Hamilton-Jacobi formalism for Linearized Gravity

M. C. Bertin[1], B. M. Pimentel[2], C. E. Valcárcel[3], G. E. R. Zambrano[4]

[1] *CMCC, Universidade Federal do ABC.*
Rua Santa Adália, 166, Santo André, SP, Brazil.

[2,3] *Instituto de Física Teórica, UNESP - São Paulo State University,*
P. O. Box 70532-2, 01156-970, São Paulo, SP, Brazil.

[4] *Departamento de Física, Universidad de Nariño,*
Calle 18 Carrera 50, San Juan de Pasto, Nariño, Colombia.

Abstract

In this work we study the theory of linearized gravity via the Hamilton-Jacobi formalism. We make a brief review of this theory and its Lagrangian description, as well as a review of the Hamilton-Jacobi approach for singular systems. Then we apply this formalism to analyze the constraint structure of the linearized gravity in instant and front-form dynamics.

Keywords: Hamilton-Jacobi formalism, Linearized gravity.

1 Introduction

Einstein's field equations in vacuum arise from a variational principle, setting to zero the first variation of the Einstein-Hilbert action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R, \tag{1}$$

with respect to the metric of space-time, where R is the Ricci's scalar. The constant $\kappa = 8\pi Gc^{-4}$ is obtained in four dimensions in the weak field approximation. Despite that General Relativity (GR) has a major difference to other fields, since it treats the gravitational phenomena as manifestations of the geometry of the space-time, it has been handled with the same tools for it's canonical quantization. However, GR as well as the other fundamental interactions is a constrained theory which requires consistent methods of constraint analysis.

In 1950 Dirac was outlining his Hamiltonian formalism for singular systems [1]. Studying the gravitational field [2], he found that a foliation of the space-time simplifies the constraint structure of gravity with the cost of abandoning the four-symmetry of the Lagrangian stage.

From a particle physicist's point of view, it would be extremely useful to have a theory of gravity in a flat space-time that maintains all the characteristics of the gravitational phenomena in a non-relativistic limit. This imposition leads us to consider massless fields with spin 0 or 2 (higher even spin fields will only be considered if the spin 2 fails describing the theory). A model of scalar gravitational field was proposed by Nordström [3], but it ended to be in contradiction with experimentation, since it does not interact with photons. It also failed when trying to compute the Mercury's perihelion.

The simplest description of gravity as a spin 2 field is the one with a massless symmetric tensor of rank 2. This model is well described by the Fierz-Pauli Lagrangian density [4],

[1] mario.bertin@ufabc.edu.br
[2] pimentel@ift.unesp.br
[3] valcarcel@ift.unesp.br
[4] gramos@udenar.edu.co

which becomes more successful when experimental confrontation comes about. Another spin 2 field in a fixed background is obtained by linearization of the GR in the weak-metric approximation, resulting in the linearized GR (LGR). In this scheme, the linearized Einstein's equations possess a gauge invariance, and we can use this symmetry to build a Lagrangian density that describes LGR as a gauge theory. Surprisingly, we obtain a one-parameter family of Lagrangian densities where the Fierz-Pauli Lagrangian appears as one of them [5].

Moreover, linearized gravity appears as an attempt to achieve a perturbative canonical quantization of gravity [2]. At principle, since these models are based on gauge invariant actions, they are good theories for the quantization programme proposed by Dirac. However, these theories still present some difficult problems, e.g. non-renormalizability in four dimensions (see [6] and references therein). On the other hand, attempts to learn key properties about quantum gravity are taken in modified models in two and three dimensions, where the theories become not only renormalizable, but at least in the two dimensional case exactly solvable [7]. In three dimensions, GR is usually modified with a topological Chern-Simons term [8], and more recently with a massive higher derivative term [9]. In these cases, the linearized theories are equivalent to massive Fierz-Pauli theories, and can be used, for example, to calculate one-loop partition functions [10].

On the other hand, there is an increasing interest in field theories in front-form dynamics [11]. This kind of dynamics reduces the number of independent degrees of freedom, which is due to the fact that the stability group of the Poincaré group in front-form has seven generators, one more than in the instant-form description. Besides, the algebra of these these generators takes its simplest form in front-form dynamics. For some important systems this feature is responsible for a complete separation of physical degrees of freedom, resulting in an excitation-free quantum vacuum. This is actually verified, e.g., in QCD [12] and spontaneous symmetry breaking models [13].

In this work we study the constraint structure of linearized gravity in instant and front-form dynamics. For this task, we employ the Hamilton-Jacobi (HJ) approach for singular systems, first developed by Güler [14], as a generalization of Carathéodory's method for regular mechanics [15]. Unlike Dirac's approach [1], which is a consistency method to build a Hamiltonian dynamics from a Lagrangian system, the HJ theory is a full formalism by itself. As necessary conditions for the existence of extremes of a given action, e.g. (1), the constraints of a theory appear as first-order partial differential equations, whose characteristics equations describe a system with several independent variables, or parameters. To be sufficient conditions as well, the so called HJ partial differential equations (PDE) must also obey integrability, i.e., they must form a complete set of involutive constraints.

The search for integrability, which is in fact the constraint analysis by itself, generally reveals two types of HJ equations, called involutive and non-involutive constraints. Involutive HJ equations are the ones that form a closed set of integrable equations. The presence of a non-involutive set indicates dependence between the parameters of the theory: they must be treated with a redefinition of the phase-space dynamics. In this context, it is shown in [16], for first-order actions, that the structure of generalized brackets (GB) appears naturally. Later, a more complete analysis of non-involutive constraints shows that the GB is a general structure [17]. Several developments and applications on the HJ formalism can be found in [18, 19, 20].

Our main goal in studying the instant and front-form dynamics of the LGR is to obtain the algebra of the involutive constraints. In instant-form there are only involutive constraints, but in front-form the structure of the dynamics in the coordinates of the light cone reveals a set of non-involutive HJ equations. This structure allows us to use the method developed in [17] to obtain the generalized brackets, which is an essential tool for canonical quantization.

The paper is structured as follows. Section 2 contains a brief review of the HJ formalism. In section 3 we introduce the linearization of the sourceless Einstein's field equations and its relation to the Fierz-Pauli Lagrangian. Then, we employ the HJ formalism to integrability analysis, first in the instant-form dynamics (section 4), next in the front-form dynamics (section 5). The last section is dedicated to final remarks.

2 The Hamilton-Jacobi formalism

Let us consider a Lagrangian function $L(x^i, \dot{x}^i, t)$, $i = 1, 2, ..., N$, whose Hessian matrix

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \tag{2}$$

is singular of rank P. This means that we have P conjugated momenta

$$p_a = \frac{\partial L}{\partial \dot{x}^a}, \quad a = 1, \ldots, P \tag{3}$$

that can be inverted in relations of the type $\dot{x}^a = \dot{x}^a\,(p, x, t)$, but $R = N - P$ relations between the canonical variables

$$p_z + H_z = 0, \quad z = 1, \ldots, R, \tag{4}$$

where $H_z = -\partial L/\partial \dot{x}^z$, correspond to canonical constraints.

The HJ equation derived from the stationary action principle with help of Carathéodory's [15] equivalent Lagrangian method has the form

$$p_0 + p_a \dot{x}^a + p_z \dot{x}^z - L = 0, \tag{5}$$

where $p_0 \equiv \partial_t S$, $p_a = \partial_a S$, and $p_z = \partial_z S$. We may define the canonical Hamiltonian as

$$H_0 \equiv p_a \dot{x}^a + p_z \dot{x}^z - L, \tag{6}$$

then we have a set of $R + 1$ Hamilton-Jacobi partial differential equations (HJ PDE)

$$H'_\alpha \equiv p_\alpha + H_\alpha = 0, \quad \alpha = 0, 1, \ldots, R, \tag{7}$$

here $x^0 = t$, and the H'_α are just called the Hamiltonian functions of the theory. In other words, the HJ approach replaces the study of R canonical constraints with the analysis of $R + 1$ HJ PDE.

Being a first-order system, we may use Cauchy's method to solve the HJ PDE, which gives us a set of total differential equations (TDE) related to them. The resultant equations are called characteristics equations,

$$dx^i = \frac{\partial H'_0}{\partial p_i} dx^0 + \frac{\partial H'_z}{\partial p_i} dx^z = \frac{\partial H'_\alpha}{\partial p_i} dt^\alpha, \tag{8a}$$

$$dp_i = -\frac{\partial H'_0}{\partial x^i} dx^0 - \frac{\partial H'_z}{\partial x^i} dx^z = -\frac{\partial H'_\alpha}{\partial x^i} dt^\alpha, \tag{8b}$$

$$dS = p_a dx^a - H_\alpha dt^\alpha, \tag{8c}$$

where we have written $t^\alpha \equiv (x^0, x^z)$ as the independent variables, or parameters, while we see that (x^a, p^a) are the dependent variables of the theory.

For any function $F = F(t^\alpha, x^a, p^a)$ we have that

$$dF = \frac{\partial F}{\partial x^a} dx^a + \frac{\partial F}{\partial p^a} dp^a + \frac{\partial F}{\partial t^\alpha} dt^\alpha = \{F, H'_\alpha\} dt^\alpha, \tag{9}$$

where we have used (8a) and (8b), as well as the extended Poison Brackets

$$\{F, G\} \equiv \frac{\partial F}{\partial x^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial x^i}\frac{\partial F}{\partial p_i} + \frac{\partial F}{\partial t}\frac{\partial G}{\partial p_0} - \frac{\partial G}{\partial t}\frac{\partial F}{\partial p_0}\,. \tag{10}$$

Let us define a vector field X_α such that for any function F defined in the phase space, $X_\alpha(F) \equiv \{F, H'_\alpha\}$. The characteristic equations for the canonical variables can be written as

$$dz^K = \{z^K, H'_\alpha\} dt^\alpha = X_\alpha(z^K) dt^\alpha, \tag{11}$$

where $z^K = (x^i, p^i)$.

The conditions that ensures the integrability of the system are the Frobenius' integrability conditions (IC), which are given by $\{H'_\alpha, H'_\beta\} = 0$. On the other hand, these IC imply $[X_\alpha, X_\beta] = 0$, i.e., the vector fields X_α must form a complete orthogonal basis on the vector space of the parameter space. Generally, Hamiltonians that obey the Lie algebra $\{H'_\alpha, H'_\beta\} = C^\gamma_{\alpha\beta} H'_\gamma$ are sufficient to assure integrability [21]. However, these IC imply

$$[X_\alpha, X_\beta]F = C^\gamma_{\beta\alpha} X_\gamma(F) + \{F, C^\gamma_{\beta\alpha}\} H'_\gamma. \tag{12}$$

If the structure coefficients $C^\gamma_{\alpha\beta}$ are field independent, the Lie algebra of the Hamiltonians is reflected in a Lie algebra of the vector fields. This is sufficient to assure the existence of a finite Lie group of transformations generated by X_α. Otherwise, if the $C^\gamma_{\alpha\beta}$ are field dependent, the last term on the right hand side of (12) spoils the algebra of the vector fields, therefore, the existence of a finite group of transformations cannot be ensured.

The analysis of IC can also be achieved through the fundamental differential (9), since

$$dH'_\alpha = \{H'_\alpha, H'_\beta\} dt^\beta = 0. \tag{13}$$

If a subset of Hamiltonians does not satisfy (13), they are non-involutive constraints, and we may apply the procedure outlined in [17], defining the matrix M with elements $M_{xy} = \{H'_x, H'_y\}$. If this matrix has rank $S \leq R$, we define the GB with the largest regular submatrix $M_{\bar{a}\bar{b}} = \{H'_{\bar{a}}, H'_{\bar{b}}\}$. In this case, there is an inverse $(M^{-1})^{\bar{a}\bar{b}}$ which is used to define the Generalized Brackets (GB)

$$\{F, G\}^* \equiv \{F, G\} - \{F, H'_{\bar{a}}\}(M^{-1})^{\bar{a}\bar{b}}\{H'_{\bar{b}}, G\}. \tag{14}$$

This expression has all the properties of the PB: it is a bilinear antisymmetric operator that obeys the Jacobi identity and the Leibniz rule. With the GB the dynamics is given by

$$dF = \{F, H'_{\bar{\alpha}}\}^* dt^{\bar{\alpha}}, \quad \bar{\alpha} = 0, S+1, \ldots, R. \tag{15}$$

The dynamical evolution of the system depends on $(R - S)$ parameters. If the system is not complete, new HJ PDE may be found by $\{H'_{\bar{z}}, H'_0\} = 0$, where $\bar{z} = S+1, \cdots, R$, and IC must be tested for these new constraints as well.

3 The Linearized Gravity

The linearized General Relativity is obtained from the weak field approximation of the Einstein's equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} \Theta_{\mu\nu}, \tag{16}$$

where $\Theta_{\mu\nu}$ is the source energy momentum tensor. Here we decompose the metric $g_{\mu\nu}$ into a Minkowski background $\eta_{\mu\nu}$, and a perturbation $\phi_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon \phi_{\mu\nu} + O(\varepsilon^2), \tag{17}$$

where ε is a small parameter introduced to maintain the correct order of the expansion series. For the LGR only linear terms in ε are considered. Under this assumptions and considering a sourceless gravitational field we obtain, from (17) in (16),

$$\eta^{\alpha\beta} \partial_\gamma \partial^\gamma \phi^\nu{}_\nu - \partial_\gamma \partial^\gamma \phi^{\alpha\beta} + \partial^\alpha \partial_\lambda \phi^{\lambda\beta} + \partial^\beta \partial_\lambda \phi^{\lambda\alpha} - \partial^\alpha \partial^\beta \phi^\lambda{}_\lambda - \eta^{\alpha\beta} \partial_\gamma \partial_\mu \phi^{\mu\gamma} = 0. \tag{18}$$

On the other hand, (18) can be obtained as the Euler-Lagrange (EL) equations for the Fierz-Pauli Lagrangian density [4]

$$\mathcal{L} = \frac{1}{4} \partial_\mu \phi^\nu{}_\nu \partial^\mu \phi^\lambda{}_\lambda - \frac{1}{4} \partial_\lambda \phi_{\mu\nu} \partial^\lambda \phi^{\mu\nu} + \frac{1}{2} \partial_\mu \phi^\mu{}_\nu \partial_\lambda \phi^{\lambda\nu} - \frac{1}{2} \partial_\mu \phi^{\mu\nu} \partial_\nu \phi^\lambda{}_\lambda. \tag{19}$$

It can be verified that (19) is invariant under the gauge transformation

$$\phi_{\alpha\beta} \to \phi_{\alpha\beta} + \partial_\alpha \Lambda_\beta + \partial_\beta \Lambda_\alpha, \tag{20}$$

where $\Lambda_\alpha = \Lambda_\alpha(x)$ are arbitrary differentiable functions. The transformation (20) is actually similar to the given in the electromagnetic field. In order to eliminate the ambiguity raised for this gauge symmetry it is customary to define a traceless tensor

$$h_{\mu\nu} \equiv \phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\phi^{\alpha}_{\ \alpha}, \tag{21}$$

which simplifies (18):

$$\partial^\mu \partial_\mu h_{\alpha\beta} - \partial^\mu \partial_\alpha h_{\beta\mu} - \partial^\mu \partial_\beta h_{\alpha\mu} + \eta_{\alpha\beta}\partial^\mu \partial^\nu h_{\mu\nu} = 0. \tag{22}$$

More important, (21) allows us to choose

$$\partial^\mu \partial_\mu \Lambda_\alpha = -\partial^\mu h_{\alpha\mu}, \tag{23}$$

from where we obtain a gauge condition

$$\partial^\mu h_{\alpha\mu} = 0, \tag{24}$$

in analogy with the Lorenz gauge from electrodynamics. Equation (24) is called de Donder gauge, or harmonic gauge. Finally, the equation of motion for $h_{\alpha\beta}$ is

$$\partial^\mu \partial_\mu h_{\alpha\beta} = 0, \tag{25}$$

which is a relativistic wave equation for a massless spin 2 field, the graviton. In the linear approximation, the graviton is the mediator of the gravitational interaction, analogous to the photon which is the mediator in QED theory. The analysis of the plane wave solution of (25), the polarization states and helicity of the graviton can be found in [6].

On the other hand, (19) is not the only Lagrangian density for the LGR. There is a one-parameter family of Lagrangians [5] that results in the same field equations (18). In the next sections we work only with the Fierz-Pauli Lagrangian (19). In the context of Dirac's formalism in front-form dynamics, this model was studied in [22].

4 LGR in instant-form

The procedure adopted in the preceding section is valid in four dimensions, but it can be easily extended for d dimensions. We adopt the mostly minus metric $\eta_{\mu\nu} = diag(+ - - - ...)$. Breaking the covariance in the Lagrangian formalism, making explicit the time variable $\tau = x^0$, we get the Lagrangian density

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\partial_i \phi_{i0}\partial_0 \phi_{00} + \left[\frac{1}{2}\partial_i \phi_{00} + \partial_j \phi_{ij} - \frac{1}{2}\partial_i \phi_{jj}\right]\partial_0 \phi_{0i} \\ &\quad + \left[\frac{1}{4}\delta_{ij}\partial_0 \phi_{kk} - \frac{1}{4}\partial_0 \phi_{ij} - \frac{1}{2}\delta_{ij}\partial_k \phi_{0k}\right]\partial_0 \phi_{ij} - \mathcal{V}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \mathcal{V} &= \frac{1}{2}\phi_{00}\left[\partial_i \partial_i \phi_{jj} - \partial_i \partial_j \phi_{ij}\right] + \frac{1}{2}\phi_{0i}\left[\partial_i \partial_j \phi_{0j} - \partial_j \partial_j \phi_{0i}\right] \\ &\quad -\frac{1}{4}(\partial_i \phi_{jk})^2 + \frac{1}{4}(\partial_i \phi_{jj})^2 + \frac{1}{2}(\partial_i \phi_{ij})^2 - \frac{1}{2}\partial_i \phi_{ij}\partial_j \phi_{kk}. \end{aligned} \tag{27}$$

Due to the symmetry of the field $\phi_{\mu\nu}$ we have that

$$\frac{\partial \phi_{\mu\nu}(x)}{\partial \phi_{\alpha\beta}(y)} \equiv \Delta^{\mu\nu}_{\alpha\beta}\delta^d(x-y) = \frac{1}{2}\left[\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha\right]\delta^d(x-y), \tag{28}$$

where $\delta^d(x-y)$ is Dirac's delta function in d dimensions. The conjugated momenta are given by

$$p^{00} = -\frac{1}{2}\partial_i\phi_{i0}, \tag{29a}$$

$$p^{0i} = \frac{1}{4}\partial_i\phi_{00} + \frac{1}{2}\partial_j\phi_{ij} - \frac{1}{4}\partial_i\phi_{jj}, \tag{29b}$$

$$p^{ij} = \frac{1}{2}\delta_{ij}\partial_0\phi_{kk} - \delta_{ij}\partial_k\phi_{0k} - \frac{1}{2}\partial_0\phi_{ij}. \tag{29c}$$

This system is singular, and we identify equations (29a) and (29b) as constraints.

It was pointed out by Anderson [23] that it is possible to simplify the canonical constraints. Particularly, we may simplify calculations by adding surface terms in the Lagrangian, with the identity

$$\partial_\rho\phi_{\alpha\beta}\partial_\gamma\phi_{\mu\nu} = \partial_\gamma\phi_{\alpha\beta}\partial_\rho\phi_{\mu\nu} + \partial_\rho(\phi_{\alpha\beta}\partial_\gamma\phi_{\mu\nu}) - \partial_\gamma(\phi_{\alpha\beta}\partial_\rho\phi_{\mu\nu}). \tag{30}$$

Then we are able to eliminate the dependence in $\partial_0\phi_{0\mu}$ and obtain

$$\mathcal{L} = \left(\partial_i\phi_{0j} - \delta_{ij}\partial_k\phi_{0k} + \frac{1}{4}\delta_{ij}\partial_0\phi_{kk} - \frac{1}{4}\partial_0\phi_{ij}\right)\partial_0\phi_{ij} - \mathcal{V}. \tag{31}$$

The new conjugated momenta are

$$\pi^{0\mu} = 0, \tag{32a}$$

$$\pi^{ij} = \frac{1}{2}\delta_{ij}\partial_0\phi_{kk} - \frac{1}{2}\partial_0\phi_{ij} + \frac{1}{2}\partial_i\phi_{0j} + \frac{1}{2}\partial_j\phi_{0i} - \delta_{ij}\partial_k\phi_{0k}. \tag{32b}$$

We have reduced the constraints (29a) and (29b) in one single constraint (32a). This fact has a close resemblance with the electromagnetic case, where the primary constraint has the form $\pi^0 = 0$. Equation (32b) is a dynamical relation, from where we get the velocities as functions of the conjugated momenta

$$\partial_0\phi_{ij} = -2\pi^{ij} + \frac{2}{(d-2)}\delta_{ij}\pi^{kk} + \partial_i\phi_{0j} + \partial_j\phi_{0i}. \tag{33}$$

We notice that (32b) is not defined in two dimensions. The canonical Hamiltonian density is given by

$$\begin{aligned}\mathcal{H}_0 \;=\; & -\left(\pi^{ij}\right)^2 + \frac{1}{(d-2)}\left(\pi^{kk}\right)^2 - 2\phi_{0i}\mathcal{C}^i + \frac{1}{2}\phi_{00}\mathcal{C}^0 \\ & -\frac{1}{4}\left(\partial_i\phi_{jk}\right)^2 + \frac{1}{4}\left(\partial_i\phi_{jj}\right)^2 + \frac{1}{2}\left(\partial_i\phi_{ij}\right)^2 - \frac{1}{2}\partial_i\phi_{ij}\partial_j\phi_{kk},\end{aligned} \tag{34}$$

where we define the functions

$$\mathcal{C}^0 \equiv \partial_i\partial_i\phi_{jj} - \partial_i\partial_j\phi_{ij}, \tag{35a}$$

$$\mathcal{C}^i \equiv \partial_j\pi^{ij}. \tag{35b}$$

In the context of the HJ formalism, we have $(d+1)$ Hamiltonian densities

$$\mathcal{H}'^{\tau} = \pi^\tau + \mathcal{H}_0 = 0, \tag{36a}$$

$$\mathcal{H}'^{0\mu} = \pi^{0\mu} = 0. \tag{36b}$$

The first relation is related to the time variable $\tau = x^0$, while the second one is related to the variables $\phi_{0\mu}$, that now stands as parameters of the theory. The fundamental PB, observing (28), are given by

$$\{\phi_{\alpha\beta}(x), \pi^{\mu\nu}(y)\} = \Delta^{\mu\nu}_{\alpha\beta}\,\delta^{d-1}(\mathbf{x}-\mathbf{y}). \tag{37}$$

All PB are computed at equal times $x^0 = y^0 = cte$.

The characteristics equations of the theory suggest the definition of the fundamental differential

$$dF = \{F, \mathcal{H}'^{\tau}\}d\tau + \{F, \mathcal{H}'^{00}\}d\phi_{00} + 2\{F, \mathcal{H}'^{0i}\}d\phi_{0i}, \tag{38}$$

where integration is implicit on the right hand side. The factor 2 in the last term is due to the symmetry of $\phi_{\mu\nu}$.

Following the next step in the HJ formalism, we test the integrability conditions for the Hamiltonian densities. We obtain

$$d\mathcal{H}'^{00} = -\frac{1}{2}\mathcal{C}^0 d\tau = 0, \tag{39a}$$

$$d\mathcal{H}'^{0i} = \mathcal{C}^i d\tau = 0. \tag{39b}$$

Then, $\mathcal{C}^\mu$ defined in (35) are new Hamiltonian densities, corresponding to the HJ equations $\mathcal{C}^\mu = 0$, and the IC have to be tested with them as well. From these new densities, the only non-zero PB is

$$\{\mathcal{C}'^0(x), \mathcal{H}'^{\tau}(y)\} = \partial_i\partial_j\pi^{ij}\ \delta^{d-1}(\mathbf{x}-\mathbf{y}) = \partial_i\mathcal{C}^i\ \delta^{d-1}(\mathbf{x}-\mathbf{y}). \tag{40}$$

This means that the IC for these Hamiltonian densities are identically satisfied and the system is considered complete.

Once we have the complete set of Hamiltonian densities (35) and (36), we are able to build the evolution of the system with the differential

$$dF = \{F, \mathcal{H}'^{\tau}\}d\tau + \{F, \mathcal{H}'^{00}\}d\phi_{00} + 2\{F, \mathcal{H}'^{0i}\}d\phi_{0i} + \{F, \mathcal{C}^\mu\}d\omega_\mu, \tag{41}$$

where ω_μ are new parameters related to the Hamiltonians $\mathcal{C}^\mu$. Again, integration is implicit on the right side. The complete set of Hamiltonian densities is in involution, i.e, the PB are identically zero or they are linear combinations of the previous Hamiltonian densities. In particular, the algebra of the generators $\mathcal{H}'^{0\mu}$ and $\mathcal{C}^\mu$ is abelian.

For this involutive system, the characteristic equations are given by (41). For $F = \phi_{\mu\nu}$ we have

$$\begin{aligned} d\phi_{\mu\nu} &= \left[-2\Delta^{ij}_{\mu\nu}\pi^{ij} + \frac{2}{d-2}\Delta^{ii}_{\mu\nu}\pi^{jj} + 2\Delta^{ij}_{\mu\nu}\partial_i\phi_{0j}\right]d\tau \\ &\quad +\Delta^{00}_{\mu\nu}d\phi_{00} + 2\Delta^{0i}_{\mu\nu}d\phi_{0i} - \Delta^{ij}_{\mu\nu}\partial_i d\omega_j. \end{aligned} \tag{42}$$

These equations reproduce the fact that $\phi_{0\mu}$ are parameters of the theory, since their velocities cannot be fixed $(d\phi_{0\mu} = d\phi_{0\mu})$. They also give us back the relation (33), as expected, apart of the term in ω_j.

For $F = \pi^{\mu\nu}$, we obtain

$$\begin{aligned} d\pi^{\mu\nu} &= \left[\frac{1}{2}\Delta^{\mu\nu}_{jj}\left(\partial_i\partial_i\phi_{kk} - \partial_i\partial_i\phi_{00} - \partial_i\partial_k\phi_{ik}\right)\right. \\ &\quad +\frac{1}{2}\Delta^{\mu\nu}_{ij}\left(\partial_i\partial_j\phi_{00} - \partial_i\partial_j\phi_{kk} - \partial_k\partial_k\phi_{ij} + 2\partial_i\partial_k\phi_{kj}\right) \\ &\quad \left.+\frac{1}{2}\Delta^{\mu\nu}_{00}\left(\partial_i\partial_j\phi_{ij} - \partial_i\partial_i\phi_{jj}\right) + 2\Delta^{\mu\nu}_{0i}\partial_j\pi^{ij}\right]d\tau \\ &\quad + \left[\Delta^{\mu\nu}_{ij}\partial_i\partial_j - \Delta^{\mu\nu}_{jj}\partial_i\partial_i\right]d\omega_0. \end{aligned} \tag{43}$$

They reproduce the EL equations (18) apart of the linear term in ω_0, as follows: the equation for π^{00} is equivalent to the first IC (35a), which is also the EL equation (18) with $\alpha = \beta = 0$. For π^{0i} the correspondent characteristic equation is equivalent to the second IC (35b), and gives the EL equation for $\alpha = 0$ and $\beta = i$. The dynamical equations of the theory are actually the equations for π^{ij}, which became the EL equation for $\alpha = i$ and $\beta = j$, when (32b) is taken account. Then, the characteristics equations are equivalent to the EL equations when appropriate parameters ω_μ are chosen.

5 LGR in front-form

In relativistic field theories we are free to choose the parameter that determines the time evolution. This freedom comes from the physical requirement of Poincaré covariance. When dealing with a field theory in flat space-time, the choice of a particular parameter τ comes with the choice of a family of surfaces $\Sigma_\tau = constant$. If we knew the configuration of the fields over one of the members of the family the field equations in canonical form should give us the evolution of this configuration on later surfaces in a unique way. It was outlined by Dirac [24] that the quantization of a relativistic field theory in instant-form is not the only kind of relativistic dynamics. In fact there are at least five inequivalent forms of Hamiltonian dynamics of relativistic field theories [25]. One of them is the front-form dynamics.

If we have a d-dimensional Minkowski space-time, the light-cone coordinates are defined by

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^{d-1}), \tag{44a}$$

$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^{d-1}), \tag{44b}$$

$$x^i = x^i \ , \ i = 1, 2, ..., d-2. \tag{44c}$$

In this, we set $\tau = x^+$ as the new time parameter, and x^- and x^i stands as spatial coordinates. The transverse coordinates are denoted by $\mathrm{x} = (x^1, ..., x^n)$, with $n = d-2$. Therefore, the dynamics of fields in this coordinate system is given by the configuration over a surface $x^+ = \tau_0$ and its evolution to later surfaces by means of a Hamiltonian function. This kind of dynamics is often called front-form, null-plane, or even light-front dynamics, and the surfaces of constant x^+ are called null-planes. Since a null-plane divides space-like and time-like vectors, the causal structure is included into the light-cone coordinates.

In order to obtain the conjugated momenta, we will separate the time and spatial coordinates from the Lagrangian density:

$$\begin{aligned}
\mathcal{L} \;=\;& \partial_+\phi_{++}\left[-\frac{1}{2}\partial_-\phi_{--}\right] + \partial_+\phi_{+-}\left[-\frac{1}{2}\partial_-\phi_{ii}\right] + \partial_+\phi_{+i}\partial_-\phi_{i-} \\
&+\partial_+\phi_{--}\left[\frac{1}{2}\partial_-\phi_{++} - \partial_i\phi_{i+} + \frac{1}{2}\partial_+\phi_{ii}\right] \\
&+\partial_+\phi_{-i}\left[-\frac{1}{2}\partial_+\phi_{-i} + \partial_k\phi_{ki} + \partial_i\phi_{+-} - \frac{1}{2}\partial_i\phi_{kk}\right] \\
&+\partial_+\phi_{ij}\left[-\frac{1}{2}\delta_{ij}\partial_-\phi_{+-} + \frac{1}{2}\delta_{ij}\partial_-\phi_{kk} - \frac{1}{2}\delta_{ik}\partial_-\phi_{kj} - \frac{1}{2}\delta_{ij}\partial_k\phi_{k-}\right] - \mathcal{V},
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
\mathcal{V} \;=\;& \phi_{++}\left[\frac{1}{2}\partial_i\partial_i\phi_{--} + \frac{1}{2}\partial_-\partial_-\phi_{ii} - \partial_-\partial_i\phi_{i-}\right] \\
&+\phi_{+-}\left[-\frac{1}{2}\partial_i\partial_i\phi_{+-} + \partial_i\partial_i\phi_{kk} + \partial_-\partial_i\phi_{+i} - \partial_i\partial_k\phi_{ik}\right] \\
&+\phi_{+i}\left[-\partial_k\partial_k\phi_{-i} - \frac{1}{2}\partial_-\partial_-\phi_{+i} + \partial_-\partial_k\phi_{ki} + \partial_i\partial_k\phi_{k-} - \partial_-\partial_i\phi_{kk}\right] \\
&+\left[\frac{1}{2}\partial_i\phi_{im}\partial_k\phi_{km} - \frac{1}{2}\partial_i\phi_{ik}\partial_k\phi_{mm} + \frac{1}{4}(\partial_i\phi_{kk})^2 - \frac{1}{4}(\partial_i\phi_{km})^2\right].
\end{aligned} \tag{46}$$

As we did in instant-form, we may perform partial integrations and eliminate surface terms in order to simplify the expressions for the momenta, obtaining the equivalent Lagrangian density

$$
\begin{aligned}
\mathcal{L} \;=\;& \partial_+\phi_{--}\left[-\partial_i\phi_{i+}+\frac{1}{2}\partial_+\phi_{ii}\right] \\
&+\partial_+\phi_{-i}\left[-\frac{1}{2}\partial_+\phi_{-i}+\partial_k\phi_{ki}+\partial_i\phi_{+-}+\partial_-\phi_{+i}\right] \\
&+\partial_+\phi_{ij}\left[-\delta_{ij}\partial_-\phi_{+-}+\frac{1}{2}\delta_{ij}\partial_-\phi_{kk}-\frac{1}{2}\partial_-\phi_{ij}-\delta_{ij}\partial_k\phi_{-k}\right]-\mathcal{V}.
\end{aligned}
\tag{47}
$$

From here we may write the momenta

$$\pi^{+\mu}=0\;, \tag{48a}$$

$$\pi^{--}=\frac{1}{2}\partial_+\phi_{ii}-\partial_i\phi_{+i}\;, \tag{48b}$$

$$\pi^{-i}=\frac{1}{2}\left(\partial_-\phi_{+i}-\partial_+\phi_{-i}+\partial_k\phi_{ik}+\partial_i\phi_{+-}\right)\;, \tag{48c}$$

$$\pi^{ij}=\frac{1}{2}\delta_{ij}\partial_+\phi_{--}-\delta_{ij}\partial_-\phi_{+-}+\frac{1}{2}\delta_{ij}\partial_-\phi_{kk}-\frac{1}{2}\partial_-\phi_{ij}-\delta_{ij}\partial_k\phi_{-k}. \tag{48d}$$

Relations (48b) and (48c) can be inverted to obtain the velocities

$$\partial_+\phi_{ii}=2\pi^{--}+2\partial_i\phi_{+i} \tag{49a}$$

$$\partial_+\phi_{-i}=-2\pi^{-i}+\partial_-\phi_{+i}+\partial_k\phi_{ik}+\partial_i\phi_{+-}. \tag{49b}$$

Relation (48d) has a peculiarity. The trace part can be inverted to obtain

$$\partial_+\phi_{--}=\frac{2}{n}\pi^{ii}-\left(\frac{n-1}{n}\right)\partial_-\phi_{ii}+2(\partial_-\phi_{+-}+\partial_i\phi_{-i}), \tag{50}$$

for $n\neq 0$ ($d\neq 2$). The traceless part, on the other hand, is a constraint

$$\bar{\pi}^{ij}+\frac{1}{2}\partial_-\bar{\phi}_{ij}=0. \tag{51}$$

Here, the bar on any tensor is defined by

$$\bar{A}_{ij}\equiv A_{ij}-\frac{1}{n}\delta_{ij}A_{kk}, \tag{52}$$

which describes its traceless part. We notice that for the four dimensional case, i.e. $n=2$, $\bar{\phi}_{ij}=h_{ij}$. Now we compute the canonical Hamiltonian density:

$$
\begin{aligned}
\mathcal{H}_\tau \;=\;& 2\pi^{-i}\left[\partial_j\bar{\phi}_{ij}+\frac{1}{n}\partial_i\phi_{kk}-\pi^{-i}\right]+\pi^{--}\left[\frac{2}{n}\pi^{kk}-\frac{n-1}{n}\partial_-\phi_{kk}+2\partial_k\phi_{-k}\right] \\
&-\phi_{++}\mathcal{C}^+-2\phi_{+-}\mathcal{C}^--2\phi_{+i}\mathcal{C}^i-\frac{1}{4}(\partial_i\bar{\phi}_{jk})^2 \\
&-\frac{1}{2}\partial_i\bar{\phi}_{ij}\partial_j\phi_{kk}+\frac{n-3}{4n}(\partial_i\phi_{jj})^2,
\end{aligned}
\tag{53}
$$

where

$$\mathcal{C}^+=\partial_i\partial_-\phi_{i-}-\frac{1}{2}\partial_-\partial_-\phi_{ii}-\frac{1}{2}\partial_i\partial_i\phi_{--}, \tag{54a}$$

$$\mathcal{C}^-=\partial_i\pi^{-i}+\partial_-\pi^{--}-\frac{1}{2}\partial_i\partial_i\phi_{jj}, \tag{54b}$$

$$\mathcal{C}^i=\partial_-\pi^{-i}+\frac{1}{n}\partial_i\left[\pi^{kk}-\frac{1}{2}\partial_-\phi_{kk}\right]-\partial_-\partial_j\bar{\phi}_{ij}+\frac{1}{2}\partial_i\partial_k\phi_{-k}+\frac{1}{2}\partial_k\partial_k\phi_{-i}. \tag{54c}$$

Following the HJ formalism we have the Hamiltonian densities

$$\mathcal{H}'^{\tau}\equiv\pi^\tau+\mathcal{H}_\tau=0, \tag{55a}$$

$$\mathcal{H}'^{+\mu}\equiv\pi^{+\mu}=0, \tag{55b}$$

$$\mathcal{Q}'^{ij}\equiv\bar{\pi}^{ij}+\frac{1}{2}\partial_-\bar{\phi}_{ij}=0. \tag{55c}$$

The first equation is related to the time parameter, the second to the $\phi_{+\mu}$ fields, and the last one to the traceless part of ϕ_{ij}. From these densities we identify the parameters of the theory and build the fundamental differential

$$\begin{aligned} dF &= \{F, \mathcal{H}'^{\tau}\}d\tau + \{F, \mathcal{H}'^{++}\}d\phi_{++} \\ &+2\{F, \mathcal{H}'^{+-}\}d\phi_{+-} + 2\{F, \mathcal{H}'^{+i}\}d\phi_{+i} + \{F, \mathcal{Q}'^{ij}\}d\bar{\phi}_{ij}. \end{aligned} \tag{56}$$

As usual, integration is implicit on the right hand side.

Now we proceed testing integrability and searching for new Hamiltonian densities. We obtain

$$d\mathcal{H}'^{++} = \mathcal{C}^{+}d\tau = 0, \tag{57a}$$

$$d\mathcal{H}'^{+-} = \mathcal{C}^{-}d\tau = 0, \tag{57b}$$

$$d\mathcal{H}'^{+i} = \mathcal{C}^{i}d\tau = 0, \tag{57c}$$

that identifies $\mathcal{C}^{+}$, $\mathcal{C}^{-}$, and $\mathcal{C}^{i}$ as new Hamiltonian densities of the system. We may write

$$\mathcal{C}^{i} = \partial_{-}\pi^{-i} + \partial_{j}\pi^{ij} + \frac{1}{2}\partial_{i}\partial_{k}\phi_{-k} + \frac{1}{2}\partial_{k}\partial_{k}\phi_{-i}, \tag{58}$$

where we have made a simplification with help of Hamiltonian (55c). The IC $d\mathcal{Q}'^{ij} = 0$ will give a relation between the parameters $\tau = x^{+}$ and $\bar{\phi}_{ij}$. This means that these parameters are not independent, and we must eliminate this dependence with apropriate GB. Testing the integrability of the generators $\mathcal{C}^{\mu}$ we may see that there are no more Hamiltonians, then the system is considered completed.

For each density (55), we have related an independent variable $(\tau, \phi_{+\mu}, \bar{\phi}_{ij})$. However, for the densities (54) we have to add a new set of variables, $(\omega_{+}, \omega_{-}, \omega_{i})$ respectively, to the theory. Therefore, we define the new fundamental differential

$$\begin{aligned} dF &= \{F, \mathcal{H}'^{\tau}\}d\tau + \{F, \mathcal{H}'^{++}\}d\phi_{++} + 2\{F, \mathcal{H}'^{+-}\}d\phi_{+-} \\ &+2\{F, \mathcal{H}'^{+i}\}d\phi_{+i} + \{F, \mathcal{Q}'^{ij}\}d\bar{\phi}_{ij} + \{F, \mathcal{C}^{\mu}\}d\omega_{\mu}. \end{aligned} \tag{59}$$

With the purpose of reducing the phase space with only the independent parameters of the theory, we have to analyze the algebra of the Hamiltonian densities. We have that $\mathcal{H}'^{+\mu}$ and $\mathcal{C}^{\mu}$ are in involution. On the other hand, the non-involutive Hamiltonian density $\mathcal{Q}'^{ij}$ satisfies

$$\{\mathcal{Q}^{ij}(x), \mathcal{Q}^{ij}(y)\} = P^{ijkl}\partial_{-}\delta(x^{-} - y^{-})\delta^{n}(\mathbf{x} - \mathbf{y}), \tag{60}$$

where P^{ijkl} is a projector tensor, since it projects any transverse tensor of rank 2 in its symmetric traceless part

$$P^{ijkl} \equiv \Delta^{ij}_{kl} - \frac{1}{n}\delta_{ij}\delta_{kl}. \tag{61}$$

This projector is not defined in the two dimensional case.

As we have mentioned, the parameters related to the non-involutive constraints can be eliminated of the dynamical evolution after we compute the GB. We start by building the matrix

$$M^{(ij,kl)}(x,y) \equiv \{\mathcal{Q}^{ij}(x), \mathcal{Q}^{kl}(y)\}\ . \tag{62}$$

The inverse is given by

$$(M^{-1})_{(ij,kl)}(x,y) = \frac{1}{2}W_{ijkl}\epsilon(x^{-} - y^{-})\delta^{n}(\mathbf{x} - \mathbf{y}) + f_{ijkl}, \tag{63}$$

where $\epsilon(x)$ is the step function and W_{ijkl} is the inverse of the projector P^{ijkl}:

$$W_{ijkl} = \frac{n}{n-1}P^{ijkl}, \tag{64}$$

which satisfies $P^{ijmn}W_{mnkl} = \Delta^{ij}_{kl}$. The existence of W_{ijkl} is assured for $n > 1$, since we can verify that P^{ijkl} is a regular matrix in this case.

The f_{ijkl} are arbitrary functions that do not depend on x^-. They appear as consequence of the null-plane dynamics because we have not specified sufficient boundary conditions to uniquely determine the evolution of the system [26]. Therefore, this inverse is not unique, but represents a family of matrices. It is possible to determine boundary conditions such that the boundary terms are zero, and a unique dynamics emerges. This behavior is characteristic of the front-form dynamics, as outlined in [27]. Let us make $f_{ijkl} = 0$, in this case the GB can be defined as

$$\begin{aligned}\{F(x), G(y)\}^* &\equiv \{F(x), G(y)\} \\ &\quad - \int dz \int dw \{F(x), \mathcal{Q}^{ij}(z)\}(M^{-1})_{(ij,kl)}(z,w)\{\mathcal{Q}^{kl}(w), G(y)\},\end{aligned} \tag{65}$$

and the fundamental GB are

$$\{\phi_{\mu\nu}, \phi_{\alpha\beta}\}^* = \frac{1}{2}\Delta^{ij}_{\mu\nu}\Delta^{kl}_{\alpha\beta}P^{ijkl}\epsilon\left(x^- - y^-\right)\delta^n\left(\mathbf{x} - \mathbf{y}\right), \tag{66a}$$

$$\left\{\phi_{\mu\nu}, \pi^{\alpha\beta}\right\}^* = \left[\Delta^{\alpha\beta}_{\mu\nu} + \frac{1}{2}\Delta^{ij}_{\mu\nu}\Delta^{\alpha\beta}_{kl}P^{ijkl}\right]\delta\left(x^- - y^-\right)\delta^n\left(\mathbf{x} - \mathbf{y}\right), \tag{66b}$$

$$\left\{\pi^{\mu\nu}, \pi^{\alpha\beta}\right\}^* = \frac{1}{2}\Delta^{\mu\nu}_{ij}\Delta^{\alpha\beta}_{kl}P^{ijkl}\partial_-\delta\left(x^- - y^-\right)\delta^n\left(\mathbf{x} - \mathbf{y}\right). \tag{66c}$$

By direct calculation, we see that these GB applied to the constraints of the theory result in a closed algebra and, therefore, all constraints become involutive: integrability is then achieved. Besides, the algebra of the involutive constraints $\mathcal{H}'^{+\mu}$ and $\mathcal{C}^\mu$ is abelian indeed. This is expected since we need the algebra and the number of involutive constraints of a relativistic theory to be independent of the choice of dynamics for a good dynamical description. This ensures that all time-preserved quantities are also independent of this choice.

With the GB, the dynamics of the system is given by the differential

$$dF = \{F, \mathcal{H}'^{\tau}\}^* d\tau + \{F, \mathcal{H}'^{+\mu}\}^* d\phi_{+\mu} + \{F, \mathcal{C}^\mu\}^* d\omega_\mu. \tag{67}$$

Then we may express the characteristics equations of the system. Let us begin with the variables $\phi_{\mu\nu}$:

$$d\phi_{+\mu} = d\phi_{+\mu}, \tag{68a}$$

$$d\phi_{--} = \left[\frac{2}{n}\pi^{kk} - \left(\frac{n-1}{n}\right)\partial_-\phi_{kk} + 2\partial_k\phi_{-k} + 2\partial_-\phi_{+-}\right]d\tau - \partial_- d\omega_-, \tag{68b}$$

$$d\phi_{-i} = \left[-2\pi^{-i} + \partial_j\phi_{ij} + \partial_i\phi_{+-} + \partial_-\phi_{+i}\right]d\tau - \frac{1}{2}\left(\partial_i d\omega_- + \partial_- d\omega_i\right). \tag{68c}$$

The first equation is expected, since the variables $\phi_{+\mu}$ are parameters related to the Hamiltonians $\mathcal{H}'^{+\mu}$. Equations (68b) and (68c) are equivalent to (50) and (49b) with proper choice of the parameters ω_- and ω_i. For the equation of the trace of ϕ_{ij} we obtain

$$d\phi_{ii} = \left[2\pi^{--} + 2\partial_m\phi_{+m}\right]d\tau - \partial_i d\omega_i, \tag{69}$$

which is just equal to equation (49a) if we set $\partial_i d\omega_i = 0$. For $i \neq j$ we have

$$\begin{aligned} d\bar{\phi}_{ij} &= \frac{1}{2}P^{ijkl}d\tau\int dy^- d^n y\,\epsilon\left(x^- - y^-\right)\delta^n\left(x - y\right) \times \\ &\quad \times \left[2\partial_-\partial_k\phi_{+l} - 2\partial_k\pi^{-l} + \frac{1}{2}\partial_k\partial_l\phi_{mm} + \frac{1}{2}\partial_m\partial_m\phi_{kl}\right] \\ &\quad - \frac{1}{2}\left(\partial_i d\omega_j + \partial_j d\omega_i\right). \end{aligned} \tag{70}$$

This is actually a dynamical equation. With some work it is possible to show that this is the equivalent EL equation (18) for $(\alpha, \beta) = (i, j)$, with $i \neq j$.

For the momenta, we have the relations

$$d\pi^{++} = \left[\partial_i\partial_-\phi_{i-} - \frac{1}{2}\partial_i\partial_i\phi_{--} - \frac{1}{2}\partial_-\partial_-\phi_{ii}\right]d\tau, \tag{71a}$$

$$d\pi^{+-} = \left[\partial_i\pi^{-i} + \partial_-\pi^{--} - \frac{1}{2}\partial_i\partial_i\phi_{kk}\right]d\tau, \tag{71b}$$

$$d\pi^{+i} = \left[\partial_-\pi^{-i} + \frac{1}{n}\partial_i\pi^{kk} - \partial_-\partial_j\phi_{ij} + \frac{1}{2n}\partial_i\partial_-\phi_{kk} + \frac{1}{2}\left(\partial_i\partial_k\phi_{-k} + \partial_k\partial_k\phi_{-i}\right)\right]d\tau. \tag{71c}$$

These equations represent the integrability conditions that give rise to the constraints $\mathcal{C}'^{\mu} = 0$. They are the non-dynamical set of EL equations.

The following equations

$$d\pi^{--} = -\frac{1}{2}\partial_k\partial_k\phi_{++}d\tau + \frac{1}{2}\partial_k\partial_k d\omega_+ \tag{72a}$$

$$d\pi^{-i} = \left[\partial_i\pi^{--} - \frac{1}{2}\partial_i\partial_-\phi_{++} + \frac{1}{2}\left(\partial_i\partial_j + \delta^i_j\partial_k\partial_k\right)\phi_{+j}\right]d\tau - \frac{1}{2}\partial_i\partial_- d\omega_+ - \frac{1}{4}\left(\partial_i\partial_j + \delta^i_j\partial_k\partial_k\right)d\omega_j \tag{72b}$$

$$d\pi^{ij} = \left[\frac{1}{2}\partial_j\pi^{-i} + \frac{1}{2}\partial_i\pi^{-j} + \frac{1}{n}\delta_{ij}\partial_k\pi^{-k} - \frac{1}{2}\partial_-\partial_j\phi_{+i} - \frac{1}{2}\partial_-\partial_i\phi_{+j} - \frac{1}{n}\delta_{ij}\partial_-\partial_k\phi_{+k} - \frac{1}{4}\partial_k\partial_k\phi_{ij} - \frac{1}{4}\partial_i\partial_j\phi_{kk} + \frac{1}{2n}\delta_{ij}\partial_k\partial_k\phi_{ll}\right]d\tau - \frac{1}{8}\partial_-\partial_j d\omega_i - \frac{1}{8}\partial_-\partial_i d\omega_j + \frac{1}{4n}\delta_{ij}\partial_-\partial_k d\omega_k, \tag{72c}$$

complete the remaining set of EL equations.

6 Final Remarks

In this work we have used the HJ formalism to analyze the constraints of linearized gravity. We found that, while the instant-form dynamics have only constraints in involution, a sub-set of Hamiltonian densities in the front-form dynamics are non-involutive. The later case becomes a good laboratory to build the GB in the context of the HJ formalism.

We have carried out the usual procedure of construction of a Lagrangian density from the properties of gauge invariance of the linearized Einstein's equations. In both forms of dynamics, we were able to modify the Lagrangian in order to obtain simplifications on the momenta, and therefore to analyze the structure of their Hamiltonian densities. Using the IC, we were able to find the complete set of Hamiltonian densities.

In instant-form, the theory has constraints that come from the IC, represented by the Hamiltonian densities (35). Together with (36), they form a complete integrable set. In particular, the densities (35) close an abelian Lie algebra with the Poisson brackets. To build the field equations, we have extended the space of parameters to embrace the independent variables related to the Hamiltonians (35). The analysis resulted to be in full accordance with the field equations (18).

In the front-form dynamics, we have found a richer structure. There was a subset of non-involutive constraints, represented by the densities (55c). With this set we have built the GB, eliminating the traceless variables $\bar{\phi}_{ij}$. As usual when describing a theory in the coordinates of the light-cone, these GB are unique only if boundary conditions are carefully chosen on a null-plane $x^- = cte$, setting to zero the arbitrary functions f_{ijkl} that appear in (63).

We also verified that the involutive constraints $\mathcal{H}'^{+\mu}$ and $\mathcal{C}^{\mu}$ obey an abelian Lie algebra, this time with respect to the generalized brackets (65). This is despite the fact that the

Hamiltonians $\mathcal{C}^i$ do not close an algebra with the Poisson brackets. It is a very good feature of the front-form description of this theory that the non-involutive constraints $\mathcal{Q}^{ij}$ are exactly those needed to ensure the correct algebra of these constraints via the definition (65). It can be seen that, for the computation of $\left\{\mathcal{C}^i, \mathcal{C}^j\right\}^*$, the second term in the right side of (65) exactly cancels the non-zero term $\left\{\mathcal{C}^i, \mathcal{C}^j\right\}$. As expected, because of the presence of the constraints $\mathcal{C}^\mu = 0$, the characteristics equations of the system have arbitrary parameters ω^μ not related to variables of the system. The fundamental differential (67) is built with the complete set of Hamiltonian functions, and gives rise to characteristics equations that are again equivalent to the field equations (18).

Acknowledgments

MCB thanks UFABC and FAPESP for partial support. BMP thanks CNPq and CAPES for partial support. CEV thanks CAPES for full support. GERZ thanks VIPRI-UDENAR for full support.

References

[1] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).
P. A. M. Dirac, Can. J. Math. **3**, 1 (1951).
P. A. M. Dirac, Proc. Roy. Soc. A **246**, 326 (1958).
P. A. M. Dirac - *Lectures on Quantum Mechanics*, Yeshiva University, New York, 1964.

[2] P. A. M. Dirac, Proc. Roy. Soc. Lond. A **19**, 333 (1958).

[3] G. Nordström, Phys. Zeit. **13**, 1126 (1912).

[4] M. Fierz, W. Pauli, Proc. Roy. Soc. Lond. A **173**, 211 (1939).

[5] T. Padmanabhan, Int. J. Mod. Phys. D **17**, 367 (2008).

[6] C. Kiefer, *Quantum Gravity*, Oxford University Press, 2^{nd} ed. 2007.

[7] D. Grumiller, W. Kummer, D.V. Vassilevich, Phys. Rep. **369**, 327 (2002).

[8] S. Deser, R. Jackiw, S. Templeton, Phys. Rev. Lett. **48**, 975 (1982).
S. Deser, R. Jackiw, S. Templeton, Ann. Phys. **140**, 372 (1982). *Erratum-ibid.* **185**, 406 (1988).

[9] E. A. Bergshoeff, O. Hohm, P. K. Townsend, Phys. Rev. Lett. **48**, 975 (2009).
E. A. Bergshoeff, O. Hohm, P. K. Townsend, Phys. Rev. D **79**, 124042 (2009).

[10] S. Giombi, A. Maloney, X. Yin, *JHEP* **08**, 007 (2008).
M. R. Gaberdiel, D. Grumiller, D. Vassilevich, *JHEP* **11**, 094 (2010).

[11] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
B. L. G. Bekker, *Forms of Relativistic Dynamics*, Lecture Notes in Physics, **572**, Springer, New York (2001).

[12] P. P. Srivastava, S. J. Brodsky, Phys. Rev. D **64**, 045006 (2001).
S. J. Brodsky, Braz. J. Phys. **34**, 157 (2004).

[13] Th. Heinzl, St. Krusche, E. Werner, Phys. Lett. B **272**, 54 (1991).

[14] Y. Güler, Il Nuovo Cimento B **100**, 251 (1987).
Y. Güler, J. Math. Phys. **30**, 785 (1992).
Y. Güler, Il Nuovo Cimento B **107**, 1398 (1992).

[15] C. Carathéodory, *Calculus of Variations and Partial Differential Equations of the First Order*, American Mathematical Society 3^{rd} ed. (1999).

[16] M. C. Bertin, B. M. Pimentel, P. J. Pompeia, Mod. Phys. Lett. A **20**, 2873 (2005).

[17] M. C. Bertin, B. M. Pimentel, C. E. Valcárcel, Ann. Phys. **323**, 3137 (2008).

[18] B. M. Pimentel, R. G. Teixeira, Il Nuovo Cimento B **111**, 841 (1996).
B. M. Pimentel, R. G. Teixeira, Il Nuovo Cimento B **113**, 805 (1998).
B. M. Pimentel, R. G. Texeira, J. L. Tomazelli, Ann. Phys. **267**, 75 (1998).
B. M. Pimentel, P. J. Pompeia, J.F. da Rocha-Neto, R.G. Teixeira, Gen. Rel. Grav. **35**, 877 (2003).
B. M. Pimentel, P. J. Pompeia, J. F. da Rocha-Neto, Il Nuovo Cimento B **120**, 981 (2005).
M. C. Bertin, B. M. Pimentel, P. J. Pompeia, Ann. Phys. **323**, 527 (2008).
M.C. Bertin, B.M. Pimentel, P.J. Pompeia, Ann. Phys. **325**, 2499 (2010).

[19] D. Baleanu, Y. Güler, Il Nuovo Cimento B **117**, 917 (2002).

[20] S.I. Muslih. Gen. Rel. Grav. **36**, 1151 (2004).

[21] A. S. Mishchenko, A. T. Fomenko, Func. Analysis Appl. **12**, 46 (1978).

[22] D. Evens, G. Kunstatter, C. Torre, Class. Quant. Grav. **4**, 1503 (1987).

[23] J. L. Anderson, Phys. Rev. **111**, 965 (1958).

[24] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[25] B. L. G. Bekker, *Forms of Relativistic Dynamics*, Lecture Notes in Physics, **572**, Springer, New York (2001).

[26] P. J. Steinhardt, Ann. Phys **128**, 425 (1980).

[27] R. Casana, B. M. Pimentel, G. E. R. Zambrano, Int. J. Mod. Phys. E **16**, 2993 (2007).
M. C. Bertin, B. M. Pimentel, C. E. Valcárcel, G. E. R. Zambrano, PoS(ISFTG) **046** (2009).
M. C. Bertin, B. M. Pimentel, G. E. R. Zambrano, PoS(ISFTG) **077** (2009).
R. Casana, B. M. Pimentel, G. E. R. Zambrano, Nucl. Phys. B (Proc. Suppl.) **119**, 219 (2010).