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# New Entropy-Based Similarity Measure between Interval-Valued Intuitionstic Fuzzy Sets

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**Abstract:** In this paper, we propose a new approach to constructing similarity measures using the entropy measure for Interval-Valued Intuitionistic Fuzzy Sets. In addition, we provide several illustrative examples to demonstrate the practicality and effectiveness of the proposed formula. Finally, we use the new proposed similarity measure to develop a new approach for solving problems of pattern recognition and multi-criteria fuzzy decision-making.

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## 1. Introduction

Atanassov [1] introduced intuitionistic fuzzy sets (IFSs) which are characterized by both a membership function and a non-membership function. In 1989, Atanassov and Gargov [2] generalised the notion of intuitionistic fuzzy sets to interval-valued intuitionistic fuzzy sets (IVIFSs), in which the membership values are intervals, rather than exact numbers. IVIFSs operations, relations, and operators concerning IVIFSs were defined by Atanassov. Recently, Yager proposed the notation of a Pythagorean fuzzy set (PFS) [3], after which Peng developed it to a generalized form called the interval-valued Pythagorean fuzzy set (IVPFS). Han and Deng [4] proposed an interval-valued Pythagorean prioritized game framework in a group decision-making process.Two important topics in the theory of fuzzy sets—entropy measures and similarity measures of IFSs—have been widely proposed.

In 1965, Zadeh [5] introduced fuzzy entropy and defined it as the fuzziness degree of a fuzzy set. De Luca and Termini [6] proposed the axiomatic construction of the entropy of fuzzy sets. In 1996, the entropy measure of IFS and IVIFS was investigated by Burillo and Bustince [7] to measure the degree of intuitionism. A non-probabilistic entropy measure of IFS and IVIFS was proposed in terms of the ratio of intuitionistic fuzzy cardinalities by Szimidt and Kacprzyk [8]. Hung and Yang [9] proposed an axiomatic definition of the entropy of IFSs and IVFSs using probability, and they proposed the axiomatic definition and its properties. In addition, they introduced two families of entropy measures and showed how their proposed measure was more reliable for measuring the degree of fuzziness. Recently, Pan and Deng [10] proposed a new entropy measure that can measure uncertainty of probability distribution. This measure is the generalization of the Shannon entropy.

The similarity measure of IFSs is used to measure the degree of similarity between two IFSs. In 2004, Szimidt and Kacprzyk [11] investigated a similarity measure between IFSs using a distance measure. They applied the new measure to assess the extent of agreement between a group of experts giving their opinions expressed by intuitionistic fuzzy preference relations [11]. Moreover, a family of similarity measures was proposed by Szimidt and Kacprzyk [12], and they compared them with

some existing similarity measures. Hong and Kim [13], Hung and Yang [14], and Xu [15] defined some other similarity measures using different distance measures for IFSs. In addition to the new measure, Hung and Yang proved some properties of the proposed measure, and compared their measure with existing measures, and the comparison showed that their proposed similarity measure is much simpler than existing measures. More recently, Garg and Kumar [16] discussed the weaknesses of some of the existing measures, and proposed some novel similarity measures between IFSs.

Zeng and Guo [17] proved that some similarity measures and entropies of IVFSs could be deduced by normalized distances of IVFSs using the axiomatic definitions. Also, they investigated some formulae to calculate the entropy and similarity measure of the IVIFSs. Zeng and Li [18] and Zhang et al. [19] showed that similarity measures and entropies of the IVFSs could be transformed by each other. Zhang and Yu et al. [20] put forward some entropy formulae of IFSs according to the relationship between entropies and similarity measures of IFSs. In 2011, Wei and Wang [21] proposed an approach to construct similarity measures using entropy measures for interval-valued intuitionistic fuzzy sets.

In this paper, we follow the procedure outlined by Wei and Wang [21] and propose a new similarity measure. We demonstrate the usefulness of our new measure for pattern recognition and multi-criteria decision-making through two case studies.

The structure of this paper is as follows. Section 2 reviews some concepts and definitions of IFSs and IVIFSs. Section 3 gives an axiomatic definition, and provides some proposed entropy formulas of IVIFSs. Section 4 investigates the relationship between the entropy and similarity measure of IVIFSs, and proves that similarity measures of IVIFSs can be constructed by entropy measures of IVIFSs based on the axiomatic definition. In particular, we define a new similarity measure of IVIFSs according to the entropy formula of IVIFSs defined in Section 3. This is followed by applications of the proposed similarity measure to pattern recognition and multi-criteria fuzzy decision-making in Section 5. This paper is concluded in Section 6.

## 2. Preliminaries

**Definition 1.** *Let X be a universe of discourse. An IFS in X is a triple, having the following form:*

$$A = \{(x, \mu_A(x), v_A(x)) | x \in X\} \tag{1}$$

*where*

$$\mu_A : X \to [0,1], v_A : X \to [0,1] \tag{2}$$

*with the condition*

$$0 \le \mu_A(x) + v_A(x) \le 1, \qquad \forall x \in X$$

*where $\mu_A(x), v_A(x)$ denote the degrees of membership and non-membership of x in A, respectively. IFS(X) denotes the set of IFSs on X.*

*We call $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ the Intuitionistic index of x in A, which denotes the hesitancy degree of x in A [1].*

*The complementary set $A^C$ of A is defined as*

$$A^C = \{(x, v_A(x), \mu_A(x)) | x \in X\} \tag{3}$$

**Definition 2.** *Let X be a universe of discourse, and int(0, 1) denotes all closed subintervals of the interval [0, 1]. An IVIFS A in X is an object having the form:*

$$A = \{(x, \mu_A(x), v_A(x)) | x \in X\}. \tag{4}$$

*where*

$$\mu_A : X \to int(0,1), v_A : X \to int(0,1), \tag{5}$$

*with the condition*

$$0 \leq sup(\mu_A(x)) + sup(v_A(x)) \leq 1.$$

*where $\mu_A$, $v_A$ denote the degrees of membership and non-membership of x in A, let $\mu_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ (where $\mu_A^-(x) = \mu_A(x) - \epsilon$ and $\mu_A^+(x) = \mu_A(x) + \epsilon$, where $\epsilon$ is an infinitesimal number which is a number that is larger than each negative real number and is smaller than each positive real number), $v_A(x) = [v_A^-(x), v_A^+(x)]$ (where $v_A^-(x) = v_A(x) - \epsilon$ and $v_A^+(x) = v_A(x) + \epsilon$, where $\epsilon$ is an infinitesimal number which is a number that is larger than each negative real number and is smaller than each positive real number). Then,*

$$A = \{(x, [\mu_A^-(x), \mu_A^+(x)]), [v_A^-(x), v_A^+(x)]) | x \in X\}.$$

*with the condition $0 \leq \mu_A^+(x) + v_A^+(x) \leq 1$. IVIFS(X) denotes the set of IVIFSs in X [2].*
*we call $\pi_A(x) = [\pi_A^-(x), \pi_A^+(x)] = [1 - \mu_A^+(x) - v_A^+(x), 1 - \mu_A^-(x) - v_A^-(x)]$ the Interval-Valued Intuitionistic index of x in A, which denotes the hesitancy degree of x in A.*

**Definition 3.** *For two IVIFSs $A(x) = \{(x, [\mu_A^-(x), \mu_A^+(x)]), [v_A^-(x), v_A^+(x)]) | x \in X\}$ and $B(x) = \{(x, [\mu_B^-(x), \mu_B^+(x)]), [v_B^-(x), v_B^+(x)] | x \in X\}$, the following relations and operations can be defined:*

1. *$A \subseteq B$ if and only if $\mu_A^-(x) \leq \mu_B^-(x)$, $\mu_A^+(x) \leq \mu_B^+(x)$, $v_A^-(x) \geq v_B^-(x)$, $v_A^+(x) \geq v_B^+(x)$, for each $x \in X$*
2. *$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$*
3. *$A^C(x) = \{(x, [v_A^-(x), v_A^+(x)], [\mu_A^-(x), \mu_A^+(x)])) | x \in X\}$*

## 3. Entropy of IVIFSs

**Definition 4.** *A real-valued function $E : IVIFS(X) \to [0,1]$ is called an entropy for IVIFSs if it satisfies the following axiomatic requirements [22]:*

1. *$E(A) = 0$ if, and only if A is a fuzzy set.*
2. *$E(A) = 1$ if, and only if $[\mu_A^-(x), \mu_A^+(x)] = [v_A^-(x), v_A^+(x)] = [0,0]$ for all $x \in X$.*
3. *$E(A) = E(A^c)$ for all $A \in IVIFSs(X)$*
4. *For two IVIFSs A and B on X, if $A \leq B$ when $\mu_A^-(x) \leq \mu_B^-(x)$ and $\mu_A^+(x) \leq \mu_B^+(x)$ and $v_A^-(x) \geq v_B^-(x)$ and $v_A^+(x) \geq v_B^+(x)$ for each $x \in X$, then $E(B) \leq E(A)$.*

**Definition 5.** *For an IVIFS A, In [10,21,23], some examples of the entropy formula were provided and defined by, for each $A \in IVIFS$,*

$$E_1(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{min\{\mu_A^-(x_i), v_A^-(x_i)\} + min\{\mu_A^+(x_i), v_A^+(x_i)\} + \pi_A^-(x_i) + \pi_A^+(x_i)}{max\{\mu_A^-(x_i), v_A^-(x_i)\} + max\{\mu_A^+(x_i), v_A^+(x_i)\} + \pi_A^-(x_i) + \pi_A^+(x_i)} \tag{6}$$

$$E_2(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{2 - |\mu_A^-(x_i) - v_A^-(x_i)| + |\mu_A^+(x_i) - v_A^+(x_i)| + \pi_A^-(x_i) + \pi_A^+(x_i)}{2 + |\mu_A^-(x_i) - v_A^-(x_i)| + |\mu_A^+(x_i) - v_A^+(x_i)| + \pi_A^-(x_i) + \pi_A^+(x_i)} \tag{7}$$

## 4. New Similarity Measure between IVIFSs

In this section, we propose a new similarity measure between IVIFS $M(A, B)$ by using an entropy measure of IVIFSs. Firstly, we give an extension of the axiomatic definition for similarity measures in ([16,24–27]) of IVIFSs.

**Definition 6.** *A real-valued function $S : IVIFS(X) \times IVIFS(X) \to [0,1]$ is called a similarity measure on IVIFS(X) if it satisfies the following axiomatic requirements [28,29]:*

*(S1)* $0 \le S(A,B) \le 1$;
*(S2)* $S(A,B) = 1$ if, and only if $A = B$;
*(S3)* $S(A,B) = S(B,A)$;
*(S4)* $A \subseteq B \subseteq C$ if $S(A,C) \subseteq S(A,B)$ and $S(A,C) \subseteq S(B,C)$.

Let $X = \{x_1, x_2, ..., x_n\}$ be a finite universe of discourse. For $A, B \in IVIFSs(X)$, we define IVIFS $M(A,B)$ by the following:

$$M(A,B) = \{(x, [\mu^-_{M(A,B)}(x), \mu^+_{M(A,B)}(x)], [v^-_{M(A,B)}(x), v^+_{M(A,B)}(x)]) | x \in X\} \tag{8}$$

where

$$\mu^-_{M(A,B)}(x) = min\{M_{AB1}(x), M_{AB2}(x)\} \tag{9}$$

$$\mu^+_{M(A,B)}(x) = max\{M_{AB1}(x), M_{AB2}(x)\} \tag{10}$$

$$v^-_{M(A,B)}(x) = min\{M_{AB3}(x), M_{AB2}(x)\} \tag{11}$$

$$v^+_{M(A,B)}(x) = max\{M_{AB3}(x_j), M_{AB2}(x)\} \tag{12}$$

and

$M_{AB1}(x) = \frac{1}{4}[2 + |\mu^-_A(x) - \mu^-_B(x)| + |v^-_A(x) - v^-_B(x)| - (v^-_A(x) - \mu^-_B(x)) - (v^-_B(x) - \mu^-_A(x)) - (\pi^+_A(x) - \pi^+_B(x) - v^+_B(x))]$

$M_{AB2}(x) = \frac{1}{4}[2 + |\mu^+_A(x) - \mu^+_B(x)| + |v^+_A(x) - v^+_B(x)| - (v^+_A(x) - \mu^+_B(x)) - (v^+_B(x) - \mu^+_A(x)) - (\pi^-_A(x) - \pi^-_B(x) - v^-_B(x))]$

$M_{AB3}(x) = \frac{1}{4}[2 - |\mu^-_A(x) - \mu^-_B(x)| + |v^-_A(x) - v^-_B(x)| + (v^-_A(x) - \mu^-_B(x)) + (v^-_B(x) - \mu^-_A(x)) + (\pi^+_A(x) - \pi^+_B(x) - v^+_B(x))]$

$M_{AB4}(x) = \frac{1}{4}[2 - |\mu^+_A(x) - \mu^+_B(x)| + |v^+_A(x) - v^+_B(x)| + (v^+_A(x) - \mu^+_B(x)) + (v^+_B(x) - \mu^+_A(x)) + (\pi^-_A(x) - \pi^-_B(x) - v^-_B(x))]$

**Theorem 1.** *Suppose that E is an entropy measure for IVIFS. The function $S(A,B)$ on IVIFS(X) defined above by $E(M(A,B))$ is a similarity measure.*

**Proof.** It is straightforward to prove that $E(M(A,B))$ satisfies the four conditions $(E1)$–$(E4)$ in Definition 4

(S1) We can easily show $0 \le E(M(A,B)) \le 1$ since $0 \le E(A) \le 1$ for all $A \in IVIFS(X)$ and $M(A,B) \in IVIFS(X)$.
(S2) With the definition of the entropy and similarity measure,
we can obtain
$E(M(A,B)) = 1$
$\iff \mu^-_{M(A,B)}(x_j) = v^-_{M(A,B)}(x_j)$ and $\mu^+_{M(A,B)}(x_j) = v^+_{M(A,B)}(x_j) \forall x_j \in X$
$\iff \mu^-_A(x_j) = \mu^-_B(x_j), v^-_A(x_j) = v^-_B(x_j), \mu^+_A(x_j) = \mu^+_B(x_j)$ and $v^+_A(x_j) = v^+_B(x_j) \forall x_j \in X$
$\iff A = B$

(S3) From the definition of M(A,B), it is straightforward to show that $M(A,B) = M(B,A)$.

Thus, it is obvious that $E(M(A, B)) = E(M(B, A))$.

(S4) Since $A \subseteq B \subseteq C$, we have $\mu_A^-(x) \leq \mu_B^-(x) \leq \mu_C^-(x)$, $\mu_A^+(x) \leq \mu_B^+(x) \leq \mu_C^+(x)$, $v_A^-(x) \geq v_B^-(x) \geq v_C^-(x)$, $v_A^+(x) \geq v_B^+(x) \geq v_C^+(x)$. Thus:

$$|\mu_A^-(x) - \mu_C^-(x)| \geq |\mu_A^-(x) - \mu_B^-(x)|,$$
$$|\mu_A^+(x) - \mu_C^+(x)| \geq |\mu_A^+(x) - \mu_B^+(x)|,$$
$$|v_A^-(x) - v_C^-(x)| \geq |v_A^-(x) - v_B^-(x)|,$$
$$|v_A^+(x) - v_C^+(x)| \geq |v_A^+(x) - v_B^+(x)|.$$

Then

$$(v_A^-(x) - \mu_C^-(x)) \leq (v_A^-(x) - \mu_B^-(x))$$
$$(v_A^+(x) - \mu_C^+(x)) \leq (v_A^+(x) - \mu_B^+(x))$$

And

$$(v_C^-(x) - \mu_A^-(x)) \leq (v_B^-(x) - \mu_A^-(x))$$
$$(v_C^+(x) - \mu_A^+(x)) \leq (v_B^+(x) - \mu_A^+(x))$$

Also

$$\pi_A^+(x) - \pi_C^+(x) - v_C^+(x)) \leq \pi_A^+(x) - \pi_B^+(x) - v_B^+(x))$$
$$\pi_A^-(x) - \pi_C^-(x) - v_C^-(x) \leq \pi_A^-(x) - \pi_B^-(x) - v_B^-(x)$$

Hence we have

$$M_{AC1}(x) \geq M_{AB1}(x), \qquad M_{AC2}(x) \geq M_{AB2}(x),$$
$$M_{AC3}(x) \geq M_{AB3}(x), \qquad M_{AC4}(x) \geq M_{AB4}(x).$$

Similarly, we can obtain:
$\mu_{M(A,C)}^-(x) \geq \mu_{M(A,B)}^-(x)$

$\mu_{M(A,C)}^+(x) \geq \mu_{M(A,B)}^+(x)$

$v_{M(A,C)}^-(x) \geq v_{M(A,B)}^-(x)$

$v_{M(A,C)}^+(x) \geq v_{M(A,B)}^+(x)$

Therefore $M(A, B) \subseteq M(A, C)$
Besides we have $\mu_{M(A,C)}^-(x) \geq v_{M(A,C)}^-(x)$, $\mu_{M(A,C)}^+(x) \geq v_{M(A,C)}^+(x)$

by the definition of $M(A, B)$. We can get:

$E(M(A, B)) \geq E(M(A, C)), S(A, B) \geq S(A, C)$

In the same way, we can also have $E(M(B, C)) \geq E(M(A, C))$.

Let $E$ be the entropy measure defined by Equation (6), that is, for $A \in IVIFSs(X)$, $E(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{min\{\mu_A^-(x_i), v_A^-(x_i)\} + min\{\mu_A^+(x_i), v_A^+(x_i)\} + \pi_A^-(x_i) + \pi_A^+(x_i)}{max\{\mu_A^-(x_i), v_A^-(x_i)\} + max\{\mu_A^+(x_i), v_A^+(x_i)\} + \pi_A^-(x_i) + \pi_A^+(x_i)}$

Then, the proposed similarity measure $S$ for $A, B \in IVIFS(X)$ looks like:

$$S(A,B) = E(M(A,B))$$
$$= \frac{1}{n} \sum_{i=1}^{n} \frac{min\{\mu_{M(A,B)}^-(x_i), v_{M(A,B)}^-(x_i)\} + min\{\mu_{M(A,B)}^+(x_i), v_{M(A,B)}^+(x_i)\} + \pi_{M(A,B)}^-(x_i) + \pi_{M(A,B)}^+(x_i)}{max\{\mu_{M(A,B)}^-(x_i), v_{M(A,B)}^-(x_i)\} + max\{\mu_{M(A,B)}^+(x_i), v_{M(A,B)}^+(x_i)\} + \pi_{M(A,B)}^-(x_i) + \pi_{M(A,B)}^+(x_i)}$$

where $\mu_{M(A,B)}^-(x_i), v_{M(A,B)}^-(x_i), \mu_{M(A,B)}^+(x_i)$ and $v_{M(A,B)}^+(x_i)$, $i = 1...n$ are defined by Equations (9)–(12) □

**Definition 7.** *Let* $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *be a weight vector of the elements* $x_i$, $i = 1, 2, ..., n$. *Then, we can define the weighted similarity measure by*

$$S(A,B) = \sum_{i=1}^{n} \omega_i \frac{min\{\mu_{M(A,B)}^-(x_i), v_{M(A,B)}^-(x_i)\} + min\{\mu_{M(A,B)}^+(x_i), v_{M(A,B)}^+(x_i)\} + \pi_{M(A,B)}^-(x_i) + \pi_{M(A,B)}^+(x_i)}{max\{\mu_{M(A,B)}^-(x_i), v_{M(A,B)}^-(x_i)\} + max\{\mu_{M(A,B)}^+(x_i), v_{M(A,B)}^+(x_i)\} + \pi_{M(A,B)}^-(x_i) + \pi_{M(A,B)}^+(x_i)} \tag{13}$$

*where* $\omega_i \geq 0$ *and* $\sum_{i=1}^{n} \omega - i = 1$. *If* $\omega = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$.

## 5. Applications

The IFS and IVIFS theory have been applied to areas such as artificial intelligence, networking, soft decision-making, programming logic, and operational research. IVIFSs and IFSs are useful for dealing with vagueness and imprecision [30]. Moreover, they are suitable for modeling and processing imperfect information. In this section, we present two applications.

### 5.1. Pattern Recognition

We apply the proposed weighted similarity measure, defined by Equation (13), to solve pattern recognition problems with IVIFS information. We adopt the same steps as in [15]:
**Step 1**: Suppose that there exist $m$ patterns represented by IVIFSs

$$A_i = \{(x_j, [\mu_{A_i}^-(x_j), \mu_{A_i}^+(x_j)], [v_{A_i}^-(x_j), v_{A_i}^+(x_j)]) | x_j \in X\}$$

for $i = 1, 2, ..., m$, in the feature space $X = \{x_1, x_2, ..., x_n\}$, and suppose that there is a sample to be recognized, which is represented by an IVIFS

$$B = \{(x_j, [\mu_B^-(x_j), \mu_B^+(x_j)], [v_B^-(x_j), v_B^+(x_j)]) | x_j \in X\}$$

**Step 2**: Calculate the weighted similarity degree $S(A_i, B)$ between $A_i$ and $B$ by formula in Equation (13).
**Step 3**: Select the largest one, denoted by $S(A_k, B)$, from $S(A_i, B)$, $i = 1, 2, ..., m$. Then, $B$ belongs to the pattern $A_k$.

Now, we consider an example of a pattern recognition problem on the classification of building materials given in [15]. We use the same data as in [15].

**Example 1.** *Assume that there are four classes of building materials* $A_i$ , $i = 1, 2, 3, 4$, *and an unknown building material B, which are represented by the IVIFSs in the feature space* $X = \{x_1, x_2, ..., x_{12}\}$ *with a weight vector* $\omega$:

$$\omega = (0.1, 0.05, 0.08, 0.06, 0.03, 0.07, 0.09, 0.12, 0.15, 0.07, 0.13, 0.05)^T :$$

*Suppose we have the following data indicated by $[\mu_{A_i}^-(x_j), \mu_{A_i}^+(x_j)]$ and $[v_{A_i}^-(x_j), v_{A_i}^+(x_j)]$ for each $A_i$ for every variable $i = 1, 2, 3, 4$, and for each $x_j$ for every variable $j = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ given in Tables 1 and 2 [15].*

**Table 1.** The four classes of building materials $A_i$ , $i = 1, 2, 3, 4$ [15].

|  |  | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| $A_1$ | $\mu$ | [0.1, 0.2] | [0.1, 0.2] | [0.5, 0.6] | [0.8, 0.9] | [0.4, 0.5] | [0.0, 0, 1] |
|  | $v$ | [0.5, 0.6] | [0.7, 0.8] | [0.3, 0.4] | [0.0, 0.1] | [0.3, 0.4] | [0.8, 0.9] |
| $A_2$ | $\mu$ | [0.5, 0.6] | [0.6, 0.7] | [1.0, 1.0] | [0.1, 0.2] | [0.0, 0.1] | [0.7, 0.8] |
|  | $v$ | [0.3, 0.4] | [0.1, 0.2] | [0.0, 0.0] | [0.6, 0.7] | [0.8, 0.9] | [0.1, 0.2] |
| $A_3$ | $\mu$ | [0.4, 0.5] | [0.6, 0.7] | [0.9, 1.0] | [0.0, 0.1] | [0.0, 0.1] | [0.6, 0.7] |
|  | $v$ | [0.3, 0.4] | [0.2, 0.3] | [0.0, 0.0] | [0.8, 0.9] | [0.8, 0.9] | [0.2, 0.3] |
| $A_4$ | $\mu$ | [1.0, 1.0] | [1.0, 1.0] | [0.8, 0.9] | [0.7, 0.8] | [0.0, 0.1] | [0.0, 0.1] |
|  | $v$ | [0.0, 0.0] | [0.0, 0.0] | [0.0, 0.1] | [0.1, 0.2] | [0.7, 0.9] | [0.8, 0.9] |
| $B$ | $\mu$ | [0.9, 1.0] | [0.9, 1.0] | [0.7, 0.8] | [0.6, 0.7] | [0.0, 0.1] | [0.1, 0.2] |
|  | $v$ | [0.0, 0.0] | [0.0, 0.0] | [0.1, 0.2] | [0.1, 0.2] | [0.8, 0.9] | [0.7, 0.8] |

**Table 2.** The four classes of building materials $A_i$ , $i = 1, 2, 3, 4$ [15].

|  |  | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $x_{11}$ | $x_{12}$ |
|---|---|---|---|---|---|---|---|
| $A_1$ | $\mu$ | [0.3, 0.4] | [1.0, 1.0] | [0.2, 0.3] | [0.4, 0.5] | [0.7, 0.8] | [0.4, 0.5] |
|  | $v$ | [0.5, 0.6] | [0.6, 0.7] | [0.0, 0.0] | [0.4, 0.5] | [0.1, 0.2] | [0.4, 0.5] |
| $A_2$ | $\mu$ | [0.5, 0.6] | [0.6, 0.7] | [1.0, 1.0] | [0.1, 0.2] | [0.0, 0.1] | [0.7, 0.8] |
|  | $v$ | [0.3, 0.4] | [0.2, 0.3] | [0.0, 0.0] | [0.7, 0.8] | [0.8, 0.9] | [0.1, 0.2] |
| $A_3$ | $\mu$ | [0.1, 0.2] | [0.2, 0.3] | [0.5, 0.6] | [1.0, 1.0] | [0.3, 0.4] | [0.0, 0.1] |
|  | $v$ | [0.7, 0.8] | [0.6, 0.7] | [0.2, 0.4] | [0.0, 0.0] | [0.4, 0.5] | [0.8, 0.9] |
| $A_4$ | $\mu$ | [0.1, 0.2] | [0.1, 0.2] | [0.4, 0.5] | [1.0, 1.0] | [0.3, 0.4] | [0.0, 0.1] |
|  | $v$ | [0.7, 0.8] | [0.7, 0.8] | [0.3, 0.4] | [0.0, 0.0] | [0.4, 0.5] | [0.8, 0.9] |
| $B$ | $\mu$ | [0.1, 0.2] | [0.1, 0.2] | [0.4, 0.5] | [1.0, 1.0] | [0.3, 0.4] | [0.0, 0.1] |
|  | $v$ | [0.7, 0.8] | [0.7, 0.8] | [0.3, 0.4] | [0.0, 0.0] | [0.4, 0.5] | [0.7, 0.9] |

*Our purpose is to distinguish which class the unknown pattern B belongs to by using the above steps and calculating the similarity degrees $S(A_i, B)$ between each $A_i$ and B by (13), where we have:*

*We obtained the results in Table 3. Clearly, the similarity degree $S(A_4, B)$ between $A_4$ and B, is the largest one. Hence, B belongs to pattern $A_4$, which is the same result as in [15]. Moreover, it is very clear that the pairs of B are closer to $A_4$ than the others.*

**Table 3.** The proposed similarity measure $S$ results.

|  | $(A_1, B)$ | $(A_2, B)$ | $(A_3, B)$ | $(A_4, B)$ |
|---|---|---|---|---|
| $S$ | 0.63022 | 0.54985 | 0.76412 | 0.83245 |

*5.2. Multiple-Criteria Decision-Making Problems*

In this subsection, we introduce a method to solve a multi-criteria fuzzy decision making problem with weights using the proposed similarity measure [21,23].

Let $M = \{M_1, M_2, ..., M_m\}$ be a set of options, and $C = \{C_1, C_2, ..., C_n\}$ be a set of criteria. Assume that $\omega_j$ is the weight of the criterion $C_j, j = 1, 2, ..., n$, where $\omega_j \in [0, 1]$ and $\sum_{i=1}^n \omega_j = 1$. The characteristics of the option $M_i$ in terms of criteria $C$ are represented by the following IVIFSs:

$$M_i = \{\langle C_j, [\mu_{ij}^-, \mu_{ij}^+], [v_{ij}^-, v_{ij}^+]\rangle\}, i = 1, 2, ..., m,$$

where $[\mu_{ij}^-, \mu_{ij}^+]$ indicates the degree that option $M_i$ satisfies criterion $C_j$, and $[v_{ij}^-, v_{ij}^+]$ indicates the degree that option $M_i$ does not satisfy criterion $C_j$.

Using the weighted similarity measure defined by Equation (13), we introduce the following approach to solving the above multi-criteria fuzzy decision-making problem:

**Step 1**: Find out the positive-ideal solution $M^+$ and negative-ideal solution $M^-$:

$$
\begin{aligned}
M^+ &= \{< [\mu_{1+}^-, \mu_{1+}^+], [v_{1+}^-, v_{1+}^+], ..., [\mu_{n+}^-, \mu_{n+}^+], [v_{n+}^-, v_{n+}^+] >\}, \\
M^- &= \{< [\mu_{1-}^-, \mu_{1-}^+], [v_{1-}^-, v_{1-}^+], ..., [\mu_{n-}^-, \mu_{n-}^+], [v_{n-}^-, v_{n-}^+] >\},
\end{aligned}
$$

where, for each $j = 1, 2, ..., n$,

$$
\begin{aligned}
\langle [\mu_{1+}^-, \mu_{1+}^+], [v_{1+}^-, v_{1+}^+] \rangle &= \langle [max_i \mu_{ij}^-, max_i \mu_{ij}^+], [min_i v_{ij}^-, min_i v_{ij}^+] \rangle \\
\langle [\mu_{1-}^-, \mu_{1-}^+], [v_{1-}^-, v_{1-}^+] \rangle &= \langle [min_i \mu_{ij}^-, min_i \mu_{ij}^+], [max_i v_{ij}^-, max_i v_{ij}^+] \rangle
\end{aligned}
$$

**Step 2**: Calculate the similarity measure $S(M_i, M^+)$ between the option $M_i (i = 1, 2, ..., m)$ and the positive-ideal solution $M_+$, and the similarity measure $S(M_i, M^-)$ between the option $M_i (i = 1, 2, ..., m)$ and the negative-ideal solution $M^-$ by using the formula in Equation (13).

**Step 3**: Calculate the relative similarity measure $S(M_i)$ of $M_i$ with respect to $M^+$ and $M^-$, where

$$
S(M_i) = \frac{S(M_i, M^+)}{S(M_i, M^-) + S(M_i, M^+)}, i = 1, 2, ..., n.
\tag{14}
$$

**Step 4**: Select the largest one, denoted by $S(M_k)$, among the values $S(M_i)$, $i = 1, 2, ..., m$. Then, $M_k$ is the best choice.

**Example 2** ([21]). *Consider a supplier selection problem for a product. Suppose that a company sets up six evaluating indices for this problem: price ($G_1$), deadline ($G_2$), quality ($G_3$), the level of technology ($G_4$), service ($G_5$), and the future cooperation ($G_6$), and suppose that the correspondent weight vector is $\omega = 0.2, 0.1, 0.25, 0.1, 0.15, 0.2$. Assume that there are five suppliers $M_j (j = 1, 2, 3, 4, 5)$. Experts evaluate these suppliers by the above indices and obtain the evaluating information in Table 4, which indicated $[\mu_{M_i}^-(G_j), \mu_{M_i}^+(G_j)]$ and $[v_{M_i}^-(G_j), v_{M_i}^+(G_j)]$ for each $M_i$ for every variable $i = 1, 2, 3, 4, 5$ and for each $G_j$ for every variable $j = 1, 2, 3, 4, 5, 6$:*

**Table 4.** The five suppliers $M_i$, $i = 1, 2, 3, 4, 5$ [21].

|        |       | $G_1$          | $G_2$          | $G_3$          | $G_4$          | $G_5$          | $G_6$          |
|--------|-------|----------------|----------------|----------------|----------------|----------------|----------------|
| $M_1$  | $\mu$ | $[0.4, 0.5]$   | $[0.6, 0.8]$   | $[0.4, 0.5]$   | $[0.8, 0.9]$   | $[0.2, 0.6]$   | $[0.5, 0.7]$   |
|        | $v$   | $[0.2, 0.3]$   | $[0.1, 0.2]$   | $[0.2, 0.4]$   | $[0.1, 0.1]$   | $[0.2, 0.3]$   | $[0.1, 0.2]$   |
| $M_2$  | $\mu$ | $[0.5, 0.7]$   | $[0.6, 0.8]$   | $[0.3, 0.4]$   | $[0.8, 0.9]$   | $[0.2, 0.5]$   | $[0.1, 0.2]$   |
|        | $v$   | $[0.1, 0.2]$   | $[0.1, 0.2]$   | $[0.4, 0.6]$   | $[0.0, 0.1]$   | $[0.3, 0.4]$   | $[0.4, 0.5]$   |
| $M_3$  | $\mu$ | $[0.2, 0.3]$   | $[0.4, 0.5]$   | $[0.7, 0.8]$   | $[0.2, 0.5]$   | $[0.7, 0.8]$   | $[0.5, 0.6]$   |
|        | $v$   | $[0.6, 0.7]$   | $[0.3, 0.4]$   | $[0.1, 0.2]$   | $[0.1, 0.2]$   | $[0.0, 0.1]$   | $[0.2, 0.4]$   |
| $M_4$  | $\mu$ | $[0.5, 0.6]$   | $[0.3, 0.4]$   | $[0.5, 0.8]$   | $[0.6, 0.7]$   | $[0.3, 0.4]$   | $[0.1, 0.2]$   |
|        | $v$   | $[0.1, 0.2]$   | $[0.2, 0.4]$   | $[0.1, 0.2]$   | $[0.1, 0.2]$   | $[0.3, 0.4]$   | $[0.7, 0.8]$   |
| $M_5$  | $\mu$ | $[0.4, 0.5]$   | $[0.8, 0.9]$   | $[0.6, 0.8]$   | $[0.8, 0.9]$   | $[0.7, 0.8]$   | $[0.5, 0.6]$   |
|        | $v$   | $[0.3, 0.4]$   | $[0.0, 0.1]$   | $[0.1, 0.2]$   | $[0.0, 0.1]$   | $[0.1, 0.2]$   | $[0.1, 0.2]$   |

*By Step 1, we obtain the positive-ideal solution $M^+$ and the negative-ideal solution $M^-$ in Table 5.*

**Table 5.** The positive-ideal solution $M^+$ and the negative-ideal solution $M^-$ [21].

|       |       | $G_1$        | $G_2$        | $G_3$        | $G_4$        | $G_5$        | $G_6$        |
|-------|-------|--------------|--------------|--------------|--------------|--------------|--------------|
| $M^+$ | $\mu$ | $[0.5, 0.7]$ | $[0.8, 0.9]$ | $[0.7, 0.8]$ | $[0.8, 0.9]$ | $[0.7, 0.8]$ | $[0.5, 0.7]$ |
|       | $v$   | $[0.1, 0.2]$ | $[0.0, 0.1]$ | $[0.1, 0.2]$ | $[0.0, 0.1]$ | $[0.0, 0.1]$ | $[0.1, 0.2]$ |
| $M^-$ | $\mu$ | $[0.2, 0.3]$ | $[0.3, 0.4]$ | $[0.3, 0.4]$ | $[0.2, 0.5]$ | $[0.2, 0.4]$ | $[0.1, 0.2]$ |
|       | $v$   | $[0.6, 0.7]$ | $[0.3, 0.4]$ | $[0.4, 0.6]$ | $[0.1, 0.2]$ | $[0.3, 0.4]$ | $[0.7, 0.8]$ |

By Step 2, we use formula in Equation (13) to calculate the similarity measure $S(M_i, M^+)$ and $S(M_i, M^-)$, respectively:

Then, we obtain the relative similarity measure $S(M_i)$ of $M_i$ with respect to $M^+$ and $M^-$ by the formula in Equation (14):

$$S(M_1) = 0.45793, S(M_2) = 0.41931, S(M_3) = 0.45233, S(M_4) = 0.42791, S(M_5) = 0.49541.$$

The results in Table 6 shows that $M_5 > M_1 > M_3 > M_4 > M_2$, so $M_5$ is the proper supplier, which is the same result as in [15]. We replicated the same procedure that was used in [21]. We obtained the result that $M_5$ is the proper supplier.

**Table 6.** The proposed similarity measure $S$ results.

| | | | |
|---|---|---|---|
| $S(M_1, M^+)$ | 0.55286 | $S(M_1, M^-)$ | 0.65444 |
| $S(M_2, M^+)$ | 0.57748 | $S(M_2, M^-)$ | 0.79973 |
| $S(M_3, M^+)$ | 0.54882 | $S(M_3, M^-)$ | 0.66450 |
| $S(M_4, M^+)$ | 0.57397 | $S(M_4, M^-)$ | 0.76737 |
| $S(M_5, M^+)$ | 0.52264 | $S(M_5, M^-)$ | 0.53232 |

## 6. Conclusions

Many similarity measures that were defined based on entropy measures have been applied to the problems based on intuitionistic fuzzy information, but they could not be used to deal with the problems based on interval-valued intuitionistic fuzzy information. In this paper, we introduced a new similarity measure between IVIFSs. This measure has been constructed using the entropy measure for IVIFSs. Moreover, we have verified the efficiency of the proposed similarity measure by applying it to two case studies. Two applications were introduced—one was to solve a pattern recognition problem, and the other was for a multiple-criteria fuzzy decision-making problem. A potential issue is that one measure will not be optimal for all data. Although our measure was effective in our two case studies, in future work we could explore its performance on other applications, such as multiple-attribute decision-making and edge detection, image segmentation, and fault-tree analysis. In addition, we will compare the performance of our measures and IFSs measures on IVIFS data in future.

## References

1. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
2. Atanassov, K.; Gargov, G. Interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349. [CrossRef]

3.    Cabrerizo, F.J.; Morente-Molinera, J.A.; Pedrycz, W.; Taghavi, A.; Herrera-Viedma, E. Granulating linguistic information in decision making under consensus and consistency. *Expert Syst. Appl.* **2018**, *99*, 83–92. [CrossRef]

4.    Han, Y.; Deng, Y.; Cao, Z.; Lin, C.T. An interval-valued Pythagorean prioritized operator-based game theoretical framework with its applications in multicriteria group decision making. *Neural Comput. Appl.* **2016**, 1–19. [CrossRef]

5.    Zedah, L. Fuzzy Sets. *Inf. Control.* **1965**, *8*, 338–353.

6.    De Luca, A.; Termini, S. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory. In *Readings in Fuzzy Sets for Intelligent Systems*; Elsevier: New York, NY, USA, 1993; pp. 197–202.

7.    Burillo, P.; Bustince, H. Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets. *Fuzzy Sets Syst.* **1996**, *78*, 305–316. [CrossRef]

8.    Szmidt, E.; Kacprzyk, J. Entropy for intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **2001**, *118*, 467–477. [CrossRef]

9.    Hung, W.L.; Yang, M.S. Fuzzy entropy on intuitionistic fuzzy sets. *Int. J. Intell. Syst.* **2006**, *21*, 443–451. [CrossRef]

10.   Pan, L.; Deng, Y. A new belief entropy to measure uncertainty of basic probability assignments based on belief function and plausibility function. *Entropy* **2018**, *20*, 842. [CrossRef]

11.   Szmidt, E.; Kacprzyk, J. A concept of similarity for intuitionistic fuzzy sets and its use in group decision making. In Proceedings of the 2004 IEEE International Conference on Fuzzy Systems (IEEE Cat. No.04CH37542), Budapest, Hungary, 25–29 July 2004; Volume 2, pp. 1129–1134.

12.   Szmidt, E.; Kacprzyk, J. Analysis of Similarity Measures for Atanassov's Intuitionistic Fuzzy Sets. In Proceedings of the Joint 2009 International Fuzzy Systems Association World Congress and 2009 European Society of Fuzzy Logic and Technology Conference, Lisbon, Portugal, 20–24 July 2009.

13.   Hong, D.H.; Kim, C. A note on similarity measures between vague sets and between elements. *Inf. Sci.* **1999**, *115*, 83–96. [CrossRef]

14.   Hung, W.L.; Yang, M.S. Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. *Pattern Recognit. Lett.* **2004**, *25*, 1603–1611. [CrossRef]

15.   Zeshui, X. On similarity measures of interval-valued intuitionistic fuzzy sets and their application to pattern recognitions. *J. Southeast Univ.* **2007**, *1*, 027.

16.   Garg, H.; Kumar, K. An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making. *Soft Comput.* **2018**, *22*, 4959–4970. [CrossRef]

17.   Zeng, W.; Guo, P. Normalized distance, similarity measure, inclusion measure and entropy of interval-valued fuzzy sets and their relationship. *Inf. Sci.* **2008**, *178*, 1334–1342. [CrossRef]

18.   Zeng, W.; Li, H. Relationship between similarity measure and entropy of interval valued fuzzy sets. *Fuzzy Sets Syst.* **2006**, *157*, 1477–1484. [CrossRef]

19.   Zhang, H.; Zhang, W.; Mei, C. Entropy of interval-valued fuzzy sets based on distance and its relationship with similarity measure. *Knowl.-Based Syst.* **2009**, *22*, 449–454. [CrossRef]

20.   Zeng, W.; Yu, X.; Yu, F.; Cui, B. Entropy of intuitionistic fuzzy set based on similarity measure. In Proceedings of the 3rd International Conference on Innovative Computing Information and Control, ICICIC'08, Dalian, China, 18–20 June 2008; p. 398.

21.   Wei, C.P.; Wang, P.; Zhang, Y.Z. Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications. *Inf. Sci.* **2011**, *181*, 4273–4286. [CrossRef]

22.   Zhang, Y.; Ma, P.; Su, X.; Zhang, C. Entropy on interval-valued intuitionistic fuzzy sets and its application in multi-attribute decision making. In Proceedings of the 14th International Conference on Information Fusion, Chicago, IL, USA, 5–8 July 2011; pp. 1–7.

23.   Meng, F.; Chen, X. Entropy and similarity measure for Atannasov's interval-valued intuitionistic fuzzy sets and their application. *Fuzzy Optim. Decis. Mak.* **2016**, *15*, 75–101. [CrossRef]

24.   Mohamed, S.S.; AbdAla, A.S. Applying a new similarity measure between general type-2 fuzzy sets to clustering. In Proceedings of the 2011 IEEE 12th International Symposium on Computational Intelligence and Informatics (CINTI), Budapest, Hungary, 21–22 November 2011; pp. 283–286.

25.   Pekala, B.; Balicki, K. Interval-valued intuitionistic fuzzy sets and similarity measure. *Iran. J. Fuzzy Syst.* **2017**, *14*, 87–98.

26. Fei, L.; Wang, H.; Chen, L.; Deng, Y. (1604-2881) A new vector valued similarity measure for intuitionistic fuzzy sets based on owa operators. *Iran. J. Fuzzy Syst.* **2018**. [CrossRef]

27. Mishra, A.R. Intuitionistic fuzzy information measures with application in rating of township development. *Iran. J. Fuzzy Syst.* **2016**, *13*, 49–70.

28. Arefi, M.; Taheri, S. Weighted similarity measure on interval-valued fuzzy sets and its application to pattern recognition. *Iran. J. Fuzzy Syst.* **2014**, *11*, 67–79.

29. Xu, Z.; Chen, J. An overview of distance and similarity measures of intuitionistic fuzzy sets. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2008**, *16*, 529–555. [CrossRef]

30. Ahmad, Y.; Husain, S.; Asthanvi, I.S. Study on the Development of Decision Making Using Intuitionistic Fuzzy Set (IFS) and Interval Valued Intuitionistic Fuzzy Set (IVIFS). *IOSR J. Eng.* **2013**, *3*, 34–42. [CrossRef]