Author's pre-print

Cite as: Barnes, A. (2019) 'Perseverance in mathematical reasoning: the role of children's conative focus in the productive interplay between cognition and affect', *Research in Mathematics Education*, 21

Title

Perseverance in mathematical reasoning: the role of children's conative focus in the productive interplay between cognition and affect

Abstract

Mathematical reasoning requires perseverance to overcome the cognitive and affective difficulties encountered whilst pursuing a reasoned line of enquiry. The aims of the study were: to understand how children's perseverance in mathematical reasoning (PiMR) manifests in reasoning activities, and to examine how PiMR can be facilitated through a focus on children's active goals. The article reports on children aged 10-11 from two English schools, purposively selected for their limited PiMR. Data relating to their cognitive and affective responses and the focus of their attention, a conative component, were collected by observation and interview. The study defines the construct perseverance in mathematical reasoning (PiMR). Conative characteristics of PiMR were used to analyse the cognitive-affective interplay during mathematical reasoning. It revealed the role that children's active goals play in restricting and enabling PiMR. The article offers new approaches to designing pedagogic interventions and collecting and analysing data relating to perseverance in vivo.

Keywords: perseverance in mathematical reasoning, affect, conative domain

Date submitted: 16 Apr 2018 **Date accepted:** 28 Feb 2019

Perseverance in mathematical reasoning: the role of children's conative focus in the productive interplay between cognition and affect

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Introduction

The importance of reasoning in mathematics learning has been established; it is critical in forming and justifying mathematical arguments and is a basic skill on which children's use of mathematics is founded (Ball & Bass, 2003). However, mathematical reasoning is not straightforward for children. In pursuing a line of reasoned enquiry, becoming stuck and having to change direction of thought is common (Mason, Burton, & Stacey, 2010), and can be accompanied by emotions such as frustration or bewilderment (Goldin, 2000). Perseverance, an aspect of conation, is required to overcome cognitive difficulties and associated feelings.

The idea of learning perseverance has gained popularity in education. Drawing on the idea of growth mindset (Dweck, 2000) to develop effective learning behaviours, teachers place value on children's effort and persistence. Their guidance commonly encourages children to push themselves and keep going. However, to support children to persevere in mathematical reasoning, is guidance to *keep going* sufficient? Given the importance of mathematical reasoning in children's learning and the difficulty of persevering in reasoning, this study sought to better understand how perseverance manifests during mathematical reasoning activities.

The emotions associated with reasoning are not simply a by-product of cognition that can be isolated and disregarded. There is bi-directional interplay between cognition and affect, with thoughts impacting on feelings and vice versa (Di Martino & Zan, 2013; Hannula, 2011). Whilst Hannula (2011, p. 35) argues that in mathematical thinking thoughts and emotions are "intrinsically interwoven", he laments that the processes involved in this are not well understood. To better understand this cognitive-affective interplay and its impact on primary (elementary) children's perseverance in mathematical reasoning, the following research question was explored:

> How does the cognitive-affective interplay impact on the capacity of children aged 10-11 to persevere in mathematical reasoning?

Conceptual framework

Three key conceptual areas arose from the research question: the cognitive and affective aspects of mathematical reasoning and perseverance in mathematical reasoning. Definitions of the construct perseverance in mathematical reasoning (PiMR) are not evident in the literature so were formulated for this study, drawing on the conative domain. Conation describes the motivational and volitional aspects of behaviour, of which perseverance is an aspect (Huitt & Cain, 2005). Whilst the distinctions between the cognitive, affective and conative domains are "a matter of emphasis rather than a true partition" (Snow and Jackson III, 1997, p.1), a tripartite psychological classification (Figure 1) was valuable in this study; it provided a lens through which to understand and analyse children's responses to activities involving mathematical reasoning. The conceptual framework is structured to reflect this.

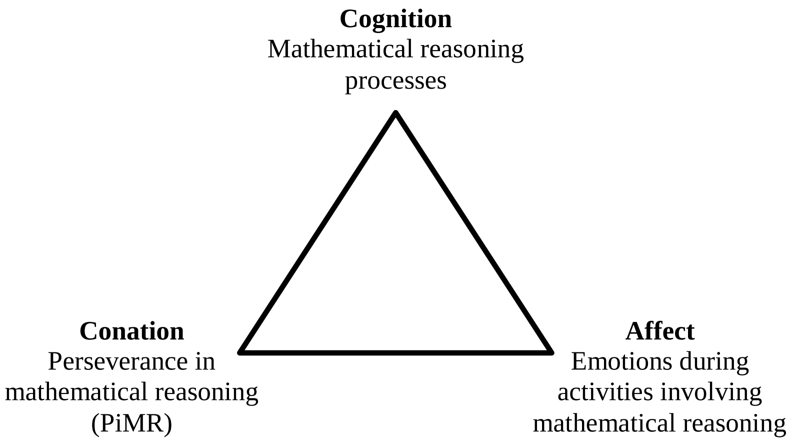


Figure 1: Tripartite psychological classification applied to mathematical reasoning

Hannula (2011) notes the need to develop a coherent, shared understanding of affect in mathematics education and to relate new affect-related studies to this. Hannula's (2012) metatheory for mathematics-related affect provides a framework (summarised in Table 1) to facilitate this shared understanding.

Table 1: Hannula's (2012) three-dimensional metatheory for mathematics-related affect

Dimension	**Illustration**
Physiological, psychological and social nature of affect	This dimension reflects what Hannula describes as a system-theoretical perspective to enable connections to be sought between neuroscientific, traditional psychological, and social frameworks. Hannula (2012) exemplifies these connections as follows: to establish a new social norm in a class (social framework), individual students change their behaviour (psychological framework) and this is echoed in neural connections (physiological framework).
State and trait aspects of affect	Hannula (2012) argues that emotions have both a rapidly fluctuating emotional state and a more stable emotional trait. For example, a student can experience many, rapidly changing emotions during mathematical tasks, e.g. pleasure, bewilderment (state emotion), but can tend to approach mathematical tasks with a more stable emotion, e.g. apprehension (trait emotion).
Cognitive, motivational and emotional aspects of affect	This dimension concerns psychological processes. Cognition concerns understanding, thinking and dealing with information. Motivation directs behaviour including goals and choices. Hannula (2012) argues that emotions (e.g. pleasure, shame) reflect success or failure in goal-directed behaviour and feedback to cognitive and motivational processes.

In relation to these dimensions, my study focused on children’s psychological responses in individual mathematics lessons and hence concerned state rather than trait aspects of cognition and affect. My study also drew on the motivation aspect of Hannula’s third dimension. State-motivation concerns “active goals” (Hannula, 2011, p.45); these are “in-the-moment” goals that engage a child for short periods during lessons (Goldin, Epstein, Schorr, & Warner, 2011, p.550). Whilst motivation is an important aspect of conation (Huitt & Cain, 2005), perseverance is also characterised by two further conative components, striving and self-regulation (discussed later). Hence, to define PiMR, the motivation aspect of Hannula’s (2012) framework was extended to include characteristics from the broader conative domain.

Mathematical reasoning: the cognitive domain

I interpreted mathematical reasoning as:

> The pursuit of a line of enquiry to produce assertions and develop an argument to reach and justify conclusions.

This draws on Pólya’s plausible reasoning and the inductive approaches involved in “mathematics in the making” (1959, p. 37).

There is consensus in research literature regarding the mathematical reasoning processes involved in pursuing a line of enquiry (e.g., Ball & Bass, 2003; Mason, et al., 2010). From this corpus, I identified five key cognitive processes: specialising (making trials), spotting patterns/relationships, conjecturing, generalising and convincing. They were significant in this study because they are indicators of children’s mathematical reasoning.

Conjecturing is the formation of an idea that appears reasonable but whose validity is not yet established (Mason, et al., 2010). Forming a conjecture requires a general rule to be inferred from specific examples, and spotting patterns is necessary in this. To create a situation in which patterns can emerge, examples need to be created. Initially, this is characterised by trying arbitrary examples using random specialisation (Mason, et al., 2010). This facilitates getting a feel for the problem when little is known. However, for patterns to emerge from which conjectures can be formed the data must be ordered and a systematic approach to specialisation (Mason, et al., 2010) supports this. This facilitates generalisation; the formation of statements about what is happening and the conditions for this.

Mulligan and Mitchelmore (2012) highlight the importance of understanding the relationship between patterns and the underlying mathematical structures. Understanding mathematical structure is significant in constructing arguments about why patterns occur and why a generalisation might be true. In a primary (elementary) school context, mathematical arguments do not necessarily have to take the form of formal logic or proof (Stylianides & Stylianides, 2006), rather “sensible” (Lithner, 2008, p. 257) reasons can support mathematical assertions. These include anchoring arguments in the mathematical properties being reasoned about (Lithner, 2008) and drawing on the mathematical data to validate the conclusion (Bergqvist & Lithner, 2012).

Mathematical reasoning: the affective domain

Goldin (2000) describes two commonly experienced, idealised pathways of state emotion that could be experienced during mathematical reasoning in a problem-solving context. Both pathways share a starting sequence in which students experience curiosity and puzzlement as they engage with a problem. This is followed by bewilderment as they seek effective problem-solving strategies. At this point, the pathways split. In one, students choose an appropriate strategy, which leads to feelings of encouragement. Further success results in pleasure and moments of elation as new insights emerge. Finally, students experience satisfaction in both the successful outcome and the approach taken.

In Goldin's alternative pathway, students' bewilderment does not lead to choosing an effective strategy and frustration sets in. If a way forward is not found, emotions become increasingly negative, and anxiety, fear and even despair are experienced. At this point, students may endeavour to comply in order to alleviate uncomfortable emotions; this may lead to the use of rote procedures or avoidance strategies. In either case, the cognitive outcome is not mathematical reasoning. In Goldin's (2000) idealised pathways, the cognitive-affective interplay during activities involving mathematical reasoning is evident.

Perseverance in mathematical reasoning: a conative construct

The conative psychological domain, discussed more commonly outside mathematics education literature, concerns volitional aspects of human behaviour (Hilgard, 1980; Snow & Jackson III, 1997). Perseverance is an aspect of the conative domain (Huitt & Cain, 2005) that involves staying power and striving to overcome difficulty or delayed success in achieving goals (Tait-McCutcheon, 2008). In this study, I interpreted perseverance in the contexts of learning and mathematical reasoning to articulate the components of PiMR. I used the notions of striving and staying power to overcome difficulty, in conjunction with my interpretation of mathematical reasoning, to define PiMR as:

striving to pursue a line of mathematical reasoning, despite difficulty or delay in achieving success.

The trait aspects of conation include internal motivation and volition (Huitt & Cain, 2005; Snow & Jackson III, 1997), and dispositions to strive and self-regulate (Tanner & Jones, 2003). The related state aspects include having active goals (Hannula, 2011), engagement and striving, and self-regulating (Huitt & Cain, 2005; Tanner & Jones, 2003). Each of these is an important aspect of PiMR; engagement and focusing attention on active goals give intent and purpose to striving, and self-regulation facilitates effective monitoring of actions, to overcome difficulties and move towards active goals. This section discusses how these state aspects were interpreted in the context of mathematical reasoning.

Children's active goals during mathematical reasoning may be evident in the focus of engagement, for example, an active goal of creating as many solutions as possible might be inferred from observable engagement in repeatedly making solutions. Fredricks, Blumenfeld and Paris (2004) argue that engagement includes concentration, attention and contributing to class discussion. With a focus on mathematical reasoning, these can be interpreted as:

- focusing attention on the mathematical:
    - concepts in which the reasoning is anchored (Lithner, 2008)
    - processes required to form a reasoned line of enquiry (e.g., Bergqvist & Lithner, 2012)
- contributing to class/group discussions stimulated by the reasoning activity and the related concepts and processes.

Striving requires effort and staying power; it is pro-active, goal-oriented (Huitt & Cain, 2005) and allied to the active goals that characterise the state aspect of motivation (Hannula, 2011). Striving towards the goal of reaching and justifying conclusions requires the formation of assertions and arguments. This results in successful PiMR in which there is observable movement between reasoning processes, from specialising and spotting patterns towards conjecturing, generalising and forming convincing arguments. Progressing from one reasoning process to another requires the learning from one process to be applied in the next; this necessitates "pro-active (not reactive or habitual) behaviour" (Tanner & Jones, 2003, p. 277). However, striving might also be interpreted as keeping going. Williams (2014) argues that keeping going, irrespective of the quality of each try, is a demonstration of persistence rather than perseverance. In mathematical reasoning, persistent behaviour may not lead to the productive use of the outcomes of trials; this could inhibit pattern spotting, conjecturing and generalising, resulting in limited movement between reasoning processes. Hence, repeated application of one or two reasoning processes is an indicator of persistence, whilst movement between reasoning processes is an indicator of PiMR.

Pro-active behaviour requires effective self-regulation. When applied to mathematical reasoning, self-regulation of cognition includes reflection on both the information generated and the value of the processes and strategies employed to inform action (Özcan, 2016). This pro-active, focused reflection facilitates progression, from making trials, towards conjecturing, generalising and forming convincing arguments.

Affect also has a self-regulatory, meta-affective component that concerns emotions about emotions and the cognitive monitoring of emotions (DeBellis &

Goldin, 2006). Engagement with mathematical reasoning, including navigating being stuck, can be accompanied by emotions such as puzzlement or fear (Goldin, 2000). Emotional awareness can facilitate meta-affective responses that enable difficult emotions to be experienced differently for cognitive gain. For example, Debellis and Goldin (2006) reason that frustration during mathematical activity can be experienced as pleasure because it is indicative of enhanced interest and challenge; this enables alternative approaches to be sought. Malmivuori (2006) describes conscious monitoring of emotions and the subsequent cognitive actions taken as active regulation of affect.

However, being aware of emotions is not an automatic catalyst for action, hence the expression of feelings relating to mathematical learning does not guarantee liberty from debilitating emotions nor progress in reasoning. Malmivuori (2006) describes an alternative meta-affective response, automatic affective regulation. This operates within a limited self-regulatory system in which habitual affective responses override self-regulation. For example, when becoming stuck, frustration may automatically be accompanied by fear, which impedes higher order mental processes. Malmivuori argues that automatic affective regulation can manifest in habitual behaviours such as defensive actions.

Goswami (2015) argues that self-regulatory processes are important in facilitating strategic control over mental processes to consciously inhibit or develop thoughts, feelings and behaviours; this is significant in PiMR because it facilitates the adjustments needed to overcome difficulties or delays encountered in constructing a reasoned line of enquiry. Whilst self-regulation is important in the development of children’s PiMR, Goswami (2015) argues it is not easy for children in the primary (elementary) phase to develop and apply. Hence there is value in developing teacher interventions to support children's self-regulation during mathematical reasoning.

Pedagogic interventions

The data reported in this article were part of a larger research project that sought to improve children's PiMR through teacher interventions. The initial pedagogic intervention adopted a provisional approach. In computing, the provisional capability of programming facilitates provisional thinking by enabling users to make swift changes to code and test alternatives. Papert's (1980) LOGO is illustrative of an environment in which provisional thinking is applied. Children create code to move a screen turtle, e.g. to form a triangle. The instructions are enacted dynamically on the screen, providing immediate, accurate feedback on the code. This can facilitate provisionality of thought through conjecturing, making trials and using the resulting data to improve the code.

Provisional thinking also has affective impact; it fosters an attitude that mathematical thinking is fallible, that it concerns trial, improvement and conjecturing rather than the pursuit of right or wrong answers. Papert (1980) argues that this approach makes children less fearful of being wrong.

In the intervention, I sought to facilitate a provisional, conjectural approach and create an enabling affective environment through giving children access to materials that could be used provisionally to:

- construct and adapt physical and written representations
- re-position representations in relation to each other.

The intervention was augmented during the study to include two further aspects. The first sought to create an explicit focus on forming generalisations and convincing arguments about why these were true. For example, by incorporating a writing task to articulate generalisations. The second afforded more time, two one-hour lessons, rather than one, on consecutive days to act on and interpret the data arising from specialising and spotting patterns.

The Study

Study design

The study took place in two year 6 classes (ages 10-11) in different schools in England and focused on eight children. This article draws on data from three children, who represent the range of cognitive, affective and conative responses of the study group at the end of the research. I worked alongside two teachers who had mathematical subject and pedagogic expertise; this provided a secure foundation for applying and developing the interventions. The study comprised a baseline lesson and two intervention cycles. The purposes of the baseline lesson were to:

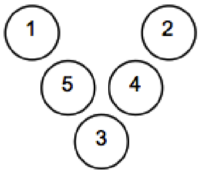
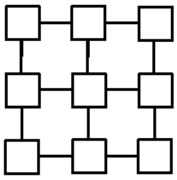
- validate the purposive selection of children by confirming that they demonstrated limited PiMR
- evaluate children’s baseline PiMR by gathering data before the intervention outlined above began
- familiarise the children with mechanisms for data collection.

Table 2 details the mathematical activities and the interventions applied. The teachers and I chose activities that:

- were appropriately pitched for the children in each class
- afforded opportunities to pursue a reasoned line of enquiry

- afforded opportunities for children to experience and respond to affect whilst engaging with activities involving mathematical reasoning.

Table 2: Mathematical activities and pedagogic intervention

Cycle	Activity	Pedagogic intervention
Baseline lesson	**Magic Vs**[1] Arrange the numbers 1–5 in a V so that each arm of the V sums to the same total. E.g.: 	Before intervention began
Cycle 1	**Paths around a pond** A square pond is surrounded by a path that is 1 unit wide. Explore what happens as the pond changes size.	Opportunities for children to use Cuisenaire rods in a provisional way
Cycle 2	**Number differences**[2] Arrange the numbers 1-9 on the grid so that the difference between joined squares is odd. 	Opportunities for children to use number cards in a provisional way Explicit focus on forming generalisations and convincing arguments. Provision of time to develop reasoning relating to one activity by affording two one-hour lessons on consecutive days.

[1] NRICH (2015a), [2] NRICH (2015b)

The teachers selected four children from his/her class to form the study group of eight. The teachers based their selection on their assessments of children who seemed to have limited PiMR, and then wrote baseline pen-portraits detailing the selected children's PiMR. Table 3 shows the pen-portraits of the three children discussed in this article.

Table 3: Baseline pen-portraits

Child*	Teacher* School*	Pen-portraits
Alice		Able but reluctant, often disinterested in maths. Always looks for quick fix.
Ruby	Mr Hall Hilltop Primary	Struggles to verbalise reasoning. Will sit and wait rather than actively attack problem. Often seems to give up.

Michelle	Ms Parry Parkside Primary	Quite nervy over maths. More abstract thinking worries her.

*All pseudonyms

Methods

Collecting and analysing data relating to the state aspect of children's cognition, affect and conation presented a challenge because these are internal mental responses that might be inferred through external behaviours. Whilst I sought observable indicators of each, exemplified in Table 4, there was no guaranteed correlation between internal process and external indicator. I sought to diminish the impact of this limitation by using multiple data collection approaches to enable triangulation. Data were generated using direct observation, audio records and photographs in mathematics lessons, and in post-lesson interviews. The resulting data were collated into synthesised transcripts. Table 4 summarises the data collection methods with examples of data generated using each method.

Table 4: Summary of data collection methods and exemplification of data generated

Data collection point	**Data collection method**	**Examples of data Cognitive**	**Affective**	**Conative**
Lessons	Observations	Use of mathematical reasoning processes, e.g., specialising	Facial expressions, e.g., raising eyebrows Body language/position, e.g., head close to work Pace of construction of representations	Focus of engagement Actions relating to focus/change of focus Movement between/stasis within reasoning processes

	Audio records	Dialogue relating to reasoning processes, e.g. generalising: “It's always…	Oral expressions and utterances, e.g. groaning Tone/pace of oral expression	Dialogue relating to: focus/change of focus, e.g. “let’s try and make….” self-regulation, e.g.: “That didn’t work, let’s try…”
	Photographs	Mathematical representations		
Interviews	Audio records	Explanations of mathematical reasoning	Discussions of emotions experienced during lessons	Explanations of: focus, rationale for focus reasons for changing focus

I applied the findings from literature on the tripartite psychological classification (Figure 1) to create three coding categories and related codes:

1. Cognitive events - mathematical reasoning processes
   Codes: specialising, spotting patterns/relationships, conjecturing, generalising, convincing (Bergqvist & Lithner, 2012; Lithner, 2008; Mason et al, 2010; Mulligan & Mitchelmore, 2012)
2. Affective events - emotions during mathematical reasoning
   Code: Demonstration of affect (Goldin, 2000)
3. Conative events: PiMR
   Codes: striving, active goals, self-regulatory processes (Debellis & Goldin, 2006; Hannula, 2011, 2012: Huitt & Cain, 2005; Malmivouri, 2006; Özcan, 2016).

The inferences made in encoding the data were theory and researcher dependent and hence open to interpretation. I sought to minimise the impact of sole-researcher interpretation by presenting and triangulating data from all sources.

PiMR results in movement between reasoning processes in response to a mathematical challenge; to support the presentation and theorisation of findings, I created diagrams in the style of Figure 2 to illustrate the children's movement between reasoning processes and their cognitive-affective-conative interplay.

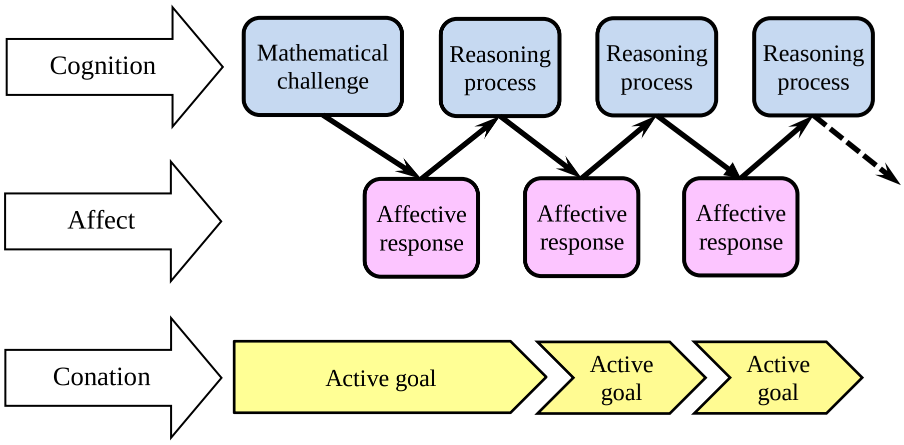


Figure 2: Representation of cognitive-affective-conative interplay during mathematical reasoning

This study offers an interpretation of events that draws on the conceptual framework to illuminate children's cognitive-affective-conative interplay during activities involving mathematical reasoning. Hence, whilst other interpretations of the data are possible, the analysis draws on particular examples to demonstrate the cognitive-affective-conative interplay in order to be able to offer a framework for future empirical research.

Findings

Baseline lesson

In this lesson, no intervention was used; the teachers applied their typical pedagogic practice. They introduced Magic Vs (Table 2) by displaying two sets of the numbers 1–5 in V-formations (Figure 3), stating that one of the formations was magic. They asked their classes to:

- identify which v-formation was magic with a rationale

- explore how to create additional magic-Vs
- form generalised statements with explanations as to why these were true.

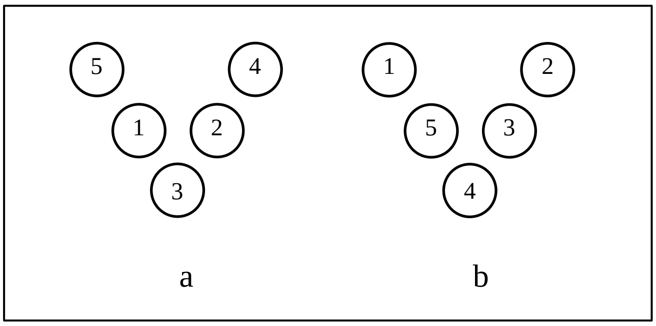


Figure 3: V-formations displayed on board

In Parkside Primary, the reason for the V in Figure 3a being magic emerged during the initial whole class discussion. Following this, Michelle began by exploring how to create additional magic-Vs. She appeared to understand that one criterion was to use only the numbers 1–5 as she said to a peer:

> We have to do 1 to 5

She then generated trials using random specialisation, which she believed to be successful as each arm of the V totalled the same value, but she used the numbers 1–6, first omitting 4, then 2 (Figure 4). Michelle's misapplication of the criterion to use the numbers 1–5 restricted her pursuit of a reasoned line of enquiry; her trials did not result in the emergence of patterns. Without the opportunity to notice patterns, she was not able to form conjectures or generalisations.

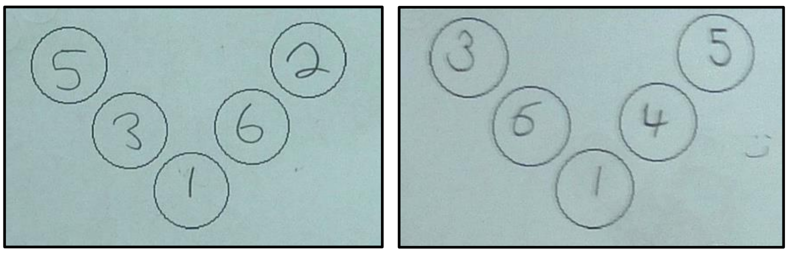


Figure 4: Michelle's trials

In Hilltop Primary, the reason why the V in Figure 3a was magic did not emerge during the initial class discussion so this formed the focus for Alice and Ruby's exploration. The girls explored which V might be magic and why. Their trials involved two ideas, summing the numbers within each V and considering the odd/even properties of this total and the numbers 1-5:

Alice	I think you need to work out the magic number - so that's 15
Ruby	Why 15?
Alice	15 is what it adds up to. But it can't work because they're both exactly the same
Alice	[Excited tones] Ah, I think I've got it [re-sums the numbers within one V]
Ruby	[Excited tones] Oh, no, no, no, wait, you can add them
Alice	[Sharp intake of breath, excited gasp] I think I know what you mean by magic—we need to try to figure out a number which is both odd and even
Alice	There's more odds
Both	[In unison] than even
Ruby	[Sharp gasp] Oh we add them
Ruby	Then see if 15 is an odd or an even
Alice	15 is odd

Following this discussion, Mr Hall demonstrated to the class that the two arms of the V in Figure 3a sum to the same total, which makes it magic. However, this did not support Ruby and Alice to construct their own magic-Vs and they revisited their earlier ideas. When Mr Hall tried to focus their attention on the odd/even properties of the numbers in the V, they remained focused on summing the numbers and exploring the odd/even properties relating to the total:

Ruby	We added them, we worked out if they were odd or even, and there are more odd than even, so then we added the numbers up
Alice	We done, 5's odd, 3's odd, 1's odd and 4 and 2 are even so only 2 even and 3 odd
Mr Hall	I like that, so we've got 3 out of 5…
Alice	[Interrupting] And we're trying to find, we thought the magic number might be something that is both odd and both even. 10 goes into it and so does 5, and 10 is even and 5 is odd

Alice and Ruby formed conjectures about the magic total being 15, and 15 having both odd and even properties, and Alice realised the limitations of these: "it can't work because they're both exactly the same"; "15 is odd". However, whilst Alice and Ruby formed conjectures and Michelle randomly specialised, none created examples that revealed patterns, generalise or form convincing arguments about a generalisation.

All three girls strived throughout the activity and appeared to have active goals: Alice and Ruby focused on understanding the properties of a magic-V and Michelle focused on making further successful trials. However, all three demonstrated limited cognitive self-regulation. Michelle did not realise that, despite her apparent creation of magic-Vs, there were no emerging patterns, hence continued to misapply the criterion to use the numbers 1-5. Alice and Ruby repeatedly revisited their two conjectures despite realising their limitations. This apparent lack of cognitive self-regulation may have inhibited their capacity to act on Mr Hall’s demonstration of the magic-V properties.

The girls’ affective responses seemed to be predominantly characterised by pleasure. For Alice and Ruby, this was indicated in excited tones in their speech, sharp intakes of breath and speaking in unison. Michelle seemed to express pleasure through giggling after completing each V, which might have been founded on her belief that she had constructed successful trials. At the end of the lesson, Alice created the drawing in Figure 5:

Figure 5: Alice's drawing

In interview, she explained the meaning of the drawing:

Alice	[Laughs] well it stands for, at the beginning I was like what is going on? And at the end, I love it
Researcher	Why did you enjoy it?
Alice	Because it was difficult, it wasn't easy

This expression of enjoyment of puzzlement arising from a difficult challenge seemed to be a meta-affect response. This may have enabled her to strive throughout the lesson despite making limited progress in mathematical reasoning.

Figure 6 summarises the pathway of reasoning responses predominantly used by the girls, alongside their affective responses and active goals. It illustrates limited PiMR, evident in the lack of movement between reasoning processes, and their apparent pleasure despite this. In his idealised affective pathway, Goldin (2000) relates pleasure to the experience of success in mathematical problem-solving; consequently, these are surprising affective outcomes as the girls expressed pleasure despite being unsuccessful. One explanation for this is that the girls' limited display of cognitive self-regulation and resulting lack of awareness about the limited extent of their reasoning may have impacted on their affective response and enabled them to experience pleasure, regardless of their limited PiMR.

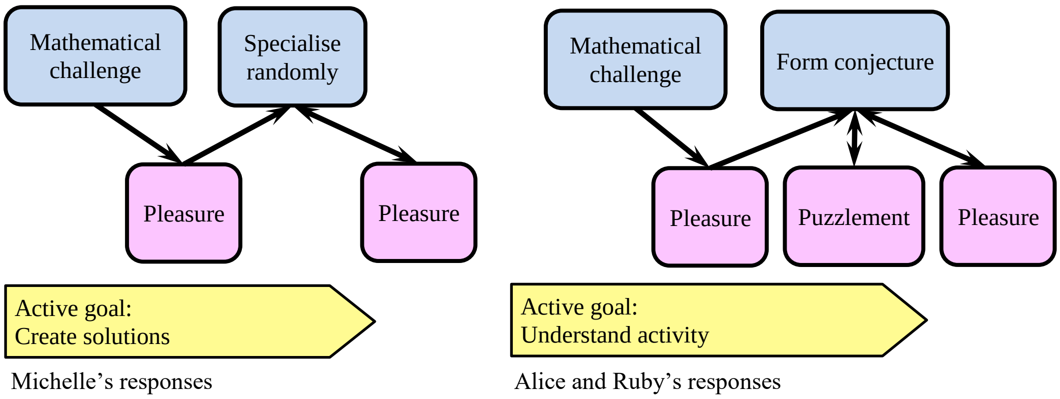


Figure 6: Cognitive-affective-conative interplay in baseline lesson

First intervention cycle

The teachers introduced Paths around the Ponds (Table 2) by modelling the 1[2] pond/path using images of Cuisenaire rods on the board (Figure 7).

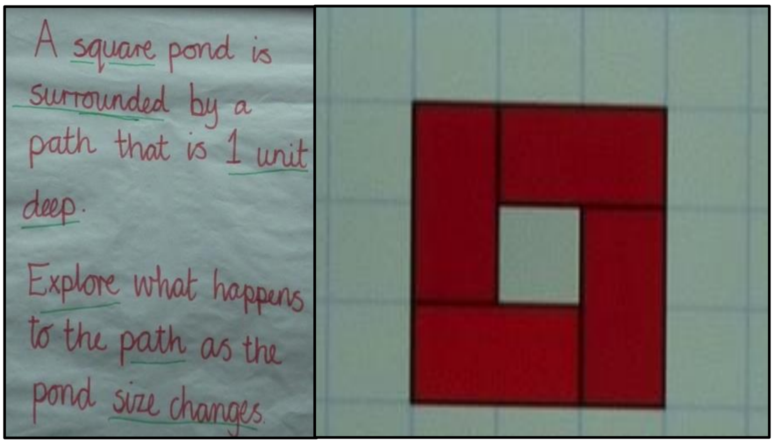


Figure 7: Activity introduction

The intervention (Table 2) provided the children with opportunities to use rods provisionally to construct representations of ponds/paths. Following construction, they were given pencils and paper to record the corresponding numeric sequence.

Initially Ruby and Alice specialised randomly by selecting four 10cm rods to represent the path and arranged these into an oblong (Figure 8). They continued this construction, creating successively smaller concentric oblongs inside the original:

Figure 8: Ruby and Alice's first trials

At this point, each girl seemed to self-regulate. Alice realised that these trials did not fit the activity criteria:

> That's really weird, it doesn’t work

Ruby, in the post-lesson interview, reflected on her approach following these unsuccessful trials:

> I put the centre first and then the outside - I think I found it easier doing it that way

This may have helped her to ensure the squareness of both pond and path. Following their initial exploration, they adopted Ruby's idea and constructed the 9^2 pond using nine 9cm rods, surrounded by a square path constructed from four 10cm rods. They then proceeded to create nine representations of ponds/paths that were systematically constructed and arranged (Figure 9).

Figure 9: Ruby and Alice's systematic construction and ordering of trials

Michelle's initial approach was to construct the 1^2 example (Figure 7), then to specialise systematically to construct and order the 2^2-4^2 examples (Figure 10):

Figure 10: Michelle's first trials

Michelle, Ruby and Alice seemed to notice and apply structural patterns; this resulted in a systematic approach to the construction of trials whereby each pond was represented by n number of rods of length n, and each path by 4 rods of length $n+1$. The resulting trials were then systematically ordered.

The children expressed pleasure and excitement when they spotted patterns:

Alice	They go up in steps [excited tones]
Alice	I've got a pattern [cheers, claps]

In the post-lesson interview, Michelle expressed enjoyment in the use of physical representations:

> [It was] really fun, because you got to like do it with props instead of just writing stuff on paper

There was notable similarity in the children's apparent affective responses in the baseline lesson and in cycle 1, with pleasure seemingly the predominant emotion.

There was scant evidence of the children verbalising conjectures. However, in the post-lesson interview, Michelle and Ruby's reflections indicate that they had formed a conjecture about the emerging colour pattern:

Michelle	On the pond before, the purple was the path, on the one before that green was the path that is now the pond.
Ruby	So the red's on the outside there [1^2 pond] so it's on the inside there [2^2 pond]. Then the green's on the outside so then it's on the inside.

During the activity, the active goal that all three girls invested their effort into and strived towards seemed to be the systematic construction of all possible examples from the Cuisenaire set. Alice again expressed pleasure at the experience of challenge:

> It was really fun because it was really challenging

This meta-affective response may have supported her to remain focused on her active goals.

Once the children's constructions were completed, the teachers explained that they should now look for numerical patterns by representing the data in a table. None of the girls explored numerical patterns or attempted to construct a table of numeric data. Michelle sat passively for the remainder of the lesson and there was no further evidence of her striving towards any goal relating to mathematical reasoning. Ruby and Alice appeared to adopt a new active goal, unrelated to the activity, and constructed towers from the rods; this did not result in any further mathematical reasoning. When asked, in the post-lesson interview, why they built towers rather than exploring numerical patterns and recording these in a table, Ruby responded:

> I thought we didn't need to do it on the paper because we'd already done it

Figure 11 summarises the pathway of reasoning responses predominantly used by the girls, alongside their affective responses and active goals.

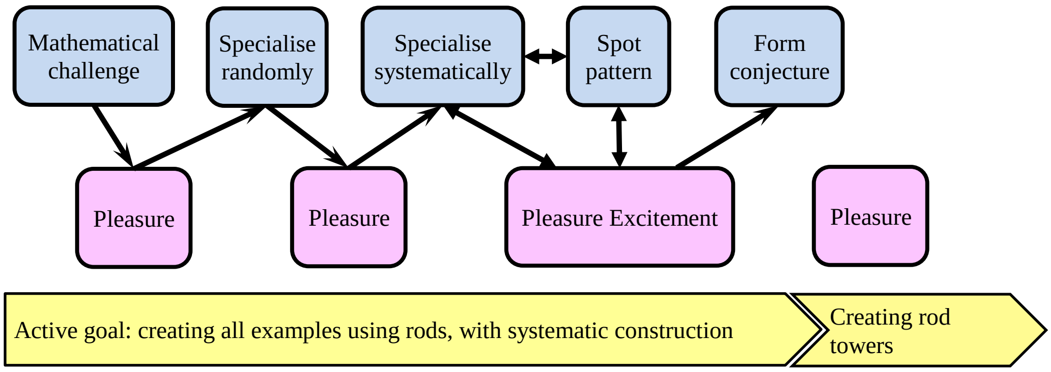


Figure 11: Cognitive-affective-conative interplay in cycle 1

The provisional use of representations facilitated the creation of systematically constructed and ordered trials and supported the children to spot patterns; this seemed to be a source of pleasure and excitement. However, there were no observed instances of forming generalisations and convincing arguments about the sequence. Whilst there were instances of conjecturing, the pleasure and excitement gained from spotting patterns led to more specialising. When the girls had constructed the 1^2–9^2 ponds/paths and were not able to make the 10^2 pond/path because of the absence of an 11cm rod, they had achieved their apparent active goal to construct all examples. Their focus on the activity then ceased. There seems to be bi-directional interplay between creating trials and expressing pleasure; Ruby and Alice gained pleasure from constructing successful trials so wanted to continue construction with the rods. This may have created the conditions for them to persist in creating further trials and, when avenues for this were exhausted, to continue to use the rods for construction, despite this being unrelated to the growing sequence.

Comparison between Figures 6 and 11 shows the development in the children's PiMR following the intervention and reveals a consistency in children's affect across these lessons.

Second intervention cycle

The teachers applied the interventions detailed in Table 2 by providing:

- opportunities to work provisionally using number cards
- explicit focus on the formation of generalisations and convincing arguments
- additional time by allocating two consecutive lessons to the activity

The teachers introduced Number Differences (Table 2) with the goals of forming conjectures with convincing arguments:

Mr Hall	[You need to] identify and explain a successful pattern, so it's not just about saying those are my numbers, I'm done
Ms Parry	Figuring out why is the big focus of the puzzle we will be doing over the next two [lessons]

In the first four minutes, Ruby and Alice created successful solutions and Ruby formed her first conjecture, expressed as an idea for specialising. When challenged by Alice, Ruby articulated a convincing argument as to why this would work that was anchored (Lithner, 2008) in the odd differences between adjacent numbers:

Ruby	We could just put them in order, 1, 2, 3, 4, 5…
Alice	That's not going to work
Ruby	Yes it is because all of them [the differences] are 1

The pair then appeared to form a conjecture that there needed to be an odd number in the middle, and their subsequent trials became increasingly systematic as they tested this:

Alice	Shall we try 9 in the middle? What number shall we put in the middle? What's odd?
Ruby	Put all the odd numbers in the middle

Following initial explorations in manipulating the number cards, Michelle appeared to be able to use the odd/even properties of numbers to support placing the number cards. Having created one successful trial, Michelle appeared to form and test a conjecture about the odd/even property of the corner numbers by beginning with an even number in the top left corner (Figure 12). She realised that she was not able to use the remaining number 7 but needed an even number in the bottom right corner to maintain an odd difference between adjacent numbers, so she used a Numicon 2[1]. However, she self-regulated by comparing this trial against the activity criteria and rejected this solution as it did not use the numbers 1–9. She then reverted to beginning the grid with an odd number in the top left corner.

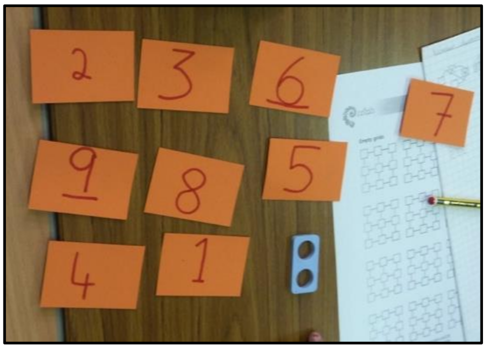


Figure 12: Michelle's trial positioning even numbers in the corners

Following this exploration time to make trials, spot patterns and form and test conjectures, both teachers asked their classes to write explanations of what they had found that also explained why it worked. Mr Hall refocused the class to support their

[1] Numicon are physical number shapes utilising a tens frame image (Griffiths, Back, & Gifford, 2017).

movement towards generalising and convincing by using the number of successful trials as a signal to change focus:

> If you have 10 solutions and a pattern that works then your job is to explain that pattern and why it works.

Ruby drafted a response to the first part, generalising how to create successful solutions:

> First we found out that the <u>odd</u> numbers go in the middle one by one. Then all the other <u>odd</u> numbers go in the corners, and the <u>even</u> numbers go in the spaces left [Ruby's emphasis].

Alice's draft written response (Figure 13) generalised the pattern and began to explain why this worked by anchoring her argument (Lithner, 2008) in the *odd difference* between odd and even numbers. Initially she drew on the odd/even property of the sum rather than the difference between adjacent numbers, but was able to notice and correct this.

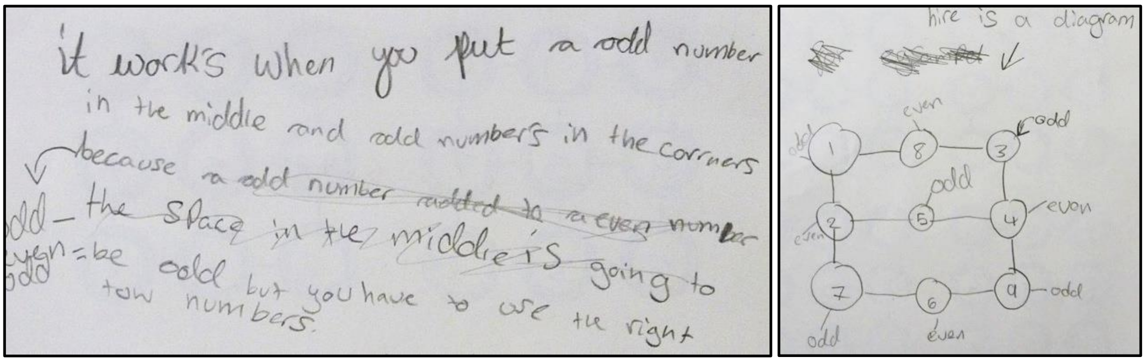


Figure 13: Alice's first draft

Throughout this first part of cycle 2, the girls' affective responses were not dissimilar to those in cycle 1. There was pleasure, perhaps in anticipation of the challenge to come:

Ms Parry	I might tease you with the main event [reveals Number Difference grid] so you know what you are working towards
Michelle	[Smiles]

There were many expressions of pleasure in creating numerous successful solutions, e.g., this exchange:

Alice	We've done 12
Ruby	It's actually been quite fun
Alice	[Laughs]

However, the children's pleasure in creating solutions shifted to disappointment when guided to move onto other reasoning processes:

Mr Hall	If you have 10 solutions and a pattern that works, then your job is to explain that pattern and why it works.
Alice	[Groans]

Having begun work on written explanations of the pattern and why it worked and despite guidance to create a maximum of 10 successful solutions, Ruby and Alice returned to creating solutions:

Alice	One more to go and then we've got 23

In the second lesson of cycle 2, Alice and Ruby stopped making further solutions and refocused on explaining why the generalised pattern worked. In their final written responses all three girls explained that an odd number must be positioned adjacent to an even number to create an odd difference:

> the odds have to be in the corners and the middle because there is more odd numbers than even numbers. If 2 odds are next to each other the diference will be even and if 2 even numbers are next to ech other the difference will be even. So there needs to be an odd and an even next to each other *[sic]*.
>
> Michelle's final written response (Alice's response was similar)

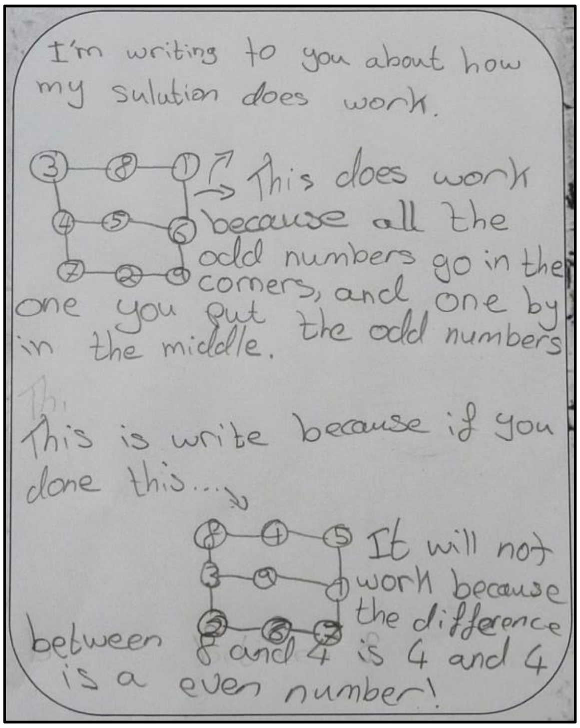


Figure 14: Ruby's final written response

In addition, Alice and Michelle applied the generalisation that the difference between an odd and even number will always be odd, constructing a convincing argument. Ruby did not anchor her argument in the generalised differences between odd and even numbers; instead she used a counter-example to illustrate that if two even numbers were in adjacent positions, their difference would be even. Michelle was the only child to construct an argument about why the odd numbers also needed to be positioned in the corners and the middle of the grid.

In the post-lesson interviews, the girls discussed their feelings about the lesson:

Alice	Well we found out how we actually understood it. [The difficulties were] trying to start it off, trying to get all those little bits of information and putting them into something bigger that explains more. I'm proud, I'm over the moon with joy.
Michelle	I feel really good. I understand it. [I'm] happy and proud that I know how to do it.
Ruby	I'm happy actually

Alice and Michelle expressed pride; there had been no expressions or indicators of pride in any lesson in the study preceding this. Their pride seemed to arise from their understanding of how to position the numbers so that the differences between adjacent numbers were odd and why this positioning worked. Ruby did not express feelings of pride at the end of cycle 2, rather, she expressed happiness with her work; this is consistent with the girls' responses in the preceding lessons. In her final written explanation (Figure 14), Ruby fully articulated the pattern of the numbers but did not utilise the generalisations about differences between odd and even numbers in her explanation. Her doubts in the merit of her writing may have reflected her difficulty in this:

> I think mine's all wrong [reviewing her writing]

Ruby's partial explanation of why the generalised pattern worked, in conjunction with her difficulty in utilising generalised differences between odd and even numbers in her reasoning, may have impacted on her affective response; it may have contributed to her expressing happiness with her work but, unlike Alice and Michelle, not describing feelings of pride.

Michelle's focus appeared to develop during the course of the activity; initially she seemed to strive towards specialising, then to establishing, generalising and applying a pattern and finally to explaining why it worked. This suggests that her overarching active goals across these foci were to establish what was happening and why. There is close alignment between these active goals and that presented by Ms Parry at the start of cycle 2: to "figure out why". Alice and Ruby seemed to begin with similar foci to Michelle, specialising then establishing, generalising and applying a pattern. They may have adopted part of Mr Hall's intended active goal, to identify the pattern, but once they had described the generalisation, they re-focused on applying this to create further solutions, with the apparent goal to make as many solutions as possible. Whilst pleasure and excitement are seemingly enabling affective responses, the pleasure Alice and Ruby appeared to gain from creating many trials may have focused their attention on this process and away from forming convincing arguments about their generalisations. In the second of the two lessons, Alice and Ruby were able to adopt the teacher's intended goal and focused on constructing convincing arguments about why their generalisation worked.

Figure 15 summarises the cognitive-affective-conative interplay predominantly observed in the girls during cycle 2.

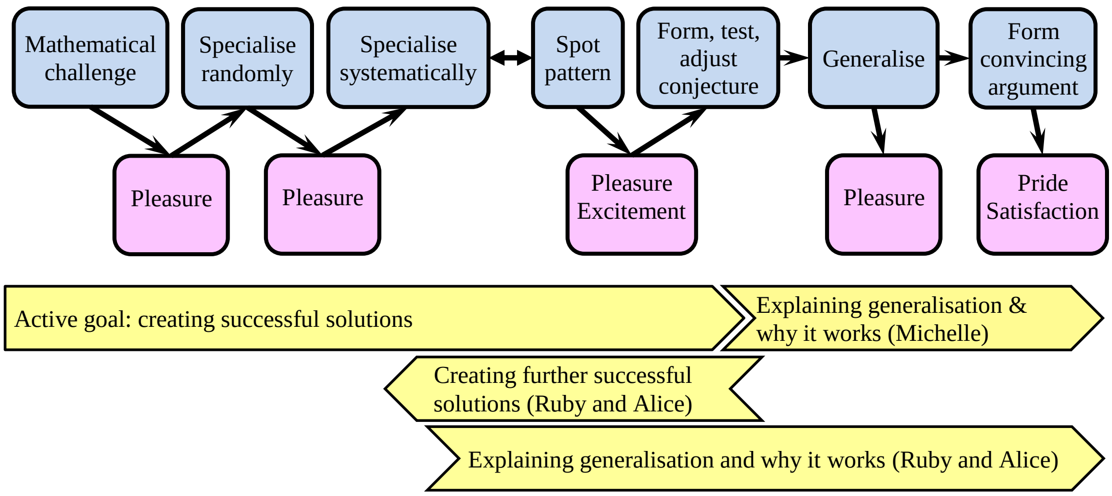


Figure 15: Cognitive-affective-conative interplay in cycle 2

Discussion

The small-scale nature of the study facilitated the collection of fine-grained data; this was valuable as it exemplified the conative aspects of children's mathematical reasoning and teachers' pedagogic interventions. Moreover, it illuminated how these might be identified and interpreted to understand cognitive-affective interplay and inform pedagogic intervention.

This study concurred with Di Martino and Zan (2013) in finding a bi-directional interplay between cognition and emotion when engaging in mathematical activities involving reasoning. For example, in cycle 2:

- the impact of Alice and Michelle's cognition on emotions was evident; they constructed convincing arguments, which they connected to feelings of pride and satisfaction
- the pleasure Ruby and Alice expressed when constructing successful solutions was subsequently followed by the creation of further solutions.

The application of a tripartite psychological classification to data collection and analysis revealed that there is cognitive-affective-conative interplay, and children's active goals, inferred through the focus of their engagement, impacted on cognitive-affective interplay.

Pleasure appeared to be the predominant affective response, across all except the final lesson in the study, impacting on the children's cognitive and conative responses. The children enjoyed engaging in activities involving challenge and mathematical reasoning and derived pleasure from creating multiple solutions. Their pleasure reinforced their active goal of creating trials. This led to persistent specialising and they did not display awareness that, as well as specialising, they were spotting patterns and conjecturing. Consequently, these processes remained incidental to their active goal.

During the cycles of persistent specialising, the resulting pleasure experienced may have inhibited the girls' capacity to reflect on both the value of the information gathered from specialising and the processes they were applying. This could have been the reason for their limited cognitive self-regulation. Their pleasure in creating solutions may also have impacted on their affective self-regulation; pleasure in these actions fostered further pleasure and excitement and these seemingly positive emotions may have impeded their capacity for meta-affective responses that could have led to higher order cognition. Debellis and Goldin (2006) argue that meta-affective responses enable difficult emotions to be monitored for cognitive gain. However, if the impeding emotion is pleasure, this may mask the need to apply meta-affective approaches to self-regulate.

Goswami (2015) cautions that the development of self-regulatory processes is not easy in the primary (elementary) phase; this study has demonstrated that children with limited PiMR might be working within weak or developing self-regulatory systems, which can result in habitual actions and emotions. However, utilising pedagogic interventions that focus on moving students from specialising to forming generalisations and convincing arguments can support them in transcending their habitual cognitive and affective responses; in this study, this resulted in successful PiMR. If students have not yet developed the self-regulation required for successful PiMR, teachers' interventions can provide them with active goals that focus their efforts and enable them to persevere in mathematical reasoning.

Implications for practice

Teachers' assessments of children's PiMR during lessons cannot be guided by their affective responses alone; whilst high levels of pleasure seem to be positive affective responses, pleasure is a poor indicator of PiMR. The following are more reliable indicators of PiMR and could be used to support teachers' assessments:

- movement between reasoning processes, rather than stasis in one or two processes

- a focus on articulating a generalisation and why it is true
- expressions of pride and satisfaction.

To be alert to the reasoning processes that children are using, teachers need to be familiar with these and how PiMR results from movement between processes towards forming generalisations and convincing arguments. Diagrammatic representations of pathways of reasoning processes could be utilised by university mathematics education tutors to raise teachers' awareness of reasoning processes, and the children's application of and movement between these processes. This could help teachers to plan, enact and assess the impact of pedagogies that facilitate movement between reasoning processes and PiMR.

Whilst persistence is seemingly of value, perseverance is more significant as it requires the development of self-regulating rather than habitual behaviours. Guidance to persevere, e.g. *push yourself, keep going* could be interpreted as *keep persisting*, irrespective of the outcome of the try. It is important that teachers are able to interpret perseverance guidance in the context of mathematical reasoning. Developing teachers' awareness of the construct PiMR, with its focus on producing assertions, developing arguments and justifying conclusions, would support this. It could raise teachers' awareness of the need to focus conative behaviours and active goals on these outcomes, rather than valuing behaviours that strive towards and focus on other goals, or valuing striving and high levels of engagement without consideration of the focus. Perseverance guidance needs to be augmented with a conative focus; e.g.:

> Push yourself to explain why the generalisation is true
> Keep going when things get difficult to convince yourself why this is true

This might be extended to learning contexts beyond mathematics by augmenting the perseverance guidance to indicate the processes that the children should be using and identifying the active goals they should be striving for.

Conclusions

This study extended the motivation aspect of Hannula's (2012) metatheoretical framework to include the broader conative domain. This enabled the construct PiMR to be defined and exemplified and offers a new approach to analysing the cognitive-affective-conative interplay in children's mathematical learning.

The study revealed the role that the children's active goals had on creating productive cognitive-affective interplay. Teachers' interventions successfully impacted on guiding children's active goals and this supported successful PiMR, even when children were operating within weak self-regulatory systems.

The study offers a framework for future research in this field. Whilst small-scale, it shows how the conative features of children's mathematical responses and teacher's interventions can be identified, and how these might be interpreted to support the development of productive cognitive and affective learning conditions.

Tables

Figures

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