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Evaluating the Uncertainty of a Boolean Formula with Belief Functions

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Abstract. In fault-tree analysis, probabilities of failure of components are often assumed to be precise and the events are assumed to be independent, but this is not always verified in practice. By giving up some of these assumptions, results can still be computed, even though it may require more expensive algorithms, or provide more imprecise results. Once compared to those obtained with the simplified model, the impact of these assumptions can be evaluated. This paper investigates the case when probability intervals of atomic propositions come from independent sources of information. In this case, the problem is solved by means of belief functions. We provide the general framework, discuss computation methods, and compare this setting with other approaches to evaluating the uncertainty of formulas.

Key words: Fault-trees, Belief functions, Boolean satisfaction

1 Introduction

One of the objectives of safety analysis is to evaluate the probabilities of dreadful events. In an analytical approach, this dreadful event is described as a Boolean function F of some atomic events, that represent the failures of the components of a system, or possibly some of its configuration states. This method requires that all probabilities of elementary component failures or configuration states be known and independent, in order to compute the probability of the dreadful event. But in real life scenarios, those assumptions are not always verified. This study takes place in the context of maintenance and dependability studies (Airbus project @MOST) in aviation business.

In this paper, we first investigate different approaches using interval computations in order to compute the probability of a Boolean expression in terms of the probabilities of its literals, a problem of direct relevance in fault-tree analysis. The usual assumptions that probabilities of literals are known and the

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corresponding events are independent are removed. We consider the situation when knowledge about probabilities is incomplete (only probability intervals are available), and envisage two assumptions about independence: first the case when no assumption is made about dependence between events represented by atoms, and then the case when the probability intervals come from independent sources of information. We more specifically investigate the use of belief functions to model the latter case, taking advantage of the fact that imprecise probabilities on a binary set are belief functions. We give results on the form of the global belief function resulting from applying Dempster rule of combination to atomic belief functions. We provide results on the computation of belief and plausibility of various kinds of propositional formulas, as found in the application to fault tree analysis. We compare the obtained results with those obtained in other scenarios (stochastic independence between atoms, and the no independence assumption).

2 Evaluation of the probability of a Boolean expression

Let $\mathcal{X}$ be a set of Boolean variables $x_1, \ldots, x_n$ such that $x_i \in \Omega_i = \{A_i, \neg A_i\}$; $A_1, \ldots, A_n$ denote atomic symbols associated to elementary faults or configuration states of a system. We denote by $\Omega = \prod_{i=1}^{n} \{A_i, \neg A_i\}$ the set of interpretations $\mathcal{X} \to \{0, 1\}$. An element $\omega \in \Omega$ is also called *minterm*, and it corresponds to a *stochastic elementary event*. It can also be interpreted as describing the state of the world at a given time. It can be written both as a maximal conjunction of literals or denoted by the set of its positive literals (it is *Herbrand's notation*). Let F be a Boolean formula expressed by means of the variables x_i: its models form a subset $[F]$ of Ω, the set of states of the world where F is true; also called the *set of minterms* of F. Hence, the probability of $F, P(F)$, can be written as the sum:

$$P(F) = \sum_{\omega \in [F]} p(\omega) \tag{1}$$

where $p(\omega)$ stands for $P(\{\omega\})$. When the independence of the x_i's is assumed (i.e. A_i independent of A_j, $\forall i \neq j$), this sum becomes:

$$P(F) = \sum_{\omega \in [F]} [\prod_{A_i \in \mathcal{L}_\omega^+} P(A_i) \prod_{A_i \in \mathcal{L}_\omega^-} (1 - P(A_i))] \tag{2}$$

where $\mathcal{L}_\omega^+$ is the set of positive literals of ω and $\mathcal{L}_\omega^-$ the set of its negative literals.

In the case where $P(A_i)$ is only known to lie in an interval, i.e. $P(A_i) \in [l_i, u_i], i = 1 \ldots n$, the problem is to compute the tightest range $[l_F, u_F]$ containing the probability $P(F)$. Let $\mathscr{P}$ be the convex probability family $\{P, \forall i \; P(A_i) \in [l_i, u_i]\}$ on Ω. In the following, we shall formally express this problem under various assumptions concerning independence.

2.1 Without any independence hypothesis

Without knowledge about the dependency between the $x_i, i = 1 \ldots n$, finding the tightest interval for the range $[l_F, u_F]$ of $P(F)$ boils down to a *linear optimization problem under constraints*. This goal is achieved by solving the two following problems:

$$l_F = \min(\sum_{\omega \models F} p(\omega)) \text{ and } u_F = \max(\sum_{\omega \models F} p(\omega))$$

under the constraints $l_i \leq \sum_{\omega \models A_i} p(\omega) \leq u_i, i = 1 \ldots n$ and $\sum p(\omega) = 1$.

Solving each of those problems can be done by linear programming with 2^n unknown variables $p(\omega)$. It is a particular case of the probabilistic satisfiability problem studied in [3], where known probabilities are attached to *sentences* instead of just atoms.

2.2 When variables x_i are stochastically independent

In the case where the independence of the $x_i, i = 1 \ldots n$, is assumed,

$$p(\omega) = \prod_{i=1}^{n} P(x_i(\omega)) \tag{3}$$

where: $x_i(\omega) = \begin{cases} A_i \; if \; \omega \models A_i \\ \neg A_i \; otherwise \end{cases}$. The corresponding probability family $\mathscr{P}_I = \{\prod_{i=1}^{n} P_i \mid P_i(\{A_i\}) \in [l_i, u_i]\}$, where P_i is a probability measure on Ω_i, is not convex. Indeed, take two probability measures P, $P' \in \mathscr{P}_I$, $P = \prod_{i=1}^{n} P_i$ and $P' = \prod_{i=1}^{n} P'_i$. For $\lambda \in [0, 1]$, the sum $\lambda \prod_{i=1}^{n} P_i + (1-\lambda) \prod_{i=1}^{n} P'_i \neq \prod_{i=1}^{n} (\lambda P_i + (1-\lambda) P'_i)$, so it is not an element of $\mathscr{P}_I$.

This assumption introduces some non-linear constraints in the previous formulation, hence the previous methods (section 2.1) cannot be applied. Instead of a linear problem with 2^n variables, we now have a non-linear optimization problem with n variables. Interval Analysis can be used to solve it [1].

3 The case of independent sources of information

When there is no knowledge about the dependency between the x_i's, but the information about $P(A_i)$ comes from independent sources, *belief functions* can be used to solve the problem of probability evaluation. The information $P(A_i) \in [l_i, u_i]$ is totally linked to its source. l_i can be seen as the degree of belief of A_i and u_i as its plausibility: $l_i = Bel(A_i)$ and $u_i = Pl(A_i)$ in the sense of Shafer.

Proposition 1. *The interval* $[l_i, u_i]$ *defines a unique* belief function *on* Ω_i*.*

Proof: To see it we must find a unique mass assignment and the solution is:
- $Bel(\{A_i\}) = l_i = m^i(\{A_i\})$;
- $Pl(\{A_i\}) = 1 - Bel(\{\neg A_i\}) = u_i \implies m^i(\{\neg A_i\}) = Bel(\{\neg A_i\}) = 1 - u_i$;
- The sum of masses is $m^i(\{A_i\}) + m^i(\{\neg A_i\}) + m^i(\Omega_i) = 1$, so $m^i(\Omega_i) = u_i - l_i$.

We call such m^i *atomic mass functions*. In order to combine two independent mass functions, Dempster rule of combination should be used.

Definition 1 (Dempster-Shafer rule).
For two masses m^1 *and* m^2*, the joint mass* $m^{1,2}$ *can be computed as follows:*
- $m^{1,2}(\varnothing) = 0$
- $m^{1,2}(S) = \dfrac{\sum_{B \cap C = S} m^1(B)m^2(C)}{1 - \sum_{B \cap C = \varnothing} m^1(B)m^2(C)}, \forall S \subseteq \Omega$

In our problem, each source gives an atomic mass function, and there are n sources, so the mass function over all Ω is : $m_\Omega = m^1 \oplus \cdots \oplus m^n$. To find this m_Ω for n atomic mass functions, we can use the associativity of Dempster rule of combination. Here, $A_i, i = 1, \ldots, n$ are atomic symbols, they are always compatible, i.e. $A_i \wedge A_j \neq \varnothing$ for all $A_i, A_j, i \neq j$. So the denominator is one in the above equation.

A focal element of m^Ω is made of a conjunction of terms of the form A_i, $\neg A_j$ and Ω_k (which is the tautology), for $i \neq j \neq k$. Hence it is a *partial model*. Let $\mathcal{P}(F)$ bet the set of partial models ϕ of a Boolean formula F, that are under the form of conjunction of elements $\lambda_i \in \{A_i, \neg A_i, \Omega_i\}$: $\phi = \underset{i=1,\ldots,n}{\wedge} \lambda_i$. Then, $\mathcal{P}(F) = \{\phi = \underset{A_i \in \mathcal{L}_\phi^+}{\wedge} A_i \underset{\neg A_i \in \mathcal{L}_\phi^-}{\wedge} \neg A_i \models F\}$, with $\mathcal{L}_\phi^+$ (resp. $\mathcal{L}_\phi^-$) the set of positive (resp. negative) literals of ϕ.

Proposition 2 (Combination of n atomic mass functions).
For n *atomic masses* $m^i, i = 1, \ldots, n$ *on* Ω_i*, the joint mass* m^Ω *on* Ω *can be computed as follows for any partial model* ϕ*:*

$$m^\Omega(\phi) = \prod_{i \in \mathcal{L}_\phi^+} l_i \prod_{i \in \mathcal{L}_\phi^-} (1 - u_i) \prod_{i \notin \mathcal{L}_\phi} (u_i - l_i) \tag{4}$$

This modeling framework differs from the usual one when atomic variables are supposed to be stochastically independent. Here, the independence assumption pertains to the sources of information, not the physical variables.

4 The belief and plausibility of a Boolean formula

The belief of a Boolean formula F, of the form $Bel(F) = \sum_{\phi \models F} m_\Omega(\phi)$, theoretically requires 3^n computations due to the necessity of enumerating the partial models for n atomic variables. Indeed, all conjunctions $\phi = \underset{i=1,\dots,n}{\wedge} \lambda_i$ must be checked for each $\lambda_i \in \{A_i, \neg A_i, \Omega_i\}$. Verifying that a partial model implies F also requires 2^n computations. Plausibility computation, given by the equation $Pl(F) = \sum_{S \wedge \phi \neq \varnothing} m_\Omega(\phi)$ requires to determine partial models not incompatible with F. From the partial models, it will need at most 2^n computation. But it can also be computed by using the duality of belief and plausibility given by:

$$Pl(F) = 1 - Bel(\neg F) \tag{5}$$

Example 1. **Belief functions of the disjunction** $F = A_1 \vee A_2$

	A_1	$\neg A_1$	Ω_1
A_2	$A_1 \wedge A_2$ $l_1 l_2$	$\neg A_1 \wedge A_2$ $(1-u_1)l_2$	A_2 $(u_1-l_1)l_2$
$\neg A_2$	$A_1 \wedge \neg A_2$ $l_1(1-u_2)$	$\neg A_1 \wedge \neg A_2$ $(1-u_1)(1-u_2)$	$\neg A_2$ $(u_1-l_1)(1-u_2)$
Ω_2	A_1 $l_1(u_2-l_2)$	$\neg A_1$ $(1-u_1)(u_2-l_2)$	Ω $(u_1-l_1)(u_2-l_2)$

Partial models that imply F are $\{A_1, A_2, A_1 \wedge \neg A_2, A_2 \wedge \neg A_1, A_1 \wedge A_2\}$, so: $Bel(F) = (u_1 - l_1)l_2 + l_1(u_2 - l_2) + l_1 l_2 + l_1(1 - u_2) + l_2(1 - u_1) = l_1 + l_2 - l_1 l_2$, that also reads $1 - (1 - l_1)(1 - l_2)$. Likewise, partial models that are compatible with F are $\{A_1 \wedge A_2, \Omega, A_1, A_2, \neg A_1, \neg A_2, A_1 \wedge \neg A_2, A_2 \wedge \neg A_1\}$, hence $Pl(F) = u_1 + u_2 - u_1 u_2 = 1 - (1 - u_1)(1 - u_2)$.

4.1 Conjunctions and Disjunctions of Literals

In the more general case, we can compute the belief and plausibility of conjunctions and disjunctions of literals indexed by $K \subseteq \{1, \dots, n\}$.

Proposition 3. *The belief of a conjunction C, and that of a disjunction D of literals $x_i, i \in K$ are respectively given by:*

$$Bel(C) = \prod_{i \in \mathcal{L}_C^+} l_i \prod_{i \in \mathcal{L}_C^-} (1 - u_i); \quad Bel(D) = 1 - \prod_{i \in \mathcal{L}_D^+} (1 - l_i) \prod_{i \in \mathcal{L}_D^-} u_i.$$

We can deduce the plausibility of conjunctions and disjunctions of literals, noticing that

$$Bel(\vee_{i \in \mathcal{L}^+} A_i \vee \vee_{i \in \mathcal{L}^-} \neg A_i) = 1 - Pl(\wedge_{i \in \mathcal{L}^+} \neg A_i \wedge \wedge_{i \in \mathcal{L}^-} A_i)$$

.

Proposition 4. *The plausibility of a conjunction C, and that of a disjunction D of literals $x_i, i \in K$ are respectively given by:*

$$Pl(C) = \prod_{i \in \mathcal{L}_C^+} (1 - l_i) \prod_{i \in \mathcal{L}_C^-} u_i; \quad Pl(D) = 1 - \prod_{i \in \mathcal{L}_D^+} l_i \prod_{i \in \mathcal{L}_D^-} (1 - u_i)$$

4.2 Application to Fault-trees

Definition 2 (Fault-tree). *A* fault-tree *is a graphical representation of chains of events leading to a dreadful event (failure).*

Classical fault-trees are a graphical representation dedicated to Boolean functions that are representable by means of two operators $\vee$(OR) and $\wedge$(AND).

Only few applications of Dempster-Shafer theory to fault-Tree Analysis are reported in literature. Limbourg et al. [2] created a Matlab toolbox where each probability is modeled by a random interval on [0,1]. Instead of Dempster rule, they use Weighted average [4] for the aggregation of the belief functions of different variables. Murtha [5] uses the same method in an application to small unmanned aerial vehicles. Another method using 3-valued logic proposed by Guth [6] is compared by Cheng to interval computation, over small examples of Fault-trees [7]. The above results can be specialized to fault trees.

A path in a fault tree links the top (dreadful) event to the leaves of the tree: it is called a *cut*. When this path has a minimal number of steps, it is said to be a *minimal cut*. Each cut is a *conjunction of atoms*. As a consequence of the above results we can compute the belief and plausibility of conjunctions and disjunction of k atoms $A_1, \ldots A_k$:

$$Bel(C) = \prod_{i=1}^{k} l_i, \qquad Pl(C) = \prod_{i=1}^{k} u_i \tag{6}$$

$$Bel(D) = 1 - \prod_{i=1}^{k}(1 - l_i), \qquad Pl(D) = 1 - \prod_{i=1}^{k}(1 - u_i). \tag{7}$$

From a Fault-tree F, an approximation can be obtained by means of minimal cuts. For a given *order* (maximal number of atoms in conjunctions), appropriate software can find the set of all Minimal Cuts that lead to the top event. The disjunction of all those Minimal Cuts will give us a partial Fault-tree which will be an approximation of F. Fig. 1 is an example of such a Partial Fault-tree.

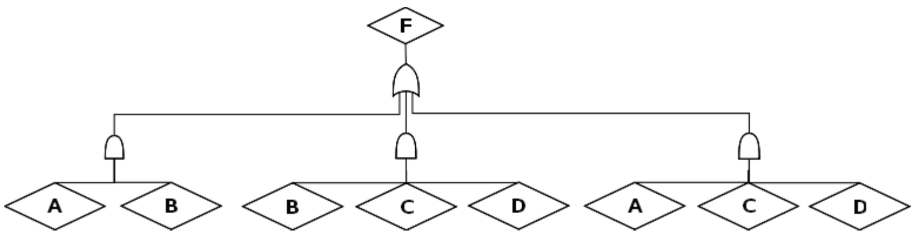

Fig. 1. Example of Partial Fault Tree

The Boolean formula F' represented by this tree will always be under the form of a disjunction of conjunctions of atoms $C_1 \vee ... \vee C_m$. The formula written in this form will be referred to as a Disjunctive Atomic Normal Form (DANF) (excluding negative literals). In order to compute the Belief function of such a formula, we should generalize the computation of the belief of a disjunction of k atoms.

Proposition 5 (Belief of a disjunctive atomic normal form (DANF)).

$$Bel(C_1 \vee ... \vee C_m) = \sum_{i=1}^{m} Bel(C_i) - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} Bel(C_i \wedge C_j)$$
$$+ \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m} Bel(C_i \wedge C_j \wedge C_k) - ... + (-1)^{m+1} Bel(C_1 \wedge ... \wedge C_m),$$

where C_i are conjunctions of atoms.

During the computation, the conjunctions of conjunctions, such as $C_i \wedge C_j \wedge C_k$ must be simplified, deleting redundant atoms. Note that this apparent additivity of a generally non-additive function is due to the specific shape of focal elements (partial models). In general, for S and T Boolean formulas, we cannot write $Bel(S \vee T) = Bel(S) + Bel(T) - Bel(S \wedge T)$, because there are focal elements in $S \vee T$ that are subsets of neither S nor T nor $S \wedge T$. Here due to the DANF form, all partial models of $C_1 \vee ... \vee C_m$ are conjunctions of literals appearing in the conjunctions.

A similar result holds for computing the plausibility of a DNF.

Proposition 6 (Pl of a disjunctive atomic normal form (DANF)).

$$Pl(C_1 \vee ... \vee C_m) = \sum_{i=1}^{m} Pl(C_i) - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} Pl(C_i \wedge C_j)$$
$$+ \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m} Pl(C_i \wedge C_j \wedge C_k) - ... + (-1)^{m+1} Pl(C_1 \wedge ... \wedge C_m),$$

where C_i are conjunctions of literals.

Thanks to the duality between Belief and Plausibility, both computations are quite similar, hence the time complexity does not increase. It is also much less time consuming than an exhaustive computation as presented in section 3.

4.3 General case

The general case of a Boolean formula with positive and negative literals is more tricky. Such a formula can appear, for example, in fault-trees representing different modes in a system, or representing exclusive failures [1]. Of course we can assume the formula is in DNF format. But it will be a conjunction of literals, and it is no longer possible to apply the two previous propositions. Indeed when conjunctions contain opposite literals, they have disjoint sets of models but their disjunctions may be implied by partial models (focal elements) that imply none of the conjuncts. For instance consider $A \vee (\neg A \wedge B)$ (which is just the disjunction $A \vee B$ we know how to deal with). It does not hold that $Bel(A \vee (\neg A \wedge B) = Bel(A) + Bel(\neg A \wedge B)$, since the latter sum neglects

$m(B)$, where B is a focal element that implies neither A nor $\neg A \wedge B$. However if $C_1 \vee ... \vee C_m$ are pairwise mutually inconsistent partial models such that no disjunction of C_i and C_j contains a partial model implying neither C_i nor C_j, the computation can be simplified since then $Bel(C_1 \vee ... \vee C_m) = \sum_{i=1}^{m} Bel(C_i)$. For instance, the belief in an exclusive OR $Bel((A_1 \wedge \neg A_2) \vee (A_2 \wedge \neg A_1))$ is of this form. More work is needed in the general case.

5 Comparison between Interval Analysis and Dempster-Shafer Theory

Table 1 summarizes the results obtained using the two methods seen in section 2.2 and 3 applied to Boolean formulas: (i) the belief functions method with the assumption that the probability values come from independent sources of information, and (ii) the full-fledged interval analysis method under the assumption that all atomic events are independent [1]. These two assumptions do not reflect the same kind of situations. In particular the independence between sources of information may be justified if elementary components in the device under study are different from one another, which often implies that the sources of information about them will be distinct. However the fact that such elementary components interact within a device tends to go against the statistical independence of their respective behaviors.

Those results are given for the basic Boolean operators with variables A, B, C and D. The probability interval used for those computations are: $P(A) \in [0.3, 0.8]$, $P(B) \in [0.4, 0.6]$, $P(C) \in [0.2, 0.4]$, and $P(D) \in [0.1, 0.5]$.

Table 1. Comparison between Interval Analysis and Dempster-Shafer Theory

Connective	Formula	Belief Functions (i)	Interval Analysis (ii)
OR	$A \vee B$	$l_F = l_A + l_B - l_A l_B = 0.58$ $u_F = u_A + u_B - u_A u_B = 0.92$	$l_F = l_A + l_B - l_A l_B = 0.58$ $u_F = u_A + u_B - u_A u_B = 0.92$
AND	$A \wedge B$	$l_F = l_A l_B = 0.12$ $u_F = u_A u_B = 0.48$	$l_F = l_A l_B = 0.12$ $u_F = u_A u_B = 0.48$
IMPLIES	$A \Rightarrow B$	$l_F = l_A + (1 - u_A)(1 - u_B) = 0.48$ $u_F = 1 - l_A(u_B - l_A) = 0.94$	$l_F = 1 - u_A + l_B u_A = 0.52$ $u_F = 1 - l_A + u_B l_A = 0.88$
ExOR	$A \triangle B$	$l_F = l_A(1 - u_B) + l_B(1 - u_A)$ $u_F = u_A + u_B - l_A l_B - u_A u_B$ $[l_F, u_F] = [0.2, 0.8]$	[0.44,0.56]
Fault-tree (Fig. 1)	F	$l_F = l_A l_B + l_B l_C l_D + l_A l_C l_D - 2 l_A l_B l_C l_D$ $u_F = u_A u_B + u_B u_C u_D + u_A u_C u_D - 2 u_A u_B u_C u_D$ $[l_F, u_F] = [0.1292, 0.568]$	$l_F = l_A l_B + (1 - l_A) l_B l_C l_D + (1 - l_B) l_A l_C l_D$ $l_F = u_A u_B + (1 - u_A) u_B u_C u_D + (1 - u_B) u_A u_C u_D$ $[l_F, u_F] = [0.1292, 0.568]$

In Interval Analysis, knowing the monotonicity of a formula makes the determination of its range straightforward. A Boolean formula is *monotonic* with respect to a variable when we can find an expression of the formula where this variable appears only in a positive or negative way. It is the case for the formulas *And*, *Or*, and *Implies*.

But when the monotonicity is not easy to study, an exhaustive computation for all intervals boundaries must be carried out, like for the *Equivalence* and the *Exclusive Or* [1].

The difference between the results varies a lot with the formula under study. Sometimes, using the Dempster-Shafer theory give the same results as interval analysis, hence, in those cases, the dependency assumption does not have a big influence on the output value; e.g in case of conjunction and disjunction of literals, but also disjunction of conjunctions of atoms (as shown in table 1). This is not surprising as focal elements also take the form of conjunctions of literals, and their masses are products of marginals. The fact that in such cases the same results are obtained does not make the belief function analysis redundant: it shows that the results induced by the stochastic independence assumption are valid even when this assumption is relaxed, for some kinds of Boolean formulas. For more general kinds of Boolean formulas, the intervals computed by using belief functions are in contrast wider than when stochastic independence is assumed.

In general, the probability family induced by the stochastic independence assumption will be included in the probability family induced by the belief functions. This proposition can be proved using the results of Fetz [9] and Couso and Moral [10]. Any probability measure in $\mathcal{P}(m) = \{P \geq Bel\}$ dominating a belief function induced by a mass function m can be written in the form: $P = \sum_{E \subseteq \Omega} m(E) \cdot P_E$ where P_E is a probability measure such that $P_E(E) = 1$ that shares the mass $m(E)$ among elements of E. For a function of two Boolean variables x_1 and x_2, with two ill-known probability values $P_1(A_1) = p_1$ and $P_1(A_2) = p_2$, p_1 is of the form $l_1 + \alpha(u_1 - l_1)$ for some $\alpha \in [0, 1]$ and p_2 is of the form $l_2 + \beta(u_2 - l_2)$ for some $\beta \in [0, 1]$. The explicit sharing, among interpretations, of the masses $m(E)$, induced by probability intervals $[l_1, u_1]$ and $[l_2, u_2]$, that enables $P = P_1 P_2$ to be recovered is:

1. The masses on interpretations bear on singletons, hence do not need to be shared.
2. The masses on partial models are shared as follows
   - $m(A_1)\beta$ is assigned to A_1A_2, $m(A_1)(1 - \beta)$ to $A_1\neg A_2$.
   - $m(A_2)\alpha$ is assigned to A_1A_2, $m(A_2)(1 - \alpha)$ to $\neg A_1A_2$.
   - $m(\neg A_1)\beta$ is assigned to $\neg A_1A_2$, $m(\neg A_1)(1 - \beta)$ to $\neg A_1\neg A_2$.
   - $m(\neg A_2)\alpha$ is assigned to $A_1\neg A_2$, $m(A_2)(1 - \alpha)$ to $\neg A_1\neg A_2$.
3. $m(\Omega)$ is shared as follows: $\alpha\beta m(\Omega)$ to A_1A_2, $(1 - \alpha)\beta m(\Omega)$ to $\neg A_1A_2$, $\alpha(1 - \beta)m(\top)$ to $A_1\neg A_2$, $(1 - \alpha)(1 - \beta)m(\top)$ to A_1A_2.

It can be checked that the joint probability P_1P_2 is the form $\sum_{E \subseteq \Omega} m(E) \cdot P_E$ using this sharing of masses. This result can be extended to more than 2 variables. It indicates that the assumptions of source independence is weaker than the one of stochastic independence, and is of course stronger than no independence assumption at all. So the belief function approach offers a useful and tractable approach to evaluate the impact of stochastic independence assumptions on the knowledge of the probability of dreadful events in fault-tree analysis.

6 Conclusion

The more faithful models are to the actual world, the more their complexity increases. When assumptions are made to simplify the model, then it is important to know how far the results stand away from reality in order to use them as appropriately as possible. Having a means to compare different kinds of models and different kinds of assumptions can be a good asset in order to make best decisions out of the models. In this paper, we have laid bare three kinds of assumptions for the calculation of the probability of some risky event in terms of probability of basic atomic formulas. We have focused on the belief function approach that assumes independence between sources of information proving imprecise probabilistic information. We did also gave a formal solution for an application to fault-tree analysis based on a DNF conversion. A practical scalable solution for handling general Boolean formulas is a topic for further research.

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