Chapter 8
Inexact Discretionary Inputs in Data Envelopment Analysis

Majid Zerafat Angiz Langroudi

Abstract In this chapter, the relationship between fuzzy concepts and the efficiency score in Data envelopment analysis (DEA) is dealt with. A new DEA model for handling crisp data using fuzzy concept is proposed. In addition, the relationship between possibility sets and the efficiency score in the traditional crisp CCR model is presented. The relationship provides an alternative perspective of viewing efficiency. With the usage of the appropriate fuzzy and possibility sets to represent certain characteristics of the input data, many DEA models involving input data with various characteristics could be studied. Furthermore, based upon the proposed models, two nondiscretionary models are introduced in which some inputs or outputs, in a fuzzy sense, are inexact discretionary variables. For this purpose, a two-stage algorithm will be presented to treat the DEA model in the presence of an inexact discretionary variable. With this relationship, a new perspective of viewing and exploring other DEA models is now made possible.

Keywords Data envelopment analysis · Fuzzy · Possibility distribution · Efficiency · Non-discretionary variables

1 Introduction

Since its inception 48 years ago, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines. Applications of fuzzy technology can be found in artificial intelligence, computer sciences, control engineering, decision theory, expert systems, logic, management sciences, operations research, robotics and others [1].

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A. Emrouznejad and M. Tavana (eds.), *Performance Measurement with Fuzzy Data Envelopment Analysis*, Studies in Fuzziness and Soft Computing 309,
DOI: 10.1007/978-3-642-41372-8_8,

Sugeno [2] defined a fuzzy measure. Banon [3] shows that very many measures with finite universe, such as probability measures, belief functions, plausibility measures and so on, are fuzzy measures in the sense of Sugeno. In recent years, some specific interpretations of fuzzy set theory have been suggested. One of them is possibility theory. In the framework of fuzzy set theory, Zadeh [4] introduced the notion of a possibility distribution and the concept of a possibility measure, which is a special type of fuzzy measure proposed by Sugeno [2]. Possibility theory focuses primarily on imprecision. Possibility theory used to correspond, roughly speaking, to the min–max version of fuzzy set theory, that is, to fuzzy set theory in which the intersection is modeled by the min operator and the union by the max operator. This interpretation of possibility theory, however, is no longer correct. Rather, it has been developed into a well-founded and comprehensive theory.

DEA researchers have begun using fuzzy concept for measuring efficiency and productivity of DMUs since 1992. Some existing approaches for solving fuzzy DEA are the tolerance approaches; the fuzzy ranking approaches; the α-level based approaches; the defuzzification approaches; and the possibility approach [5].

Guo et al. [6] were pioneers in using the fuzzy DEA models based on possibility and necessity measures. Alp [7] further extended the concept of possibilistic DEA by considering problems in handling real data such as fuzziness, impreciseness and incompleteness. In such cases, the difficulty in model building can be overcome by using fuzzy set theory and concepts. In his study, only situations of incomplete data were considered and a new method for possibilistic DEA was introduced. Saati et al. [8] introduced a mathematical programming approach for measuring technical efficiency in a possibilistic environment.

Lertworasirikul et al. [9] further expanded the possibility approach to the Fuzzy DEA model from an optimistic viewpoint. Lertworasirikul et al. [10] further developed a fuzzy BCC model where the possibility and credibility approaches are provided and compared with a $\overrightarrow{\tau}Z$ level based approach for solving the fuzzy DEA models. Using the possibility approach, they revealed the relationship between the primal and dual models of fuzzy BCC. Using the credibility approach they showed how the efficiency value for each *DMU* can be obtained as a representative of its possible range.

Liu and Chuang [11] developed a method to find the fuzzy efficiency measures embedded with the assurance region (AR) concept when some observations are fuzzy numbers. They utilized Zadeh's extension principle to transform a fuzzy DEA-AR model into a family of crisp DEA-AR models to calculate the lower and upper bounds of efficiency scores at a specific level. From different possibility levels, a membership function is derived accordingly.

This research, however, is not about fuzzy DEA. It does not deal with fuzzy input or output data. It is about the use of the fuzzy concept to handle crisp data in DEA. The usage of fuzzy concepts in handling certain crisp mathematical modeling situations has resulted in the formulation of creative and efficient procedures. This can be seen for example in the work of Zerafat Angiz et al. [12] and Emrouznejad et al. [13]. Motivated by these results, in this research an alternative interpretation of the CCR model using the possibility set is presented. The interpretation opens a window for viewing efficiency in a new perspective. This new perspective is very useful in certain applications of the DEA models. An example application is shown in the handling of non-discretionary data.

Banker and Morey [14] initially introduced a DEA model in which the inputs are divided into two non-discretionary and discretionary variables. A number of approaches have been proposed for handling non-discretionary DEA models. Among them are the works of Golany and Roll [15], Ruggiero [16, 17], Muniz [18], Muniz et al. [19] and Cordero-Ferrera [20].

The remainder of this research is organized as follows. Section. 2 provides some background information about the possibility sets. The relationship between efficiency in DEA and the possibility sets together with some interpretations of efficiency are presented in Sect. 3. An alternative view of efficiency based on fuzzy concepts is presented in Sect. 4. A case study is illustrated in Sect. 5. In Sect. 6, a new non-discretionary DEA model is introduced. Finally, the conclusion is presented in Sect. 7.

2 Fuzzy Events via Possibility Measures

In the framework of fuzzy set theory, Zadeh [21] proposed possibility theory which is a special type of fuzzy measure for modeling and characterizing situations involving uncertainty. The following definitions have been extracted from Lertworasirikul et al. [9, 10]:

Let $P(X_i)(i = 1, 2, \ldots, n)$ be the power set of a set $X_i\,(i = 1, 2, \ldots, n)$. A possibility measure is a function $\pi : P(X_i) \to [0, 1]$ with the properties

$$\begin{aligned}
&1. \quad \pi(\emptyset) = 0, \quad \pi(X_i) = 1\\
&2. \quad A \subset B \Rightarrow \quad \pi(A) \le \pi(B)\\
&3. \quad \pi\left(\bigcup_{i\in I} A_i\right) = \sup_{i\in I} \pi(A_i) \quad A_i \subset P(X_i) \text{ with an index set } I;
\end{aligned} \tag{1}$$

$(X_i, P(X_i), \pi_i)$ is called the possibility space.

Based on the possibility measure, Zadeh [21] defines a fuzzy variable $\tilde{\xi}$ as follows:

$$\begin{aligned}
\mu_{\tilde{\xi}}(s) &= \pi\left(\left\{x_i \in X_i \middle| \tilde{\xi}(x_i) = s\right\}\right)\\
&= \sup_{x_i \in X_i}\left\{\pi\{x_i\} \middle| \tilde{\xi}(x_i) = s\right\}, \ \forall s \in R.
\end{aligned} \tag{2}$$

The Cartesian product of the possibility space $(X, P(X), \pi)$ in which $X = X_1 \times X_2 \times \cdots \times X_n$, is defined as follows:

$$\pi(A) = \min_{i=1,2,\ldots,n} \{\pi_i(A_i) | A = A_1 \times A_2 \times \cdots \times A_n, A_i \in P(X_i)\}.$$

Consider $\tilde{a}$ and $\tilde{b}$ as fuzzy variables on the possibility spaces $(X_1, P(X_1), \pi_1)$ and $(X_2, P(X_2), \pi_2)$, respectively. On the product possibility space $(X = X_1 \times X_2, P(X), \pi)$, the fuzzy event $\tilde{a} \leq \tilde{b}$ is defined as

$$\pi\left(\tilde{a} \leq \tilde{b}\right) = \sup_{\substack{x_1 \in X_1 \\ x_2 \in X_2}} \left\{\pi\left\{(x_1, x_2) \middle| \tilde{a}(x_1) \leq \tilde{b}(x_2)\right\}\right\}$$

$$=$$

The possibility of fuzzy event $\tilde{a} \leq \tilde{b}$ is obtained from Expression (2) as follows:

$$\pi\left(\tilde{a} \leq \tilde{b}\right) = \sup_{s,t \in R}\left\{\min\left(\mu_{\tilde{a}}(s), \mu_{\tilde{b}}(t)\right) | s \leq t\right\}.$$

Furthermore, the possibility of the fuzzy event $a \leq \tilde{b}$ in which a is a crisp value is given as

$$\pi\left(a \leq \tilde{b}\right) = \sup_{t \in R}\left\{\mu_{\tilde{b}}(t) | s \leq t\right\}.$$

Lertworasirikul et al. [10] proved the following Lemma using the possibility of fuzzy events concept.

Lemma 1 *Let $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n$ be fuzzy variables with normal and convex membership functions. Then,*

$$\pi(\tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n \geq b) \geq \alpha \text{ if only if } (\tilde{a}_1)^U_\alpha + (\tilde{a}_2)^U_\alpha + \cdots + (\tilde{a}_n)^U_\alpha \geq b. \quad (3)$$

where the symbol $(.)^U_\alpha$ denotes the upper bound of the α-level set of $\tilde{a}_i$ $i = 1, 2, \ldots n$.

If the fuzzy number $\tilde{r}_i = \left((\tilde{r}_i)^L_0, (\tilde{r}_i)^L_1, (\tilde{r}_i)^U_1, (\tilde{r}_i)^U_0\right)$, $i = 1, 2, \ldots, n$ is trapezoidal, for any level of α such that $0 \leq \alpha \leq 1$, the following is true:

$$\begin{aligned} &\pi(\tilde{r}_1 + \tilde{r}_2 + \cdots + \tilde{r}_n \geq b) \geq \alpha \text{ if only if} \\ (1 - \alpha)&\left((\tilde{r}_1)^U_0 + \cdots + (\tilde{r}_n)^U_0\right) + \alpha\left((\tilde{r}_1)^U_1 + \cdots (\tilde{r}_n)^U_1\right) \geq b \end{aligned} \quad (4)$$

Figure 1 shows the trapezoidal fuzzy number $\tilde{r} = \left((\tilde{r})^L_0, (\tilde{r})^L_1, (\tilde{r})^U_1, (\tilde{r})^U_0\right)$.

3 Interpretation of the CCR Model Using the Possibility Set

In this section, we present an alternative interpretation of the efficiency concept in data envelopment analysis. To begin with, we start with the CCR model and after a few substitutions and changes a possibility form of the model is obtained. A graphical illustration and explanation is given next and followed by some discussion.

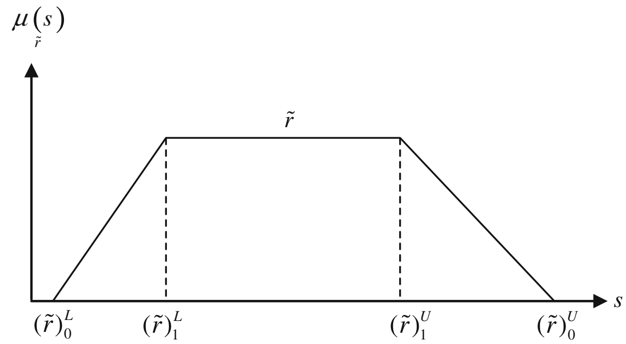

Fig. 1 Trapezoidal fuzzy number $\tilde{r} = \left((\tilde{r})_0^L, (\tilde{r})_1^L, (\tilde{r})_0^U, (\tilde{r})_1^U\right)$

3.1 Derivation of the Possibility Model

Consider the following DEA Model introduced by Charnes et al. [22]:

$$\begin{aligned} & \min \quad \theta \\ s.t. \quad & \theta x_{ip} - \sum_{j=1}^{n} \lambda_j x_{ij} \geq 0 \quad i = 1, 2, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} \quad r = 1, 2, \ldots, s \\ & \lambda_j \geq 0 \quad j = 1, 2, \ldots, n. \end{aligned} \tag{5}$$

In Model (5), $x_{ij}(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ represents the quantity of input i consumed by DMU_j and $y_{rj}(r = 1, 2, \ldots, s;\ j = 1, 2, \ldots, n)$ is the quantity of output r produced by DMU_j. The variable θ measures the efficiency of DMU_p, and $\lambda_j(j = 1, 2, \ldots, n)$ are the raw weights assigned to the peer DMUs when solving the DEA model.

By adding $\theta \sum_{j=1}^{n} \lambda_j x_{ij} - \theta \sum_{j=1}^{n} \lambda_j x_{ij}$ to inequality (5), the following mathematical programming model is obtained:

$$\begin{aligned} \min \quad & \theta \\ s.t. \quad & \theta x_{ip} - \theta \sum_{j=1}^{n} \lambda_j x_{ij} + \theta \sum_{j=1}^{n} \lambda_j x_{ij} - \sum_{j=1}^{n} \lambda_j x_{ij} \geq 0 \quad i = 1, 2, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} \quad r = 1, 2, \ldots, s \\ & \lambda_j \geq 0 \quad j = 1, 2, \ldots, n. \end{aligned} \tag{6}$$

After some rearrangement, Model (6) is written as follows:

$$
\begin{array}{lll}
\min & \theta & \\
s.t. & \theta(x_{ip} - \sum\limits_{j=1}^{n} \lambda_j x_{ij}) + (1-\theta)(0 - \sum\limits_{j=1}^{n} \lambda_j x_{ij}) \geq 0 & i = 1, 2, \ldots, m \\
& \sum\limits_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} & r = 1, 2, \ldots, s \\
& \lambda_j \geq 0 & j = 1, 2, \ldots, n
\end{array}
\tag{7}
$$

Assume that $\left(x_{ip}\right)_1^U = 0$ and $\left(x_{ip}\right)_0^U = x_{ip}$. In addition, consider the following trapezoidal fuzzy number:

$$
\tilde{x}_{ip} = \left(\left(\tilde{x}_{ip}\right)_0^L, \left(\tilde{x}_{ip}\right)_1^L, \left(\tilde{x}_{ip}\right)_1^U, \left(\tilde{x}_{ip}\right)_0^U \right) = \left(-x_{ip}, -x_{ip}, 0, x_{ip}\right).
$$

Figure 2 illustrates such a fuzzy number.

Using the fuzzy number $\tilde{x}_{ip}$, Model (7) is written as follows:

$$
\begin{array}{lll}
& \min \quad \theta & \\
s.t. & \theta\left(\left(x_{ip}\right)_0^U - \sum\limits_{j=1}^{n} \lambda_j x_{ij} \right) + (1-\theta)\left(\left(x_{ip}\right)_1^U - \sum\limits_{j=1}^{n} \lambda_j x_{ij} \right) \geq 0 & i = 1, 2, \ldots, m \\
& \sum\limits_{j=1}^{n} \lambda_j y_{{rj}_{rp}} \qquad r = 1, 2, \ldots, s & \\
& \lambda_j \geq 0 \qquad j = 1, 2, \ldots, n. &
\end{array}
\tag{8}
$$

By assuming that $\theta = 1 - \beta$, Model (8) is converted to the following non-linear programming problem:

$$
\begin{array}{lll}
& 1 - \max \beta & \\
s.t. & (1-\beta)\left(\left(x_{ip}\right)_0^U - \sum\limits_{j=1}^{n} \lambda_j x_{ij} \right) + \beta\left(\left(x_{ip}\right)_1^U - \sum\limits_{j=1}^{n} \lambda_j x_{ij} \right) \geq 0 & i = 1, 2, \ldots, m \\
& \sum\limits_{j=1}^{n} \lambda_j y_{{rj}_{rp}} & r = 1, 2, \ldots, s \\
& \lambda_j \geq 0 & j = 1, 2, \ldots, n.
\end{array}
\tag{9}
$$

Based on Expression (4), the possibility programming formulation of the DEA model is thus obtained as follows:

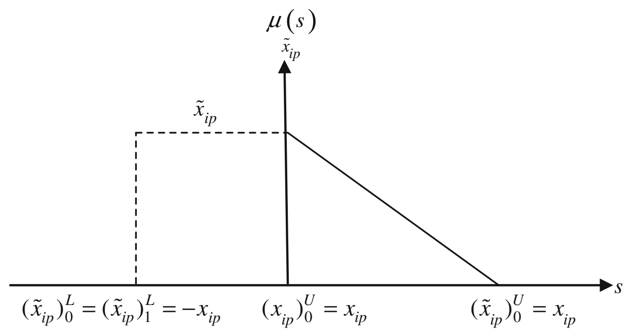

Fig. 2 Membership function of the fuzzy number $\tilde{x}_{ip}$

$$
\begin{aligned}
& \max \quad \beta \\
s.t. \quad & (\pi(\tilde{x}_{ip} - \sum_{j=1}^{n} \lambda_j x_{ij}) \geq 0) \geq \beta \qquad i = 1, 2, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} \quad r = 1, 2, \ldots, s \\
& \lambda_j \geq 0 \qquad j = 1, 2, \ldots, n.
\end{aligned}
\tag{10}
$$

Afterwards, the above model is called the Possibilistic CCR model. Obviously, if equation $\sum_{j=1}^{n} \lambda_j = 1$ is added to Model (5), the BCC model is obtained. Since the equation does not play any role in the inequalities associated with the possibility sets of Model (10), the above derivation is also valid for the input oriented BCC model.

4 A Traditional CCR Model Basis on the Fuzzy Concept

In this section an alternative interpretation of efficiency based on the fuzzy concept is presented. For this end, a new DEA model for handling crisp data using the fuzzy concept is proposed. Assume that all postulates to construct the production possibility set corresponding to constant return to scale are satisfied. Therefore, the production possibility set corresponding to the CCR model and the proposed model are similar. To present the new model, we introduce a triangular fuzzy number with the following membership function.

In Model (5) consider inputs x_{ip} and outputs y_{rp} related to the DMU under evaluation, say DMU_p. Then, the membership function of fuzzy number $\tilde{x}_{ip}$ is considered as follows:

$$
\mu_{\tilde{x}_{ip}}(\overline{x}_{ip}) = \frac{x_{ip} - \overline{x}_{ip}}{x_{ip}} \qquad x_{ip} \leq \overline{x}_{ip} \quad i = 1, 2, \ldots, m \tag{11}
$$

The following linear programming model is proposed:

$$
\begin{aligned}
& \text{max}-\text{min model} \\
& \max\left\{\min\left\{\mu_{\tilde{x}_{ip}}\left(\bar{x}_{ip}\right)\right\}(i=1,2,\ldots,m)\right\} \\
& s.t. \\
& \qquad \sum_{j=1}^{n} \lambda_j x_{ij} \le \bar{x}_{ip} \quad i=1,2,\ldots,m \\
& \qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp} \quad j=1,2,\ldots,s \\
& \qquad \bar{x}_{ip} \le x_{ip} \qquad i=1,2,\ldots,m \\
& \qquad \lambda_j \ge 0 \qquad j=1,2,\ldots,n
\end{aligned}
\tag{12}
$$

where $\bar{x}^{*}_{ip}$ indicates the inputs that are necessary for the DMU_p to be efficient.

Obviously, Model (12) is always feasible and the value of optimum objective function in (12) is non-negative and less than or equal to 1.

Since the objective functions are the membership functions, the optimal value in (12) does not exceed the maximum value of the membership values, i.e. a value of 1. The values $\lambda_p = 1, \lambda_j = 0 (j \neq p)$ and $\bar{x}_{ip} = x_{ip}$ are the feasible solutions in Model (12); thus it is always feasible.

Assume that $\alpha = \min\left\{\mu_{\tilde{x}_{ip}}\left(\bar{x}_{ip}\right)\right\}$, then:

$$
\begin{aligned}
& \max \quad \alpha \\
& s.t. \\
& C1 \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \bar{x}_{ip} \quad i=1,2,\ldots,m \\
& C2 \quad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp} \\
& C3 \quad \alpha \le \frac{x_{ip} - \bar{x}_{ip}}{x_{ip}} \quad i=1,2,\ldots,m \\
& C4 \quad x_{ip} \le \bar{x}_{ip} \qquad i=1,2,\ldots,m \\
& \qquad \lambda_j \ge 0 \quad j=1,2,\ldots,n \\
& \qquad \alpha \ge 0
\end{aligned}
\tag{13}
$$

This problem is now solved for each DMU.

The main idea of our proposal is that there exists a relationship between the efficiency scores in Model (5) and the membership values of the fuzzy numbers $\tilde{x}_{ip}$. The lesser the membership values the higher the efficiency score of the DMU under evaluation.

The following theorem indicates equivalency between the efficiency scores of Models (5) and (13).

Theorem 1 *Assume that* $\theta^* = \min\theta$ *and* $\alpha^* = \max\alpha$ *are the optimal values of* (5) and (13), *respectively. Then,* $\theta^* = 1 - \alpha^*$. *In other words, the treatment of Model* (13) *and the CCR model are similar.*

In Model (6), consider $\alpha \leq \frac{x_{ip} - \bar{x}_{ip}}{x_{ip}}$, implying that $\alpha x_{ip} \leq x_{ip} - \bar{x}_{ip}$, and

$$\bar{x}_{ip} \leq x_{ip} - \alpha x_{ip} = (1 - \alpha) x_{ip} \tag{14}$$

There is a one-to-one correspondence between the values $\bar{x}_{ip}$ and $(1 - \alpha)x_{ip}$, and therefore we can consider Eq. (14). Replacing the last inequality (14) in C1 in Model (13), the following linear programming problem is obtained:

$$\begin{array}{ll}
\max & \alpha \\
s.t. & \\
C\prime 1 & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \bar{x}_{ip} \leq (1-\alpha) x_{ip} \quad i = 1, 2, \ldots, m \\
C'2 & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} \\
C'3 & \alpha \leq \dfrac{x_{ip} - \bar{x}_{ip}}{x_{ip}} \qquad i = 1, 2, \ldots, m \\
C'4 & \bar{x}_{ip} \leq x_{ip} \qquad i = 1, 2, \ldots, m \\
 & \lambda_j \geq 0 \qquad j = 1, 2, \ldots, n \\
 & \alpha \geq 0
\end{array} \tag{15}$$

Assume $\theta = 1 - \alpha$. Then Model (15) is converted to the following:

$$\begin{array}{ll}
\max & 1 - \theta \\
s.t. & \\
C''1 & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \bar{x}_{ip} \leq \theta x_{ip} \quad i = 1, 2, \ldots, m \\
C''2 & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} \\
C''3 & 1 - \theta \leq \dfrac{x_{ip} - \bar{x}_{ip}}{x_{ip}} \quad i = 1, 2, \ldots, m \\
C''4 & \bar{x}_{ip} \leq x_{ip} \qquad i = 1, 2, \ldots, m \\
 & \lambda_j \geq 0 \quad j = 1, 2, \ldots, n \\
 & \theta \geq 0
\end{array} \tag{16}$$

In $C''3$, $1-\theta \leq \frac{x_{ip}-\bar{x}_{ip}}{x_{ip}}$ implies that $\theta x_{ip} \leq \bar{x}_{ip}$, thus, $C''3$ is implied by $C''1$. Since $0 \leq \frac{x_{ip}-\bar{x}_{ip}}{x_{ip}} \leq 1$, $0 \leq \theta \leq 1$ is obtained from $C''3$. On the other hand, considering $\max(1-\theta) = 1 - \min\theta$, Model (16) is written as follows:

$$
\begin{aligned}
& 1 - \min\,\theta \\
& s.t. \quad C''1 \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ip} \quad i = 1, 2, \ldots, m \\
& \qquad C''2 \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} \\
& \qquad \lambda_j \geq 0 \quad j = 1, 2, \ldots, n \\
& \qquad \theta \geq 0
\end{aligned}
\tag{17}
$$

This means that $\theta^* = 1 - \alpha^*$.

By some substitutions, Model (5) implies Model (13). Simply put, Model (13) demonstrates a fuzzy interpretation of the CCR model. The efficiency score in this model is the membership value of the point located at the intersection of the fuzzy interval and the efficiency frontier.

4.1 Graphical Illustration

For a better understanding of the relationship between θ and $1-\alpha$ and also the relationship between the efficiency in the CCR model and the possibility set, we refer to Fig. 3a and b which provide two two-dimensional diagrams of a simple efficiency case study in which only a single input (x) is used to produce a single output (y). There are a number of DMUs in this case study, however, only the data

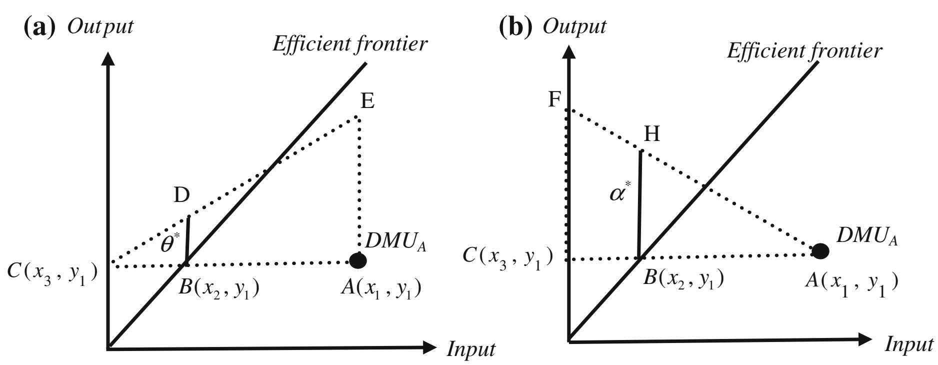

Fig. 3 **a** A fuzzy number related to efficiency. **b** An alternative fuzzy number related to efficiency

for DMU_A is plotted on the diagrams. Triangles *CAE* and *CAF* are the membership functions of the fuzzy numbers which correspond to the input of DMU_A. In Fig. 3a, the membership function is in the form of the possibility aspect, whereas in Fig. 3b, it is in the impossibility aspect. These membership functions are superimposed onto the graphs. Notice that, the shape of the membership functions reflects the characteristic of the input variables in the DEA (i.e., the smaller the level of the input the more efficient is the DMU).

In referring to Fig. 3a, *BD* is parallel to *AE*; therefore the two triangles *CEA* and *CDB* are similar. So, we have

$$\frac{CB}{CA} = \frac{BD}{AE} \tag{18}$$

It is well known that the value $\frac{CB}{CA}$ is equivalent to the technical efficiency of DMU_A. Thus, since $AE = 1$, $\frac{BD}{AE} = \theta^*$ is the technical efficiency of DMU_A which is also the membership value corresponding to point *B*, the reference point of DMU_A on the efficient frontier.

Furthermore, we consider Fig. 3b in which triangle *CAF* is similar to triangle *BAH*. Therefore, based on Thales theorem or the Basic Proportionality theorem which states that the line drawn parallel to one side of a triangle divides the other two sides in the same ratio, we have

$$\frac{BA}{CA} = \frac{BH}{CF} \quad (CF = 1) \tag{19}$$

And thus,

$$1 - \frac{BA}{CA} = \frac{CA - BA}{CA} = \frac{CB}{CA} \tag{19.1}$$

Obviously, the following expressions are obtained from expressions (18), (19), and (20):

$$BD = 1 - BH \quad \text{or} \quad \theta^* = 1 - \alpha^* \tag{20}$$

In fact, the above equations illustrate the relationship between the efficiency in DEA and the optimal α- level of the new fuzzy number.

4.2 Discussion

We refer to Fig. 3a and b again which provide two two-dimensional diagrams illustrating the possibility interpretation of the efficiency of DMU_A. For this purpose, two antonym keywords, production possibility and production impossibility, are utilized. It is clear that if we consider $0 \leq \beta \leq 1$ as a measure of the production possibility, then $1 - \beta$ is the measure of the production impossibility. In Fig. 3b, since point A is an observed value for DMU_A, the value of the production impossibility corresponding to point A is considered zero. In other words, output y_1 can certainly be produced by input x_1. If the input level is decreased to $x_3 = 0$, producing y_1 is impossible. Therefore, the production impossibility of $C(x_3, y_1)$ is 1. The production impossibility for producing output y_1 using inputs between $x_3 = 0$ and x_1 is a fuzzy concept which is illustrated by a triangular fuzzy number demonstrated by the dotted lines. At point B, which is on the efficient frontier, the production impossibility value is $BH = \alpha^*$. This value is related to the efficiency score of DMU_A.

In Fig. 3a an alternative interpretation is presented based on the production possibility aspect. By reducing the input from the observed value at point A, we reached point B, which is on the efficient frontier. The production possibility value at this point is $BD = \theta^*$, which indicates the efficiency score of DMU_A.

5 An Illustration Example

The following example demonstrates the correspondence between the proposed fuzzy and the possibilistic methodologies and the CCR model.

Assume that a major organization consists of six branches called DMUs. Due to the recession, adjusting the budget is the organization's agenda. Since the management is interested in maintaining the current level of production, DEA as a powerful tool is chosen to determine the decrease in inputs by maintaining the current level of production. In this case, the budget allocated to each branch is divided into sub budgets, the budget related to the employee's salary and the allocated budget associated with other affairs. In Table 1 a list of 6 DMUs with two inputs and two outputs measurements is given. Inputs I_1 (in \$100,000) and I_2 (in \$1,000,000) are the allocated budget associated with other affairs and the budget regarding the employee's salary, respectively. Outputs O_1 (in 10,000 tons)

Table 1 The list of DMUs with two inputs and two outputs

DMUs	I_1	I_2	O_1	O_2
1	1.50	1.50	1.40	0.35
2	4.00	0.70	1.40	2.10
3	3.20	1.20	4.20	1.05
4	5.20	2.00	2.80	4.20
5	3.50	1.20	1.90	2.50
6	3.20	0.70	1.40	1.50

Table 2 Efficiency scores for CCR and proposed models

DMUs	CCR	The result of possibilistic and fuzzy models (9) and (15)
	θ^*	α^* and β^*
1	0.711111	0.288889
2	1	0
3	1	0
4	1	0
5	0.988488	0.011512
6	0.893962	0.106038

and O_2 (in 10,000 tons) indicate the amount of the two products produced in the branches. A ranking of these DMUs based on their efficiency scores is necessary. In this section, the data listed in Table 1 are ranked in order so that the proposed methodology can be compared with the CCR model.

The efficiency of CCR and the proposed models are given in Table 2.

Since $\theta^* = 1 - \alpha^* = 1 - \beta^*$ the results of the new proposed methods presented in Sects. 3 and 4, and the CCR model are the same. After evaluation, management is suggested to decrease the inputs of inefficient DMUs 1, 5 and 4 as follows:

DMU_1 should decrease I_1 and I_2 from 1.50 and 1.50 to 1.07 and 1.07, respectively.
DMU_5 should decrease I_1 and I_2 from 3.50 and 1.20 to 3.46 and 1.19, respectively.
DMU_6 should decrease I_1 and I_2 from 3.20 and 0.70 to 2.86 and 0.63, respectively.

6 Inexact Discretionary Variables

The uses of fuzzy concepts in handling certain crisp mathematical modelling situations have resulted in the formulation of creative and efficient procedures. This section describes another use of the fuzzy concept and the possibility in a crisp situation. To this end, an alternative application of the new fuzzy approaches in handling non-discretionary data is presented. For the newly proposed non-discretionary models, the usage of membership function replaces the need to determine the discretionary index of a non-discretionary variable. The discretionary index concepts are used in some of the existing non-discretionary models. In real life applications, discretionary indexes are usually not known and are arbitrarily determined by decision makers.

6.1 A Non-discretionary DEA Model Using the Fuzzy Concept

In this section, an application of the possibility set concept that has been discussed in the previous section is presented. The context of the application is in the handling of non-discretionary variables.

One of the significant concepts in data envelopment analysis is the use of non-discretionary variables. An input or output is called a non-discretionary variable if it cannot be varied at the discretion of management or other users. Banker and Morey [14] were pioneers in this study by including non-discretionary variables in the input-oriented DEA model. The Banker and Morey model, considering constant return to scale, is given by the following mathematical programming model:

$$
\begin{aligned}
\min \quad & \varphi \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \varphi x_{ip} \quad i \in D \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{ip} \quad i \in ND \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_i^+ = y_{rp} \quad j = 1, 2, \ldots, s \\
& \lambda_j \geq 0 \quad j = 1, 2, \ldots, n
\end{aligned}
\tag{21}
$$

In Model (19.1), s_i^- and s_i^+ are the slack variables for the i-th input and r-th output, respectively, and the symbols D and ND refer to the discretionary and non-discretionary variables, respectively.

6.2 Inexact Discretionary Variables Using Possibility Sets

Golany and Roll [15] pointed out that in many real-life efficiency studies, a factor is neither fully controllable nor totally uncontrollable. For example, managers can make marginal alterations in personnel scheduling. However they have to comply with general guidelines of their organisation in many other aspects involving the use of their human resources. In other words, the factor is partially controllable. To incorporate this factor into a DEA model, an index taking on the values between 0 and 1 is used to represent the degree of discretion that the DMU has with respect to the factor. In this research, such a factor will be called an *inexact discretionary* variable and since a membership function (a fuzzy number concept) will be used to describe the factor instead of the discretionary index, the term *fuzzy non-discretionary* (FND) variable will also be used.

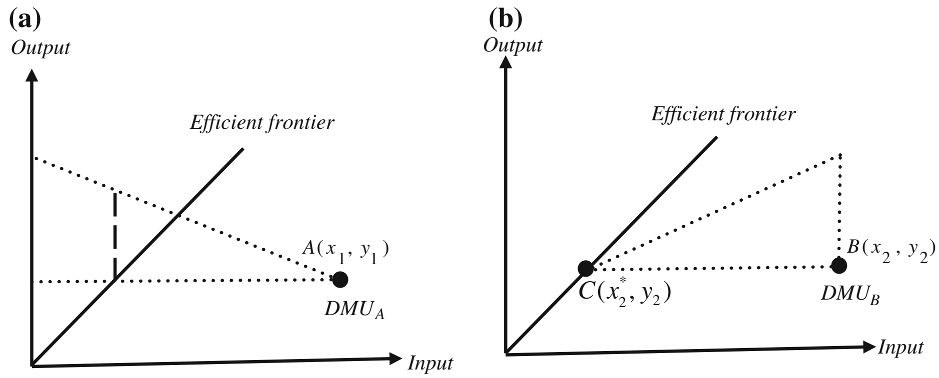

Fig. 4 **a** Discretionary input. **b** Inexact discretionary input

Figure 4a and b illustrate two separate DEA studies with a single input and a single output, in which the input variable corresponding to A is discretionary and the input variable corresponding to B is inexact discretionary. The appropriate membership functions are superimposed onto the graphs. Notice that, for the membership function of the discretionary variable, more weight is given for the smaller values, reflecting a characteristic of the input variable in a traditional DEA model, which is, that the smaller the level of input the more efficient is the DMU. However, for the membership function of the inexact discretionary variable, more weight is given for the observed/current value to reflect the reluctance on the part of the DMU to reduce the value.

The value x_2^* is the input level at point C, which is the projection of point B onto the efficient frontier. In fact, $x_2^* = \theta^* x_2$ where θ^* is the optimal solution of the CCR model related to DMU_B (i.e., Model (1) where all the inputs are treated as discretionary variables). The inclusion of the inexact discretionary variables or the *fuzzy non-discretionary* (FND) variables into Model (19.1) resulted in the following non-linear programming models:

$$1 - \max \varphi \tag{22}$$

$$s.t. (1-\varphi)\Big((x_{ip})_0^U - \sum_{j=1}^{n} \lambda_j x_{ij} \Big) + \varphi\Big((x_{ip})_1^U - \sum_{j=1}^{n} \lambda_j x_{ij} \Big) \geq 0 \qquad i \in I_D \tag{22.1}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ip} \qquad i \in I_{ND} \tag{22.2}$$

$$(1-\varphi)\Big((x_{ip})_0^U - \sum_{j=1}^{n} \lambda_j x_{ij} \Big) + \varphi\Big(\big(x_{ip}^*\big)_1^U - \sum_{j=1}^{n} \lambda_j x_{ij} \Big) \geq 0 \qquad i \in I_{FND} \tag{22.3}$$

$$\begin{aligned} \sum_{j=1}^{n} \lambda_j y_{rj} &\geq y_{rp} \\ \lambda_j \geq 0 \qquad j &= 1,2,\ldots n \\ \varphi &\geq 0 \end{aligned} \tag{22.4}$$

$$\min \quad \delta \tag{23}$$

$$s.t.C1 \qquad \delta\Big(\big(x_{ip}\big)_0^U - \sum_{j=1}^{n}\lambda_j x_{ij}\Big) + (1-\delta)\Big(\big(x_{ip}\big)_1^U - \sum_{j=1}^{n}\lambda_j x_{ij}\Big) \geq 0 \qquad i \in I_D \tag{23.1}$$

$$C2 \sum_{j=1}^{n}\lambda_j x_{ij} \leq x_{ip} \qquad i \in I_{ND} \tag{23.2}$$

$$C3\, \delta\Big(\big(x_{ip}\big)_0^U - \sum_{j=1}^{n}\lambda_j x_{ij}\Big) + (1-\delta)\bigg(\Big(x_{ip}^*\Big)_1^U - \sum_{j=1}^{n}\lambda_j x_{ij}\bigg) \geq 0 \qquad i \in I_{FND} \tag{23.3}$$

$$\begin{aligned} C4 \sum_{j=1}^{n} \lambda_j y_{rj} &\geq y_{rp} \\ \lambda_j \geq 0 \qquad j &= 1,2,\ldots,n \\ \delta &\geq 0 \end{aligned} \tag{23.4}$$

The above models are obtained from Models (8) and (9) and can be solved using a two-stage method. In the first stage, all variables are treated as discretionary variables and the traditional CCR model (5) or the newly proposed possibility programming model (10) is used to find the efficiency score of each DMU (i.e. θ^*). These efficiency scores are then used to determine variables $\left(x_{ip}^*\right)_1^U$ of Equations (22.3) and (23.3). Variables $\left(x_{ip}^*\right)_1^U$ are the projections of the inexact discretionary variables x_{ip} onto the efficient frontier. They are determined using $\left(x_{ip}^*\right)_1^U = \theta^* x_{ip} = x_{ip}^*$. An example of such a variable is the value x_2^* in Fig. 4b. In the second stage, the non-linear programming model (23) is transformed into a linear programming model and solved.

Once again, consider the case study in Sect. 5. The result of the model recommends the management to adjust the inputs to gain an efficient status. After getting this feedback from the DEA evaluation, the decision maker realized that such a decrease in the employees' salary may make trouble for organization (period). The management is aware that reducing the employee's salary (I_2) can lead to dangerous consequences. For instance, such adjustment can cause the

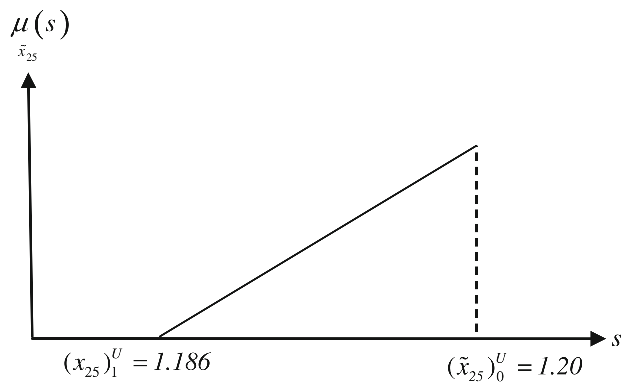

Fig. 5 Membership function of the fuzzy number $\widetilde{x}_{25}$

employee's dissatisfaction which in turn influences the efficiency of organization. So, there is an inverse relationship between salary and employee's satisfaction which can be described by a fuzzy number. Thus, they decide to gradually adjust the reduction in the employee's salary.

So we assume that the second input is an inexact discretionary variable.

In the first stage of the proposed method where all variables are treated as discretionary variables and the traditional CCR model or the newly proposed possibility set equivalent model is used to find the efficiency score of each DMU, the result has been shown in Table 2. From the result, the values for variables $\left(x_{ip}^*\right)_1^U$ of Equation (23.3) are determined. As an example, for DMU_5, $\left(x_{25}^*\right)_1^U = \theta^* x_{25} = 0.988488 * 1.2 = 1.186$. Based on the result, the management is suggested to decrease the salary budget from \$1,200,000 to \$1,186,000. But management is aware that in these circumstances, such a reduction is not possible. Thus, the following fuzzy number, based on the relationship between salary deduction and staff satisfaction, is designed (Fig. 5):

Then, the following linear programming model for finding the efficiency of DMU_5 in stage 2 is solved:

$$\begin{aligned}
&\min \varphi \\
&s.t. \\
&1.50\lambda_1 + 4.00\lambda_2 + 3.20\lambda_3 + 5.20\lambda_4 + 3.50\lambda_5 + 3.20\lambda_6 \le 3.50\varphi \\
&1.50\lambda_1 + 0.70\lambda_2 + 1.20\lambda_3 + 2.00\lambda_4 + 1.20\lambda_5 + 0.70\lambda_6 \le 1.20\varphi + 1.186185(1-\varphi) \\
&1.40\lambda_1 + 1.40\lambda_2 + 4.20\lambda_3 + 2.80\lambda_4 + 1.90\lambda_5 + 1.40\lambda_6 \ge 1.90 \\
&0.35\lambda_1 + 2.10\lambda_2 + 1.05\lambda_3 + 4.20\lambda_4 + 2.50\lambda_5 + 1.50\lambda_6 \ge 2.50
\end{aligned} \tag{24}$$

The overall result of the efficiency analysis when input 2 is inexact discretionary is shown in Table 3. Note that there are some suggestions for improvement

Table 3 Result of the CCR model in the presence of inexact discretionary variable

DMU_j			Inexact non-discretionary
	$\bar{x}^*_{1j}$	$\bar{x}^*_{2j}$	φ^*
1	1.067	1.375	0.711
2	–	–	1
3	–	–	1
4	–	–	1
5	3.397	1.199	0.971
6	2.581	0.686	0.807

of both the input variables. As an example, for DMU_5, reducing the input variable 1 from 3.50 to $\bar{x}^*_{15} = 3.397$ and the input variable 2 from 1.20 to $\bar{x}^*_{25} = 1.199$ is suggested. This means, the reduction in salary budget is adjusted to \$1,199,000 instead of \$1,186,000. As is seen in Table 3, the most adjusted salary reduction is applied to DMU_1 that is, $\bar{x}^*_{21} = 1.375$. This means that the salary budget associated with DMU_1 will be reduced from \$1,500,000 to \$1,375,000.

Efficiency scores corresponding to DMU_1 in Tables 2 and 3, have not changed, because the efficient DMU associated with it, lies on the weak efficiency frontier. We can see this by solving the following dual form:

$$\begin{aligned}
Max\,Z = 1.40u_1 + 0.35u_2 \\
\text{s.t.} \\
1.50v1 + 1.50v_2 = 1 \\
1.40u_1 + 0.35u_2 - (1.50v_1 + 1.50v_2) \le 0 \\
140u_1 + 2.10u_2 - (4.00v_1 + 0.70v_2) \le 0 \\
4.20u_1 + 1.05u_2 - (3.20v_1 + 1.20v_2) \le 0 \\
2.80u_1 + 4.20u_2 - (5.20v_1 + 2.00v_2) \le 0 \\
1.90u_1 + 2.50u_2 - (3.50v_1 + 1.20v_2) \le 0 \\
1.40u_1 + 1.50u_2 - (3.20v_1 + 0.70v_2) \le 0 \\
v_1, v_2, u_1, u_2 \ge 0
\end{aligned}$$

The optimal solution $(u_1 = 0.004, u_2 = 0.006, v_1 = 0.007, v_2 = 0.000)$ confirms such a claim.

6.3 Inexact Discretionary Variables the Using Fuzzy Concept

In this sub section another approach based on the fuzzy concept is presented to analyze an inexact discretionary variable. The Fig. 6a and b illustrate two DMUs A and B with a single input and a single output, in which the input variable

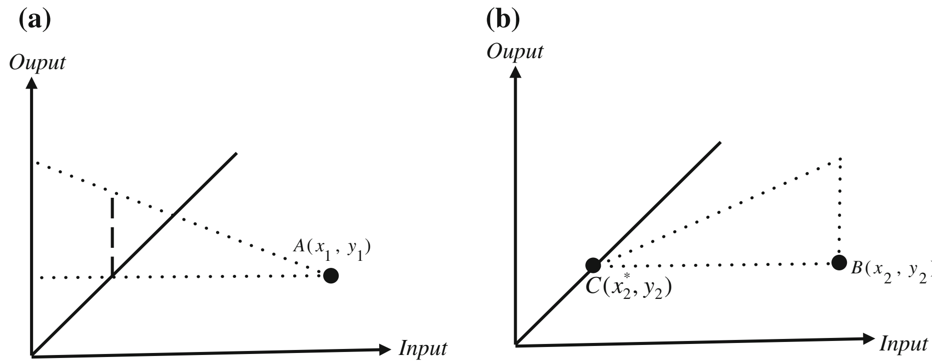

Fig. 6 **a** Efficiency in Fuzzy view of CCR. **b** Inexact discretionary input

corresponding to A is discretionary and the input variable corresponding to B is inexact discretionary.

The membership functions concerned with the input variables of the discretionary A and the inexact discretionary B are defined as follows:

$$\mu_{\tilde{x}_1}(\bar{x}_1) = \frac{x_1 - \bar{x}_1}{x_1} \quad x_1 \le \bar{x}_1 \text{ and } \mu_{\tilde{x}_2}(\bar{x}_2) = \frac{x_2 - \bar{x}_2}{x_2 - x_2'} \qquad x_2 \ge \bar{x}_2 \tag{25}$$

The value x_2^* shows the input associated with the input variable of DMU_C. DMU_C is the decision-making unit corresponding to DMU_B on the efficiency frontier. In fact, $x_2^* = \theta^* x_2$ in which θ^* is the optimal solution of the CCR model related to DMU_B. Furthermore, assume that FND is the set of inexact discretionary. By adding the constraints related to FDN in Model (23), the following multi-objective linear programming problem is proposed:

$$\begin{array}{lll}
\max & \beta_i & i \in I_{FND} \\
 & & \max \quad \alpha \\
 & & \text{s.t.} \\
C1 & \sum_{j=1}^{n} \lambda_j x_{ij} \le \bar{x}_{ip} & i \in I_D \\
C2 & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ip} & i \in I_{ND} \\
C3 & \sum_{j=1}^{n} \lambda_j x_{ij} \le \bar{x}_{ip} & i \in I_{FND} \\
C4 & \sum_{j=1}^{n} \lambda_j y_{rj_{rp}} & \\
C5 & \alpha \le \frac{x_{ip} - \bar{x}_{ip}}{x_{ip}} & i \in I_D \\
C6 & \beta_i \le \frac{x_{ip}^u - \bar{x}_{ip}}{x_{ip}^u - x_{ip}^l} & i \in I_{FND} \\
\multicolumn{3}{l}{\alpha \ge 0, \quad \beta_i \ge 0 \ i \in I_{FND}, \ \lambda_j \ge 0 \ j = 1, 2, \ldots, n}
\end{array} \tag{26}$$

The above model is solved in a two-stage algorithm. At first, to the find membership function related to C6, the point corresponding with the inexact discretionary input in the efficiency frontier should be recognized. x_{ip}^{l}, in C6, shows such a point and is obtained as $x_{ip}^{l} = \theta x_{ip} = x_{ip}^{*}$. In Fig. 6a, the input x_2^* is representative of such a point over efficiency frontier that is corresponding to the inexact discretionary input x_2. To convert the above multi-objective programming to a linear programming problem, the following mathematical programming problem is proposed:

$$
\begin{array}{lll}
 & \max \quad \rho & \\
 & \text{s.t.} & \\
C1 & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \bar{x}_{ip} & i \in I_D \\
C2 & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ip} & i \in I_{ND} \\
C3 & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \bar{x}_{ip} & i \in I_{FND} \\
C4 & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} & \\
C5 & \rho \leq \alpha \leq \frac{x_{ip} - \bar{x}_{ip}}{x_{ip}} & i \in I_D \\
C6 & \rho \leq \beta_i \leq \frac{x_{ip}^{u} - \bar{x}_{ip}}{x_{ip}^{u} - x_{ip}^{l}} & i \in I_{FND} \\
 & \lambda_j \geq 0 \quad j = 1, 2, \ldots, n & \\
 & \alpha \geq 0 & \\
 & \rho \geq 0 & \\
 & \beta_i \geq 0 & i \in I_{FND}
\end{array}
\tag{27}
$$

We refer again to the case study given in Sect. 5. Assume again that the second input in the case study is an inexact discretionary variable. The fuzzy number associated with this variable is defined according to C6. The fuzzy numbers associated with the second input is defined as $\mu_{\tilde{x}_{52}}(\bar{x}_{52}) = \frac{1.20 - \bar{x}_{52}}{1.20 - 1.186185}$ and the linear programming problem related to DMU_5 is written as follows:

Table 4 Result of the CCR model in the presence of inexact discretionary variable

	$\bar{x}_{11}^{*}$	$\bar{x}_{12}^{*}$	α^{*}	$1-\alpha^{*}$
DMU			Inexact non-discretionary	
1	1.066667	1.163793	0.2241378	0.7758622
2	–	–	0	1
3	–	–	0	1
4	–	–	0	1
5	3.460165	1.186342	0.01138147	0.9886185
6	2.839548	0.632889	0.0958722	0.9041278

$$
\begin{aligned}
&\max \rho \\
&\text{s.t.} \\
&1.50\lambda_1 + 4.00\lambda_2 + 3.20\lambda_3 + 5.20\lambda_4 + 3.50\lambda_5 + 3.20\lambda_6 \le \bar{x}_{51} \\
&1.50\lambda_1 + 0.70\lambda_2 + 1.20\lambda_3 + 2.00\lambda_4 + 1.20\lambda_5 + 0.70\lambda_6 \le \bar{x}_{52} \\
&1.40\lambda_1 + 1.40\lambda_2 + 4.20\lambda_3 + 2.80\lambda_4 + 1.90\lambda_5 + 1.40\lambda_6 \le 1.90 \\
&0.35\lambda_1 + 2.10\lambda_2 + 1.05\lambda_3 + 4.20\lambda_4 + 2.50\lambda_5 + 1.50\lambda_6 \le 2.50 \\
&\rho \le \frac{3.50 - \bar{x}_{51}}{3.50} \\
&\rho \le \frac{1.20 - \bar{x}_{52}}{1.20 - 1.186185} \\
&0 \le \bar{x}_{51} \le 3.50 \\
&0 \le \bar{x}_{52} \le 1.20
\end{aligned}
\tag{28}
$$

In the above model, the amount 1.186185 comes from $x_2^l = \theta x_2 = 0.988488 * 1.2$ that indicates the point corresponding to the number 1.2 located in the efficiency frontier. Table 4 indicates the efficiency scores, considering the input variable 2, as an inexact discretionary variable. It is noteworthy that there are suggestions for improvement for both the input variables. In this case, reducing the input variable 1 from 3.50 to 3.397 and the input variable 2 from 1.20 to 1.1996 is suggested. The efficiency score in the above mentioned problem is 0.9706094 and it is lower than the efficiency score of the CCR model which is 0.9884877.

7 Conclusion

The relationship between possibility sets and efficiency score in the traditional crisp CCR model has thus been presented. The relationship provides a new perspective of viewing efficiency. With the usage of the appropriate possibility sets to represent certain characteristics of the input data, many DEA models involving input data with various characteristics could be studied. This paper described a case involving non-discretionary input data. The usage of the possibility sets replaces the need to determine the discretionary index of a non-discretionary variable. The discretionary index concepts are used in some of the existing non-discretionary models. In real life applications, discretionary indexes are usually not known and are arbitrarily determined by decision makers.

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