Hamilton-Jacobi formalism for Podolsky's electromagnetic theory on the null-plane

M. C. Bertin[1], B. M. Pimentel[2], C. E. Valcárcel[3], G. E. R. Zambrano[4]

[1]*Instituto de Física, Universidade Federal da Bahia,*
Câmpus Universitário de Ondina, 40210-340, Salvador - BA, Brazil.

[2]*Instituto de Física Teórica, UNESP - São Paulo State University,*
Caixa Postal 70532-2, 01156-970, São Paulo, SP, Brazil.

[3]*Centro de Matemática, Computação e Cognição,*
Universidade Federal do ABC, 09210-170, Santo André, SP, Brazil

[4]*Departamento de Física, Universidad de Nariño,*
Calle 18 Carrera 50, San Juan de Pasto, Nariño, Colombia.

Abstract

We develop the Hamilton-Jacobi formalism for Podolsky's electromagnetic theory on the null-plane. The main goal is to build the complete set of Hamiltonian generators of the system, as well as to study the canonical and gauge transformations of the theory.

Contents

[1]mbertin@ufba.br
[2]pimentel@ift.unesp.br
[3]carlos.valcarcel@ufabc.edu.br
[4]gramos@udenar.edu.co

1 Introduction

In this paper, we analyse the null-plane canonical structure of the generalised Podolsky's electrodynamics *via* the Hamilton-Jacobi (HJ) formalism for singular systems. Since we are dealing with three separate subjects, each one deserving proper treatment by itself, we separate the work in three main parts.

In the first part, section 2, it is presented the general Hamilton-Jacobi theory for singular and higher-order lagrangian systems, as it is the case of Podolsky's theory. The HJ formalism is well known from classical mechanics to be a road for the study of integrability of classical systems, but its fundamental role in dynamical systems was discovered only after Carathéodory's work [1] on variational principles and first-order partial differential equations (PDEs). Carathéodory built the HJ theory directly from Hamilton's principle, showing that it is actually the theory that relates first-order PDEs, first-order ordinary differential equations (ODEs), and lagrangian variational problems. This unifying point of view is called the *complete figure of the variational calculus*, and congregates all analytic, algebraic, and geometric pictures of mechanics [2].

Considering singular systems, which are theories whose Lagrange function has singular Hessian matrices, the HJ theory does not exhibit the same fundamental problems seen when the purely hamiltonian picture is considered [3, 4]. Carathéodory's complete figure is still valid, and a general treatment of constraints is naturally emergent [5]. In the HJ theory, canonical constraints are seen as a set of Hamilton-Jacobi partial differential equations (HJPDE), and the integrability of this set is achieved by an integrability theorem, leading to Frobenius' integrability conditions [6, 7]. Advantages of the HJ approach come from the fact that it provides a full theoretical stage to the canonical formulation of singular systems, rather than a consistency construction. One of the advantages is the absence of the so called Dirac's conjecture. Also, gauge fixing is not required for equivalence between the canonical and lagrangian descriptions at the classical level. Moreover, second and higher derivative lagrangian systems were treated *via* the HJ formalism in [8]. High-order theories were introduced by Ostrogradsky [9], and have been used since for many

purposes. Among them, we may cite developments in high-order gauge theories [10], attempts to solve the problem of renormalisation of the gravitational field [11], and recently quantum gravity massive theories [12].

Podolsky's theory has developed by Bopp [13], and independently by Podolsky and Schwed [14] as a second-order gauge theory for the electromagnetic field, in order to treat the r^{-1} dependency in the electrostatic potential. Because of this behaviour, the energy necessary to assemble a system of point charges has infinite contribution of the charge's self-energy. This problem is related to divergences in the infrared sector of the quantum electrodynamics (QED), as well to the problem of infinite vacuum polarisation current. The theory proposed is described by the lagrangian density

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}a^2\partial_\lambda F^{\mu\lambda}\partial^\gamma F_{\mu\gamma}, \tag{1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as the components of the electromagnetic tensor field, and a being a parameter with dimension of the inverse of mass. The second-order derivative term in (1) results in a well defined electrostatic potential for $r = 0$, so the self-energy contribution may be computed, it is just proportional to q^2/a for each charge. The theory may be interpreted as an effective theory for short distances [15], as a way to get rid of the problems related to the $r = 0$ singularity in QED. In this case, the parameter a becomes a cut-off distance for the theory. When $r \gg a$, the theory would become experimentally indistinguishable from Maxwell's. In this point of view, a is linked to an effective radius for the electron.

If interpreted as a fundamental theory, the parameter a is related to a sector of massive photons, which can be seen by the dispersion relation

$$p^2\left(p^2 - m_\gamma^2\right) = 0, \qquad p^2 \equiv p_\mu p^\mu, \qquad m_\gamma^2 \equiv 1/a^2, \tag{2}$$

taken from the generalised wave equation, result of the field equations of the lagrangian (1). This relation indicates two kinds of photons, with modes $p^2 = 0$ e $p^2 = m_\gamma^2$ respectively. The first mode corresponds to massless photons, the second mode is linked to photons with mass parameter m_γ. Massive photons are not observed in nature, so it is not generally believed that Podolsky's theory is fundamental. Yet, experimental attempts to provide upper limits to m_γ are taken from time to time [16]. The best result, however, is given by Luo *et al* [17], which has $m_f < 2.1 \times 10^{-51}g$.

In section 3 we discuss forms of relativistic dynamics, specially how to project generators of the Poincaré group when a given form of dynamics is defined in Minkowski space-time. The first attempts of quantisation of Podolsky's field were made in instant-form, where time is defined by the x^0 axis as the evolution parameter. There are five different forms of such hamiltonian dynamics, each one related to different decompositions of the Poincaré group [18]. The dynamics on the null-plane, also called the front-form dynamics, is the hamiltonian dynamics of fields over a null-plane $x^0 + x^3 = cte$. The evolution parameter is chosen to be the coordinate $x^+ \equiv 1/\sqrt{2}\left(x^0 + x^3\right)$, where the classical (quantum) evolution of the system is given by the definition of appropriate fundamental "equal-time" brackets (commutators), defined on a null-plane of constant x^+, plus a special set of initial-

boundary data.

There are some good reasons, both physical and mathematical, to analyse field theories on the null-plane. One of them is the fact that this kind of dynamics usually reduces the number of independent degrees of freedom necessary to describe a field [19, 20]. This is closely related to the fact that the stability group of the Poincaré group in front-form, which is the sub-group of transformations that relates field configurations in a single surface $x^+ = cte$, has seven generators, one more than the six kinematic generators in instant-form. Besides, the algebra of these kinematic generators takes its simplest form in front-form dynamics. For some important systems this feature is responsible for a complete separation of physical degrees of freedom, resulting in a clean and excitation-free quantum vacuum.

We let the application of the null-plane HJ formalism for section 4. The canonical structure of Podolsky's theory was already studied in [21], in instant-form, and in [22] on the null-plane, both using Dirac's method. However, these papers had the goal of analysing proper gauge conditions for the theory, in order to clarify the physical degrees of freedom. Our main purpose here, using the HJ theory, is to obtain a complete set of involutive generators, or hamiltonians. We see that the presence of a non-involutive subset of constraints demands modification of the dynamics of the system, with the introduction of generalised brackets (GBs). With these brackets, we are able to find the complete set of generators, which close a Lie algebra with the GBs, assuring integrability as stated by Frobenius' theorem. On the other hand, any complete set of hamiltonians generate canonical transformations, so we also present a way of defining these transformations and relate them to the gauge transformations of Podolsky's lagrangian. This is done without any mention to Dirac's conjecture, or even the necessity of ad-hoc methods, such as the case of Castellani's procedure [23]. In section 5 we present our final remarks.

2 The HJ theory for singular systems and higher-order actions

2.1 Higher-order theories

Let us start with a system described by n generalised coordinates q^i. In general, the coordinates should be at least of class C^{2k} in a time parameter τ, but for now we may define the variables q_I^i as the I-th derivative of q^i with respect to τ, where $I = \{0, 1, \cdots, k\}$ for a given $k \in \mathbb{N}$. A high-order lagrangian theory, in this case of order k, is a theory described by the action

$$A[\gamma] = \int_\gamma d\tau L\left[\tau, q_I^i(\tau)\right],$$ (3)

which is a functional of curve segments $\gamma : q^i = q^i(\tau)$ in the space of generalised coordinates.

Supposing $\delta\tau \equiv \bar{\tau} - \tau$ and $\delta q^i \equiv \bar{q}^i(\bar{\tau}) - q^i(\tau)$ general first-order variations of the time

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and coordinates, we have the first-order variation,

$$\delta A = \int d\tau \left[\frac{\delta L}{\delta q^i} \left(\delta - \delta\tau \frac{d}{d\tau} \right) q^i + \frac{d}{d\tau} \left(\frac{\delta L}{\delta q^i_{(I+1)}} \delta q^i_I - H\delta\tau \right) \right]. \tag{4}$$

In (4), the derivatives

$$\frac{\delta}{\delta q^i} \equiv \sum_{I=0}^{k} (-1)^I \frac{d^I}{d\tau^I} \frac{\partial}{\partial q^i_I}, \tag{5}$$

are known as the Lagrange derivatives, and

$$H \equiv \frac{\delta L}{\delta q^i_{(I+1)}} q^i_{(I+1)} - L, \tag{6}$$

is defined as the hamiltonian function of the system. Summation is implicit for all repeated indexes, except in some few expressions, as (5). If the action $A[\gamma]$ has an extreme configuration γ_0, this configuration is a solution of the Euler-Lagrange (EL) equations

$$\frac{\delta L}{\delta q^i} = 0, \tag{7}$$

which are the equations of motion of the action (3).

2.2 Equivalent lagrangians

The HJ formalism in Carathéodory's point of view [1] requires a way of defining equivalent actions of the functional (3). This is done by supposing a point-transformation $q^i \to q'^i$ with generating function $S\left(\tau, q^i_{I'}\right)$. With this transformation, the new lagrangian is related to the old one by $L' = L - dS/d\tau$. Here we introduce another index $I' = \{0, 1, \cdots, k-1\}$, which will be useful ahead. Adding a total derivative to the lagrangian does not change the equations of motion, so if γ_0 is an extreme configuration of the action $\int L d\tau$, it is also an extreme configuration of the transformed action $\int L' d\tau$. If this point-transformation leads to $L' = 0$ for $\gamma = \gamma_0$, and $L' > 0$ or $L' < 0$ for any $\gamma \neq \gamma_0$ in a close neighbourhood of γ_0, $A[\gamma]$ has, respectively, a local minimum ($L' > 0$) or a local maximum ($L' < 0$) in γ_0. Expanding L' in a Taylor series around γ_0, we see that the new lagrangian is approximated by a lower-order term that is quadratic positive (or negative) definite in δq^i_I. The necessary conditions for the extreme configuration become

$$\left. \frac{\partial L'}{\partial q^i_I} \right|_{\gamma=\gamma_0} = 0, \tag{8}$$

which result in

$$\frac{\partial S}{\partial q^i_{I'}} = \frac{\partial L}{\partial q^i_{(I'+1)}} - \frac{d}{d\tau} \left(\frac{\partial S}{\partial q^i_{(I'+1)}} \right), \tag{9}$$

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at $\gamma = \gamma_0$.

If we take $I' = k - 1$, (9) becomes

$$\frac{\partial S}{\partial q^i_{(k-1)}} = \frac{\partial L}{\partial q^i_k}, \tag{10}$$

For $I' = k - 2$ we have

$$\frac{\partial S}{\partial q^i_{(k-2)}} = \frac{\partial L}{\partial q^i_{(k-1)}} - \frac{d}{d\tau}\left(\frac{\partial S}{\partial q^i_{(k-1)}}\right).$$

With (10), the result is

$$\frac{\partial S}{\partial q^i_{(k-2)}} = \frac{\partial L}{\partial q^i_{(k-1)}} - \frac{d}{d\tau}\frac{\partial L}{\partial q^i_k} = \frac{\delta L}{\delta q^i_{(k-1)}},$$

where we recognise the Lagrange derivative on the right. Following this iterative process, the general relations are found to be

$$\frac{\partial S}{\partial q^i_{I'}} = \frac{\delta L}{\delta q^i_{(I'+1)}}. \tag{11}$$

On the other hand, $L' = L - dS/d\tau$ immediately leads to

$$L - \frac{\partial S}{\partial \tau} - q^i_{(I'+1)}\frac{\partial S}{\partial q^i_{I'}} = 0. \tag{12}$$

Therefore, the existence of a function $S\left(\tau, q^i_{I'}\right)$ that obeys the conditions (11) and (12) is a necessary condition for the existence of an extreme configuration of the action (3).

2.3 The HJ equation

The HJ formalism may be reached if we make of eq. (12) a PDE for the function S. In principle, we expect solving this problem with expressions of the type $q^i_{(I'+1)} = \phi^i_{(I'+1)}\left(\tau, q^i_{I'}, \partial S/\partial q^i_{I'}\right)$, that would be taken from (11). This, however, is not the case. Let us take again the expression (11) for $I' = k - 1$:

$$\frac{\partial S}{\partial q^i_{(k-1)}} = \frac{\delta L}{\delta q^i_k} = \frac{\partial L}{\partial q^i_k} = \psi_i\left(\tau, q^i_{I'}, q^i_k\right),$$

where ψ is function of τ and the variables q^i_I. This equation can indeed be inverted to produce the expression

$$q^i_k = \phi^i_k\left(\tau, q^i_{I'}, \frac{\partial S}{\partial q^i_{(k-1)}}\right), \tag{13}$$

if the Hessian condition

$$\det\left(\frac{\partial\psi_i}{\partial q_k^j}\right) = \det\left(\frac{\partial^2 L}{\partial q_k^i \partial q_k^j}\right) \neq 0 \tag{14}$$

is satisfied.

However, eq. (11) for $I' = k - 2$ yields

$$\frac{\partial S}{\partial q_{(k-2)}^i} = \frac{\partial L}{\partial q_{(k-1)}^i} - \frac{\partial^2 L}{\partial\tau\partial q_k^i} - q_{(I'+1)}^j \frac{\partial^2 L}{\partial q_{I'}^j \partial q_k^i} - q_{(k+1)}^j \frac{\partial^2 L}{\partial q_k^j \partial q_k^i}, \tag{15}$$

where a derivative superior to k appears in the last term. This equation provides

$$q_{(k+1)}^j = \phi_{(k+1)}^j \left(\tau, q_{I'}^i, \frac{\partial S}{\partial q_{(k-1)}^i}, \frac{\partial S}{\partial q_{(k-2)}^i}\right), \tag{16}$$

if, again, (14) is satisfied. All orders have the same behaviour. Particularly, the lowest order $I' = 0$ results in

$$q_{(2k)}^i = \phi_{(2k)}^i \left(\tau, q_{I'}^i, \frac{\partial S}{\partial q_{I'}^i}\right).$$

Therefore, the best we can do is to invert (11) to obtain expressions for the variables of the theory from order k to $2k$. The condition for that is the Hessian matrix

$$W_{ij} \equiv \frac{\partial^2 L}{\partial q_k^i \partial q_k^j} \tag{17}$$

to be non-singular.

This is not a problem if we consider the variables $q_{I'}^i$ independent coordinates of a configuration space $\mathbb{Q}$. This interpretation is also suggested by the end-point term in (4), which defines the structure of the canonically conjugated variables of the theory: that sum is a linear combination of independent variations $\delta q_{I'}^i$ only up to order $I = k - 1$, since L depends of the coordinates up to order k. In this case, (12) is just written by

$$\frac{\partial S}{\partial\tau} + \frac{\partial S}{\partial q_{(I'-1)}^i} q_{I'}^i + \frac{\partial S}{\partial q_{(k-1)}^i} \phi_k^i - L\left[\tau, q_{I'}^i, \phi_k^i\right] = 0, \tag{18}$$

and no other equation is needed.

Now let us observe the hamiltonian function defined in (6), which can be written using (11) by

$$H = \frac{\delta L}{\delta q_{I'}^i} q_{I'}^i + \frac{\delta L}{\delta q_k^i} \phi_k^i - L = \frac{\partial S}{\partial q_{(I'-1)}^i} q_{I'}^i + \frac{\partial S}{\partial q_{(k-1)}^i} \phi_k^i - L \tag{19}$$

for $I' > 0$. Substituting (13), we may write (18) as

$$\frac{\partial S}{\partial \tau} + H\left(\tau, q_{I'}^i, \frac{\partial S}{\partial q_{I'}^i}\right) = 0. \tag{20}$$

Eq. (20) is the expected Hamilton-Jacobi equation.

The introduction of canonical variables is straightforward. In the HJ formalism, the momenta conjugated to the variables $q_{I'}^i$ are defined as the components of the gradient of the function S,

$$p_i^{I'} \equiv \frac{\partial S}{\partial q_{I'}^i} = \frac{\delta L}{\delta q_{(I'+1)}^i}. \tag{21}$$

This definition results in the canonical hamiltonian

$$H_0\left(\tau, q_{I'}^i, p_i^{I'}\right) \equiv p_i^{(I'-1)} q_{I'}^i + p_i^{(k-1)} \phi_k^i - L, \tag{22}$$

with all q_k^i substituted by ϕ_k^i as functions of the momenta. It also results in the canonical HJ equation

$$\Phi_0\left(\tau, p_\tau, q_{I'}^i, p_i^{I'}\right) \equiv \pi_0 + H_0\left(\tau, q_{I'}^i, p_i^{I'}\right) = 0, \tag{23}$$

where we introduce the notation $\pi_0 \equiv \partial S / \partial \tau$.

2.4 The HJ equations for singular systems

The Hessian condition appears in the HJ formalism as a condition for the existence of the PDE (20). But in the case of violation of the Hessian condition (14), implying the existence of constraints, it is still straightforward to define a HJ approach. Let us suppose the $n \times n$ Hessian matrix W_{ij} to be singular of rank $m < n$. Then, there is a regular sub-matrix $m \times m$ and a null-space of dimension $r = n - m$, splitting the configuration space in two subspaces: the space of the variables $q_{I'}^a$, for $a = \{1, \cdots, m\}$, which will be called $\mathbb{Q}_m$, and the space of the variables $t_{I'}^z \equiv q_{I'}^z$ with $z = \{1, \cdots, r\}$, which will be called Γ_r. The former are variables belonging to the regular space of the Hessian, related to the matrix

$$W_{ab} \equiv \frac{\partial^2 L}{\partial q_k^a \partial q_k^b}, \tag{24}$$

which obeys $\det W_{ab} \neq 0$. The later are variables belonging to the null-space of the same Hessian matrix.

In the singular case, eq. (11) produce the $m \cdot k$ relations

$$q_{(k+I')}^a = \phi_{(k+I')}^a\left(\tau, t_{I'}^z, q_{I'}^a, p_a^{I'}\right), \qquad p_a^{I'} \equiv \frac{\partial S}{\partial q_{I'}^a}, \tag{25}$$

but also the $r \cdot k$ identities

$$\frac{\partial S}{\partial t_{I'}^z} = \frac{\delta L}{\delta q_{(I'+1)}^z} \equiv -H_z^{I'} \left(\tau, t_{I'}^z, q_{I'}^a, p_a^{I'} \right). \tag{26}$$

In canonical form, (26) become

$$\Phi_z^{I'} \left(\tau, t_{I'}^y, q_{I'}^a, \pi_y^{I'}, p_a^{I'} \right) \equiv \pi_z^{I'} + H_z^{I'} = 0, \quad \pi_z^{I'} \equiv \frac{\partial S}{\partial t_{I'}^z}. \tag{27}$$

Eqs. (27) form a set of $r \cdot k$ canonical constraints, and also a set of $r \cdot k$ first-order PDEs.

We notice that the canonical hamiltonian function (19) does not depend on the variables t_k^z, so the definition of this function is not dependent of the Hessian condition, and the HJ equation (23) is still valid. Let us introduce the notation $t_{I'}^\alpha = \left(\tau, t_{I'}^z \right)$, then $\alpha = \{0, 1, \cdots, r\}$. Since the momentum conjugated to τ is named π_0, we may also include these as new variables in the theory as $\pi_\alpha^{I'} = \left(\pi_0, \pi_z^{I'} \right)$. Then, eqs. (23) and (24) may be written in a unified way,

$$\Phi_\alpha^{I'} \left(t_{I'}^\beta, q_{I'}^a, \pi_\beta^{I'}, p_a^{I'} \right) \equiv \pi_\alpha^{I'} + H_\alpha^{I'} \left(t_{I'}^\beta, q_{I'}^a, p_a^{I'} \right) = 0, \tag{28}$$

where we also use $H_\alpha^{I'} = \left(H_0, H_z^{I'} \right)$. Eqs. (28) form a set of Hamilton-Jacobi first-order partial differential equations.

2.5 Integrability and characteristic equations

The HJ equations (28) are necessary, but still not sufficient for the existence of extreme configurations of the action $A[\gamma]$. It is still necessary that (28) provides at least one complete solution for the S function. These equations are generally a set of $(r+1) \cdot k$ non-linear coupled PDEs of the first-order, so we expect that a complete solution contains $(r+1) \cdot k$ constants of integration related to $(r+1) \cdot k$ linearly independent parameters. In other words, we expect that a complete solution has the form $S = S \left[t_{I'}^\alpha, q_{I'}^a \left(t_{I'}^\alpha \right) \right]$ for a set of $(r+1) \cdot k$ parameters $t_{I'}^\alpha$ and a set of $m \cdot k$ variables $q_{I'}^a \left(t_{I'}^\alpha \right)$. Of course, this is possible only if $\Phi_\alpha^{I'}$ form a set of $(r+1) \cdot k$ linearly independent equations. For the following, there is no need to carry the I' index. We just write the generators $\Phi_\alpha \equiv \Phi_\alpha^{I'}$, in which now $\alpha = 0, \cdots, (r+1) \cdot k$, and use the compact notation $q^i \equiv \left(t_{I'}^\alpha, q_{I'}^a \right) = (t^\alpha, q^a)$ and $p_i \equiv \left(\pi_\alpha^{I'}, p_a^{I'} \right) = (\pi_\alpha, p_a)$.

The necessary and sufficient conditions for complete integrability are the Frobenius' integrability conditions [7]

$$\{\Phi_\alpha, \Phi_\beta\} = C_{\alpha\beta}{}^\gamma \Phi_\gamma, \tag{29}$$

in which $C_{\alpha\beta}{}^\gamma$ is a set of structure coefficients. The brackets are the complete Poisson brackets (PB)

$$\{A, B\} \equiv \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q^i} \frac{\partial A}{\partial p_i}. \tag{30}$$

If (29) holds, Φ_α form a complete set of constraints in involution with PB operation. We name these set involutive constraints. Therefore, Frobenius' conditions are resumed in the fact that Φ_α are generators of a Lie algebra with the PB.

If $F\left(q^i, p_i\right)$ is an observable of the complete phase-space, its dynamics is given by the fundamental differential

$$dF = \{F, \Phi_\alpha\}\, dt^\alpha, \tag{31}$$

so the dynamics takes place in the complete phase-space of the variables $\xi^i \equiv \left(t^\alpha_{I'}, q^a_{I'}, \pi^{I'}_\alpha, p^{I'}_a\right)$, where $\Phi^{I'}_\alpha\left(\xi\right)$ are the generators, and $t^\alpha_{I'}$ are the evolution parameters. This phase-space is actually degenerate. Let us calculate (now with explicit indexes I')

$$dq^a_{I'} = \left\{q^a_{I'}, \Phi^{J'}_\alpha\right\} dt^\alpha_{J'} = \left\{q^a_{I'}, p^{K'}_b\right\} \frac{\partial \Phi^{J'}_\alpha}{\partial p^{K'}_b} dt^\alpha_{J'}. \tag{32}$$

Using the definition (30), we see that $\left\{q^a_{I'}, p^{K'}_b\right\} = \delta^a_b \delta^{K'}_{I'}$, so

$$dq^a_{I'} = \frac{\partial \Phi^{J'}_\alpha}{\partial p^{I'}_a} dt^\alpha_{J'}. \tag{33}$$

On the other hand,

$$dp^{I'}_a = \left\{p^{I'}_a, \Phi^{J'}_\alpha\right\} dt^\alpha_{J'} = \left\{p^{I'}_a, q^b_{K'}\right\} \frac{\partial \Phi^{J'}_\alpha}{\partial q^b_{K'}} dt^\alpha_{J'} = -\frac{\partial \Phi^{J'}_\alpha}{\partial q^a_{I'}} dt^\alpha_{J'}. \tag{34}$$

Eqs. (33) and (34) are the generalisation of Hamilton's equations for singular systems. They are generally called the characteristic equations of the system (28). A third fundamental characteristic equation is given by

$$dS = \left\{S, \Phi^{J'}_\alpha\right\} dt^\alpha_{J'} = p^{I'}_a \, dq^a_{I'} + \pi^{I'}_\alpha \, dt^\alpha_{I'} - \Phi^{I'}_\alpha \, dt^\alpha_{I'}, \tag{35}$$

and results in an integral equations for S if (33) and (34) provide solutions for $p^{I'}_a\left(t^\alpha_{I'}\right)$ and $q^a_{I'}\left(t^\alpha_{I'}\right)$. The CE for $t^\alpha_{I'}$ and $\pi^{I'}_\alpha$ are just identities, which reveals the degenerate character of the complete phase space.

There is, however, a reduced phase-space which is not degenerate. It is the space of the variables $\xi^A \equiv \left(q^a_{I'}, p^{I'}_a\right)$. In fact, it is shown in [7] that the sector of the variables $\left(t^\alpha_{I'}, \pi^{I'}_\alpha\right)$ in the complete phase space has zero volume element. The reason for that is the fact that $t^\alpha_{I'}$ form a parameter space which is isomorphic to a complete affine vector space. In other words, the solutions of the characteristic equations are trajectories in the reduced phase-space, of the type $\xi^A = \xi^A\left(t^\alpha_{I'}\right)$, parametrised by the variables $t^\alpha_{I'}$. For this reason, we call ξ^A the dependent variables, and $t^\alpha_{I'}$ the independent variables of the theory. Usually, the dependent variables are related to the "true" degrees of freedom of a physical system.

2.6 The generalised brackets

The Frobenius' theorem implies the HJ equations $\Phi_\alpha = 0$ must be complete and linearly independent. However, in physical examples this condition is not usually satisfied. If the set $\Phi = \{\Phi_\alpha\}$ is not integrable, two possible reasons are: (1) there may be other HJ equations not included in this set, case in which the equations are not complete, or (2) a subset of Φ is not linearly independent. In this case, since a physical system must be integrable, the aim is to discover a complete set of involutive constraints. For that, we use the procedure outlined in [6].

If the HJ equations $\Phi_\alpha = 0$ are valid, Frobenius' condition $\{\Phi, \Phi\} \subset \Phi$ implies the generators are dynamical invariants, in other words,

$$d\Phi_\alpha = \{\Phi_\alpha, \Phi_\beta\} dt^\beta = 0. \tag{36}$$

Eq. (36) is a set of linear equations for the differential of the independent variables,

$$M_{\alpha\beta} dt^\beta = 0, \tag{37}$$

where we introduced the matrix $M_{\alpha\beta} \equiv \{\Phi_\alpha, \Phi_\beta\}$. Remember that, in our case, this matrix has dimension $(r+1) \cdot k$.

We suppose the general case in which $M_{\alpha\beta}$ has rank $p \leq (r+1) \cdot k$. Then, there is an invertible sub-matrix M_{xy} whose entries are the PB between a subset $\{\Phi_x\} \subset \Phi$, with $x = 1, \cdots, p$. In this case, for $\alpha = x$ (37) becomes

$$M_{x\beta} dt^\beta = 0 \quad \Longrightarrow \quad M_{xy} dt^y = -M_{x\alpha'} dt^{\alpha'}, \tag{38}$$

in which $\alpha' = 1, \cdots, q$ is the index of the null space of $M_{\alpha\beta}$. Then, (37) can be written as

$$dt^x = -\left(M^{-1}\right)^{xy} \{\Phi_y, \Phi_{\alpha'}\} dt^{\alpha'}, \tag{39}$$

where $\left(M^{-1}\right)^{xy}$ is the inverse matrix related to M_{xy}. Therefore, if the set Φ is not integrable, the former independent variables $t^\alpha \equiv \left(t^x, t^{\alpha'}\right)$ are not mutually independent, due to the fact that the matrix $M_{\alpha\beta}$ has non-zero rank. In fact, (39) can be written for any set of non involutive constraints, not only in the case of maximal rank.

Let us write (37) for $\alpha = \alpha'$,

$$M_{\alpha'\beta} dt^\beta = 0 \quad \Longrightarrow \quad M_{\alpha'x} dt^x + M_{\alpha'\beta'} dt^{\beta'} = 0. \tag{40}$$

Using (39), we have

$$\left[\{\Phi_{\alpha'}, \Phi_{\beta'}\} - \{\Phi_{\alpha'}, \Phi_x\} \left(M^{-1}\right)^{xy} \{\Phi_y, \Phi_{\beta'}\}\right] dt^{\beta'} = 0. \tag{41}$$

Now we introduce the generalised brackets (GB) between two observables A and B:

$$\{A, B\}^* \equiv \{A, B\} - \{A, \Phi_x\} \left(M^{-1}\right)^{xy} \{\Phi_y, B\}, \tag{42}$$

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then we may write (40) as

$$\left\{\Phi_{\alpha'}, \Phi_{\beta'}\right\}^* dt^{\beta'} = 0. \tag{43}$$

It is straightforward to show that the GB are good brackets. They are antisymmetric bilinear differential operators that obey the Jacobi identity.

Suppose we have chosen a subset $\{\Phi_x\} \subset \Phi$ that is not of maximal rank. In this case, (43) yields again two sets of equations, one of them will give rise to another generalised brackets, since not all linear combinations of the independent variables was discovered. The other set is just (43) again. When all linear combinations of the parameters are eliminated by a final GB, the differentials $dt^{\alpha'}$ in (43) are truly linearly independent. Then, the integrability conditions are just given by

$$\left\{\Phi_{\alpha'}, \Phi_{\beta'}\right\}^* = 0. \tag{44}$$

Eqs. (44) may give two results. It may be identically satisfied for some generators, but it may result in relations between the phase-space variables, which should be taken as new constraints. If found, these new HJ equations should be added to the set $\Phi' \equiv \{\Phi_{\alpha'}\}$. Eqs. (43) should be used again, since it is possible that the new set Φ' is non-involutive, and new GB may be defined. This process must be repeated until (44) gives no more constraints, completing the system of generators, which will happen if the system is really integrable. In this case, the remaining set of constraints obeys the conditions

$$\left\{\Phi_{\alpha'}, \Phi_{\beta'}\right\}^* = C_{\alpha'\beta'}{}^{\gamma'} \Phi_{\gamma'}, \tag{45}$$

this time with the GB. Therefore, if the system is integrable we must be able to reduce it to a set of complete involutive constraints with a GB.

With a complete involutive set, the fundamental differential of an observable F is given by

$$dF = \{F, \Phi_\alpha\} dt^\alpha = \{F, \Phi_{\alpha'}\}^* dt^{\alpha'}, \tag{46}$$

with use of (39). Note that the generators belonging to the invertible part of the matrix $M_{\alpha\beta}$, the ones that are not involutive, are now completely eliminated since the GB of a non involutive constraint with any observable of the phase-space is identically zero. If a system has non involutive constraints, the characteristic equations must be calculated from (46). Therefore, a subset of non involutive constraints is responsible to modify the dynamics of the system.

2.7 Canonical and lagrangian symmetries

If Φ is a complete set of involutive constraints, it is shown in [7] that each member of the set is a generator of an active canonical transformation on the reduced phase-space, called a characteristic flow (CF). This is better seen in a geometrical framework: the symplectic

structure of the complete phase-space is a degenerate 2-form $\omega = -d\theta$, where $\theta = dS$ is the fundamental lepagian 1-form defined by the CE (35). The symplectic form is degenerate since it can be written as the sum of a regular 2-form $\omega_P = dq^a \wedge dp_a$, in canonical coordinates, and a singular 2-form $a = dt^\alpha \wedge d\pi_\alpha$ when the HJ equations $\Phi_\alpha = 0$ are satisfied. In fact, the 2-form a is identically a zero form when the Frobenius' conditions (45) are satisfied. The 2-form ω_P defines the symplectic structure of the reduced phase-space.

A canonical transformation is any change in the phase-space that preserves its symplectic structure. To relate each Φ_α to a canonical transformation, we define a vector field $X_\alpha \equiv \{\bullet, \Phi_\alpha\}^*$ to each Φ_α (the $\bullet$ symbol takes the place of a phase-space observable). If the set Φ is completely integrable, the Lie algebra (45) of the generators implies a Lie algebra of the vector fields X_α,

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha = f_{\alpha\beta}{}^\gamma X_\gamma, \tag{47}$$

if the structure coefficients $f_{\alpha\beta}{}^\gamma = -C_{\alpha\beta}{}^\gamma$ are independent of the complete phase-space variables. Supposing (47) holds, any operator with the form $T \equiv \exp(\Delta t^\alpha X_\alpha)$ is a member of the group of canonical transformations if, and only if, $[\omega, T] = 0$, which actually is the case. Even if (47) is not obeyed, it is shown that $[X_\alpha, \omega] = 0$, which implies the symplectic structure is preserved by the infinitesimal first-order transformation

$$\delta\xi = \delta t^\alpha X_\alpha \xi. \tag{48}$$

We call (48) a characteristic flow generated by the vector fields X_α. Note that the CE (33), (34) themselves have the form of characteristic flows $d\xi = dt^\alpha X_\alpha \xi$. Therefore, the Frobenius' theorem implies each Φ_α generates its own CF, related to an independent variable t^α, and each of these flows is independent of the others. Moreover, a general CF is a linear combination of the flows generated by each X_α, with the form (48).

A special CF is defined as follows. Suppose $\delta t^0 = \delta\tau = 0$ and $\delta t^z = \epsilon^z$, where ϵ^z form a set of LI constant parameters. In this case,

$$\delta\xi = \epsilon^z X_z \xi = \{\xi, \Phi_z\}^* \epsilon^z. \tag{49}$$

The transformation (49) is a canonical flow if the generators Φ_z close themselves a Lie algebra $\{\Phi_x, \Phi_y\}^* = C_{xy}{}^z \Phi_z$. On the other hand, the only way to assure this algebra is if it is restricted to the reduced phase-space, where $\Phi_\alpha = 0$ holds. In this case, all algebrae become abelian. The transformation (49), restricted to the reduced phase-space where $\{\Phi_x, \Phi_y\}^* = 0$ is a characteristic flow, which was called "point transformations" by Dirac, in his hamiltonian picture [3]. The generator of the point transformation (49) is given by $G = \Phi_z \epsilon^z$, since $\delta\xi = \{\xi, G\}$ reproduces (49) when $\Phi_z = 0$.

If the original system has lagrangian symmetries, e.g. local gauge symmetries in field theories, we expect these symmetries manifest themselves as canonical symmetries, in this case characteristic flows in the HJ picture. In this case, eq. (4) for $\delta\tau = 0$ yields the Lie

equation

$$\delta L = \frac{\delta L}{\delta q^i} \delta q^i + \frac{d}{d\tau} \left(\frac{\delta L}{\delta q^i_{,(I+1)}} \delta q^i_I \right) = 0, \tag{50}$$

and it has to be satisfied by the point transformation (49) for $\delta q^i = \epsilon^z X_z q^i = \left\{ q^i, G \right\}^*$.

3 Forms of relativistic dynamics

3.1 Poincaré generators in field theories

In sec. 2, we revised the HJ theory for singular systems in a classical mechanical background. But since we propose the study of Podolsky's electrodynamics, our particular interest rests in second-order relativistic field theories, described by the functional

$$A[\phi] \equiv \int_\Omega L \left[x^\mu, \phi^a(x), \phi^a_\mu(x), \phi^a_{\mu\nu}(x) \right] d\omega. \tag{51}$$

In (51), Ω is a volume of an $(d+1)$-dimensional Minkowski space-time $\mathcal{M}$ spanned by a rectangular coordinate system $x^\mu = \left(x^0, x^1, x^2, \cdots, x^d \right)$ with volume element $d\omega \equiv dx^0 dx^1 \cdots dx^d$. The Lagrangian density L depends generally on the space-time coordinates x^μ, the fields $\phi^a(x)$ at each point $x \in \Omega$, its first $\left(\phi^a_\mu \equiv \partial_\mu \phi^a \right)$ and second $\left(\phi^a_{\mu\nu} \equiv \partial_\mu \partial_\nu \phi^a \right)$ derivatives. If we are dealing with a relativistic field, L must be Lorentz invariant. The action (51) is distinct from (3) because the domain is a set of fields of several variables, and the integral itself is a multiple integral. For this paper, the metric has signature $(+ - --)$.

With (51), the natural path for a HJ formalism in Carathéodory's point of view would be to define equivalent actions. But unlike the case of classical mechanics, there is not a unique way of implementing Carathéodory's programme in a covariant way. The simplest method, introduced by Weyl [24], is by adding to A a simple boundary term. A second way, developed by Carathéodory himself [2], would be adding a determinant to the Lagrangian density. Each method leads to distinct definitions of geodesic fields, resulting in distinct HJ theories. Carathéodory's choice leads to a Hamiltonian function defined by the determinant of the energy-momentum tensor. Weyl's choice is the usual one, because it is analog to the classical mechanical method. Moreover, it does not suffer from important limitations of the alternative approach, e.g. the requirement of positivity of the Lagrangian density. But even if we choose Weyl's definition of geodesic fields, an important characteristic of relativistic theories makes impracticable the construction of a covariant HJ theory with constraints. These theories are invariant under any particular choice of global evolution parameter, or time. In this case, the HJ equation becomes an identity, not a PDE. An attempt to solve this problem is also done in [2], but it relies on unphysical assumptions if applied to fundamental fields. Until now, no satisfactory way of defining a covariant HJ formalism for constrained systems is at hand.

This problem is closely related to the problem of construction of covariant Hamiltonian formalisms with constraints. Dirac was the first to conclude that a consistent Hamiltonian

approach for field theories requires a choice of relativistic dynamics [18]. Such choice is essentially a choice of "time" parameter τ as a linear combination of the space-time axes, defined in the direction of a vector field $u(x)$. This vector may be related to the world line of a physical observer in a referential frame, but this is not necessarily the case: a given frame of reference can use different forms of dynamics to describe a physical system. Along with the vector u, one chooses a family Σ of d-dimensional surfaces orthogonal to $u(x)$ at each point $x \in \Omega$. Each member $\Sigma_\tau \in \Sigma$ must be labeled uniquely by a value of τ, then we must choose the family so that the world line of a given observer intersects each family member once.

When the vector u and the family Σ are chosen, the symmetry group of the space-time may be decomposed. If $T_{\mu\nu}$ are the components of the energy-momentum tensor the time evolution generator, the hamiltonian function, is defined by the double projection

$$H \equiv \int_{\Sigma_\tau} d\sigma \, u_\mu T^\mu_\nu u^\nu, \tag{52}$$

where $d\sigma_x$ is the d-volume element of Σ_τ. If w is a unit vector orthogonal to u,

$$P_w \equiv \int_{\Sigma_\tau} d\sigma \, u_\mu T^\mu_\nu w^\nu \tag{53}$$

defines the linear momentum in the direction of w, which generates translations in Σ_τ in the w direction.

Let $\{v_i(x)\}$ be a base of vector fields defined in Σ_τ for a given point x, where $i, j = 1, \cdots, d$. In this case,

$$L_{ij} \equiv \int_{\Sigma_\tau} d\sigma \, u_\mu \left(T^\mu_\alpha x_\beta - T^\mu_\beta x_\alpha \right) v_i^\alpha v_j^\beta \tag{54}$$

is the orbital angular momentum matrix, and

$$S_{ij} \equiv - \int_{\Sigma_\tau} d\sigma \, u_\mu \left[\frac{\delta L}{\delta \phi^a_\mu} \phi^a_\nu \left(J_{\alpha\beta} \right)^\nu_\gamma x^\gamma \right] v_i^\alpha v_j^\beta \tag{55}$$

is the spin matrix of the fields, where $\left(J_{\alpha\beta} \right)^\nu_\gamma$ are the generators of the Lorentz group. The matrix $M_{ij} = L_{ij} + S_{ij}$ generates rotations in Σ_τ. We also have boosts, generated by

$$B_j \equiv \int_{\Sigma_\tau} d\sigma \, u_\mu \left[T^\mu_\alpha x_\beta - T^\mu_\beta x_\alpha - \frac{\delta L}{\delta \phi^a_\mu} \phi^a_\nu \left(J_{\alpha\beta} \right)^\nu_\gamma x^\gamma \right] u^\alpha v_j^\beta, \tag{56}$$

which are pseudo-rotations in the plane defined by v_j and u.

3.2 Null-plane dynamics

Because of the causal structure of Minkowski's metric, there is not a unique way of defining a hamiltonian dynamics. The most simple and used one was called by Dirac [18] the instant-form dynamics. It is the dynamics of fields in 3-dimensional euclidian spaces orthogonal to the time axis $t = x^0/c$, the time measured by a clock at rest with respect to the

laboratory. In this case, u is given by the components $(1,0,0,0)$, while the base vectors v_i are the usual $\{x^1, x^2, x^3\}$ axes. The hamiltonian function is given by $H = \int_{\Sigma_\tau} d\sigma\, T_{00}$, and the linear momenta are just $P_i = \int_{\Sigma_\tau} d\sigma\, T_{0i}$. Generators of rotations and boosts are also immediately calculated from (54), (55) and (56).

The null-plane dynamics, on the other hand, is the form of dynamics where u lies in the light-cone. We define time as the parameter

$$x^+ = \frac{1}{\sqrt{2}} \left(x^0 + x^3 \right), \tag{57}$$

so u takes the form of the axis

$$u^\mu = \frac{1}{\sqrt{2}} \left(1, 0, 0, 1 \right). \tag{58}$$

The equation $x^+ = const.$ defines a characteristic hyper-surface Σ_{x^+} orthogonal to the u axis. This surface is called the null-plane, and it represents an electromagnetic wave front in vacuum.

With this choice of evolution parameter and characteristic surfaces it is convenient to choose an appropriate coordinate system. Let x^μ be the rectangular coordinates $\left(x^0, x^1, x^2, x^3 \right)$, and let us consider the following transformation

$$y^\mu = \Gamma^\mu_{\ \nu} x^\nu, \tag{59}$$

with

$$\Gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \mathbf{I}_{2 \times 2}\sqrt{2} & 0 \end{pmatrix}, \tag{60}$$

in which $\mathbf{I}$ is the 2×2 identity matrix, and y^μ is defined by the set $\left(x^+, x^-, x^1, x^2 \right)$. We see that $x^\pm = 1/\sqrt{2} \left(x^0 \pm x^3 \right)$, while the remaining coordinates are unchanged. The coordinates y^μ are called null-plane coordinates.

The null-plane dynamics was mistaken, for some time, with a limiting process known as the infinite momentum frame [20]. But the transformation (59) is not a reference choice. It is actually a parameter choice with a combining coordinate system. It is not even a Lorentz transformation, since the metric in y is given by

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\mathbf{I}_{2 \times 2} \end{pmatrix}. \tag{61}$$

With this metric, the norm of a Lorentz vector is not a quadratic form in the temporal $(+)$ and longitudinal $(-)$ components, but remains quadratic in the transverse components 1

and 2:

$$A^2 = \eta_{\mu\nu} A^\mu A^\nu = 2A^+ A^- - \left(A^i \right)^2, \qquad i = 1, 2.$$

We stress that the metric (61) is the metric used to raise and lower indexes in the null-plane coordinates.

For example, the D'Lambertian operator assumes the form

$$\Box \equiv \partial_\mu \partial^\mu = \partial_+ \partial^+ + \partial_- \partial^- - \nabla^2 = 2\partial_+ \partial_- - \nabla^2, \tag{62}$$

where $\nabla \equiv \partial_i \partial_i$. In this case, the null-plane dynamics is distinct of the usual instant-form. The Klein-Gordon equation, for instance,

$$\left(\Box + m^2 \right) \phi = 0,$$

has the null-plane form

$$\partial_+ \partial_- \phi = \frac{1}{2} \left(\nabla^2 - m^2 \right) \phi.$$

This is a first-order equation in the temporal coordinate x^+, therefore the initial value problem is not a Cauchy problem, but a characteristic value problem. To fix a unique solution, it is necessary to provide a field configuration in a plane $x^+ = const.$, and a second configuration in some $x^- = const.$ plane, for example.

In the null-plane we may write expressions for the hamiltonian function and the linear momenta, based on eqs. (52) and (53). The x^+ evolution generator is given by

$$H = \int_\Sigma d\sigma \, T^+_{\ +} = \int_\Sigma d\sigma \, T_{-+}, \quad d\sigma = dx^- dx^1 dx^2, \tag{63}$$

while translations are generated by

$$P_- = \int_\Sigma d\sigma \, T^+_{\ -} = \int_\Sigma d\sigma \, T_{--}, \quad P_i = \int_\Sigma d\sigma \, T^+_{\ i} = \int_\Sigma d\sigma \, T_{-i}. \tag{64}$$

Generators of rotations and boosts may be calculated with eqs. (54), (55) and (56).

4 Podolsky's electromagnetic theory on the null-plane

4.1 Conjugated momenta

Podolsky's electromagnetic theory is described by the Lagrangian density

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} a^2 \partial_\lambda F^{\mu\lambda} \partial^\gamma F_{\mu\gamma}, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{65}$$

in which A_μ are the fundamental fields, components of a $U(1)$ gauge connection. The fields $F_{\mu\nu}$ are components of the curvature field strength related to the connection field.

Therefore, the density (65) is second-order in the fields A_μ, and preserves the $U(1)$ gauge symmetry. In fact, all relativistic second-order Lagrangians with the $U(1)$ symmetry are equivalent to (65) [10].

The action related to (65) is given by

$$A = \int_\Omega d\omega \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} a^2 \partial_\lambda F^{\mu\lambda} \partial^\gamma F_{\mu\gamma} \right), \tag{66}$$

where Ω is a 4-volume in Minkowski space-time with 4-volume element $d\omega$. The general first variation of this functional, analogue to eq. (4), is given by

$$\begin{aligned} \delta A &= \int_\Omega d\omega \frac{\delta L}{\delta A_\mu} \left(\delta - \delta x^\gamma \partial_\gamma \right) A_\mu \\ &\quad + \int_\Omega d\omega \partial_\gamma \left[\frac{\delta L}{\delta (\partial_\gamma A_\mu)} \delta A_\mu + \frac{\delta L}{\delta (\partial_\gamma \partial_\nu A_\mu)} \delta (\partial_\nu A_\mu) - H^\gamma_\mu \delta x^\mu \right], \end{aligned} \tag{67}$$

where

$$\frac{\delta}{\delta A_\mu} = \frac{\partial}{\partial A_\mu} - \partial_\gamma \frac{\partial}{\partial (\partial_\gamma A_\mu)} + \partial_\gamma \partial_\lambda \frac{\partial}{\partial (\partial_\gamma \partial_\lambda A_\mu)} \tag{68}$$

is the Lagrange derivative up to the second-order derivative term, and

$$H^\alpha_\beta \equiv \frac{\delta L}{\delta (\partial_\alpha A_\mu)} \partial_\beta A_\mu + \frac{\delta L}{\delta (\partial_\alpha \partial_\gamma A_\mu)} \partial_\beta \partial_\gamma A_\mu - L \delta^\alpha_\beta \tag{69}$$

is the general form of the energy-momentum tensor density.

Observing (69), we have the following covariant momenta

$$\pi^{\mu\nu} = \frac{\delta L}{\delta (\partial_\mu A_\nu)} = F^{\mu\nu} - 2a^2 \eta^{\omega\gamma} \Upsilon^{\alpha\mu}_{\beta\omega} \Delta^{\nu\lambda}_{\alpha\gamma} \partial_\lambda \partial_\rho F^{\beta\rho}, \tag{70a}$$

$$\pi^{\mu\nu\lambda} = \frac{\delta L}{\delta (\partial_\mu \partial_\nu A_\lambda)} = 2a^2 \eta^{\omega\gamma} \Upsilon^{\alpha\mu}_{\beta\omega} \Delta^{\nu\lambda}_{\alpha\gamma} \partial_\rho F^{\beta\rho}, \tag{70b}$$

where we use the symbols

$$\Upsilon^{\alpha\beta}_{\mu\nu} \equiv \frac{1}{2} \left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu \right), \qquad \Delta^{\alpha\beta}_{\mu\nu} \equiv \frac{1}{2} \left(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu \right). \tag{71}$$

In this case, we may also write

$$H_{\alpha\beta} = F^\mu_\alpha A_{\mu,\beta} + a^2 \eta_{\alpha\tau} \Upsilon^{\tau\mu}_{\epsilon\nu} \partial^\epsilon \partial_\lambda F^{\nu\lambda} A_{\mu,\beta} + 2a^2 \eta^{\gamma\nu} \eta_{\alpha\phi} \Delta^{\lambda\phi}_{\mu\nu} \Upsilon^{\mu\epsilon}_{\psi\gamma} \partial_\tau F^{\psi\tau} A_{\epsilon,\lambda\beta} - \eta_{\alpha\beta} L, \tag{72}$$

as the non symmetric energy-momentum tensor density. The field equations are written by

$$\frac{\delta L}{\delta A_\mu} = \left[1 + a^2 \Box \right] \partial_\alpha F^{\mu\alpha} = 0. \tag{73}$$

In the null-plane dynamics, where the time axis is the unit vector $u^\alpha = (1, 0, 0, 0)$ in

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null-plane coordinates, the Hamiltonian function is given by the expression

$$H = \int_\Sigma d\sigma H^\alpha_\beta u_\alpha u^\beta = \int_\Sigma d\sigma H^+_+. \tag{74}$$

Again, Σ is a 3-surface of constant x^+, and $d\sigma \equiv dx^- dx^1 dx^2$ its respective volume element. The Hamiltonian density takes the form

$$\begin{aligned}
\mathcal{H}_c &\equiv H^+_+ = \pi^{\mu+} A_{\mu,+} + \pi^{\mu+\nu} A_{\mu,+\nu} - L \\
&= \left[\pi^{\mu+} - \partial_-\pi^{\mu+-} - \partial_i\pi^{\mu+i}\right] A_{\mu,+} + \pi^{\mu++} A_{\mu,++} - L,
\end{aligned} \tag{75}$$

where $i = 1, 2$. This Hamiltonian is precisely of the form $\mathcal{H}_c = p^\mu \bar{A}_\mu + \pi^\mu \left(\partial_+\bar{A}_\mu\right) - L$, with

$$p^\mu \equiv \pi^{\mu+} - \partial_-\pi^{\mu+-} - \partial_i\pi^{\mu+i}, \tag{76a}$$

$$\pi^\mu \equiv \pi^{\mu++}. \tag{76b}$$

Then, we identify (p^μ, π^μ) as the momenta conjugated to the variables $\left(A_\mu, \bar{A}_\mu\right)$, respectively. They have the explicit expressions

$$\pi^\mu = a^2 \eta^{\mu+} \partial_\rho F^{+\rho}, \tag{77a}$$

$$p^\mu = F^{\mu+} + \partial_+\pi^\mu + 2a^2 \left(\delta^\mu_\beta \partial^+ - \frac{1}{4}\delta^+_\beta \partial^\mu - \frac{1}{2}\eta^{\mu\alpha}\Delta^{+\lambda}_{\alpha\beta}\partial_\lambda\right) \partial_\rho F^{\beta\rho}. \tag{77b}$$

4.2 The HJ equations

If we write (77) in the null-plane components, we have the following conjugated momenta:

$$\pi^+ = 0, \tag{78a}$$

$$\pi^i = 0, \tag{78b}$$

$$\pi^- = a^2 \partial_\lambda F^{+\lambda}, \tag{78c}$$

$$p^+ = \partial_-\pi^-, \tag{78d}$$

$$p^- = F^{-+} + 2a^2 \partial^+ \partial_\lambda F^{-\lambda}, \tag{78e}$$

$$p^i = F^{i+} + 2a^2 \partial^+ \partial_\lambda F^{i\lambda} - \partial_i\pi^-. \tag{78f}$$

These are the variables conjugated to $\bar{A}_+$, $\bar{A}_i$, $\bar{A}_-$, A_+, A_- and A_i respectively. We observe that the momenta π^+, π^i and p^+ are not invertible, thus, we associate them to the canonical constraints

$$\pi^+ = 0, \quad \pi^i = 0, \quad p^+ - \partial_-\pi^- = 0. \tag{79a}$$

From p^-, on the other hand, we obtain an expression for $\partial_+\bar{A}_-$,

$$\partial_+\bar{A}_- = \left(2a^2\partial_-\right)^{-1}\left[p^- - \bar{A}_- + \left(1 - 2a^2\nabla^2\right)\partial_-A_+ + 2a^2\partial_-\partial_-\bar{A}_+ + 2a^2\partial_-\partial_i\bar{A}_i\right], \tag{80}$$

where $\left(2a^2\partial_-\right)^{-1}$ represents a Green's function of the operator $2a^2\partial_-$, which depends on a

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set of initial/boundary conditions, the characteristic data of the fields. Without specifying these data, (80) is actually a family of solutions. We will need to fix the characteristic data later when analysing the HJ equations of the system, and in fact the expression (80) is not required in any of the following analysis. In this case, we let the initial/boundary analysis for later.

Now, the expression for p^i can be written in the form

$$p^i = F_{-i} + \partial_i \pi^- + 2a^2 \partial_- \left(2\partial_- \bar{A}_i - \partial_i \bar{A}_- - \partial_- \partial_i \bar{A}_+ + \partial_j F_{ij} \right). \tag{81}$$

This expression does not provide any velocity, therefore, (79) and (81) constitute the following set of constraints

$$\phi_1 = \pi^+, \tag{82a}$$

$$\phi_2^i = \pi^i, \tag{82b}$$

$$\phi_3 = p^+ - \partial_- \pi^-, \tag{82c}$$

$$\phi_4^i = p^i - \partial_i \pi^- + F_{i-} + 2a^2 \partial_- \left[\partial_i \bar{A}_- - 2\partial_- \bar{A}_i + \partial_i \partial_- A_+ - \partial_j F_{ij} \right]. \tag{82d}$$

Since

$$\pi^\mu \equiv \frac{\partial S}{\partial \bar{A}_\mu}, \quad p^\mu \equiv \frac{\partial S}{\partial A_\mu}, \tag{83}$$

eqs. (82) are a set of HJ equations.

Another HJ equation is necessary, the one related to the canonical Hamiltonian density (75), which has the explicit form

$$\mathcal{H}_c = p^\mu \bar{A}_\mu + \pi^- \left(\partial_- \bar{A}_+ + \partial_i \bar{A}_i - \nabla^2 A_+ \right) - \frac{1}{2} a^2 \partial_\lambda F^{i\lambda} \partial^\gamma F_{i\gamma} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{84}$$

where $\nabla^2 = \partial_i \partial_i$. Note that this density has no explicit dependency of any velocity, the same happening with the densities in (82). Then, there is no need for (80) in the construction of the constraints. In this case, we simply write the set of HJ equations

$$\phi_0 = p_0 + \mathcal{H}_c, \tag{85a}$$

$$\phi_1 = \pi^+, \tag{85b}$$

$$\phi_2^i = \pi^i, \tag{85c}$$

$$\phi_3 = p^+ - \partial_- \pi^-, \tag{85d}$$

$$\phi_4^i = p^i - \partial_i \pi^- + F_{i-} + 2a^2 \partial_- \left[\partial_i \bar{A}_- - 2\partial_- \bar{A}_i + \partial_- \partial_i A_+ - \partial_j F_{ij} \right], \tag{85e}$$

where $p_0 \equiv \partial S / \partial x^+$ is the momentum conjugated to the time variable x^+.

4.3 Generalised Brackets

To be a complete integrable system, (85) must be a complete involutive set with the complete Poisson brackets

$$
\{F, G\} \equiv \int_\Sigma d\sigma_x \left[\frac{\partial F}{\partial A_\mu(x)} \frac{\partial G}{\partial p^\mu(x)} - \frac{\partial G}{\partial A_\mu(x)} \frac{\partial F}{\partial p^\mu(x)} + \right.
$$
$$
\left. + \frac{\partial F}{\partial \bar{A}_\mu(x)} \frac{\partial G}{\partial \pi^\mu(x)} - \frac{\partial G}{\partial \bar{A}_\mu(x)} \frac{\partial F}{\partial \pi^\mu(x)} \right], \tag{86}
$$

for any observables F and G of the complete phase space of the variables $\left(A_\mu, \bar{A}_\mu, p^\mu, \pi^\mu\right)$. Clearly, the fundamental PB are

$$
\left\{A_\mu(x), p^\nu(y)\right\} = \left\{\bar{A}_\mu(x), \pi^\nu(y)\right\} = \delta_\mu^\nu \delta^3(x - y), \tag{87}
$$

in which $\delta^3(x - y) \equiv \delta(x^- - y^-) \delta(x^1 - y^1) \delta(x^2 - y^2)$ is the appropriate Dirac's delta.

When calculating the matrix $M \equiv \{\phi_A, \phi_B\}$, with $\phi_A = \left(\phi_0, \phi_1, \phi_2^i, \phi_3, \phi_4^i\right)$, we see that ϕ_1 is in involution, but not the remaining constraints. Particularly, the subset $\left(\phi_2^i, \phi_3, \phi_4^i\right)$ is non involutive, but it gives rise to a singular sub-matrix, indicating that a particular linear combination of these constraints is integrable. However, we have the subset $\left(\phi_2^i, \phi_4^i\right)$, which obeys the relations

$$
\left\{\phi_2^i(x), \phi_4^j(y)\right\} = -4a^2 \eta^{ij} \partial_-^x \partial_-^x \delta^3(x - y), \tag{88a}
$$
$$
\left\{\phi_4^i(x), \phi_4^j(y)\right\} = 2\eta^{ij} \left[1 - 2a^2 \nabla_x^2\right] \partial_-^x \delta^3(x - y). \tag{88b}
$$

This particular subset of HJ equations give rise to the matrix

$$
M_{IJ}^{ij} \equiv 2\eta^{ij} \begin{pmatrix} 0 & -2a^2 \partial_-^x \partial_-^x \\ 2a^2 \partial_-^x \partial_-^x & \partial_-^x \left[1 - 2a^2 \nabla_x^2\right] \end{pmatrix} \delta^3(x - y), \quad I, J = 2, 4. \tag{89}
$$

A matrix $\left(M^{-1}\right)_{ij}^{IJ}$ obeying

$$
\int d^3 z M_{IJ}^{ij}(x, z) \left(M^{-1}\right)_{jk}^{JK}(z, y) = \int d^3 z \left(M^{-1}\right)_{jk}^{JK}(x, z) M_{IJ}^{ij}(z, y) = \delta_k^i \delta_I^K \delta^3(x - y), \tag{90}
$$

if unique, is defined as the inverse matrix of M_{IJ}^{ij}. A solution for (90) is given by

$$
\left(M^{-1}\right)_{ij}^{IJ}(x, y) \equiv \frac{1}{2} \eta_{ij} \begin{pmatrix} \alpha(x, y) & \beta(x, y) \\ \gamma(x, y) & 0 \end{pmatrix}, \tag{91}
$$

with

$$
\alpha(x, y) = \frac{1}{4a^2} \left|x^- - y^-\right|^2 \epsilon(x^- - y^-) \left[1 - 2a^2 \nabla_x^2\right] \delta^2(\mathbf{x} - \mathbf{y}), \tag{92a}
$$
$$
\beta(x, y) = -\gamma(x, y) = \frac{1}{a^2} \left|x^- - y^-\right| \delta^2(\mathbf{x} - \mathbf{y}). \tag{92b}
$$

In (92) we have $\delta^2 (\mathbf{x} - \mathbf{y}) \equiv \delta (x^1 - y^1) \delta (x^2 - y^2)$, and the sign function

$$\epsilon (x - y) \equiv \begin{cases} 1 & x > y \\ 0 & x = y \\ -1 & x < y \end{cases} . \tag{93}$$

To each of the functions (92) we may add arbitrary functions of x^+, x^1, and x^2, and polynomials of x^- up to second-order. Therefore, we do not have a unique inverse. These functions can be fixed using proper boundary conditions for the fields in a null plane of constant x^-. We choose these conditions to be $\partial_- A_\mu = \partial_-^2 A_\mu = \partial_-^3 A_\mu = 0$ for $x^- \to -\infty$. In this case, we may treat (91) as an appropriate inverse matrix.

The GB related to the subset (ϕ_2^i, ϕ_4^i) are given by

$$\{F, G\}^* \equiv \{F, G\} - \iint_\Sigma d\sigma_x d\sigma_y \{F, \phi_I^i (x)\} (M^{-1})_{ij}^{IJ} (x, y) \{\phi_J^j (y), G\}, \tag{94}$$

and result in the fundamental relations

$$\{A_\mu (x), \bar{A}_\nu (y)\}^* = -\frac{1}{2} \eta_{\mu j} \delta_\nu^j \beta (x, y), \tag{95a}$$

$$\{A_\mu (x), p^\nu (y)\}^* = \delta_\mu^\nu \delta^3 (x - y), \tag{95b}$$

$$\{\bar{A}_\mu (x), \bar{A}_\nu (y)\}^* = \frac{1}{2} \eta_{\mu j} \delta_\nu^j \alpha (x, y) + \Delta_{\mu\nu}^{j-} \partial_j^x \beta (x, y), \tag{95c}$$

$$\{\bar{A}_\mu (x), p^\nu (y)\}^* = \frac{1}{2} \delta_\mu^j \delta_-^\nu \partial_i^x \beta (x, y) \tag{95d}$$

$$- a^2 \delta_\mu^i \left[\delta_k^\nu \partial_k^x \partial_i^x - \delta_+^\nu \partial_i^x \partial_- + \delta_i^\nu \left(\frac{1}{2a^2} - \nabla_x^2 \right) \right] \partial_-^x \beta (x, y), \tag{95e}$$

$$\{\bar{A}_\mu (x), \pi^\nu (y)\}^* = \delta_\mu^\nu \delta^3 (x - y) + a^2 \delta_\mu^i \left[\delta_-^\nu \partial_i^x - 2 \delta_i^\nu \partial_-^x \right] \partial_-^x \beta (x, y). \tag{95f}$$

Since the subset (ϕ_2^i, ϕ_4^i) is eliminated, we have $\{\phi_1, \phi_3\}^* = 0$ and $\{\phi_0, \phi_1\}^* = \phi_3$. However,

$$\{\phi_0 (x), \phi_3 (y)\}^* = - \left[\partial_-^x p^- (x) + \partial_i^x p^i (x) \right] \delta^3 (x - y). \tag{96}$$

Eq. (96) implies a new HJ equation, named

$$\Phi (x) \equiv -\partial_-^x p^- (x) - \partial_i^x p^i (x) = 0. \tag{97}$$

Now we have the extended, and renamed set of constraints

$$\Phi_0 \equiv p_0 + \mathcal{H}_c, \tag{98a}$$

$$\Phi_1 \equiv \pi^+ (x), \tag{98b}$$

$$\Phi_2 \equiv p^+ (x) - \partial_-^x \pi^- (x), \tag{98c}$$

$$\Phi_3 \equiv -\partial_-^x p^- (x) - \partial_i^x p^i (x). \tag{98d}$$

Calculating the GB between them, we see that this set is complete and in involution with

the GB operation. Therefore, we have completed the task at hand, which was to find a complete integrable set of HJ equations for the system.

4.4 The field equations

With the GB (94), the fundamental differential

$$dF\left(x\right) = \int_{\Sigma} d\sigma_y \left\{F\left(x\right), \Phi_A\left(y\right)\right\}^* dt^A\left(y\right), \qquad A = \left(0, 1, 2, 3\right), \tag{99}$$

gives the dynamical evolution of any observable $F\left(A, \bar{A}, p, \pi\right)$ of the phase space of the system. The set of HJ equations (98), represented here by the set $\Phi_A\left(A, \bar{A}, p, \pi\right) = 0$, is completely integrable due to the analysis made in sec. 4.3, and provides a set of dynamical generators for the evolution of the system. They may also be called Hamiltonians of the system. Each Hamiltonian Φ_A is related to an independent variable (or parameter) t^A, which is completely arbitrary in principle. We know, of course, that some of the parameters must be related to some of the phase space variables, and, particularly, $t^0 = x^+$ is found when $F = x^+$ in (99). The parameter t^3, on the other hand, is not part of the original set of independent variables, since Φ_3 is a generator found later in the integrability analysis, and no phase space variable can be related to it, in principle. Nevertheless, the parameter space is expanded to contain as many parameters as the number of involutive constraints of the system, as long as integrability is assured. Moreover, the Frobenius' theorem implies linear independence of the independent variables, so the evolution in a given "direction" does not depend on other "directions" of the parameter space.

For A_μ, we have the characteristic equations

$$dA_\mu = \bar{A}_\mu dt^0 + \delta_\mu^+ dt^2 + \left(\delta_\mu^- \partial_- + \delta_\mu^i \partial_i\right) dt^3. \tag{100}$$

Since $t^0 = x^+$, time evolution is given by

$$\partial_+ A_+ = \bar{A}_+, \quad \partial_+ A_- = \bar{A}_-, \quad \partial_+ A_i = \bar{A}_i, \tag{101}$$

as expected. The characteristic equation associated to $\bar{A}_\mu$ is

$$d\bar{A}_\mu = \delta_\mu^- \left[\partial_- \bar{A}_+ + \partial_i \bar{A}_i - \nabla^2 A_+\right] dt^0 + \delta_\mu^+ dt^1 + \left[\delta_\mu^- \partial_- + \delta_\mu^i \partial_i\right] dt^2. \tag{102}$$

Time evolution alone yields

$$\partial_+ \bar{A}_+ = 0, \quad \partial_+ \bar{A}_i = 0, \quad \partial_+ \bar{A}_- = \partial_- \bar{A}_+ + \partial_i \bar{A}_i - \nabla^2 A_+, \tag{103}$$

which are expected. Particularly, the $\partial_+ \bar{A}_-$ equation agrees with (80) when the constraints are used.

The characteristic equations associated to the canonical momenta p^μ and π^μ are ob-

tained from (99), resulting in the expressions

$$\partial_+ p^+ = \partial_\lambda F^{\lambda +} - a^2 \partial_i \partial_- \partial_\lambda F^{\lambda i} - a^2 \nabla^2 \partial_\lambda F^{\lambda +}, \tag{104a}$$

$$\partial_+ p^i = \partial_- F^{-i} + \left(1 - a^2 \square\right) \partial_j F^{ij}, \tag{104b}$$

$$\partial_+ p^- = \partial_i F^{i-}. \tag{104c}$$

Similarly, for π^μ we derive

$$\partial_+ \pi^+ = \phi_3, \tag{105a}$$

$$\partial_+ \pi^+ = F_{+-} - p^- + a^2 \partial_i \partial_\lambda F^{\lambda i}, \tag{105b}$$

$$\partial_+ \pi^i = -\phi_4^i. \tag{105c}$$

The relations associated to π^+ and π^- indicate what we saw from the integrability conditions, i.e., ϕ_3 and ϕ_4^i result from the conditions $d\phi_1 = 0$ and $d\phi_2^i = 0$. Using the fact that $\partial_+ \pi^+ = 0$, $\partial_+ \pi^i = 0$ and $\pi^- = a^2 \partial_\lambda F^{+\lambda}$, we obtain

$$p^+ = a^2 \partial_- \partial_\lambda F^{+\lambda}, \tag{106a}$$

$$p^- = F^{-+} + a^2 \partial_\lambda F^{-\lambda}, \tag{106b}$$

$$p^i = F^{+i} - a^2 \partial_i \partial_+ \partial_\lambda F^{\lambda +} - 2a^2 \partial_- \partial_+ \partial_\lambda F^{\lambda i}. \tag{106c}$$

Replacing (106) in (105), we deduce the field equations

$$\left(1 + a^2 \square\right) \partial_\lambda F^{\mu \lambda} = 0. \tag{107}$$

Therefore, the time evolution sector of the characteristic equations are equivalent to the Lagrangian field equations.

4.5 Symmetries

Let us go back to the complete set of dynamical generators (98), with the canonical Hamiltonian density (84). The Hamiltonians Φ_A, as shown in [7], are generators of canonical transformations in the complete phase space. The x^+ "time" evolution studied above is a particular case. As usual in constraint analysis, we are also interested in the canonical transformations which represent Lagrangian gauge transformations. In this case, we consider the following infinitesimal transformations

$$\delta F(x) = \int_\Sigma d\sigma_y \left\{F(x), \Phi_a(y)\right\}^* \delta\omega^a(y), \qquad a = (1, 2, 3), \tag{108}$$

which are taken with $\delta x^+ = 0$. The generator is given by

$$G \equiv \int_\Sigma d\sigma_x \Phi_a(x) \delta\omega^a(x), \qquad a = 1, 2, 3. \tag{109}$$

with $\Phi_a = 0$.

Now we calculate a fixed point transformation on the Lagrangian density (65), given by

$$\delta L = \delta A_\mu \frac{\delta L}{\delta A_\mu} = \delta A_\mu \left(\Box + m^2\right) \partial_\alpha F^{\mu\alpha}, \qquad m^2 \equiv 1/a^2, \tag{110}$$

apart of a total divergence. Remember that $\delta L/\delta A_\mu$ is the Lagrange derivative (68). On the other hand, δA_μ is generated by G *via* the generalised brackets, which gives the result

$$\delta A_\mu = \{A_\mu, G\}^* = \delta_\mu^+ \delta\omega^2 - \left(\delta_\mu^- \partial_- + \delta_\mu^i \partial_i\right) \delta\omega^3. \tag{111}$$

Apart of divergence terms, substitution of (111) in (110) yields

$$\delta L = \delta\omega^2 \left(\Box + m^2\right) \partial_\alpha F^{+\alpha} - \delta\omega^3 \left(\Box + m^2\right) \left(\partial_- \partial_\alpha F^{-\alpha} + \partial_i \partial_\alpha F^{i\alpha}\right).$$

Note that the identity $\partial_\alpha \partial_\beta F^{\alpha\beta} = 0$ results in

$$\partial_- \partial_\alpha F^{-\alpha} + \partial_i \partial_\alpha F^{i\alpha} = -\partial_+ \partial_\alpha F^{+\alpha},$$

so, after some calculation,

$$\delta L = \left(\delta\omega^2 + \partial_+ \delta\omega^3\right) \left(\Box + m^2\right) \partial_\alpha F^{+\alpha}. \tag{112}$$

The system is gauge invariant under the transformation (111) if $\delta\mathcal{L}$ is zero in Ω. Of course, the field equations themselves lead to $\delta\mathcal{L} = 0$, but if the symmetry is understood to be valid outside the solutions of the variational problem, we must have

$$\delta\omega^2 = -\partial_+ \delta\omega^3. \tag{113}$$

Supposing $\Lambda \equiv \delta\omega^3$ the gauge parameter, we have

$$\delta\omega^2 = -\partial_+ \Lambda, \tag{114}$$

so (111) gives

$$\delta A_\mu = -\delta_\mu^+ \partial_+ \Lambda - \delta_\mu^- \partial_- \Lambda - \delta_\mu^i \partial_i \Lambda = -\partial_\mu \Lambda. \tag{115}$$

Therefore, we recover the correct gauge transformations $A_\mu \to A_\mu - \partial_\mu \Lambda$.

On the other hand, taking $\partial_+ \Lambda \equiv \dot{\Lambda}$,

$$\delta \bar{A}_\mu = \delta_\mu^+ \delta\omega^1 - \delta_\mu^- \partial_- \dot{\Lambda} - \delta_\mu^i \partial_i \dot{\Lambda},$$

and $\delta\omega^1$ is still arbitrary. However, if the gauge transformation for A_μ is given by $\delta A_\mu = -\partial_\mu \Lambda$, the same transformation for $\bar{A}_\mu$ should be $\delta \bar{A}_\mu = -\partial_\mu \dot{\Lambda}$ for the sake of consistency.

Therefore, we should fix $\delta\omega^1 = -\ddot{\Lambda}$, so

$$\delta\bar{A}_\mu = -\delta_\mu^+ \ddot{\Lambda} - \delta_\mu^- \partial_-\dot{\Lambda} - \delta_\mu^i \partial_i\dot{\Lambda} = -\partial_\mu\dot{\Lambda}. \tag{116}$$

After some calculation, the generator of gauge transformations (109) becomes

$$G = \int_\Sigma d\sigma_x \left(\partial_\mu\pi^\mu \dot{\Lambda} + \partial_\mu p^\mu \Lambda \right). \tag{117}$$

In fact, the generator (117) reproduces the relations (115) and (116).

5 Final remarks

In this paper, we analysed the null-plane canonical structure of Podolsky's generalised electrodynamics *via* the Hamilton-Jacobi formalism. The essence of the HJ approach is to understand canonical constraints as generators of transformations on the phase space. Each generator is a hamiltonian function responsible to the evolution of the system along a respective evolution parameter, as the canonic hamiltonian function which is the generator of time evolution. The flows generated by the hamiltonians are solutions of the characteristic equations taken from the fundamental differential (46). If complete integrability is assured by application of Frobenius' conditions, these generators are in involution with the generalised brackets, closing themselves a Lie algebra. A complete set of involutive constraints generates transformations that preserve the symplectic structure of the phase space, therefore becoming canonical transformations, whose flows are called characteristic flows. In this case, we understand the HJ formalism as the search for a complete set of involutive hamiltonians of a singular system.

Applied to Podolsky's theory, the HJ approach starts with the definition of the constraints (85). Among these constraints, we identify two sets of non-involutive ones, leading to the GB (94), with fundamental relations calculated in (95). These brackets are defined only with the application of proper boundary conditions, $\partial_- A_\mu = \partial_-^2 A_\mu = \partial_-^3 A_\mu = 0$ for $x^- \to -\infty$. The GB eliminate the set $\left(\phi_2^i, \phi_4^i\right)$, but reveals a new constraint given by (97). The complete set of hamiltonians, written in (98), is in involution with the GB, therefore completing the task of finding the complete set of generators of the system.

In section 4.4 the canonical field equations are calculated. It is shown that these characteristic equations, in the temporal sector, are equal to the Euler-Lagrange equations of Podolsky's lagrangian.

On the other hand, the evolution of the system in the direction of the remaining independent variables ω^a, with $\delta x^+ = 0$, is analysed in section 4.5. In this case, the hamiltonians Φ_a generate characteristic flows defined by (109), and these flows are symmetries of the system. Since Φ_a form a complete set of compatible observables in the reduced phase space, we expect that these transformations are related to the gauge transformations defined by an invariant field strength $F_{\mu\nu}$. In fact, $\delta L = 0$ leads directly to the relation (113) between the former independent variables. Choosing $\Lambda \equiv \delta\omega^3$ as the gauge parameter, we

see that ω^2 must have the form $\delta\omega^2 = -\dot{\Lambda}$, while ω^1 must obey $\delta\omega^1 = -\ddot{\Lambda}$. The correct gauge transformations (115) and (116) are generated by the generating function (117).

Here, we stress the fact that the application of the HJ theory not only provides straightforward results for the complete set of generators and consistent canonical field equations, but also resulted in the correct generators of the gauge transformations, *via* the correct relation among the independent variables of the theory. This last result was always of some controversy in the literature, since Dirac's method incorporate an unproven assumption, Dirac's conjecture, which is the statement that all first-class constraints (correspondent to our involutive constraints) must contribute to the construction of the generator. Moreover, no ad-hoc method, as Castellani's procedure [23], was necessary. In fact, it seems that Dirac's conjecture is actually not a defined problem in the HJ formalism. It is our intent, however, to study the relation between Frobenius' theorem and this conjecture in the near future.

Acknowledgements

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