath) = -\frac{\eta_{yh}(\varepsilon-1)\bar{Y}}{2(\eta_{yh}+(1-\eta_{yh})\varepsilon)}\Bigg(\left(\frac{\eta_{yh}+(1-\eta_{yh}+\eta_{wh})\varepsilon}{\eta_{yh}}\right)\rho_t(\imath) - \left(\frac{1-\eta_{yh}+\eta_{wh}+\eta_{yh}\eta_{ic}}{\eta_{yh}}\right)\mathsf{Y}_t + \left(\frac{1+\eta_{wh}}{\eta_{yh}}\right)\mathsf{A}_{it} - \epsilon_{it}\Bigg)^2 + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.5.13}$$

By comparing the equation above to the expression for the profit-maximizing flexible relative price $\hat{\rho}_{it}$ given in (B.3.2), equation (B.5.13) is equivalent to

$$F_t(\imath) = -\frac{\eta_{yh}(\varepsilon-1)\bar{Y}}{2(\eta_{yh}+(1-\eta_{yh})\varepsilon)\eta_\epsilon^2}(\rho_t(\imath) - \hat{\rho}_{it})^2 + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.5.14}$$

where η_ϵ is defined in (B.3.3).

When a firm in industry i is using a price at time τ chosen in period t, the level of profits is denoted by $F_{i,\tau|t} \equiv F(R_{it}/P_\tau; \varrho_{i\tau}, Y_\tau, A_{i\tau}, w_{i,\tau|t}, \varepsilon_{i\tau})$. From equation (B.5.14), the second-order approximation of this level of profits is:

$$F_{i,\tau|t} = -\frac{\eta_{yh}(\varepsilon-1)\bar{Y}}{2(\eta_{yh}+(1-\eta_{yh})\varepsilon)\eta_\epsilon^2}(\mathsf{R}_{it} - \mathsf{P}_\tau - \hat{\rho}_{it})^2 + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.5.15}$$

The function $\mathcal{F}_{it}$ defined in (3.2.6) represents the market value of profits earned by a firm in industry i setting a price at time t over the period for which the price is in use. This function is sum of current and expected future profits $F_{i,\tau|t}$ in (B.5.15):

$$\mathcal{F}_{it} \equiv \sum_{\tau=t}^{\infty}(\theta_{i,\tau-t}/\theta_{i0})\mathbb{E}_t[\mathfrak{M}_{\tau|t}F_{i,\tau|t}] \tag{B.5.16}$$

As the asset-pricing kernel $\mathfrak{M}_{\tau|t}$ satisfies $\mathfrak{M}_{\tau|t} = \beta^{\tau-t} + \mathcal{O}(||\boldsymbol{v}_t||)$, a second-order approximation of $\mathcal{F}_{it}$ can be obtained from equation (B.5.15) as follows:

$$\mathcal{F}_{it} = -\frac{\eta_{yh}(\varepsilon-1)\bar{Y}}{2(\eta_{yh}+(1-\eta_{yh})\varepsilon)\eta_\epsilon^2}\sum_{\tau=t}^{\infty}\left(\frac{\beta^{\tau-t}\theta_{i,\tau-t}}{\sum_{j=0}^{\infty}\beta^j\theta_{ij}}\right)\mathbb{E}_t[(\mathsf{R}_{it} - \mathsf{P}_\tau - \hat{\rho}_{i\tau})^2] + \text{t.i.d.} + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{B.5.17}$$

Comparing this with the definition of the sequence $\{\vartheta_{ij}\}_{j=0}^{\infty}$ in (3.2.9) confirms that the statement of the lemma is true.

B.6 Proof of Lemma 6

The $n \times n$ matrix polynomial $\boldsymbol{\chi}(z)$ introduced in (C.1.11) is constructed as follows,

$$\boldsymbol{\chi}(z) \equiv \boldsymbol{\Omega}\boldsymbol{\phi}^{-1}\boldsymbol{\Upsilon}(z) + \boldsymbol{\Omega}\boldsymbol{\mathcal{R}} \tag{B.6.1}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\Upsilon}(z)$ are defined in (C.1.7), and the matrix $\boldsymbol{\mathcal{R}}$ in (C.1.9). The matrix polynomial $\boldsymbol{\Upsilon}(z)$ is itself defined in terms of $\boldsymbol{\Phi}(z)$, which is a diagonal matrix polynomial built up from the sequences of recursive coefficients $\{\phi_{ij}\}_{j=1}^{m+1}$ in (2.4.7). From (C.1.1) and (C.1.3), $\boldsymbol{\Phi}(z)$ has degree $m+1$,

$$\boldsymbol{\Phi}(z) = \mathbf{I} - \sum_{j=1}^{m+1} \boldsymbol{\Phi}_j z^j \quad , \quad \boldsymbol{\Phi}_j \equiv \mathrm{diag}\{\phi_{1j}, \ldots, \phi_{nj}\} \tag{B.6.2}$$

and all the $n \times n$ matrices $\boldsymbol{\Phi}_j$ are diagonal. Since $\boldsymbol{\Phi}(z)$ has degree $m+1$, equation (C.1.7) implies that $\boldsymbol{\Upsilon}(z)$ must have $m+1$ positive and negative powers of z:

$$\boldsymbol{\Upsilon}(z) = \sum_{j=-(m+1)}^{m+1} \boldsymbol{\Upsilon}_j z^j \tag{B.6.3}$$

All the $\boldsymbol{\Upsilon}_j$ matrices are diagonal because all the $\boldsymbol{\Phi}_j$ in (B.6.2) are diagonal. Using its definition in (C.1.7), the matrix polynomial $\boldsymbol{\Upsilon}(z)$ has the following property,

$$\boldsymbol{\Upsilon}(\beta z^{-1}) = \boldsymbol{\Phi}(\beta z^{-1})\boldsymbol{\Phi}(\beta(\beta z^{-1})^{-1}) - \boldsymbol{\Phi}(1)\boldsymbol{\Phi}(\beta) = \boldsymbol{\Phi}(z)\boldsymbol{\Phi}(\beta z^{-1}) - \boldsymbol{\Phi}(1)\boldsymbol{\Phi}(\beta) = \boldsymbol{\Upsilon}(z) \tag{B.6.4}$$

which since $\boldsymbol{\Upsilon}(z) = \boldsymbol{\Upsilon}(z)'$ means that $\boldsymbol{\Upsilon}(z)$ is discounted para-Hermitian, that is, $\boldsymbol{\Upsilon}(z) = \boldsymbol{\Upsilon}(\beta z^{-1})'$. Comparing (B.6.4) with (B.6.3), it follows that $\boldsymbol{\Upsilon}_{-j} = \beta^j \boldsymbol{\Upsilon}_j$. Therefore, $\boldsymbol{\Upsilon}(z)$ can be written as:

$$\boldsymbol{\Upsilon}(z) = \sum_{j=1}^{m+1} \boldsymbol{\Upsilon}_j z^j + \boldsymbol{\Upsilon}_0 + \sum_{j=1}^{m+1} \beta^j \boldsymbol{\Upsilon}_j z^{-j} \tag{B.6.5}$$

From the definition of $\boldsymbol{\chi}(z)$ in (B.6.1), equation (B.6.5) implies that $\boldsymbol{\chi}(z)$ also has $m+1$ positive and negative powers of z:

$$\boldsymbol{\chi}(z) = \sum_{j=1}^{m+1} \boldsymbol{\chi}_j z^j + \boldsymbol{\chi}_0 + \sum_{j=1}^{m+1} \beta^j \boldsymbol{\chi}_j z^{-j} \quad , \quad \boldsymbol{\chi}_j = \begin{cases} \boldsymbol{\Omega}\boldsymbol{\phi}^{-1}\boldsymbol{\Upsilon}_0 + \boldsymbol{\Omega}\boldsymbol{\mathcal{R}} & \text{if } j = 0 \\ \boldsymbol{\Omega}\boldsymbol{\phi}^{-1}\boldsymbol{\Upsilon}_j & \text{if } j = 1, 2, \ldots, m+1 \end{cases} \tag{B.6.6}$$

The formula for $\boldsymbol{\mathcal{R}}$ in (C.1.9) implies that $\boldsymbol{\Omega}\boldsymbol{\mathcal{R}} = \boldsymbol{\Omega} - \boldsymbol{\omega}\boldsymbol{\omega}'$, which is a symmetric matrix since $\boldsymbol{\Omega}$ is diagonal. Furthermore, $\boldsymbol{\phi}$ is diagonal, as are all of $\boldsymbol{\Upsilon}_j$. This means that all the $\boldsymbol{\chi}_j$ matrices in (B.6.6) are symmetric. Hence, the matrix polynomial $\boldsymbol{\chi}(z)$ also has the discounted para-Hermitian property

$$\boldsymbol{\chi}(z) = \boldsymbol{\chi}(\beta z^{-1})' \tag{B.6.7}$$

for all $z \in \mathbb{C}\backslash\{0\}$ that is claimed in the lemma.

In proving the second and third parts of the proposition it is useful to introduce the following $n \times n$ matrix polynomial $\boldsymbol{\mathcal{Q}}(z)$:

$$\boldsymbol{\mathcal{Q}}(z) \equiv \boldsymbol{\Theta}(z)\boldsymbol{\Theta}(\beta z^{-1})\boldsymbol{\Theta}(1)^{-1}\boldsymbol{\Theta}(\beta)^{-1} \tag{B.6.8}$$

$\boldsymbol{\mathcal{Q}}(z)$ is stated in terms of $\boldsymbol{\Theta}(z)$, which is itself defined in (C.1.3), and from (C.1.1) it can be seen to have infinitely many positive powers of z in general, each having a diagonal matrix as its coefficient:

$$\boldsymbol{\Theta}(z) = \sum_{j=0}^{\infty} \boldsymbol{\Theta}_j z^j \quad , \quad \boldsymbol{\Theta}_j \equiv \operatorname{diag}\{\theta_{1j}, \ldots, \theta_{nj}\} \tag{B.6.9}$$

It follows from (B.6.8) that $\boldsymbol{\mathcal{Q}}(z)$ has infinitely many positive and negative powers of z, and each again has a diagonal matrix coefficient:

$$\boldsymbol{\mathcal{Q}}(z) \equiv \sum_{j \to -\infty}^{\infty} \boldsymbol{\mathcal{Q}}_j z^j \tag{B.6.10}$$

From its definition (B.6.8) and the fact that $\boldsymbol{\Theta}(z)$ is a diagonal matrix polynomial, it is clear that $\boldsymbol{\mathcal{Q}}(z) = \boldsymbol{\mathcal{Q}}(\beta z^{-1})$, and so from (B.6.10) that $\boldsymbol{\mathcal{Q}}_{-j} = \beta^j \boldsymbol{\mathcal{Q}}_j$. It follows that $\boldsymbol{\mathcal{Q}}(z)$ can be written in the form:

$$\boldsymbol{\mathcal{Q}}(z) = \boldsymbol{\mathcal{Q}}_0 + \sum_{j=1}^{\infty} \boldsymbol{\mathcal{Q}}_j (z^j + \beta^j z^{-j}) \tag{B.6.11}$$

As each sequence $\{\theta_{ij}\}_{j=0}^{\infty}$ is a probability distribution, all the matrices $\boldsymbol{\Theta}_j$ in (B.6.9) are positive semi-definite, $\boldsymbol{\Theta}(1) = \mathbf{I}$ and $\boldsymbol{\Theta}(\beta)$ is a positive definite matrix. Moreover, equation (2.4.5) implies that $\boldsymbol{\Theta}_0$ and $\boldsymbol{\Theta}_1$ are positive definite. Hence, all the matrices $\boldsymbol{\mathcal{Q}}_j$ are positive semi-definite and $\boldsymbol{\mathcal{Q}}_1$ is positive definite. From its definition (B.6.8) it is apparent that $\boldsymbol{\mathcal{Q}}(1) = \mathbf{I}$, so (B.6.11) implies that the matrix $\boldsymbol{\mathcal{Q}}_0$ is given by:

$$\boldsymbol{\mathcal{Q}}_0 = \mathbf{I} - \sum_{j=1}^{\infty} \boldsymbol{\mathcal{Q}}_j (1 + \beta^j) \tag{B.6.12}$$

Now consider the matrix polynomial $\boldsymbol{\mathcal{Q}}(z)$ evaluated at some specific value of $z \in \mathbb{C}\backslash\{0\}$. Write this number in polar form as $z = |z|e^{\mathrm{i}\varpi}$, where $\mathrm{i} \equiv \sqrt{-1}$, and $|z|$ and ϖ are the modulus and argument of z respectively. Since $e^{\mathrm{i}\varpi} = \cos\varpi + \mathrm{i}\sin\varpi$, the evaluated matrix polynomial $\boldsymbol{\mathcal{Q}}(|z|e^{\mathrm{i}\varpi})$ can be split into real and imaginary parts using (B.6.11):

$$\boldsymbol{\mathcal{Q}}(|z|e^{\mathrm{i}\varpi}) = \left(\boldsymbol{\mathcal{Q}}_0 + \sum_{j=1}^{\infty} \boldsymbol{\mathcal{Q}}_j (|z|^j + \beta^j |z|^{-j}) \cos(j\varpi) \right) + \mathrm{i} \left(\sum_{j=1}^{\infty} \boldsymbol{\mathcal{Q}}_j (|z|^j - \beta^j |z|^{-j}) \sin(j\varpi) \right) \tag{B.6.13}$$

Using the definition of $\boldsymbol{\Upsilon}(z)$ in (C.1.7) and the link between $\boldsymbol{\Theta}(z)$ and $\boldsymbol{\Phi}(z)$ in (C.1.4), the polynomial $\boldsymbol{\Upsilon}(z)$ can be expressed in terms of $\boldsymbol{\mathcal{Q}}(z)$ and $\boldsymbol{\phi}$ using the definitions (C.1.7) and (B.6.8):

$$\boldsymbol{\Upsilon}(z) = \boldsymbol{\phi}(\boldsymbol{\mathcal{Q}}(z)^{-1} - \mathbf{I}) \tag{B.6.14}$$

From the equations (B.6.1) and (C.1.9) defining $\boldsymbol{\chi}(z)$ and $\boldsymbol{\mathcal{R}}$, the expression for $\boldsymbol{\Upsilon}(z)$ in (B.6.14) can be used to provide two alternative representations of $\boldsymbol{\chi}(z)$:

$$\boldsymbol{\chi}(z) = \boldsymbol{\Omega}\boldsymbol{\mathcal{Q}}(z)^{-1}(\mathbf{I} - \boldsymbol{\mathcal{Q}}(z)) + \boldsymbol{\Omega}\boldsymbol{\mathcal{R}} \quad , \quad \boldsymbol{\chi}(z) = \boldsymbol{\Omega}\boldsymbol{\mathcal{Q}}(z)^{-1} - \boldsymbol{\omega}\boldsymbol{\omega}' \tag{B.6.15}$$

Attention is now turned to establishing that $\boldsymbol{\chi}(z)$ is positive definite if $|z| = \sqrt{\beta}$. The matrix $\boldsymbol{\Omega}$ is obviously positive definite since $\omega_i > 0$ for all i. Using (C.1.7) and (C.1.4) the diagonal matrix $\boldsymbol{\phi}$ is equal to

$$\boldsymbol{\phi} = (\boldsymbol{\Theta}(0)\boldsymbol{\Theta}(1)^{-1})(\boldsymbol{\Theta}(0)\boldsymbol{\Theta}(\beta)^{-1}) \tag{B.6.16}$$

The matrices $\boldsymbol{\Theta}(1) = \mathbf{I}$ and $\boldsymbol{\Theta}(\beta)$ are known to be positive definite. Since $\theta_{i0} > 0$, so is $\boldsymbol{\Theta}(0)$, implying that $\boldsymbol{\phi}$ too is positive definite. Next observe that $\boldsymbol{\mathcal{R}}$ in (C.1.9) is idempotent,

$$\boldsymbol{\mathcal{R}}^2 = (\mathbf{I} - \boldsymbol{\iota}\boldsymbol{\omega}')^2 = \mathbf{I} - \boldsymbol{\iota}\boldsymbol{\omega}' - \boldsymbol{\iota}\boldsymbol{\omega}' + \boldsymbol{\iota}(\boldsymbol{\omega}'\boldsymbol{\iota})\boldsymbol{\omega}' = \mathbf{I} - \boldsymbol{\iota}\boldsymbol{\omega}' = \boldsymbol{\mathcal{R}} \tag{B.6.17}$$

because $\boldsymbol{\omega}'\boldsymbol{\iota} = 1$. Any idempotent matrix is automatically positive semi-definite.

The real component of $\boldsymbol{\mathcal{Q}}(z)$ is denoted by $\Re(\boldsymbol{\mathcal{Q}}(z))$, and can be obtained from (B.6.12) and (B.6.13):

$$\Re(\boldsymbol{\mathcal{Q}}(z)) = \mathbf{I} - \sum_{j=1}^{\infty}\{(1+\beta^j) - (|z|^j + \beta^j|z|^{-j})\cos(j\varpi)\}\boldsymbol{\mathcal{Q}}_j \tag{B.6.18}$$

Because $\cos(j\varpi) \leq 1$ for all ϖ, and each matrix $\boldsymbol{\mathcal{Q}}_j$ is positive semi-definite, a matrix inequality for the real component of $\boldsymbol{\mathcal{Q}}(z)$ can be derived,

$$\Re(\boldsymbol{\mathcal{Q}}(z)) \leqq \mathbf{I} - \sum_{j=1}^{\infty}(1-|z|^j)(1-\beta^j|z|^{-j})\boldsymbol{\mathcal{Q}}_j \tag{B.6.19}$$

where $\leqq$ signifies that the right-hand side is equal to the left-hand side plus some positive semi-definite matrix. The imaginary component of $\boldsymbol{\mathcal{Q}}(z)$, denoted by $\Im(\boldsymbol{\mathcal{Q}}(z))$, is obtained in a similar way from (B.6.13):

$$\Im(\boldsymbol{\mathcal{Q}}(z)) = \sum_{j=1}^{\infty}(|z|^j - \beta^j|z|^{-j})\sin(j\varpi)\boldsymbol{\mathcal{Q}}_j \tag{B.6.20}$$

Now it is shown that $\boldsymbol{\mathcal{Q}}(z)$ is positive definite on the circle in the complex plane with radius $\sqrt{\beta}$. Set $|z| = \sqrt{\beta}$ and consider an arbitrary ϖ. The restriction on the modulus of z implies that $|z|^j = \beta^j|z|^{-j} = \beta^{j/2}$, which from the expression for the imaginary component of $\boldsymbol{\mathcal{Q}}(z)$ in (B.6.20) means that $\Im(\boldsymbol{\mathcal{Q}}(z)) = \mathbf{0}$. Furthermore, since $\beta^{j/2} < 1$ for all $j = 1, 2, \ldots$, and all $\boldsymbol{\mathcal{Q}}_j$ are positive semi-definite with at least $\boldsymbol{\mathcal{Q}}_1$ being positive definite, the inequality in (B.6.19) implies that $\Re(\mathbf{I} - \boldsymbol{\mathcal{Q}}(z)) \gg \mathbf{0}$, that is, the real component of $\mathbf{I} - \boldsymbol{\mathcal{Q}}(z)$ is a positive definite matrix. Since the imaginary component of $\boldsymbol{\mathcal{Q}}(z)$ is zero, $\Im(\mathbf{I} - \boldsymbol{\mathcal{Q}}(z)) = \mathbf{0}$. As $\mathbf{I} - \boldsymbol{\mathcal{Q}}(z) = \Re(\mathbf{I} - \boldsymbol{\mathcal{Q}}(z)) + \mathrm{i}\Im(\mathbf{I} - \boldsymbol{\mathcal{Q}}(z))$, the matrix $\mathbf{I} - \boldsymbol{\mathcal{Q}}(z)$ is positive definite whenever $|z| = \sqrt{\beta}$.

Next, note that the definition of $\boldsymbol{\mathcal{Q}}(z)$ in (B.6.8) and the relationship between $\boldsymbol{\Theta}(z)$ and $\boldsymbol{\Phi}(z)$ given in (C.1.4) imply that the inverse of the matrix polynomial $\boldsymbol{\mathcal{Q}}(z)$ is equal to:

$$\boldsymbol{\mathcal{Q}}(z)^{-1} = \boldsymbol{\Phi}(z)\boldsymbol{\Phi}(\beta z^{-1}) \tag{B.6.21}$$

Let $z^\dagger$ denote the complex conjugate of z, so $z^\dagger = |z|e^{-\mathrm{i}\varpi}$. The inverse of z can be written in terms of the complex conjugate as $z^{-1} = (1/|z|^2)z^\dagger$. So when $|z| = \sqrt{\beta}$, βz^{-1} is equal to the complex conjugate $z^\dagger$. Hence, using (B.6.21) and the definition of the diagonal matrix polynomial $\boldsymbol{\Phi}(z)$ in (C.1.3):

$$\boldsymbol{\mathcal{Q}}(z)^{-1} = \boldsymbol{\Phi}(z)\boldsymbol{\Phi}(z^\dagger) = \mathrm{diag}\{\phi_1(z)\phi_1(z^\dagger), \ldots, \phi_n(z)\phi_n(z^\dagger)\} \tag{B.6.22}$$

Because the coefficients in the polynomial $\phi_i(z)$ defined in (C.1.1) are real numbers, $\phi_i(z^\dagger) = \phi_i(z)^\dagger$. And note that the product of any number with its conjugate, such as $\phi_i(z)\phi_i(z)^\dagger$, is non-negative and only equal to zero when $\phi_i(z) = 0$. This case can be ruled out by noting that $|z| = \sqrt{\beta} < 1$, which would imply that $\phi_i(z)$ has a root strictly inside the unit circle if $\phi_i(z) = 0$. But it is known that $\phi_i(z)$ has all its roots strictly outside the unit circle. So by putting these results together it is deduced that $\phi_i(z)\phi_i(z^\dagger) > 0$ if $|z| = \sqrt{\beta}$, and thus the diagonal matrix $\boldsymbol{\mathcal{Q}}(z)^{-1}$ in (B.6.22) is positive definite.

Therefore, in summary, all of $\boldsymbol{\Omega}$, $\boldsymbol{\mathcal{Q}}(z)^{-1}$ and $\mathbf{I} - \boldsymbol{\mathcal{Q}}(z)$ are positive definite when $|z| = \sqrt{\beta}$. As $\boldsymbol{\mathcal{R}}$ is positive semi-definite, the first expression for $\boldsymbol{\chi}(z)$ given in (B.6.15) implies that $\boldsymbol{\chi}(z)$ is a positive definite matrix when z lies on the circle in the complex plane with radius $\sqrt{\beta}$.

The next claim in the lemma to be checked is that no root of $\boldsymbol{\chi}(z)$ has modulus strictly between β and 1. Let $\zeta \in \mathbb{C}$ be any root of the equation $|\boldsymbol{\chi}(z)| = 0$. By definition, there must exist a non-zero vector $\mathbf{v} \in \mathbb{C}^n$ such that $\boldsymbol{\chi}(\zeta)\mathbf{v} = \mathbf{0}$. The vector $\mathbf{v}$ is referred to as the nullspace vector associated with the root ζ. In what follows, it is supposed for contradiction that $\beta < |\zeta| < 1$.

A necessary condition for ζ to be root of $\boldsymbol{\chi}(z)$ is now derived in the case where $\beta \leq |\zeta| \leq 1$, which obviously includes the case just introduced. Using the weights in the vector $\boldsymbol{\omega}$, construct a weighted average of the elements of $\mathbf{v}$, denoted by $\bar{\mathrm{v}} \equiv \boldsymbol{\omega}'\mathbf{v}$. Using this definition, the second expression for $\boldsymbol{\chi}(z)$ in (B.6.15) implies that $\boldsymbol{\chi}(\zeta)\mathbf{v} = \mathbf{0}$ is equivalent to:

$$\boldsymbol{\Omega}\boldsymbol{\mathcal{Q}}(\zeta)^{-1}\mathbf{v} = \bar{\mathrm{v}}\boldsymbol{\omega} \tag{B.6.23}$$

Premultiplying (B.6.23) by $\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\Omega}^{-1}$ yields $\mathbf{v} = \bar{\mathrm{v}}\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota}$, and then averaging both sides using the weights in $\boldsymbol{\omega}$ implies:

$$\bar{\mathrm{v}}\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota} = \bar{\mathrm{v}} \tag{B.6.24}$$

First, consider the possibility that $\bar{\mathrm{v}}$ might be zero. By setting $\bar{\mathrm{v}} = 0$ in (B.6.23) and using (B.6.21) this entails:

$$\boldsymbol{\Phi}(\zeta)\boldsymbol{\Phi}(\beta\zeta^{-1})\mathbf{v} = \mathbf{0} \tag{B.6.25}$$

Since $\mathbf{v} \neq \mathbf{0}$ and the matrix polynomial $\boldsymbol{\Phi}(z)$ in (C.1.3) is diagonal, it is clear that there is some $i = 1, \ldots, n$ such that $v_i \neq 0$ and $\phi_i(\zeta)\phi_i(\beta\zeta^{-1})v_i = 0$. Hence, it must be the case that either $\phi_i(\zeta) = 0$ or $\phi_i(\beta\zeta^{-1}) = 0$. But as $\beta \leq |\zeta| \leq 1$, both $|\zeta| \leq 1$ and $|\beta\zeta^{-1}| \leq 1$, so $\phi_i(z)$ would have to have a root on or inside the unit circle. But it is known that $\phi_i(z)$ has no such root. Therefore, the case of $\bar{\mathrm{v}} = 0$ can be ruled out, and when $\bar{\mathrm{v}}$ is non-zero, it can be cancelled from both sides of equation (B.6.24):

$$\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota} = 1 \tag{B.6.26}$$

The earlier inequality for $\Re(\boldsymbol{\mathcal{Q}}(\zeta))$ in (B.6.19) implies that:

$$\Re(\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota}) \leq 1 - \sum_{j=1}^{\infty} \boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}_j\boldsymbol{\iota}(1 - |\zeta|^j)(1 - \beta^j|\zeta|^{-j}) \tag{B.6.27}$$

Since $\beta < |\zeta| < 1$, it is clear that $1 - |\zeta|^j > 0$ and $1 - \beta|\zeta|^{-j} > 0$ for all $j = 1, 2, \ldots$. And as each element of the diagonal matrix $\boldsymbol{\mathcal{Q}}_j$ is non-negative, and each diagonal element of $\boldsymbol{\mathcal{Q}}_1$ is strictly positive, it follows that $\Re(\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota}) < 1$. But equation (B.6.26) shows that $\Re(\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota}) = 1$ is a necessary condition for ζ to be a root. This contradiction of the original supposition shows that there is no root ζ of $\boldsymbol{\chi}(z)$ with $\beta < |\zeta| < 1$.

It is now shown that both $z = 1$ and $z = \beta$ are roots of $\boldsymbol{\chi}(z)$. Note that the definition of $\boldsymbol{\Upsilon}(z)$ in (C.1.7) implies that $\boldsymbol{\Upsilon}(1) = \boldsymbol{\Upsilon}(\beta) = \mathbf{0}$. Next, observe from the definition of $\boldsymbol{\mathcal{R}}$ in (C.1.9) that $\boldsymbol{\mathcal{R}}\boldsymbol{\iota} = \mathbf{0}$. It follows that $\boldsymbol{\chi}(1)\boldsymbol{\iota} = \boldsymbol{\chi}(\beta)\boldsymbol{\iota} = \mathbf{0}$, establishing that both $z = 1$ and $z = \beta$ are always roots of $\boldsymbol{\chi}(z)$ with corresponding nullspace vector $\boldsymbol{\iota}$.

Now suppose there exists a root with modulus $|\zeta| = 1$ or $|\zeta| = \beta$ exactly, but $\zeta \neq 1$ and $\zeta \neq \beta$, so ζ is not a positive real number. Note that in either case, $|\zeta|^j + \beta^j|\zeta|^{-j} = 1 + \beta^j$ for all $j = 1, 2, \ldots$, so using the expression for $\Re(\boldsymbol{\mathcal{Q}}(z))$ in (B.6.18), $\Re(\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota})$ can be written as follows:

$$\Re(\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota}) = 1 - \sum_{j=1}^{\infty}(1 + \beta^j)(1 - \cos(j\varpi))\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}_j\boldsymbol{\iota} \tag{B.6.28}$$

As ζ is not a positive real number, the argument ϖ of ζ must be such that $\cos(\varpi) < 1$. Each diagonal matrix $\boldsymbol{\mathcal{Q}}_j$ is positive semi-definite, and at least $\boldsymbol{\mathcal{Q}}_1$ is positive definite. It follows that $\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}_1\boldsymbol{\iota} > 0$, and $(1 - \cos(j\varpi))\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}_j\boldsymbol{\iota} \geq 0$ for all $j = 2, 3, \ldots$. So it must be the case that $\Re(\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota} < 1$. But since

$\beta \leq |\zeta| \leq 1$, the necessary condition (B.6.26) for ζ to be a root requires $\Re(\boldsymbol{\omega}'\boldsymbol{\mathcal{Q}}(\zeta)\boldsymbol{\iota}) = 1$. Thus $z = 1$ and $z = \beta$ are the only roots of $\boldsymbol{\chi}(z)$ with modulus 1 or β exactly.

Finally, it is shown that the roots of $\boldsymbol{\chi}(z)$ $z = 1$ and $z = \beta$ have multiplicity one. To do this, denote the determinant of matrix polynomial $\boldsymbol{\chi}(z)$ by $\mathscr{D}(z) \equiv |\boldsymbol{\chi}(z)|$. The expression for $\boldsymbol{\chi}(z)$ in (C.1.11) means that the determinant can be written as follows:

$$\mathscr{D}(z) = |\boldsymbol{\Omega}| \left|\boldsymbol{\phi}^{-1}\boldsymbol{\Upsilon}(z) + \boldsymbol{\mathcal{R}}\right| \tag{B.6.29}$$

As both $z = 1$ and $z = \beta$ are roots, $\mathscr{D}(1)$ and $\mathscr{D}(\beta)$ are necessarily zero. The multiplicity of the roots can be determined by calculating the first derivative of $\mathscr{D}(z)$ and evaluating it at $z = 1$ and $z = \beta$.

To calculate the derivative of $\mathscr{D}(z)$ in (B.6.29), note that $\boldsymbol{\phi}^{-1}\boldsymbol{\Upsilon}(z)$ is a diagonal matrix polynomial, and the matrix $\boldsymbol{\mathcal{R}}$ is independent of z. Using the formula for the derivative of a determinant, $\mathscr{D}'(z)$ is equal to

$$\mathscr{D}'(z) = |\boldsymbol{\Omega}| \sum_{i=1}^{n} \left| \left[\boldsymbol{\phi}^{-1}\boldsymbol{\Upsilon}(z) + \boldsymbol{\mathcal{R}}\right]_{-i} \right| \left[\boldsymbol{\phi}^{-1}\boldsymbol{\Upsilon}'(z)\right]_{i} \tag{B.6.30}$$

where $[\cdot]_{-i}$ denotes the matrix obtained after deleting the i-th row and i-th column, and $[\cdot]_i$ denotes the (i,i)-th element of the matrix. From definition (C.1.7), the matrix polynomial $\boldsymbol{\Upsilon}(z)$ satisfies $\boldsymbol{\Upsilon}(1) = \mathbf{0}$, so hence the derivative $\mathscr{D}'(z)$ in (B.6.30) evaluated at $z = 1$ is equal to:

$$\mathscr{D}'(1) = |\boldsymbol{\Omega}| \sum_{i=1}^{n} \left|[\boldsymbol{\mathcal{R}}]_{-i}\right| \left[\boldsymbol{\phi}^{-1}\boldsymbol{\Upsilon}'(1)\right]_{i} \tag{B.6.31}$$

The matrix $\boldsymbol{\mathcal{R}}$ has been shown to be idempotent in (B.6.17), and thus is positive semi-definite. A consequence of this is that all the determinants of the principal submatrices of $\boldsymbol{\mathcal{R}}$ must be non-negative, which means that $\left|[\boldsymbol{\mathcal{R}}]_{-i}\right| \geq 0$. Now suppose one of these determinants is zero. There must then exist a non-zero vector $\mathbf{v}_{-i} \in \mathbb{R}^{n-1}$ such that $[\boldsymbol{\mathcal{R}}]_{-i}\,\mathbf{v}_{-i} = \mathbf{0}$. The definition of $\boldsymbol{\mathcal{R}}$ in (C.1.9) implies that $[\boldsymbol{\mathcal{R}}]_{-i}$ can be written as,

$$[\boldsymbol{\mathcal{R}}]_{-i} = \mathbf{I} - \boldsymbol{\iota}[\boldsymbol{\omega}]'_{-i} \tag{B.6.32}$$

where $[\boldsymbol{\omega}]_{-i}$ denotes the vector $\boldsymbol{\omega}$ with the i-th element removed. So if $[\boldsymbol{\mathcal{R}}]_{-i}\,\mathbf{v}_{-i} = \mathbf{0}$ then (B.6.32) implies $\mathbf{v}_{-i} = ([\boldsymbol{\omega}]'_{-i}\mathbf{v}_{-i})\boldsymbol{\iota}$. Consequently, as the vector $\mathbf{v}_{-i}$ is non-zero, it follows that $[\boldsymbol{\omega}]'_{-i}\mathbf{v}_{-i} \neq 0$. Premultiplying by $[\boldsymbol{\omega}]'_{-i}$ and cancelling the non-zero factor $[\boldsymbol{\omega}]'_{-i}\mathbf{v}_{-i}$ from both sides implies that $[\boldsymbol{\omega}]'_{-i}\boldsymbol{\iota} = 1$. But $[\boldsymbol{\omega}]'_{-i}\boldsymbol{\iota}$ is obviously less than 1 because:

$$[\boldsymbol{\omega}]'_{-i}\,\boldsymbol{\iota} = \left(\sum_{j=1}^{n} \omega_j\right) - \omega_i = 1 - \omega_i < 1 \tag{B.6.33}$$

Therefore, the null space of $[\boldsymbol{\mathcal{R}}]_{-i}$ is empty for all i. This means that the determinant $\left|[\boldsymbol{\mathcal{R}}]_{-i}\right|$ is non-zero. Hence, as this determinant has also been shown to be non-negative, $\left|[\boldsymbol{\mathcal{R}}]_{-i}\right| > 0$ for all i.

The derivative of the matrix polynomial $\boldsymbol{\Upsilon}(z)$ defined in (C.1.7) is given by

$$\boldsymbol{\Upsilon}'(z) = \boldsymbol{\Phi}'(z)\boldsymbol{\Phi}(\beta z^{-1}) - \beta z^{-2}\boldsymbol{\Phi}(z)\boldsymbol{\Phi}'(\beta z^{-1}) \tag{B.6.34}$$

and the derivative $\boldsymbol{\Phi}'(z)$ can be written in terms of $\boldsymbol{\Theta}(z)$ using (C.1.4):

$$\boldsymbol{\Phi}'(z) = -\boldsymbol{\Theta}(0)\boldsymbol{\Theta}'(z)\boldsymbol{\Theta}(z)^{-2} \tag{B.6.35}$$

Equation (B.6.35) is substituted into (B.6.34) to get an expression for the derivative $\boldsymbol{\Upsilon}'(z)$:

$$\boldsymbol{\Upsilon}'(z) = -\left(\boldsymbol{\Theta}(0)\boldsymbol{\Theta}(z)^{-1}\right)\left(\boldsymbol{\Theta}(0)\boldsymbol{\Theta}(\beta z^{-1})^{-1}\right)\left\{\left(\boldsymbol{\Theta}'(z)\boldsymbol{\Theta}(z)^{-1}\right) - \beta z^{-2}\left(\boldsymbol{\Theta}'(\beta z^{-1})\boldsymbol{\Theta}(\beta z^{-1})^{-1}\right)\right\} \tag{B.6.36}$$

By defining a new matrix polynomial $\mathfrak{T}(z)$,

$$\mathfrak{T}(z) \equiv z\boldsymbol{\Theta}'(z)\boldsymbol{\Theta}(z)^{-1} \tag{B.6.37}$$

and by using the definitions in (C.1.7) and (C.1.4), the expression for $\boldsymbol{\Upsilon}'(z)$ in (B.6.36) evaluated at $z = 1$ can be simplified as follows:

$$\boldsymbol{\Upsilon}'(1) = -\boldsymbol{\phi}\left\{\mathfrak{T}(1) - \mathfrak{T}(\beta)\right\} \tag{B.6.38}$$

An expression for the derivative of the determinant $\mathscr{D}(z)$ evaluated at $z = 1$ is thus obtained from (B.6.31) and (B.6.38):

$$\mathscr{D}'(1) = -\sum_{i=1}^{n}\left|[\boldsymbol{\mathcal{R}}]_{-i}\right|[\mathfrak{T}(1) - \mathfrak{T}(\beta)]_i \tag{B.6.39}$$

The relationship between the terms $\mathfrak{T}(1)$ and $\mathfrak{T}(\beta)$ in (B.6.38) is studied by computing the derivative of $\mathfrak{T}(z)$, using its definition in (B.6.37):

$$\mathfrak{T}'(z) = z^{-1}\left\{z\boldsymbol{\Theta}'(z)\boldsymbol{\Theta}(z)^{-1} + z^2\boldsymbol{\Theta}''(z)\boldsymbol{\Theta}(z)^{-1} - \left(z\boldsymbol{\Theta}'(z)\boldsymbol{\Theta}(z)^{-1}\right)^2\right\} \tag{B.6.40}$$

Introduce the following sequences of matrix functions $\mathfrak{N}_j(z)$ specified as follows

$$\mathfrak{N}_j(z) \equiv z^j\boldsymbol{\Theta}_j\left(\sum_{k=0}^{\infty} z^k\boldsymbol{\Theta}_k\right)^{-1} \tag{B.6.41}$$

where the matrices $\boldsymbol{\Theta}_j$ are drawn from equation (B.6.9), and the sequence is well defined for $0 < z \leq 1$. Each $n \times n$ matrix $\mathfrak{N}_j(z)$ is thus diagonal and positive semi-definite. The sum of $\mathfrak{N}_j(z)$ for $j = 0, 1, \ldots$ is the identity matrix $\mathbf{I}$ for any value of z. The expressions appearing in the formula for the derivative $\mathfrak{T}'(z)$ in (B.6.40) can be stated in terms of the infinite series $\{\mathfrak{N}_j(z)\}_{j=0}^{\infty}$ from (B.6.41) by noting that (B.6.9) implies $\mathfrak{N}_j(z) = z^j\boldsymbol{\Theta}_j\boldsymbol{\Theta}(z)^{-1}$:

$$z\boldsymbol{\Theta}'(z)\boldsymbol{\Theta}(z)^{-1} = \sum_{j=0}^{\infty} j\mathfrak{N}_j(z) \quad , \quad z\boldsymbol{\Theta}'(z)\boldsymbol{\Theta}(z)^{-1} + z^2\boldsymbol{\Theta}''(z)\boldsymbol{\Theta}(z)^{-1} = \sum_{j=0}^{\infty} j^2\mathfrak{N}_j(z) \tag{B.6.42}$$

By substituting these into equation (B.6.40),

$$\mathfrak{T}'(z) = z^{-1}\left\{\sum_{j=0}^{\infty} j^2\mathfrak{N}_j(z) - \left(\sum_{j=0}^{\infty} j\mathfrak{N}_j(z)\right)^2\right\} \tag{B.6.43}$$

and noting that $\sum_{j=0}^{\infty}\mathfrak{N}_j(z) = \mathbf{I}$, the following expression for $\mathfrak{T}'(z)$ is obtained:

$$\mathfrak{T}'(z) = z^{-1}\sum_{j=0}^{\infty}\mathfrak{N}_j(z)\left(j\mathbf{I} - \sum_{k=0}^{\infty} k\mathfrak{N}_k(z)\right)^2 \tag{B.6.44}$$

Equation (2.4.5) implies that both diagonal matrices $\boldsymbol{\Theta}_0$ and $\boldsymbol{\Theta}_1$ have strictly positive elements on their main diagonals. This translates into diagonal matrix polynomials $\mathfrak{N}_0(z)$ and $\mathfrak{N}_1(z)$ from (B.6.41) being positive definite for all $0 < z \leq 1$. All the other matrices $\mathfrak{N}_j(z)$ are also diagonal and at least positive semi-definite. So the matrix within parentheses in (B.6.44) is diagonal, and when squared becomes positive semi-definite for all j. It is clear that the term in parentheses for $j=1$ differs from that term when $j=0$ by the identity matrix $\mathbf{I}$, and since both $\mathfrak{N}_0(z)$ and $\mathfrak{N}_1(z)$ are positive definite, the sum of the terms $j=0$ and $j=1$ in (B.6.44) must be a positive definite matrix. As all the remaining terms are at least positive semi-definite, $\mathfrak{T}'(z) \gg \mathbf{0}$ for $0 < z \leq 1$, that is, $\mathfrak{T}'(z)$ is positive definite.

Since every diagonal element of $\mathfrak{T}'(z)$ is positive for all $0 < z \leq 1$, it follows that each diagonal element of $\mathfrak{T}(z)$ is a strictly increasing function of z for $0 < z \leq 1$. Therefore, because $\beta < 1$, each diagonal element of $\mathfrak{T}(1) - \mathfrak{T}(\beta)$ is strictly positive. And from (B.6.39), together with the positive definiteness of $[\boldsymbol{\mathcal{R}}]_{-i}$, this means that $\mathscr{D}'(1) < 0$, so the root $z=1$ must have multiplicity one.

For the other root $z=\beta$, the equivalent of expression (B.6.38) for the derivative of $\boldsymbol{\Upsilon}(z)$ evaluated at β can be obtained from (B.6.36) and (B.6.37):

$$\boldsymbol{\Upsilon}'(\beta) = \beta^{-1}\boldsymbol{\phi}^{-1}\{\mathfrak{T}(1) - \mathfrak{T}(\beta)\} \tag{B.6.45}$$

And an expression for the derivative of determinant $\mathscr{D}(z)$ evaluated at $z=\beta$ can be found from (B.6.30) by noting that $\boldsymbol{\Upsilon}(\beta) = \mathbf{0}$ and combining this with (B.6.45):

$$\mathscr{D}'(\beta) = \beta^{-1}\sum_{i=1}^{n} \left|[\boldsymbol{\mathcal{R}}]_{-i}\right| [\mathfrak{T}(1) - \mathfrak{T}(\beta)]_i \tag{B.6.46}$$

The positive definiteness of $\mathfrak{T}(1) - \mathfrak{T}(\beta)$ and $[\boldsymbol{\mathcal{R}}]_{-i}$ ensure that $\mathscr{D}'(\beta) > 0$. Thus $z=\beta$ is also found to have multiplicity one, completing the proof of the claims of the lemma.

B.7 Proof of Lemma 7

Start from the definition of the $n \times n$ matrix polynomial $\boldsymbol{\chi}(z)$ in (C.1.11), and suppose this has all the properties stated in Lemma 6. Let $\mathfrak{B}(z)$ be equal to $\boldsymbol{\chi}(z)$ evaluated at $\sqrt{\beta}z$:

$$\mathfrak{B}(z) \equiv \boldsymbol{\chi}(\sqrt{\beta}z) \tag{B.7.1}$$

As can be seen from equation (B.6.6), the matrix polynomial $\boldsymbol{\chi}(z)$ has $m+1$ positive and $m+1$ negative powers of z, so this must also be a property of $\mathfrak{B}(z)$ in (B.7.1):

$$\mathfrak{B}(z) = \sum_{j=-(m+1)}^{m+1} \mathfrak{B}_j z^j \tag{B.7.2}$$

From definition (B.7.1), the coefficients of $\mathfrak{B}(z)$ in (B.7.2) are linked to those of $\boldsymbol{\chi}(z)$ by $\mathfrak{B}_j = \beta^{\frac{j}{2}}\boldsymbol{\chi}_j$. Since Lemma 6 establishes that $\boldsymbol{\chi}(z)$ is a discounted para-Hermitian matrix, $\boldsymbol{\chi}(z) = \boldsymbol{\chi}(\beta z^{-1})'$, its matrix coefficients satisfy $\boldsymbol{\chi}_{-j} = \beta^j \boldsymbol{\chi}_j'$, which implies $\mathfrak{B}_{-j} = \mathfrak{B}_j$. From (B.7.2) this means that $\mathfrak{B}(z) = \mathfrak{B}(z^{-1})'$ for all $z \in \mathbb{C}\backslash\{0\}$, and thus the matrix polynomial $\mathfrak{B}(z)$ is para-Hermitian.

Lemma 6 establishes that $\boldsymbol{\chi}(z)$ is a positive definite matrix when evaluated at any z with $|z| = \sqrt{\beta}$. An immediate consequence of definition (B.7.1) is that $\mathfrak{B}(z)$ is a positive definite matrix whenever $|z|=1$. This also means that there exists a z_0 such that $|\mathfrak{B}(z_0)| \neq 0$, and thus $|\mathfrak{B}(z)|$ is not identically zero. The definitions of $\boldsymbol{\chi}(z)$ and $\mathfrak{B}(z)$ in (C.1.11) and (B.7.1) show that coefficients matrices $\boldsymbol{\chi}_j$ and $\mathfrak{B}_j$ are real valued.

According to a result proved by Youla (1961), if $\mathfrak{B}(z)$ is a real-valued matrix polynomial with $m+1$ positive and negative powers of z, which is para-Hermitian, and which is positive-semi definite everywhere on the unit circle, and has a determinant which is not identically zero, then there exists a spectral factorization of $\mathfrak{B}(z)$. All of these properties are met, so there exists a matrix polynomial $\mathfrak{C}(z)$ such that

$$\mathfrak{B}(z) = \mathfrak{C}(z^{-1})'\mathfrak{C}(z) \tag{B.7.3}$$

which is referred to as the spectral factorization of $\mathfrak{B}(z)$. The matrix polynomial $\mathfrak{C}(z)$ has no roots strictly inside the unit circle, has only $m+1$ positive powers of z,

$$\mathfrak{C}(z) \equiv \sum_{j=0}^{m+1} \mathfrak{C}_j z^j \tag{B.7.4}$$

where all the matrices $\mathfrak{C}_j$ are real valued. This spectral factorization of $\mathfrak{B}(z)$ is used to construct a unique discounted spectral factorization of $\chi(z)$.

The first step is to note that $\mathfrak{C}_0$ must be an invertible matrix, otherwise $\mathfrak{C}(z)$ would have a root at $z = 0$, which is inside the unit circle. The invertibility of $\mathfrak{C}_0$ allows a new matrix polynomial $\boldsymbol{\Lambda}(z)$ and a non-singular matrix $\aleph$ to be defined as follows:

$$\boldsymbol{\Lambda}(z) \equiv \mathfrak{C}_0^{-1}\mathfrak{C}\left(\frac{1}{\sqrt{\beta}}z\right) \quad , \quad \aleph \equiv (\mathfrak{C}_0'\mathfrak{C}_0)^{-1} \tag{B.7.5}$$

The definition of $\aleph$ in (B.7.5) ensures that it is real-valued, invertible, symmetric and positive definite. By construction, $\boldsymbol{\Lambda}(z)$ is an $n \times n$ matrix polynomial satisfying $\boldsymbol{\Lambda}(0) = \mathbf{I}$. Using (B.7.4) and (B.7.5) it is clear that $\boldsymbol{\Lambda}(z)$ has only $m+1$ positive powers of z, and can be written explicitly as

$$\boldsymbol{\Lambda}(z) \equiv \mathbf{I} - \sum_{j=1}^{m+1} \mathbf{\Lambda}_j z^j \quad , \quad \mathbf{\Lambda}_j \equiv -\mathfrak{C}_0^{-1}\mathfrak{C}_j \tag{B.7.6}$$

with real-valued coefficient matrices $\mathbf{\Lambda}_j$.

The spectral factorization of $\mathfrak{B}(z)$ in (B.7.3) and the definitions in (B.7.5) imply that,

$$\mathfrak{B}\left(\frac{1}{\sqrt{\beta}}z\right) = \mathfrak{C}\left(\left(\frac{1}{\sqrt{\beta}}z\right)^{-1}\right)'\mathfrak{C}\left(\frac{1}{\sqrt{\beta}}z\right) = (\mathfrak{C}_0\boldsymbol{\Lambda}(\beta z^{-1}))'(\mathfrak{C}_0\boldsymbol{\Lambda}(z)) \tag{B.7.7}$$

which combined with the definition of $\mathfrak{B}(z)$ in (B.7.1) yields a discounted spectral factorization of $\chi(z)$:

$$\chi(z) = \boldsymbol{\Lambda}(\beta z^{-1})'\aleph^{-1}\boldsymbol{\Lambda}(z) \tag{B.7.8}$$

As it is known that $\mathfrak{C}(z)$ has no roots with modulus $|z| < 1$, it follows from definition (B.7.5) that $\boldsymbol{\Lambda}(z)$ has no roots with modulus $|z| < \sqrt{\beta}$. Moreover, if $\boldsymbol{\Lambda}(z)$ were to have a root with modulus $\sqrt{\beta} \leq |z| < 1$, then it would follow from (B.7.8) that $\chi(z)$ also shares this root. But any such a root has a modulus $\beta < |z| < 1$, since $0 < \beta < 1$ implies $\beta < \sqrt{\beta}$. This is contrary to the result of Lemma 6 that $\chi(z)$ has no roots in this range. Therefore, $\boldsymbol{\Lambda}(z)$ must have no roots strictly inside the unit circle.

Now the question of the uniqueness of discounted spectral factorization (B.7.8) of $\chi(z)$ is addressed. The spectral factorization of $\mathfrak{B}(z)$ in (B.7.3) is known not to be unique, so take two matrix polynomials $\mathfrak{C}(z)$ and $\tilde{\mathfrak{C}}(z)$ which can both be used in (B.7.3) and satisfy the properties identified earlier. Let $\tilde{\aleph}$ and $\tilde{\boldsymbol{\Lambda}}(z)$ be constructed according to (B.7.5) using $\tilde{\mathfrak{C}}(z)$ instead of $\mathfrak{C}(z)$.

Equation (B.7.3) and the requirement that neither $\mathfrak{C}(z)$ nor $\tilde{\mathfrak{C}}(z)$ has any roots inside the unit circle mean that both matrix polynomials $\mathfrak{C}(z)$ and $\tilde{\mathfrak{C}}(z)$ must share the same set of roots and nullspace vectors. As they must both have exactly $m+1$ positive powers of z and no negative powers, the fact that they share the same set of roots and nullspace vectors implies that $\tilde{\mathfrak{C}}(z)$ can be obtained from $\mathfrak{C}(z)$ by premultiplication of a non-singular matrix. So let $\tilde{\mathfrak{C}}(z) = \mathcal{W}\mathfrak{C}(z)$ for some invertible matrix $\mathcal{W}$. Because (B.7.3) implies that $\mathfrak{C}(z^{-1})'\mathfrak{C}(z) = \tilde{\mathfrak{C}}(z^{-1})'\tilde{\mathfrak{C}}(z)$ for all $z \in \mathbb{C}\backslash\{0\}$, the matrix $\mathcal{W}$ must be unitary, that is, $\mathcal{W}'\mathcal{W} = \mathbf{I}$. Then from the definitions of $\tilde{\mathbf{\Lambda}}_j$ and $\tilde{\aleph}$ in (B.7.5) and (B.7.6) it is clear that,

$$\tilde{\mathbf{\Lambda}}_j = -(\mathcal{W}\mathfrak{C}_0)^{-1}(\mathcal{W}\mathfrak{C}_j) = \mathbf{\Lambda}_j \quad , \quad \tilde{\aleph} = ((\mathcal{W}\mathfrak{C}_0)'(\mathcal{W}\mathfrak{C}_0))^{-1} = \aleph \tag{B.7.9}$$

establishing the uniqueness of matrix polynomial $\boldsymbol{\Lambda}(z)$ and matrix $\aleph$ in the discounted spectral factorization (C.1.12).

Suppose $\zeta \in \mathbb{C}$ is a root of $\boldsymbol{\chi}(z)$ with $|\zeta| \geq 1$. This means that $|\boldsymbol{\chi}(\zeta)| = 0$, or equivalently, there exists a non-zero vector $\mathbf{v} \in \mathbb{C}^n$ such that $\boldsymbol{\chi}(\zeta)\mathbf{v} = \mathbf{0}$. By using the factorization (B.7.8):

$$\boldsymbol{\chi}(\zeta)\mathbf{v} = \boldsymbol{\Lambda}(\beta\zeta^{-1})\aleph^{-1}\boldsymbol{\Lambda}(\zeta)\mathbf{v} = \mathbf{0} \tag{B.7.10}$$

As $|\beta\zeta^{-1}| < 1$, $|\boldsymbol{\Lambda}(\beta\zeta^{-1})| \neq 0$ since $\boldsymbol{\Lambda}(z)$ has no roots inside the unit circle. Cancelling this non-singular matrix along with $\aleph^{-1}$ from (B.7.10) implies that $\boldsymbol{\Lambda}(\zeta)\mathbf{v} = \mathbf{0}$, so ζ is a root of $\boldsymbol{\Lambda}(z)$ with the same nullspace vector $\mathbf{v}$.

Conversely, suppose $\zeta \in \mathbb{C}$ is a root of $\boldsymbol{\Lambda}(z)$ and $\mathbf{v}$ the corresponding nullspace vector. Then $\boldsymbol{\Lambda}(\zeta)\mathbf{v} = \mathbf{0}$. The factorization (B.7.8) clearly implies $\boldsymbol{\chi}(\zeta)\mathbf{v} = \mathbf{0}$ also. So ζ is a root of $\boldsymbol{\chi}(z)$ with nullspace vector $\mathbf{v}$. This establishes all the claims of the lemma.

B.8 Proof of Lemma 8

Take the coefficient matrices $\mathbf{\Lambda}_j$ of the matrix polynomial $\boldsymbol{\Lambda}(z)$ from (C.1.13) and define new $n \times n$ matrices $\mathbf{\Gamma}_j$ for $j = 1, \ldots, m$ and the matrix polynomial $\boldsymbol{\Gamma}(z)$ as follows:

$$\boldsymbol{\Gamma}(z) \equiv \mathbf{I} - \sum_{j=1}^{m} \mathbf{\Gamma}_j z^j \quad , \quad \mathbf{\Gamma}_j \equiv -\sum_{k=j+1}^{m+1} \mathbf{\Lambda}_k \tag{B.8.1}$$

Using these definitions it follows that:

$$\boldsymbol{\Gamma}(z)(1-z) + \boldsymbol{\Lambda}(1)z = \mathbf{I} - \left(\sum_{j=1}^{m+1} \mathbf{\Lambda}_j + \mathbf{\Gamma}_1\right) z - \sum_{j=2}^{m} (\mathbf{\Gamma}_j - \mathbf{\Gamma}_{j-1})z^j + \mathbf{\Gamma}_m z^{m+1} \tag{B.8.2}$$

Equation (B.8.1) implies that $\sum_{j=1}^{m+1} \mathbf{\Lambda}_j + \mathbf{\Gamma}_1 = \mathbf{\Lambda}_1$, $\mathbf{\Gamma}_j - \mathbf{\Gamma}_{j-1} = \mathbf{\Lambda}_j$ for all $j = 2, \ldots, m$, and $\mathbf{\Gamma}_m = -\mathbf{\Lambda}_{m+1}$. Hence $\boldsymbol{\Lambda}(z) = \boldsymbol{\Gamma}(z)(1-z) + \boldsymbol{\Lambda}(1)z$ for all z.

Taking the definition of $\boldsymbol{\Lambda}(z)$ in (C.1.13), and the decomposition $\mathbf{P}_t = \boldsymbol{\iota}\mathsf{P}_t + \boldsymbol{\rho}_t$ of the price-level vector into economy-wide price level and relative price components:

$$\mathbf{P}_t - \sum_{j=1}^{m+1} \mathbf{\Lambda}_j \mathbf{P}_{t-j} = \boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}_t = \boldsymbol{\Lambda}(\mathbb{L})\boldsymbol{\iota}\mathsf{P}_t + \boldsymbol{\Lambda}(\mathbb{L})\boldsymbol{\rho}_t \tag{B.8.3}$$

Let $\Delta \equiv \mathbb{I} - \mathbb{L}$ be the first-difference operator. By substituting the expression for $\boldsymbol{\Lambda}(z)$ for that in terms of $\boldsymbol{\Gamma}(z)$, the first term on the right-hand side of (B.8.3) can be written in terms of economy-wide inflation $\pi_t = \Delta\mathsf{P}_t$,

$$\boldsymbol{\Lambda}(\mathbb{L})\boldsymbol{\iota}\mathsf{P}_t = \{\boldsymbol{\Gamma}(\mathbb{L})\Delta + \boldsymbol{\Lambda}(1)\mathbb{L}\}\boldsymbol{\iota}\mathsf{P}_t = \boldsymbol{\Gamma}(\mathbb{L})\boldsymbol{\iota}\pi_t \tag{B.8.4}$$

where the fact that $\boldsymbol{\Lambda}(1)\boldsymbol{\iota} = \mathbf{0}$ has been used, which is obtained from the results of Lemmas 6 and 7. Next, the definitions of the $\boldsymbol{\Lambda}(z)$ and $\boldsymbol{\Gamma}(z)$ matrix polynomials are written out explicitly in the following:

$$\boldsymbol{\Gamma}(\mathbb{L})\boldsymbol{\iota}\pi_t + \boldsymbol{\Lambda}(\mathbb{L})\boldsymbol{\rho}_t = \left(\boldsymbol{\iota}\pi_t - \sum_{j=1}^{m} \boldsymbol{\Gamma}_j \boldsymbol{\iota}\pi_{t-j}\right) + \left(\boldsymbol{\rho}_t - \sum_{j=1}^{m+1} \boldsymbol{\Lambda}_j \boldsymbol{\rho}_{t-j}\right) \tag{B.8.5}$$

The definitions of the vector of inflation rates $\boldsymbol{\pi}_t$ and the vector of relative prices $\boldsymbol{\rho}_t$ imply that $\boldsymbol{\iota}\pi_t + \boldsymbol{\rho}_t = \boldsymbol{\pi}_t + \boldsymbol{\rho}_{t-1}$. Putting this together with (B.8.3), (B.8.4) and (B.8.5) yields,

$$\mathbf{P}_t - \sum_{j=1}^{m+1} \boldsymbol{\Lambda}_j \mathbf{P}_{t-j} = \boldsymbol{\pi}_t - \left(\sum_{j=1}^{m} \boldsymbol{\Gamma}_j \boldsymbol{\iota}\pi_{t-j} - \boldsymbol{\rho}_{t-1} + \sum_{j=1}^{m+1} \boldsymbol{\Lambda}_j \boldsymbol{\rho}_{t-j}\right) \tag{B.8.6}$$

confirming the claim in (C.1.20).

The linear homogeneous difference equation (C.1.21) is equivalent to $\boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}_t = \mathbf{0}$ in terms of matrix polynomials and lag operators. The general solution is found by examining the roots and nullspace vectors of the $n \times n$ matrix polynomial $\boldsymbol{\Lambda}(z)$ with degree $m+1$, as given in (C.1.13). The determinant $|\boldsymbol{\Lambda}(z)|$ is scalar polynomial of degree up to mn. For $k = 1, \ldots, mn$, let ζ_k denote the k-th root of $|\boldsymbol{\Lambda}(z)| = 0$. This may be a complex number, and when the polynomial $|\boldsymbol{\Lambda}(z)|$ has degree less than mn, it is necessary to allow for some roots at infinity, so ζ_k belongs to the set $\mathbb{C} \cup \{\infty\}$ in general. Lemma 7 shows that $\boldsymbol{\Lambda}(z)$ has no roots strictly inside the unit circle, so $|\zeta_k| \geq 1$ for all k.

Suppose ζ_k is a finite root with multiplicity one. There must exist a non-zero nullspace vector $\mathbf{v}_k \in \mathbb{C}^n$ associated with this root. Define the $n \times 1$ vector-valued function $\mathbf{f}_k(\tau)$ as follows:

$$\mathbf{f}_k(\tau) \equiv (\zeta_k^{-1})^{\tau} \mathbf{v}_k \tag{B.8.7}$$

Multiplying the function $\mathbf{f}_k(\tau)$ by matrix polynomial $\boldsymbol{\Lambda}(\mathbb{L})$ yields $\boldsymbol{\Lambda}(\mathbb{L})\mathbf{f}_k(\tau) = (\zeta_k^{-1})^{\tau}\boldsymbol{\Lambda}(\zeta_k)\mathbf{v}_k = \mathbf{0}$ because $\boldsymbol{\Lambda}(\zeta_k)\mathbf{v}_k = \mathbf{0}$ is the definition of a root–nullspace vector pair. Thus a multiple of $\mathbf{f}_k(\tau)$ must be part of the general solution of $\boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}_t = \mathbf{0}$.

When ζ_k does not have multiplicity one, or is not finite, the form of the function $\mathbf{f}_k(\tau)$ may need to be different. It can be shown that if ζ_k is a finite repeated root then $\mathbf{f}_k(\tau) = t^q(\zeta_k^{-1})^{\tau-q}\tilde{\mathbf{v}}_k$ where q is a finite positive number not more than the multiplicity of root ζ_k. It may also be necessary to use a vector $\tilde{\mathbf{v}}_k$ that is not one of the nullspace vectors. When ζ_k is an infinite (and possibly repeated root), then $\mathbf{f}_k(\tau) = \mathfrak{I}(\tau = q)\tilde{\mathbf{v}}_k$ where $\mathfrak{I}(\cdot)$ is the indicator function, and q and $\tilde{\mathbf{v}}_k$ are as just described. But no matter which expression it is necessary to use, all of the functions $\mathbf{f}_k(\tau)$ satisfy $\boldsymbol{\Lambda}(\mathbb{L})\mathbf{f}_k(\tau) = \mathbf{0}$. And if $|\zeta_k| > 1$ then all of the $\mathbf{f}_k(\tau)$, including (B.8.7), have the property that $\lim_{\tau\to\infty} \mathbf{f}_k(\tau) = \mathbf{0}$.

A general solution of linear homogeneous difference $\boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}_t = \mathbf{0}$ is obtained by taking a linear combination of all the vector-valued functions $\mathbf{f}_k(t - t_0)$ for $k = 1, \ldots, mn$,

$$\mathbf{P}_t = \sum_{k=1}^{mn} \mathfrak{c}_k \mathbf{f}_k(t - t_0) \tag{B.8.8}$$

where $\mathfrak{c}_1, \ldots, \mathfrak{c}_{mn}$ are arbitrary constants and t_0 is some base time period. Lemma 6 shows that $\boldsymbol{\chi}(z)$ has only one root $z = 1$ on the unit circle, and this root has multiplicity one. The corresponding nullspace vector is $\boldsymbol{\iota}$. The results of Lemma 7 show that this root is inherited by matrix polynomial $\boldsymbol{\Lambda}(z)$ (with the same multiplicity), and that $\boldsymbol{\Lambda}(z)$ cannot have any other root on the unit circle. Without loss of generality let this root be ordered first so that $\zeta_1 = 1$ and $\mathbf{v}_1 = \boldsymbol{\iota}$. Since this root has multiplicity one, the formula in (B.8.7) can be used to conclude that $\mathbf{f}_1(t - t_0) = \boldsymbol{\iota}$. Note that all the other roots must lie outside the unit circle, so $|\zeta_k| > 1$ for $k = 2, \ldots, mn$. Now define the following $n \times 1$ vector-valued function:

$$\mathbf{f}(\tau; \mathfrak{c}_2, \ldots, \mathfrak{c}_{mn}) \equiv \sum_{k=2}^{mn} \mathfrak{c}_k \mathbf{f}_k(\tau) \tag{B.8.9}$$

This has limit $\mathbf{0}$ as $\tau \to \infty$ since each function $\mathbf{f}_k(\tau)$ tends to zero as $\tau \to \infty$. Therefore, the claim in equation (C.1.21) is proved by substituting (B.8.9) into (B.8.8).

C Proofs of propositions

C.1 Proof of Proposition 1

The first step in deriving the system of pricing equations in (4.1.2) is to represent the information contained in the industry-specific hazard functions (2.4.1) using polynomials. Knowledge of a particular hazard function $\{\alpha_{ij}\}_{j=1}^{\infty}$ is equivalent to specifying the distribution of the duration of price stickiness $\{\theta_{ij}\}_{j=0}^{\infty}$ for that industry in (2.4.3), which is in turn determined by the recursion (2.4.7) based on the sequence $\{\phi_{ij}\}_{j=1}^{m+1}$. These latter two sequences are represented by polynomials using the z-transform:

$$\theta_i(z) \equiv \sum_{j=0}^{\infty} \theta_{ij} z^j \quad , \quad \phi_i(z) \equiv 1 - \sum_{j=1}^{m+1} \phi_{ij} z^j \tag{C.1.1}$$

From (2.4.7), the industry i polynomials $\theta_i(z)$ and $\phi_i(z)$ are related to each other as follows:

$$\phi_i(z) = \frac{\theta_i(0)}{\theta_i(z)} \quad , \quad \theta_i(z) = \frac{\phi_i(1)}{\phi_i(z)} \tag{C.1.2}$$

One property of the series $\{\theta_{ij}\}_{j=0}^{\infty}$ is that $\sum_{j=0}^{\infty} \theta_{ij} = 1$, so θ_{ij} must converge to zero as $j \to \infty$. This means the recursive polynomial $\phi_i(z)$ must have all its roots strictly outside the unit circle, otherwise the θ_{ij} would not tend to zero. And Lemma 1 in Sheedy (2007b) shows that if the hazard function satisfies the assumptions in (2.4.2) then the polynomial $\theta_i(z)$ must also have no roots on or inside the unit circle. Therefore, $\theta_i(z)$ and $\phi_i(z)$ are well-defined, analytic for $|z| \leq 1$ and have no roots on or inside the unit circle.

The potentially different polynomials in (C.1.1) for each industry are collected together in matrix polynomials, that is, matrices in which every element is a polynomial in z. The $n \times n$ matrix polynomials $\boldsymbol{\Theta}(z)$ and $\boldsymbol{\Phi}(z)$ contain the scalar polynomials $\theta_i(z)$ and $\phi_i(z)$ from (C.1.1) respectively on their principal diagonals:

$$\boldsymbol{\Theta}(z) \equiv \operatorname{diag}\{\theta_1(z), \ldots, \theta_n(z)\} \quad , \quad \boldsymbol{\Phi}(z) \equiv \operatorname{diag}\{\phi_1(z), \ldots, \phi_n(z)\} \tag{C.1.3}$$

By using the link between the scalar polynomials in (C.1.2) and the fact that $\boldsymbol{\Theta}(z)$ and $\boldsymbol{\Phi}(z)$ in (C.1.3) are diagonal matrices for each value of z, a correspondence between the two matrix polynomials is obtained:

$$\boldsymbol{\Phi}(z) = \boldsymbol{\Theta}(z)^{-1}\boldsymbol{\Theta}(0) \quad , \quad \boldsymbol{\Theta}(z) = \boldsymbol{\Phi}(z)^{-1}\boldsymbol{\Phi}(1) \tag{C.1.4}$$

The rationale for the introduction of these scalar and matrix polynomials is to allow the use of lag operator analysis to study the dynamics of the pricing equations. The lag operator $\mathbb{L}$ applied to any time series shifts back the elements of the time series by one period. Correspondingly, the forward operator $\mathbb{F}$ shifts time forward by one period. These operators are inverses of each other, so $\mathbb{F} = \mathbb{L}^{-1}$ and $\mathbb{L} = \mathbb{F}^{-1}$. In what follows, both operators are defined with respect to a particular information set. The particular information set intended should be apparent from the context in which the operators are used. This means the operators will shift only the time subscripts of variables, not the time subscripts of any conditional expectation operator.

If $\mathbf{P}_t$ and $\mathbf{R}_t$ are $n \times 1$ vectors of industry-specific price levels P_{it} and reset prices R_{it} respectively, then the equations (3.2.11) and (3.2.9) determining these variables can be written in terms of the matrix polynomial $\boldsymbol{\Theta}(z)$,

$$\mathbf{P}_t = \boldsymbol{\Theta}(\mathbb{L})\mathbf{R}_t \quad , \quad \boldsymbol{\Theta}(\beta)\mathbf{R}_t = \mathbb{E}_t[\boldsymbol{\Theta}(\beta\mathbb{F})(\boldsymbol{\iota}\boldsymbol{\omega}'\mathbf{P}_t + \hat{\boldsymbol{\rho}}_t)] \tag{C.1.5}$$

where $\boldsymbol{\omega}$ is the $n \times 1$ vector of industry sizes ω_i, $\boldsymbol{\iota}$ is an $n \times 1$ vector of 1s, and $\hat{\boldsymbol{\rho}}_t$ is the vector of profit-maximizing flexible relative prices defined in (4.1.1). Using (C.1.4), recursive versions of pricing equations (C.1.5) can be found in terms of the matrix polynomial $\boldsymbol{\Phi}(z)$:

$$\boldsymbol{\Phi}(\mathbb{L})\mathbf{P}_t = \boldsymbol{\Phi}(1)\mathbf{R}_t \quad , \quad \mathbb{E}_t[\boldsymbol{\Phi}(\beta\mathbb{F})\mathbf{R}_t] = \boldsymbol{\Phi}(\beta)(\boldsymbol{\iota}\boldsymbol{\omega}'\mathbf{P}_t + \hat{\boldsymbol{\rho}}_t) \tag{C.1.6}$$

A single set of equations for the price vector $\mathbf{P}_t$ is obtained by eliminating the vector of reset prices $\mathbf{R}_t$ from (C.1.6). To this end, define an $n \times n$ matrix $\boldsymbol{\phi}$ and matrix polynomial $\boldsymbol{\Upsilon}(z)$ as follows:

$$\boldsymbol{\Upsilon}(z) \equiv \boldsymbol{\Phi}(z)\boldsymbol{\Phi}(\beta z^{-1}) - \boldsymbol{\Phi}(1)\boldsymbol{\Phi}(\beta) \quad , \quad \boldsymbol{\phi} \equiv \boldsymbol{\Phi}(1)\boldsymbol{\Phi}(\beta) \tag{C.1.7}$$

Substituting the reset-price equation into the price-level equation in (C.1.6), and using the definitions in (C.1.7) yields

$$\mathbb{E}_t[\boldsymbol{\Upsilon}(\mathbb{L})\mathbf{P}_t] = \boldsymbol{\phi}(\hat{\boldsymbol{\rho}}_t - \boldsymbol{\rho}_t) \tag{C.1.8}$$

where $\boldsymbol{\rho}_t \equiv \mathbf{P}_t - \boldsymbol{\iota}\mathsf{P}_t$ is the $n \times 1$ vector of industry relative prices. As $\mathsf{P}_t = \boldsymbol{\omega}'\mathbf{P}_t$, the relative price vector $\boldsymbol{\rho}_t$ is a linear combination of the vector of industry price levels $\mathbf{P}_t$:

$$\boldsymbol{\rho}_t = \boldsymbol{\mathcal{R}}\mathbf{P}_t \quad , \quad \boldsymbol{\mathcal{R}} \equiv \mathbf{I} - \boldsymbol{\iota}\boldsymbol{\omega}' \tag{C.1.9}$$

Multiplication by the $n \times n$ matrix $\boldsymbol{\mathcal{R}}$ subtracts the weighted average of a vector from each element of the vector. Hence (C.1.9) allows (C.1.8) to be written exclusively in terms of the money price vector $\mathbf{P}_t$ and vector of profit-maximizing flexible relative prices $\hat{\boldsymbol{\rho}}_t$:

$$\mathbb{E}_t[\{\boldsymbol{\Upsilon}(\mathbb{L}) + \boldsymbol{\phi}\boldsymbol{\mathcal{R}}\}\mathbf{P}_t] = \boldsymbol{\phi}\hat{\boldsymbol{\rho}}_t \tag{C.1.10}$$

Equation (2.4.5) ensures that $\theta_{i0} > 0$ for all industries i, which means that the matrix $\boldsymbol{\Theta}(0)$ is invertible. The sequence $\{\theta_{ij}\}_{j=0}^{\infty}$ represents a probability distribution, so $\boldsymbol{\Theta}(1) = \mathbf{I}$ and $\boldsymbol{\Theta}(\beta)$ has strictly positive elements on its main diagonal. From the relationship in (C.1.4), it follows that both $\boldsymbol{\Phi}(1) = \boldsymbol{\Theta}(1)^{-1}\boldsymbol{\Theta}(0)$ and $\boldsymbol{\Phi}(\beta) = \boldsymbol{\Theta}(\beta)^{-1}\boldsymbol{\Theta}(0)$ are diagonal matrices with strictly positive elements on the main diagonal. They are thus non-singular. As $\omega_i > 0$ for all industries, the diagonal matrix $\boldsymbol{\Omega} \equiv \operatorname{diag}\{\omega_1, \ldots, \omega_n\}$ containing the industry sizes is also invertible. Therefore, an equivalent system of equations is obtained by multiplying both sides of (C.1.10) by $\boldsymbol{\Omega}\boldsymbol{\phi}^{-1}$. A new matrix polynomial $\boldsymbol{\chi}(z)$ is defined to be the coefficient of the price vector $\mathbf{P}_t$ in the resulting equation:

$$\mathbb{E}_t[\boldsymbol{\chi}(\mathbb{L})\mathbf{P}_t] = \boldsymbol{\Omega}\hat{\boldsymbol{\rho}}_t \quad , \quad \boldsymbol{\chi}(z) \equiv \boldsymbol{\Omega}\boldsymbol{\phi}^{-1}\boldsymbol{\Upsilon}(z) + \boldsymbol{\Omega}\boldsymbol{\mathcal{R}} \tag{C.1.11}$$

This system of expectational difference equations fully characterizes the profit-maximizing behaviour of firms with sticky prices, conditional on the profit-maximizing relative prices $\hat{\boldsymbol{\rho}}_t$ that firms would like to choose were prices completely flexible.

The system of equations (C.1.11) contains both leads and lags of the price vector $\mathbf{P}_t$ because the construction of the matrix polynomial $\boldsymbol{\chi}(z)$ in (C.1.7) and (C.1.11) involves both positive and negative powers of z. The next step is to find a discounted spectral factorization of $\boldsymbol{\chi}(z)$ which separates positive and negative powers of z. This corresponds to a breakdown of price and inflation dynamics into backward- and forward-looking components. In order for such a spectral factorization to exist, it is necessary to verify that the matrix polynomial $\boldsymbol{\chi}(z)$ has certain technical properties.

Lemma 6 *The $n \times n$ (real) matrix polynomial $\boldsymbol{\chi}(z)$ defined in equation (C.1.11) has the following properties:*

(i) $\boldsymbol{\chi}(z)$ is discounted para-Hermitian, that is, $\boldsymbol{\chi}(z) = \boldsymbol{\chi}(\beta z^{-1})'$ for all $z \in \mathbb{C}\backslash\{0\}$;

(ii) $\boldsymbol{\chi}(z)$ is a positive definite matrix if $|z| = \sqrt{\beta}$;

(iii) $\boldsymbol{\chi}(z)$ has no roots with modulus $\beta < |z| < 1$;

(iv) Both $z = 1$ and $z = \beta$ are roots of $\boldsymbol{\chi}(z)$ with corresponding nullspace vector $\boldsymbol{\iota}$. Both of these roots have multiplicity one, and they are the only roots with modulus 1 or β.

Proof. See appendix B.6. ■

Given the properties of matrix polynomial $\boldsymbol{\chi}(z)$ established in Lemma 6, a discounted spectral factorization exists.

Lemma 7 *If the $n \times n$ (real) matrix polynomial $\boldsymbol{\chi}(z)$ defined in equation (C.1.11) satisfies all of the properties stated in Lemma 6, then there exists a unique $n \times n$ (real) matrix polynomial $\boldsymbol{\Lambda}(z)$ and an $n \times n$ (real) symmetric, positive definite matrix $\boldsymbol{\aleph}$ such that the following hold:*

(i) $\boldsymbol{\Lambda}(z)$ spectrally factorizes the matrix polynomial $\boldsymbol{\chi}(z)$:

$$\boldsymbol{\chi}(z) = \boldsymbol{\Lambda}(\beta z^{-1})'\boldsymbol{\aleph}^{-1}\boldsymbol{\Lambda}(z) \tag{C.1.12}$$

(ii) $\boldsymbol{\Lambda}(z)$ is a matrix polynomial with $m+1$ non-negative powers of z and satisfies $\boldsymbol{\Lambda}(0) = \mathbf{I}$:

$$\boldsymbol{\Lambda}(z) \equiv \mathbf{I} - \sum_{j=1}^{m+1} \boldsymbol{\Lambda}_j z^j \tag{C.1.13}$$

(iii) $\boldsymbol{\Lambda}(z)$ has no roots with modulus $|z| < 1$;

(iv) Any root of $\boldsymbol{\Lambda}(z)$ is a root of $\boldsymbol{\chi}(z)$; any root of $\boldsymbol{\chi}(z)$ with $|z| \geq 1$ is a root of $\boldsymbol{\Lambda}(z)$. When $\boldsymbol{\Lambda}(z)$ and $\boldsymbol{\chi}(z)$ share a root, they also share the corresponding nullspace vector.

Proof. See appendix B.7. ■

By substituting equation (C.1.12) into the pricing equations (C.1.11) the following expression is obtained:

$$\mathbb{E}_t[\boldsymbol{\Lambda}(\beta\mathbb{F})'\boldsymbol{\aleph}^{-1}\boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}_t] = \boldsymbol{\Omega}\hat{\boldsymbol{\rho}}_t \tag{C.1.14}$$

As $\boldsymbol{\Lambda}(z)$ has no roots strictly inside the unit circle, and since $0 < \beta < 1$, it follows that matrix polynomial $\boldsymbol{\Lambda}(\beta z)$ has no roots on or inside the unit circle. Therefore the inverse matrix polynomial $(\boldsymbol{\Lambda}(\beta z)')^{-1}$ is analytic and has a convergent Taylor series expansion for all $z \in \mathbb{C}$ with $|z| \leq 1$. This polynomial is denoted by $\boldsymbol{\Xi}(z)$ and has an expansion in non-negative powers of z written in terms of a sequence of $n \times n$ (real) matrices $\{\boldsymbol{\Xi}_j\}_{j=0}^{\infty}$:

$$\boldsymbol{\Xi}(z) \equiv (\boldsymbol{\Lambda}(\beta z)')^{-1} \quad , \quad \boldsymbol{\Xi}(z) \equiv \sum_{j=0}^{\infty} \boldsymbol{\Xi}_j z^j \tag{C.1.15}$$

The leading term $\boldsymbol{\Xi}_0$ is the identity matrix $\mathbf{I}$. By inverting the matrix polynomial $\boldsymbol{\Lambda}(\beta\mathbb{F})'$ in (C.1.14) using the definition of $\boldsymbol{\Xi}(z)$, and then multiplying both sides by matrix $\boldsymbol{\aleph}$, a separation of equation (C.1.11) into backward- and forward-looking components is obtained:

$$\boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}_t = \aleph\mathbb{E}_t[\boldsymbol{\Xi}(\mathbb{F})\boldsymbol{\Omega}\hat{\boldsymbol{\rho}}_t] \tag{C.1.16}$$

When written out explicitly using the definitions of matrix polynomials $\boldsymbol{\Lambda}(z)$ and $\boldsymbol{\Xi}(z)$ in (C.1.13) and (C.1.15), the system of equations (C.1.16) becomes:

$$\mathbf{P}_t = \sum_{j=1}^{m+1} \mathbf{\Lambda}_j \mathbf{P}_{t-j} + \aleph \sum_{j=0}^{\infty} \mathbf{\Xi}_j \boldsymbol{\Omega} \mathbb{E}_t \hat{\boldsymbol{\rho}}_{t+j} \tag{C.1.17}$$

It is desirable to express this system of pricing equations in terms of inflation rates $\boldsymbol{\pi}_t$ and relative prices $\boldsymbol{\rho}_t$. It is known from Lemmas 6 and 7 that $\boldsymbol{\chi}(z)$ has a unit root, and that this root is inherited by $\boldsymbol{\Lambda}(z)$. The next result shows how this root can be factored out of matrix polynomial $\boldsymbol{\Lambda}(z)$:

Lemma 8 *If $\boldsymbol{\Lambda}(z)$ is the matrix polynomial described in Lemma 7 then there exists an $n \times n$ (real) matrix polynomial $\boldsymbol{\Gamma}(z)$ with the following properties:*

(i) $\boldsymbol{\Gamma}(z)$ has m non-negative powers of z and satisfies $\boldsymbol{\Gamma}(0) = \mathbf{I}$:

$$\boldsymbol{\Gamma}(z) \equiv \mathbf{I} - \sum_{j=1}^{m} \mathbf{\Gamma}_j z^j \tag{C.1.18}$$

(ii) $\boldsymbol{\Gamma}(z)$ is the matrix associated with divisor $(1-z)$ of $\boldsymbol{\Lambda}(z)$:

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Gamma}(z)(1-z) + \boldsymbol{\Lambda}(1)z \tag{C.1.19}$$

(iii) The price level $\mathbf{P}_t$ is related to inflation $\boldsymbol{\pi}_t$ and relative prices $\boldsymbol{\rho}_t$ according to,

$$\mathbf{P}_t - \sum_{j=1}^{m+1} \mathbf{\Lambda}_j \mathbf{P}_{t-j} = \boldsymbol{\pi}_t - \left(\sum_{j=1}^{m} \gamma_j \pi_{t-j} - \boldsymbol{\rho}_{t-1} + \sum_{j=1}^{m+1} \mathbf{\Lambda}_j \boldsymbol{\rho}_{t-j} \right) \tag{C.1.20}$$

where $\boldsymbol{\gamma}_j \equiv \mathbf{\Gamma}_j \boldsymbol{\iota}$.

(iv) The general solution of the homogeneous linear difference equation $\mathbf{P}_t - \sum_{j=1}^{m+1} \mathbf{\Lambda}_j \mathbf{P}_{t-j} = \mathbf{0}$ is of the following form

$$\mathbf{P}_t = \mathfrak{c}_1 \boldsymbol{\iota} + \mathbf{f}(t - t_0; \mathfrak{c}_2, \ldots, \mathfrak{c}_{mn}) \tag{C.1.21}$$

where $\mathfrak{c}_1, \ldots, \mathfrak{c}_{mn}$ are mn arbitrary constants, t_0 is some base time period, $\boldsymbol{\iota}$ is an $n \times 1$ vectors of 1s, and $\mathbf{f}(\tau; \cdots)$ is an $n \times 1$ vector-valued function such that $\lim_{\tau \to \infty} \mathbf{f}(\tau; \cdots) = \mathbf{0}$ for any choice of $\mathfrak{c}_2, \ldots, \mathfrak{c}_{mn}$.

Proof. See appendix B.8. ■

By combining pricing equations (C.1.17) with (C.1.20) and the expression for $\hat{\boldsymbol{\rho}}_t$ in (4.1.1), the result (4.1.2) is obtained. This proves the claim of the proposition.

C.2 Proof of Proposition 2

According to the hypothesis, all the hazard functions (2.4.1) for price adjustment in each industry are identical. Formally, this requires $\alpha_{ij} = \alpha_j$ for all i and j. From (2.4.4), this implies there is a common distribution of price stickiness durations $\{\theta_j\}_{j=0}^{\infty}$ in (2.4.3) applicable to all industries, so $\theta_{ij} = \theta_j$. And of course, this means there is also a common recursive representation (2.4.7) with $\phi_{ij} = \phi_j$. Let the function $\phi(z) = \phi_i(z)$ be the polynomial defined similarly to (C.1.1) using these common recursive coefficients. And let the scalar polynomial $\Upsilon(z)$ and scalar ϕ be defined analogously to their matrix equivalents in (C.1.7) as $\Upsilon(z) \equiv \phi(z)\phi(\beta z^{-1}) - \phi(1)\phi(\beta)$ and $\phi \equiv \phi(1)\phi(\beta)$. The corresponding matrix polynomial $\boldsymbol{\Upsilon}(z)$ and matrix $\boldsymbol{\phi}$ can be obtained from

$$\boldsymbol{\Upsilon}(z) = \Upsilon(z)\mathbf{I} \quad , \quad \boldsymbol{\phi} = \phi\mathbf{I} \tag{C.2.1}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. By substituting these expressions into the definition of $\boldsymbol{\chi}(z)$ in (C.1.11) the following is obtained:

$$\boldsymbol{\chi}(z) = \boldsymbol{\Omega}\{\phi^{-1}\Upsilon(z)\mathbf{I} + \boldsymbol{\mathcal{R}}\} \tag{C.2.2}$$

Now consider the diagonalization of relative price matrix $\boldsymbol{\mathcal{R}}$ defined in (C.1.9). It is known from (B.6.17) that $\boldsymbol{\mathcal{R}}$ is idempotent and so has eigenvalues equal to either 0 or 1. Since $\boldsymbol{\mathcal{R}}\boldsymbol{\iota} = \mathbf{0}$, it is clear that the vector of 1s is an eigenvector associated with the eigenvalue 0. The i-th eigenvector of $\boldsymbol{\mathcal{R}}$ is denoted by $n \times 1$ vector $\mathbf{v}_i$. Without loss of generality, let the eigenvalue 0 be ordered first, so $\mathbf{v}_1 = \boldsymbol{\iota}$. Now consider an eigenvector $\mathbf{v}$ associated with eigenvalue 1. It is clear from the definition of $\boldsymbol{\mathcal{R}}$ in (C.1.9) that this is equivalent to $\boldsymbol{\omega}'\mathbf{v} = 0$, which means that $\mathbf{v}$ must lie in the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ comprising vectors that are orthogonal to $\boldsymbol{\omega}$. Any basis for this subspace provides a set of $n-1$ linearly independent eigenvectors $\mathbf{v}_2, \ldots, \mathbf{v}_n$. Denote the matrix of eigenvalues by $\mathbf{V} \equiv (\ \mathbf{v}_1\ ,\ \ldots\ ,\ \mathbf{v}_n\ )$. As the set of vectors $\{\ \mathbf{v}_1\ ,\ \mathbf{v}_2\ ,\ \ldots\ ,\ \mathbf{v}_n\ \}$ is linearly independent, the matrix $\mathbf{V}$ is invertible, and so $\boldsymbol{\mathcal{R}}$ can be diagonalized. The diagonal matrix of eigenvalues is denoted by $\mathbf{J}$ and is related to $\boldsymbol{\mathcal{R}}$ as follows:

$$\mathbf{V}^{-1}\boldsymbol{\mathcal{R}}\mathbf{V} = \mathbf{J} \quad , \quad \mathbf{J} \equiv \mathrm{diag}\{0, 1, \ldots, 1\} \tag{C.2.3}$$

Since $\Upsilon(z) = \Upsilon(\beta z^{-1})$, the scalar polynomials $\phi^{-1}\Upsilon(z)$ and $\phi^{-1}\Upsilon(z) + 1$ have the equivalent of the para-Hermitian property discussed in Lemma 7. Using results analogous to those for matrices in Lemmas 6 and 7, there exist positive constants $\mathfrak{X}_p$ and $\mathfrak{X}_\varrho$, and scalar polynomials $\mathcal{L}_p(z)$ and $\mathcal{L}_\varrho(z)$ such that,

$$\phi^{-1}\Upsilon(z) = \mathcal{L}_p(\beta z^{-1})\mathfrak{X}_p\mathcal{L}_p(z) \quad , \quad \phi^{-1}\Upsilon(z) + 1 = \mathcal{L}_\varrho(\beta z^{-1})\mathfrak{X}_\varrho\mathcal{L}_\varrho(z) \tag{C.2.4}$$

The polynomials $\mathcal{L}_p(z)$ and $\mathcal{L}_\varrho(z)$ have $m+1$ non-negative powers of z, have no roots strictly inside the unit circle, and are normalized so that $\mathcal{L}_p(0) = \mathcal{L}_\varrho(0) = 1$. Using the matrix of eigenvectors $\mathbf{V}$, an $n \times n$ matrix $\boldsymbol{\mathfrak{X}}$ and an $n \times n$ matrix polynomial $\boldsymbol{\mathcal{L}}(z)$ can be defined as follows:

$$\boldsymbol{\mathcal{L}}(z) \equiv \mathbf{V}\mathrm{diag}\{\mathcal{L}_p(z), \mathcal{L}_\varrho(z), \ldots, \mathcal{L}_\varrho(z)\}\mathbf{V}^{-1} \quad , \quad \boldsymbol{\mathfrak{X}} \equiv \mathbf{V}\mathrm{diag}\{\mathfrak{X}_p, \mathfrak{X}_\varrho, \ldots, \mathfrak{X}_\varrho\}\mathbf{V}^{-1} \tag{C.2.5}$$

Using these definitions of $\boldsymbol{\mathcal{L}}(z)$ and $\boldsymbol{\mathfrak{X}}$ in (C.2.5), the definition of $\mathbf{J}$ in (C.2.3), and the factorizations in (C.2.4), it can be seen that $\boldsymbol{\mathcal{L}}(\beta z^{-1})\boldsymbol{\mathfrak{X}}\boldsymbol{\mathcal{L}}(z) = \mathbf{V}(\phi^{-1}\Upsilon(z)\mathbf{I} + \mathbf{J})\mathbf{V}^{-1}$. An alternative expression for the discounted spectral factorization of $\boldsymbol{\chi}(z)$ can be obtained from this result, together with the equation for $\boldsymbol{\chi}(z)$ in (C.2.2) and the diagonalization of $\boldsymbol{\mathcal{R}}$ in (C.2.3):

$$\boldsymbol{\mathcal{L}}(\beta z^{-1})\boldsymbol{\mathfrak{X}}\boldsymbol{\mathcal{L}}(z) = \boldsymbol{\Omega}^{-1}\boldsymbol{\chi}(z) \tag{C.2.6}$$

The usual discounted spectral factorization of $\boldsymbol{\chi}(z)$ is given in equation (C.1.12). By substituting this into (C.2.6), it can be compared with the new factorization:

$$\boldsymbol{\mathcal{L}}(\beta z^{-1})\boldsymbol{\mathfrak{X}}\boldsymbol{\mathcal{L}}(z) = \boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}(\beta z^{-1})'\boldsymbol{\aleph}^{-1}\boldsymbol{\Lambda}(z) \tag{C.2.7}$$

It is apparent from the definition of $\boldsymbol{\mathcal{L}}(z)$ in (C.2.5) and the properties of the scalar polynomials introduced in (C.2.4) that $\boldsymbol{\mathcal{L}}(z) = \mathbf{I} - \sum_{j=1}^{m+1} \boldsymbol{\mathcal{L}}_j z^j$ has the same form as $\boldsymbol{\Lambda}(z)$ in (C.1.13). And $\boldsymbol{\mathcal{L}}(z)$ has the same set of roots as its constituent scalar polynomials $\mathcal{L}_p(z)$ and $\mathcal{L}_\varrho(z)$, neither of which has any roots strictly inside the unit circle. Thus, equation (C.2.7) implies that both $\boldsymbol{\Lambda}(z)$ and $\boldsymbol{\mathcal{L}}(z)$ have the same set of roots. Because both have the same degree, and both satisfy $\boldsymbol{\Lambda}(0) = \boldsymbol{\mathcal{L}}(0) = \mathbf{I}$, the two polynomials $\boldsymbol{\Lambda}(z)$ and $\boldsymbol{\mathcal{L}}(z)$ must be identical, and thus from (C.2.5), $\boldsymbol{\Lambda}(z) = \mathbf{V}\text{diag}\{\mathcal{L}_p(z), \mathcal{L}_\varrho(z), \ldots, \mathcal{L}_\varrho(z)\}\mathbf{V}^{-1}$.

If $\boldsymbol{\mathcal{L}}(z) = \boldsymbol{\Lambda}(z)$ then (C.2.7) implies that $\boldsymbol{\mathcal{L}}(\beta z^{-1})\boldsymbol{\mathfrak{X}} = \boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}(\beta z^{-1})'\boldsymbol{\aleph}^{-1}$. So using the definition of matrix polynomial $\boldsymbol{\Xi}(z)$ in (C.1.15), $\boldsymbol{\aleph}\boldsymbol{\Xi}(z)\boldsymbol{\Omega} = (\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}(\beta z)'\boldsymbol{\aleph}^{-1})^{-1} = (\boldsymbol{\mathcal{L}}(\beta z)\boldsymbol{\mathfrak{X}})^{-1}$, and hence from (C.2.5) the following result is obtained:

$$\boldsymbol{\aleph}\boldsymbol{\Xi}(z)\boldsymbol{\Omega} = \mathbf{V}\text{diag}\{(\mathcal{L}_p(\beta z)\mathfrak{X}_p)^{-1}, (\mathcal{L}_\varrho(\beta z)\mathfrak{X}_\varrho)^{-1}, \ldots, (\mathcal{L}_\varrho(\beta z)\mathfrak{X}_\varrho)^{-1}\}\mathbf{V}^{-1} \tag{C.2.8}$$

Using the definitions of the eigenvectors in $\mathbf{V}$, $\boldsymbol{\omega}'\mathbf{v}_1 = \boldsymbol{\omega}'\boldsymbol{\iota} = 1$ and $\boldsymbol{\omega}'\mathbf{v}_i = 0$ for $i = 2, \ldots, n$ since these latter eigenvectors belong to a subspace that is orthogonal to $\boldsymbol{\omega}$. Therefore, $\boldsymbol{\omega}'\mathbf{V} = (\ 1\ ,\ 0\ ,\ \ldots\ ,\ 0\ )$. Now take any vector $\mathfrak{c}$ such that $\boldsymbol{\omega}'\mathfrak{c} = 0$. As $\mathfrak{c}$ belongs to the subspace orthogonal to $\boldsymbol{\omega}$, $\mathfrak{c}$ is a linear combination only of the vectors $\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}$ since these are a basis for that subspace. Thus $\mathbf{V}^{-1}\mathfrak{c} = (\ 0\ ,\ \mathfrak{k}_2\ ,\ \ldots\ ,\ \mathfrak{k}_n\ )'$ for some constants $\mathfrak{k}_2, \ldots, \mathfrak{k}_n$.

Hence if $\boldsymbol{\omega}'\mathfrak{c} = 0$, by putting the above results together with $\boldsymbol{\Lambda}(z) = \boldsymbol{\mathcal{L}}(z)$ and (C.2.5), it is shown that:

$$\boldsymbol{\omega}'\boldsymbol{\Lambda}(z)\mathfrak{c} = (\ 1\ ,\ 0\ ,\ \ldots\ ,\ 0\ )\text{diag}\{\mathcal{L}_p(z), \ldots, \mathcal{L}_\varrho(z)\}(\ 0\ ,\ \mathfrak{k}_2\ ,\ \ldots\ ,\ \mathfrak{k}_n\ )' = 0 \tag{C.2.9}$$

Similarly, using these results and the expression for $\boldsymbol{\aleph}\boldsymbol{\Xi}(z)\boldsymbol{\Omega}$ in (C.2.8), it follows that

$$\boldsymbol{\omega}'\boldsymbol{\aleph}\boldsymbol{\Xi}(z)\boldsymbol{\Omega}\mathfrak{c} = (\ 1\ ,\ 0\ ,\ \ldots\ ,\ 0\ )\text{diag}\{(\mathcal{L}_p(\beta z)\mathfrak{X}_p)^{-1}, \ldots, (\mathcal{L}_\varrho(\beta z)\mathfrak{X}_\varrho)^{-1}\}(\ 0\ ,\ \mathfrak{k}_2\ ,\ \ldots\ ,\ \mathfrak{k}_n\ )' = 0 \tag{C.2.10}$$

for any $\mathfrak{c}$ with $\boldsymbol{\omega}'\mathfrak{c} = 0$. Since $\boldsymbol{\omega}'\boldsymbol{\rho}_t = 0$ because of the definition of the relative price vector, and $\boldsymbol{\omega}'\boldsymbol{\rho}^*_t = 0$ because of equation (3.1.2), the claim of the proposition is established by (C.2.9) and (C.2.10).

C.3 Proof of Proposition 3

The policymaker's loss function derived in Lemma 2 is stated in vector form in equation (4.3.1). Here, attention is focused solely on the second and third terms, which correspond to inter-industry and intra-industry price distortions respectively. No further simplifications can be made to the first term involving the output gap. The loss resulting from price distortions collectively is denoted by $\mathfrak{P}_t$ and the following expression for it is taken from (4.3.1):

$$\mathfrak{P}_t \equiv \frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t[(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}^*_\tau)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}^*_\tau) + \boldsymbol{\sigma}'_\tau\boldsymbol{\Omega}\boldsymbol{\sigma}_\tau] \tag{C.3.1}$$

The first part of the proposition requires an expression be obtained for $\mathfrak{P}_t$ that connects it to the loss functions of firms as derived in Lemma 5.

The vector $\boldsymbol{\sigma}_t$ in (C.3.1) contains the cross-sectional standard deviations of prices within each of the n industries. Formally, the cross-sectional variance σ^2_{it} for industry i at time t is defined as follows,

$$\sigma^2_{it} \equiv \mathbb{V}_{\Omega_i}[\mathsf{P}_t(\imath,\jmath)] = \mathbb{E}_{\Omega_i}[(\mathsf{P}_t(\imath,\jmath) - \mathbb{E}_{\Omega_i}[\mathsf{P}_t(\imath,\jmath)])^2] \quad , \quad \mathbb{E}_{\Omega_i}[\cdot] \equiv \frac{1}{\omega_i}\int_{\Omega_i}\int_{\Omega}\cdot\, d\jmath d\imath \tag{C.3.2}$$

where Ω_i denotes the set of firms in industry i, and $\mathbb{E}_{\Omega_i}[\cdot]$ and $\mathbb{V}_{\Omega_i}[\cdot]$ are the cross-sectional expectation and variance operators respectively. First note that the equation for the price index P_{it} in (2.2.2) and the definition of the operator $\mathbb{E}_{\Omega_i}[\cdot]$ imply that P_{it} is equal to $\mathbb{E}_{\Omega_i}[\mathsf{P}_t(\imath,\jmath)]$ up to terms that are second- or higher-order in the exogenous shocks. This yields the following second-order accurate approximation of the cross-sectional variance:

$$\sigma_{it}^2 = \mathbb{E}_{\Omega_i}[(\mathsf{P}_t(\imath,\jmath) - \mathsf{P}_{it})^2] + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{C.3.3}$$

By adding and subtracting $\mathsf{P}_t + \hat{\rho}_{it}$ and using the definition of the relative price $\rho_{it} \equiv \mathsf{P}_{it} - \mathsf{P}_t$, the cross-sectional expectation in (C.3.3) is algebraically identical to:

$$\mathbb{E}_{\Omega_i}[(\mathsf{P}_t(\imath,\jmath) - \mathsf{P}_{it})^2] = \mathbb{E}_{\Omega_i}[((\mathsf{P}_t(\imath,\jmath) - \mathsf{P}_t - \hat{\rho}_{it}) - (\rho_{it} - \hat{\rho}_{it}))^2] \tag{C.3.4}$$

Because $\rho_{it} - \hat{\rho}_{it}$ is constant on the set Ω_i and $\mathbb{E}_{\Omega_i}[\mathsf{P}_t(\imath,\jmath)]$ is equivalent to P_{it} up to second-order terms, equation (C.3.4) can be used to deduce:

$$\mathbb{E}_{\Omega_i}[((\mathsf{P}_t(\imath,\jmath) - \mathsf{P}_{it})^2] = \mathbb{E}_{\Omega_i}[(\mathsf{P}_t(\imath,\jmath) - \mathsf{P}_t - \hat{\rho}_{it})^2] - (\rho_{it} - \hat{\rho}_{it})^2 + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{C.3.5}$$

By combining (C.3.3) and (C.3.5), a second-order accurate expression is obtained for the cross-sectional variance of prices in industry i:

$$\sigma_{it}^2 = \mathbb{E}_{\Omega_i}[(\mathsf{P}_t(\imath,\jmath) - \mathsf{P}_t - \hat{\rho}_{it})^2] - (\rho_{it} - \hat{\rho}_{it})^2 + \mathcal{O}(||\boldsymbol{v}_t||^3) \tag{C.3.6}$$

Putting (C.3.6) back into the loss function (C.3.1) produces a new expression for $\mathfrak{P}_t$,

$$\begin{aligned}\mathfrak{P}_t \equiv \frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\Bigg[&(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}_\tau^*)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}_\tau^*) - (\boldsymbol{\rho}_\tau - \hat{\boldsymbol{\rho}}_\tau)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau - \hat{\boldsymbol{\rho}}_\tau)\\ &+\sum_{i=1}^{n}\omega_i\mathbb{E}_{\Omega_i}[(\mathsf{P}_\tau(\imath,\jmath) - \mathsf{P}_\tau - \hat{\rho}_{i\tau})^2]\Bigg]\end{aligned} \tag{C.3.7}$$

where since this is a second-order approximation, third- and higher-order terms $\mathcal{O}(||\boldsymbol{v}_t||^3)$ are suppressed. Using the cross-sectional distribution $\{\theta_{ij}\}_{j=0}^{\infty}$ of the duration of price stickiness in industry i from (2.4.3), the cross-sectional expectation in (C.3.7) can be replaced by a summation over past reset prices:

$$\mathbb{E}_{\Omega_i}[(\mathsf{P}_t(\imath,\jmath) - \mathsf{P}_t - \hat{\rho}_{it})^2] = \sum_{j=0}^{\infty}\theta_{ij}(\mathsf{R}_{i,\tau-j} - \mathsf{P}_\tau - \hat{\rho}_{i\tau})^2 \tag{C.3.8}$$

And by substituting (C.3.8) into the final term of (C.3.7):

$$\begin{aligned}\mathfrak{P}_t \equiv \frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\Bigg[&(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}_\tau^*)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}_\tau^*) - (\boldsymbol{\rho}_\tau - \hat{\boldsymbol{\rho}}_\tau)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau - \hat{\boldsymbol{\rho}}_\tau)\\ &+\sum_{i=1}^{n}\omega_i\sum_{j=0}^{\infty}\theta_{ij}(\mathsf{R}_{i,\tau-j} - \mathsf{P}_\tau - \hat{\rho}_{i\tau})^2\Bigg]\end{aligned} \tag{C.3.9}$$

An alternative, but equivalent, expression for $\mathfrak{P}_t$ in terms of firms' loss functions can be obtained by changing the order of summation in the final term of (C.3.9):

$$\begin{aligned}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\sum_{j=0}^{\infty}\theta_{ij}(\mathsf{R}_{i,\tau-j} - \mathsf{P}_\tau - \hat{\rho}_{i\tau})^2\right] &= \sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j\theta_{ij}(\mathsf{R}_{i\tau} - \mathsf{P}_{\tau+j} - \hat{\rho}_{i,\tau+j})^2\right]\\ &+\sum_{\tau\to-\infty}^{t-1}\beta^{\tau-t}\mathbb{E}_t\left[\sum_{j=t-\tau}^{\infty}\beta^j\theta_{ij}(\mathsf{R}_{i\tau} - \mathsf{P}_{\tau+j} - \hat{\rho}_{i,\tau+j})^2\right]\end{aligned} \tag{C.3.10}$$

Therefore, using the definition of firms' loss functions $\mathfrak{F}_{it}$ and $\mathfrak{F}_{i,t|T}$ in (4.3.2), and summing (C.3.10) over all industries yields,

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\sum_{i=1}^{n}\omega_i\mathbb{E}_{\Omega_i}[(\mathsf{P}_\tau(\imath,\jmath)-\mathsf{P}_\tau-\hat{\rho}_{i\tau})^2]\right]=\sum_{i=1}^{n}\frac{\omega_i\theta_{i0}}{\vartheta_{i0}}\mathbb{E}_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathfrak{F}_{i\tau}+\sum_{\tau\to-\infty}^{t-1}\beta^{\tau-t}\mathfrak{F}_{i,t|\tau}\right] \tag{C.3.11}$$

where the sequence $\{\vartheta_{ij}\}_{j=0}^{\infty}$ is defined in (3.2.9). In summary, an expression for the total loss to the benevolent policymaker arising from price distortions is obtained by combining (C.3.7) and (C.3.11):

$$\begin{aligned}\mathfrak{P}_t=&\frac{1}{2}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[(\boldsymbol{\rho}_t-\boldsymbol{\rho}_t^*)'\boldsymbol{\Omega}(\boldsymbol{\rho}_t-\boldsymbol{\rho}_t^*)-(\boldsymbol{\rho}_\tau-\hat{\boldsymbol{\rho}}_\tau)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau-\hat{\boldsymbol{\rho}}_\tau)\right]\\&+\sum_{i=1}^{n}\frac{\omega_i\theta_{i0}}{\vartheta_{i0}}\mathbb{E}_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathfrak{F}_{i\tau}+\sum_{\tau\to-\infty}^{t-1}\beta^{\tau-t}\mathfrak{F}_{i,t|\tau}\right]\end{aligned} \tag{C.3.12}$$

As $\mathfrak{P}_t$ is equal to $\mathfrak{U}_t$ apart from the first output-gap term in (4.3.1), this equation confirms that (4.3.3) is true.

The next step is to compute the partial derivatives of (C.3.12). Starting with the reset price vector $\mathbf{R}_\tau$, since $\mathsf{R}_{i\tau}$ satisfies first-order condition (3.2.9), $\mathsf{R}_{i\tau}$ minimizes firms' loss function $\mathfrak{F}_{i\tau}$. As this reset price does not appear in any of the other terms of (C.3.11), it follows that the derivative of $\mathfrak{P}_t$ with respect to $\mathbf{R}_\tau$ is zero if firms are maximizing profits (that is, minimizing their loss functions). The derivative with respect to the relative price vector $\boldsymbol{\rho}_\tau$ can be obtained by differentiating the quadratic forms on the first line of (C.3.12) since that is the only place this vector occurs:

$$\frac{\partial\mathfrak{P}_t}{\partial\mathbf{R}_\tau}=\mathbf{0}\quad,\quad\frac{\partial\mathfrak{P}_t}{\partial\boldsymbol{\rho}_\tau}=\beta^{\tau-t}\boldsymbol{\Omega}(\hat{\boldsymbol{\rho}}_\tau-\boldsymbol{\rho}_\tau^*) \tag{C.3.13}$$

The derivatives of $\mathfrak{P}_t$ with respect to the other endogenous variables P_τ and y_τ are best calculated with the equation for $\mathfrak{P}_t$ in (C.3.9) before the order of summation was changed. Differentiating with respect to P_τ yields

$$\frac{\partial\mathfrak{P}_t}{\partial\mathsf{P}_\tau}=-\beta^{\tau-t}\boldsymbol{\omega}'(\boldsymbol{\Theta}(\mathbb{L})\mathbf{R}_\tau-\boldsymbol{\iota}\mathsf{P}_\tau-\hat{\boldsymbol{\rho}}_\tau)=\beta^{\tau-t}\boldsymbol{\iota}'\boldsymbol{\Omega}(\hat{\boldsymbol{\rho}}_\tau-\boldsymbol{\rho}_\tau^*) \tag{C.3.14}$$

where the equivalence up to second-order terms of $\boldsymbol{\Theta}(\mathbb{L})\mathbf{R}_\tau$ and $\mathbf{P}_\tau$ from (C.1.5) has been used, along with the definition of the price level $\mathsf{P}_\tau=\boldsymbol{\omega}'\mathbf{P}_\tau$ and the fact that $\boldsymbol{\omega}'\boldsymbol{\rho}_\tau^*=0$. Finally, consider the output gap y_τ. This clearly has an effect on $\mathfrak{U}_t$ through the first term in (4.3.1), but the result below shows that it has no indirect effect on price distortions through $\hat{\boldsymbol{\rho}}_\tau$ as given in equation (4.1.1). Since $\partial\hat{\boldsymbol{\rho}}_\tau/\partial\mathsf{y}_\tau=\eta_x\boldsymbol{\iota}$, the derivative of (C.3.9) with respect to y_τ is,

$$\frac{\partial\mathfrak{P}_t}{\partial\mathsf{y}_\tau}=\eta_x\beta^{\tau-t}\boldsymbol{\iota}'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau-\hat{\boldsymbol{\rho}}_\tau)-\eta_x\beta^{\tau-t}\boldsymbol{\iota}'\boldsymbol{\Omega}(\boldsymbol{\Theta}(\mathbb{L})\mathbf{R}_\tau-\boldsymbol{\iota}\mathsf{P}_\tau-\hat{\boldsymbol{\rho}}_\tau)=0 \tag{C.3.15}$$

where again the fact that $\boldsymbol{\Theta}(\mathbb{L})\mathbf{R}_\tau-\boldsymbol{\iota}\mathsf{P}_\tau$ is equal to $\boldsymbol{\rho}_\tau$ up to second-order terms has been used. The derivative of $\mathfrak{P}_t$ with respect to price vector $\mathbf{P}_\tau$ can be obtained by noting that $\mathsf{P}_\tau=\boldsymbol{\omega}'\mathbf{P}_\tau$, and $\boldsymbol{\rho}_\tau=\boldsymbol{\mathcal{R}}\mathbf{P}_\tau$ from (C.1.9), and then applying the chain rule,

$$\frac{\partial\mathfrak{P}_t}{\partial\mathbf{P}_\tau}=\frac{\partial\boldsymbol{\rho}_\tau'}{\partial\mathbf{P}_\tau}\frac{\partial\mathfrak{P}_t}{\partial\boldsymbol{\rho}_\tau}+\frac{\partial\mathsf{P}_\tau}{\partial\mathbf{P}_\tau}\frac{\partial\mathfrak{P}_t}{\partial\mathsf{P}_\tau}=\beta^{\tau-t}\boldsymbol{\Omega}(\hat{\boldsymbol{\rho}}_\tau-\boldsymbol{\rho}_\tau^*) \tag{C.3.16}$$

together with $\partial\boldsymbol{\rho}_\tau'/\partial\mathbf{P}_\tau=\boldsymbol{\mathcal{R}}'$ and $\partial\mathsf{P}_\tau/\partial\mathbf{P}_\tau=\boldsymbol{\omega}$, and using the definition of $\boldsymbol{\mathcal{R}}$ in (C.1.9) to simplify the expression. This confirms the second part of the proposition.

The third claim of the proposition requires that the losses from price distortions be reduced to a single quadratic form involving the inflation vector $\boldsymbol{\pi}_t$. Notice that the equation for the price index P_{it} in (3.2.11) and the definition (C.3.2) of the cross-sectional variance of log prices in industry i imply that

$$\sigma_{it}^2 = \left(\sum_{j=0}^{\infty} \theta_{ij} \mathsf{R}_{i,t-j}^2\right) - \mathsf{P}_{it}^2 + \mathcal{O}(||\boldsymbol{\upsilon}_t||^3) \tag{C.3.17}$$

where $\{\theta_{ij}\}_{j=0}^{\infty}$ is the age distribution of prices for industry i defined in (2.4.3). The expected discounted sum of current and future cross-sectional variances in (C.3.17) can be written as

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \sigma_{i\tau}^2 = \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\theta_i(\beta) \mathsf{R}_{i\tau}^2 - \mathsf{P}_{i\tau}^2\right] + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{\upsilon}_t||^3) \tag{C.3.18}$$

where the definition of $\theta_i(z)$ in (C.1.1) has been used and where "t.i.p." denotes terms independent of policy (exogenous or predetermined at time t). Using (C.1.3) and (C.3.18) the second term in the expression for $\mathfrak{P}_t$ from (C.3.1) corresponding to the discounted sum of total intra-industry price distortions is given by:

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\boldsymbol{\sigma}_\tau' \boldsymbol{\Omega} \boldsymbol{\sigma}_\tau\right] = \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[(\boldsymbol{\Phi}(1)\mathbf{R}_\tau)' \left(\boldsymbol{\Phi}(1)^{-1} \boldsymbol{\Omega} \boldsymbol{\Theta}(\beta)\right) \mathbf{R}_\tau - \mathbf{P}_\tau' \boldsymbol{\Omega} \mathbf{P}_\tau\right] + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{\upsilon}_t||^3) \tag{C.3.19}$$

By making use of (C.1.4) and (C.1.6) equation (C.3.19) becomes:

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\boldsymbol{\sigma}_\tau' \boldsymbol{\Omega} \boldsymbol{\sigma}_\tau\right] = \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[(\boldsymbol{\Phi}(\mathbb{L})\mathbf{P}_\tau)' \left(\boldsymbol{\Omega} \boldsymbol{\Phi}(\beta)^{-1}\right) \mathbf{R}_\tau - \mathbf{P}_\tau' \boldsymbol{\Omega} \mathbf{P}_\tau\right] + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{\upsilon}_t||^3) \tag{C.3.20}$$

A change in the order of summation in (C.3.20) yields the following equivalent expression:

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\boldsymbol{\sigma}_\tau' \boldsymbol{\Omega} \boldsymbol{\sigma}_\tau\right] = \sum_{\tau=t-m-1}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\mathbf{P}_\tau' \left(\boldsymbol{\Omega} \boldsymbol{\Phi}(\beta)^{-1} \boldsymbol{\Phi}(\beta\mathbb{F}) \mathbf{R}_\tau\right) - \mathbf{P}_\tau' \boldsymbol{\Omega} \mathbf{P}_\tau\right] + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{\upsilon}_t||^3) \tag{C.3.21}$$

Then using the first part of (C.1.6) again shows that (C.3.21) can be written as:

$$\begin{aligned}\sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\boldsymbol{\sigma}_\tau' \boldsymbol{\Omega} \boldsymbol{\sigma}_\tau\right] = \sum_{\tau=t-m-1}^{\infty} &\beta^{\tau-t} \mathbb{E}_t \left[\mathbf{P}_\tau' \left(\boldsymbol{\Omega} \boldsymbol{\Phi}(1)^{-1} \boldsymbol{\Phi}(\beta)^{-1} \boldsymbol{\Phi}(\beta\mathbb{F}) \boldsymbol{\Phi}(\mathbb{L}) \mathbf{P}_\tau\right) - \mathbf{P}_\tau' \boldsymbol{\Omega} \mathbf{P}_\tau\right] \\ &+ \text{t.i.p.} + \mathcal{O}(||\boldsymbol{\upsilon}_t||^3)\end{aligned} \tag{C.3.22}$$

The expression in (C.3.22) can be written in the simpler way by using the definitions of $\boldsymbol{\Upsilon}(z)$ and $\boldsymbol{\phi}$ in (C.1.7):

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\boldsymbol{\sigma}_\tau' \boldsymbol{\Omega} \boldsymbol{\sigma}_\tau\right] = \sum_{\tau=t-m-1}^{\infty} \beta^{\tau-t} \mathbb{E}_t \left[\mathbf{P}_\tau' \left(\boldsymbol{\Omega} \boldsymbol{\phi}^{-1} \boldsymbol{\Upsilon}(\mathbb{L}) \mathbf{P}_\tau\right)\right] + \text{t.i.p.} + \mathcal{O}(||\boldsymbol{\upsilon}_t||^3) \tag{C.3.23}$$

Now consider the first term in the expression for $\mathfrak{P}_t$ in (C.3.1) corresponding to inter-industry price distortions. Since efficient relative prices $\boldsymbol{\rho}_t^*$ are always independent of monetary policy and satisfy $\boldsymbol{\omega}' \boldsymbol{\rho}_t^* = 0$, the quadratic form in relative prices $\boldsymbol{\rho}_t$ can be written as

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t [(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}_\tau^*)' \boldsymbol{\Omega} (\boldsymbol{\rho}_\tau - \boldsymbol{\rho}_\tau^*)] = \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathbb{E}_t [\mathbf{P}_\tau' \boldsymbol{\Omega} \boldsymbol{\mathcal{R}} \mathbf{P}_\tau - 2\mathbf{P}_\tau' \boldsymbol{\Omega} \boldsymbol{\rho}_\tau^*] + \text{t.i.p.} \tag{C.3.24}$$

using the fact that $\boldsymbol{\rho}_t^* = \boldsymbol{\mathcal{R}} \boldsymbol{\rho}_t^*$, $\boldsymbol{\rho}_t = \boldsymbol{\mathcal{R}} \mathbf{P}_t$ and $\boldsymbol{\mathcal{R}}' \boldsymbol{\Omega} \boldsymbol{\mathcal{R}} = \boldsymbol{\Omega} \boldsymbol{\mathcal{R}}$ from the definition of $\boldsymbol{\mathcal{R}}$ in (C.1.9). By taking the result from equation (C.4.4), terms in $\wp_t$ appearing in (C.3.24) can be substituted for those in $\boldsymbol{\rho}_t^*$ as follows:

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t[(\boldsymbol{\rho}_\tau-\boldsymbol{\rho}^*_\tau)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau-\boldsymbol{\rho}^*_\tau)]=\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\mathbf{P}'_\tau\boldsymbol{\varOmega}\boldsymbol{\mathcal{R}}\mathbf{P}_\tau-2\mathbf{P}'_\tau\left(\boldsymbol{\varLambda}(\beta\mathbb{F})'\boldsymbol{\aleph}^{-1}\wp_\tau\right)\right]+\text{t.i.p.} \tag{C.3.25}$$

Because all terms dated earlier than t are predetermined with respect to monetary policy decisions made from time t onwards, (C.3.25) is equivalent to:

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t[(\boldsymbol{\rho}_\tau-\boldsymbol{\rho}^*_\tau)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau-\boldsymbol{\rho}^*_\tau)]=\sum_{\tau=t-m-1}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\mathbf{P}'_\tau\boldsymbol{\varOmega}\boldsymbol{\mathcal{R}}\mathbf{P}_\tau-2\mathbf{P}'_\tau\left(\boldsymbol{\varLambda}(\beta\mathbb{F})'\boldsymbol{\aleph}^{-1}\wp_\tau\right)\right]+\text{t.i.p.} \tag{C.3.26}$$

Using the definition of the matrix polynomial $\boldsymbol{\varLambda}(z)$ from (C.1.13), the symmetry of the matrix $\boldsymbol{\aleph}$ as established in Lemma 7, and by changing the order of summation in the second right-hand side term of (C.3.26):

$$\begin{aligned}\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t[(\boldsymbol{\rho}_\tau-\boldsymbol{\rho}^*_\tau)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau-\boldsymbol{\rho}^*_\tau)]=&\sum_{\tau=t-m-1}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\mathbf{P}'_\tau\boldsymbol{\varOmega}\boldsymbol{\mathcal{R}}\mathbf{P}_\tau\right]\\&-2\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[(\boldsymbol{\varLambda}(\mathbb{L})\mathbf{P}_\tau)'\,\boldsymbol{\aleph}^{-1}\wp_\tau\right]+\text{t.i.p.}\end{aligned} \tag{C.3.27}$$

By combining the results in equations (C.3.23) and (C.3.27) and using the definition of matrix polynomial $\boldsymbol{\chi}(z)$ in (C.1.11), the expression in (C.3.1) for the loss $\mathfrak{P}_t$ created by relative-price distortions is given by:

$$\mathfrak{P}_t=\sum_{\tau=t-m-1}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\mathbf{P}'_\tau\left(\boldsymbol{\chi}(\mathbb{L})\mathbf{P}_\tau\right)\right]-2\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[(\boldsymbol{\varLambda}(\mathbb{L})\mathbf{P}_\tau)'\,\boldsymbol{\aleph}^{-1}\wp_\tau\right]+\text{t.i.p.}+\mathcal{O}(||\boldsymbol{\upsilon}_t||^3) \tag{C.3.28}$$

Now substitute the factorization of $\boldsymbol{\chi}(z)$ from (C.1.12) as established by Lemma 7 into (C.3.28) and use the result $\boldsymbol{\varLambda}(\mathbb{L})\mathbf{P}_t=\boldsymbol{\pi}_t-\boldsymbol{\text{ш}}_t$ from (C.1.20) to deduce:

$$\begin{aligned}\mathfrak{P}_t=\sum_{\tau=t-m-1}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[\mathbf{P}'_\tau\left(\boldsymbol{\varLambda}(\beta\mathbb{F})'\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_\tau-\boldsymbol{\text{ш}}_\tau)\right)\right]-2\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[(\boldsymbol{\pi}_\tau-\boldsymbol{\text{ш}}_\tau)'\boldsymbol{\aleph}^{-1}\wp_\tau\right]\\+\text{t.i.p.}+\mathcal{O}(||\boldsymbol{\upsilon}_t||^3)\end{aligned} \tag{C.3.29}$$

Next, change the order of summation in the first term of (C.3.29) to obtain the following:

$$\begin{aligned}\mathfrak{P}_t=\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[(\boldsymbol{\varLambda}(\beta\mathbb{L})\mathbf{P}_\tau)'\,\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_\tau-\boldsymbol{\text{ш}}_\tau)\right]-2\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[(\boldsymbol{\pi}_\tau-\boldsymbol{\text{ш}}_\tau)'\boldsymbol{\aleph}^{-1}\wp_\tau\right]\\+\text{t.i.p.}+\mathcal{O}(||\boldsymbol{\upsilon}_t||^3)\end{aligned} \tag{C.3.30}$$

Finally, note that $\boldsymbol{\varLambda}(\mathbb{L})\mathbf{P}_t=\boldsymbol{\pi}_t-\boldsymbol{\text{ш}}_t$ and that $\wp_t$ is independent of monetary policy to deduce the following quadratic form for $\mathfrak{P}_t$ from (C.3.30):

$$\mathfrak{P}_t=\sum_{\tau=t}^{\infty}\beta^{\tau-t}\mathbb{E}_t\left[(\boldsymbol{\pi}_\tau-\boldsymbol{\text{ш}}_\tau-\wp_\tau)'\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_\tau-\boldsymbol{\text{ш}}_\tau-\wp_\tau)\right]+\text{t.i.p.}+\mathcal{O}(||\boldsymbol{\upsilon}_t||^3) \tag{C.3.31}$$

By comparing (4.3.5) and (C.3.1) and substituting (C.3.31) the third and final part of the proposition is proved.

C.4 Proof of Proposition 4

The expression for the policymaker's loss function $\mathfrak{U}_t$ in (4.3.1) reveals that the terms associated with price distortions from time period t_0 onwards are:

$$\mathfrak{P}_{t_0} \equiv \frac{1}{2}\sum_{\tau=t_0}^{\infty}\beta^{\tau-t_0}\mathbb{E}_{t_0}\left[(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}^*_\tau)'\boldsymbol{\Omega}(\boldsymbol{\rho}_\tau - \boldsymbol{\rho}^*_\tau) + \boldsymbol{\sigma}'_\tau\boldsymbol{\Omega}\boldsymbol{\sigma}_\tau\right] \tag{C.4.1}$$

By comparing this with the equation for $\mathfrak{U}_t$ in equation (4.3.5), an equivalent expression for $\mathfrak{P}_{t_0}$ is:

$$\mathfrak{P}_{t_0} = \frac{1}{2}\sum_{\tau=t_0}^{\infty}\beta^{\tau-t_0}\mathbb{E}_{t_0}\left[(\boldsymbol{\pi}_\tau - \boldsymbol{\text{ш}}_\tau - \wp_\tau)'\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_\tau - \boldsymbol{\text{ш}}_\tau - \wp_\tau)\right] \tag{C.4.2}$$

As it is shown in Lemma 7 that the matrix $\boldsymbol{\aleph}$ is positive definite, it follows that $\mathfrak{P}_{t_0} = 0$ if and only if $\boldsymbol{\pi}_t = \boldsymbol{\text{ш}}_t + \wp_t$ for all $t \geq t_0$. This establishes the equivalence of the first two statements in the proposition.

To establish the equivalence of the second and third statements, note that the combination of equations (C.1.14), (C.1.20), and (4.2.2) implies:

$$\mathbb{E}_t[\boldsymbol{\varLambda}(\beta\mathbb{F})'\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t)] = \boldsymbol{\Omega}\hat{\boldsymbol{\rho}}_t \tag{C.4.3}$$

The definitions of the efficient cost-push shock vector $\wp_t$ in (4.2.4) and the matrix polynomial $\boldsymbol{\varXi}(z)$ in (C.1.15) imply the following:

$$\mathbb{E}_t[\boldsymbol{\varLambda}(\beta\mathbb{F})'\boldsymbol{\aleph}^{-1}\wp_t] = \boldsymbol{\Omega}\boldsymbol{\rho}^*_t \tag{C.4.4}$$

Then subtracting (C.4.4) from (C.4.3) and multiplying both sides by the inverse of $\boldsymbol{\Omega}$ yields:

$$\hat{\boldsymbol{\rho}}_t - \boldsymbol{\rho}^*_t = \mathbb{E}_t[\boldsymbol{\Omega}^{-1}\boldsymbol{\varLambda}(\beta\mathbb{F})'\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \wp_t)] \tag{C.4.5}$$

Because the matrix polynomial $\boldsymbol{\varLambda}(\beta z)$ has all its roots strictly outside the unit circle, equation (C.4.5) can be converted into the following using the definition of $\boldsymbol{\varXi}(z)$ in (C.1.15):

$$\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \wp_t = \mathbb{E}_t[\boldsymbol{\aleph}\boldsymbol{\varXi}(\mathbb{F})\boldsymbol{\Omega}(\hat{\boldsymbol{\rho}}_t - \boldsymbol{\rho}^*_t)] \tag{C.4.6}$$

Therefore, it is clear from (C.4.5) and (C.4.6) that $\boldsymbol{\pi} = \boldsymbol{\text{ш}}_t + \wp_t$ for all $t \geq t_0$ implies and is implied by $\hat{\boldsymbol{\rho}}_t = \boldsymbol{\rho}^*_t$ for all $t \geq t_0$, completing the proof.

C.5 Proof of Proposition 5

First consider the problem of finding the time-path of prices assuming that the policymaker minimizes price distortions from period t_0 onwards. Start by defining the stochastic process $\mathbf{P}^{**}_{t|t_0}$ as follows,

$$\mathbf{P}^{**}_{t|t_0} = \sum_{j=1}^{m+1}\boldsymbol{\Lambda}_j\mathbf{P}^{**}_{t-j|t_0} + \wp_t \tag{C.5.1}$$

using the efficient cost-push shock process $\wp_t$ from (4.2.4) and with $m+1$ initial conditions given by $\mathbf{P}^{**}_{t_0-1|t_0} = \mathbf{0}$, $\ldots$, $\mathbf{P}^{**}_{t_0-(m+1)|t_0} = \mathbf{0}$. With these initial conditions it is clear that (C.5.1) defines a unique stochastic process $\{\mathbf{P}^{**}_{t|t_0}\}_{t=t_0}^{\infty}$ such that $\boldsymbol{\varLambda}(\mathbb{L})\mathbf{P}^{**}_{t|t_0} = \wp_t$ for all $t \geq t_0$. Using the results of Lemma 8, in particular equation (B.8.6), it is seen that the difference equation (C.5.1) is equivalent to the system in (4.3.7).

It has been shown in Proposition 4 that $\hat{\boldsymbol{\rho}}_t = \boldsymbol{\rho}^*_t$ when $t \geq t_0$ is equivalent to $\boldsymbol{\pi}_t = \boldsymbol{\text{ш}}_t + \wp_t$ for $t \geq t_0$.

Let the resulting path of the price vector be denoted by $\mathbf{P}^*_{t|t_0}$. The initial conditions are given by the actual history of prices prior to t_0, so $\mathbf{P}^*_{t|t_0} = \mathbf{P}_t$ if $t < t_0$. Since equations (4.2.2) and (C.1.20) show that $\boldsymbol{\pi}_t - \mathbf{m}_t = \boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}_t$, the price-level path $\mathbf{P}^*_{t|t_0}$ is determined by $\boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}^*_{t|t_0} = \wp_t$. Let $\mathbf{g}_{t|t_0} \equiv \mathbf{P}^*_{t|t_0} - \mathbf{P}^{**}_{t|t_0}$, and note that the construction of the vector $\mathbf{P}^{**}_{t|t_0}$ in (C.5.1) implies the price path can be determined by solving $\boldsymbol{\Lambda}(\mathbb{L})\mathbf{g}_{t|t_0} = \mathbf{0}$ with initial conditions $\mathbf{g}_{t|t_0} = \mathbf{P}_t$ for $t < t_0$. By using (4.2.2), (4.2.6) and (C.1.20), it can be seen that the unique solution for $\mathbf{g}_{t|t_0}$ is given by the time-path of intrinsic inflation, $\mathbf{g}_{t|t_0} = \tilde{\mathbf{P}}_t(\mathcal{H}_{t_0})$. Hence, $\mathbf{P}^*_{t|t_0} = \tilde{\mathbf{P}}_t(\mathcal{H}_{t_0}) + \mathbf{P}^{**}_{t|t_0}$. This proves the first part of the proposition.

Now consider the case where price distortions are minimized in all time periods, not just after some date t_0. From Proposition 4 this is equivalent to $\hat{\boldsymbol{\rho}}_t = \boldsymbol{\rho}^*_t$ for all t. The solution for the price vector in this case is denoted by $\mathbf{P}^*_t$. By using equations (C.1.16) and (4.2.4), $\mathbf{P}^*_t$ is determined by the following difference equation

$$\boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}^*_t = \wp_t \tag{C.5.2}$$

for all t. Suppose $\{\mathbf{P}^{**}_t\}$ is a particular solution of (C.5.2), and let $\{\mathbf{P}^*_t\}$ be any other solution. Denote the difference between these solutions by $\mathbf{g}_t \equiv \mathbf{P}^*_t - \mathbf{P}^{**}_t$. It is clear from (C.5.2) that $\mathbf{g}_t$ must satisfy the following linear homogeneous difference equation for all t:

$$\boldsymbol{\Lambda}(\mathbb{L})\mathbf{g}_t = \mathbf{0} \tag{C.5.3}$$

With arbitrary initial conditions given for some date t_0, the result in Lemma 8 shows that the solution would take the form,

$$\mathbf{g}_{t|t_0} = \bar{\mathsf{P}}\boldsymbol{\iota} + \mathbf{f}(t - t_0; \mathfrak{c}_2, \ldots, \mathfrak{c}_{mn}) \tag{C.5.4}$$

for some constants $\bar{\mathsf{P}}$ and $\mathfrak{c}_2, \ldots, \mathfrak{c}_{mn}$. As (C.5.3) must hold in all periods, take the limit $t_0 \to -\infty$ to get the class of solutions for which $\mathbf{g}_t$ is well-defined in all time periods. As Lemma 8 shows that $\lim_{\tau\to\infty} \mathbf{f}(\tau; \mathfrak{c}_2, \ldots, \mathfrak{c}_{mn}) = \mathbf{0}$, this class of solutions is given by $\mathbf{g}_t = \bar{\mathsf{P}}\boldsymbol{\iota}$. Therefore, the range of possible solutions for $\mathbf{P}^*_t$ is given by $\mathbf{P}^*_t = \bar{\mathsf{P}}\boldsymbol{\iota} + \mathbf{P}^{**}_t$, where $\bar{\mathsf{P}}$ is an arbitrary constant, and $\mathbf{P}^{**}_t$ is the particular solution of (C.5.2) normalized so that $\mathsf{P}^{**}_{t_0} = 0$ in some time period t_0. The process $\mathbf{P}^{**}_t$ depends only on the exogenous cost-push shocks $\wp_t$ and an arbitrary constant $\bar{\mathsf{P}}$, which cancels out from the implied inflation rate, ensuring that it is uniquely determined by the exogenous cost-push shocks $\wp_t$. This establishes the second part of the proposition.

Finally, under the hypothesis that all industry pricing hazard functions are identical, the results of Proposition 2 can be applied. If $\boldsymbol{\omega}'\mathfrak{c} = 0$ then equation (C.2.9) shows that $\boldsymbol{\omega}'\boldsymbol{\Lambda}(z)\mathfrak{c} = 0$. Use the definition of the matrix polynomial $\boldsymbol{\Lambda}(z)$ in (C.1.13) to write this explicitly as:

$$(\boldsymbol{\omega}'\mathfrak{c}) - \sum_{j=1}^{m+1} (\boldsymbol{\omega}'\boldsymbol{\Lambda}_j\mathfrak{c})z^j = 0 \tag{C.5.5}$$

Equation (C.5.5) is a polynomial in z that is identically equal to zero when $\boldsymbol{\omega}'\mathfrak{c} = 0$. So it must be the case that all coefficients of the powers of z are zero, that is, $\boldsymbol{\omega}'\boldsymbol{\Lambda}_j\mathfrak{c} = 0$ for all $j = 1, \ldots, m+1$. Now pre-multiply both sides of (C.5.1) by $\boldsymbol{\omega}'$ to obtain:

$$\boldsymbol{\omega}'\mathbf{P}^{**}_{t|t_0} = \sum_{j=1}^{m+1} \boldsymbol{\omega}'\boldsymbol{\Lambda}_j\mathbf{P}^{**}_{t-j|t_0} + \boldsymbol{\omega}'\wp_t \tag{C.5.6}$$

The results of Proposition 2 directly show that under the hypothesis of identical hazard functions it must be the case that $\boldsymbol{\omega}'\wp_t = 0$ for all realizations of the cost-push shocks. Now suppose that $\boldsymbol{\omega}'\mathbf{P}^{**}_{t-j|t_0} = 0$ for some t and all $j = 1, \ldots, m+1$. It follows that $\boldsymbol{\omega}'\boldsymbol{\Lambda}_j\mathbf{P}^{**}_{t-j|t_0} = 0$ for all j. Since $\boldsymbol{\omega}'\wp_t = 0$, equation

(C.5.6) implies $\boldsymbol{\omega}'\mathbf{P}^{**}_{t|t_0} = 0$ as well. As the initial conditions for (C.5.1) guarantee that $\boldsymbol{\omega}'\mathbf{P}^{**}_{t_0-j|t_0}$ for all $j \geq 1$, the result $\boldsymbol{\omega}'\mathbf{P}^{**}_{t|t_0} = 0$ follows for all $t \geq t_0$ by induction. The third part of the proposition is a direct consequence of this statement, completing the proof.

D Proofs of theorems

D.1 Proof of Theorem 1

Let the vector $\mathbf{g}_t \equiv \boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \boldsymbol{\wp}_t$ be the deviation of the inflation vector from the sum of current intrinsic inflation and efficient cost-push shocks, as defined in (4.2.2) and (4.2.4). The system of pricing equations in (C.1.14) together with (C.1.20), (4.1.1) and definitions (4.2.2) and (4.2.4) imply that:

$$\boldsymbol{\aleph}^{-1}\mathbf{g}_t - \sum_{j=1}^{m+1} \beta^j \boldsymbol{\Lambda}'_j \boldsymbol{\aleph}^{-1}\mathbb{E}_t\mathbf{g}_{t+j} = \eta_x \boldsymbol{\omega}\mathsf{y}_t + \boldsymbol{\Omega}\boldsymbol{\epsilon}_t \tag{D.1.1}$$

The expression for the policymaker's loss function in (4.3.5) can also be written in terms of $\mathbf{g}_t$ and y_t:

$$\mathfrak{U}_t = \frac{1}{2}\sum_{\tau=t}^{\infty} \beta^{\tau-t}\mathbb{E}\left[\frac{\eta_x}{\varepsilon}\mathsf{y}_\tau^2 + \mathbf{g}'_\tau \boldsymbol{\aleph}^{-1}\mathbf{g}_\tau\right] \tag{D.1.2}$$

The system of equations (D.1.1) and loss function (D.1.2) suggest that a natural discretionary equilibrium will make both $\mathbf{g}_t$ and y_t functions of the current state of the exogenous stochastic process $\{\boldsymbol{\epsilon}_t\}$, denoted by $\mathfrak{S}_t$. Attention is focused only on such Markovian equilibria.

At time t, suppose that the public believes that such a Markovian equilibrium will prevail in all future time periods from $t+1$ onwards. This means that the system of pricing equations in (D.1.1) is equivalent to

$$\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \boldsymbol{\wp}_t) - \eta_x\boldsymbol{\omega}\mathsf{y}_t = \eta_\epsilon \boldsymbol{\Omega}\boldsymbol{\epsilon}_t + \daleth(\mathfrak{S}_t) \tag{D.1.3}$$

where $\daleth(\cdot)$ is some $n \times 1$ vector-valued function of exogenous state $\mathfrak{S}_t$. Likewise, the belief that this Markovian equilibrium will prevail in the future means that the loss function (D.1.2) can be written as

$$\mathfrak{U}_t = \frac{1}{2}\left(\frac{\eta_x}{\varepsilon}\mathsf{y}_t^2 + (\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \boldsymbol{\wp}_t)'\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \boldsymbol{\wp}_t)\right) + \gimel(\mathfrak{S}_t) \tag{D.1.4}$$

for some function $\gimel(\cdot)$ of the exogenous state $\mathfrak{S}_t$.

As the policymaker can control aggregate demand by setting the interest rate i_t in (3.1.4), and since this interest rate does not enter the loss function, the pricing equations in (D.1.3) are the only effective constraint. Thus, the Lagrangian for minimizing the loss function (D.1.4) subject to (D.1.3) is

$$\begin{aligned}\mathfrak{L}^d_t = &\frac{1}{2}\left(\frac{\eta_x}{\varepsilon}\mathsf{y}_t^2 + (\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \boldsymbol{\wp}_t)'\boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \boldsymbol{\wp}_t)\right) + \gimel(\mathfrak{S}_t)\\ &+ \boldsymbol{\ell}^{d\prime}_t\left(\eta_x\boldsymbol{\omega}\mathsf{y}_t - \boldsymbol{\aleph}^{-1}(\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \boldsymbol{\wp}_t) + \eta_\epsilon\boldsymbol{\Omega}\boldsymbol{\epsilon}_t + \daleth(\mathfrak{S}_t)\right)\end{aligned} \tag{D.1.5}$$

where $\boldsymbol{\ell}^d_t$ is an $n \times 1$ vector of Lagrangian multipliers. The first-order conditions of Lagrangian (D.1.5) with respect to y_t and $\boldsymbol{\pi}_t$ are:

$$\frac{\partial \mathfrak{L}^d_t}{\partial \mathsf{y}_t} = \eta_x\left\{\frac{1}{\varepsilon}\mathsf{y}_t + \boldsymbol{\omega}'\boldsymbol{\ell}^d_t\right\} = 0 \tag{D.1.6a}$$

$$\frac{\partial \mathfrak{L}^d_t}{\partial \boldsymbol{\pi}_t} = \boldsymbol{\aleph}^{-1}\left\{(\boldsymbol{\pi}_t - \boldsymbol{\text{ш}}_t - \boldsymbol{\wp}_t) - \boldsymbol{\ell}^d_t\right\} = \mathbf{0} \tag{D.1.6b}$$

Equation (D.1.6b) implies that the Lagrangian multipliers are given by $\boldsymbol{\ell}_t^d = \boldsymbol{\pi}_t - \boldsymbol{\mathrm{m}}_t - \boldsymbol{\wp}_t$. By substituting this into (D.1.6a), the Lagrangian multipliers are eliminated from the first-order conditions (D.1.6) leaving the following optimality condition:

$$\boldsymbol{\omega}'\mathbf{g}_t + \frac{1}{\varepsilon}\mathsf{y}_t = 0 \tag{D.1.7}$$

Substituting this back into the original system of pricing equations (D.1.1) implies that $\mathbf{g}_t$ is determined by the following:

$$(\boldsymbol{\aleph}^{-1} + \eta_x\varepsilon\boldsymbol{\omega}\boldsymbol{\omega}')\mathbf{g}_t - \sum_{j=1}^{m+1}\boldsymbol{\Lambda}_j'\boldsymbol{\aleph}^{-1}\mathbb{E}_t\mathbf{g}_{t+j} = \eta_\epsilon\boldsymbol{\Omega}\boldsymbol{\epsilon}_t \tag{D.1.8}$$

The equation above involves only the current and expected future values of $\mathbf{g}_t$ and the current vector of disturbances $\boldsymbol{\epsilon}_t$. Hence, there exists a solution for $\{\mathbf{g}_t\}$ in which $\mathbf{g}_t$ depends only on the current exogenous state $\mathfrak{S}_t$. From (D.1.7) it follows that there is also a solution for $\{\mathsf{y}_t\}$ in which y_t depends exclusively on $\mathfrak{S}_t$. This confirms the supposition under which (D.1.3) and (D.1.4) were derived. Thus, a discretionary Markovian equilibrium exists in which (D.1.7) is satisfies. This equilibrium requires that the policymaker adjust interest rates to ensure that:

$$\boldsymbol{\omega}'(\boldsymbol{\pi}_t - \boldsymbol{\mathrm{m}}_t - \boldsymbol{\wp}_t) + \frac{1}{\varepsilon}\mathsf{y}_t = 0 \tag{D.1.9}$$

This clearly implies equation (5.1.1), so the claim of the theorem is verified.

D.2 Proof of Theorem 2

Pick an initial date t_0 when the binding commitment comes into force. Take the time-series $\{\mathbf{P}_{t|t_0}^{**}\}$ constructed in equation (C.5.1) which has the properties that $\boldsymbol{\Lambda}(\mathbb{L})\mathbf{P}_{t|t_0}^{**} = \boldsymbol{\wp}_t$ for all $t \geq t_0$ and $\mathbf{P}_{t|t_0}^{**} = \mathbf{0}$ if $t < t_0$. The definitions of $\boldsymbol{\wp}_t$ in (4.2.4) and the matrix polynomial $\boldsymbol{\Xi}(z)$ in (C.1.15) imply that $\mathbb{E}_t[\boldsymbol{\Lambda}(\beta\mathbb{F})'\boldsymbol{\aleph}^{-1}\boldsymbol{\wp}_t] = \boldsymbol{\Omega}\boldsymbol{\rho}_t^*$, and it follows that $\mathbf{P}_{t|t_0}^{**}$ satisfies

$$\mathbb{E}_t[\boldsymbol{\chi}(\mathbb{L})\mathbf{P}_{t|t_0}^{**}] = \boldsymbol{\Omega}\boldsymbol{\rho}_t^* \tag{D.2.1}$$

for all $t \geq t_0$. At time t_0, the constraints from pricing equations $\mathbb{E}_t[\boldsymbol{\chi}(\mathbb{L})\mathbf{P}_t] = \boldsymbol{\Omega}\hat{\boldsymbol{\rho}}_t$ in (C.1.11) apply in all time periods from t_0 onwards. By subtracting (D.2.1) from both sides and using (4.1.1), these constraints can be equivalently stated as

$$\mathbb{E}_t[\boldsymbol{\chi}(\mathbb{L})(\mathbf{P}_t - \mathbf{P}_{t|t_0}^{**})] = \eta_x\boldsymbol{\omega}\mathsf{y}_t + \boldsymbol{\Omega}\boldsymbol{\epsilon}_t \tag{D.2.2}$$

for all $t \geq t_0$.

The problem of minimizing loss function $\mathfrak{U}_{t_0}$ from (4.3.1) subject to all the constraints in (D.2.2) can be solved by setting up the following Lagrangian function

$$\mathfrak{L}_{t_0}^c \equiv \mathfrak{U}_{t_0} + \sum_{\tau=t_0}^{\infty}\beta^{\tau-t_0}\mathbb{E}_{t_0}\left[\boldsymbol{\ell}_{\tau|t_0}^{c}{}'\{\eta_x\boldsymbol{\omega}\mathsf{y}_\tau - \boldsymbol{\chi}(\mathbb{L})(\mathbf{P}_\tau - \mathbf{P}_{\tau|t_0}^{**}) + \eta_\epsilon\boldsymbol{\Omega}\boldsymbol{\epsilon}_\tau\}\right] \tag{D.2.3}$$

where $\boldsymbol{\ell}_{t|t_0}^c$ is an $n \times 1$ vector of Lagrangian multipliers associated with the constraints in (D.2.2) at time t. Future Lagrangian multipliers are multiplied by the subjective discount factor β for convenience. The result contained in equation (4.3.4) implies that $\partial\mathfrak{U}_{t_0}/\partial\mathbf{P}_t = \beta^{t-t_0}\boldsymbol{\Omega}(\hat{\boldsymbol{\rho}}_t - \boldsymbol{\rho}_t^*)$, so the first-order conditions of the Lagrangian (D.2.3) are

$$\frac{\partial\mathfrak{L}_{t_0}^c}{\partial\mathsf{y}_t} = \eta_x\beta^{t-t_0}\left\{\frac{1}{\varepsilon}\mathsf{y}_t + \boldsymbol{\omega}'\boldsymbol{\ell}_{t|t_0}^c\right\} = 0 \tag{D.2.4a}$$

$$\frac{\partial\mathfrak{L}_{t_0}^c}{\partial\mathbf{P}_t} = \beta^{t-t_0}\left\{(\hat{\boldsymbol{\rho}}_t - \boldsymbol{\rho}_t^*) - \mathbb{E}_t\left[\boldsymbol{\chi}(\beta\mathbb{F})'\boldsymbol{\ell}_{t|t_0}^c\right]\right\} = \mathbf{0} \tag{D.2.4b}$$

for all $t \geq t_0$, and where the Lagrangian multiplier $\boldsymbol{\ell}^c_{t|t_0}$ has been set to $\mathbf{0}$ when $t < t_0$ because the pricing equations from periods prior to t_0 do not act as constraints.

Using the expression for $\hat{\boldsymbol{\rho}}_t$ and $\boldsymbol{\rho}^*_t$ in (4.1.1) and equation (D.2.2) it can be seen that $\hat{\boldsymbol{\rho}}_t - \boldsymbol{\rho}^*_t = \mathbb{E}_t[\boldsymbol{\Lambda}(\mathbb{L})(\mathbf{P}_t - \mathbf{P}^{**}_{t|t_0})]$ for all $t \geq t_0$. Together with the discounted para-Hermitian property of matrix polynomial $\boldsymbol{\chi}(z)$, that is $\boldsymbol{\chi}(z) = \boldsymbol{\chi}(\beta z^{-1})'$, the first-order condition (D.2.4b) is equivalent to the following for all $t \geq t_0$:

$$\mathbb{E}_t\left[\boldsymbol{\chi}(\mathbb{L})(\mathbf{P}_t - \mathbf{P}^{**}_{t|t_0} - \boldsymbol{\ell}^c_{t|t_0})\right] = \mathbf{0} \tag{D.2.5}$$

The factorization of the matrix polynomial $\boldsymbol{\chi}(z) = \boldsymbol{\Lambda}(\beta z^{-1})'\aleph^{-1}\boldsymbol{\Lambda}(z)$ given in (C.1.12) and the knowledge that $\boldsymbol{\Lambda}(\beta z)$ has all its roots strictly outside the unit circle mean that first-order condition (D.2.5) is also equivalent to

$$\boldsymbol{\Lambda}(\mathbb{L})(\mathbf{P}_t - \mathbf{P}^{**}_{t|t_0} - \boldsymbol{\ell}^c_{t|t_0}) = \mathbf{0} \tag{D.2.6}$$

again for $t \geq t_0$. Define $\mathbf{g}_{t|t_0}$ as $\mathbf{g}_{t|t_0} \equiv \mathbf{P}_t - \mathbf{P}^{**}_t - \boldsymbol{\ell}^c_{t|t_0}$, so equation (D.2.6) can be stated as $\boldsymbol{\Lambda}(\mathbb{L})\mathbf{g}_{t|t_0} = \mathbf{0}$. Since both $\mathbf{P}^{**}_{t|t_0}$ and $\boldsymbol{\ell}^c_{t|t_0}$ are $\mathbf{0}$ for $t < t_0$, this difference equations has initial conditions given by $\mathbf{g}_{t|t_0} = \mathbf{P}_t$ for $t < t_0$. By comparing it to equation (C.1.20), the definition of intrinsic inflation $\boldsymbol{\text{Ш}}_t$ in (4.2.2), it is clear that the time-path $\tilde{\mathbf{P}}_t(\mathcal{H}_{t_0})$ of the price level implied by intrinsic inflation, defined in (4.2.6), provides the unique solution $\mathbf{g}_{t|t_0} = \tilde{\mathbf{P}}_t(\mathcal{H}_{t_0})$ of this difference equation. Hence,

$$\mathbf{P}_t - \mathbf{P}^{**}_t - \boldsymbol{\ell}^c_{t|t_0} = \tilde{\mathbf{P}}_t(\mathcal{H}_{t_0}) \tag{D.2.7}$$

holds for all t.

The long-run target price path is taken from Proposition 4, which defines $\mathbf{P}^*_{t|t_0} \equiv \tilde{\mathbf{P}}_t(\mathcal{H}_{t_0}) + \mathbf{P}^{**}_{t|t_0}$. From (D.2.7), the Lagrangian multipliers are thus equal to the difference between the actual price path and this long-run target path, $\boldsymbol{\ell}^c_{t|t_0} = \mathbf{P}_t - \mathbf{P}^*_{t|t_0}$. Combined with the first-order condition (D.2.4a) associated with the output gap, this yields the following optimality condition

$$\frac{1}{\varepsilon}\mathsf{y}_t + \boldsymbol{\omega}'(\mathbf{P}_t - \mathbf{P}^*_{t|t_0}) = 0 \tag{D.2.8}$$

for all $t \geq t_0$. In terms of the economy-wide price level P_t and long-run price-level target $\mathsf{P}^*_{t|t_0}$, equation (D.2.8) becomes

$$\mathsf{P}_t + \frac{1}{\varepsilon}\mathsf{y}_t = \mathsf{P}^*_{t|t_0} \tag{D.2.9}$$

again for all $t \geq t_0$. As $\mathsf{P}_t = \pi_t + \sum_{\tau=t_0}^{t-1}\pi_\tau + \mathsf{P}_{t_0-1}$, $\mathsf{P}^*_{t|t_0} = \pi^*_{t|t_0} + \sum_{\tau=t_0}^{t-1}\pi^*_{\tau|t_0} + \mathsf{P}^*_{t_0-1|t_0}$ and $\mathsf{P}_{t_0-1} = \mathsf{P}^*_{t_0-1|t_0}$, equation (D.2.9) implies the targeting rule stated in (5.2.1), confirming the claim of the theorem.

D.3 Proof of Theorem 3

Fix an arbitrary long-run price level target $\bar{\mathsf{P}}$, and define the price-level path $\mathbf{P}^*_t = \bar{\mathsf{P}}\boldsymbol{\iota} + \mathbf{P}^{**}_t$ in accordance with that constructed in Proposition 4. The unique time-series $\{\mathbf{P}^{**}_t\}$ from Proposition 4 depends only on the exogenous stochastic process $\{\boldsymbol{\rho}^*_t\}$. It is clear from equations (4.2.4), (C.1.12) and (C.5.2) that $\mathbb{E}_t[\boldsymbol{\chi}(\mathbb{L})\mathbf{P}^*_t] = \boldsymbol{\Omega}\boldsymbol{\rho}^*_t$ for all t.

Putting the above property of $\mathbf{P}^*_t$ together with equations (C.1.11) and (4.1.1), the system of pricing equations (4.1.2) can be equivalently expressed as:

$$\mathbb{E}_t[\boldsymbol{\chi}(\mathbb{L})(\mathbf{P}_t - \mathbf{P}^*_t)] = \eta_x\boldsymbol{\omega}\mathsf{y}_t + \eta_\epsilon\boldsymbol{\Omega}\boldsymbol{\epsilon}_t \tag{D.3.1}$$

As $\boldsymbol{\omega}'\mathsf{P}_t = \pi_t + \mathsf{P}_{t-1}$ and $\boldsymbol{\omega}'\mathbf{P}^*_t = \pi^*_t + \mathsf{P}^*_{t-1}$, the conjectured targeting rule in (5.3.1) can be written as:

$$\frac{1}{\varepsilon}\mathsf{y}_t = -\boldsymbol{\omega}'(\mathbf{P}_t - \mathbf{P}^*_t) \tag{D.3.2}$$

By substituting the targeting rule (D.3.2) into (D.3.1), a system of equations is produced that can be used to solve for the difference $\mathbf{P}_t - \mathbf{P}^*_t$:

$$\mathbb{E}_t\left[(\boldsymbol{\chi}(\mathbb{L}) + \eta_x\varepsilon\boldsymbol{\omega}\boldsymbol{\omega}')(\mathbf{P}_t - \mathbf{P}^*_t)\right] = \eta_\epsilon\boldsymbol{\Omega}\boldsymbol{\epsilon}_t \tag{D.3.3}$$

Examination of the matrix polynomial $\boldsymbol{\chi}(z)$ in (B.6.6) shows that (D.3.3) has $m+1$ lags of $\mathbf{P}_t$ and $\mathbf{P}^*_t$. The history $\mathcal{H}_t$ in (4.2.1) comprises knowledge of $m+1$ lags of $\mathbf{P}_t$. Similarly, define a history of the exogenous stochastic process $\mathbf{P}^*_t$ as $\mathcal{H}^*_t \equiv \{\ \mathbf{P}^*_{t-1}\ ,\ \ldots\ ,\ \mathbf{P}^*_{t-m-1}\ \}$. Equation (D.3.3) also contains the current value of the exogenous disturbance $\boldsymbol{\epsilon}_t$, and this exogenous process has current state $\mathfrak{S}_t$. Therefore, there exists a solution $\mathbf{P}_t - \mathbf{P}^*_t = \tilde{\mathbf{g}}_t(\mathcal{H}_t, \mathcal{H}^*_t, \mathfrak{S}_t)$ of (D.3.3) which has the following form,

$$\tilde{\mathbf{g}}_t(\mathcal{H}_t, \mathcal{H}^*_t, \mathfrak{S}_t) = \sum_{j=1}^{m+1} \boldsymbol{\mathcal{B}}_j(\mathbf{P}_{t-j} - \mathbf{P}^*_{t-j}) + \boldsymbol{\daleth}(\mathfrak{S}_t) \tag{D.3.4}$$

where the $\boldsymbol{\mathcal{B}}_j$ are $n \times n$ coefficient matrices, and $\boldsymbol{\daleth}(\cdot)$ is an $n \times 1$ vector-valued function of the current exogenous state $\mathfrak{S}_t$. This solution gives the current value of $\mathbf{P}_t - \mathbf{P}^*_t$ in terms of both predetermined and exogenous variables.

Consider the problem of choosing an optimal commitment starting from some date t_0. The policymaker faces pricing constraints (D.3.1) in all periods from t_0 onwards. In addition, there are $m+1$ vectors of constraints requiring that the policymaker initially allow inflation to evolve as it would were the conjectured targeting rule (5.3.1) followed. This evolution of inflation is given by the solution for $\mathbf{P}_t - \mathbf{P}^*_t$ in (D.3.4). Thus, these constraints are of the form $\mathbf{P}_t = \mathbf{P}^*_t + \tilde{\mathbf{g}}_t(\mathcal{H}_t, \mathcal{H}^*_t, \mathfrak{S}_t)$ for $t = t_0, t_0+1, \ldots, t_0+m$. It is known that there is a price-level and output gap path consistent with all these constraints because solution (D.3.4) was derived on the assumption that both the pricing equations (D.3.1) and the targeting rule (5.3.1) held from t_0 onwards.

The Lagrangian for this problem is obtained from loss function (4.3.1), pricing constraints (D.3.1), time-consistency constraints derived from (D.3.4),

$$\begin{aligned}\mathfrak{L}^p_{t_0} = \mathfrak{U}_{t_0} &+ \sum_{\tau=t_0}^{\infty} \beta^{\tau-t_0}\mathbb{E}_{t_0}\left[\boldsymbol{\ell}^{p\,\prime}_{\tau|t_0}\left\{\eta_x\boldsymbol{\omega}\mathsf{y}_\tau - \boldsymbol{\chi}(\mathbb{L})(\mathbf{P}_\tau - \mathbf{P}^*_\tau) + \eta_\epsilon\boldsymbol{\Omega}\boldsymbol{\epsilon}_\tau\right\}\right] \\ &+ \sum_{\tau=t_0}^{t_0+m} \beta^{\tau-t_0}\mathbb{E}_{t_0}\left[\boldsymbol{\varkappa}'_{\tau|t_0}\left\{\sum_{j=1}^{m+1}\boldsymbol{\mathcal{B}}_j(\mathbf{P}_{\tau-j} - \mathbf{P}^*_{\tau-j}) - (\mathbf{P}_\tau - \mathbf{P}^*_\tau) + \boldsymbol{\daleth}(\mathfrak{S}_\tau)\right\}\right]\end{aligned} \tag{D.3.5}$$

where $\boldsymbol{\ell}^p_{t|t_0}$ and $\boldsymbol{\varkappa}_{t|t_0}$ are the vectors of Lagrangian multipliers associated with the pricing and time-consistency constraints respectively. The first-order condition of the Lagrangian (D.3.5) with respect to the output gap y_t is

$$\frac{\partial\mathfrak{L}^p_{t_0}}{\partial\mathsf{y}_t} = \eta_x\beta^{t-t_0}\left\{\frac{1}{\varepsilon}\mathsf{y}_t + \boldsymbol{\omega}'\boldsymbol{\ell}^p_{t|t_0}\right\} = 0 \tag{D.3.6a}$$

for all $t \geq t_0$. For $t = t_0, \dots, t_0 + m$ the first-order condition with respect the price level vector $\mathbf{P}_t$ is

$$\begin{aligned}\frac{\partial \mathfrak{L}^p_{t_0}}{\partial \mathbf{P}_t} = \beta^{t-t_0}\Bigg(&\sum_{j=0}^{m+1} \beta^j \boldsymbol{\chi}_j \mathbb{E}_t[\mathbf{P}_{t+j} - \mathbf{P}^*_{t+j} - \boldsymbol{\ell}^p_{t+j|t_0}] + \sum_{j=1}^{t-t_0} \boldsymbol{\chi}_j(\mathbf{P}_{t-j} - \mathbf{P}^*_{t-j} - \boldsymbol{\ell}^p_{t-j|t_0}) \\ &+ \sum_{j=t-t_0+1}^{m+1} \boldsymbol{\chi}_j(\mathbf{P}_{t-j} - \mathbf{P}^*_{t-j}) - \boldsymbol{\varkappa}_{t|t_0} + \sum_{j=1}^{m-(t-t_0)} \boldsymbol{\mathcal{B}}'_j \mathbb{E}_t \boldsymbol{\varkappa}_{t+j|t_0}\Bigg) = \mathbf{0}\end{aligned} \tag{D.3.6b}$$

but takes a different form for $t \geq t_0 + m + 1$:

$$\frac{\partial \mathfrak{L}^p_{t_0}}{\partial \mathbf{P}_t} = \mathbb{E}_t\left[\boldsymbol{\chi}(\mathbb{L})(\mathbf{P}_t - \mathbf{P}^*_t - \boldsymbol{\ell}^p_{t|t_0})\right] = \mathbf{0} \tag{D.3.6c}$$

In both (D.3.6b) and (D.3.6c), the result that $\partial \mathfrak{U}_{t_0}/\partial \mathbf{P}_t = \beta^{t-t_0}\boldsymbol{\Omega}(\hat{\boldsymbol{\rho}}_t - \boldsymbol{\rho}^*_t) = \beta^{t-t_0}\mathbb{E}_t[\boldsymbol{\chi}(\mathbb{L})(\mathbf{P}_t - \mathbf{P}^*_t)]$ from (4.3.4), (4.1.1) and (D.3.1), and the para-Hermitian property $\boldsymbol{\chi}(z) = \boldsymbol{\chi}(\beta z^{-1})$ from (B.6.7) have been used.

Suppose the Lagrangian multipliers $\boldsymbol{\ell}^p_{t|t_0}$ are such that $\boldsymbol{\ell}^p_{t|t_0} = \mathbf{P}_t - \mathbf{P}^*_t$. This is a time-consistent expression, and when combined with the first-order condition (D.3.6a), it implies equation (D.3.2), which is equivalent to the conjectured targeting rule (5.3.1) for $t \geq t_0$. If this targeting rule is used in all periods after t_0, then the price level vector evolves according to the solution in (D.3.4), which means that all the constraints on the policymaker are satisfied. Substituting the expression for $\boldsymbol{\ell}^p_{t|t_0}$ into (D.3.6b) yields

$$\boldsymbol{\varkappa}_{t|t_0} = \sum_{j=1}^{m-(t-t_0)} \boldsymbol{\mathcal{B}}'_j \mathbb{E}_t \boldsymbol{\varkappa}_{t+j|t_0} + \sum_{j=t-t_0+1}^{m+1} \boldsymbol{\chi}_j(\mathbf{P}_{t-j} - \mathbf{P}^*_{t-j}) \tag{D.3.7}$$

for all $t = t_0, \dots, t_0 + m$. This can be solved recursively for $\boldsymbol{\varkappa}_{t|t_0}$ starting from $t_0 + m$. All the Lagrangian multipliers $\boldsymbol{\varkappa}_{t|t_0}$ are functions of predetermined values of $\mathbf{P}_t$ and $\mathbf{P}^*_t$ at t_0. Hence, values of the Lagrangian multipliers $\boldsymbol{\ell}^p_{t|t_0}$ and $\boldsymbol{\varkappa}_{t|t_0}$ have been found such that the first-order conditions (D.3.6) and constraints are satisfied by the evolution of prices and the output gap implied by the conjectured targeting rule (5.3.1). This proves the claim of the theorem.

CENTRE FOR ECONOMIC PERFORMANCE
Recent Discussion Papers

839	Christopher A. Pissarides	The Unemployment Volatility Puzzle: Is Wage Stickiness the Answer?
838	Kevin D. Sheedy	Inflation Persistence when Price Stickiness Differs between Industries
837	Kevin D. Sheedy	Intrinsic Inflation Persistence
836	Andrew E. Clark Ed Diener Yannis Georgellis Richard E. Lucas	Lags and Leads in Life Satisfaction: A Test of the Baseline Hypothesis
835	Jean-Baptiste Michau	Creative Destruction with On-the-Job Search
834	Nikolaus Wolf	Scylla and Charybodis: the European Economy and Poland's Adherence to Gold, 1928-1936
833	Sharon Belenzon Tomer Berkovitz	Innovation in Business Groups
832	Espen R. Moen Åsa Rosén	Incentives in Competitive Search Equilibrium
831	Eran Yashiv	U.S. Labor Market Dynamics Revisited
830	Laura Alfaro Andrew Charlton	Growth and the Quality of Foreign Direct Investment: Is All FDI Equal?
829	Richard Layard David Clark Martin Knapp Guy Mayraz	Cost-Benefit Analysis of Psychological Therapy
828	Emma Tominey	Maternal Smoking During Pregnancy and Early Child Outcomes
827	Christos Genakos Tommaso Valletti	Testing the “Waterbed” Effect in Mobile Telephony
826	Luis Garicano Paul Heaton	Information Technology, Organization, and Productivity in the Public Sector: Evidence from Police Departments
825	Laura Alfaro Andrew Charlton	Intra-Industry Foreign Direct Investment

824	Miklós Koren Silvana Tenreyro	Technological Diversification
823	Régis Barnichon	The Shimer Puzzle and the Correct Identification of Productivity Shocks
822	Carlos Thomas	Search Frictions, Real Rigidities and Inflation Dynamics
821	Giulia Faggio Kjell G. Salvanes John Van Reenen	The Evolution of Inequality in Productivity and Wages: Panel Data Evidence
820	Luis Araujo Emanuel Ornelas	Trust-Based Trade
819	Régis Barnichon	Productivity, Aggregate Demand and Unemployment Fluctuations
818	Rachel Griffith Sokbae Lee John Van Reenen	Is Distance Dying at Last? Falling Home Bias in Fixed Effects Models of Patent Citations
817	Tsutomu Miyagawa YoungGak Kim	Measuring Organization Capital in Japan: An Empirical Assessment Using Firm-Level Data
816	Andreas Georgiadis Alan Manning	Spend It Like Beckham? Inequality and Redistribution in the UK, 1983-2004
815	Guy Michaels Xiaojia Zhi	Freedom Fries
814	Ralph Ossa	Trade Liberalization, Outsourcing, and Firm Productivity
813	Karsten Kohn Alexander C. Lembcke	Wage Distributions by Bargaining Regime: Linked Employer-Employee Data Evidence from Germany
812	Yannis M. Ioannides Henry G. Overman Esteban Rossi-Hansberg Kurt Schmidheiny	The Effect of Information and Communication Technologies on Urban Structure
811	Guy Michaels	The Division of Labor, Coordination, and the Demand for Information Processing
810	Maia Güell Jose V. Rodriguez Mora Chris Telmer	Intergenerational Mobility and the Informative Content of Surnames

The Centre for Economic Performance Publications Unit
Tel 020 7955 7673 Fax 020 7955 7595 Email info@cep.lse.ac.uk
Web site http://cep.lse.ac.uk