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Donald L. Snyder
Utah State University

Rabindra N. Bhattacharyya
Utah State University

Utah State University
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Donald L. Snyder
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The Faustmann model has played a key role in the determination of optimal forest rotations. Faustmann introduced a simple and deterministic competitive economic model, the objective of which was to maximize the present value of perpetual returns to the fixed factor of production, a unit of timber land. The optimal rotation problem, as viewed by him, is a timber management problem abstracting from the important multiple use characteristics of forest land. Hartman (1976) and Strang (1983) developed a modified Faustmann model where the forest resource stock 'per se' is assumed to have consumptive value in the form of "recreation", a general term used to capture non-timber forest uses.

An important issue having a bearing on the problem of optimal forest rotation remains still to be explored. Hartman points out that in any realistic model, regeneration costs and the costs of making recreational services accessible to users would have to be explicitly considered. The required management decision is based on net values. Therefore, recreational as well as timber values should be considered net of their costs of production and/or maintenance. While regeneration costs have been accounted for in part by some authors, recreation costs in the context of the rotation problem have received little attention.

This paper represents an attempt to account for these costs in a general way. This analysis extends the earlier work completed by Hart-

*Authors are, respectively, Associate Professor and Ph.D. Candidate, Economics Department, Utah State University, Logan, Utah.

man and Strang by incorporating recreation cost into a more generalized Faustmann model. Our analysis shows that the decision of "never cutting" the forest as the global maximum as derived by Strang is valid only under very restrictive assumptions. When recreational costs and values are adequately accounted for, rotation patterns other than "never cutting" are the general rule. In our approach, rotation period decisions, in the sense of local maximum differ from the Hartman-Strang formulation.

Setting of the Problem

As is well known, the problem of determining the optimal rotation of a forest is fundamentally a problem in capital theory. Although the growing forest stock may be considered as an asset in the form of goods in process or inventory, a standing forest may be treated as a special kind of durable equipment providing a flow of services. This model incorporates both the commercial value of timbers when the forest is harvested as well as the value of services flowing from a standing forest. Hence, both the concepts of forest asset are relevant here.

The distinctive feature of economic activity involving capital is that it takes place at more than one point in time. Both the holding of inventories and the management of durable equipment may be treated in a unified manner through the temporal theory of production. For both, a stock of productive goods may be represented as an input to the stock-holding process when it is acquired. Output/service levels of the stockholding activity depend not only on acquiring a stock of productive goods, but also on various other inputs of material and services that represent production and maintenance activity (Jorgenson, et al.).

In this forestry problem, timber production and recreational services involve regeneration inputs, inputs required for preparing campgrounds, maintaining mountain rescue teams, generating wildlife habitat improvement programs, and providing program administration. Maintenance activity involves inputs related to preserving the flow of services of a standing forest besides preserving the stock of trees.

The objective of a harvesting or maintenance policy is to find a sequence of times for harvesting successive forest stands that maximize the discounted total "net" benefits over the life of the investment process. Any time sequence for harvesting constitutes a rotation policy; a sequence that maximizes the total net benefits is an "optimal rotation policy."

In the following analysis, the forest resource is assumed to be owned by a hypothetical competitive firm operating in an environment of certainty. Further, a given plot of land is considered, with all trees harvested simultaneously (clear cutting as opposed to selective cutting). Individual trees are assumed to be identical when they are regenerated. Rotation restores the investment and regeneration process to its original state.

The Objective Function and Existence
of an Optimal Rotation Age

This section formulates the appropriate objective function to be maximized under the above assumptions and examines the existence of the optimal harvesting age for two specific situations.

Following Hartman and Strang, let G(t) denote the stumpage value in a forest of age t. This can be thought of as the value of the timber less the cost of harvesting. G(t) is assumed to be bounded and, unlike

standard durable equipment, has the following growth curve shape: appreciating in value at an increasing rate, then at a decreasing rate, reaching a maximum, depreciating, leveling off, and finally, again gradually falling. The Hartmann-Strang G(t) curve does not exhibit this last possible eventuality. Natural biological decay is likely to over- whelm the steady-state forest at a very old age. Further, harvesting cost may be an increasing function of forest age. Taken together, these imply the ultimate falling phase of G(t). The value of the flow of services of the standing forest at age t (e.g., wildlife habitat, flood control, viewing, and hunting), will be referred to as F(t) or recre- ational services. F(t) is assumed to be bounded and that initially F(t) rises at an increasing rate, then at a decreasing rate, reaching a maximum, and eventually declines gradually. This characterization of F(t) again contrasts with the Hartman-Strang F(t) function that asymp- totically approaches a maximum and never decreases with age. But in the present analysis, it is plausible to assume that old growth trees are subject to "wear out," defined as the decline in the recreational value or quality of the standing forest attributable to the normal forest aging process. Hence, F(t) eventually declines. Figures 1 and 2 depict the assumed characteristics of G(t) and F(t) respectively. (The sub- script H is used to depict the Hartman-Strang specifications.)

F(t) may be considered as the flow of the gross value of recrea- tional services. In contrast, this analysis highlights the impact of net values associated with the life of a forest on the optimal rotation time. So, the costs associated with the producing and maintaining the flow of recreational services are introduced to derive the flow of net value.
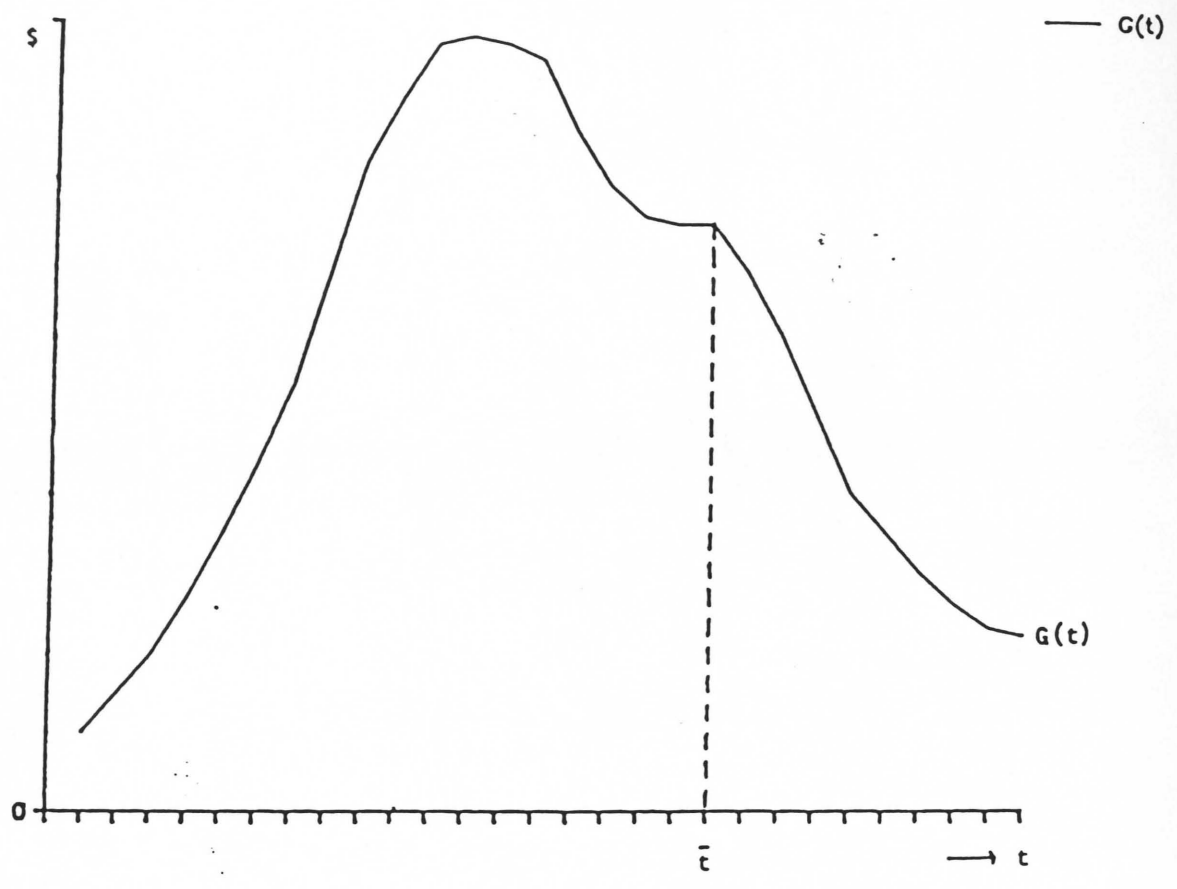
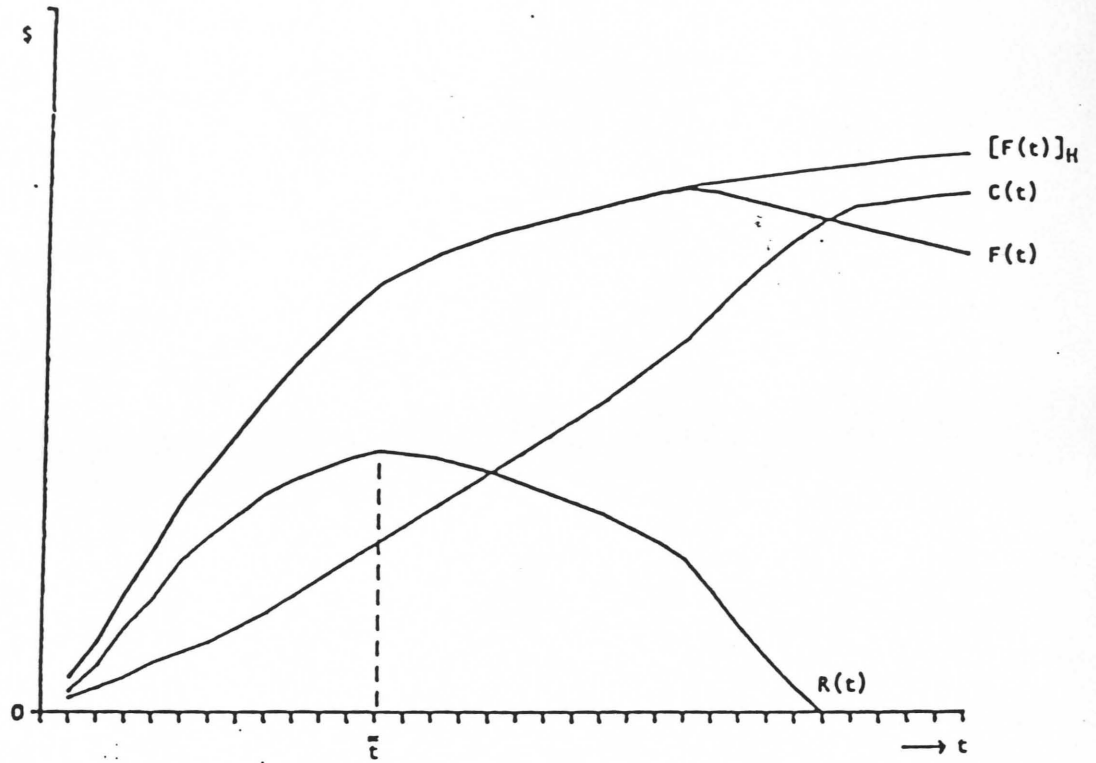Figure 1. Stumpage Value Growth Curve.

Figure 2. Value (F(t)), Net Value (R(t)), and Cost (C(t)) Curves of Recreational Services.

Consider a forest stand consisting of a stock of homogeneous trees planted and used along with other cooperating factors (such as road development and maintenance, campground preparation and clean-up, wildlife habitat improvement programs) for producing a flow of recreational services, Q. Over time, Q is made available in a competitive market. Let q_t denote the flow of Q at instant t. The corresponding value of recreational service flow is F_t. The forest stand is regenerated in an initially barren land at time t=0 at a fixed regeneration costs, C_o^R. The input cost flow to produce and make recreational services accessible to prospective users, C_t^I, is a function of q_t. The maintenance cost flow for the tree stock and other durable co-operating inputs, C_t^M, is a function of both the flow of services and of the age of the forest (assuming that ages of other inputs are linearly related to age of the forest). Consequently,

$$C_t^I = C^I(q_t); \qquad C_t^M = C^M(q_t, t);$$

$$C_t = C_t^I + C_t^M = C(q_t, t) \tag{2.1}$$

where C_t may be called the variable cost function. It seems reasonable to assume that C^I and C^M and hence C are nondecreasing and continuous. It is also assumed that C is bounded.

The forest could be harvested and timber could be sold in a competitive market whenever the entrepreneur decides to capture the rents associated with the standing forest from time t=0 through t=T, the stumpage value of the tree stock at time t=T, G_T, is a function of the age of the forest:

$$G_T = G(T) \tag{2.2}$$

where G(T), as assumed earlier, is bounded and continuous, $G'(T) \gtreqless 0$ as shown in Figure 1. The derivatives $G'(T)>0$, $G'(T)<0$ and $G'(T) = 0$, give, respectively, the rate of gain (appreciation), the rate of loss (depreciation), and the steady-state stumpage value from continuing to keep the forest on land.

The entrepreneur's optimization problem under such a situation can be separated into two parts: (1) determining optimal input and output (recreational services) levels for each point in time while the forest is standing, and (2) determining optimal lives (rotation age) of forests for one or more cycles. The optimal input and output levels are considered first. Then the appropriate objective functions are formulated to examine the existence of and criteria for an optimal rotation age for a single cycle and for an infinite chain of cycles.

Given that the entrepreneur has decided to operate a forest from time $t = 0$ through $t = T$, the initial cost and stumpage value may be ignored. The firm's problem is to maximize the present value of the quasi-rent flow from the standing forest, i.e., the difference between the present value of revenue from recreational services $F(t)$ and the present value of the variable costs $C(t)$. Since, the value of recreational services and costs at different points in time are independent in the case considered here, the firm can maximize the present value of its quasi-rent flow over the cutting cycle by maximizing the rate of discounted quasi-rent flow at each point in time (Henderson and Quandt). Furthermore, since the discount factor e^{-rt} is a constant for any fixed value of t and assuming r is given, the firm can achieve the desired result by maximizing the rate of quasi-rent flow at each point in time without discounting.

The firms rate of quasi-rent flow at instant t, R_t is

$$R_t = F_t - C_t = F_t - C^I(q_t) - C^M(q_t, t) . \qquad (2.3)$$

Setting the derivate of R_t with respect to q_t equal to zero implies that

$$\frac{dF_t}{dq_t} = \frac{dC^I}{dq_t} + \frac{\partial C^M}{\partial q_t} . \qquad (2.4)$$

The firm equates its rate of marginal cost flow which, in this case, is a sum of input and maintenance costs, to its fixed rate of marginal revenue flow (since the market is competitive), $\dfrac{dF_t}{dqt}$. The second-order condition

$$- \frac{d^2 C^I}{dq_t^2} - \frac{\partial^2 C^M}{\partial q_t^2} < 0 \qquad (2.5)$$

implies that the sum of the marginal costs increases with output.

We assume that (2.4) may be solved for the optimum value of q_t as a function of t. Substituting this function into equation (2.3), an optimal quasi-rent stream may be expressed as a function of t:

$$R_t = R(t). \qquad (2.6)$$

Similar substitution in equation (2.1) leads to

$$C_t = C(t) \qquad (2.1')$$

the optimal variable cost flow as a function of t.

Since F and C are bounded and continuous, R is also bounded and continuous (Buck). Figure 2 depicts the shape of the R(t) function.

The quasi-rent function gives the maximum quasi-rent obtainable at each point in time from operating a standing forest. It is based upon the underlying optimal combination of inputs and output. The quasi-rent function holds for all values of t, and its form is unaffected by the choice of a particular value for rotation length. Thus, the quasi-rent function may be used for analyzing the rotation length without the explicit introduction of outputs (recreational services), value of services F_t, and costs.

The existence of an optimal rotation age is treated under two specific situations: (1) under the Fisherian one-cycle and (2) under the Faustmann many cycles. For this, we utilize the logical steps developed by Jorgenson, et al.

Fisherian one-cycle situation

This situation concerns when the planning horizon runs through only one cutting of the forest. The present value of net return from the operation of a forest from t = 0 through t = T is the present value of quasi-rent stream minus the initial regeneration cost plus the present value of the receipt from the stumpage when the forest is cut at t = T at the termination of one cycle, or

$$V_1(T) = \int_0^T R(t)e^{-rt}dt - C_o^R + G(T)e^{-rT} \tag{2.7}$$

where r > o, is the discount rate.

The firms objective is to maximize $V_1(T)$ with respect to the choice variable T.

Assumptions made about R, G and r imply that V_1 is bounded and continuous (Buck). To determine the existence of an optimal rotation

age T, V_1 is differentiated with respect to T:

$$V_1^!(T) = [R(T) - rG(T) + G'(T)]e^{-rT}. \tag{2.8}$$

It can be shown that under certain reasonable assumptions, an optimal rotation age, say $\hat{T}$, does exist, and that it is not zero. First, since R and G are bounded and monotonic in the relevant intervals $[\overline{\overline{t}}, \infty)$ and $[\overline{t}, \infty)$ respectively (in Figure 2 and Figure 1), the limits

$$\lim_{t \longrightarrow \infty} R(t) = R(\infty)$$

$$\lim_{t \longrightarrow \infty} G(t) = G(\infty) \tag{2.9}$$

exist. Further, it is assumed that

$$\lim_{t \longrightarrow \infty} G'(t) = 0 . \tag{2.10}$$

With these assumptions and conclusions, it follows from (2.8) that V_1' tends to zero as T gets larger; however,

$$\lim_{t \longrightarrow \infty} e^{rt} V_1^!(T) = R(\infty) - rG(\infty) . \tag{2.11}$$

If the limit (2.11) were negative, then $V_1^!(T)$ would be negative for sufficiently large T, and hence $V_1(t)$ would be decreasing for sufficiently large T. The limit (2.11) is negative if and only if

$$rG(\infty) > R(\infty)$$

which implies that

$$G(\infty) > \frac{R(\infty)}{r} = \int_o^\infty R(\infty)e^{-rt} \, dt . \tag{2.12}$$

The inequality (2.12) is interpreted as follows: the left-hand side expression is the net stumpage value derived by starting initially with an infinitely old forest and cutting it down immediately. The right-hand side is the discounted quasi-rent derived from starting with an infinitely old forest stand and never cutting it. In light of this interpretation and noting that $R(t) < F(t)$, it is assumed that (2.12) holds. Hence $V_1(T)$ is decreasing for T larger than say T_0. This is also intuitively plausible since both $R(t)$ and $G(t)$ are falling when T is large. Thus, since V_1 is continuous, it attains a maximum on the interval $[0, T_0]$; a fortiori it attains a maximum on $[0, \infty]$ for some $T \leq T_0$.

In the Hartman-Strang formulation with $R(t)$ replaced by $F(t) > R(t)$, it is not improbable that for a single-cycle $G(\infty) \leq \frac{F(\infty)}{r}$, if the value of standing forest is relatively high. This possibility exists because of the nondecreasing $F(t)$ function in their models. In that case, $V_1(T)$ is nondecreasing and any finite solution T for rotation age may not exist. Mathematically it is inappropriate to suggest (as Strang did) that $V_1(T)$ has global maximum at infinity (Glaister). Of course, never cutting a forest may well be a consequence of such a result.

Let us now examine the possibility that $T = 0$. From (2.7) and (2.8),

$$V_1(0) = -C_0^R \text{ and} \tag{2.13}$$

$$V_1'(0) = R(0) - rG(0) + G'(0) = 0$$

since a nonexisting forest can earn neither quasi-rent nor stumpage

value equation (2.13), with $V_1(0) < 0$, implies that zero cannot be an optimal value of T.

Thus, under the assumptions made, the maximum net return is attained at a finite, positive rotation age (which may be more than one).

Faustmann many-cycle situation

Let us consider a firm which plans for an infinite horizon and an infinite chain of identical forests succeeding one another. We assume that the quasi-rent function, the initial regeneration cost, and the stumpage value function are the same for each rotation cycle. The present value of net return from the first cycle is given by (2.7). The present value of the net return from the second and third-cycle forest are respectively,

$$V_2(T) = \int_T^{2T} R(t-T)e^{-rt}dt - C_o^R e^{-rT} + G(T)e^{-r2T}$$

$$= V_1(T)e^{-rT} \tag{2.14}$$

and

$$V_3(T) = \int_{2T}^{3T} R(t-2T)e^{-rt}dt - C_o^R e^{-r2T} + G(T)e^{-r3T}$$

$$= V_1(T)e^{-r2T} . \tag{2.15}$$

In general

$$V_k(T) = [\int_o^T R(t)e^{-rt}dt - C_o^R + G(T)e^{-rt}] \, e^{-r(k-1)T}$$

$$= V_1(T)e^{-r(k-1)T} . \tag{2.16}$$

Consequently, the present value of the aggregate net return from an infinite chain of forest cycles is

$$V(T) = \sum_{k=1}^{\infty} V_k(T) = \frac{\int_0^T R(t) \, e^{-rt} \, dt - C_o^R + G(T)e^{-rT}}{1 - e^{-rT}}$$

$$= \frac{V_1(T)}{1 - e^{-rT}} \qquad (2.17)$$

which can alternatively be written as

$$V(T) = \int_o^T R(t) \, e^{rt}dt - C_o^R + G(T) \, e^{-rT} + V(T)e^{-rT} . \qquad (2.18)$$

Again the assumptions made about R, G, and r imply that function V is bounded and continuous.

To know about the existence of an optimal rotation age, V in (2.18) is differentiated with respect to T.

$$V'(T) = R(T)e^{-rT} - rG(T)e^{-rT} + G'(t)e^{-rT} - rV(T) \, e^{-rT} + V'(T)e^{-rT}$$

$$= \left[\frac{e^{-rT}}{1 - e^{-rT}}\right] [R(T) + G'(T) - rG(T) - rV(T)] . \qquad (2.19)$$

It follows from (2.19) that V'(T) approaches zero as T gets larger. However, if the limits (2.9) exist and (2.10) is valid, then

$$\lim_{T \longrightarrow \infty} e^{rT} \, V'(T) = R(\infty) - rG(\infty) - r[\int_o^T R(t)e^{-rt}dt - C_o^R] . \qquad (2.20)$$

If the limit (2.20) were negative, then V'(T) would be negative for sufficiently large T, and hence V(T) would be decreasing for a sufficiently large T. The limit (2.20) is negative if and only if,

$$rG(\infty) + r \left[\int_0^\infty R(t)e^{-rt}dt - C_0^R\right] > R(\infty)$$

which implies

$$G(\infty) + \int_0^\infty R(t)e^{-rt}dt - C_0^R > \frac{R(\infty)}{r}. \tag{2.21}$$

The inequality (2.21) can be interpreted as follows: the left-hand side is the total discounted net return obtained by starting at time zero with an infinitely old forest, cutting it immediately to get the stumpage value $G(\infty)$, replanting the forest immediately incurring a regeneration cost C_0^R without ever cutting it again to derive a discounted flow of quasi-rent $\int_0^\infty R(t)e^{-rt}dt$. The right-hand side is the total discounted quasi-rent stream derived from starting with an infinitely old forest and never harvesting it, since

$$\frac{R(\infty)}{r} = \int_0^\infty R(\infty)e^{-rt}dt . \tag{2.22}$$

Given the nature of the quasi-rent function (2.6), as shown in Figure 2; the $G(t)$ function (2.2), as shown in Figure 1; and the above interpretation; it is assumed that (2.21) holds.

Hence $V(T)$ is decreasing for T larger than, say T_0. Again, as in the single-cycle case, since $V(T)$ is continuous, it attains a maximum on the interval $[0, T_0]$; _a fortiori_ it attains a maximum on $[0, \infty)$ for some $T \leq T_0$.

The "never cut" situation of Hartman-Strang implies the reverse of the inequality (2.21) with the R function replaced by a larger valued F function and $C_0^R = 0$. That is

$$G(\infty) + F(t)e^{-rt}dt \leq \frac{F(\infty)}{r} \tag{2.21'}$$

in that case V(T) is nondecreasing for T larger than T_o and any finite solution T for optimal rotation age may not exist. Here again, of course, never cutting may be a consequence of (2.21'). The situation characterizing (2.21') depends crucially on the assumptions of steady state G(t)as $t \longrightarrow \infty$, never decreasing F(t), and noninclusion of variable and regeneration costs. Taken together, they imply nondecreasing V(t) as $t \longrightarrow \infty$ and hence (2.21'). The situation (2.21'), though not improbable, can occur only under very restrictive situations.

Thus, under the more general situation considered and the assumptions made, the maximum net return is obtained at a finite and positive rotation age, though there may be more than one local maximum.

It is to be noted that the never cutting decision is more likely under the one-cycle problem because its alternative (cutting the trees) is more limited in value in the one-cycle than in many-cycle case (Strang).

A Formal Solution and Comparison with Alternative Formulations

This section provides a formal solution of the models formulated in the previous section for optimal rotation age in terms of certain criteria. Here, again, two cases are considered: the Fisherian one-cycle case and the Faustmann many-cycle case. The former is considered for the sake of its more intuitive appeal and the help it provides for later comparisons among contending formulations.

Fisherian one-cycle solution

At an optimal rotation age $\hat{T}$, the first and second order conditions for an interior maximum are $V_1'(T) = 0$ and $V_1''(T) < 0$ respectively.

Thus, setting (2.8) equal to zero

$$R(T) + G'(T) = rG(T) \tag{2.23}$$

or

$$\frac{G'(T)}{G(T)} = r - \frac{R(T)}{G(T)} . \tag{2.24}$$

The second-order condition is (after simplification)

$$R'(T) + G''(T) < rG'(T) . \tag{2.25}$$

Hence, for an interior maximum $R(t) + G'(T)$ must intersect $rG(t)$ from above (Figure 3).

The optimality condition (2.23) can be interpreted easily. On the right is the interest foregone by postponing forest harvesting for one period. On the left is the gain from postponing the harvest one period: it consists of the quasi-rent flow (net recreational value) during the period plus (minus) the value of the timber growth (decay) over the period. Thus, for optimality, the marginal gain from postponing the harvest one period must equal the marginal loss of postponment.

In the absence of costs associated with providing recreational services $C(t) = 0$ and (2.24) reduces to the Hartman-Strang result

$$\frac{G'(T)}{G(T)} = r - \frac{F(T)}{G(T)} . \tag{2.24'}$$

Furthermore, in the absence of _net_ recreational value (quasi-rent), $R(t) = 0$, and (2.24) simply reduces to the well-known Fisherian result

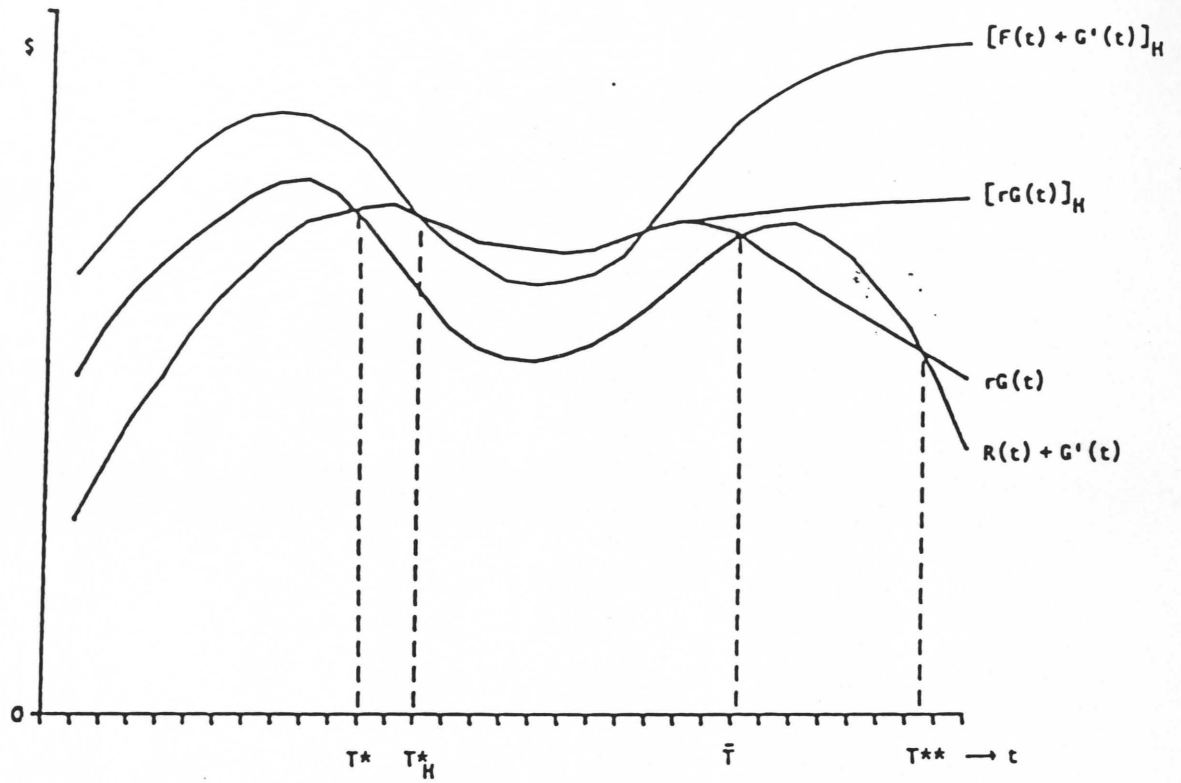$$\frac{G'(T)}{G(T)} = r, \tag{2.26}$$

Figure 3. Marginal Benefits and Marginal Costs of Not Harvesting Under Alternative Assumptions.

a forest should be harvested when its rate of growth equals the discount rate. With recreational value only, $F(T)/G(T) > 0$, and therefore (2.24') suggests that the forest should be harvested when the rate of growth is less than the discount rate. This is achieved by delaying the harvest. For similar reasons, (2.24) suggests delayed harvesting. But the quasi-rent, $R(T)$ in our formulation, is less than $F(T)$ for Hartman-Strang, implying $R(T)/G(T) < F(T)/G(T)$. Hence $[r - R(T)]/G(T) > [r - F(T)]/G(T)$. This suggests that the optimal rotation age in the presence of costs for providing recreational services will be shorter than that .in the presence of recreational benefits alone (Hartman-Strang solution), but longer than Fisherian solution. Thus our result is a further generalization of the generalized Fisherian solution of Hartman-Strang.

$R(t)/G(t)$ is the ratio of __net__ recreational value per time period of the standing forest to the stock value of harvested timber. If this ratio is greater than the discount rate, then the right-hand side of (2.24) is negative. The first-order condition (2.23), as Hartman pointed out, does not necessarily imply that $G'(t) > 0$ at the optimum. Moreover, the second-order condition will be satisfied for $G'(t)$ negative, provided $G''(t)$ is a large enough negative value. Hence an optimum may occur at a long enough time involving a negative rate of growth. Finally, if the $R(t)$ function is large enough (a distinct possibility in Hartman's formulation but rather unlikely in our formulation since $R(t) < F(t)$ and declining in the interval $[\bar{\bar{t}}, \infty)$) relative

to G(t), there may be no definitive solution to (2.23). The most likely general case is shown in Figure 3.[1]

Faustmann many-cycle solution

An optimal rotation age T under the many-cycle Faustmann case requires, $V'(T) = 0$ and $V''(T) < 0$. Thus, from (2.17) and (2.19) and setting $V'(T)$ equal to zero

$$V'(T) = \frac{e^{-rT}}{1-e^{-rT}} \left([R(T) - rG(T) + G'(T)] - \frac{r[\int_0^T R(t)e^{-rt}dt + G(T)e^{-rT} - C_0^R]}{1-e^{-rT}} \right) = 0$$

(2.27)

which implies that

$$R(T) + G'(T) + \frac{rC_0^R}{1-e^{-rT}} = rG(T) + \frac{r\int_0^T R(t)e^{-rt}dt}{1-e^{rT}} + \frac{rG(T)e^{-rT}}{1-e^{-rT}}$$

(2.28)

which for simplified expression can be written as

$$R(T) + G'(T) = \frac{1}{\lambda} [\int_0^T R(t)e^{-rt}dt - C_0^R + G(T)]$$

(2.29)

where $\lambda = \dfrac{1-e^{-rT}}{r} = \int_0^T e^{-rt}dt$ is the present value of a dollar stream of return for T years.

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[1] It is relevant to note that Figure 3 of Strang seems to be in error. The falling portion of F(t) + G'(t) curve implies G'(t) < 0 and large enough since F(t) is nondecreasing. But on the same time interval, his rG(t) curve is shown rising. With G'(t) < 0, G(t) and hence rG(t) should be falling. However, this does not have much bearing on his conclusion.

Equation (2.28) can be rearranged as

$$\frac{G'(T)}{G(T)} = r\left[\frac{1}{1-e^{-rT}} + \frac{\int_0^T R(t)e^{-rt}dt}{G(T)(1-e^{-rT})} - \frac{C_0^R}{G(T)(1-e^{-rT})}\right] - \frac{R(T)}{G(T)}.$$

$$(2.30)$$

The first-order condition of interior maximum expressed in the form of (2.29) can be interpreted as: a forest is harvested when its marginal rate of quasi-rent flow per period plus (minus) appreciation (depreciation) equals the present value of the average quasi-rent return per period of a regenerated forest net of its regeneration cost plus the stumpage value of the previous forest stand just harvested. The bracketed term on the right-hand side of (2.29) gives a net return for T years. Division by λ converts it to an annual basis. The second-order condition $V''(T) < 0$ requires, under this interpretation, that the marginal net return on the old forest cut be decreasing more rapidly than the average net return on the regenerated new forest.

Equation (2.28) also provides a useful interpretation. On the left-hand side is the gain from postponing the harvest for one period. It consists of the quasi-rent flow during the period plus (minus) the value of the timber growth (decay) over the period plus the gain in interest on capitalized value of regeneration cost for not harvesting and thus not incurring the regeneration cost in a sequence of infinite cutting cycles. On the right is the interest foregone by postponing harvesting the forest for one period.

In the absence of costs associated with recreational services and the cost of regeneration, $C(t) = C_0^R = 0$, and therefore equation (2.30) reduces to the Hartman-Strang result

$$\frac{G'(T)}{G(T)} = r\left[\frac{1}{1-e^{-rT}} + \frac{\int_0^T F(t)e^{-rt}dt}{G(T)\,(1-e^{-rT})}\right] - \frac{F(T)}{G(T)}\ . \tag{2.30'}$$

Except for the term in the brackets, (2.30') is the same as (2.24'). Loosely speaking, and following Hartman, the term in the brackets acts as a "correction factor" for the interest rate. $1-e^{-rT}$ lies between zero and one, and therefore, $1/1 - e^{-rT}$ is greater than one. Further $G(t)$ and $\int_0^T e^{-rt}F(t)dt$, are both positive. Thus, the expression in the brackets is greater than one giving rise to an "effective interest rate" (the interest rate multiplied by the "correction factor"), which is greater than the interest rate appearing in (2.24'). This has the effect of reducing the optimal harvest age relative to the model with a one-harvest horizon. For identical reasons, (2.30) has the effect of reducing the optimal harvest age relative to our model with a one-harvest horizon and indicated by (2.24). Of course, this conclusion is contingent on the assumption that the bracketed term on the right of (2.30) is positive and greater than one. This requires a very plausible assumption that the present value of the quasi-rent flow for T years net of regeneration cost is positive, i.e.,

$$\frac{r \int_0^T R(t)e^{-rt}dt}{G(T)\,(1-e^{-rT})} - \frac{r\,C_o^R}{G(T)\,(1-e^{-rT})} > 0 \tag{2.31}$$

Similar comparisons between the optimal rotation lengths implied by the solution of (2.30) and the solution of the Hartman-Strang rule (2.31') is not that intuitive. To make a comparison, we adopt the following step by step procedure, where each step implies, by the

preceeding logic, a particular optimal rotation age. This is also shown in Figure 4. We take the simple Fisherian solution as our point of reference. A review of our previous discussion implies the following rotation lengths:

$$\frac{G'(T)}{G(T)} = r \Rightarrow \hat{T}_o, \tag{2.32}$$

Fisherian one-cycle solution T;

$$\frac{G'(T)}{G(T)} = r - \frac{F(T)}{G(T)} \Rightarrow \hat{T}_1, \tag{2.33}$$

the Fisherian solution of Hartman with recreational value added;

$$\frac{G'(T)}{G(T)} = r \left[\frac{1}{1-e^{-rT}} + \frac{\int_o^T F(t)e^{-rt}dt}{G(T)(1-e^{-rT})} \right] - \frac{F(T)}{G(T)} \Rightarrow \hat{T}_2, \tag{2.34}$$

the generalized Faustmann solution of Hartman;

$$\frac{G'(T)}{G(T)} = r \left[\frac{1}{1-e^{-rT}} + \frac{\int_o^T F(t)e^{-rt}dt}{G(T)(1-e^{-rT})} \right] - \frac{R(T)}{G(T)} \Rightarrow \hat{T}_3, \tag{2.35}$$

a hypothetical solution with F(T) replaced by R(T) < F(T) in the last term; and

$$\frac{G'(T)}{G(T)} = r \left[\frac{1}{1-e^{-rT}} + \frac{\int_o^T R(t)e^{-rt}dt - C_o^R}{G(T)(1-e^{-rT})} \right] - \frac{R(T)}{G(T)} \Rightarrow \hat{T}_4, \tag{2.36}$$
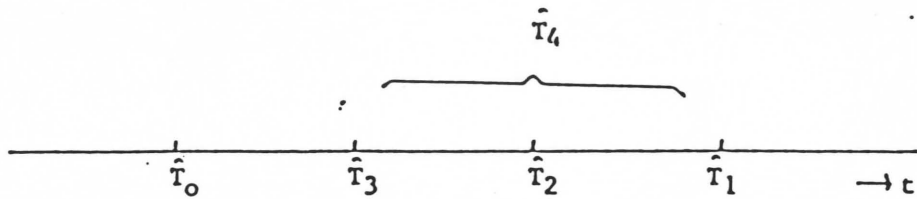
our more generalized Faustmann solution.

Figure 4. Comparative Optimal Rotation Time.

Equation (2.34) implying $\hat{T}_2$ and (2.36) implying $\hat{T}_4$ are the solutions for (2.30') and (2.30) respectively. This suggests that $\hat{T}_4$ can be less than, equal to, or greater than $\hat{T}_2$. These alternative possibilities are explored below:

Using the R(t) function defined in (2.3) and (2.6), equation (2.30) i.e., (2.36) can be reexpressed as

$$\frac{G'(T)}{G(T)} = r \left[\frac{1}{1-e^{-rT}} + \frac{\int_o^T F(t)e^{-rt}t\,dt}{G(T)(1-e^{-rT})} - \frac{\int_o^T C(t)e^{-rt}dt}{G(T)(1-e^{-rT})} \right.$$

$$\left. - \frac{c_o^R}{G(T)(1-e^{-rT})} \right] - \frac{F(T)}{G(T)} + \frac{C(T)}{G(T)}$$

$$= r \left[\frac{1}{1-e^{-rT}} + \frac{\int_o^T F(t)e^{-rt}dt}{G(T)(1-e^{-rT})} \right] - \frac{F(T)}{G(T)}$$

$$+ \left(\frac{C(T)}{G(T)} - \frac{1}{\lambda} \left[\frac{\int_o^T C(t)e^{-rt}dt}{G(T)} + \frac{c_o^R}{G(T)} \right] \right) \qquad (2.37)$$

where $\lambda = 1-e^{-rT}/r = \int_o^T e^{-rt}dt$, is as defined before, the present value of a dollar stream for T years.

Now excepting the second term within the parentheses, (2.37) is exactly the same as (2.30') or (2.34). Thus, the length of rotation $\hat{T}_4$

as compared to $\hat{T}_2$ will depend on whether the term of cost components within the parentheses is positive, zero, or negative, i.e., whether

$$C(T) - \frac{1}{\lambda} [\int_0^T C(t)e^{-rt}dt + C_0^R] \gtrless 0 . \qquad (2.38)$$

Here $C(T)$ is the amount of variable costs incurred to provide the recreational services from the forest stand at the instant T (when the forest is harvested); $\int_0^T C(t)e^{-rt}dt$ is the present value of variable costs incurred over the period $t = 0$ to $t = T$; and C_0^R is the initial regeneration cost of the stand. The term within the brackets may then be interpreted as the total cost associated with the forest stand for T years. Division by converts it to an annual total cost. Thus following the logic developed earlier,

$$\hat{T}_2 = \hat{T}_4, \qquad \text{if } C(T) = \frac{1}{\lambda} [\int_0^T C(t)e^{-rt}dt + C_0^R], \qquad (2.39)$$

$$\hat{T}_2 > \hat{T}_4, \qquad \text{if } C(T) > \frac{1}{\lambda} [\int_0^T C(t)e^{-rt}dt + C_0^R], \qquad (2.40)$$

and

$$\hat{T}_2 < \hat{T}_4, \qquad \text{if } C(T) < \frac{1}{\lambda} [\int_0^T C(t)e^{-rt}dt + C_0^R], \qquad (2.41)$$

In summary, the difference between the finite rotation lengths suggested by the Hartman-Strang formulation and the formulation developed here will depend crucially on the difference between the variable costs of recreational services and the annual total costs of the forest stand at the instant T. The differences in costs will be reflected in the differences in "effective" interest rate and hence in the optimal rotation lengths.

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