DEPARTMENT OF MATHEMATICS AND STATISTICS

On Extensions and Variants of Dependence Logic

— A study of intuitionistic connectives in the team semantics setting

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Abstract

Dependence logic is a new logic which incorporates the notion of "dependence", as well as "independence" between variables into first-order logic. In this thesis, we study extensions and variants of dependence logic on the first-order, propositional and modal level. In particular, the role of intuitionistic connectives in this setting is emphasized.

We obtain, among others, the following results:

- First-order dependence logic extended with intuitionistic and linear connectives characterizes all second-order downwards monotone properties.
- First-order independence logic extended with intuitionistic and linear connectives, and first-order inclusion logic extended with maximal implication are both equivalent to the full second-order logic over sentences.
- Complete axiomatizations for propositional dependence logic, propositional intuitionistic dependence logic, propositional independence logic extended with non-empty atom.
- Intuitionistic connectives are definable, but not *uniformly* definable in propositional dependence logic.
- Modal intuitionistic dependence logic has a connection with modal intuitionistic logic.
- Model checking problem for modal intuitionistic dependence logic is PSPACE-complete.

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Contents

Introduction

In this thesis, we study extensions and variants of dependence logic, in particular, the role of intuitionistic connectives in this setting will be emphasized.

The two central notions of this thesis are "dependence" and "independence". Dependence and independence are common phenomena in many fields: from computer science (databases, software engineering, knowledge representation, AI) to social sciences (human history, stock markets). Formally, one encounters the issue of dependence and independence when a first-order sentence such as

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \phi(x_1, x_2, y_1, y_2)$$

is being considered: the sentence is true on a given first-order model M if

> *for all values a_1 of x_1, there exists a value b_1 of y_1 (*depending on a_1*) such that for all values a_2 of x_2, there exists a value b_2 of y_2 (*depending on a_1 and a_2*) such that $\phi(a_1, a_2, b_1, b_2)$ is true on M.*

That is, in first-order logic, the value of an existentially quantified variable depends only on the values of *all* universally quantified variables that come *before* it. Such a built-in *linear* dependence relation between variables makes the familiar first-order logic a weak and restricted logic in terms of dependence and independence. The research of developing an appropriate logical formalism for dependence and independence has been active in recent years.

The first step in this direction dates back to 1960's when Henkin [46] characterized dependence between first-order variables by extending classical first-order logic with *partially ordered quantifiers*, called *branching quantifiers* or *Henkin quantifiers*. A typical sentence with a branching quantifier is as follows:

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} \phi(x_1, x_2, y_1, y_2), \tag{0.1}$$

whose semantics is given via Skolem functions: the sentence is true on a given first-order model M if the Skolem expression

$$\exists f \exists g \forall x_1 \forall x_2 \phi(x_1, x_2, f(x_1), g(x_2))$$

is true on M. Enderton [19] and Walkoe [83] showed that first-order logic with Henkin quantifiers has the same expressive power as Σ^1_1, the existential second-order logic.

Table 1: An example of a team

	x_2	y_2
s_1	0	1
s_2	0	0
s_3	1	0

In the second step, Hintikka and Sandu [48], [49] (see also [69]) developed the so-called *independence-friendly logic* (IF-logic), which can be easily proved to have the same expressive power as Σ^1_1 over sentences too. IF-logic adds into first-order logic slashed (linear) quantifiers. For example, the formula in (0.1) is expressed by

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 / \{x_1\} \phi(x_1, x_2, y_1, y_2), \tag{0.2}$$

which is true on a given first-order model M if intuitively

> *for all values a_1 of x_1, there exists a value b_1 of y_1 such that for all values a_2 of x_2, there exists a value b_2 of y_2,* independent of a_1, *such that $\phi(a_1, a_2, b_1, b_2)$ is true on M.*

The game-theoretical semantics of IF-logic is defined with respect to imperfect information games, which is a generalization of the standard game-theoretical semantics of first-order logic. However, such semantics is non-compositional, in particular, open formulas of the logic do not have meanings. It was only until [50], [52], Hodges introduced compositional semantics for IF-logic, called *trump semantics* and later also *team semantics*. The crucial innovation of the semantics is that the satisfaction relation is defined with respect to sets of assignments, called *teams*, instead of single assignments as in the usual semantics of first-order logic. Cameron and Hodges showed in [6] that it is indeed not possible to obtain compositional semantics for IF-logic with respect to single assignments.

Based on team semantics, Väänänen [78] introduced *first-order dependence logic*, which is the topic of this thesis. Dependence logic singles out the dependence between variables from the use of quantifiers, and incorporates in first-order logic a new type of atomic formulas

$$=\!(x_1, \ldots, x_{n-1}, x_n),$$

called *dependence atomic formulas* or *dependence atoms*. Intuitively, the above atom is true on a first-order model if the value of the first-order variable x_n is *functionally determined* by the values of the first-order variables $x_1, \ldots, x_{n-1}$. As we shall discuss in details in Section 1.1 that such functional dependence do not manifest in a single assignment, but in a set of assignments (a *team*). Given a first-order model M with domain $\{0, 1\}$. The dependence atom $=\!(x_2, y_2)$ is not true on the set $X = \{s_1, s_2, s_3\}$ of assignments (a team) represented by Table 1. This is because the value of y_2 is *not* functionally determined by the value of x_2, as when the value of x_2 is 0, the variable y_2 may have two different values, namely 1 (under s_1) and 0 (under s_2).

Over sentences, dependence logic also has the same expressive power as Σ^1_1, therefore it is equivalent to both first-order logic with Henkin quantifiers and IF-logic. For instance, the formula (0.1) or (0.2) can be expressed in dependence logic as:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (=(x_2, y_2) \wedge \phi(x_1, x_2, y_1, y_2)). \tag{0.3}$$

Over formulas, dependence logic is *downwards closed*, meaning that if a formula is true on a team X, then it is true also on all subteams of X. Making use of the downwards closure property, Kontinen and Väänänen [60] proved that open formulas of dependence logic characterize all downwards monotone Σ_1^1 properties with respect to non-empty teams in a certain sense.

An important extension of first-order dependence logic, namely *first-order independence logic*, was developed recently by Grädel and Väänänen [39]. Instead of emphasizing *dependence*, independence logic emphasizes *independence*. It adds into first-order logic a new type of atomic formulas, namely *independence atoms* of the form

$$\bar{x} \perp_{\bar{z}} \bar{y}.$$

Intuitively, given a first-order model, the above atom is true on a team if with respect to a fixed value of $\bar{z}$ the terms $\bar{x}$ are totally independent of the terms $\bar{y}$ in the sense that knowing the value of $\bar{x}$ does not tell us anything about the value of $\bar{y}$. Clearly, independence logic is *not* downwards closed. Galliani proved in [31] that independence logic characterize all Σ_1^1 properties with respect to non-empty teams in a certain sense. In particular, dependence atoms are definable by independence atoms (see Expression (1.1) in Section 1.2), implying that dependence logic is a sublogic of independence logic. Other interesting sublogics of independence logic are *inclusion logic* [31], [35] and *exclusion logic* [31].

The main focus of this thesis is the so-called *BID-logic* introduced by Abramsky and Väänänen [1]. BID-logic is an extension of dependence logic obtained by a careful analysis of team semantics in the presence of the downwards closure property. The propositional fragment of BID-logic turns out to correspond to the *logic of bunched implications* [72], [74] (BI) aiming at providing a semantically based logic of resources. In particular, the algebraic counterpart of BID-logic is both a commutative quantale (which carries an interpretation of linear logic) and a complete Heyting algebra (which carries an interpretation of intuitionistic logic). New connectives corresponding to the operations in such an algebraic structure are then introduced into BID-logic, namely, the *intuitionistic implication*, the *intutionistic disjunction* and the *linear implication*. In this thesis, we study the properties of these connectives, mainly the intuitionistic ones, on the first-order logic level (chapters 2-3), the propositional logic level (chapters 4-5), as well as the modal logic level (chapters 6-7).

The thesis is organized as follows:

In "Note on notation", we declare basic knowledge and facts on logic that are taken for granted in the thesis, and fix our choices for commonly used symbols and notations.

Chapter 1 introduces the known properties of all of the first-order logics mentioned above, namely first-order BID-logic, dependence logic, independence logic, inclusion logic and exclusion logic.

In Chapter 2, we study the expressive power of first-order BID-logic. In the preparation sections (sections 2.1-2.2), the so-called *first-order intuitionistic dependence logic*

(**ID**), which is first-order dependence logic extended with intuitionistic connectives is defined. We point out that **ID** satisfies all axioms of propositional intuitionistic logic, as well as all axioms of Maksimova's Logic ND ([68]) and all axioms of Kreisel-Putnam Logic KP ([63]), together with the double negation law for classical atomic formulas. This also paves the way for Section 4.2 of Chapter 4 on propositional intuitionistic dependence logic.

In the main part of Chapter 2, we show that over sentences, BID-logic, or already first-order intuitionistic dependence logic (a fragment of BID-logic), is equivalent to the full second-order logic. The content of this section is based on the publication [87]. Generalizing the method of [60], we also prove that open formulas of BID-logic characterize all downwards monotone second-order properties with respect to all teams (including the empty team) in a certain sense. Recall that dependence logic itself is equivalent to Σ^1_1, which is not closed under *classical negation*. Dependence logic extended with classical negation is called *team logic*, and team logic is also equivalent to the full second-order logic over sentences ([59]). The importance of our result lies in that BID-logic is a characterization of the full second-order logic without having the classical negation in the language.

In Chapter 3, we study first-order independence logic with implications. Lacking of the downwards closure property, intuitionistic implication and linear implication do not behave the same way in independence logic as in dependence logic. However, independence logic extended with the two implications is still equivalent to the full second-order logic over sentences, and its open formulas *define* (not *characterize*) all second-order empty set-closed properties. Moreover, we study the *maximal implication*, introduced in [59]. We prove that inclusion logic extended with maximal implication is equivalent to the full second-order logic.

Chapter 4 is devoted to the study of propositional dependence logic, propositional independence logic and their variants. We introduce these logics as natural propositional variants of their first-order counterparts. We give concrete axiomatizations and prove completeness theorems for these logics.

For a fixed number of propositional variables, there are only finitely many distinct teams. A consequence of this fact, discussed in Section 4.2, is that in the presence of intuitionistic disjunction, dependence atoms are eliminatable. Therefore the so-called *propositional intuitionistic dependence logic* is essentially equivalent to *inquisitive logic* [13]. Such a surprising connection opens the door for future research.

In Chapter 5, we investigate the uniform definability issue in propositional dependence logic. It follows from Theorem 4.4.1 that all *instances* of any connective is definable in propositional dependence logic, however, we show in this chapter that not all connectives are *uniformly definable* in the logic. For instance, intuitionistic implication and intuitionistic disjunction are two such connectives. This work is inspired by [32]. As a consequence of [13], this phenomenon also occurs in propositional intuitionistic dependence logic or inquisitive logic.

In Chapter 6, we introduce modal intuitionistic dependence logic, which is a variant of *modal dependence logic* [79] having intuitionistic connectives. We prove some preliminary model-theoretic results of the logic, including a translation from modal dependence logic into the logic. We reveal a connection between modal intuitionistic dependence logic and intuitionistic modal logic **IK** defined independently in by Edwald [22], Fischer

Servi [26] and Plotkin and Stirling [73], and show that modal intuitionistic dependence logic is complete with respect to a certain set of finite bi-relation Kripke models. We, by no means, claim that the work in Chapter 6 is complete in any sense for the investigation of modal intuitionistic dependence logic, however we present this chapter with the hope that these results will throw some light on the future research in this area.

Chapter 7 is based on the publication [18] in which we analyze the computational complexity of the model checking problem for modal intuitionistic dependence logic and its fragments built by restricting the operators allowed in the logics. In particular, we show that the model checking problem for modal intuitionistic dependence logic in general is PSPACE-complete and that for propositional intuitionistic dependence logic is coNP-complete.

Note on Notation

Throughout the thesis, we assume the standard Tarskian semantics of classical first-order logic (**FO**), the standard semantics of second-order logic (**SO**), the truth table semantics of classical propositional logic (**CPL**), the Kripke semantics of modal logic (**M**) and the Kripke semantics of intuitionistic logic (**IPL**), as well as that of intermediate logic. Readers who are not familiar with these are referred to, e.g., [43], [81], [64], [9], [4], [80], etc.

We consider first-order logic with equality. The negated equation $\neg(t_1 = t_2)$ is often written as $t_1 \neq t_2$. For the treatment of first-order models, we follow [51]. Unless otherwise specified, we use $a, b, c, d, \dots$ with or without subscripts or superscripts to stand for constants, $R, S, \dots$ with or without subscript or superscripts to stand for relations, and $f, g, h, \dots$ with or without subscripts or superscripts to stand for functions. A first-order *signature* L is a set of constant symbols, relation symbols and function symbols. We write $L(R)$ for the signature expanded from L by adding a new relation symbol R (whose arity is always clear from the context); similarly for the expanded signature $L(f)$. An L-model M is a first-order model with signature L. We write c^M for the interpretation of the constant symbol c in the model M; similarly for R^M and f^M. If M is an L-model, we sometimes write $(M, c_1^M, \dots, c_k^M, f_1^M, \dots, f_m^M, R_1^M, \dots, R_n^M)$ for the expansion of the L-model M obtained by adding the symbols $c_1, \dots, c_k, f_1, \dots, f_m, R_1, \dots, R_n$ into the signature L.

With some abuse of notation, we write M for both the model and its domain. An *assignment* s on a model M is a function from a finite set $dom(s)$ of variables into M. The set $dom(s)$ is called the *domain* of s, and it will be always clear from the context. We sometimes use tables to represent assignments, for example, Table 1 in Introduction contains three assignments s_1, s_2, s_3 defined in the obvious way. Let s be an assignment on a model M, A a set of variables, a an element of M. We write $s \upharpoonright A$ for the assignment s restricted to the domain $A \cap dom(s)$. We write $s(a/x)$ for the assignment with $dom(s(a/x)) = dom(s) \cup \{x\}$ which agrees with s everywhere except that it maps x to a. If $\{x_1, \dots, x_n\} \subseteq dom(s)$, then we write $s(\overline{x})$ for the sequence $\langle s(x_1), \dots, s(x_n)\rangle$. If t is a first-order term, then we write $t\langle s\rangle$ for the interpretation of t on M under the assignment s.

We say that M is a *suitable* model and s is a *suitable* assignment on M for a formula ϕ if all constant, relation and function symbols occurring in ϕ are in the signature of M, and all free variables of ϕ are in the domain of s.

A sequence $\langle o_1, \dots, o_n\rangle$ of objects is abbreviated as $\bar{o}$, and the length of the sequence will always be clear from the context or does not matter. That is, we write $\bar{t}$, $\bar{f}$, $\bar{R}$, $\bar{c}$, $\bar{p}$

for sequences of first-order terms, functions, relations, elements of models, propositional variables, and so forth. We use the standard abbreviation $\forall\overline{x}$ to stand for a sequence of universal quantifiers $\forall x_1 \dots \forall x_n$ (the length of $\overline{x}$ is always clear from the context or does not matter); similarly for existential quantifiers.

A formula in *negation normal form* is a formula in which negations occur only in front of atomic formulas. Any (classical) first-order, propositional or modal formula is equivalent to a formula in negation normal form. Let $\phi(\psi_1,\dots,\psi_k)$ be a formula (first-order, propositional or modal) which has $\psi_1,\dots,\psi_k$ as subformulas. We write $\phi(\theta_1/\psi_1,\dots,\theta_k/\psi_k)$ for the formula obtained by uniformly replacing each ψ_i in ϕ with θ_i for each $1 \leq i \leq k$.

The empty set is denoted by $\emptyset$; the empty sequence is denoted by $\langle\rangle$ or ϵ. The concatenation of two sequences $\bar{x}$ and $\bar{y}$ is denoted by $\bar{x}^\frown\bar{y}$ or simply $\bar{x}\bar{y}$. The notation $x \subseteq y$ means that x is a subset of y or $x = y$; the notation $x \subset y$ or $x \subsetneq y$ means that x is a proper subset of y.

Natural numbers are defined inductively as:

- $0 := \emptyset$;
- $n+1 := n \cup \{n\}$.

We denote the set of all natural numbers by ω or $\mathbb{N}$. For any set A, denote the cardinality of A by the standard notation $|A|$.

The normal form of every second-order Σ^1_n-formula is

$$\exists\bar{f}_1\forall\bar{f}_2\cdots Q\bar{f}_{n-1}\,\overline{Q}\,\bar{f}_nQ\overline{x}\psi,$$

where ψ is quantifier free, Q is $\exists$ or $\forall$ depending on the parity of n, and $\overline{Q}$ is the dual of Q. Similarly, the normal form of every second-order Π^1_n-formula is

$$\forall\bar{f}_1\exists\bar{f}_2\cdots Q\bar{f}_{n-1}\,\overline{Q}\,\bar{f}_nQ\overline{x}\psi.$$

We use the standard notations P, PSPACE, EXPTIME in computational complexity theory to stand for the classes of decision problems whose solutions can be determined by a deterministic Turing machine in polynomial time, polynomial space, exponential time, respectively; we write NP, NPSPACE, NEXPTIME for the classes of decision problems whose solutions can be determined by a non-deterministic Turing machine in polynomial time, polynomial space, exponential time, respectively.

Chapter 1

First-order logics of dependence and independence

In this chapter, we give a brief introduction to the first-order logics of dependence and independence considered in this thesis. In Section 1.1, we introduce first-order dependence logic [78] and its extension BID-logic [1], as well as team semantics. In Section 1.2, we present the definitions and basic results of first-order independence logic [39], inclusion logic and exclusion logic [31].

1.1 First-order dependence logic and BID-logic

In this section, we define first-order dependence logic and BID-logic, as well as team semantics. As discussed in Introduction, *dependence logic* (**D**) was developed by Väänänen in [78] as an alternative approach to *independence friendly logic* (IF-logic) [48], [49] (see also [69]). It adds into first-order logic a new type of atomic formulas

$$=\!(x_1,\dots,x_{n-1},x_n),$$

called *dependence atomic formulas* or *dependence atoms*. Such a formula, as well as other formulas of **D**, are evaluated on a model with respect to a set of assignments (called *teams*). A team X satisfies the above formula if the value of the first-order term x_n is *functionally determined* by the values of the first-order terms $x_1,\dots,x_{n-1}$, that is, for all assignments $s,s'\in X$,

$$\big[\,s(x_1)=s'(x_1),\,\dots,\,s(x_{n-1})=s'(x_{n-1})\,\big]\Longrightarrow s(x_n)=s'(x_n);$$

Such semantics is called *team semantics* (or *trump semantics*), which was originally introduced by Hodges [50], [52] as a compositional semantics for IF-logic.

A basic property of **D** is that it is *downwards closed*, i.e., if a team X satisfies a formula, then every subteam of X also satisfy the formula. In [1] , Abramsky and Väänänen studied team semantics with the downwards closure property, and defined an extension of **D**, called *BID-logic* (**BID**). In a general construction of Hodges' team semantics, the propositional fragment of **BID** corresponds to the *logic of bunched implications* [72],

[74] (BI), which is a semantically based logic of resources. The algebraic counterpart of **BID** is both a commutative quantale (which carries an interpretation of linear logic) and a complete Heyting algebra (which carries an interpretation of intuitionistic logic). Accordingly, **BID** has both connectives of linear logic, i.e., $\otimes$ (the additive conjunction) and $\multimap$ (linear implication), and connectives of intuitionistic logic, i.e., $\vee$ (the *intuitionistic disjunction*) and $\to$ (the *intuitionistic implication*). The connective $\otimes$ was the *disjunction* of **D** (which was inherited from the *classical disjunction* of first-order logic)[1], and we shall call it *tensor disjunction* in this thesis, although it corresponds to additive *conjunction* in **BID**.

In this thesis, we will treat **D** as a fragment or sublogic of **BID**, Below we define the syntax of **BID** and **D**.

Definition 1.1.1. Well-formed formulas of *BID-logic* (**BID**) are given by the following grammar

$$\phi ::= \alpha \mid \neg\alpha \mid =\!(x_1,\dots,x_n) \mid \bot \mid \phi\wedge\phi \mid \phi\otimes\phi \mid \phi\vee\phi \mid \phi\to\phi \mid \phi\multimap\phi \mid \exists x\phi \mid \forall x\phi$$

where α is a first-order atomic formula and $n \geq 1$.

Note that in the above definition of the syntax of **BID**, negations apply only to first-order atomic formulas. Indeed, in this thesis, we do *not* view such atomic negation as a *connective* of the logic, that is, the expression $\neg\alpha$ should be understood as a whole object (c.f. Definition 5.2.1). On the other hand, every formula of **BID** does have its *intuitionistic negation* defined in the following Convention. The issues about negation will be studied in Section 2.1.

Convention 1.1.2. *For any formulas ϕ and ψ of* **BID***, we define*

(i) $\neg\phi := \phi\to\bot$[2]

(ii) $\top := \neg\bot$

(iii) $\phi\leftrightarrow\psi := (\phi\to\psi)\wedge(\psi\to\phi)$

For technical simplicity, we use the expression $=\!(\cdots)$ to stand for a special type of *operator* which acts on variables. Any dependence atom $=\!(x_1,\dots,x_n)$ $(n\geq 1)$ is a result of an application of $=\!(\cdots)$. Similarly, the expression $=\!(\cdot)$ is an *operator* applying to single variables only, that is, the resulting dependence atoms can only be of the form $=\!(x_1)$. Moreover, all of the connectives, quantifiers and the constant $\bot$ of **BID** are viewed as *operators* as well. Next, we define our notations for sublogics or extensions of a logic built by restricting or extending the set of eligible operators of the logic.

Definition 1.1.3. Let L be a logic and Ω a set of operators of L.

(i) We write $\mathsf{L}[\Omega]$ for the sublogic of L built from literals of L using quantifiers and the operators only from Ω. We sometimes write simply $\mathsf{L}[o_1,o_2,\dots,o_n]$ instead of $\mathsf{L}[\{o_1,o_2,\dots,o_n\}]$.

[1]In the literature of dependence logic, the disjunction of the logic is usually denoted by the same symbol $\vee$ as the classical disjunction of first-order logic. However, in this thesis, for the purpose of distinction, we use the symbol $\otimes$ for the disjunction of dependence logic, and the symbol $\vee$ for the intuitionistic disjunction.

[2]We will show in Lemma 1.1.8 that for first-order atomic formulas α, under the team semantics, the negated atomic formula $\neg\alpha$ from Definition 1.1.1 has the same meaning as $\alpha\to\bot$.

(ii) We write $\mathsf{L}^{[\Omega]}$ for the logic extended from L by adding the operators in Ω. We sometimes write simply $\mathsf{L}^{[o_1,o_2,\ldots,o_n]}$ instead of $\mathsf{L}^{[\{o_1,o_2,\ldots,o_n\}]}$.

Now, we define **D** as a sublogic of **BID**.

Definition 1.1.4. *First-order dependence logic* (**D**) is the sublogic

$$\mathbf{BID}[=\!(\cdots),\wedge,\otimes,\exists,\forall]$$

of **BID**. In other words, well-formed formulas of **D** are given by the following grammar

$$\phi ::= \alpha \mid \neg\alpha \mid =\!(x_1,\ldots,x_n) \mid \phi\wedge\phi \mid \phi\otimes\phi \mid \exists x\phi \mid \forall x\phi$$

where α is a first-order atomic formula and $n \geq 1$.

The set $\text{Fv}(\phi)$ of free variables of a formula ϕ of **BID** is defined in the standard way except that for the case of dependence atoms, we have the following definition:

$$\text{Fv}(=\!(x_1,\ldots,x_n)) = \{x_1,\ldots,x_n\}.$$

We call a formula ϕ of **BID** a *sentence* in case $\text{Fv}(\phi) = \emptyset$.

As discussed, the satisfaction relation of the logic **BID** is defined with respect to *teams*. We now give the formal definition of a team.

Definition 1.1.5. A *team* X of a first-order model M is a set of assignments on M with the same domain. Denote the domain by $dom(X)$.

If A is a set of variables, then we define a team X restricted to A as

$$X \upharpoonright A = \{s \upharpoonright A \mid s \in X\}.$$

For example, Table 1 in Introduction represents a team X consisting of three assignments s_1, s_2, s_3 with domain $\{x_2, y_2\}$, i.e., $X = \{s_1, s_2, s_3\}$. In particular, the following sets are teams of a model M:

$$\emptyset \text{ and } \{s\},$$

where s is an assignment on M. For the empty domain, there is one and only one assignment on M, namely the empty assignment $\emptyset$. The singleton of the empty assignment

$$\{\emptyset\}$$

is a team of M.

To give semantics for the existential and universal quantifiers of **BID**, we need to define the following two operations on teams.

Definition 1.1.6. For any team X of M, and any function $F : X \to M$, the team

$$X(F/x) = \{s(F(s)/x) : s \in X\}$$

is called the *supplement team* of X by $F$and the team

$$X(M/x) = \{s(a/x) : a \in M,\ s \in X\}$$

is called the *duplicate team* of X.We abbreviate the supplement team

$$X(F_1/x_1)\dots(F_n/x_n)$$

as $X(F_1/x_1,\dots,F_n/x_n)$, and the duplicate team

$$X(M/x_1)\dots(M/x_n)$$

as $X(M/x_1,\dots,x_n)$ or $X(M/\bar{x})$.

Now, we give the semantics for **BID**. Such semantics is called *team semantics*. The logic **D** also has an equivalent game-theoretical semantics, however, in this thesis, we will not go into this direction. Interested readers are referred to [78] for details.

Definition 1.1.7. We inductively define the notion of a formula ϕ of **BID** being *satisfied* in a suitable first-order model M on a suitable team X of M, denoted by $M \models_X \phi$, as follows:

- $M \models_X \alpha$ with α a first-order atomic formula iff for all $s \in X$, $M \models_s \alpha$ in the usual sense;
- $M \models_X \neg\alpha$ with α a first-order atomic formula iff for all $s \in X$, $M \models_s \neg\alpha$ in the usual sense;
- $M \models_X {=}(x_1,\dots,x_n)$ iff for all $s, s' \in X$
$$\big[s(x_1) = s'(x_1),\dots,s(x_{n-1}) = s'(x_{n-1})\big] \Longrightarrow s(x_n) = s'(x_n);$$
- $M \models_X \bot$ iff $X = \emptyset$;
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$;
- $M \models_X \phi \otimes \psi$ iff there exist teams $Y, Z \subseteq X$ with $X = Y \cup Z$ such that
$$M \models_Y \phi \text{ and } M \models_Z \psi;$$
- $M \models_X \phi \vee \psi$ iff $M \models_X \phi$ or $M \models_X \psi$;
- $M \models_X \phi \to \psi$ iff for any team $Y \subseteq X$,
$$M \models_Y \phi \Longrightarrow M \models_Y \psi;$$
- $M \models_X \phi \multimap \psi$ iff for any team Y with $dom(Y) = dom(X)$,
$$M \models_Y \phi \Longrightarrow M \models_{X \cup Y} \psi;$$
- $M \models_X \exists x\phi$ iff $M \models_{X(F/x)} \phi$ for some function $F : X \to M$;
- $M \models_X \forall x\phi$ iff $M \models_{X(M/x)} \phi$.

If $M \models_X \phi$ holds for all suitable models M and all suitable teams X of M, then we say that ϕ is *valid* in **BID**, denoted by $\models_{\mathbf{BID}} \phi$ or simply $\models \phi$. Sentences have no free variable and the empty assignment $\emptyset$ is the only assignment for sentences. We say that a sentence ϕ is *true* in M if the team $\{\emptyset\}$ of the empty assignment satisfies ϕ, i.e. $M \models_{\{\emptyset\}} \phi$.

Let ϕ and ψ be two formulas of **BID**. If for any suitable model M and any suitable team X of M,

$$M \models_X \phi \Longrightarrow M \models_X \psi,$$

then we say that ψ is a *logical consequence* of ϕ, in symbols $\phi \models \psi$. If $\phi \models \psi$ and $\psi \models \phi$, then we say that ϕ and ψ are *logically equivalent*, in symbols $\phi \equiv \psi$.

The above defined team semantics deserve some comments. Most importantly, a team, as in Table 1 in Introduction, can be viewed as a relational database in the obvious way. With this setting, dependence atoms $=\!(x_1,\dots,x_{n-1},x_n)$ correspond exactly the *functional dependencies* $\{x_1,\dots,x_{n-1}\} \to x_n$ in database theory (see e.g. [23] for an early overview). In particular, dependence atoms satisfy Armstrong's axioms [2]:

(i) $=\!(x,x)$;

(ii) if $=\!(x,y,z)$, then $=\!(y,x,z)$;

(iii) if $=\!(y,z)$, then $=\!(x,y,z)$;

(iv) if $=\!(x,y)$ and $=\!(y,z)$, then $=\!(x,z)$.

Following database theory, the *implication problem* for dependence atoms asks that for a finite set Γ of dependence atoms and a dependence atom γ, whether

$$\Gamma \models \gamma$$

holds. From [2], we know that implication problem for dependence atoms can be axiomatized by Armstrong's axioms. We will come back to Armstrong's axioms in Chapter 4 in the context of propositional dependence logic, where in Example 4.4.11 we will derive these axioms in the natural deduction system of the logic.

It is possible to allow dependence atoms of the form $=\!(t_1,\dots,t_n)$, where $t_1,\dots,t_n$ are first-order terms in the syntax of **BID**, as it is done in [78]. Such an atom is satisfied by a team X if and only if (naturally) for all $s,s' \in X$,

$$\big[\,t_1\langle s\rangle = t_1\langle s'\rangle, \dots, t_{n-1}\langle s\rangle = t_{n-1}\langle s'\rangle\,\big] \Longrightarrow t_n\langle s\rangle = t_n\langle s'\rangle.$$

One sees easily that

$$=\!(t_1,\dots,t_n) \equiv \exists x_1 \dots \exists x_n (=\!(x_1,\dots,x_n) \wedge (x_1 = t_1) \wedge \dots \wedge (x_n = t_n)),$$

therefore for simplicity, in this thesis, we will restrict our attention to dependence atoms of the form $=\!(x_1,\dots,x_n)$ only.

The connectives of **BID** are of special interests for the following reasons. As discussed in [1] and the introduction of this section, for a model M and a set V of variables, taking all of the downwards closed subsets of $\wp(M^V)$, one forms the algebra of the underlying propositional logic of **BID**. Such a structure is a BI algebra, that is, both a *commutative quantale* and a *complete Heyting algebra*. The connectives $\wedge$, $\otimes$, $\vee$ and $\to$ of **BID** correspond exactly to conjunction, multiplicative conjunction, intuitionistic disjunction and intuitionistic implication of the algebraic structure, respectively. In Chapter 4, we will study the underlying propositional logic of **BID** in details.

We invite the reader to check the following lemma which justifies our choice of notations in Convention 1.1.2.

Lemma 1.1.8.

(i) $\neg\alpha \equiv \alpha \to \bot$, *whenever* α *is a first-order atom (c.f. Convention 1.1.2);*

(ii) $\bot \equiv \neg{=}(x_1,\dots,x_n) \equiv \big((x = x) \wedge (x \neq x)\big)$;[3]

(iii) $\top \equiv \forall x(x = x)$.

We now list some known basic properties of **BID**, all of which also hold for the sublogic **D**. All of these properties were proved in [78] and [1].

Theorem 1.1.9 (Locality)**.** *The truth of a formula* ϕ *of* **BID** *on a team of a model* M *depends only on the assignments of the variables occurring free in* ϕ*. That is, for any teams* X,Y *of* M *satisfying* $X \restriction \mathrm{Fv}(\phi) = Y \restriction \mathrm{Fv}(\phi)$*,*

$$M \models_X \phi \iff M \models_Y \phi.$$

Theorem 1.1.10 (Downwards Closure)**.** *For any formula* ϕ *of* **BID***, any suitable model* M *and any suitable teams* X,Y *of* M*,*

$$\big[M \models_X \phi \text{ and } Y \subseteq X\big] \Longrightarrow M \models_Y \phi.$$

Definition 1.1.11 (Flatness)**.** A formula ϕ of **BID** is said to be *flat* if for all suitable models M and all suitable teams X,

$$M \models_X \phi \iff (M \models_{\{s\}} \phi \text{ for all } s \in X).$$

Lemma 1.1.12. *Sentences of* **BID** *are flat.*

Proof. To evaluate sentences with no free variables, one only considers the singleton team $\{\emptyset\}$ of the empty assignment $\emptyset$. ■

We call the logic

$$\mathbf{BID}[\wedge,\otimes,\exists,\forall]$$

first-order logic (of **BID***)*, denoted by $\mathbb{FO}$[4], and formulas of the logic are called *first-order formulas* or *classical formulas* of **BID**, i.e., classical formulas are built with only first-order literals, $\wedge$, $\otimes$, $\exists$ and $\forall$.

[3] In some literature of dependence logic (e.g. [78]), formulas of the form $\neg{=}(x_1,\dots,x_n)$ are treated as the (primitive) negation of the dependence atom ${=}(x_1,\dots,x_n)$, and they are satisfied only by the empty team (i.e. $\neg{=}(x_1,\dots,x_n) \equiv \bot$). In this thesis, as in Definition 1.1.4, we do not allow **D** to have formulas of the form $\neg{=}(x_1,\dots,x_n)$, and in **BID** we view $\neg{=}(x_1,\dots,x_n)$ as an abbreviation of the intuitionistic negation ${=}(x_1,\dots,x_n) \to \bot$. The result here shows that both readings of the same formula $\neg{=}(x_1,\dots,x_n)$ induce actually the same semantical meaning.

[4] Note that the notation "$\mathbb{FO}$" is different from the notation "**FO**" we chose for the usual first-order logic.

Lemma 1.1.13. *Formulas of* **BID** *which do not contain dependence atoms or intuitionistic disjunctions are flat. In particular, classical formulas are flat.*

Proof. We show that the lemma by induction on the complexity of formulas ϕ described in the lemma. The only interesting case is the case $\phi = \psi \multimap \chi$. In this case, by the downwards closure property, it suffices to show that

$$\big[\forall s \in X,\ M \models_{\{s\}} \psi \multimap \chi\big] \Longrightarrow M \models_X \psi \multimap \chi$$

for all suitable models M and suitable teams X of M.

Suppose Y is a suitable team of M with $M \models_Y \psi$. By induction hypothesis, χ is flat, thus to show $M \models_{X\cup Y} \chi$, it suffices to show that for any $s_0 \in X \cup Y$, $M \models_{\{s_0\}} \chi$. Moreover, as χ is downwards closed, this reduces to showing that for any $s \in X$, $s' \in Y$, $M \models_{\{s,s'\}} \chi$.

Now, since $M \models_Y \psi$, by the downwards closure property, $M \models_{\{s'\}} \psi$. On the other hand, by assumption, $M \models_{\{s\}} \psi \multimap \chi$. It follows that $M \models_{\{s,s'\}} \chi$, as required. ∎

Lemma 1.1.14 (Empty Team Property)**.** *Formulas ϕ of* **BID** *which do not contain the linear implication $\multimap$ have the* empty team property*, that is, for any suitable model M, the empty team satisfies ϕ, i.e. $M \models_\emptyset \phi$.*

However, the full **BID** does not have the empty team property, as for example, for any model M, $M \not\models_\emptyset (x = x) \multimap (x \neq x)$. In this thesis, we will mainly focus on **BID** without linear implication, we denote this sublogic of **BID** by **BID**$^-$.

In the sequel, when comparing the expressive power of logics in the setting of team semantics, we will use the terminology "expressibility" defined as follows.

Definition 1.1.15.

(i) Let L_1 and L_2 be two logics with team semantics.

- **(a)** We say that a formula ϕ of L_1 is *expressible* in L_2, if there exists a formula ψ of L_2 such that $\phi \equiv \psi$.
- **(b)** We say that the logics L_1 and L_2 are *equivalent*, denoted by $\mathsf{L}_1 = \mathsf{L}_2$, if every formula of L_1 is expressible in L_2, and vice versa.

(ii) Let $\mathsf{L}_{\mathbf{SO}}$ be a sublogic of second-order logic and L a logic with team semantics.

- A sentence ϕ of L is *expressible* in $\mathsf{L}_{\mathbf{SO}}$, if there exists a sentence ψ of $\mathsf{L}_{\mathbf{SO}}$ such that for any suitable model M,

$$M \models \psi \Longleftrightarrow M \models_{\{\emptyset\}} \phi.$$

- A sentence ψ of $\mathsf{L}_{\mathbf{SO}}$ is *expressible* in L, if there exists an sentence ϕ of L such that for any suitable model M,

$$M \models \psi \Longleftrightarrow M \models_{\{\emptyset\}} \phi.$$

It was proved in [30] that intuitionistic disjunction $\vee$ is actually uniformly definable[5] in **D** assuming the models always have cardinality greater than 1. As we do not make this assumption in this thesis, below we present a slightly different definition.

Lemma 1.1.16. *For any formulas ϕ,ψ of* **D**, *putting*

$$\theta_0 := \forall x\exists y(x\neq y)\otimes(\phi\otimes\psi),$$

$$\theta_1 := \exists x\forall y(x=y)\otimes\exists w\exists u\big(=\!(w)\wedge=\!(u)\wedge\big((w=u)\otimes\phi\big)\wedge\big((w\neq u)\otimes\psi\big)\big),$$

where $w,u\notin \text{Fv}(\phi)\cup\text{Fv}(\psi)$, we have that $\phi\vee\psi\equiv\theta_0\wedge\theta_1$.

Proof. C.f. [30]. The formula θ_0 deals with the case when the model has cardinality 1, and the formula θ_1 deals with the case when the model has cardinality greater than 1. ∎

Corollary 1.1.17. $\mathbf{D}=\mathbf{D}^{[\vee]}$.

Next, we list the known results concerning expressive power of **D**. We will investigate the expressive power of **BID** in Chapter 2.

Theorem 1.1.18 ([78])**.** *Sentences of* **D** *are expressible in Σ^1_1, and vice versa.*

Definition 1.1.19. Let R be a k-ary relation symbol and $\phi(R)$ a second order $L(R)$-sentence. We say that $\phi(R)$ is *downwards monotone* with respect to R if for all $L(R)$-model (M,Q) and $Q'\subseteq Q$,

$$(M,Q)\models\phi(R)\Longrightarrow(M,Q')\models\phi(R).$$

The next lemma gives a syntactical characterization of downwards monotone sentences of Σ^1_1. In Lemma 2.4.1, we will generalize this result to the full second order logic.

Lemma 1.1.20 ([60])**.** *An L-sentence ϕ of Σ^1_1 is downwards monotone with respect to a predicate R iff there exists an equivalent $L(R)$-sentence ψ of Σ^1_1 in which R occurs only negatively.*

Notation 1.1.21. *Let X be a team of a model M with domain $\{x_1,\cdots,x_k\}$. The set*

$$rel(X)=\{(s(x_1),\cdots,s(x_k))\mid s\in X\}.$$

defines a k-ary relation of M corresponding to X.

Theorem 1.1.22 ([78], [60])**.**

(i) *For any L-formula $\phi(\bar{x})$ of* **D**, *there exists an $L(R)$-sentence $\psi(R)$ of Σ^1_1 which is downwards monotone with respect to a new predicate R such that for any suitable L-model M and any suitable team X of M,*

$$M\models_X\phi(\bar{x})\Longleftrightarrow(M,rel(X))\models\psi(R).$$

[5] A connective is *uniformly definable* in **D** if it is uniformly definable in the sense of [32]. In chapter 5, we investigate uniform definability in the context of propositional dependence logic.

(ii) *For any $L(R)$-sentence $\psi(R)$ of Σ_1^1 which is downwards monotone with respect to a predicate R, there exists an L-formula $\phi(\bar{x})$ of* **D** *such that for any suitable L-model M and any suitable non-empty team X of M,*

$$M \models_X \phi(\bar{x}) \iff (M, rel(X)) \models \psi(R).$$

If a logic L with team semantics and a sublogic $\mathsf{L}_{\mathbf{SO}}$ of second-order logic satisfy item (ii) of the above theorem with respect to a set T of teams, then we say that formulas of L *define* $\mathsf{L}_{\mathbf{SO}}$ with respect to T; in case items (i) and (ii) are both satisfied, then we say that formulas of L *characterize* $\mathsf{L}_{\mathbf{SO}}$ with respect to T. Formulas of **D** characterize all Σ_1^1 downwards monotone properties with respect to non-empty teams.

As **D** is equivalent to Σ_1^1, "truth" is definable in **D**, therefore by the Undefinability of Truth argument of Tarski, logical validity in **D** is non-arithmetical, and **D** cannot have any (effective) complete axiomatization. But it is possible to obtain weak completeness by making certain restrictions on the semantical consequence relation. Along this line, Galliani [33] defined a type of general models for **D** and proved the completeness theorem for **D** with respect to these models, and Kontinen and Väänänen [62] axiomatized the first-order consequences of **D**.

We know from Fagin [24] that NP corresponds exactly to Σ_1^1 over finite structures, therefore **D** also characterizes NP over finite structures. Other computational issues of **D** are studied in [57], [16], [58], [38], etc.

Many extensions and variants of **D** are investigated in recent years, including dependence logic with classical (contradictory) negation (*team logic*) [59], probabilistic dependence logic [28], [36], dynamic dependence logic [29], dependence logic with generalized quantifiers [20], [21], etc. In particular, the modal variant of dependence logic, namely *modal dependence logic* was introduced in [79]. Work on this topic include [76], [66], [67], [17], etc. In this thesis, we will devote chapters 6-7 to an extension of modal dependence logic, called *modal intuitionistic dependence logic*. In chapters 4-5, we will study the underlying propositional logic of first-order and modal dependence logic.

1.2 First-order independence, inclusion and exclusion logics

In this section, we define independence logic, inclusion logic and exclusion logic, and survey recent developments of these logics.

Dependence logic is a logic with team semantics which highlights the notion of *dependence* between variables. As mentioned in Introduction, other logics defined along this line (i.e., first-order logic with Henkin quantifiers [46], independence friendly logic [48], [49]) emphasize the notion of *independence* instead. In [39], Grädel and Väänänen introduced *independence atoms* into the team semantics setting and defined *independence logic*. A typical independence atom is as follows:

$$x \perp_z y.$$

A team X satisfies the above formula if the value of x is *totally independent* of the value of y, given a fixed value of z. This is formulated as, for all assignments $s, s' \in X$ with

$s(z) = s'(z)$, there exists $s'' \in X$ such that $s''(z) = s(z) = s'(z)$,

$$s''(x) = s(x) \text{ and } s''(y) = s'(y).$$

For example, the team $\{s_1, s_2, s_3, s_4, s_5\}$ of Table 1.1 satisfies $x \perp_z y$, and in the team, fixing a value for z, the value of x is completely undetermined by y as the restricted table actually induces a Cartesian product with respect to x and y.

One observes from this example that independence logic is *not* downwards closed, as, e.g., the subteam $\{s_1\}$ does not satisfy the same independence atom $x \perp_z y$. On the other hand, we will see that the downwards closed dependence logic is actually a sublogic of independence logic.

As mentioned in Section 1.1, teams can be viewed as databases. In such context, independence atoms correspond to *embedded multivalued dependencies* in database theory. Moreover, introducing *inclusion dependencies* [25] and *exclusion dependencies* [7], [8] of database theory into the team semantics setting, Galliani [31] (see also [30]) defined *inclusion logic* and *exclusion logic*. Inclusion logic adds *inclusion atoms* of the form $\bar{x} \subseteq \bar{y}$ and exclusion logic adds *exclusion atoms* of the form $\bar{x} \mid \bar{y}$ into first-order logic. Consider a team or a database represented in Table 1.2. Typical inclusion atom and exclusion atom are as follows:

$$\mathsf{Father} \subseteq \mathsf{Name} \quad \text{and} \quad \mathsf{Name} \mid \mathsf{Place\_of\_death}.$$

Intuitively, the inclusion atom $\mathsf{Father} \subseteq \mathsf{Name}$ is satisfied by X if in X, every value of Father is also a value of Name, or formally, for all $s \in X$, there exists $s' \in X$ such that

$$s(\mathsf{Father}) = s'(\mathsf{Name}).$$

One sees that this is not the case for the team of Table 1.2. On the other hand, the exclusion atom $\mathsf{Name} \mid \mathsf{Date\_of\_birth}$ is satisfied by X if no value of Name is a value of $\mathsf{Date\_of\_birth}$ and vice versa, or formally, for all $s, s' \in X$,

$$s(\mathsf{Name}) \neq s'(\mathsf{Place\_of\_death}).$$

This is clearly the case for the team of Table 1.2, as the data in the attribute Name is of different *type* from that in the attribute $\mathsf{Place\_of\_death}$.

Next, we give formal definitions of all these logics. In the sequel, we call all of the logics with team semantics mentioned so far *logics of dependence and independence*.

Table 1.1: An example of a team satisfying $x \perp_z y$

	z	x	y
s_1	0	1	0
s_2	0	0	1
s_3	0	1	1
s_4	0	0	0
s_5	1	1	0

	Name	Father	Place_of_death
s_1	Isaac	Abraham	Canaan
s_2	Jacob	Isaac	Canaan
s_3	Joseph	Jacob	Egypt
s_4	Judah	Jacob	Egypt

Table 1.2: An example of a database

Definition 1.2.1. Let α be any first-order atomic formula, $\bar{x}, \bar{y}, \bar{z}$ tuples of first-order terms, where $\bar{x}$ and $\bar{y}$ are non-empty and of the same length.

- Well-formed formulas of *first-order independence logic* (**Ind**) are given by the following grammar

$$\phi ::= \alpha \,\Big|\, \neg\alpha \,\Big|\, \bar{x} \perp_{\bar{z}} \bar{y} \,\Big|\, \phi \wedge \phi \,\Big|\, \phi \otimes \phi \,\Big|\, \exists x\phi \,\Big|\, \forall x\phi$$

- Well-formed formulas of *first-order inclusion logic* (**Inc**) are given by the following grammar

$$\phi ::= \alpha \,\Big|\, \neg\alpha \,\Big|\, \bar{x} \subseteq \bar{y} \,\Big|\, \phi \wedge \phi \,\Big|\, \phi \otimes \phi \,\Big|\, \exists x\phi \,\Big|\, \forall x\phi$$

- Well-formed formulas of *first-order exclusion logic* (**Exc**) are given by the following grammar

$$\phi ::= \alpha \,\Big|\, \neg\alpha \,\Big|\, \bar{x} \mid \bar{y} \,\Big|\, \phi \wedge \phi \,\Big|\, \phi \otimes \phi \,\Big|\, \exists x\phi \,\Big|\, \forall x\phi$$

- The union of inclusion and exclusion logic is called *inclusion/exclusion logic* (**I/E**).

Formulas of the forms $\bar{x} \perp_{\bar{z}} \bar{y}$, $\bar{x} \subseteq \bar{y}$ and $\bar{x} \mid \bar{y}$ are called *(conditional) independence atoms*, *inclusion atoms* and *exclusion atoms*, respectively. We write the independence atom $\bar{x} \perp_{\langle\rangle} \bar{y}$ simply as $\bar{x} \perp \bar{y}$, and call such independence atoms *unconditional independence atoms*. We call the sublogic of independence logic in which only unconditional independence atoms are allowed *pure (first-order) independence logic*.

As mentioned, all of the above logics have team semantics defined as follows. A game-theoretic semantics for inclusion and exclusion logic was introduced in [31], but in this thesis, we will not go into this direction.

Definition 1.2.2. We inductively define the notion of a formula ϕ of **Ind**, **Inc** or **Exc** being *satisfied* in a suitable model M by a suitable team$X$ of M, denoted by $M \models_X \phi$. All the cases are the same as those of **BID** as defined in Definition 1.1.7 except the following:

- $M \models_X \bar{x} \perp_{\bar{z}} \bar{y}$ iff for all $s, s' \in X$,

$$s(\bar{z}) = s'(\bar{z}) \Longrightarrow \exists s'' \in X \text{ such that } s''(\bar{z}) = s(\bar{z}) = s'(\bar{z}),$$
$$s''(\bar{x}) = s(\bar{x}) \text{ and } s''(\bar{y}) = s'(\bar{y}).$$

- $M \models_X \bar{x} \subseteq \bar{y}$ iff for all $s \in X$, there exists $s' \in X$ such that $s'(\bar{y}) = s(\bar{x})$.
- $M \models_X \bar{x} \mid \bar{y}$ iff for all $s, s' \in X$, $s(\bar{x}) \neq s'(\bar{y})$.
- $M \models_X \exists x\phi$ iff there exists $F : X \to \wp(M) \setminus \{\emptyset\}$ such that $M \models_{X[F/x]} \phi$, where
$$X[F/x] = \{s(a/x) \mid s \in X,\ a \in F(s)\}.$$

Note that for the sake of *Locality*, we modify the semantics of existential quantifier. The semantics for existential quantifier as above is called *lax semantics*, while the semantics for existential quantifier as in Definition 1.1.7 is called *strict semantics*. For logics having the downwards closure property (e.g., dependence logic), lax and strict semantics coincide, however, for logics lacking of the downwards closure property, only lax semantics respects Locality. For further discussions on lax and strict semantics, see [31] and [30]. With lax semantics defined as above, all of the logic **Ind**, **Inc** and **Exc** are *local*, namely Theorem 1.1.9 holds for these logics. Moreover, all of these logics have the *empty team property*, namely Theorem 1.1.14 holds for these logics.

As mentioned in the introduction, the independence atom

$$x_1 \cdots x_m \perp_{z_1 \cdots z_k} y_1 \cdots y_m$$

corresponds exactly in database theory to the *embedded multivalued dependency*

$$\{z_1, \ldots, z_k\} \twoheadrightarrow \{x_1, \ldots, x_m\} \mid \{y_1, \ldots, y_m\}.$$

Atoms $\bar{x} \twoheadrightarrow \bar{y}$ (called *multivalued dependence atom*) which correspond to *multivalued dependencies* in database theory were introduced in [20], and dependence logic with multivalued dependence atom is proved in [30] to be equivalent to independence logic.

An independence atom $\bar{x} \perp_{\bar{z}} \bar{y}$ in which the underlying set of variables of the tuples $\bar{x}$, $\bar{y}$ and $\bar{z}$ are pairwise disjoint is said to be *normal*. Any independence atom is shown in [30] to be expressible by a normal one.

By [37], the implication problem for unconditional independence atoms can be axiomatized by the following Geiger-Paz-Pearl's axioms:

(i) if $\bar{x} \perp \bar{y}$, then $\bar{x} \perp \bar{y}$;

(ii) if $\bar{x} \perp \bar{y}$ and $\bar{z}$ is a subsequence of $\bar{x}$, then $\bar{z} \perp \bar{y}$;

(iii) if $\bar{u}$ is a permutation of $\bar{x}$, $\bar{v}$ is a permutation of $\bar{y}$, and $\bar{x} \perp \bar{y}$, then $\bar{u} \perp \bar{v}$;

(iv) if $\bar{x} \perp \bar{y}$ and $\bar{x}\bar{y} \perp \bar{z}$, then $\bar{x} \perp \bar{y}\bar{z}$.

In Example 4.7.9 of Chapter 4, we will derive these axioms in the context of propositional independence logic in the natural deduction system of the logic.

Dependence atoms, as observed in [39], are easily definable by independence atoms as follows:

$$=\!(x_1, \ldots, x_{n-1}, x_n) \equiv x_n \perp_{x_1 \cdots x_{n-1}} x_n. \tag{1.1}$$

Therefore **D** is a sublogic of **Ind**. From this, one easily derives that **Ind** $=$ **Ind**$^{[\vee]}$, as it is not hard to see that the formula $\theta_0 \wedge \theta_1$ in Lemma 1.1.16 can also work as a definition for $\phi \vee \psi$ in the logic **Ind**. Basic results concerning the expressive power of **Ind** are listed in the following two theorems.

Theorem 1.2.3 ([39]). *Sentences of* **Ind** *are expressible in* Σ_1^1*, and vice versa.*

Theorem 1.2.4 ([31]).

(i) *For any* L*-formula* $\phi(\bar{x})$ *of* **Ind***, there exists an* $L(R)$*-sentence* $\psi(R)$ *of* Σ_1^1 *with a new predicate* R *such that for any* L*-model* M *and any suitable team* X *of* M*,*

$$M \models_X \phi(\bar{x}) \Longleftrightarrow (M, rel(X)) \models \psi(R).$$

(ii) *For any* $L(R)$*-sentence* $\psi(R)$ *of* Σ_1^1 *with a predicate* R*, there exists an* L*-formula* $\phi(\bar{x})$ *of* **Ind** *such that for any* L*-model* M *and any suitable non-empty team* X *of* M*,*

$$M \models_X \phi(\bar{x}) \Longleftrightarrow (M, rel(X)) \models \psi(R).$$

It follows from the equivalence of independence logic and Σ_1^1 that independence logic is not (effectively) axiomatizable either. In [42] however, the first-order consequences of **Ind** were axiomatized. Computational issues concerning hierarchies in **Ind** are studied in [34].

Inclusion and exclusion logics are clearly expressible in Σ_1^1, therefore they are both sublogics of **Ind**. Moreover, as shown in [31], **Exc** is in fact equivalent to **D**. As a consequence, **Exc** is downwards closed and is equivalent to Σ_1^1, namely Theorem 1.1.10, Theorem 1.1.18 and Theorem 1.1.22 hold for **Exc**.

As observed in [31], inclusion logic is closed under unions, that is the following theorem holds.

Theorem 1.2.5 ([31]). *For any formula* ϕ *of* **Inc***, any suitable model* M*, any collection of suitable teams* $\{X_i\}_{i\in I}$ *of* M*,*

$$\forall i \in I,\ M \models_{X_i} \phi \Longrightarrow M \models_{\bigcup_{i\in I} X_i} \phi.$$

Recent result by Galliani and Hella [35] shows that **Inc** is equivalent to the positive greatest fixed point logic.

Theorem 1.2.6 ([35]). *For any formula* $\phi(R, \bar{x})$ *of the positive greatest fixed point logic, there exists a formula* $\psi(\bar{x})$ *of* **Inc***, and vice versa, such that for all suitable models* M *and all suitable teams* X *of* M*,*

$$M \models_X \psi(\bar{x}) \iff (M, rel(X)) \models_s \phi(R, \bar{x}) \text{ for all } s \in X.$$

In particular, over sentences, **Inc** *and positive greatest fixed point logic (*GFP^+*) have the same expressive power.*

Corollary 1.2.7 ([35]). *On finite structures,* **Inc** *and least fixed point logic (*LFP*) have the same expressive power. In particular, on ordered finite structures,* **Inc** *captures* PTIME.

Proof. It is well-known that over finite structures, LFP is equivalent to GFP^+. By Immerman [54] and Vardi [82], on ordered finite structures, LFP captures PTIME. ∎

Below we summarize the most important results concerning expressive powers of the logics of dependence and independence obtained in the research area so far.

Figure 1.1: First-order logics of dependence and independence

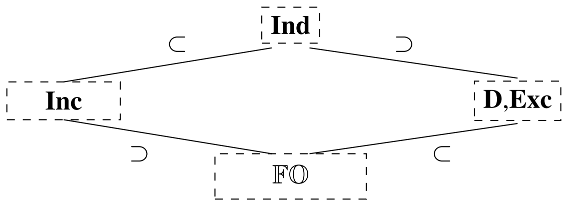


Table 1.3: Expressive power of logics of dependence and independence

Logic	Expressive power	Over finite structures
Ind	Σ^1_1	NP
D	downwards monotone Σ^1_1	NP
Exc	downwards monotone Σ^1_1	NP
Inc	GFP^+	PTIME

Theorem 1.2.8 ([39], [31], [35])**.** *Relationships between the logics of dependence and independence considered in this section are as depicted in Figure 1.1 and Table 1.3.*

Chapter 2

First-oder dependence logic with implications

In this chapter, we investigate the expressive power of first-order dependence logic extended with intuitionistic and linear implications, or that of BID-logic (as by Corollary 1.1.16, intuitionistic disjunction is eliminatable in the logic). In Section 2.1, we sort out interesting properties concerning negation, flat formulas and singleton teams of the logic $\mathbf{BID}^-$ i.e., **BID** without linear implication (which has the empty team property, by Lemma 1.1.14). Section 2.2 introduces an important sublogic of **BID**, called *first-order intuitionistic dependence logic*. In Section 2.3, we prove that over sentences, **BID** and first-order intuitionistic dependence logic are equivalent to the full second-order logic. The content of this section is based on the publication [87]. Section 2.4 proves that formulas of **BID** characterize all second order downwards monotone properties. In Section 2.5, we make some concluding remarks.

Intuitionistic implication and linear implication are of particular interests in the context of **BID**, because, among other things, as pointed out in [1], there are Galois connections between intuitionistic implication $\to$ and conjunction $\wedge$, linear implication $\multimap$ and tensor disjunction $\otimes$, that is, for all formulas ϕ, ψ, χ of **BID**,

$$\phi \wedge \psi \models \chi \Longleftrightarrow \phi \models \psi \to \chi,$$

$$\phi \otimes \psi \models \chi \Longleftrightarrow \phi \models \psi \multimap \chi.$$

In this thesis, we will mainly focus on the properties of intuitionistic implication. As the name suggested, axioms of intuitionistic propositional logic (**IPL**) are valid in $\mathbf{BID}^-$, namely the following axiom schemes are valid in $\mathbf{BID}^-$:

1. $\phi \to (\psi \to \phi)$
2. $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$
3. $\phi \wedge \psi \to \phi$
4. $\phi \wedge \psi \to \psi$

5. $\phi \to \phi \vee \psi$

6. $\psi \to \phi \vee \psi$

7. $\phi \to (\psi \to (\phi \wedge \psi))$

8. $(\phi \to \chi) \to ((\psi \to \chi) \to (\phi \vee \psi \to \chi))$

9. $\bot \to \phi$

Note that the above axioms 1-8 are valid also in **BID**, while axiom 9 (ex falso) is not valid in the presence of linear implication $\multimap$, as the axiom requires the empty team property.

2.1 Negation, flat formulas and singleton teams

In this section, we investigate negation, flat formulas of $\mathbf{BID}^-$, and the behavior of $\mathbf{BID}^-$ formulas under singleton teams.

As pointed out in the discussion after Definition 1.1.1, **BID** or any of its sublogic does not have negation as a preliminary connective. Consider the *classical (contradictory) negation* (denoted by $\sim$) whose team semantics is defined as

- $\mathfrak{M} \models_X \sim \phi$ iff $\mathfrak{M} \not\models_X \phi$.

for any formula ϕ of **BID**, any suitable team X of any suitable model $\mathfrak{M}$. As **BID** is downwards closed, the classical negation is clearly not definable in **BID**[1]. In this section, by the term "negation", we mean the intuitionistic negation $\phi \to \bot$ of formulas ϕ of **BID**. Recall that in Convention 1.1.2, we have reserved the usual negation notation $\neg\phi$ for such intuitionistic negations. For issues concerning negations in dependence logic, the reader is referred to [61],[5].

First of all, one observes easily that the (intuitionistic) negation of **BID** does not satisfy *law of excluded middle* either for tensor disjunction $\otimes$ or for intuitionistic disjunction $\vee$, as, e.g., $\not\models_{\mathbf{BID}} =\!(x) \otimes \neg =\!(x)$ and $\not\models_{\mathbf{BID}} =\!(x) \vee \neg =\!(x)$. On the other hand, the following negation-related formulas are derivable in **IPL**, thus valid in $\mathbf{BID}^-$:

1. $(\phi \to \psi) \to (\neg\psi \to \neg\phi)$

2. $\phi \to \neg\neg\phi$

3. $\neg\neg\neg\phi \leftrightarrow \neg\phi$

Next, we prove an easy but useful fact about flat formulas of $\mathbf{BID}^-$ (recalling Definition 1.1.11 for flatness).

Fact 2.1.1. *Let ϕ and ψ be formulas of* $\mathbf{BID}^-$. *If ψ is flat, then $\phi \to \psi$ is flat.*

---

[1] Suppose ϕ is a formula of **BID** which is equivalent to $\sim \bot$. For any suitable model M and any suitable non-empty team X, since $M \not\models_X \bot$, we have that $M \models_X \phi$. As the formula ϕ of **BID** is downwards closed, we must have that $M \models_\emptyset \phi$, implying $M \not\models_\emptyset \bot$, which is not the case.

Proof. Suppose ψ is flat. It suffices to show that for any suitable model M, any suitable team X of M,

$$\forall s \in X,\ M \models_{\{s\}} \phi \to \psi \Longrightarrow M \models_X \phi \to \psi.$$

Assume the antecedent. Let $Y \subseteq X$ be a non-empty team such that $M \models_Y \phi$. For any $s \in Y$, by the downwards closure property, we have that $M \models_{\{s\}} \phi$, thus $M \models_{\{s\}} \psi$ by assumption. Since ψ is flat, it follows that $M \models_Y \psi$, as required. ■

An immediate corollary of the above fact is that negated formulas of $\mathbf{BID}^-$ are flat. This simple result will play an essential role in Section 4.2 on propositional intuitionistic dependence logic.

Corollary 2.1.2. *Negated formulas are flat, that is $\neg\phi$ (i.e. $\phi \to \bot$) is flat for any formula ϕ of* $\mathbf{BID}^-$.

The double negation law clearly holds for negated formulas ϕ of $\mathbf{BID}^-$, as we have that $\models \neg\phi \leftrightarrow \neg\neg(\neg\phi)$. Next, we show that the validity of the double negation law for a formula is actually a necessary and sufficient condition for the formula being flat.

Lemma 2.1.3. *A formula ϕ of* $\mathbf{BID}^-$ *is flat if and only if it satisfies the double negation law (i.e., $\models \neg\neg\phi \leftrightarrow \phi$ holds). In particular, double negation law holds for classical formulas (or first-order formulas).*

Proof. As $\models \phi \to \neg\neg\phi$ always holds, it suffices to show that for all formulas ϕ of $\mathbf{BID}^-$,

$$\phi \text{ is flat } \Longleftrightarrow \models \neg\neg\phi \to \phi.$$

"$\Longrightarrow$": If ϕ is flat, then $\neg\neg\phi \to \phi$ is flat by Fact 2.1.1. It is easy to see that $M \models_{\{s\}} \neg\neg\phi \to \phi$ holds for any suitable model M and any suitable singleton team $\{s\}$ of M, thus by flatness we obtain that $M \models_X \neg\neg\phi \to \phi$ holds for any suitable team X of M.

"$\Longleftarrow$": Suppose ϕ is not flat. Then there exists a suitable model M and a suitable team X such that

$$M \models_{\{s\}} \phi \text{ for all } s \in X, \text{ but } M \not\models_X \phi.$$

For any $s \in X$, we have that $M \models_{\{s\}} \neg\neg\phi$, thus as $\neg\neg\phi$ is flat, we obtain that $M \models_X \neg\neg\phi$. By assumption, $M \not\models_X \phi$, thus $M \not\models_X \neg\neg\phi \to \phi$. ■

Lemma 2.1.3 gave a characterization of flat formulas of $\mathbf{BID}^-$. An interesting fact (Fact 4.1.11) which we will not be able to state rigouosly until Chapter 4 is that the underlying propositional logic of $\mathbf{BID}^-$ is not closed under uniform substitution, as simply, double negation law fails for non-flat formulas of the logic.

By definition, to determine whether a flat formula ϕ is satisfied by a team X on a model M, it is sufficient to check whether ϕ is satisfied by all singleton teams $\{s\}$ for $s \in X$. In this sense, flat formulas of $\mathbf{BID}^-$ are simple. The next lemma shows that on singleton teams, quantifier-free $\mathbf{BID}^-$ formulas behave actually as classical formulas.

Lemma 2.1.4. *Let ϕ and ψ be formulas of* $\mathbf{BID}^-$. *Let M be a suitable model and s a suitable assignment of M.*

(i) $M \models_{\{s\}} \phi(\top/{=}(x_1,\dots,x_n)) \iff M \models_{\{s\}} \phi$. *In particular,*

$$M \models_{\{s\}} {=}(x_1,\dots,x_n)$$

always holds.

(ii) $M \models_{\{s\}} \neg\phi \iff M \not\models_{\{s\}} \phi$. *In particular, if ϕ is a sentence, then*

$$M \models_{\{\emptyset\}} \neg\phi \iff M \not\models_{\{\emptyset\}} \phi.$$

(iii) $M \models_{\{s\}} \phi \otimes \psi \iff M \models_{\{s\}} \phi \vee \psi \iff M \models_{\{s\}} \neg\phi \to \psi$.

(iv) $M \models_{\{s\}} \phi \to \psi \iff M \models_{\{s\}} \neg\phi \otimes \psi$.

Proof. Straightforward. ■

In Lemma 2.1.4 (i), every occurrence of any dependence atom in the formula ϕ is replaced by the constant $\top$. The resulting formula is written as ϕ^{f} in [78] and such a procedure is called *flattening*. For formulas ϕ of dependence logic **D**, it is shown that $\phi \models \phi^{\mathsf{f}}$. However, this result *cannot* be generalized to $\mathbf{BID}^-$, as, e.g., $\phi = \forall x {=}(x) \to \bot$ is satisfied by all teams of models with cardinality > 1 (since $\forall x {=}(x)$ is never satisfied by any team in such cases), but the flattened formula $\phi^{\mathsf{f}} = \forall x \top \to \bot$ is not satisfied by non-empty teams. Nevertheless, item (i) of the above lemma shows that in $\mathbf{BID}^-$, on singleton teams, ϕ indeed implies ϕ^{f}. On the other hand, in $\mathbf{BID}^-$ the flat formula $\neg\neg\phi$ can be viewed as a type of *flattening* of ϕ, and we know that $\phi \models \neg\neg\phi$.

Adding the "classical (contradictory) negation" into $\mathbf{BID}^-$, we obtain the so-called *team logic* ([78]). Definable team properties of team logic correspond exactly to all second-order properties with respect to non-empty teams, in particular, sentences of team logic have the same expressive power as sentences of the full second-order logic ([59], see also [71]). We have pointed out that classical negation is not definable in $\mathbf{BID}^-$, but Lemma 2.1.4 (ii) illustrates that restricted to singleton teams, the intuitionistic negation behaves as the classical negation.

Moreover, Lemma 2.1.4 (iii) means that on singleton teams, tensor disjunction $\otimes$ is definable using intuitionistic implications $\to$. This result can be strengthened as follows.

Lemma 2.1.5. *For any formulas ϕ and ψ of $\mathbf{BID}^-$, if ϕ is flat, then*

$$\phi \otimes \psi \equiv \neg\phi \to \psi.$$

Proof. It suffices to show that for any suitable model M and any suitable team X of M, it holds that

$$M \models_X \phi \otimes \psi \iff M \models_X \neg\phi \to \psi.$$

$\Longrightarrow$: Suppose $M \models_X \phi \otimes \psi$. Then there exist two teams Y, Z with $X = Y \cup Z$ such that $M \models_Y \phi$ and $M \models_Z \psi$. For any non-empty team $U \subseteq X$ with $M \models_U \neg\phi$, the downwards closure property gives that for any $s \in U$, $M \models_{\{s\}} \neg\phi$, i.e. $M \not\models_{\{s\}} \phi$. Since $M \models_Y \phi$, in view of the the downwards closure property we conclude that $s \notin Y$, thus $U \subseteq Z$, which implies $M \models_U \psi$ by the downwards closure property.

$\Longleftarrow$: Suppose $M \models_X \neg\phi \to \psi$. Define

$$Y = \{s \in X \mid M \models_{\{s\}} \phi\} \text{ and } Z = \{s \in X \mid M \not\models_{\{s\}} \phi\}.$$

Clearly, $X = Y \cup Z$. Since ϕ is flat, we have that $Y \models \phi$. On the other hand, for every $s \in Z$, $M \models_{\{s\}} \neg\phi$, which implies that $M \models_Z \neg\phi$, as $\neg\phi$ is flat by Corollary 2.1.2. Thus, since $M \models_X \neg\phi \to \psi$, we obtain that $M \models_Z \psi$. Hence $M \models_X \phi \otimes \psi$. ■

Corollary 2.1.6. *Law of excluded middle with respect to tensor disjunction holds for flat formulas ϕ of* **BID**$^-$*, i.e., $\models \phi \otimes \neg\phi$ holds whenever ϕ is flat. In particular, law of excluded middle holds for classical formulas.*

Proof. Clearly, $\models \neg\phi \to \neg\phi$, which implies $\models \phi \otimes \neg\phi$ whenever ϕ is flat. ■

We have shown in Lemma 1.1.16 that intuitionistic disjunction $\vee$ is uniformly definable in **D**, therefore it is also definable uniformly in **BID**$^-$ in terms of other connectives. Lemma 2.1.5 defines tensor disjunction $\otimes$ uniformly under certain constraint. We now prove in the next lemma that with an essential use of intuitionistic implication $\to$, tensor disjunction $\otimes$ is uniformly definable in **BID**$^-$ in terms of the other connectives.

Lemma 2.1.7. *For any formulas ϕ, ψ of* **BID**$^-$*, putting*

$$\theta_2 := \exists x \forall y (x = y) \to (\neg\phi \to \psi),$$

$$\theta_3 := \forall x \exists y (x \neq y) \to \exists w \exists u ((w = u \to \phi) \wedge (w \neq u \to \psi)),$$

where $w, u \notin \mathrm{Fv}(\phi) \cup \mathrm{Fv}(\psi)$, we have that $\phi \otimes \psi \equiv \theta_2 \wedge \theta_3$

Proof. If a suitable model M has cardinality 1, then θ_3 is trivially satisfied. On M, there is a unique assignment s. By Lemma 2.1.4(iii),

$$M \models_{\{s\}} \phi \otimes \psi \iff M \models_{\{s\}} \neg\phi \to \psi \iff M \models_{\{s\}} \theta_2 \wedge \theta_3.$$

If M has cardinality > 1, then θ_2 is trivially satisfied. It is then sufficient to show that for any suitable team X of M,

$$M \models_X \exists w \exists u ((w = u \to \phi) \wedge (w \neq u \to \psi)) \iff M \models_X \phi \otimes \psi.$$

We leave the proof of the above expression for the reader. ■

2.2 First-order intuitionistic dependence logic

In this section, we define an important sublogic of **BID**, *first-order intuitionistic dependence logic*, and show some of its basic properties.

Definition 2.2.1. First-order intuitionistic dependence logic (**ID**) is the sublogic

$$\mathbf{BID}[\bot, =(\cdots), \wedge, \vee, \to, \exists, \forall]$$

of **BID**. In other words, well-formed formulas of **ID** are given by the following grammar

$$\phi ::= \alpha \mid \neg\alpha \mid =(x_1, \ldots, x_n) \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \exists x \phi \mid \forall x \phi$$

where α is a first-order atomic formula and $n \geq 1$.

By the team semantics, for a dependence atom $=\!(x)$ with a single variable x,

$$M \models_X =\!(x) \text{ iff for all } s, s' \in X,\ s(x) = s'(x). \tag{2.1}$$

That is, the variable x has a constant value under all assignments in X. We call such dependence atoms *constancy dependence atoms*. First-order dependence logic with constancy dependence atoms only is called *constancy first-order dependence logic*.

Theorem 2.2.2 ([31])**.** *Constancy first-order dependence logic is equivalent to first-order logic (of* **BID***) over sentences.*

The next lemma shows that dependence atoms of the form $=\!(x_1, \ldots, x_n)$ can be decomposed to constancy dependence atoms using intuitionistic implication.

Lemma 2.2.3 ([1])**.** $=\!(x_1, \ldots, x_n) \equiv \big(=\!(x_1) \wedge \cdots \wedge =\!(x_{n-1})\big) \to =\!(x_n)$.

Alternatively, as shown in [32], one can also decompose dependence atoms using the so-called *announcement operator* δx, since

$$=\!(x_1, \ldots, x_n) \equiv \delta x_1 \ldots \delta x_{n-1} =\!(x_n).$$

Putting together all the properties regarding the expressive power of **ID** we have so far, we obtain the following equivalent definitions of **ID**.

Corollary 2.2.4.

$$\begin{aligned} \mathbf{ID} &= \mathbf{ID}[\bot, =\!(\cdot), \wedge, \to, \exists, \forall] \\ &= \mathbf{D}^{[\to]} \\ &= \textit{constancy } \mathbf{D}^{[\to]} \\ &= \mathbf{BID}^{-}. \end{aligned}$$

Proof. Follows from Lemma 1.1.16, Lemma 2.1.7 and Lemma 2.2.3. ∎

Corollary 2.2.5. $\mathbf{BID} = \mathbf{D}^{[\to, \multimap]}$.

In [60], two weak quantifiers $\exists^1$ and $\forall^1$ are introduced. Their team semantics are defined as follows:

Definition 2.2.6. Let M be a suitable model, and X a suitable team of M for a formula ϕ of **BID**. Define

- $M \models_X \exists^1 x\phi$ iff there exists $a \in M$ such that $M \models_{X(a/x)\phi}$, where
$$X(a/x) = \{s(a/x) : s \in X\}.$$
- $M \models_X \forall^1 x\phi$ iff for all $a \in M$, $M \models_{X(a/x)\phi}$.

First-order dependence logic extended with $\exists^1$ and $\forall^1$ is shown in [60] to be equivalent to **D**, that is,

$$\mathbf{D}^{[\exists^1, \forall^1]} = \mathbf{D}.$$

As a consequence, every instances of the formulas $\exists^1 x\phi$ and $\forall^1 x\phi$ are expressible in **D**. A *uniform definition* for the weak existential quantifier $\exists^1$ was given in [60]:

$$\exists^1 x\phi \equiv \exists x(=\!(x) \wedge \phi).$$

On the other hand, Galliani proved in [32] that the weak universal quantifier $\forall^1$ is not *uniformly definable* in **D**. But now, in **ID**, with an essential use of intuitionistic implication, $\forall^1$ *is* uniformly definable, as shown in the following fact.

Fact 2.2.7. *For any formula ϕ of* **ID***, we have that*

$$\forall^1 x\phi \equiv \forall x(=\!(x) \to \phi).$$

Proof. Easy, by the downwards closure property. ∎

Corollary 2.2.8. $\mathbf{ID}^{[\exists^1,\forall^1]} = \mathbf{ID}$.

Another interesting observation is that constancy dependence atoms are definable using the weak existential quantifier $\exists^1$, as shown in the following fact.

Fact 2.2.9. $=\!(x) \equiv \exists^1 y(y = x)$.

Proof. Easy. ∎

ID with weak quantifiers only, i.e., (with some abuse of notation) the logic

$$\mathbf{WID} := \mathbf{ID}[\bot, =\!(\cdots), \wedge, \vee, \to, \exists^1, \forall^1],$$

can be viewed as a weak form of **ID**. An easy consequence of Fact 2.2.9 is that in the weak **ID**, dependence atoms can be eliminated, as recorded in the following corollary.

Corollary 2.2.10. $\mathbf{WID} = \mathbf{ID}[\bot, \wedge, \vee, \to, \exists^1, \forall^1]$.

Proof. By Fact 2.2.9 and Lemma 2.2.3. ∎

We have illustrated in Corollary 2.1.4 that on singleton teams, formulas of **ID** behave classically. As an easy consequence of this fact, over sentences, **WID** is in fact equivalent to first-order logic. However, the expressible power of open formulas of **WID** is unknown.

Theorem 2.2.11. *Sentence of* **WID** *are expressible in first-order logic, and vice versa.*

Proof. By Corollary 2.2.10, it suffices to prove the theorem for first-order logic and the logic $\mathbf{ID}[\bot, \wedge, \vee, \to, \exists^1, \forall^1]$.

Let ϕ be a formula of $\mathbf{ID}[\bot, \wedge, \vee, \to, \exists^1, \forall^1]$, and ϕ^* the formula ϕ viewed as a first-order formula (view $\exists^1$ as the first-order existential quantifier, etc.). We show by induction on ϕ that for any suitable model M, any suitable assignment s of M,

$$M \models_{\{s\}} \phi \iff M \models_s \phi^*. \tag{2.2}$$

We only prove the interesting cases. If $\phi = \exists^1 x\psi(x)$, then

$$\begin{aligned} M \models_{\{s\}} \exists^1 x\psi(x) &\iff \text{there is } a \in M \text{ such that } M \models_{\{s(a/x)\}} \psi(x) \\ &\iff \text{there is } a \in M \text{ such that } M \models_{s(a/x)} \psi(x) \\ &\qquad \text{(by induction hypothesis)} \\ &\iff M \models_s \exists x\psi(x). \end{aligned}$$

The case $\phi = \forall^1 x\psi(x)$ is proved similarly.

It follows from (2.2) that for ϕ being a sentence,

$$M \models_{\{\emptyset\}} \phi \iff M \models_{\mathbf{FO}} \phi^*.$$

Therefore every **WID** sentence is expressible in first-order logic.

Conversely, again by (2.2), every sentence ψ of first-order logic is expressible by the sentence ψ^* of **WID**, where ψ^* is ψ viewed as a sentence of **WID** (view the first-order existential quantifier $\exists$ as $\exists^1$, etc.). ■

Returning to the logic **ID**, in addition to axioms of intuitionistic propositional logic (**IPL**), it is easy to check that axioms of intuitionistic first-order predicate logic (**IQL**) are also valid in **ID**, that is, axioms of **IPL** together with the following ones:

1. $\forall x\phi(x) \to \phi(t)$, where t a first-order term such that no occurrence of any variable in t becomes bound in $\phi(t)$;
2. $\phi(t) \to \exists x\phi(x)$, where t a first-order term such that no occurrence of any variable in t becomes bound in $\phi(t)$.

As for inference rules, Modus Ponens, Generlization Rules of **IQL** listed as follows are valid rules in **ID**:

- $\dfrac{\phi \to \psi \qquad \psi}{\psi}$ (MP)
- $\dfrac{\phi \to \psi(x)}{\phi \to \forall x\psi(x)}$, ($\forall$Gen)

  where x is a variable which does not occur free in ϕ;
- $\dfrac{\psi(x) \to \phi}{\exists x\psi(x) \to \phi(x)}$, ($\exists$Gen)

  where x is a variable which does not occur free in ϕ.

However, we will see in Fact 4.1.11 of Chapter 4 that the underlying propositional logic of **ID** is *not* closed under Uniform Substitution.

Moreover, intuitionistic implication admits Deduction Theorem in **ID**.

Theorem 2.2.12 (Deduction Theorem)**.** *For any formulas ϕ and ψ of* **ID***,*

$$\phi \models \psi \iff \models \phi \to \psi.$$

Proof. By definition and the downwards closure property of **ID**. ■

We end this section by pointing out that axioms schemes of two intermediate logics, Maksimova's Logic ND ([68]) and Kreisel-Putnam Logic KP ([63]), are all valid in **ID**. These axioms in the team semantics setting were first studied in [13] in the context of *inquisitive logic*, which is essentially equivalent to *propositional intuitionistic dependence logic*. We will discuss these axioms in details in the context of propositional intuitionistic dependence logic in Section 4.2.

Fact 2.2.13. *The following axioms are valid in* **ID***.*

ND$_k$ $\big(\neg\phi \to \bigvee_{1\leq i\leq k} \neg\psi_i\big) \leftrightarrow \bigvee_{1\leq i\leq k}(\neg\phi \to \neg\psi_i)$ *for all $k \in \mathbb{N}$;*

KP $(\neg\phi \to \psi \vee \chi) \to \big((\neg\phi \to \psi) \vee (\neg\phi \to \chi)\big)$.

2.3 Expressive power of first-order intuitionistic dependence logic

In this section, we prove that first-order intuitionistic dependence logic is equivalent to the full second-order logic over sentences. The content of this section is based on the publication [87].

It was shown in [1] that **BID** formulas can be translated into second-order logic (**SO**). We record the theorem with a proof sketch as follows.

Theorem 2.3.1 ([1]). *For any L-formula $\phi(\bar{x})$ of* **BID***, there exists a second-order $L(R)$-sentence $\tau_R(\phi) = \psi(R)$ which is downwards monotone with respect to a new predicate R such that for any L-model M and any suitable team X of M,*

$$M \models_X \phi(\bar{x}) \Longleftrightarrow (M, rel(X)) \models \psi(R).$$

Proof. (sketch) We define the translation $\tau_R(\phi)$ by induction on ϕ. Let

$$\begin{aligned}
\tau_R(\theta_0 \vee \theta_1) =& \tau_R(\theta_0) \vee \tau_R(\theta_1),\\
\tau_R(\theta_0 \to \theta_1) =& \forall S(\forall \bar{x}(S(\bar{x}) \to R(\bar{x})) \to (\tau_S(\theta_0) \to \tau_S(\theta_1))),\\
\tau_R(\theta_0 \multimap \theta_1) =& \forall S(\tau_S(\theta_0) \to \forall T(\forall \bar{x}(T(\bar{x}) \leftrightarrow (S(\bar{x}) \vee R(\bar{x}))) \to \tau_T(\theta_1))).
\end{aligned}$$

Other cases are defined in the same way as in [78]. ∎

It follows from the above theorem that **BID** sentences are expressible in second-order logic. We now proceed to prove that the other direction of this statement also holds, namely, there is a translation from the sentences of the full second-order logic into **BID**. From this proof, it will follow that **ID** is equivalent to the full second-order logic over sentences too.

A translation from Σ_1^1-sentences into **D**, which is a sublogic of **BID**, was given in [78]. Below we include a sketch of the proof of this translation (which is the nontrivial direction of Theorem 1.1.18). The idea of this proof will be generalized in the sequel.

Theorem 2.3.2 ([78]). *Σ_1^1-sentences are expressible in* **D**.

Proof. (idea) Without loss of generality, we may assume every Σ_1^1 sentence ϕ is of the following special Skolem normal form

$$\exists f_1 \cdots \exists f_n \forall x_1 \cdots \forall x_m \psi,$$

where ψ is a quantifier-free formula of first-order logic, and for each $1 \leq i \leq n$, every occurrence of the function symbol f_i is of the same form $f_i x_{i_1} \ldots x_{i_q}$ for some fixed sequence $\langle x_{i_1}, \ldots, x_{i_q} \rangle$ of variables from the set $\{x_1, \ldots, x_m\}$. We find a sentence ϕ^* of **D** which expresses ϕ. The idea behind the sentence ϕ^* is that in ϕ, we replace each occurrence of the function symbol f_i by a new variable y_i, and add a dependence atom to specify that y_i is functionally determined by the arguments $x_{i_1}, \ldots, x_{i_q}$ of f_i. This can be done because we have required that each occurrence of f_i is of the same form $f_i x_{i_1} \ldots x_{i_q}$. To be precise, the sentence ϕ^* of **D** is defined as follows:

$$\begin{aligned}
\phi^* := \forall x_1 \cdots \forall x_m \exists y_1 \cdots \exists y_n (=\!(x_{1_1}, \ldots, x_{1_q}, y_1) \wedge \\
\cdots \wedge =\!(x_{n_1}, \ldots, x_{n_q}, y_n) \wedge \psi'),
\end{aligned} \tag{2.3}$$

where ψ' is a formula of **D** obtained from the formula ψ of first-order logic by replacing everywhere the classical disjunction by $\otimes$, and $f_i x_{i_1} \dots x_{i_q}$ by a new variable y_i for each i. In ϕ^*, the dependence atoms together with the existential quantifiers enable us to pick exactly those functions corresponding to the functions assigned to the existentially quantified function variables $f_1, \dots, f_n$ in ϕ. ■

Remark 2.3.3. *Equation (2.3) with the first-order quantifier-free formula ψ' in conjunctive normal form is a normal form for sentences of* **D**.

In the remaining part of this section, we give a direction translation from sentences of the full second-order logic into **BID**. First, we recall the normal form of second-order logic formulas.

Theorem 2.3.4 (Normal Form of **SO**). *Every second-order formula is equivalent to a formula of the form*

$$\forall \overline{f^1} \exists \overline{f^2} \cdots \forall \overline{f^{2n-1}} \exists \overline{f^{2n}} \forall \overline{x} \psi,$$

where ψ is quantifier-free, and we assume without loss of generality that for the corresponding $Q \in \{\forall, \exists\}$, each $Q\overline{f^i} = Qf^i_1 \cdots Qf^i_p$ and each f^i_j is of arity q.

The basic idea of the translation for sentences of the full second-order logic is generalized from that of the proof of Theorem 2.3.2 for Σ^1_1-sentences. For each second-order sentence in a special normal form (to be clarified in Lemma 2.3.7), we replace each function variable by a new variable and specify the functionality of the new variable by adding the corresponding dependence atoms. We have seen in the proof of Theorem 2.3.2 that dependence atoms together with existential quantifiers enable us to simulate existentially quantified function variables; on the other hand, universally quantified function variables can also be simulated using dependence atoms and intuitionistic implications. In this way, we will be able to express all second-order sentences in **BID**.

To make this idea work, we need to first turn every second-order sentence ϕ into a better normal form than the one given in Theorem 2.3.4, that is we need to guarantee that for each q-ary function variable f^i_j, every occurrence of f^i_j in ϕ is of the same form $f^i_j x_{i,j_1} \dots x_{i,j_q}$ for some fixed sequence $\langle x_{i,j_1} \dots x_{i,j_q} \rangle$ of variables (this normal form is inspired by the Σ^1_1 normal form in Theorem 2.3.2, see Section 6.3 in [78] for detailed discussions). To this end, we need three lemmas.

The first lemma removes nesting of function symbols in a formula.

Lemma 2.3.5. *Let $\phi(ft_1 \dots t_q)$ be any first-order formula, in which the q-ary function symbol f has an occurrence of the form $ft_1 \dots t_q$ for some terms $t_1 \dots t_q$. Then we have that*

$$\models \phi(ft_1 \dots t_q) \leftrightarrow \forall x_1 \cdots \forall x_q ((t_1 = x_1) \wedge \cdots \wedge (t_q = x_q) \to \phi(fx_1 \dots x_q)),$$

where $x_1, \dots, x_q$ are new variables and $\phi(fx_1 \dots x_q)$ is the formula obtained from $\phi(ft_1 \dots t_q)$ by replacing everywhere $ft_1 \dots t_q$ by $fx_1 \dots x_q$.

Proof. Easy. ■

The second lemma unifies the arguments of function symbols in a formula.

Lemma 2.3.6. *Let $\phi(fx_1 \dots x_q, fy_1 \dots y_q)$ be a first-order formula, in which the q-ary function symbol f has an occurrence of the form $fx_1 \dots x_q$ and an occurrence of the form $fy_1 \dots y_q$ with $\{x_1, \dots, x_q\} \cap \{y_1, \dots, y_q\} = \emptyset$. Then we have that*

$$\begin{aligned}&\models \forall x_1 \cdots \forall x_q \forall y_1 \cdots \forall y_q \phi(fx_1 \dots x_q, fy_1 \dots y_q)\\ &\quad \leftrightarrow \exists g \forall x_1 \cdots \forall x_q \forall y_1 \cdots \forall y_q \big(\phi(fx_1 \dots x_q, gy_1 \dots y_q)\\ &\quad \wedge ((x_1 = y_1) \wedge \cdots \wedge (x_q = y_q) \rightarrow (fx_1 \dots x_q = gy_1 \dots y_q))\big),\end{aligned}$$

where $\phi(fx_1 \dots x_q, gy_1 \dots y_q)$ is the first-order formula obtained from the formula $\phi(fx_1 \dots, x_q, fy_1 \dots y_q)$ by replacing everywhere $fy_1 \dots y_q$ by $gy_1 \dots y_q$.

Proof. Easy. ■

The next lemma gives the intended normal form for second-order formulas.

Lemma 2.3.7. *Every second-order formula is equivalent to a formula ϕ of the form*

$$\forall f_1^1 \cdots \forall f_p^1 \exists f_1^2 \cdots \exists f_p^2 \cdots \cdots \forall f_1^{2n-1} \cdots \forall f_p^{2n-1} \exists f_1^{2n} \cdots \exists f_p^{2n} \forall x_1 \cdots \forall x_m \psi,$$

where

- *ψ is quantifier free;*
- *each function symbol f_j^i is of arity q, and its every occurrence is of the same form $f_j^i \mathbf{x}^{i,j}$, where*
$$\mathbf{x}^{i,j} = \langle x_{i,j_1}, \dots, x_{i,j_q} \rangle$$
with $\{x_{i,j_1}, \dots, x_{i,j_q}\} \subseteq \{x_1, \dots, x_m\}$.

Proof. Start with a formula in the normal form described in Theorem 2.3.4, apply Lemma 2.3.5 and Lemma 2.3.6 several times, and add dummy quantifiers. ■

The next lemma states that under the right valuations, the behavior of functions can be simulated by new variables. This technical lemma will play a role in the proof of Theorem 2.3.9.

Lemma 2.3.8. *Let $\psi(\overline{f}, \overline{x})$ be any quantifier-free formula of first-order logic with*

$$\bar{f} = \langle f_1, \dots, f_p \rangle \text{ and } \bar{x} = \langle x_1, \dots, x_m \rangle,$$

where each function symbol f_j is of arity q, and its every occurrence is of the same form

$$f_j x_{j_1} \dots x_{j_q} \text{ with } \{x_{j_1} \dots x_{j_q}\} \subseteq \{x_1, \dots, x_m\}.$$

Let $(M, \overline{F})$ be any suitable model with function symbols $f_1, \dots, f_p$ interpreted as $F_1, \dots, F_p$, respectively. Consider the classical formula ψ' of **BID** *obtained from ϕ by replacing everywhere the classical disjunction by $\otimes$, and $f_j x_{j_1} \dots x_{j_q}$ by a new variable y_j for each $1 \leq j \leq p$. Let s be a suitable assignment for ψ' such that for all $1 \leq j \leq p$,*

$$s(y_j) = F_j(s(x_{j_1}), \dots, s(x_{j_q})). \tag{2.4}$$

Then

$$(M, \overline{F}, s(\overline{x})) \models \psi(\overline{f}, \overline{x}) \Longleftrightarrow M \models_{\{s\}} \psi'.$$

Proof. It is easy to show by induction that for any term t, $t\langle s\rangle = t'\langle s\rangle$, where t' is obtained from t by replacing everywhere $f_j x_{j_1} \ldots x_{j_q}$ by y_j for each $1 \leq j \leq p$. Next, we show the lemma by induction on ψ. The only interesting case is the case $\psi = \theta_0 \vee \theta_1$. In this case, we have that

$$\begin{aligned}(M, \overline{F}, s(\overline{x})) \models \theta_0 \vee \theta_1 &\Longleftrightarrow (M, \overline{F}, s(\overline{x})) \models \theta_0 \text{ or } (M, \overline{F}, s(\overline{x})) \models \theta_1\\ &\Longleftrightarrow M \models_{\{s\}} \theta_0' \text{ or } M \models_{\{s\}} \theta_1'\\ &\qquad \text{(by induction hypothesis)}\\ &\Longleftrightarrow M \models_{\{s\}} \theta_0' \otimes \theta_1'\\ &\qquad \text{(since } \{s\} = \{s\} \cup \{s\} = \{s\} \cup \emptyset).\end{aligned}$$

■

Now we are in a position to give the translation from second-order sentences into **BID**. In the proof of the following theorem, we abbreviate a sequence of the form $\langle u_{i,1}, \ldots, u_{i,p}\rangle$ by $\overline{u_i}$, and $\langle F_1^i, \ldots, F_p^i\rangle$ by $\overline{F^i}$.

Theorem 2.3.9. *Second-order sentences are expressible in* **BID** *and vice versa.*

Proof. It follows from Theorem 2.3.1 that **BID** sentences are expressible in second-order logic. For the other direction, without loss of generality, we may assume that every second-order sentence ϕ is of the form described in Lemma 2.3.7. For each pair $\langle i, j\rangle$ ($1 \leq i \leq 2n$, $1 \leq j \leq p$), pick a new variable $u_{i,j}$ not occurring in ϕ. We inductively define formulas δ_i of **BID** for $2n \geq i \geq 1$ as follows:

- let
$$\delta_{2n} := \exists u_{2n,1} \cdots \exists u_{2n,p}(\Theta_{2n} \wedge \psi');$$
- for $2n > i \geq 1$, let
$$\delta_i := \begin{cases} \Theta_i \to \delta_{i+1}, & \text{if } i \text{ is odd;}\\ \exists u_{i,1} \cdots \exists u_{i,p}(\Theta_i \wedge \delta_{i+1}), & \text{if } i \text{ is even,}\end{cases}$$

  where
$$\Theta_i = \bigwedge_{j=1}^{p} =(\mathbf{x}^{i,j}, u_{i,j})$$
  and ψ' is the classical formula of **BID** obtained from the formula ψ of first-order logic by replacing everywhere the classical disjunction by $\otimes$, and each $f_j^i \mathbf{x}^{i,j}$ by $u_{i,j}$.

Let

$$\phi^* = \forall u_{1,1} \cdots \forall u_{1,p} \forall u_{3,1} \cdots \forall u_{3,p} \cdots \cdots \forall u_{2n-1,1} \cdots \forall u_{2n-1,p} \forall \overline{x} \delta_1 \tag{2.5}$$

$$\Big[\text{i.e. } \phi^* = \forall u_{1,1} \cdots \forall u_{1,p} \forall u_{3,1} \cdots \forall u_{3,p} \cdots \cdots \forall u_{2n-1,1} \cdots \forall u_{2n-1,p} \forall \overline{x}\\ (\Theta_1 \to \exists u_{2,1} \cdots \exists u_{2,p}(\Theta_2 \wedge (\Theta_3 \to \exists u_{4,1} \cdots \exists u_{4,p}(\Theta_4 \wedge \cdots \cdots\\ \cdots \cdots \wedge (\Theta_{2n-1} \to \exists u_{2n,1} \cdots \exists u_{2n,p}(\Theta_{2n} \wedge \psi' \underbrace{)) \cdots \cdots))))}_{2n}\Big].$$

The general idea behind the **BID** formula ϕ^* is that the δ_i's for i odd, simulate the $\forall\overline{f^i}$'s, and the δ_i's for i even, simulate the $\exists\overline{f^i}$'s in the second-order sentence ϕ. The rest of the proof is devoted to show that such sentence ϕ^* does express the second-order sentence ϕ, i.e. to show that for any suitable model M,

$$M \models \phi \Longleftrightarrow M \models_{\{\emptyset\}} \phi^*.$$

"$\Longrightarrow$": Suppose $M \models \phi$. Then for any sequence of functions

$$F_1^1, \dots, F_p^1 : M^q \to M,$$

there exists a sequence of functions (depending on $\overline{F^1}$)

$$F_1^2{}_{(\overline{F^1})}, \dots, F_p^2{}_{(\overline{F^1})} : M^q \to M$$

such that for any ... ... for any sequence of functions

$$F_1^{2n-1}, \dots, F_p^{2n-1} : M^q \to M,$$

there exists a sequence of functions (depending on $\overline{F^1}, \overline{F^3} \dots, \overline{F^{2n-1}}$)

$$F_1^{2n}{}_{(\overline{F^1},\dots,\overline{F^{2n-1}})}, \dots, F_p^{2n}{}_{(\overline{F^1},\dots,\overline{F^{2n-1}})} : M^q \to M$$

such that

$$(M, \overline{F^1}, \dots, \overline{F^{2n}}) \models \forall\overline{x}\psi(\overline{f^1}, \dots, \overline{f^{2n}}). \tag{2.6}$$

Let Y_1 be a non-empty subteam of

$$X = \{\emptyset\}(M/\overline{u_1}, \overline{u_3}, \dots, \overline{u_{2n-1}}, \overline{x})$$

such that $M \models_{Y_1} \Theta_1$. It suffices to show that

$$M \models_{Y_1} \delta_2, \text{ i.e. } M \models_{Y_1} \exists u_{2,1} \cdots \exists u_{2,q}(\Theta_2 \wedge \delta_3). \tag{2.7}$$

The team Y_1 corresponds to a sequence of functions $F_1^1{}_{(Y_1)}, \dots, F_p^1{}_{(Y_1)} : M^q \to M$ defined as follows: for any $1 \leq j \leq p$, and for some fixed $a_0 \in M$, let

$$F_j^1(\overline{d}) = \begin{cases} s(u_{1,j}), & \text{if there exists } s \in Y_1 \text{ such that } s(\mathbf{x}^{1,j}) = \overline{d}; \\ a_0 \in M, & \text{otherwise.} \end{cases}$$

Each F_j^1 is well-defined. Indeed, for any $\overline{d} \in M^q$, any $s, s' \in Y_1$ such that

$$s(\mathbf{x}^{1,j}) = \overline{d} = s'(\mathbf{x}^{1,j}),$$

since $M \models_{Y_1} =\!(\mathbf{x}^{1,j}, u_{1,j})$, we must have that

$$s(u_{1,j}) = s'(u_{1,j}).$$

Now, using the functions $F_1^2{}_{(\overline{F^1})}, \dots, F_p^2{}_{(\overline{F^1})}$ given by the assumption, we define a sequence of functions $\alpha_{2,1}{}_{(F_1^2)}, \dots, \alpha_{2,p}{}_{(F_p^2)}$ from the corresponding supplement teams of

Y_1 to M such that the supplement team $Y_1(\alpha_{2,1}/u_{2,1})\dots(\alpha_{2,p}/u_{2,p})$ satisfies $\Theta_2 \wedge \delta_3$. For each $1 \leq j \leq p$, define the function

$$\alpha_{2,j} : Y_1(\alpha_{2,1}/u_{2,1})\dots(\alpha_{2,j-1}/u_{2,j-1}) \to M$$

corresponding to $F^2_{j}(\overline{F^1})$ by taking

$$\alpha_{2,j}(s) = F^2_j(s(\mathbf{x}^{2,j})).$$

Put

$$Y_2 = Y_1(\alpha_{2,1}/u_{2,1})\dots(\alpha_{2,p}/u_{2,p}).$$

It suffices to show that $M \models_{Y_2} \Theta_2$ and

$$M \models_{Y_2} \delta_3, \text{ i.e. } M \models_{Y_2} \Theta_3 \to \delta_4. \tag{2.8}$$

The former is obvious by the definitions of Y_2 and $\overline{\alpha_2}$. To show the latter, repeat the same argument and construction $n-1$ times, and it then suffices to show that for any non-empty subteams Y_3 of Y_2, Y_5 of Y_4, … , Y_{2n-1} of Y_{2n-2} such that

$$M \models_{Y_3} \Theta_3,\ M \models_{Y_5} \Theta_5,\ \dots,\ M \models_{Y_{2n-1}} \Theta_{2n-1},$$

it holds that

$$M \models_{Y_4} \Theta_4, M \models_{Y_6} \Theta_6, \dots, M \models_{Y_{2n}} \Theta_{2n} \tag{2.9}$$

and $M \models_{Y_{2n}} \psi'$, where $Y_4, Y_6 \dots, Y_{2n}$ are supplement teams defined in the same way as above. Clause (2.9) follows immediately from the definitions of $Y_4, Y_6, \dots, Y_{2n}$ and $\overline{\alpha_4}, \overline{\alpha_6}, \dots \overline{\alpha_{2n}}$. To show $M \models_{Y_{2n}} \psi'$, since ψ' is flat (classical), it suffices to show $M \models_{\{s\}} \psi'$ holds for all $s \in Y_{2n}$.

For the functions $\overline{F^1}(Y_1), \overline{F^2}(\overline{F^1}), \dots, \overline{F^{2n-1}}(Y_{2n-1}), \overline{F^{2n}}(\overline{F^{2n-1}})$ obtained as above, by (2.6) we have that

$$(M, \overline{F^1}, \dots, \overline{F^{2n}}, s(\overline{x})) \models \psi(\overline{f^1}, \dots, \overline{f^{2n}}, \overline{x}).$$

Now, it follows from the definitions of $\overline{F^1}, \dots, \overline{F^{2n}}$ that condition (2.4) in Lemma 2.3.8 is satisfied for each F^i_j, hence, an application of Lemma 2.3.8 gives the desired result that $M \models_{\{s\}} \psi'$.

"$\Longleftarrow$": Suppose $M \models_{\{\emptyset\}} \phi^*$. Then

$$M \models_X \Theta_1 \to \delta_2,$$

where

$$X = \{\emptyset\}(M/\overline{u_1}, \overline{u_3}, \dots, \overline{u_{2n-1}}, \overline{x}).$$

Let $F^1_1, \dots, F^1_p : M^q \to M$ be an arbitrary sequence of functions. Take a subteam $Y_1(\overline{F^1})$ of X which corresponds to these functions by putting

$$Y_1 = \{s \in \{\emptyset\}(M/\overline{u_1}, \overline{u_3} \dots \overline{u_{2n-1}}, \overline{x}) \\ \mid s(u_{1,1}) = F^1_1(s(\mathbf{x}^{1,1})), \dots, s(u_{1,p}) = F^1_p(s(\mathbf{x}^{1,p}))\}.$$

Clearly, $M \models_{Y_1} \Theta_1$ holds, thus we have that $M \models_{Y_1} \delta_2$ holds (i.e., (2.7) holds). So there exist functions

$$\alpha_{2,1(\overline{F^1})} : Y_1 \to M, \ldots\ldots, \alpha_{2,p(\overline{F^1})} : Y_1(\alpha_{2,1}/u_{2,1})\ldots(\alpha_{2,p-1}/u_{2,p-1}) \to M$$

depending on $\overline{F^1}$ such that $M \models_{Y_2} \Theta_2$ and $M \models_{Y_2} \delta_3$ holds (i.e., (2.8) holds), where

$$Y_2 = Y_1(\alpha_{2,1}/u_{2,1})\ldots(\alpha_{2,p}/u_{2,p}).$$

Now, we define functions $F^2_{1}(\overline{F^1}), \ldots, F^2_{p}(\overline{F^1}) : M^q \to M$, which simulate $\alpha_{2,1}, \ldots, \alpha_{2,p}$ as follows: for each $1 \le j \le p$ and for any $\overline{d} \in M^q$, let

$$F^2_j(\overline{d}) = s(u_{2,j}) \text{ for some } s \in Y_2 \text{ such that } s(\mathbf{x}^{2,j}) = \overline{d}.$$

Note that the definition of Y_2 guarantees such s in the above definition always exists, and moreover, each F^2_j is well-defined since for any $s, s' \in Y_2$ with

$$s(\mathbf{x}^{2,j}) = \overline{d} = s'(\mathbf{x}^{2,j}),$$

as $M \models_{Y_2} {=}(\mathbf{x}^{2,j}, u_{2,j})$, we must have that

$$s(u_{2,j}) = s'(u_{2,j}).$$

Repeat the same argument and construction $n-1$ times to define inductively for any sequences of functions $\overline{F^3}, \overline{F^5}, \ldots, \overline{F^{2n-1}}$, the subteams Y_3 of Y_2, … , Y_{2n} of Y_{2n-1} such that

$$M \models_{Y_3} \Theta_3,\ M \models_{Y_5} \Theta_5, \ldots, M \models_{Y_{2n-1}} \Theta_{2n-1},$$

and the supplement teams $Y_4, Y_6, \ldots, Y_{2n}$ satisfy

$$M \models_{Y_4} \Theta_4,\ M \models_{Y_6} \Theta_6, \ldots, M \models_{Y_{2n-2}} \Theta_{2n-2},\ M \models_{Y_{2n}} \Theta_{2n} \wedge \psi',$$

and to define inductively the sequences of functions

$$\overline{F^4}, \overline{F^6} \ldots, \overline{F^{2n}} : M^q \to M,$$

according to the functions $\overline{\alpha_4}, \overline{\alpha_6}, \ldots, \overline{\alpha_{2n}}$ obtained from the existential quantifiers $\exists\overline{u_4}$, $\exists\overline{u_6}, \ldots, \exists\overline{u_{2n}}$. It then suffices to show that

$$(M, \overline{F^1}, \ldots, \overline{F^{2n}}) \models \forall\overline{x}\psi(\overline{f^1}, \ldots, \overline{f^{2n}}).$$

Let $\overline{a}$ be an arbitrary sequence in M of the same length as that of $\overline{x}$. By the construction of Y_{2n}, there must exists $s \in Y_{2n}$ such that $s(\overline{x}) = \overline{a}$. Since $M \models_{Y_{2n}} \psi'$, by the downwards closure property, we have that $M \models_{\{s\}} \psi'$. Note that by the definitions of $\overline{F^1}, \ldots, \overline{F^{2n}}$, condition (2.4) in Lemma 2.3.8 is satisfied for each F^i_j, hence, an application of Lemma 2.3.8 gives the desired result that

$$(M, \overline{F^1}, \ldots, \overline{F^{2n}}, s(\overline{x})) \models \psi(\overline{f^1}, \ldots, \overline{f^{2n}}, \overline{x}).$$

■

Theorem 2.3.10. *Second-order sentences are expressible in* **ID**, *and vice versa.*

Proof. It follows from Theorem 2.3.1 that **ID** sentences are expressible in second-order logic. For the other direction, note that in the proof of Theorem 2.3.9, the linear implication and intuitionistic disjunction did not play any role, thus it follows that second-order sentences are expressible in $\mathbf{D}^{[\multimap]}$, which is equivalent in **ID** by Corollary 2.2.4. ■

Remark 2.3.11. *In fact, Lemma 2.3.7 gives a normal form for Π^1_{2n}-sentences ($n \in \omega$), therefore the formula in Equation (2.5) can be viewed as an* **ID***-normal form for Π^1_{2n}-sentences.*

Moreover, every Σ^1_{2n}-sentence ϕ is equivalent to a sentence $\neg\psi$, where ψ is a Π^1_{2n}-sentence. Taking ψ^* to be the formula in Equation (2.5) for ψ, by Lemma 2.1.4, we have that

$$M \models \phi \iff M \not\models \psi \iff M \not\models_{\{\emptyset\}} \psi^* \iff M \models_{\{\emptyset\}} \neg\psi^*,$$

namely the sentence $\neg\psi^*$ of **ID** *is the translation of the Σ^1_{2n}-sentence ϕ.*

Obviously, applying the trick of Lemma 2.3.7, one can also obtain a nice normal form for Σ^1_{2n-1} sentence, so the above observation holds for Σ^1_{2n-1}- and Π^1_{2n-1}-sentences as well. In particular, the proof of Theorem 2.3.2 (for Σ^1_1-sentences) can then be viewed as a special case of the proof of Theorem 2.3.9.

As mentioned, *team logic* [78] (**TL**), which is dependence logic extended with classical negation is also equivalent to the full second-order logic over sentences ([59], see also [71]). The significance of our result here is that the equivalence of **BID** (or already **ID**) and the full **SO** on the sentence level is established without the presence of the logical connective classical negation.

We summarize the results of the expressive power of sublogics of **BID** over sentences we have obtained so far in Figure 2.1.

Figure 2.1: Expressive power of sublogics of **BID** over sentences

SO	**BID**, **ID**, $\mathbf{D}^{[\multimap]}$, constancy $\mathbf{D}^{[\multimap]}$, **TL**
Σ^1_1	**D**
FO	constancy **D**

2.4 Definability in BID-logic

Formulas of **D** characterize Σ^1_1 downwards monotone properties with respect to non-empty teams ([78],[60], or Theorem 1.1.22). In this section, we show that formulas of **BID** characterize all second-order downwards monotone properties with respect to all teams (including the empty team). The argument of this section is divided into two parts: in the first part, we deal with these properties over non-empty teams with formulas of $\mathbf{BID}^-$, while in the second part, we use linear implication to obtain the missing piece with respect to the empty team. The method presented in this section combines those in [60] and in Section 2.3.

We start with giving a syntactical characterization for second order downwards monotone sentences (recall Definition 1.1.19). This generalizes Lemma 1.1.20 ([60]) with essentially the same proof as that of Proposition 4.7 in [60].

Lemma 2.4.1 (due to [60])**.** *A second-order $L(R)$-sentence $\phi(R)$ is downwards monotone with respect to a predicate R iff there exists an equivalent second-order $L(R)$-sentence ψ in which R occurs only negatively.*

Proof. "$\Longleftarrow$": Assume $\phi(R)$ has only negative occurrences of R and $\phi(R)$ is in negation normal form. It suffices to show by induction on subformulas $\theta(R)$ of $\phi(R)$ that $\theta(R)$ is downwards monotone. The only interesting case $\theta = \neg R\bar{t}$ is easily verified.

"$\Longrightarrow$": Suppose $\phi(R)$ is downwards monotone with respect to R. Let $\phi(S)$ be the formula obtained from ϕ by replacing every occurrence of R by a new predicate-variable S. Letting

$$\psi(R) = \exists S(\phi(S) \wedge \forall \bar{x}(R\bar{x} \to S\bar{x})),$$

where R occurs only negatively, it is straightforward to verify by downwards monotonicity that $\models \phi(R) \leftrightarrow \psi(R)$. ■

Next, we generalize Lemma 2.3.7 and obtain a normal form for every second-order downwards monotone sentence.

Lemma 2.4.2. *Every second-order downwards monotone sentence with respect to a predicate R is equivalent to a formula of the form*

$$\exists g_0 \exists g_1 \forall f_1^1 \cdots \forall f_p^1 \exists f_1^2 \cdots \exists f_p^2 \cdots \cdots \exists f_1^{2n} \cdots \exists f_p^{2n} \forall \bar{x} \forall \bar{y}(\psi \wedge (R\bar{y} \to (g_0\bar{y} = g_1\bar{y}))),$$

where

- *ψ is quantifier-free and does not contain the predicate R;*
- *each function symbol f_j^i is of arity q, and its every occurrence is of the same form $f_j^i\mathbf{x}^{i,j}$, where*
$$\mathbf{x}^{i,j} = \langle x_{i,j_1}, \ldots, x_{i,j_q} \rangle$$
with $\{x_{i,j_1}, \ldots, x_{i,j_q}\} \subseteq \{x_1, \ldots, x_m\}$;
- *every occurrence of the function symbol g_l ($l = 0, 1$) is of the same form $g_l\bar{y}$.*

Proof. Let $\phi(R)$ be a sentence as described. First, apply Lemma 2.3.7 to obtain an equivalent sentence $\theta(R)$ in the normal form described in the lemma. By Lemma 2.4.1, $\theta(R)$ is equivalent to

$$\exists S(\theta(S) \wedge \forall \bar{y}(R\bar{y} \to S\bar{y})),$$

where S is a new predicate symbol. This sentence is equivalent to

$$\exists g_0 \exists g_1(\theta(g_0\bar{t} = g_1\bar{t}) \wedge \forall \bar{y}(R\bar{y} \to (g_0\bar{y} = g_1\bar{y}))),$$

where $\theta(g_0\bar{t} = g_1\bar{t})$ is obtained from $\theta(S)$ by replacing every occurrence of $S\bar{t}$ by $(g_0\bar{t} = g_1\bar{t})$, which is clearly equivalent to a formula of the required form. ■

Let X be a team of M and s an assignment on M such that $dom(X)\cap dom(s)=\emptyset$. We write Xs for the set

$$Xs=\{t^\frown s\mid t\in X\}.$$

Next, we prove the main theorem of this section that every second-order downwards monotone property is definable *with respect to non-empty teams* by a formula of $\mathbf{BID}^-$.

Theorem 2.4.3. *For any second-order $L(R)$-sentence $\phi(R)$ which is downwards monotone with respect to a k-ary predicate R, there exists a formula $\phi^*(w_1,\cdots,w_k)$ of $\mathbf{BID}^-$ such that for any L-model M and any non-empty team X of M with domain $\{w_1,\cdots,w_k\}$,*

$$(M,rel(X))\models\phi(R)\Longleftrightarrow M\models_X\phi^*(\bar{w}).\tag{2.10}$$

Proof. We may assume that every downwards monotone $L(R)$-sentence $\phi(R)$ is of the normal form described in Lemma 2.4.2. The required formula $\phi^*(w_1,\cdots,w_k)$ is the same as Formula (2.5) in the proof of Theorem 2.3.9, except we now let

$$\delta_{2n}(w_1,\dots,w_k)\ :=\ \exists u_{2n,1}\cdots\exists u_{2n,p}\Big(\Theta_{2n}\wedge\psi'\wedge\big(\bigotimes_{i=1}^{k}(w_i\neq y_i)\otimes(v_0=v_1)\big)\Big),$$

where ψ' is the classical formula of $\mathbf{BID}^-$ obtained from the formula ψ of first-order logic by replacing everywhere

- the classical disjunction by $\otimes$,
- each $f_j^i\mathbf{x}^{i,j}$ by $u_{i,j}$,
- each $g_l y_1\dots y_k$ by v_l,

and

$$\begin{aligned}\phi^*(\bar{w}):=\forall u_{1,1}\cdots\forall u_{1,p}\forall u_{3,1}\cdots\forall u_{3,p}\cdots\cdots\forall u_{2n-1,1}\cdots\forall u_{2n-1,p}\forall\overline{x}\forall\overline{y}&\\ \exists v_0\exists v_1(=(\bar{y},v_0)\wedge=(\bar{y},v_1)\wedge\delta_1(\bar{w})).&\end{aligned}\tag{2.11}$$

It remains to show that for any suitable model M, any non-empty team X of M with domain $\{w_1,\dots,w_k\}$, (2.10) holds. The proof goes through a very similar argument to that in the proof of Theorem 2.3.9. We will only show here the different steps.

For the direction "$\Longrightarrow$", assuming $(M,rel(X))\models\phi(R)$ for $X\neq\emptyset$, let Y_{2n+1} be the team obtained by the same argument as that in the proof of Theorem 2.3.9. We will show that

$$M\models_{Y_{2n+1}}\psi'\wedge\big(\bigotimes_{i=1}^{k}(w_i\neq y_i)\otimes(v_0=v_1)\big).$$

Since R does not occur in ψ and ψ' is flat, as in the proof of Theorem 2.3.9, $M\models_{Y_{2n+1}}\psi'$ follows from Lemma 2.3.8. It then remains to show that

$$M\models_{Y_{2n+1}}\bigotimes_{i=1}^{k}(w_i\neq y_i)\otimes(v_0=v_1).$$

Observe that the above formula is classical, thus it is flat. It then suffices to show that for all $s \in Y_{2n+1}$,

$$M \models_{\{s\}} \bigotimes_{i=1}^{k} (w_i \neq y_i) \otimes (v_0 = v_1). \tag{2.12}$$

Indeed, if $s(\bar{w}) = s(\bar{y})$, then

$$s(\bar{y}) = s(\bar{w}) \in rel(X).$$

By assumption, there exist functions $G_0, G_1 : M^k \to M$ such that

$$(M, rel(X), G_0, G_1, s(\bar{y})) \models R\bar{y} \to (g_0\bar{y} = g_1\bar{y}),$$

thus

$$s(v_0) = G_0(s(\bar{y})) = G_1(s(\bar{y})) = s(v_1).$$

Hence (2.12) is obtained.

Conversely, for the direction "$\Longleftarrow$", assuming $M \models_X \phi^*$ for $X \neq \emptyset$, let G_0, G_1 be the team obtained by the same argument as that in the proof of Theorem 2.3.9. Let $\overline{a}$ and $\overline{b}$ be arbitrary sequences of elements in M of the same lengths as $\bar{x}$ and $\bar{y}$, respectively. By the construction of Y_{2n+1}, there must exists an assignment

$$s : (dom(Y_{2n+1}) \setminus dom(X)) \to M$$

such that $Xs \subseteq Y_{2n+1}$ and

$$s(\overline{x}) = \overline{a} \text{ and } s(\overline{y}) = \overline{b}.$$

We show that

$$(M, G_0, G_1, \overline{F^1}, \ldots, \overline{F^{2n}}, s(\overline{x}), s(\overline{y})) \models \psi(g_0, g_1, \overline{f^1}, \ldots, \overline{f^{2n}})$$

and

$$(M, rel(X), G_0, G_1, s(\overline{x}), s(\overline{y})) \models R\bar{y} \to (g_0\bar{y} = g_1\bar{y}).$$

As in the proof of Theorem 2.3.9, the former follows from Lemma 2.3.8. It then remains to show the latter. Indeed, since

$$M \models_{Y_{2n+1}} \bigotimes_{i=1}^{k} (w_i \neq y_i) \otimes (v_0 = v_1),$$

there are $V, W_1, \ldots, W_k \subseteq Y_{2n+1}$ such that $Y_{2n+1} = V \cup W_1 \cup \cdots \cup W_k$,

$$M \models_V v_0 = v_1 \text{ and } M \models_{W_i} w_i \neq y_i,$$

for each $1 \leq i \leq k$. Now, assume

$$(M, rel(X), G_0, G_1, s(\overline{x}), s(\overline{y})) \models g_0\bar{y} \neq g_1\bar{y},$$

then $G_0(s(\bar{y})) \neq G_1(s(\bar{y}))$, which by the construction of Y_{2n+1} means that

$$s(v_0) = G_0(s(\bar{y})) \neq G_1(s(\bar{y})) = s(v_1).$$

It follows that $Xs \nsubseteq V$. Noting that by the construction,

$$Y_{2n+1} \upharpoonright dom(X) = X,$$

we have that

$$Xs \subseteq W_1 \cup \cdots \cup W_k,$$

which means that for each $t \in X$, there exists some $1 \leq i_t \leq k$ such that $ts \in W_{i_t}$. Hence

$$t(w_{i_t}) \neq s(y_{i_t}), \text{ so } t(\bar{w}) \neq s(\bar{y}),$$

for each $t \in X$, which means that $s(\bar{y}) \notin rel(X)$, i.e.

$$(M, rel(X), G_0, G_1, s(\bar{x}), s(\bar{y})) \models \neg R\bar{y},$$

as required. ■

Theorem 2.4.4. *Formulas of* **BID**$^-$ *characterize second-order downwards monotone properties with respect to non-empty teams.*

Proof. Follows from Theorem 2.4.3 and Theorem 2.3.1. ■

The proof of Theorem 2.4.3 does not work for non-empty teams. In the remaining part of this section, we investigate the expressive power of the full logic **BID**, in which linear implication is present. The empty team property is lost in the full **BID**, however, in this case we do obtain a similar theorem to Theorem 2.4.3 for **BID** which holds with respect to all teams, including the empty one.

The empty team $\emptyset$ can be viewed as a team of any model M with any domain $\{x_1, \ldots, x_k\}$, and we have that

$$rel(\emptyset) = \{(s(x_1), \ldots, s(x_k)) \mid s \in \emptyset\} = \emptyset.$$

In case a predicate R occurring in a second-order sentence $\phi(R)$ is interpreted as the empty set $\emptyset$ (or $rel(\emptyset)$) in a model, one can replace each occurrence of $R\bar{t}$ in $\phi(R)$ by the constant $\perp$ (*falsum*) without affecting the truth value of the formula in the model. We check this observation in the next lemma.

Lemma 2.4.5. *Let $\phi(R)$ be a second-order $L(R)$-sentence with a k-ary predicate R. Then for any suitable L-model M,*

$$(M, \emptyset) \models \phi(R) \iff M \models \phi(\perp),$$

where $\phi(\perp)$ is obtained from $\phi(R)$ by replacing everywhere $R\bar{t}$ by $\perp$.

Proof. We show by induction on subformulas $\psi(R)$ of $\phi(R)$ that for any suitable model M, any suitable assignment s on M

$$(M, \emptyset) \models_s \psi(R) \iff M \models_s \psi(\perp).$$

The only interesting case is the case $\psi(R) = R(\bar{t})$ for some terms $t_1, \ldots, t_k$. As $(t_1\langle s\rangle, \ldots, t_k\langle s\rangle) \notin \emptyset$, we always have that $(M, \emptyset) \not\models_s R(\bar{t})$; on the other hand, $M \not\models_s \perp$, thus the lemma holds for this case. ■

Next, we show that with an essential use of linear implication, formulas of **BID** define second-order properties with respect to the empty team. Note that we do not require these properties to be downwards monotone. Indeed, this lemma will be re-used in Section 3.1 on independence logic without the downwards closure property.

Lemma 2.4.6. *For any second-order $L(R)$-sentence $\phi(R)$, there is an L-formula $\top \multimap \phi^\star$ of* **BID** *such that for any L-model M,*

$$(M,\emptyset) \models \phi(R) \iff M \models_\emptyset \top \multimap \phi^\star.$$

Proof. Noting that $\phi(\bot)$ is an L-sentence, we let $\phi^\star$ be the L-sentence of **BID**$^-$ obtained from Theorem 2.3.9 satisfying

$$M \models \phi(\bot) \iff M \models_{\{\emptyset\}} \phi^\star$$

for all L-model M. It follows that

$$\begin{aligned}(M,\emptyset) \models \phi(R) &\iff M \models \phi(\bot) \text{ (by Lemma 2.4.5)}\\ &\iff M \models_{\{\emptyset\}} \phi^\star\\ &\iff M \models_\emptyset \top \multimap \phi^\star\\ &\qquad \text{(by Locality, since Fv}(\phi^\star) = \text{Fv}(\top) = \emptyset).\end{aligned}$$

∎

Finally, we combine the results of Theorem 2.4.3 and Lemma 2.4.6 to show that formulas of **BID** define all second-order downwards monotone properties with respect to all teams.

Theorem 2.4.7. *For any second-order $L(R)$-sentence $\phi(R)$ downwards monotone with respect to a k-ary predicate R, there is an L-formula $\psi(w_1,\ldots,w_k)$ of* **BID** *such that for any L-model M and any team X of M (including the empty team) with domain $\{w_1,\ldots,w_k\}$,*

$$(M,rel(X)) \models \phi(R) \iff M \models_X \psi(\bar{w}).$$

Proof. Let $\phi^*(\bar{w})$ be the **BID**$^-$ formula obtained from Theorem 2.4.3 and $\top \multimap \phi^\star$ the sentence obtained from Lemma 2.4.6. Let

$$\psi(\bar{w}) := (\bot \wedge (\top \multimap \phi^\star)) \otimes \phi^*(\bar{w}).$$

It suffices to show that for any L-model M and any team X of M with domain $\{w_1,\ldots,w_k\}$,

$$(M,rel(X)) \models \phi(R) \iff M \models_X (\bot \wedge (\top \multimap \phi^\star)) \otimes \phi^*(\bar{w}).$$

"$\Longrightarrow$": Suppose $(M,rel(X)) \models \phi(R)$. If $X \neq \emptyset$, then by Theorem 2.4.3, $M \models_X \phi^*(\bar{w})$. Since $\emptyset \subseteq rel(X)$ and $\phi(R)$ is downwards monotone with respect to R, we have that $(M,\emptyset) \models \phi(R)$, thus by Lemma 2.4.6, $M \models_\emptyset \bot \wedge (\top \multimap \phi^\star)$. Hence $M \models_X \psi(\bar{w})$.

If $X = \emptyset$, then $M \models_\emptyset \bot \wedge (\top \multimap \phi^\star)$ by Lemma 2.4.6. On the other hand, by the empty team property of **BID**$^-$, $M \models_\emptyset \phi^*(\bar{w})$. Hence, we obtain $M \models_\emptyset \psi(\bar{w})$.

Conversely, suppose $M \models_X \psi(\bar{w})$. Then, we must have that

$$M \models_{\emptyset} \bot \wedge (\top \multimap \phi^{\star}) \text{ and } M \models_X \phi^{*}(\bar{w}).$$

In case $X = \emptyset$, that $(M, rel(\emptyset)) \models \phi(R)$ follows from Lemma 2.4.6; in case $X \neq \emptyset$, that $(M, rel(X)) \models \phi(R)$ follows from Theorem 2.4.3. ■

Theorem 2.4.8. *Formulas of* **BID** *characterize second-order downwards monotone properties.*

Proof. Follows from Theorem 2.4.7 and Theorem 2.3.1. ■

One should not confuse the above result with that of team logic in [59]: formulas of team logic are proved to characterize all second-order properties with respect to non-empty teams. We summarize the results on expressive power of formulas of sublogics of **BID** we have obtained so far in Figure 2.2.

Figure 2.2: Expressive power of sublogics of **BID** over formulas

SO w.r.t. non-empty teams	**TL**
downwards monotone **SO**	**BID**, $\mathbf{D}^{[\to,\multimap]}$
downwards monotone **SO** w.r.t. non-empty teams	$\mathbf{BID}^{-}$, **ID**, $\mathbf{D}^{[\to]}$
downwards monotone Σ_1^1 w.r.t. non-empty teams	**D**

2.5 Concluding remarks

The logic **D** is equivalent to Σ_1^1, therefore it characterizes NP. By the result in sections 2.4-2.5, the logic **ID** or **BID** characterizes the Polynomial Hierarchy PH. An **ID**-normal form for Π_n^1-formulas (or Σ_n^1-formulas) is obtained in Section 2.4 (see Remark 2.3.11), but a syntactical characterization for the fragment of **ID** that is equivalent to Π_n^1 (or Σ_n^1) is unknown. Or further, an **ID**-characterization of the complexity classes Π_n^{P} and Σ_n^{P} is unknown.

Independence friendly logic (IF-logic) [47][48] is equivalent to Σ_1^1, thus to **D**, on the level of sentences. This indicates a possibility of obtaining a similar result with that of sections 2.3-2.4 for an extension of IF-logic. However, the argument we presented in these two sections relies heavily on the role the intuitionistic and linear implications play in the translation; that is, it is based on a deep understanding of the general framework of Hodges' team semantics. Since the original semantics of IF-logic was given by means of imperfect information games ([48]), to obtain a similar result for a reasonable extension of IF-logic, one may have to seek for different notions, the game-theoretic ones, which correspond to the intuitionistic and linear implications in the team semantics context.

On the other hand, in the literature, there is *dependence friendly logic* (DF-logic) whose semantics is given by imperfect information games as well. DF-logic emphasizes the *dependence* between variables with quantifiers with back slashes instead. Formulas of DF-logic are built from first-order literals using conjunction, disjunction and back-slashed existential and universal quantifiers. Intuitively, the formula

$$\exists x \backslash x_1, \ldots, x_n \phi$$

means "there exists an x, dependent completely of $x_1, \ldots, x_n$ such that ϕ"; analogously, the formula

$$\forall x \backslash x_1, \ldots, x_n \phi$$

means "for all x, dependent completely of $x_1, \ldots, x_n$, we have ϕ". In [1], a compositional team semantics for the two back-slashed quantifiers are given as: for all suitable models M, all suitable teams X of M,

- $M \models_X (\exists x \backslash x_1, \ldots, x_n)\phi \iff_{\mathsf{def}} M \models_X \exists x(=(x_1, \ldots, x_n, x) \wedge \phi)$;
- $M \models_X (\forall x \backslash x_1, \ldots, x_n)\phi \iff_{\mathsf{def}} M \models_X \forall x(=(x_1, \ldots, x_n, x) \to \phi)$.

Applying a similar translation with that in the proof of Theorem 2.3.1, DF-logic are easily seen to be expressible in second-order logic. Conversely, by a similar argument with that of the proof of Theorem 2.3.9, every second-order sentence is expressible in DF-logic. Therefore we have the following theorem.

Theorem 2.5.1. *DF-logic is equivalent to the full second-order logic over sentences.*

Proof. By Theorem 2.3.9, every second-order sentences is equivalent to a formula ϕ^* of **BID**. Observe that all occurrences of dependence atoms and intuitionistic implication in this formula are (essentially) of the forms

$$\exists y_1 \cdots \exists y_k (=(x_{1,1}, \ldots, x_{1,n_1}, y_1) \wedge \cdots \wedge =(x_{k,1}, \ldots, x_{k,n_k}, y_k) \wedge \theta)$$

and

$$\forall y_1 \cdots \forall y_k (=(x_{1,1}, \ldots, x_{1,n_1}, y_1) \wedge \cdots \wedge =(x_{k,1}, \ldots, x_{k,n_k}, y_k) \to \theta)^2$$

The former can be replaced equivalently by

$$(\exists y_1 \backslash x_{1,1}, \ldots, x_{1,n_1}) \cdots (\exists y_k \backslash x_{k,1}, \ldots, x_{k,n_k})\theta,$$

and the latter can be replaced equivalently by

$$(\forall y_1 \backslash x_{1,1}, \ldots, x_{1,n_1}) \cdots (\forall y_k \backslash x_{k,1}, \ldots, x_{k,n_k})\theta.$$

■

[2] In the formula ϕ^*, all of the universal quantifiers are in the front, but each $\forall u_{i,j}$ for i odd can be moved to the front of δ_i.

Chapter 3

First-order independence logic with implications

First-order independence logic (**Ind**) is equivalent to Σ^1_1 over sentences ([39]), and over formulas, it characterizes all Σ^1_1 properties with respect to non-empty teams ([31]). As in Chapter 2, in this chapter, we will obtain the expressive power of the full second-order logic by adding implications to the logic **Ind**. However, **Ind** is not downwards closed, so intuitionistic and linear implications do not behave the same way in **Ind** as in **BID**. In Section 3.1, we show by a similar argument with that in Chapter 2 that over formulas, **Ind** extended with intuitionistic implication still characterizes all second-order properties with respect to non-empty teams; and with respect to all teams, **Ind** extended with both intuitionistic and linear implication defines (not characterizes) all second-order empty set-closed properties. In Section 3.2, we study the *maximal implication* introduced in [59], and show that first-order inclusion logic extended with maximal implication is equivalent to the full second-order logic. In Section 3.3, we make some concluding remarks and list the main open problems.

3.1 Definability in first-order independence logic with intuitionistic and linear implications

Based on the downwards closure property, intuitionistic and linear implications were defined in the context of first-order dependence logic [1]. The lack of the downwards closure property in first-order independence logic makes it less interesting for the study of these two implications in the context of this logic. However, in this section we show by a similar argument with that in Chapter 2 that **Ind** extended with the two implications does have some nice properties: $\mathbf{Ind}^{[\rightarrow]}$ still characterizes all second-order properties with respect to non-empty teams; and with respect to all teams, $\mathbf{Ind}^{[\rightarrow,\multimap]}$ defines (not characterizes) all second-order empty set-closed properties.

We start with clarifying some basic facts about intuitionistic and linear implications in the context of **Ind**.

Fact 3.1.1. *In the logic* $\mathbf{Ind}^{[\to,\multimap]}$,

$$\phi \otimes \psi \models \chi \iff \phi \models \psi \multimap \chi,$$

but

$$\phi \wedge \psi \models \chi \not\Longleftrightarrow \phi \models \psi \to \chi.$$

Proof. We only give a counterexample to the second clause. Consider the model $M = \{0,1\}$, and the four assignments $s_0, s_1, s_2, s_3 : \{x,y\} \to M$ defined by Table 3.1. Clearly, $(x \perp y) \wedge \top \models x \perp y$. However,

$$M \models_{\{s_0,s_1,s_2,s_3\}} x \perp y \text{ but } M \not\models_{\{s_0,s_1,s_2,s_3\}} \top \to (x \perp y),$$

as , e.g., $M \not\models_{\{s_0,s_1\}} x \perp y$. ■

In the remaining part of this section, we investigate the expressive power of the logic $\mathbf{Ind}^{[\to,\multimap]}$. First of all, we know from Chapter 2 that $\mathbf{D}^{[\to,\multimap]}$ is equivalent to the full second-order logic over sentences. By [39], dependence atoms are expressible in **Ind**, therefore we obtain the following immediate corollary.

Corollary 3.1.2. *Second-order sentences are expressible in* $\mathbf{Ind}^{[\to,\multimap]}$*, and vice versa.*

Proof. Follows from Theorem 2.3.10 and [39]. ■

Next, we proceed to generalize the argument in Section 2.4 to determine the expressive power of open formulas of $\mathbf{Ind}^{[\to,\multimap]}$. We first show that formulas of $\mathbf{Ind}^{[\to]}$ characterize all second-order properties (not necessarily downwards monontone) with respect to non-empty teams. This is proved by generalizing Theorem 2.4.4 and the result in [31].

By [39], [1], every formula of $\mathbf{Ind}^{[\to]}$ can be translated into second-order logic (c.f. Theorem 1.2.4 and Theorem 2.3.1).

Theorem 3.1.3. *For any L-formula $\phi(\bar{x})$ of* $\mathbf{Ind}^{[\to,\multimap]}$*, there exists a second-order $L(R)$-sentence $\psi(R)$ with a new predicate R such that for any L-model M and any team X of M,*

$$M \models_X \phi(\bar{x}) \iff (M, rel(X)) \models \psi(R).$$

Proof. Follows from [39], [1]. ■

To give the translation of the other direction, analogous to Section 2.4, we need to obtain a normal form for any second-order sentences with a new relation symbol that can occur both positively and negatively.

Table 3.1

	x	y
s_0	0	1
s_1	1	0
s_2	0	0
s_3	1	1

Lemma 3.1.4. *Let $\phi(R)$ be any second-order $L(R)$-sentence. Then*

$$\models \phi(R) \leftrightarrow \exists S(\phi(S) \wedge \forall \bar{y}(R\bar{y} \leftrightarrow S\bar{y})).$$

where $\phi(S)$ is obtained from $\phi(R)$ by replacing everywhere R by a new predicate symbol S.

Proof. Easy. C.f. Lemma 2.4.1. ■

Lemma 3.1.5. *Every second-order $L(R)$-sentence is equivalent to a formula of the form*

$$\exists g_0 \exists g_1 \forall f_1^1 \cdots \forall f_p^1 \exists f_1^2 \cdots \exists f_p^2 \cdots \cdots \exists f_1^{2n} \cdots \exists f_p^{2n} \forall \bar{x} \forall \bar{y}(\psi \wedge (R\bar{y} \leftrightarrow (g_0\bar{y} = g_1\bar{y}))),$$

where

- *ψ is quantifier-free and does not contain the predicate R;*
- *each function symbol f_j^i is of arity q, and its every occurrence is of the same form $f_j^i \mathbf{x}^{i,j}$, where*
$$\mathbf{x}^{i,j} = \langle x_{i,j_1}, \ldots, x_{i,j_q} \rangle$$
with $\{x_{i,j_1}, \ldots, x_{i,j_q}\} \subseteq \{x_1, \ldots, x_m\}$;
- *every occurrence of the function symbol g_l ($l = 0, 1$) is of the same form $g_l\bar{y}$.*

Proof. Apply Lemma 2.3.7 and Lemma 3.1.4. C.f. Lemma 2.4.2. ■

Now, we prove that formulas of $\mathbf{Ind}^{[\to]}$ define all second-order properties with respect to non-empty teams. Recall that first-order dependence logic and inclusion logic are sublogics of **Ind** (see Section 1.2, or Figure 1.1), therefore in the following proof, we will freely use dependence atoms and inclusion atoms in the constructed formula ϕ^*. Readers can also view these atoms as shorthands for the equivalent formulas in the language of **Ind**.

Theorem 3.1.6. *For any second-order $L(R)$-sentence $\phi(R)$ with a k-ary predicate R, there exists a formula $\phi^*(w_1, \cdots, w_k)$ of $\mathbf{Ind}^{[\to]}$ such that for any L-model M and any non-empty team X of M with domain $\{w_1, \cdots, w_k\}$,*

$$(M, rel(X)) \models \phi(R) \Longleftrightarrow M \models_X \phi^*(\bar{w}). \tag{3.1}$$

Proof. We may assume that every $L(R)$-sentence $\phi(R)$ is of the normal form described in Lemma 3.1.5. The required formula $\phi^*(w_1, \cdots, w_k)$ is the same as Formula (2.5) in the proof of Theorem 2.3.9, except we now let

$$\begin{aligned}\delta_{2n} \ := \ \exists u_{2n,1} \cdots \exists u_{2n,p} \Big(\Theta_{2n} \wedge \psi' \wedge \big(\bigotimes_{i=1}^{k} (w_i \neq y_i) \otimes (v_0 = v_1) \big) \\ \wedge \big((\bar{y} \subseteq \bar{w}) \otimes (v_0 \neq v_1) \big) \Big),\end{aligned}$$

where ψ' is the classical formula of **Ind** obtained from the formula ψ of first-order logic by replacing everywhere

- the classical disjunction by $\otimes$,
- each $f^i_j\mathbf{x}^{i,j}$ by $u_{i,j}$,
- each $g_l y_1 \dots y_k$ by v_l,

and

$$\begin{aligned}\phi^*(\bar{w}) := \forall u_{1,1} \cdots \forall u_{1,p} \forall u_{3,1} \cdots \forall u_{3,p} \cdots \cdots \forall u_{2n-1,1} \cdots \forall u_{2n-1,p} \forall \bar{x} \forall \bar{y} \\ \exists v_0 \exists v_1 (=\!(\bar{y}, v_0) \wedge =\!(\bar{y}, v_1) \wedge \delta_1(\bar{w})).\end{aligned} \tag{3.2}$$

It remains to show that for any suitable model M, any non-empty team X of M with domain $\{w_1, \dots, w_k\}$, (3.1) holds. The proof goes through a very similar argument to those in the proofs of Theorem 2.3.9 and Theorem 2.4.3.[1] We will only show here the different steps.

For the direction "$\Longrightarrow$", assuming $(M, rel(X)) \models \phi(R)$ for $X \neq \emptyset$, let Y_{2n+1} be the team obtained by the same argument as that in the proof of Theorem 2.3.9. We will show that

$$M \models_{Y_{2n+1}} \psi' \wedge \big(\bigotimes_{i=1}^{k}(w_i \neq y_i) \otimes (v_0 = v_1)\big) \wedge \big((\bar{y} \subseteq \bar{w}) \otimes (v_0 \neq v_1)\big).$$

By the construction, we have that

$$\begin{aligned}(M, rel(X), G_0, G_1, \overline{F^1}, \dots, \overline{F^{2n}}) \models \forall \bar{x} \forall \bar{y} \big(\psi(g_0, g_1, \overline{f^1}, \dots, \overline{f^{2n}}) \\ \wedge (R\bar{y} \leftrightarrow (g_0\bar{y} = g_1\bar{y}))\big).\end{aligned} \tag{3.3}$$

Thus, that

$$M \models_{Y_{2n+1}} \psi' \wedge \big(\bigotimes_{i=1}^{k}(w_i \neq y_i) \otimes (v_0 = v_1)\big)$$

follows from the corresponding part in the proof of Theorem 2.4.3. It remains to show that

$$M \models_{Y_{2n+1}} (\bar{y} \subseteq \bar{w}) \otimes (v_0 \neq v_1).$$

Define

$$V = \{s \in Y_{2n+1} \mid s(v_0) \neq s(v_1)\},$$

and $W = Y_{2n+1} \setminus V$. Clearly, $M \models_V v_0 \neq v_1$. We show that $M \models_W \bar{y} \subseteq \bar{w}$. Let $s \in W$ be arbitrary. Since $M \models_W v_0 = v_1$, we have that $s(v_0) = s(v_1)$, thus by the definition of Y_{2n+1},

$$G_0(s(\bar{y})) = s(v_0) = s(v_1) = G_1(s(\bar{y})).$$

[1] The proofs of Theorem 2.3.9 and Theorem 2.4.3 do not reply on the downwards closure property of the logic, except the fact that the classical formula ψ' is downwards closed is used, but classical formulas are downwards closed in all logics based on team semantics.

Moreover, note that in this proof, lax semantics is applied to existential quantifiers. But this does not give rise to an essential difference in the proof, since in the direction "$\Longrightarrow$" of the proof, one can define each function $\alpha_{i,j}$ by taking $\alpha_{i,j}(s) = \{F^i_j(s(\mathbf{x}^{i,j}))\}$; for the other direction "$\Longleftarrow$" of the proof, when defining each function F^i_j, the construction of $\phi^*(\bar{w})$ guarantees that the corresponding $s(u_{i,j})$ has always a single value.

Now, by (3.3), we have that

$$(M, rel(X), G_0, G_1, s(\bar{y})) \models (g_0\bar{y} = g_1\bar{y}) \to R\bar{y},$$

hence $(M, rel(X), G_0, G_1, s(\bar{y})) \models R\bar{y}$, i.e.

$$s(\bar{y}) \in rel(X).$$

It follows that there exists $t \in X$ such that $t(\bar{w}) = s(\bar{y})$. Now, let s_0 be s restricted to $dom(Y_{2n+1}) \setminus dom(X)$. By the construction of Y_{2n+1}, we know that $Xs_0 \subseteq Y_{2n+1}$, thus $ts_0 \in Y_{2n+1}$ and

$$ts_0(\bar{w}) = t(\bar{w}) = s(\bar{y}),$$

as required.

Conversely, for the direction "$\Longleftarrow$", assuming $M \models_X \phi^*$ for $X \neq \emptyset$, let G_0, G_1 be the team obtained by the same argument as that in the proof of Theorem 2.3.9. Let $\overline{a}, \overline{b}$ be arbitrary sequences in M with the same length as $\bar{x}$ and $\bar{y}$, respectively. By the construction of Y_{2n+1}, there must exists an assignment

$$s : (dom(Y_{2n+1}) \setminus dom(X)) \to M$$

such that $Xs \subseteq Y_{2n+1}$ and

$$s(\overline{x}) = \overline{a} \text{ and } s(\overline{y}) = \overline{b}.$$

We show that

$$(M, G_0, G_1, \overline{F^1}, \ldots, \overline{F^{2n}}, s(\overline{x}), s(\overline{y})) \models \psi(g_0, g_1, \overline{f^1}, \ldots, \overline{f^{2n}}),$$

$$(M, rel(X), G_0, G_1, s(\overline{x}), s(\overline{y})) \models R\bar{y} \to (g_0\bar{y} = g_1\bar{y}),$$

and

$$(M, rel(X), G_0, G_1, s(\overline{x}), s(\overline{y})) \models (g_0\bar{y} = g_1\bar{y}) \to R\bar{y}.$$

The first two of the above three expressions follow from the corresponding part in the proof of Theorem 2.4.3. It remains to show the last expression.

Since

$$M \models_{Y_{2n+1}} (\bar{y} \subseteq \bar{w}) \otimes (v_0 \neq v_1),$$

there are $V, W \subseteq Y_{2n+1}$ such that $Y_{2n+1} = V \cup W$,

$$M \models_V v_0 \neq v_1 \text{ and } M \models_W \bar{y} \subseteq \bar{w}.$$

Now, assume

$$(M, rel(X), G_0, G_1, s(\overline{x}), s(\overline{y})) \models g_0\bar{y} = g_1\bar{y},$$

then $G_0(s(\bar{y})) = G_1(s(\bar{y}))$, which by the construction of Y_{2n+1} means that

$$s(v_0) = G_0(s(\bar{y})) = G_1(s(\bar{y})) = s(v_1).$$

Thus, $Xs \nsubseteq V$ and $Xs \subseteq W$. For any $ts \in Xs$, since $M \models_W \bar{y} \subseteq \bar{w}$, there exists $s' \in W$ such that

$$s'(\bar{w}) = ts(\bar{y}) = s(\bar{y}).$$

Note that by the construction of Y_{2n+1}, we have that

$$Y_{2n+1} = \bigcup_{s \in Y_{2n+1}} X s_0,$$

where s_0 is s restricted to $dom(Y_{2n+1}) \setminus dom(X)$. Hence, we must have that

$$s' = t' s_0',$$

for some $t' \in X$. Thus for such $t' \in X$

$$t'(\bar{w}) = s'(\bar{w}) = ts(\bar{y}) = s(\bar{y}), \text{ so } s(\bar{y}) \in rel(X),$$

i.e.

$$(M, rel(X), G_0, G_1, s(\bar{x}), s(\bar{y})) \models R\bar{y},$$

as required. ■

Theorem 3.1.7. $\mathbf{Ind}^{[\to]}$ *formulas characterize all second-order properties with respect to non-empty teams.*

Proof. Follows from Theorem 3.1.6 and [39] [1]. ■

Next, we generalize Theorem 2.4.7 and show that formulas of $\mathbf{Ind}^{[\to,\multimap]}$ *define* all second-order properties that are closed under empty set.

Definition 3.1.8. Let R be a k-ary relation symbol and $\phi(R)$ a second order $L(R)$-sentence. We say that $\phi(R)$ is *closed under empty set* with respect to R if for all $L(R)$-model (M,Q),

$$(M,Q) \models \phi(R) \Longrightarrow (M,\emptyset) \models \phi(R).$$

Theorem 3.1.9. *For any second-order $L(R)$-sentence $\phi(R)$ closed under empty set with respect to the predicate R, there is an L-formula $\psi(\bar{w})$ of* $\mathbf{Ind}^{[\to,\multimap]}$ *such that for any L-model M and any team X of M (including the empty team),*

$$(M, rel(X)) \models \phi(R) \iff M \models_X \psi(\bar{w}).$$

Proof. Apply Lemma 2.4.6 (noting that $\mathbf{BID} \subseteq \mathbf{Ind}^{[\to,\multimap]}$) and Theorem 3.1.6, use a similar argument to that of the proof of Theorem 2.4.7. ■

However, in the absence of the downwards closure property, the logic $\mathbf{Ind}^{[\to,\multimap]}$ does not *characterize*[2] the empty set-closed second-order properties. The following Fact illustrates that the converse of the above theorem fails.

Fact 3.1.10. *There exists an L-formula $\phi(\bar{w})$ of* $\mathbf{Ind}^{[\to,\multimap]}$ *such that the second order $L(R)$-sentence $\psi(R)$ obtained from Theorem 3.1.3 is not closed under empty set with respect to R.*

Proof. It suffices to find a formula ϕ, a model M, a team X of M such that $M \models_X \phi$ but $M \not\models_\emptyset \phi$. Let $M = \{0,1\}$ and the four assignments $s_0, s_1, s_2, s_3 : \{x,y\} \to M$ defined by Table 3.1. Clearly, $M \models_{\{s_0,s_1,s_2,s_3\}} \top \multimap (x \perp y)$, but $M \not\models_\emptyset \top \multimap (x \perp y)$. ■

[2] See the discussion after Theorem 1.1.22 of Chapter 1 for the difference between "*defining*" and "*characterizing*".

3.2 First-order inclusion logic with maximal implication

In this section, we show that inclusion logic (**Inc**) extended with maximal implication (introduced in [59]) is equivalent to the full second-order logic.

It is pointed out in [30] that maximal implication can be interpreted as *minimal skeptical implication* in the framework of the logic of belief revision (see e.g. [3]), but in this thesis, we will not go into this direction.

We first recall the definition of maximal implication from [59].

Definition 3.2.1 (Maximal implication)**.** The binary connective $\hookrightarrow$ is called the *maximal implication*, and its team semantics is defined as follows. For any formulas ϕ, ψ of first-order independence logic, for any suitable model M, any suitable team X of M,

- $M \models_X \phi \hookrightarrow \psi$ iff for all maximal $Y \subseteq X$ such that $M \models_Y \phi$, it holds that $M \models_Y \psi$.

The logic **Ind**$^{[\hookrightarrow]}$ clearly has the empty team property. By the above definition, if the family

$$\mathscr{F} = \{Y \subseteq X \mid M \models_Y \phi\}$$

does not have any maximal element, then $M \models_X \phi \hookrightarrow \psi$ is trivially true. Now, we show that the maximal implication is nontrivial when applied to formulas closed under unions of chains.

Definition 3.2.2. A formula ϕ with team semantics is said to be *closed under unions of chains* if for any suitable model M, any suitable teams $\{X_i\}_{i<\alpha}$ of M with $X_i \subseteq X_j$ for all $i < j < \alpha$, it holds that

$$M \models_{X_i} \phi \text{ for all } i < \alpha \Longrightarrow M \models_{\bigcup_{i<\alpha} X_i} \phi.$$

Lemma 3.2.3. *(Axiom of Choice) Let ϕ be a formula which is closed under unions of chains. If $M \models_Y \phi$ for some $Y \subseteq X$, then there always exists a maximal extension $Z \subseteq X$ of Y such that $M \models_Z \phi$.*

Proof. Suppose $M \models_Y \phi$ for some $Y \subseteq X$. Consider the family

$$\mathscr{F} = \{Z \subseteq X \mid M \models_Z \phi \text{ and } Z \supseteq Y\}.$$

Clearly, $(\mathscr{F}, \subseteq)$ forms a partial order. Since ϕ is closed under unions of chains, every chain $\{X_i\}_{i<\alpha}$ in $\mathscr{F}$ has an upper bound $\bigcup_{i<\alpha} X_i$. Therefore by Zorn's lemma, $\mathscr{F}$ has a maximal element. ■

Corollary 3.2.4. *For formulas of first-order inclusion logic, Lemma 3.2.3 holds and the maximal extension is unique.*

Proof. By Theorem 1.2.5, formulas of **Inc** are closed under unions, thus closed under unions of chains. Moreover, in this case, the family $\mathscr{F}$ in the proof of Lemma 3.2.3 has a unique upper bound $\bigcup \mathscr{F}$. ■

Lemma 3.2.5. *Lemma 3.2.3 holds for formulas of* **Ind** *or* **D** *without any occurrence of* $\otimes$ *and* $\exists$*.*

Proof. It suffices to show that the formulas satisfying the requirement in the lemma are closed under unions of chains. We will show by induction on ϕ that for any suitable model M, any suitable teams $\{X_i\}_{i<\alpha}$ of M with $X_i \subseteq X_j$ for all $i < j < \alpha$, it holds that

$$M \models_{X_i} \phi \text{ for all } i < \alpha \Longrightarrow M \models_X \phi,$$

where $X = \bigcup_{i<\alpha} X_i$.

We only show the non-trivial cases. If $\phi = \bar{x} \perp_{\bar{z}} \bar{y}$, then for any $s, s' \in X$ with $s(\bar{z}) = s'(\bar{z})$, there exists $k < \alpha$ such that $s, s' \in X_k$. Since $M \models_{X_k} \bar{x} \perp_{\bar{z}} \bar{y}$, there exists $s'' \in X_k \subseteq X$ such that

$$s''(\bar{z}) = s(\bar{z}) = s'(\bar{z}),$$
$$s''(\bar{x}) = s(\bar{x}) \text{ and } s''(\bar{y}) = s'(\bar{x}).$$

Hence $M \models_X \bar{x} \perp_{\bar{z}} \bar{y}$.

The case $=\!(\overline{x})$ is checked similarly as the above case.

If $\phi = \forall x \psi$, then for each $i < \alpha$, since $M \models_{X_i} \forall x \psi$, $M \models_{X_i(M/x)} \psi$. For any $i < j < \alpha$, as $X_i \subseteq X_j$, we have that $X_i(M/x) \subseteq X_j(M/x)$. Note that

$$\bigcup_{i<\alpha} X_i(M/x) = (\bigcup_{i<\alpha} X_i)(M/x) = X(M/x).$$

Hence, by induction hypothesis, $M \models_{X(M/x)} \psi$ holds, thereby $M \models_X \forall x \psi$. ■

The next fact shows that maximal implication is transitive only with respect to valid formulas.

Fact 3.2.6. *For any formulas ϕ, ψ, χ of* $\mathbf{Ind}^{[\hookrightarrow]}$,

$$[\models \phi \hookrightarrow \psi \text{ and } \models \psi \hookrightarrow \chi\,] \Longrightarrow \models \phi \hookrightarrow \chi.$$

However,

$$(\phi \hookrightarrow \psi) \wedge (\psi \hookrightarrow \chi) \not\models \phi \hookrightarrow \chi.$$

Proof. Suppose $\models \phi \hookrightarrow \psi$ and $\models \psi \hookrightarrow \chi$. We show that for any suitable model M, any suitable team X of M, $M \models_X \phi \hookrightarrow \chi$. Let Y be a maximal subset of X such that $M \models_Y \phi$. Since $M \models_X \phi \hookrightarrow \psi$, $M \models_Y \psi$. On the other hand, since $M \models_Y \psi \hookrightarrow \chi$ and Y is obviously the maximal subset of Y satisfying $M \models_Y \psi$, we obtain that $M \models_Y \chi$, as required.

However, consider the model $M = \{0,1\}$, and the four assignments $s_0, s_1, s_2, s_3 : \{x,y,z\} \to M$ defined by the following table:

	x	y	z
s_0	0	1	0
s_1	0	0	0
s_2	0	0	1
s_3	1	0	0

Clearly,

$$M \models_{\{s_0,s_1,s_2,s_3\}} (x = z) \hookrightarrow (z \subseteq y) \text{ and } M \models_{\{s_0,s_1,s_2,s_3\}} (z \subseteq y) \hookrightarrow (y \subseteq x),$$

but $M \not\models_{\{s_0,s_1,s_2,s_3\}} (x = z) \hookrightarrow (y \subseteq x)$. ■

Dependence atoms of the form $=\!(x_1, \ldots, x_n)$ are definable uniformly from constancy dependence atoms using either the announcement operator ([32]) or intuitionistic implication (see [1] or Lemma 2.2.3). However, such definitions make heavy use of the downwards closure property of first-order dependence logic, therefore it cannot be generalized directly to the case of first-order independence logic. In the next lemma, we present a decomposition of conditional independence atoms to unconditional ones using the maximal implication instead. A simple form of this decomposition can be found in Section 7.5 of [30].

Lemma 3.2.7. *If* $\bar{z} = \langle z_1, \ldots, z_k \rangle$, *then*

$$\bar{x} \perp_{\bar{z}} \bar{y} \equiv \left(\bigwedge_{i=1}^{k} (z_i \perp z_i) \right) \hookrightarrow (\bar{x} \perp \bar{y}).$$

Proof. Suppose $M \models_X \bar{x} \perp_{\bar{z}} \bar{y}$. Let $Y \subseteq X$ be a maximal subteam such that

$$M \models_Y \bigwedge_{i=1}^{k} (z_i \perp z_i).$$

For any $s, s' \in Y$, we have that $s(\bar{z}) = s'(\bar{z})$, thus by assumption, there exists $s'' \in X$ such that

$$s''(\bar{z}) = s(\bar{z}) = s'(\bar{z}),$$

$$s''(\bar{x}) = s(\bar{x}) \text{ and } s''(\bar{y}) = s'(\bar{y}). \tag{3.4}$$

It follows that

$$M \models_{Y \cup \{s''\}} \bigwedge_{i=1}^{k} (z_i \perp z_i),$$

thus by the maximality of Y, we must have that $s'' \in Y$, as required.

Conversely, suppose $M \models_X \left(\bigwedge_{i=1}^{k} (z_i \perp z_i)\right) \hookrightarrow (\bar{x} \perp \bar{y})$. Let $s, s' \in X$ be arbitrary elements such that $s(\bar{z}) = s'(\bar{z})$. Then we have that

$$M \models_{\{s,s'\}} \bigwedge_{i=1}^{k} (z_i \perp z_i).$$

By Lemma 3.2.5, $\{s, s'\}$ has a maximal extension $Y \subseteq X$ such that

$$M \models_Y \bigwedge_{i=1}^{k} (z_i \perp z_i),$$

thus by the assumption, $M \models_Y \bar{x} \perp \bar{y}$, which implies that there exists $s'' \in Y \subseteq X$ such that (3.4) holds. ■

Next, we show that independence atoms, exclusion atoms, dependence atoms are all expressible in the logic $\mathbf{Inc}^{[\hookrightarrow]}$.

X:

	$\bar{x}$	$\bar{y}$
s_0	$\bar{a}$	$\bar{b}$
s_1	$\bar{c}$	$\bar{d}$

Y:

	$\bar{x}$	$\bar{y}$	$\bar{w}$	$\bar{v}$
s_{00}	$\bar{a}$	$\bar{b}$	$\bar{a}$	$\bar{b}$
s_{01}	$\bar{a}$	$\bar{b}$	$\bar{a}$	$\bar{d}$
s_{02}	$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{b}$
s_{03}	$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{d}$
s_{10}	$\bar{c}$	$\bar{d}$	$\bar{a}$	$\bar{b}$
s_{11}	$\bar{c}$	$\bar{d}$	$\bar{a}$	$\bar{d}$
s_{12}	$\bar{c}$	$\bar{d}$	$\bar{c}$	$\bar{b}$
s_{13}	$\bar{c}$	$\bar{d}$	$\bar{c}$	$\bar{d}$

Table 3.2

Lemma 3.2.8. *Let $\bar{x} = \langle x_1, \dots, x_k \rangle$, $\bar{y} = \langle y_1, \dots, y_k \rangle$ be two tuples. Let $\bar{w} = \langle w_1, \dots, w_k \rangle$, $\bar{v} = \langle v_1, \dots, v_k \rangle$ be two other tuples, neither of which has common variables with $\bar{x}$ and $\bar{y}$.*

(i) $\bar{x} \mid \bar{y} \equiv \forall \bar{w} \forall \bar{v} \Big(\big((\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y}) \big) \hookrightarrow \bigotimes_{i=1}^{k} (w_i \neq v_i) \Big)$

(ii) $\bar{x} \perp \bar{y} \equiv \forall \bar{w} \forall \bar{v} \Big(\big((\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y}) \big) \hookrightarrow (\bar{w}\bar{v} \subseteq \bar{x}\bar{y}) \Big)$

(iii) $=\!(x) \equiv \forall y \big((y \subseteq x) \hookrightarrow (x = y) \big)$

(iv) $=\!(x_1, \dots, x_m, y) \equiv \big(=\!(x_1) \wedge \dots \wedge =\!(x_m) \big) \hookrightarrow =\!(y)$

Proof. For any team X, and any sequence $\bar{z} = \langle z_1, \dots, z_m \rangle$ of variables in the domain of X, define

$$X \upharpoonright \bar{z} := \{ s(\bar{z}) \mid s \in X \} = \{ \langle s(z_1), \dots, s(z_m) \rangle \mid s \in X \}.$$ [3]

We first prove the following claim.

Claim: Let M be any model, X a team of M with $\{x_1, y_1, \dots, x_k, y_k\} \subseteq dom(X)$. Then there exists a unique maximal subteam $Y \subseteq X(M/\bar{w}, \bar{v})$ such that

$$M \models_Y (\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y}), \tag{3.5}$$

and we have that

$$(Y \upharpoonright \bar{x}) \times (Y \upharpoonright \bar{y}) = Y \upharpoonright \bar{w}\bar{v} \tag{3.6}$$ [4]

(see Table 3.2).

Proof of Claim. The existence and uniqueness of the required maximal team Y are guaranteed by Corollary 3.2.4. It remains to check (3.6).

For any $s(\bar{w}\bar{v}) \in Y \upharpoonright \bar{w}\bar{v}$, by (3.5), there exist $s_0, s_1 \in Y$ such that

$$s_0(\bar{x}) = s(\bar{w}) \text{ and } s_1(\bar{y}) = s(\bar{v}).$$

[3] Note that $X \upharpoonright \bar{z}$ is different from $X \upharpoonright \{z_1, \dots, z_m\}$ defined in Definition 1.1.5.

[4] Here, with some abuse of notation, we identify a pair $(\bar{a}, \bar{b})$ of sequences with the concatenation $\bar{a}\bar{b}$ of the two sequences.

Since $(s_0(\bar{x}), s_1(\bar{y})) \in (Y \upharpoonright \bar{x}) \times (Y \upharpoonright \bar{y})$, we have that

$$s(\bar{w})s(\bar{v}) = s_0(\bar{x})s_1(\bar{y}) = (s_0(\bar{x}), s_1(\bar{y})) \in (Y \upharpoonright \bar{x}) \times (Y \upharpoonright \bar{y}).$$

Conversely, for any $(s_0(\bar{x}), s_1(\bar{y})) \in (Y \upharpoonright \bar{x}) \times (Y \upharpoonright \bar{y})$, letting $s : dom(X) \cup \{w_1, v_1, \dots, w_k, v_k\} \to M$ be any assignment satisfying

$$s(\bar{w}) = s(\bar{x}) = s_0(\bar{x}), \;\; s(\bar{v}) = s(\bar{y}) = s_1(\bar{y}),$$

we have that

$$M \models_{Y \cup \{s\}} (\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y}).$$

It then follows from the maximality of Y that $s \in Y$, thus

$$s_0(\bar{x})s_1(\bar{y}) = s(\bar{w})s(\bar{v}) \in Y \upharpoonright \bar{w}\bar{v}.$$

$\dashv$

Now, let M be any suitable model, and X a suitable team of M. We proceed to prove the lemma.

(i) It suffices to show that

$$M \models_X \bar{x} \mid \bar{y} \iff M \models_X \forall \bar{w} \forall \bar{v} \Big(\big((\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y}) \big) \hookrightarrow \bigotimes_{i=1}^{k} (w_i \neq v_i) \Big).$$

"$\Longrightarrow$": Suppose $M \models_X \bar{x} \mid \bar{y}$. Let $Y \subseteq X(M/\bar{w}, \bar{v})$ be the maximal subteam such that $M \models_Y (\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y})$. It suffices to show that $M \models_Y \bigotimes_{i=1}^{k} (w_i \neq v_i)$.

For any $s \in Y$, since $M \models_Y (\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y})$, there exist $s_0, s_1 \in Y$ satisfying

$$s(\bar{w}) = s_0(\bar{x}) \text{ and } s(\bar{v}) = s_1(\bar{y}). \tag{3.7}$$

As $M \models_X \bar{x} \mid \bar{y}$, we must have that $s_0(\bar{x}) \neq s_1(\bar{y})$, thus $s(\bar{w}) \neq s(\bar{v})$. It follows that there exists $1 \leq i \leq k$ such that $s(w_i) \neq s(v_i)$. Hence $M \models_Y \bigotimes_{i=1}^{k} (w_i \neq v_i)$.

"$\Longleftarrow$": Suppose $M \models_X \forall \bar{w} \forall \bar{v} \Big(\big((\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y}) \big) \hookrightarrow \bigotimes_{i=1}^{k} (w_i \neq v_i) \Big)$. By Claim, there exists a unique maximal $Y \subseteq X(M/\bar{w}, \bar{v})$ such that (3.5) and (3.6) hold. For any $s_0, s_1 \in X$, by (3.6), there exists $s \in Y$ such that (3.7) holds. By assumption, $M \models_Y \bigotimes_{i=1}^{k} (w_i \neq v_i)$, thus $s(\bar{w}) \neq s(\bar{v})$, thereby $s_0(\bar{x}) \neq s_1(\bar{y})$. Hence $M \models_X \bar{x} \mid \bar{y}$.

(ii) It suffices to show that

$$M \models_X \bar{x} \perp \bar{y} \iff M \models_X \forall \bar{w} \forall \bar{v} \Big(\big((\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y}) \big) \hookrightarrow (\bar{w}\bar{v} \subseteq \bar{x}\bar{y}) \Big).$$

"$\Longrightarrow$": Suppose $M \models_X \bar{x} \perp \bar{y}$. Let $Y \subseteq X(M/\bar{w}, \bar{v})$ be the maximal subteam such that $M \models_Y (\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y})$. It suffices to show that $M \models_Y \bar{w}\bar{v} \subseteq \bar{x}\bar{y}$.

For any $s \in Y$, since $M \models_Y (\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y})$, there exist $s_0, s_1 \in Y$ such that (3.7) holds. As $M \models_X \bar{x} \perp \bar{y}$, there must exists $s' \in X$ such that

$$s'(\bar{x}) = s_0(\bar{x}) \text{ and } s'(\bar{y}) = s_1(\bar{y}).$$

It follows that there exists $s'' \in Y$ such that

$$s''(\bar{x})s''(\bar{y}) = s'(\bar{x})s'(\bar{y}) = s_0(\bar{x})s_1(\bar{y}) = s(\bar{w})s(\bar{v}).$$

Hence $M \models_Y \bar{w}\bar{v} \subseteq \bar{x}\bar{y}$.

"$\Longleftarrow$": Suppose $M \models_X \forall\bar{w}\forall\bar{v}\Big(\big((\bar{w} \subseteq \bar{x}) \wedge (\bar{v} \subseteq \bar{y})\big) \hookrightarrow (\bar{w}\bar{v} \subseteq \bar{x}\bar{y})\Big)$. By Claim, there exists a unique maximal $Y \subseteq X(M/\bar{w},\bar{v})$ such that (3.5) and (3.6) hold. For any $s_0, s_1 \in X$, by (3.6), there exists $s \in Y$ such that (3.7) holds. By assumption, $M \models_Y \bar{w}\bar{v} \subseteq \bar{x}\bar{y}$, thus there exists $s' \in Y$ such that

$$s'(\bar{x}\bar{y}) = s(\bar{w}\bar{v}),$$

which implies that form some $s'' \in X$,

$$s''(\bar{x}) = s'(\bar{x}) = s(\bar{w}) = s_0(\bar{x}) \text{ and } s''(\bar{y}) = s'(\bar{y}) = s(\bar{v}) = s_1(\bar{y}).$$

Hence $M \models_X \bar{x} \perp \bar{y}$.

(iii) It suffices to show that

$$M \models_X =\!(x) \iff M \models_X \forall y\big((y \subseteq x) \hookrightarrow (x = y)\big).$$

"$\Longrightarrow$": Suppose $M \models_X =\!(x)$. Let $Y \subseteq X(M/y)$ be the maximal subteam such that $M \models_Y y \subseteq x$. It suffices to show that $M \models_Y x = y$.

Since $M \models_X =\!(x)$, there exists $a \in M$ such that for all $s_0 \in X$, $s_0(x) = a$. Now, for any $s \in Y$, since $M \models_Y y \subseteq x$, there exists $s' \in Y$ such that $s(y) = s'(x) = a = s(x)$. Hence $M \models_Y x = y$.

"$\Longleftarrow$": Suppose $M \models_X \forall y\big((y \subseteq x) \hookrightarrow (x = y)\big)$. Let $Y \subseteq X(M/y)$ be the maximal subteam such that $M \models_Y y \subseteq x$. For any $s_0, s_1 \in X$, letting $s : dom(X) \cup \{y\} \to M$ be any assignment satisfying

$$s(x) = s_0(x), \ \ s(y) = s_1(x),$$

we have that $M \models_{Y \cup \{s\}} y \subseteq x$. By the maximality of Y, $s \in Y$. By assumption, $M \models_Y x = y$, thus

$$s_0(x) = s(x) = s(y) = s_1(x).$$

Hence $M \models_X =\!(x)$.

(iv) Easy, c.f. Lemma 2.2.3. ■

We now proceed to investigate the expressive power of first-order inclusion logic extended with maximal implication. First, we give a translation from $\mathbf{Inc}^{[\hookrightarrow]}$ into second-order logic.

Theorem 3.2.9. *For any L-formula $\phi(\bar{x})$ of* $\mathbf{Inc}^{[\hookrightarrow]}$*, there exists a second-order $L(R)$-sentence $\tau_R(\phi) = \psi(R)$ with a new predicate R such that for any L-model M and any suitable team X of M,*

$$M \models_X \phi(\bar{x}) \iff (M, rel(X)) \models \psi(R).$$

Proof. (sketch) We define the translation $\tau_R(\phi)$ of ϕ by induction. For the only interesting case, we let

$$\tau_R(\theta_0 \hookrightarrow \theta_1) = \forall S\Big(\big(\tau_S(\theta_0) \wedge \forall S'(\tau_{S'}(\theta_0) \to \exists \bar{x}(S\bar{x} \wedge \neg S'\bar{x}))\big) \to \tau_S(\theta_1)\Big).$$

■

By examining the role the intuitionistic implications play in the proof of Theorem 3.1.6, one obtains the expressive power of inclusion logic extended with maximal implication.

Theorem 3.2.10. *Formulas of* **Inc**$^{[\hookrightarrow]}$ *characterize all second-order properties with respect to non-empty teams. In particular,* **Inc**$^{[\hookrightarrow]}$ *sentences are expressible in* **SO***, and vice versa.*

Proof. In the proof of Theorem 3.1.6, if one replace all the intuitionistic implications in the formula $\phi^*(\bar{w})$ by maximal implications, the argument still works. The direction "$\Longrightarrow$" of the proof still works because $\phi \to \psi \models \phi \hookrightarrow \psi$. The other direction "$\Longleftarrow$" still works because the Y_i's in the proof of Theorem 3.1.6 are in fact maximal teams satisfying $M \models_{Y_i} \Theta_i$.

Moreover, by Lemma 3.2.8, all dependence atoms in the formula $\phi^*(\bar{w})$ can be replaced equivalently by a formula in **Inc**$^{[\hookrightarrow]}$.

Putting the arguments together, we conclude that the theorem holds. ■

3.3 Concluding remarks

We conclude the main results obtained in Chapter 2 and Chapter 3 concerning the expressive power of logics of dependence and independence extended with linear and intuitionistic implication in Figure 3.1 and Figure 3.2.

Below we list two main open problems of this chapter:

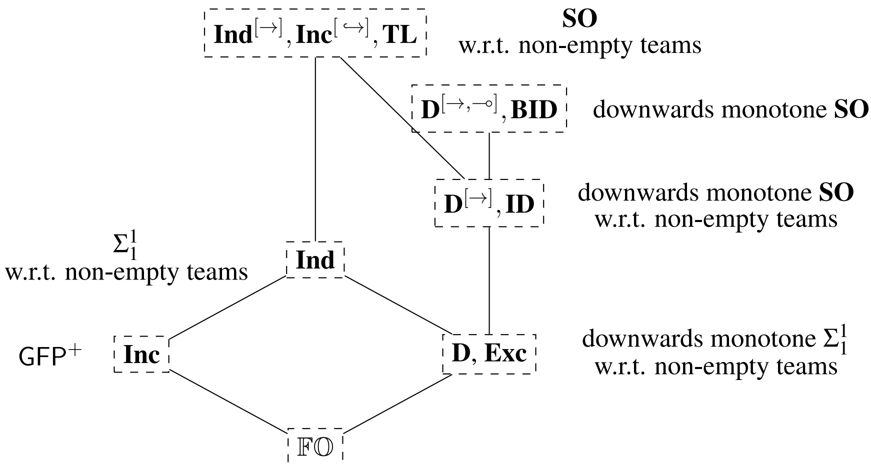
1. In Theorem 3.1.9, we proved that the logic **Ind**$^{[\to,\multimap]}$ *defines* all empty set-closed second-order properties. But the precise expressive power of formulas of the logic, i.e., the properties that the logic *characterizes*, is unkown.

2. Maximal implication is transitive only with respect to valid formulas (Fact 3.2.6). The properties of maximal implication need to be further studied.

Figure 3.1: Expressive power of logics over sentences

SO	**BID**, **ID**, $\mathbf{D}^{[\to]}$, **TL**, $\mathbf{Ind}^{[\to]}$, $\mathbf{Inc}^{[\hookrightarrow]}$
Σ_1^1	**D**, **Ind**
FO	constancy **D**

Figure 3.2: Expressive power of logics over formulas



Chapter 4

Propositional dependence, independence logics and their variants

In this chapter, we consider the logics of dependence and independence concepts in *propositional logic*. We define the underlying propositional logic of first-order dependence logic, intuitionistic dependence logic, independence logic, as well as first-order inclusion logic and exclusion logic. We give axioms and prove completeness theorems for these logics and their variants.

In Section 4.1, we discuss motivations and philosophical backgrounds of the propositional logics of dependence and independence. Formal definitions and basic properties of propositional dependence and independence logic are given in this section. In sections 4.2-4.5, we study downwards closed propositional logics with team semantics. In Section 4.2, we introduce propositional intuitionistic dependence logic. We reveal its surprising connections with inquisitive logic ([13]): the two logics are essentially equivalent. As a consequence, propositional intuitionistic dependence logic is complete with respect to the axioms given in [13] for inquisitive logic, and one extra axiom for dependence atoms. We also point out that it is a maximal downwards closed logic. In Section 4.3, we axiomatize another natural maximal downwards closed logic, namely propositional dependence logic extended with intuitionistic disjunction. Based on this, in Section 4.4, we axiomatize propositional dependence logic. Moreover, this section contains a proof (due to Taneli Huuskonen) that propositional dependence logic is also a maximal downwards closed logic. In Section 4.5, we generalize the method used in Section 4.4 to axiomatize propositional exclusion logic. In sections 4.6-4.8, we study propositional logics with team semantics which are not downwards closed. In Section 4.6, we introduce and axiomatize a natural maximal such logic, namely propositional dependence logic extended with intuitionistic disjunction and non-empty atom (a new atom that is satisfied only by non-empty teams). Based on this, in Section 4.7, we axiomatize propositional independence logic extended with non-empty atom. In Section 4.8, we generalize the method in Section 4.8 to axiomatize propositional inclusion logic extended with non-empty atom. Finally, in Section 4.9, we list some open problems of this chapter.

4.1 Introduction

Studying the logics of dependence and independence concepts in propositional logic is similar to the case of predicate logic in that we use the method of teams. A team in this case is defined to be a set of valuations. There are, however, also grave differences. Most importantly, propositional dependence and independence logics are *decidable* because for any given formula of the logics with n propositional variables, there are in total 2^n valuations and 2^{2^n} teams. The method of truth tables has its analogue in these logics, but the size of such tables grows exponentially faster than in the case of traditional propositional logic, rendering it virtually inapplicable. This emphasizes the role of the axioms and the completeness theorem in providing a manageable alternative for establishing logical consequence.

Classical propositional logic is based on propositions of the form

p
not p
p or q
If p, then q

and more generally

$$\text{If } p_{i_1}, \ldots, p_{i_k}, \text{ then } q. \tag{4.1}$$

We present extensions of classical propositional calculus in which one can express, in addition to the above, propositions of the form "q depends on p" and "q is independent of p", or more generally

$$q \text{ depends on } p_{i_1}, \ldots, p_{i_k}, \tag{4.2}$$

and

$$p_{i_1}, \ldots, p_{i_k} \text{ are independent of } p_{j_1}, \ldots, p_{j_m}. \tag{4.3}$$

In our setting, both (4.2) and (4.3) are expressed as atomic facts. The former is characterized formly by a new atomic formula

$$=\!(p_{i_1}, \ldots, p_{i_k}, q),$$

called *dependence atom*, while the latter by the so-called *independence atom*

$$p_{i_1}, \ldots, p_{i_k} \perp p_{j_1}, \ldots, p_{j_m}.$$

Intuitively, (4.2) means that to know whether q holds it is sufficient to consult $p_{i_1}, \ldots, p_{i_k}$. Note that as in the first-order dependence logic case, (4.2) says nothing about the *way* in which $p_{i_1}, \ldots, p_{i_k}$ are logically related to q. It may be that $p_{i_1} \wedge \ldots \wedge p_{i_k}$ logical implies q, or that $\neg p_{i_1} \wedge \ldots \wedge \neg p_{i_k}$ logical implies $\neg q$, or anything in between. Technically speaking, this is to say:

$$\text{The truth value of } q \text{ is a function of the truth values of } p_{i_1}, \ldots, p_{i_k}. \tag{4.4}$$

Given the huge amounts of data available nowadays, arising from DNA, astronomical data, Google data, etc, with no clear picture what the functions in action are, it seems—and we suggest—that the propositions (4.2) and their logic would deserve a mathematical treatment just as the simpler propositions (4.1) have deserved in classical propositional logic.

Examples of natural language sentences of this kind are the following:

1. *Whether it rains depends completely on whether it is winter or summer.*
2. *Whether you end up in the town depends entirely on whether you turn here left or right.*
3. *I will be absent depending on whether he shows up or not.*
4. *Whether the earth will be destroyed depends only on whether there is another planet that crashes into the earth.*

Another basic ingredient of classical propositional calculus is, as in (4.3), the a priori *independence* of the atomic propositions. Knowing the truth value of the sequence $p_{i_1},\ldots,p_{i_k}$ gives no information of the truth value of $p_{j_1},\ldots,p_{j_m}$. Any individual valuation s fixes the true value of both $p_{i_1},\ldots,p_{i_k}$ and $p_{j_1},\ldots,p_{j_m}$, but if we have a set of valuations (called a *team*), the truth value of neither of the two sequences needs to be fixed, and we can ask are these truth values independent of each other in the sense that knowing one does not reveal, in the light of the given team, the other. This is, of course, the matter in the *maximal* team of *all* valuations s for all relavant propositional variables. The maximal team represents the world of all logical possibilities. However, in practice we may be interested in a particular team and the manifestation of independence in *that* team.

For example, if we have a pool of human chromosomes arising from a group of actual people, we may ask whether certain traits are independent of each other in *this* pool of chromosomes. Knowing that they would be independent, if *all* logically possible chromosomes were present, would be of no interest what so ever. Of course, such a team of all logically possible chromosomes would densely fill every cubic millimeter of the physical universe.

Here are some other examples:

1. *Whether it rains is completely independent on whether it is winter or summer.*
2. *As to whether you end up in the town or not it makes no difference whether you turn here left or right.*
3. *I will decide whether I come to the party independently of whether he decides to show up or not.*
4. *Whether the earth will be destroyed is independent of whether the sea level rises over 50 cm or not.*

In this chapter we give exact mathematical meanings to "dependence" and "independence". Below we give formal definitions of propositional dependence and independence logic.

Definition 4.1.1. Let $p_i,p_{i_1},\ldots,p_{i_k},p_{j_1},\ldots,p_{j_m}$ be propositional variables, and $k,m \geq 1$.

- Well-formed formulas of *propositional dependence logic* (**PD**) are given by the following grammar

$$\phi ::= p_i \mid \neg p_i \mid =\!(p_{i_1},\dots,p_{i_k}) \mid \phi\wedge\phi \mid \phi\otimes\phi$$

- Well-formed formulas of *propositional independence logic* (**PInd**) are given by the following grammar:

$$\phi ::= p_i \mid \neg p_i \mid p_{i_1}\dots p_{i_k}\perp p_{j_1}\dots p_{j_m} \mid \phi\wedge\phi \mid \phi\otimes\phi,$$

Definition 4.1.2.

(i) A *valuation* s is a function $s:\mathbb{N}\to\{0,1\}$.[1] For any $n\in\mathbb{N}$, an *n-valuation* s_0 on N is a restriction of a valuation s to an n-element subset $N\subseteq\mathbb{N}$, that is, $s_0 = s\restriction N$ with $|N| = n$.

(ii) A *team* is a set of valuations. An n-team on N is a set of n-valuations on N.

(iii) We write $\phi(p_{i_1},\dots,p_{i_n})$ to mean that the propositional variables occurring in the formula ϕ are among $p_{i_1},\dots,p_{i_n}$. A formula of the form $\phi(p_{i_1},\dots,p_{i_n})$ is called an *n-formula*.

Fix an n-element subset $N\subseteq\mathbb{N}$, there are in total 2^n distinct n-valuations, and 2^{2^n} distinct n-valuations, among which there exists a maximal team consisting of all of the n-teams on N, denoted by $\mathbf{2^n}$.

Definition 4.1.3. We inductively define the notion of a formula ϕ of **PD** or **PInd** being *true* on a team X, denoted by $X\models\phi$, as follows:

- $X\models p_i$ iff for all $s\in X$, $s(i)=1$;
- $X\models\neg p_i$ iff for all $s\in X$, $s(i)=0$;
- $X\models\,=\!(p_{i_1},\dots,p_{i_k})$ iff for all $s,s'\in X$

$$\langle s(i_1),\dots,s(i_{k-1})\rangle = \langle s'(i_1),\dots,s'(i_{k-1})\rangle \implies s(i_k)=s'(i_k);$$

- $X\models p_{i_1}\dots p_{i_k}\perp p_{j_1}\dots p_{j_m}$ iff for all $s,s'\in X$, there exists $s''\in X$ such that

$$\langle s''(i_1),\dots,s''(i_k)\rangle = \langle s(i_1),\dots,s(i_k)\rangle$$

and

$$\langle s''(j_1),\dots,s''(j_m)\rangle = \langle s'(j_1),\dots,s'(j_m)\rangle.$$

- $X\models\phi\wedge\psi$ iff $X\models\phi$ and $X\models\psi$;

[1] In literature of propositional logics, a valuation is usually a function s from a set of propositional variables to $\{0,1\}$. In this thesis, for technical reasons, we choose to define valuations as in this definition. Each natural number in the set $\mathbb{N}$ stands for an index of a propositional variable.

- $X \models \phi \otimes \psi$ iff there exist teams $Y, Z \subseteq X$ with $X = Y \cup Z$ such that

$$Y \models \phi \text{ and } Z \models \psi;$$

Let L be the logic **PD** or **PInd**. For any formula ϕ of L, if $X \models \phi$ holds for all teams X, then we say that ϕ is *valid* in the logic, denoted by $\models_{\mathsf{L}} \phi$ or simply $\models \phi$. The notions of *logical consequence* and *logical equivalence* are defined analogously to the first-order case.

We call the independence atom $p_{i_1} \dots p_{i_k} \perp p_{j_1} \dots p_{j_m}$ an *unconditional* independence atoms. As in the first-order case, we can also define *conditional independence atoms* of the form $p_{i_1} \dots p_{i_a} \perp_{p_{k_1} \dots p_{k_c}} p_{j_1} \dots p_{j_b}$, whose team semantics is as follows: for any team X,

- $X \models p_{i_1} \dots p_{i_a} \perp_{p_{k_1} \dots p_{k_c}} p_{j_1} \dots p_{j_b}$ iff for all $s, s' \in X$ with $s(k_1) \cdots s(k_c) = s'(k_1) \cdots s'(k_c)$, there exists $s'' \in X$ such that

$$\langle s''(k_1), \dots, s''(k_c) \rangle = \langle s(k_1), \dots, s(k_c) \rangle = \langle s'(k_1), \dots, s'(k_c) \rangle,$$

$$\langle s''(i_1), \dots, s''(i_a) \rangle = \langle s(i_1), \dots, s(i_a) \rangle$$

and

$$\langle s''(j_1), \dots, s''(j_b) \rangle = \langle s'(j_1), \dots, s'(j_b) \rangle.$$

But in this case, conditional independence atoms are definable by unconditional ones, as the next lemma shows, where if p_i is a propositional variable, then we let

$$p_i^1 := p_i \text{ and } p_i^0 := \neg p_i.$$

Lemma 4.1.4.

$$p_{j_1} \cdots p_{j_a} \perp_{p_{i_1} \dots p_{i_c}} p_{k_1} \cdots p_{k_b} \equiv \bigotimes_{s \in \mathbf{2^c}} \big(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_c}^{s(i_c)} \wedge (p_{j_1} \dots p_{j_a} \perp p_{k_1} \cdots p_{k_b}) \big),$$

where $\mathbf{2^c}$ *is the maximal c-team on* $\{i_1, \dots, i_c\}$.

Proof. Easy. ∎

As in the first-order logic case, dependence atoms are definable in terms of conditional independence atoms as follows:

$$=\!(p_{i_1}, \dots, p_{i_{k-1}}, p_{i_k}) \equiv p_{i_k} \perp_{p_{i_1}, \dots, p_{i_{k-1}}} p_{i_k} \tag{4.5}$$

(c.f. Equation (1.1)), thus by Lemma 4.1.4, they are definable by unconditional independence atoms as well.

The team semantics of the above defined logics is a natural adaption of the first-order team semantics, therefore, not surprisingly, many of the relevant properties of the first-order dependence logics are true for the propositional dependence logics. Most importantly, analogous to the first-order case, all of the above defined logics have the empty team property, locality property, and **PD** has the downwards closure property.

Lemma 4.1.5 (Empty Team Property)**.** **PD** *and* **PInd** *have the* empty team property*, that is,* $\emptyset \models \phi$ *for every formula* ϕ *in any of the logics.*

Proof. Easy, by induction on ϕ. ∎

Lemma 4.1.6 (Locality)**.** *Let* $\phi(p_{i_1},\dots,p_{i_n})$ *be an* n*-formula of* **PD** *or* **PInd**. *For any teams* X,Y *such that* $X \upharpoonright \{i_1,\dots,i_n\} = Y \upharpoonright \{i_1,\dots,i_n\}$, *we have that*

$$X \models \phi \iff Y \models \phi.$$

Proof. Easy, by induction on ϕ. ∎

Theorem 4.1.7 (Downward Closure)**.** *For any formula* ϕ *of* **PD***, any teams* X,Y*,*

$$[\,X \models \phi \text{ and } Y \subseteq X\,] \Longrightarrow Y \models \phi.$$

Proof. Easy, by induction on ϕ. ∎

Corollary 4.1.8. *For any downwards closed* n*-formula* $\phi(p_{i_1},\dots,p_{i_n})$*,*

$$\mathbf{2^n} \models \phi \iff \models \phi,$$

where $\mathbf{2^n}$ *is the maximal* n*-team on* $\{i_1,\dots,i_n\}$.

Proof. The direction "$\Longleftarrow$" follows from Locality. For the direction "$\Longrightarrow$", if $\mathbf{2^n} \models \phi(p_{i_1},\dots,p_{i_n})$, then $X \models \phi$ for all n-teams X on $\{i_1,\dots,i_n\}$, since $X \subseteq \mathbf{2^n}$ and ϕ is downwards closed. By Locality, this means that $\models \phi$. ∎

Fix an n-element set $N = \{i_1,\dots,i_n\} \subseteq \mathbb{N}$. Let $\mathbf{2^n}$ be the maximal n-team on N. The *semantic truth set* of an n-formula $\phi(p_{i_1},\dots,p_{i_n})$ of **PD** or **PInd** is defined as the set $[\![\phi]\!]$ of all n-teams satisfying ϕ, namely

$$[\![\phi]\!] := \{X \subseteq \mathbf{2^n} \mid X \models \phi\}.$$

Clearly, for any two n-formulas $\phi(p_{i_1},\dots,p_{i_n})$ and $\psi(p_{i_1},\dots,p_{i_n})$, $\phi \equiv \psi$ if and only if $[\![\phi]\!] = [\![\psi]\!]$. Let ∇_N be the family of all non-empty downwards closed collections of n-teams on N, i.e.

$$\nabla_N = \{\mathcal{K} \subseteq 2^{\mathbf{2^n}} \mid \mathcal{K} \neq \emptyset, \text{ and } X \in \mathcal{K},\ Y \subseteq X \text{ imply } Y \in \mathcal{K}\}.$$

For any n-formula $\phi(p_{i_1},\dots,p_{i_n})$ of **PD**, since it is downwards closed, $[\![\phi]\!] \in \nabla_N$.

As in the first-order case, formulas ϕ satisfying

$$X \models \phi \iff \forall s \in X,\ \{s\} \models \phi$$

for all teams X are called *flat* formulas. A formula built from propositional variables and negated propositional variables by conjunction $\wedge$ and tensor disjunction $\otimes$ is called a *classical formula*. That is, a classical formula of **PD** or **PInd** is a formula that does not contain dependence atoms or independence atoms, or a formula of the logic **PD**$[\wedge,\otimes]$ or **PInd**$[\wedge,\otimes]$.

Lemma 4.1.9. *Classical formulas of* **PD** *or* **PInd** *are flat.*

Proof. Easy, by induction. ■

Classical formulas behaves classically on singleton teams, as the following lemma shows.

Lemma 4.1.10. *If ϕ is a classical formula of* **PD** *or* **PInd**, *then identifying tensor disjunction $\otimes$ with the classical disjunction $\vee$, for any valuation s,*

$$s \models_{\mathbf{CPL}} \phi \iff \{s\} \models \phi.$$

Proof. We prove the lemma by induction on ϕ. The only interesting case is the case $\phi = \psi \otimes \chi$. In this case, we have that

$$\begin{aligned}\{s\} \models \psi \otimes \chi &\iff \{s\} \models \psi \text{ or } \{s\} \models \chi\\ &\iff s \models_{\mathbf{CPL}} \psi \text{ or } s \models_{\mathbf{CPL}} \chi \text{ (by induction hypotheis)}\\ &\iff s \models_{\mathbf{CPL}} \psi \vee \chi.\end{aligned}$$

■

We end this section by pointing out that none of the logics **PD** and **PInd** is closed under *Uniform Substitution*:

$$\frac{\phi(p_{i_1},\dots,p_{i_n})}{\phi(\psi_1/p_{i_1},\dots,\psi_n/p_{i_n})}\;(\mathsf{Sub})$$

We will discuss this fact in the next section and Section 5.1.

Fact 4.1.11. *Neither of* **PD** *and* **PInd** *is closed under* Sub.

Proof. In the definition of the syntax of the logics **PD** and **PInd**, we only allow negations occur in front propositional variables, therefore strictly speaking, for example the formula $p_i \otimes \neg p_i$ *cannot* have substitution instances of the form $\phi \otimes \neg\phi$ in the logics **PD** and **PInd**, as $\neg\phi$ is simply not a well-formed formula. Even if we define $\neg\phi$ as the formula obtained by pushing negation all the way to the front of atomic formulas and define

$$\neg{=}(p_{i_1},\dots,p_{i_k}) := \bot \text{ and } \neg(p_{i_1}\dots p_{i_k} \perp p_{j_1}\dots p_{j_m}) := \bot,$$

still we have that $\models p_i \otimes \neg p_i$, but

$$\not\models {=}(p_i) \otimes \neg{=}(p_i) \text{ and } \not\models (p_i \perp p_i) \otimes \neg(p_i \perp p_i).$$

■

4.2 Propositional intuitionistic dependence logic and inquisitive logic

Before we investigate propositional dependence and independence logic, in this section, we introduce a natural and interesting variant of propositional dependence logic, namely propositional intuitionistic dependence logic.

As in the first-order logic case, "classical (contradictory) negation" is not definable in the downwards closed propositional dependence logic (c.f. Footnote 1 of Section 2.1). This raises the question of how to define *implications*, or how to express *conditional statements* in **PD**. One natural solution is to intrepret the conditional statement

$$\text{“}\textit{if } \phi, \textit{ then } \psi\text{”} \tag{4.6}$$

as

$$\phi \subseteq \psi := \phi^- \otimes \psi,$$

where ϕ^- stands for the *literal negation* of ϕ, that is the formula $\neg\phi$ with negation $\neg$ pushed inside ϕ all the way to the front of atomic formulas. This way, for example, "if $(p \wedge \neg q)$, then r" is expressed by the formula

$$(p \wedge \neg q) \subseteq r := (\neg p \otimes q) \otimes r.$$

However, despite of the intuitive meaning of this treatment for conditional statements, this solution has a technical drawback: it is not able to express conditionals of dependence statements. For example, the following conditional statement

> *If whether the earth will be destroyed depends only on whether there is another planet that crashes into the earth, then whether the human being will migrate to other planets depends only on whether the crash will occur.*

will be interpreted as

$$=\!(p,q) \subseteq =\!(p,r) := \big(\neg =\!(p,q)\big) \otimes =\!(p,r).$$

But $\neg =\!(p,q)$ is (by definition) equivalent to $\bot$ (see Footnote 3 in Lemma 1.1.8), thus we have

$$\big(=\!(p,q) \subseteq =\!(p,r)\big) \equiv =\!(p,r),$$

which is certainly unreasonable.

A better treatment of conditional statements is, as we suggest, to read (4.6) as

$$\phi \models \psi \ \ (\phi \text{ logically implies } \psi). \tag{4.7}$$

Given that the logic is downwards closed (as with **PD**), the above expression is (by the Deduction Theorem of intuitionistic implication, c.f. Theorem 2.2.12) equivalent to

$$\models \phi \to \psi \ \ (\text{“}\phi \text{ intuitionistically implies } \phi\text{” is valid}).$$

In view of this, we propose to interpret (4.6) as

$$\phi \to \psi,$$

where $\to$ is the intuitionistic implication studied in the preceding chapters.

In this section, we define *propositional intuitionistic dependence logic* (**PID**), in which conditional statements about dependence can have a reasonable interpretation. Moreover, we will reveal a surprising connection between **PID** and inquisitive logic [13]: the two logics are essentially equivalent. We present a complete axiomatization of **PID** due to [13].

Below we give formal definition of propositional intuitionistic dependence logic.

Definition 4.2.1. Well-formed formulas of *propositional intuitionistic dependence logic* (**PID**) are given by the following grammar:

$$\phi ::= p_i \mid \neg p_i \mid \bot \mid =\!(p_i) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi$$

To simplify notations, we apply Convention 1.1.2 to **PID** too, in particular, $\phi \to \bot$ is abbreviated as $\neg\phi$ for any formula ϕ.

Definition 4.2.2. We inductively define the notion of a formula ϕ of **PID** being *true* on a team X, denoted by $X \models \phi$. All the cases are the same as those of **PD** as defined in Definition 4.1.3 except the following:

- $X \models \bot$ iff $X = \emptyset$;
- $X \models =\!(p_i)$ iff for all $s, s' \in X$, $s(i) = s'(i)$;
- $X \models \phi \vee \psi$ iff $X \models \phi$ or $X \models \psi$;
- $X \models \phi \to \psi$ iff for any team $Y \subseteq X$,

$$Y \models \phi \Longrightarrow Y \models \psi.$$

It is straightforward to verify that **PID** has the empty team property, the locality property and the downwards closure property defined in Section 4.1. Next we show that **PID** is not closed under Sub.

Fact 4.2.3. PID *is not closed under* Sub.

Proof. We have that $\models \neg\neg p_i \to p_i$, but by Lemma 2.1.3, $\models \neg\neg\phi \to \phi$ fails for non-flat formulas (e.g. $\not\models \neg\neg=\!(p_i) \to =\!(p_i)$). ■

One observes that propositional intuitionistic dependence logic is the underlying propositional logic of first-order intuitionistic dependence logic. As a consequence, **PID** inherits all relevant properties from **ID**, including the following:

- Dependence atoms of the form $=\!(p_i)$ are called *constancy dependence atoms*. By the same proof as that of Lemma 2.2.3, non-constancy dependence atoms are definable by constancy ones, as

$$=\!(p_{i_1}, \dots, p_{i_k}) \equiv \big(=\!(p_{i_1}) \wedge \dots \wedge =\!(p_{i_{k-1}})\big) \to =\!(p_{i_k}).$$

- All axioms of intuitionistic propositional calculus (**IPL**), all axioms of Maksimova's Logic ND, all axioms of Kreisel-Putnam Logic KP are valid in **PID**, i.e., Fact 2.2.13 holds for **PID** as well.

- Deduction Theorem holds in **PID** (c.f. Theorem 2.2.12).
- A formula is flat iff it satisfies the double negation law; in particular, negated formulas are flat. That is Lemma 2.1.3 and Corollary 2.1.2 hold for **PID** as well.
- Lemma 2.1.4 and Lemma 2.1.5 are true for **PID** and tensor disjunction.

The disjunction of **IPL** has the so-called *disjunction property*, the same is true for **PID**, as shown in the next theorem.

Theorem 4.2.4 (Disjunction Property)**.** *For any formulas ϕ and ψ of* **PID***,*

$$\text{if} \models \phi \vee \psi, \text{ then} \models \phi \text{ or} \models \psi.$$

Proof. By Corollary 4.1.8. ■

Unlike in **ID**, where intuitionistic disjunction is superfluous as it is uniformly definable (Lemma 1.1.16), intuitionistic disjunction of **PID** turns out to be so useful that in its presence, dependence atoms are, in fact, eliminatable.

Lemma 4.2.5. $=\!(p_i) \equiv p_i \vee \neg p_i$*;*

Proof. Easy. ■

The above lemma shows that **PID** is equivalent to **PID**$[\bot, \wedge, \vee, \to]$, the fragment of **PID** which has no occurrences of dependence atoms. Dick de Jongh and Tadeusz Litak observed[2] that this fragment of **PID** is essentially equivalent to *propositional inquisitive logic* [13] (see also [11]).

Inquisitive logic is a new logic based on the so-called *inquisitive semantics*. Inquisitive semantics is a new type of formal semantics, first conceived by Groenendijk [41] and Mascarenhas [70]. It develops a new notion of semantic meaning that directly reflects the use of language in exchanging information. The central aim of inquisitive semantics is to develop a new notion of semantic meaning that captures both informative and inquisitive content of natural language. This enriched notion of meaning is intended to provide a new foundation for the analysis of discourse that is aimed at the exchange of information.

Early work on inquisitive logic was done by Mascarenhas [70] and Sano [75]. A generalized formal system for inquisitive semantics and logic was developed by Groenendijk [40], Ciardelli and Roelofsen [13]. One of the basic formal notions of this generalized inquisitive semantics is the notion of *information states*. An information state is a set of models for the language, that is, a set of possible worlds. One thinks of an information state as the set of configurations that the subject considers possible for the actual world. It turns out that in terms of mathematical content, an information state is essentially a *team*. Exactly as in team semantics, the satisfaction relation

$$\text{“}s \models \phi\text{”}$$

of inquisitive logic (called the “support” relation) is defined between teams (i.e. information states) s and formulas ϕ. In the setting of inquisitive semantics, an information state s embodies a potential update of the common ground, and the proposition expressed by

[2] In a private conversation with the author in September 2011.

a sentence ϕ captures both inquisitive and informative content. That an information state s supports a sentence ϕ is interpreted as "s settles the issue raised by ϕ". This way, the nature of information exchange is reflected in the semantics as a cooperative process of raising and resolving issues, achieving the goal of inquisitive semantics.

As seen from the above, although inquisitive logic has a completely different motivation from that of dependence logic, it uses essentially and independently *team semantics*. Moreover, propositional inquisitive logic (**InqL**) [13] has exactly the same syntax as the logic **PID**$[\bot,\wedge,\vee,\to]$, and it interprets the corresponding atomic formulas and logical constants the same ways as in **PID**. This way, **InqL** and **PID** are essentially equivalent.

Below we present important properties and a complete axiomatization of **PID** obtained essentially in [13] and [11]. For simplicity, we will stick on our notations and refer to **InqL** only indirectly.

In **PID**, we stipulate $\bigvee\emptyset := \bot$. Next lemma shows that every n-team X is definable up to subteams by an n-formula Ψ_X of **PID** (or **InqL**).

Lemma 4.2.6 (due to [13])**.** *Let X be an n-team on $N=\{i_1,\dots,i_n\}$. Define a formula Ψ_X of* **PID** *(or* **InqL***) as*

$$\Psi_X := \neg\neg \bigvee_{s\in X} (p_{i_1}^{s(i_1)} \wedge\cdots\wedge p_{i_n}^{s(i_n)}).$$

Then for any n-team Y on N,

$$Y \models \Psi_X \iff Y \subseteq X.$$

Proof. Easy, or see [13]. ■

PID (or **InqL**) is a maximal downwards closed logic in the sense of the following definition.

Definition 4.2.7. A logic L with team semantics is called a *maximal downwards closed logic* if for every n-element set $N=\{i_1,\dots,i_n\}\subseteq\mathbb{N}$,

$$\nabla_N = \{[\![\phi]\!] : \phi(p_{i_1},\dots,p_{i_n}) \text{ is an } n\text{-formula of } \mathsf{L}\}$$

Theorem 4.2.8 (due to [13])**.** **PID** *(or* **InqL***) is a maximal downwards closed logic.*

Proof. It suffices to show "$\supseteq$". For every $\mathcal{K}\in\nabla_N$, noting that $\mathcal{K}$ is finite (it has at most 2^{2^n} elements), we obtain by Lemma 4.2.6 that

$$Y \models \bigvee_{X\in\mathcal{K}} \Psi_X \iff \exists X\in\mathcal{K}(Y\subseteq X) \iff Y\in\mathcal{K},$$

i.e., $[\![\bigvee_{X\in\mathcal{K}}\Psi_X]\!] = \mathcal{K}$. ■

The above theorem also shows that every formula of **PID** (or **InqL**) can be expressed as a formula in the so-called *disjunctive-negative normal form*:

$$\bigvee_{j\in J} \neg\neg \bigvee_{s\in X_j} (p_{i_1}^{s(i_1)} \wedge\cdots\wedge p_{i_n}^{s(i_n)}),$$

where J is a set of indices and each X_j is an n-team on $\{i_1, \ldots, i_n\}$. Interested readers are referred to [13] for more details on this normal form, we only remark here that in the above formula in the normal form, each (double) negated formula is flat (Corollary 2.1.2).

A set L of formulas is called a *weak intermediate logic* if **IPL** $\subseteq \mathsf{L} \subseteq$ **CPL** and L is closed under MP ([13]); a weak intermediate logic L is called an *intermediate logic* if L is closed under Sub. Using the normal form, it was proved in [13] that **InqL** is a *weak intermediate logic*.

Definition 4.2.9 ([13])**.**

- Let ϕ be a formula. The *negative variant* $\phi^\neg$ of ϕ is obtained from ϕ by replacing any occurrence of a propositional variable p with $\neg p$.
- The *negative variant* $\mathsf{L}^\neg$ of a logic L is defined as $\mathsf{L}^\neg = \{\phi \mid \phi^\neg \in \mathsf{L}\}$.

The logic **InqL**, as shown in [13], is equivalent to the negative variant of Maksimova's logic [68]

$$\mathsf{ND} = \textbf{IPL} \oplus \{\mathrm{ND}_k \mid k \in \mathbb{N}\},^3$$

and also to the negative variant of Kreisel-Putnam logic [63]

$$\mathsf{KP} = \textbf{IPL} \oplus \mathrm{KP},$$

where ND_k and KP are defined in Fact 2.2.13.

Theorem 4.2.10 ([13])**.** **InqL** $= \mathsf{ND}^\neg = \mathsf{KP}^\neg$.

This is then also true for **PID**, in particular, the logic **PID** is complete with respect to the following Hilbert style deductive system:

Definition 4.2.11 (A Deductive System for **PID**)**.** We write $\vdash_{\textbf{PID}} \phi$ if the **PID** formula ϕ is derivable from the following axioms using the following rules:

Axioms:

1. all substitution instances of **IPL** axioms
2. $\neg\neg p_i \to p_i$ for all propositional variables p_i
3. axiom schemes of ND_k for all $k \in \mathbb{N}$:

$$(\mathrm{ND}_k) \qquad \big(\neg\phi \to \bigvee_{1\leq i\leq k} \neg\psi_i\big) \to \bigvee_{1\leq i\leq k} (\neg\phi \to \neg\psi_i).$$

   or axiom scheme of

$$(\mathrm{KP}) \qquad (\neg\phi \to (\psi \vee \chi)) \to ((\neg\phi \to \psi) \vee (\neg\phi \to \chi)).$$

4. $=\!(p_i) \leftrightarrow (p_i \vee \neg p_i)$

Rules:

[3] Denote by $\mathsf{L}_1 \oplus \mathsf{L}_2$ the smallest set of formulas containing all axioms of the two propositional logics L_1 and L_2 and is closed under MP and Sub.

Modus Ponens: $\dfrac{\phi \to \psi \qquad \psi}{\psi}$ (MP)

Theorem 4.2.12 (Completeness Theorem of **PID**). *For any formula ϕ of,*

$$\vdash_{\mathbf{PID}} \phi \iff \models_{\mathbf{PID}} \phi.$$

Proof. Axiom 4 eliminates dependence atoms. The rest of the proof is due to [13]. ∎

An important feature of **PID** (or **InqL**), as pointed out in Fact 4.1.11, is that it is not closed under uniform substitution, however, it is closed under the so-called *flat substitution* described in the following lemma.

Lemma 4.2.13 (due to [13]). **PID** *is closed under flat substitution, that is, for any formula $\phi(p_{i_1},\dots,p_{i_n})$ of* **PID**,

$$\models \phi(p_{i_1},\dots,p_{i_n}) \implies \models \phi(\psi_1/p_{i_1},\dots,\psi_n/p_{i_n}),$$

whenever $\psi_1,\dots,\psi_n$ are flat formulas of **PID**.

Proof. See [13]. ∎

It was proved in [11] that **InqL** is strongly complete with respect to negative saturated intuitionistic Kripke models. This is then also true for **PID**. We will come back to this issue in Chapter 6.

Definition 4.2.14. An *intuitionistic Kripke model* is a triple $\mathfrak{M} = (W, \geq, V)$ consisting of a non-empty set W, a partial ordering $\geq$ on W, a function (a *valuation*) $V : \text{Prop} \to \wp(W)$ satisfying *monotonicity* with respect to $\geq$, that is,

$$[w \in V(p) \text{ and } w \geq v] \Longrightarrow v \in V(p).$$

The pair $\mathfrak{F} = (W, \geq)$ is called the underlying frame of $\mathfrak{M}$.

Definition 4.2.15. A point w in an intuitionistic Kripke model $\mathfrak{M} = (W, \geq, V)$ is called an *endpoint* iff there is no point $v \in W$ such that $w \geq v$. Denote the set of all endpoints seen from w by E_w, i.e.,

$$E_w = \{v \in W \mid w \geq v \text{ and } v \text{ is an endpoint}\}.$$

Definition 4.2.16. An intuitionistic Kripke model $\mathfrak{M} = (W, \geq, V)$ is called a *negative saturated model* iff

- the valuation V is *negative*, namely

$$\mathfrak{M}, w \models p \iff \mathfrak{M}, w \models \neg\neg p;$$

- the underlying frame of $\mathfrak{M}$ is *saturated*, that is, for every point $w \in W$,
  - $E_w \neq \emptyset$;
  - for every non-empty subset $E \subseteq E_w$, there exists $v \leq w$ such that $E_v = E$.

Theorem 4.2.17 (due to [11]). **PID** *is strongly complete with respect to negative saturated intuitionistic Kripke models.*

Theorem 4.2.18 (Strong Completeness Theorem). *Let Γ be a set of formulas and ϕ a formula of* **PID**. *Then*

$$\Gamma \vdash_{\mathbf{PID}} \phi \iff \Gamma \models_{\mathbf{PID}} \phi.$$

Proof. Due to [11]. ∎

Theorem 4.2.19 (Compactness Theorem). *For any set Γ of formulas and any formula ϕ of* **PID**, *if $\Gamma \models \phi$, then there exists a finite set $\Delta \subseteq \Gamma$ such that $\Delta \models \phi$.*

4.3 Axiomatizing propositional dependence logic with intuitionistic disjunction

We showed in Theorem 4.2.8 that **PID** is a maximal downwards closed logic. In this section, we study another naturally arisen maximal downwards closed logic, that is propositional dependence logic extended with intuitionistic disjunction ($\mathbf{PD}^{[\vee]}$). We give axioms and prove a completeness theorem for the logic.

We start with analyzing the expressive power of $\mathbf{PD}^{[\vee]}$. For logics based on team semantics which have the empty team property (such as **PD**, $\mathbf{PD}^{[\vee]}$ and **PID**), we define

- $\bigotimes\emptyset := \bigvee\emptyset := \bot$,

where $\bot$ is a shorthand for $p_i \wedge \neg p_i$ for any p_i. In the next lemma, we prove that all n-teams X are definable up to subteams in $\mathbf{PD}^{[\vee]}$ by a very similar formula Θ_X to the formula Ψ_X of **PID** in Lemma 4.2.6.

Lemma 4.3.1. *Let X be an n-team on $N = \{i_1, \dots, i_n\}$. Define a formula Θ_X of $\mathbf{PD}^{[\vee]}$ as*

$$\Theta_X := \bigotimes_{s \in X} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)}).$$

Then for any n-team Y on N,

$$Y \models \Theta_X \iff Y \subseteq X.$$

Proof. "$\Longrightarrow$": Suppose $Y \models \Theta_X$. If $X = \emptyset$, then $\Theta_X := \bot$ and we must have that $Y = \emptyset = X$. Otherwise, if $X \neq \emptyset$, then for each $s \in X$, there exists Y_s such that

$$Y = \bigcup_{s \in X} Y_s \text{ and } Y_s \models p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)}.$$

Then, either $Y_s = \emptyset$ or $Y_s = \{s\}$ implying $Y \subseteq X$.

"$\Longleftarrow$": By the downwards closure property, it suffices to show that $X \models \Theta_X$. Clearly, if $X = \emptyset$, then $\Theta_X := \bot$ and $X \models \Theta_X$. Otherwise, clearly, for each $s \in X$, we have that $\{s\} \models p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)}$, which implies that $X \models \Theta_X$. ∎

Now we show that $\mathbf{PD}^{[\vee]}$ is a maximal downwards closed logic by a very similar argument to that in the proof of Theorem 4.2.8.

Theorem 4.3.2. **PD**$^{[\vee]}$ *is a maximal downwards closed logic.*

Proof. It suffices to show that

$$\nabla_N = \{[\![\phi]\!] : \phi(p_{i_1}, \ldots, p_{i_n}) \text{ is an } n\text{-formula of } \mathbf{PD}^{[\vee]}\}$$

for any $N = \{i_1, \ldots, i_n\} \subseteq \mathbb{N}$. The direction "$\supseteq$" follows from the fact that **PD**$^{[\vee]}$ is downwards closed. For the other inclusion "$\subseteq$", for every $\mathcal{K} \in \nabla_N$, $\mathcal{K} = [\![\bigvee_{X \in \mathcal{K}} \Theta_X]\!]$ (note that $\mathcal{K}$ is finite). ■

Corollary 4.3.3. **PD**$^{[\vee]}$ $=$ **PID** $=$ **InqL**.

Proof. By Theorem 4.3.2 and Theorem 4.2.8. ■

The proof of Theorem 4.3.2 shows that every n-formula $\phi(p_{i_1}, \ldots, p_{i_n})$ of **PD**$^{[\vee]}$ is logically equivalent to a formula in the *normal form*

$$\bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)}), \tag{4.8}$$

where F is a finite set of indices and each X_f is an n-team on $\{i_1, \ldots, i_n\}$. It is worthwhile to point out that a similar normal form (of typically infinite size) for first-order dependence logic extended with intuitionistic disjunction was suggested already in [1]. The formula in the normal form (4.8) does not contain dependence atoms, this means that dependence atoms are expressible in **PD**$^{[\vee]}$. We prove this in the following lemma.

Lemma 4.3.4. $=\!(p_{j_0}, \ldots, p_{j_k}) \equiv \bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \cdots \wedge p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}\right)$, *where* $\mathbf{2^k}$ *is the maximal* n*-team on* $K = \{j_0, \ldots, j_{k-1}\}$.

Proof. It suffices to show that for any team X,

$$X \models =\!(p_{j_0}, \ldots, p_{j_k}) \iff X \models \bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \cdots \wedge p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}\right).$$

Suppose $X \models =\!(p_{j_0}, \ldots, p_{j_k})$. Define $f : \mathbf{2^k} \to 2$ by taking

$$f(s) = \begin{cases} t(j_k), & \text{if } \exists t \in X \text{ such that } t \upharpoonright K = s, \\ 1, & \text{otherwise.} \end{cases}$$

The function f is well-defined, since for any $t_0, t_1 \in X$ such that

$$t_0 \upharpoonright K = t_1 \upharpoonright K, \tag{4.9}$$

the assumption guarantees that $t_0(j_k) = t_1(j_k)$.

For each $s \in \mathbf{2^k}$, define

$$X_s = \{t \in X : t \upharpoonright K = s\}.$$

It is easy to see that $X = \bigcup_{s \in \mathbf{2^k}} X_s$ and

$$X_s \models p_{j_0}^{s(j_0)} \wedge \cdots \wedge p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}. \tag{4.10}$$

Hence $X \models \bigotimes_{s \in \mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \cdots \wedge p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)} \right)$.

Conversely, for "$\Longleftarrow$", by assumption, there exists a function $f : \mathbf{2^k} \to 2$ and for each $s \in \mathbf{2^k}$ there exists a team $X_s \subseteq X$ such that $X = \bigcup_{s \in \mathbf{2^k}} X_s$ and (4.10) holds. For any $t_0, t_1 \in X$ such that (4.9) holds, there exists $s \in \mathbf{2^k}$ satisfying

$$s = t_0 \upharpoonright K = t_1 \upharpoonright K.$$

Clearly, $t_0, t_1 \in X_s$. By (4.10), $t_0(j_k) = f(s) = t_1(j_k)$, as required. ■

We now present a natural deduction system for $\mathbf{PD}^{[\vee]}$ for which the normal form (4.8) can be obtained proof-theoretically. The main goal of this section is to prove the completeness theorem for this system.

Definition 4.3.5 (A natural deduction system for $\mathbf{PD}^{[\vee]}$)**.** The rules are given as follows:

1. Conjunction Introduction: $\dfrac{\phi \qquad \psi}{\phi \wedge \psi}\,(\wedge\mathsf{I})$

2. Conjunction Elimination: $\dfrac{\phi \wedge \psi}{\phi}\,(\wedge\mathsf{E}) \qquad \dfrac{\phi \wedge \psi}{\psi}\,(\wedge\mathsf{E})$

3. Intuitionistic Disjunction Introduction: $\dfrac{\phi}{\phi \vee \psi}\,(\vee\mathsf{I}) \qquad \dfrac{\psi}{\phi \vee \psi}\,(\vee\mathsf{I})$

4. Intuitionistic Disjunction Elimination:

$$\dfrac{\phi \vee \psi \qquad \begin{matrix}[\phi]\\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\chi}\,(\vee\mathsf{E})$$

5. Tensor Disjunction Introduction: $\dfrac{\phi}{\phi \otimes \psi}\,(\otimes\mathsf{I})$

6. Weak Tensor Disjunction Elimination:

$$\dfrac{\phi \otimes \psi \qquad \begin{matrix}[\phi]\\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\chi}\,(\otimes\mathsf{WE})$$

   whenever χ is a classical formula.

7. Tensor Disjunction Substitution:

$$\dfrac{\phi \otimes \psi \qquad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\phi \otimes \chi}\ (\otimes\mathsf{Sub})$$

8. Commutative and Associative Laws for Tensor Disjunction:

$$\dfrac{\phi \otimes \psi}{\psi \otimes \phi}\ (\mathsf{Com}\otimes) \qquad \dfrac{\phi \otimes (\psi \otimes \chi)}{(\phi \otimes \psi) \otimes \chi}\ (\mathsf{Ass}\otimes)$$

9. Contradiction Elimination: $\dfrac{\phi \otimes (p_i \wedge \neg p_i)}{\phi}\ (\bot\mathsf{E})$

10. Atomic Excluded Middle: $\dfrac{}{p_i \otimes \neg p_i}\ (\mathsf{EM}_0)$

11. Dependence Atom Introduction:

$$\dfrac{\bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \cdots \wedge \dots p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}\right)}{=\!(p_{j_0}, \dots, p_{j_{k-1}}, p_{j_k})}\ (\mathsf{DepI})$$

where $\mathbf{2^k}$ is the maximal k-team on the set $\{j_0, \dots, j_{k-1}\}$.

12. Dependence Atom Elimination:

$$\dfrac{=\!(p_{j_0}, \dots, p_{j_{k-1}}, p_{j_k})}{\bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \cdots \wedge \dots p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}\right)}\ (\mathsf{DepE})$$

where $\mathbf{2^k}$ is the maximal k-team on the set $\{j_0, \dots, j_{k-1}\}$.

13. Distributive Laws:

$$\dfrac{\phi \otimes (\psi \vee \chi)}{(\phi \otimes \psi) \vee (\phi \otimes \chi)}\ (\mathsf{Dstr}\otimes\vee) \qquad \dfrac{\phi \wedge (\psi \otimes \chi)}{(\phi \wedge \psi) \otimes (\phi \wedge \chi)}\ (\mathsf{Dstr}\wedge\otimes)$$

$$\dfrac{(\phi \otimes \psi) \vee (\phi \otimes \chi)}{\phi \otimes (\psi \vee \chi)}\ (\mathsf{Dstr}\otimes\vee\otimes)$$

If a formula ϕ of $\mathbf{PD}^{[\vee]}$ is derivable in the system, then we write $\vdash_{\mathbf{PD}^{[\vee]}} \phi$ or simply $\vdash \phi$. If $\phi \vdash \psi$ and $\psi \vdash \phi$, then we say that ϕ and ψ are *provably equivalent*, in symbols $\phi \dashv\vdash \psi$.

In the above system, all of the substitution, commutative, associative and distributive rules involving tensor disjunction $\otimes$ are necessary, as we only have the *weak* elimination rule for tensor disjunction ($\otimes$WE), whereas the *strong* elimination rule:

$$\frac{\phi \otimes \psi \quad \begin{matrix}[\phi] \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\psi] \\ \vdots \\ \chi\end{matrix}}{\chi}$$

is not valid (since e.g. $\models =\!(p_i) \otimes =\!(p_i)$ but $\not\models =\!(p_i)$). Moreover, not all usual distributive laws are valid in **PD**$^{[\vee]}$, for example, the following distributive law:

$$\frac{(\phi \vee \psi) \otimes (\phi \vee \chi)}{\phi \vee (\psi \otimes \chi)}$$

is not valid even for classical formulas, since e.g.,

$$(p \vee \neg p) \otimes (p \vee \neg p) \not\models p \vee (\neg p \otimes \neg p).$$

Interesting derivable rules are listed in the next lemma.

Corollary 4.3.6. *The following are derivable rules:*

1. *ex falso:* $\dfrac{p_i \wedge \neg p_i}{\phi}$ (*ex falso*)

2. *Distributive Laws:*

$$\frac{\phi \otimes (\psi \wedge \chi)}{(\phi \otimes \psi) \wedge (\phi \otimes \chi)} \ (\mathsf{Dstr}\otimes\wedge) \qquad \frac{\phi \vee (\psi \otimes \chi)}{(\phi \vee \psi) \otimes (\phi \vee \chi)} \ (\mathsf{Dstr}\vee\otimes)$$

$$\frac{(\phi \otimes \psi) \wedge (\phi \otimes \chi)}{\phi \otimes (\psi \wedge \chi)} \ (*) \ (\mathsf{Dstr}^*\otimes\wedge\otimes)$$

$$\frac{(\phi \wedge \psi) \otimes (\phi \wedge \chi)}{\phi \wedge (\psi \otimes \chi)} \ (*) \ (\mathsf{Dstr}^*\wedge\otimes\wedge)$$

($*$) whenever ϕ is a classical formula.

3. *Commutative and Associative Rules for Conjunction and Intuitionistic Disjunction:*

$$\frac{\phi \wedge \psi}{\psi \wedge \phi} \ (\mathsf{Com}\wedge) \qquad \frac{\phi \vee \psi}{\psi \vee \phi} \ (\mathsf{Com}\vee)$$

$$\frac{(\phi \wedge \psi) \wedge \chi}{\phi \wedge (\psi \wedge \chi)} \ (\mathsf{Ass}\wedge) \qquad \frac{(\phi \vee \psi) \vee \chi}{\phi \vee (\psi \vee \chi)} \ (\mathsf{Ass}\vee)$$

4. *Substitution Rule for Intuitionistic Disjunction and Conjunction:*

$$\frac{\phi \vee \psi \quad \begin{matrix}[\psi] \\ \vdots \\ \chi\end{matrix}}{\phi \vee \chi} \ (\vee\mathsf{Sub}) \qquad \frac{\phi \wedge \psi \quad \begin{matrix}[\psi] \\ \vdots \\ \chi\end{matrix}}{\phi \wedge \chi} \ (\wedge\mathsf{Sub})$$

5. *Distributive Laws for Intuitionistic Disjunction and Conjunction:*

$$\dfrac{\phi\wedge(\psi\vee\chi)}{(\phi\wedge\psi)\vee(\phi\wedge\chi)}\,\textit{(Dstr)} \qquad \dfrac{\phi\vee(\psi\wedge\chi)}{(\phi\vee\psi)\wedge(\phi\vee\chi)}\,\textit{(Dstr)}$$

$$\dfrac{(\phi\vee\psi)\wedge(\phi\vee\chi)}{\phi\vee(\psi\wedge\chi)}\,\textit{(Dstr)} \qquad \dfrac{(\phi\wedge\psi)\vee(\phi\wedge\chi)}{\phi\wedge(\psi\vee\chi)}\,\textit{(Dstr)}$$

Proof. The rules in Items 3-5 are derived as usual. It remains to derive all the other rules.
For (ex falso):

$$\dfrac{\dfrac{p_i\wedge\neg p_i}{\phi\otimes(p_i\wedge\neg p_i)}\,(\otimes\mathsf{I})}{\phi}\,(\bot\mathsf{E})$$

For (Dstr $\otimes\wedge$):

$$\dfrac{(\otimes\mathsf{Sub})\,\dfrac{\phi\otimes(\psi\wedge\chi)\qquad \dfrac{[\psi\wedge\chi]}{\psi}\,(\wedge\mathsf{E})}{\phi\otimes\psi} \qquad \dfrac{\phi\otimes(\psi\wedge\chi)\qquad \dfrac{[\psi\wedge\chi]}{\chi}\,(\wedge\mathsf{E})}{\phi\otimes\chi}\,(\otimes\mathsf{Sub})}{(\phi\otimes\psi)\wedge(\phi\otimes\chi)}\,(\wedge\mathsf{I})$$

For (Dstr $\vee\otimes$):

$$\dfrac{\phi\vee(\psi\otimes\chi)\qquad \dfrac{\dfrac{[\phi]}{\phi\otimes\phi}\,(\otimes\mathsf{I})}{(\phi\vee\psi)\otimes(\phi\vee\chi)}\,(\vee\mathsf{I},\otimes\mathsf{Sub}) \qquad \dfrac{[\psi\otimes\chi]}{(\phi\vee\psi)\otimes(\phi\vee\chi)}\,(\vee\mathsf{I},\otimes\mathsf{Sub})}{(\phi\vee\psi)\otimes(\phi\vee\chi)}\,(\vee\mathsf{E})$$

For (Dstr$^*\otimes\wedge\otimes$): If ϕ is a classical formula, then we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\phi\otimes\psi)\wedge(\phi\otimes\chi)}{\big((\phi\otimes\psi)\wedge\phi\big)\otimes\big((\phi\otimes\psi)\wedge\chi\big)}\,(\mathsf{Dstr}\wedge\otimes)}{\phi\otimes\big((\phi\otimes\psi)\wedge\chi\big)}\,(\wedge\mathsf{E},\otimes\mathsf{Sub})}{\phi\otimes(\phi\wedge\chi)\otimes(\psi\wedge\chi)}\,(\mathsf{Dstr}\wedge\otimes)}{\phi\otimes\phi\otimes(\psi\wedge\chi)}\,(\wedge\mathsf{E},\otimes\mathsf{Sub})}{\phi\otimes(\psi\wedge\chi)}\,(\otimes\mathsf{WE},\otimes\mathsf{Sub})$$

For (Dstr$^*\wedge\otimes\wedge$): If ϕ is a classical formula, then we have the following derivation:

$$\dfrac{\dfrac{\dfrac{(\phi\wedge\psi)\otimes(\phi\wedge\chi)}{\phi\otimes\phi}\,(\wedge\mathsf{E},\otimes\mathsf{Sub})}{\phi}\,(\otimes\mathsf{WE}) \qquad (\wedge\mathsf{E},\otimes\mathsf{Sub})\,\dfrac{(\phi\wedge\psi)\otimes(\phi\wedge\chi)}{\psi\otimes\chi}}{\phi\wedge(\psi\otimes\chi)}\,(\wedge\mathsf{I})$$

■

Next, we prove the Soundness Theorem for the above deductive system.

Theorem 4.3.7 (Soundness Theorem). *For any formulas ϕ and ψ of* $\mathbf{PD}^{[\vee]}$*,*

$$\phi \vdash \psi \Longrightarrow \phi \models \psi.$$

Proof. It suffices to show that all of the deductive rules are valid. The rules 1-5, 7-10 are easy to verify. The validity of Dependence Atom Introduction and Elimination rules follows from Lemma 4.3.4. It remains to verify the validity of the rules 4,5 and 13.

For ($\otimes$WE), it suffices to show that

$$\phi \models \chi \text{ and } \psi \models \chi \Longrightarrow \phi \otimes \psi \models \chi,$$

whenever χ is a classical formula. By Lemma 4.1.9, the assumption implies that χ is flat. For any team X such that $X \models \phi \otimes \psi$, there are teams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$. Since $\phi \models \chi$ and $\psi \models \chi$, we have that $Y \models \chi$ and $Z \models \chi$, which imply that $X \models \chi$, as χ is flat.

For (Dstr$\otimes\vee$), it suffices to show that $\phi \otimes (\psi \vee \chi) \models (\phi \otimes \psi) \vee (\phi \otimes \chi)$. For any team X such that $X \models \phi \otimes (\psi \vee \chi)$, there are teams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi \vee \chi$. It follows that $Y \cup Z \models \phi \otimes \psi$ or $Y \cup Z \models \phi \otimes \chi$, hence $X \models (\phi \otimes \psi) \vee (\phi \otimes \chi)$.

For (Dstr$\wedge\otimes$), it suffices to show that $\phi \wedge (\psi \otimes \chi) \models (\phi \wedge \psi) \otimes (\phi \wedge \chi)$. For any team X such that $X \models \phi \wedge (\psi \otimes \chi)$, we have that $X \models \phi$ and $X \models \psi \otimes \chi$. The latter implies that there are teams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \psi$ and $Z \models \chi$. By the downwards closure property, $Y \models \phi$ and $Z \models \chi$. It follows that $Y \models \phi \wedge \psi$ and $Z \models \phi \wedge \chi$, thus $X \models (\phi \wedge \psi) \otimes (\phi \wedge \chi)$.

For (Dstr$\otimes\vee\otimes$), it suffices to show that $(\phi \otimes \psi) \vee (\phi \otimes \chi) \models \phi \otimes (\psi \vee \chi)$. For any team X such that $X \models (\phi \otimes \psi) \vee (\phi \otimes \chi)$, we have that $X \models \phi \otimes \psi$ or $X \models \phi \otimes \chi$. In the former case, there are teams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$, which implies that $Z \models \psi \vee \chi$, thereby $X \models \phi \otimes (\psi \vee \chi)$. By a similar argument, one derives $X \models \phi \otimes (\psi \vee \chi)$ in the latter case as well. ∎

Next, we show that every $\mathbf{PD}^{[\vee]}$ formula is provably equivalent to a formula in the intended normal form (4.8).

Theorem 4.3.8. *Any n-formula $\phi(p_{i_1}, \ldots, p_{i_n})$ of* $\mathbf{PD}^{[\vee]}$ *is provably equivalent to a formula of the form*

$$\bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)}),$$

where F is a finite set of indices, and each X_f is an n-team on $N = \{i_1, \ldots, i_n\}$.

Proof. Let $\mathbf{2^n}$ be the maximal n-team on N. We prove the theorem by induction on $\phi(p_{i_1}, \ldots, p_{i_n})$.

Case $\phi(p_{i_1}, \ldots, p_{i_n}) = p_{i_k}$ for $1 \leq k \leq n$. If $N = \{i_k\}$, then $p_{i_k} \dashv\vdash \bigotimes_{s \in \{\mathbf{1}\}} p_{i_k}^{s(i_k)}$, where $\mathbf{1} : \{i_k\} \to \{0, 1\}$ is defined as $\mathbf{1}(i_k) = 1$.

Now, assume $N \supset \{i_k\}$. We prove that $p_{i_k} \dashv\vdash \theta$, where

$$\theta := \bigotimes_{s \in \mathbf{2^n}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_k} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)}).$$

For $p_{i_k} \vdash \theta$, we have the following derivation:

$(1) p_{i_k}$

$(2) (p_{i_0} \otimes \neg p_{i_0}) \wedge \cdots \wedge (p_{i_{k-1}} \otimes \neg p_{i_{k-1}}) \wedge (p_{i_{k+1}} \otimes \neg p_{i_{k+1}}) \wedge \cdots \wedge (p_{i_n} \otimes \neg p_{i_n})$
(EM$_0$, $\wedge$I)

$(3) \bigotimes_{s \in \mathbf{2^n}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)})$ (Dstr$\wedge\otimes$, $\otimes$I, $\otimes$Sub)

$(4) p_{i_k} \wedge \bigotimes_{s \in \mathbf{2^n}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)})$ ((1), (3), $\wedge$I)

$(5) \bigotimes_{s \in \mathbf{2^n}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_k} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)})$
(Dstr$\wedge\otimes$, Com$\wedge$, $\otimes$Sub)

For the other direction $\theta \vdash p_{i_k}$, we have the following derivation:

$(1) \theta$

$(2) \bigotimes_{s \in \mathbf{2^n}} \big(p_{i_k} \wedge (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)})\big)$ (Com$\wedge$, $\otimes$Sub)

$(3) p_{i_k} \wedge \bigotimes_{s \in \mathbf{2^n}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)})$ (Dstr$^*\wedge\otimes\wedge$, $\otimes$Sub)

$(4) p_{i_k}$ ($\wedge$E)

Case $\phi(p_{i_1}, \ldots, p_{i_n}) = \neg p_{i_k}$ $(1 \leq k \leq n)$ is proved similarly with the above case.

Case $\phi = {=\!(}p_{i_{j_0}}, \cdots, p_{i_{j_k}})(p_{i_1}, \ldots, p_{i_n})$. Put $K = \{i_{j_0}, \ldots, i_{j_{k-1}}\}$. If $N \setminus \{i_{j_k}\} = K$, then by (DepE) and (DepI), we derive

$$={(}p_{i_{j_0}}, \cdots, p_{i_{j_k}})(p_{i_{j_0}}, \cdots, p_{i_{j_k}}) \dashv\vdash \bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^k}} (p_{i_{j_0}}^{s(i_{j_0})} \wedge \cdots \wedge \ldots p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s)}).$$

Now, assume $N \setminus \{i_{j_k}\} \supset K$. We show that ${=\!(}p_{i_{j_0}}, \cdots, p_{i_{j_k}}) \dashv\vdash \theta$, where

$$\theta := \bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s \upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)}),$$

where $\mathbf{2^k}$ is the maximal k-team on K and $\mathbf{2^{n-1}}$ is the maximal $(n-1)$-team on $N \setminus \{i_{j_k}\}$[4]. Let $M = N \setminus (K \cup \{i_{j_k}\})$, $|M| = m$ and $\mathbf{2^m}$ be the maximal m-team on M. For ${=\!(}p_{i_{j_0}}, \cdots, p_{i_{j_k}}) \vdash \theta$, we have the following derivation:

$(1) ={(}p_{i_{j_0}}, \cdots, p_{i_{j_k}})$

$(2) \bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^k}} (p_{i_{j_0}}^{s(i_{j_0})} \wedge \cdots \wedge \ldots p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s)})$ (DepE)

[4]Note that here $\mathbf{2^{n-1}}$ can be viewed as the subset $\{s \cup \{(i_{j_k}, 1)\} \mid s \in \mathbf{2^{n-1}}\}$ of the maximal n-team $\mathbf{2^n}$ on $\{i_1, \ldots, i_n\}$.

(3) $\bigvee_{f\in 2^{\mathbf{2^k}}}\bigotimes_{s\in\mathbf{2^k}}\Big(p_{i_{j_0}}^{s(i_{j_0})}\wedge\cdots\wedge\ldots p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})}\wedge p_{i_{j_k}}^{f(s)}\wedge\Big(\bigotimes_{t\in\mathbf{2^m}}\bigwedge_{a\in M}p_{i_a}^{t(i_a)}\Big)\Big)$

(EM$_0$, Dstr$\wedge\otimes$)

(4) $\bigvee_{f\in 2^{\mathbf{2^k}}}\bigotimes_{s\in\mathbf{2^k}}\bigotimes_{t\in\mathbf{2^m}}\Big(p_{i_{j_0}}^{s(i_{j_0})}\wedge\cdots\wedge\ldots p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})}\wedge p_{i_{j_k}}^{f(s)}\wedge\Big(\bigwedge_{a\in M}p_{i_a}^{t(i_a)}\Big)\Big)$ (Dstr $\wedge\otimes$)

(5) $\bigvee_{f\in 2^{\mathbf{2^k}}}\bigotimes_{s\in\mathbf{2^{n-1}}}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})}\wedge p_{i_{j_k}}^{f(s\upharpoonright K)}\wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})}\wedge\cdots\wedge p_{i_n}^{s(i_n)})$

For the other direction $\theta\vdash{=}(p_{i_{j_0}},\cdots,p_{i_{j_k}})$, we have the following derivation:

(1) $\bigvee_{f\in 2^{\mathbf{2^k}}}\bigotimes_{s\in\mathbf{2^{n-1}}}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})}\wedge p_{i_{j_k}}^{f(s\upharpoonright K)}\wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})}\wedge\cdots\wedge p_{i_n}^{s(i_n)})$

(2) $\bigvee_{f\in 2^{\mathbf{2^k}}}\bigotimes_{s\in\mathbf{2^{n-1}}}(p_{i_{j_0}}^{s(i_{j_0})}\wedge\cdots\wedge\ldots p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})}\wedge p_{i_{j_k}}^{f(s\upharpoonright K)})$ ($\wedge$E, $\otimes$Sub)

(3) $\bigvee_{f\in 2^{\mathbf{2^k}}}\bigotimes_{s\in\mathbf{2^k}}(p_{i_{j_0}}^{s(i_{j_0})}\wedge\cdots\wedge\ldots p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})}\wedge p_{i_{j_k}}^{f(s)})$ ($\otimes$WE)

(4) ${=}(p_{i_{j_0}},\cdots,p_{i_{j_k}})$ (DepI)

Case $\phi(p_{i_1},\ldots,p_{i_n})=\psi(p_{i_1},\ldots,p_{i_n})\vee\chi(p_{i_1},\ldots,p_{i_n})$. By induction hypothesis, we have that

$$\psi(p_{i_1},\ldots,p_{i_n})\dashv\vdash\bigvee_{f\in F}\bigotimes_{s_0\in X_f}(p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)}),$$

$$\chi(p_{i_1},\ldots,p_{i_n})\dashv\vdash\bigvee_{g\in G}\bigotimes_{s_1\in X_g}(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)}),\tag{4.11}$$

where each $X_f,X_g\subseteq\mathbf{2^n}$. If $F,G\neq\emptyset$, then it follows from ($\vee$E) and ($\vee$I) that

$$\psi\vee\chi\dashv\vdash\bigvee_{h\in F\cup G}\bigotimes_{s\in X_h}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}).$$

If $F=\emptyset$ or $G=\emptyset$, then $\psi\dashv\vdash p_{i_1}\wedge\neg p_{i_1}$ or $\chi\dashv\vdash p_{i_1}\wedge\neg p_{i_1}$. In the former case, we derive by (ex falso), ($\vee$E) and ($\vee$I) that

$$\begin{aligned}\psi\vee\chi&\dashv\vdash(p_{i_1}\wedge\neg p_{i_1})\vee\Big(\bigvee_{g\in G}\bigotimes_{s_1\in X_g}(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)})\Big)\\&\dashv\vdash\bigvee_{g\in G}\bigotimes_{s_1\in X_g}(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)}).\end{aligned}$$

Similarly, in the latter case, we derive $\psi\vee\chi\dashv\vdash\bigvee_{f\in F}\bigotimes_{s_0\in X_f}(p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)})$.

Case $\phi(p_{i_1},\dots,p_{i_n}) = \psi(p_{i_1},\dots,p_{i_n}) \otimes \chi(p_{i_1},\dots,p_{i_n})$. By induction hypothesis, we have (4.11). If $F = \emptyset$ or $G = \emptyset$, then $\psi \dashv\vdash p_{i_1} \wedge \neg p_{i_1}$ or $\chi \dashv\vdash p_{i_1} \wedge \neg p_{i_1}$. In the former case, we derive by (⊥E) and (⊗I) that

$$\psi \otimes \chi \dashv\vdash (p_{i_1} \wedge \neg p_{i_1}) \otimes \Big(\bigvee_{g\in G} \bigotimes_{s_1 \in X_g} (p_{i_1}^{s_1(i_1)} \wedge \dots \wedge p_{i_n}^{s_1(i_n)}) \Big)$$
$$\dashv\vdash \bigvee_{g\in G} \bigotimes_{s_1 \in X_g} (p_{i_1}^{s_1(i_1)} \wedge \dots \wedge p_{i_n}^{s_1(i_n)}).$$

Similarly, in the latter case, we derive $\psi \otimes \chi \dashv\vdash \bigvee_{f\in F} \bigotimes_{s_0 \in X_f} (p_{i_1}^{s_0(i_1)} \wedge \dots \wedge p_{i_n}^{s_0(i_n)})$.

Now, assume $F, G \neq \emptyset$. We show that $\psi \otimes \chi \dashv\vdash \theta$, where

$$\theta := \bigvee_{f\in F} \bigvee_{g\in G} \bigotimes_{s \in X_f \cup X_g} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)}).$$

For $\psi \otimes \chi \vdash \theta$, we have the following derivation:

(1) $\psi \otimes \chi$

(2) $\Big(\bigvee_{f\in F} \bigotimes_{s_0 \in X_f} (p_{i_1}^{s_0(i_1)} \wedge \dots \wedge p_{i_n}^{s_0(i_n)}) \Big) \otimes \Big(\bigvee_{g\in G} \bigotimes_{s_1 \in X_g} (p_{i_1}^{s_1(i_1)} \wedge \dots \wedge p_{i_n}^{s_1(i_n)}) \Big)$

(3) $\bigvee_{f\in F} \Big(\Big(\bigotimes_{s_0 \in X_f} (p_{i_1}^{s_0(i_1)} \wedge \dots \wedge p_{i_n}^{s_0(i_n)}) \Big) \otimes \Big(\bigvee_{g\in G} \bigotimes_{s_1 \in X_g} (p_{i_1}^{s_1(i_1)} \wedge \dots \wedge p_{i_n}^{s_1(i_n)}) \Big) \Big)$

(Dstr ⊗ ∨)

(4) $\bigvee_{f\in F} \bigvee_{g\in G} \Big(\Big(\bigotimes_{s_0 \in X_f} (p_{i_1}^{s_0(i_1)} \wedge \dots \wedge p_{i_n}^{s_0(i_n)}) \Big) \otimes \Big(\bigotimes_{s_1 \in X_g} (p_{i_1}^{s_1(i_1)} \wedge \dots \wedge p_{i_n}^{s_1(i_n)}) \Big) \Big)$

(Dstr ⊗ ∨)

(5) $\bigvee_{f\in F} \bigvee_{g\in G} \bigotimes_{(s,a) \in (X_f \times \{0\}) \cup (X_g \times \{1\})} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)})$

(6) $\bigvee_{f\in F} \bigvee_{g\in G} \bigotimes_{s \in X_f \cup X_g} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)})$ (⊗WE)

The other direction $\theta \vdash \psi \otimes \chi$ is proved symmetrically using (⊗I) and (Dstr ⊗ ∨⊗).

Case $\phi(p_{i_1},\dots,p_{i_n}) = \psi(p_{i_1},\dots,p_{i_n}) \wedge \chi(p_{i_1},\dots,p_{i_n})$. By induction hypothesis, we have (4.11). If $F = \emptyset$ or $G = \emptyset$, then $\psi \dashv\vdash p_{i_1} \wedge \neg p_{i_1}$ or $\chi \dashv\vdash p_{i_1} \wedge \neg p_{i_1}$. In this case, we derive $\psi \wedge \chi \dashv\vdash p_{i_1} \wedge \neg p_{i_1}$, i.e., $\psi \wedge \chi \dashv\vdash \bigvee \emptyset$, by (∧E) and (ex falso).

Now, assume $F, G \neq \emptyset$. We show that $\psi \wedge \chi \dashv\vdash \theta$, where

$$\theta := \bigvee_{f\in F} \bigvee_{g\in G} \bigotimes_{s \in X_f \cap X_g} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)}).$$

For $\psi \wedge \chi \vdash \theta$, we have the following derivation:

(1) $\psi \wedge \chi$

$$(2)\Big(\bigvee_{f\in F}\bigotimes_{s_0\in X_f}(p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)})\Big)\wedge\Big(\bigvee_{g\in G}\bigotimes_{s_1\in X_g}(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)})\Big)$$

$$(3)\bigvee_{f\in F}\bigvee_{g\in G}\Big(\Big(\bigotimes_{s_0\in X_f}(p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)})\Big)\wedge\Big(\bigotimes_{s_1\in X_g}(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)})\Big)\Big)$$

(Dstr)

$$(4)\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{s_0\in X_f}\bigotimes_{s_1\in X_g}\Big((p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)})\wedge(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)})\Big)$$

(Dstr $\wedge\otimes$, $\otimes$Sub)

$$(5)\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{(s_0,s_1)\in X_f\times X_g}\big((p_{i_1}^{s_0(i_1)}\wedge p_{i_1}^{s_1(i_1)})\wedge\cdots\wedge(p_{i_n}^{s_0(i_n)}\wedge p_{i_n}^{s_1(i_n)})\big)$$

(Com $\otimes$, Ass $\otimes$)

$$(6)\bigvee_{f\in F}\bigvee_{g\in G}\Big(\Big(\bigotimes_{\substack{(s_0,s_1)\in X_f\times X_g\\ s_0=s_1}}\big((p_{i_1}^{s_0(i_1)}\wedge p_{i_1}^{s_1(i_1)})\wedge\cdots\wedge(p_{i_n}^{s_0(i_n)}\wedge p_{i_n}^{s_1(i_n)})\big)\Big)$$

$$\otimes\Big(\bigotimes_{\substack{(s_0,s_1)\in X_f\times X_g\\ s_0\neq s_1}}\big((p_{i_1}^{s_0(i_1)}\wedge p_{i_1}^{s_1(i_1)})\wedge\cdots\wedge(p_{i_n}^{s_0(i_n)}\wedge p_{i_n}^{s_1(i_n)})\big)\Big)\Big)$$

$$(7)\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{\substack{(s_0,s_1)\in X_f\times X_g\\ s_0=s_1}}\big((p_{i_1}^{s_0(i_1)}\wedge p_{i_1}^{s_1(i_1)})\wedge\cdots\wedge(p_{i_n}^{s_0(i_n)}\wedge p_{i_n}^{s_1(i_n)})\big)\quad(\wedge\mathsf{E},\bot\mathsf{E})$$

$$(8)\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{(s,s)\in X_f\times X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)})\quad(\wedge\mathsf{E})$$

$$(9)\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{s\in X_f\cap X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)})\quad(\otimes\mathsf{WE})$$

For the other direction $\theta\vdash\psi\wedge\chi$, we have the following derivation:

$$(1)\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{s\in X_f\cap X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)})$$

$$(2)\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{\substack{s_0,s_1\in X_f\cap X_g\\ s_0=s_1}}\Big((p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)})\wedge(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)})\Big)\quad(\wedge\mathsf{I})$$

$$(3)\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{s_0,s_1\in X_f\times X_g}\Big((p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)})\wedge(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)})\Big)\quad(\otimes\mathsf{I})$$

$$(4)\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{s_0\in X_f}\bigotimes_{s_1\in X_g}\big((p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)})\wedge(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)})\big)$$

$$(5)\bigvee_{f\in F}\bigvee_{g\in G}\Big(\Big(\bigotimes_{s_0\in X_f}(p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)})\Big)\wedge\Big(\bigotimes_{s_1\in X_g}(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)})\Big)\Big)$$

(Dstr$^*\wedge\otimes\wedge$, $\otimes$Sub)

$$(6)\Big(\bigvee_{f\in F}\bigotimes_{s_0\in X_f}(p_{i_1}^{s_0(i_1)}\wedge\cdots\wedge p_{i_n}^{s_0(i_n)})\Big)\wedge\Big(\bigvee_{g\in G}\bigotimes_{s_1\in X_g}(p_{i_1}^{s_1(i_1)}\wedge\cdots\wedge p_{i_n}^{s_1(i_n)})\Big)$$

(Dstr)

$(7)\psi\wedge\chi$

■

The next lemma is crucial in proof of the completeness theorem.

Lemma 4.3.9. *For any finite non-empty collections $\{X_f \mid f\in F\}$, $\{Y_g \mid g\in G\}$ of n-teams on an n-element set $N\subseteq\mathbb{N}$, the following are equivalent:*

(a) $\bigvee_{f\in F}\Theta_{X_f}\models\bigvee_{g\in G}\Theta_{Y_g}$*;*

(b) *for each $f\in F$, we have that $X_f\subseteq Y_{g_f}$ for some $g\in G$.*

Proof. (a)⇒(b): For each $f\in F$, by Lemma 4.3.1,

$$X_f\models\Theta_{X_f},\text{ thus } X_f\models\bigvee_{f\in F}\Theta_{X_f},$$

which by (a) implies that $X_f\models\bigvee_{g\in G}\Theta_{Y_g}$. It follows that there exists $g_f\in G$ such that $X_f\models\Theta_{Y_{g_f}}$. Hence by Lemma 4.3.1, $X_f\subseteq Y_{g_f}$.

(b)⇒(a): Suppose X is any n-team on N satisfying $X\models\bigvee_{f\in F}\Theta_{X_f}$. Then $X\models\Theta_{X_f}$ for some $f\in F$, which by Lemma 4.3.1 and (b) means that $X\subseteq X_f\subseteq Y_{g_f}$ for some $g_f\in G$. Since by Lemma 4.3.1, $Y_{g_f}\models\Theta_{Y_{g_f}}$, it follows from the downwards closure property of $\Theta_{Y_{g_f}}$ that

$$X\models\Theta_{Y_{g_f}},\text{ thereby } X\models\bigvee_{g\in G}\Theta_{Y_g},$$

as required. ■

Now, we are in a position to prove the completeness theorem for $\mathbf{PD}^{[\vee]}$.

Theorem 4.3.10 (Completeness Theorem)**.** *For any $\mathbf{PD}^{[\vee]}$ formulas ϕ and ψ,*

$$\phi\models\psi\Longrightarrow\phi\vdash\psi.$$

Proof. Suppose $\phi\models\psi$, where $\phi=\phi(p_{i_1},\dots,p_{i_n})$ and $\psi=\psi(p_{i_1},\dots,p_{i_n})$. By Theorem 4.3.8, we have that

$$\phi\dashv\vdash\bigvee_{f\in F}\Theta_{X_f},\quad \psi\dashv\vdash\bigvee_{g\in G}\Theta_{Y_g}.$$

for some finite sets $\{X_f\mid f\in F\}$ and $\{Y_g\mid g\in G\}$ of n-teams on $\{i_1,\dots,i_n\}$. Then, by the Soundness Theorem, we have that

$$\bigvee_{f\in F}\Theta_{X_f}\models\bigvee_{g\in G}\Theta_{Y_g}.$$

If $F = \emptyset$, then $\phi \dashv\vdash p_{i_1} \wedge \neg p_{i_1}$, thus, by (ex falso), we obtain that $\phi \vdash \psi$. If $G = \emptyset$, then $\psi \dashv\vdash p_{i_1} \wedge \neg p_{i_1}$, thus we must have that $\phi \dashv\vdash p_{i_1} \wedge \neg p_{i_1}$, hence $\phi \vdash \psi$.

If $F, G \neq \emptyset$, then Lemma 4.3.9, for each $f \in F$, we have that $X_f \subseteq Y_{g_f}$ for some $g_f \in G$. Thus, we have the following derivation:

$$
\begin{array}{lll}
(1) & \Theta_{X_f} & \\
(2) & \bigotimes_{s \in X_f}(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)}) & \\
(3) & (\bigotimes_{s \in X_f}(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)})) \otimes (\bigotimes_{s \in Y_{g_f} \setminus X_f}(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)})) & (\otimes\mathsf{I}) \\
(4) & \bigotimes_{s \in Y_{g_f}}(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)}) & \\
(5) & \Theta_{Y_{g_f}} & \\
(6) & \bigvee_{g \in G} \Theta_{Y_g} & (\vee\mathsf{I})
\end{array}
$$

Thus, $\Theta_{X_f} \vdash \bigvee_{g \in G} \Theta_{Y_g}$ for each $f \in F$, which by ($\vee$E) implies that

$$\bigvee_{f \in F} \Theta_{X_f} \vdash \bigvee_{g \in G} \Theta_{Y_g}, \text{ namely } \phi \vdash \psi.$$

■

Corollary 4.3.11. *For any* $\mathbf{PD}^{[\vee]}$ *formula* $\phi(p_{i_1}, \ldots, p_{i_n})$*,*

$$\models \phi \implies \vdash \phi.$$

Proof. Since $\models p_{i_1} \otimes \neg p_{i_1}$, by Theorem 4.3.10 we have that

$$\models \phi \implies p_{i_1} \otimes \neg p_{i_1} \models \phi \implies p_{i_1} \otimes \neg p_{i_1} \vdash \phi.$$

Thus, we obtain $\vdash \phi$ by (EM_0). ■

Theorem 4.3.12 (Strong Completeness Theorem)**.** *Let* Γ *be a set of formulas and* ϕ *a formula of* $\mathbf{PD}^{[\vee]}$*. Then*

$$\Gamma \vdash \phi \iff \Gamma \models \phi.$$

Proof. The direction "$\Longrightarrow$" follows from Soundness Theorem. For the other direction "$\Longleftarrow$": Since $\mathbf{PD}^{[\vee]} = \mathbf{PID}$ (Corollary 4.3.3) and Compactness Theorem holds for **PID** (Theorem 4.2.19), we know that $\mathbf{PD}^{[\vee]}$ is also compact. Thus, if $\Gamma \models \phi$, then there exists a finite set $\Delta \subseteq \Gamma$ such that $\Delta \models \phi$. Taking

$$\theta = \bigwedge_{\psi \in \Delta} \psi,$$

we have $\theta \models \phi$. Hence by Theorem 4.3.10, we obtain $\theta \vdash \phi$, thereby $\Gamma \vdash \phi$. ■

4.4 Axiomatizing propositional dependence logic

In this section, we study the expressive power of propositional dependence logic (**PD**), and give a complete axiomatization of the logic based on the method used in Section 4.3 for $\mathbf{PD}^{[\vee]}$.

We proved in Theorem 4.3.2 that $\mathbf{PD}^{[\vee]}$ is a maximal downwards closed logic. It turns out that **PD** is also a maximal downwards closed logic, although apparently it appears to be less expressive than $\mathbf{PD}^{[\vee]}$ (as the latter has an extra connective, the intuitionistic disjunction). This result is due to Taneli Huuskonen (2012, unpublished). Below we present the proof with his permission.

Theorem 4.4.1 (T. Huuskonen)**.** **PD** *is a maximal downwards closed logic.*

Proof. It suffices to show that for every set $N = \{i_1, \dots, i_n\} \subseteq \mathbb{N}$, every collection $\mathcal{K} \in \nabla_N$, there is an n-formula $\phi(p_{i_1}, \dots, p_{i_n})$ of **PD** such that $\mathcal{K} = [\![\phi]\!]$.

We define formulas α_k for each $k \in \omega$ as follows:

- $\alpha_0 := p_{i_1} \wedge \neg p_{i_1}$;
- $\alpha_1 := {=}(p_{i_1}) \wedge \cdots \wedge {=}(p_{i_n})$;
- $\alpha_k := \bigotimes_{i=1}^{k} \alpha_1$, for $k > 1$.

Claim 1. For any n-team X on N, $X \models \alpha_k$ iff $|X| \leq k$.

Proof of Claim 1. Clearly,

$$X \models \alpha_0 \iff X = \emptyset \iff |X| \leq 0$$

and

$$X \models \alpha_1 \iff |X| \leq 1.$$

For $k > 1$, we have that

$$\begin{aligned} X \models \alpha_k &\iff X = \bigcup_{i=1}^{k} X_i \text{ with } X_i \models \alpha_1 \\ &\iff X = \bigcup_{i=1}^{k} X_i \text{ with } |X_i| \leq 1 \\ &\iff |X| \leq k \end{aligned}$$

⊣

Let Y be a non-empty n-team with $|Y| = k+1$ and $\mathbf{2^n}$ the maximal n-team on N . By Lemma 4.3.1,

$$X \models \Theta_{\mathbf{2^n} \setminus Y} \iff X \subseteq \mathbf{2^n} \setminus Y \iff X \cap Y = \emptyset. \tag{4.12}$$

Define

$$\Theta^{\star}_{Y} := \alpha_k \otimes \Theta_{2^n \setminus Y}.$$

Claim 2. For any n-team X on N, we have that $X \models \Theta^\star_Y$ iff $Y \nsubseteq X$.

Proof of Claim 2. We have that

$$\begin{aligned} X \models \Theta^\star_Y &\iff X = U \cup V \text{ such that } U \models \alpha_k \text{ and } V \models \Theta_{2^n \setminus Y} \\ &\iff X = U \cup V \text{ such that } |U| \le k \text{ and } V \cap Y = \emptyset \\ &\quad\;\; \text{(by Claim 1 and (4.12)).} \qquad (*) \end{aligned}$$

If $Y \nsubseteq X$, then $|Y \setminus X| \ge 1$. We have that $X = (Y \cap X) \cup (X \setminus Y)$. Clearly $(X \setminus Y) \cap Y = \emptyset$. On the other hand,

$$|Y \cap X| = |Y \setminus (Y \setminus X)| = |Y| - |Y \setminus X| \le (k+1) - 1 = k.$$

Thus, by $(*)$, we conclude that $X \models \Theta^\star_Y$.

Conversely, suppose $X \models \Theta^\star_Y$. By $(*)$, $X = U \cup V$ such that $|U| \le k < k+1 = |Y|$ and $V \cap Y = \emptyset$. It follows that there exists $s \in Y$ such that $s \notin U \cup V = X$, thus $Y \nsubseteq X$, as required. ⊣

Now, let $\mathcal{K} \in \nabla_N$. Consider the finite collection

$$2^{\mathbf{2^n}} \setminus \mathcal{K} = \{Y_j \mid j \in J\}$$

of n-teams on N which are not in $\mathcal{K}$.

Claim 3. For any n-team X on N,

$$X \in \mathcal{K} \iff Y_j \nsubseteq X \text{ for all } j \in J.$$

Proof of Claim 3. If $X \notin \mathcal{K}$, then by definition, $X = Y_{j_0}$ for some $j_0 \in J$, so $Y_{j_0} \subseteq X$. Conversely, if $X \in \mathcal{K}$, then as $\mathcal{K}$ is downwards closed, for all $Y \subseteq X$, $Y \in \mathcal{K}$. Thus for all $Y_j \notin \mathcal{K}$, we must have that $Y_j \nsubseteq X$. ⊣

Finally, since $\emptyset \in \mathcal{K}$, we have that $Y_j \neq \emptyset$ for any $j \in J$. Hence by Claim 2 and Claim 3, we obtain that for any n-team X on N,

$$X \models \bigwedge_{j \in J} \Theta^\star_{Y_j} \iff Y_j \nsubseteq X \text{ for all } j \in J \iff X \in \mathcal{K},$$

i.e., $\mathcal{K} = [\![\bigwedge_{j \in J} \Theta^\star_{Y_j}]\!]$, as required. ■

Corollary 4.4.2. $\mathbf{PD} = \mathbf{PD}^{[\vee]} = \mathbf{PID} = \mathbf{InqL}$.

Proof. By Theorem 4.4.1 and Corollary 4.3.3. ■

Propositional logic can be viewed as first-order logic over a first-order model **2** with a two-element domain $\{0,1\}$. An n-valuation on $N = \{i_1, \dots, i_n\}$ can be viewed as a first-order assignment from an n-element set $\{x_{i_1}, \dots, x_{i_n}\}$ of first-order variables into the first-order model **2**. In this sense, Corollary 4.4.2 implies that for a fixed number n, **PD** (viewed as a special kind of first-order dependence logic) can characterize all downwards closed collections K of first-order teams of **2** with an n-element domain. These collections K are called $(2,n)$-*suits* in [6], and the function value $f(2,n)$ represents the number of distinct collections K for the fixed n. It then follows straightforwardly from the counting result in [6] that there is no compositional semantics for **PD** in which the semantic truth set of an n-formula ϕ is a subset of $\mathbf{2}^n$. This justifies that the team semantics given in this chapter is an appropriate compositional semantics for **PD**.

The proof of Theorem 4.4.1 shows that every n-formula $\phi(p_{i_1},\ldots,p_{i_n})$ of **PD** is logically equivalent to a formula in the *normal form*

$$\bigwedge_{j\in J}\Theta^{\star}_{Y_j},$$

where J is a finite set of indices and each Y_j is an n-team on $\{i_1,\ldots,i_n\}$. Using this normal form, one can define a natural deduction system for **PD** and prove the completeness theorem. However, as this normal form is complex, we will not take this approach to the axiomatization of **PD**. Instead, we define a similar natural deduction system with that of $\mathbf{PD}^{[\vee]}$ and prove the completeness theorem by a similar technique.

One of the crucial steps in the argument is that dependence atoms will be eliminated in a certain way. In the case of $\mathbf{PD}^{[\vee]}$, we made use of the following equivalence (Lemma 4.3.4)

$$=\!(p_{j_0},\ldots,p_{j_k})\equiv\bigvee_{f\in 2^{\mathbf{2^k}}}\bigotimes_{s\in\mathbf{2^k}}\left(p_{j_0}^{s(j_0)}\wedge\cdots\wedge p_{j_{k-1}}^{s(j_{k-1})}\wedge p_{j_k}^{f(s)}\right)\tag{4.13}$$

and added the corresponding rules (DepI) and (DepE) in the natural deduction system. However, the intuitionistic disjunction, which plays a crucial role in the equivalence (4.13), is not an eligible connective in the logic **PD**. To overcome this problem, we make the following observations.

Consider a formula ϕ of **PD**, which contains an occurrence of the dependence atom $=\!(p_{j_0},\ldots,p_{j_k})$. For any team X such that $X\models\phi$, there exists a subteam $Y\subseteq X$ such that $Y\models\ =\!(p_{j_0},\ldots,p_{j_k})$[5]. By the team semantics, there exists a function $f:\mathbf{2^k}\to 2$ such that for all $s\in Y$,

$$s(j_k)=f(s\upharpoonright\{j_0,\ldots,j_{k-1}\}),$$

where $\mathbf{2^k}$ is the maximal k-team on $\{j_0,\ldots,j_{k-1}\}$. In ϕ, replace the occurrence of the formula $=\!(p_{j_0},\ldots,p_{j_k})$ by

$$(=\!(p_{j_0},\ldots,p_{j_k}))^*_f:=\bigotimes_{s\in\mathbf{2^k}}\left(p_{j_0}^{s(j_0)}\wedge\cdots\wedge p_{j_{k-1}}^{s(j_{k-1})}\wedge p_{j_k}^{f(s)}\right),$$

and denote the resulting formula by ϕ^*_f. The formula ϕ^*_f can be viewed as an *approximation* of ϕ, and we will see in Lemma 4.4.3 that $X\models\phi^*_f$.

More generally, suppose the following are all the occurrences of all dependence atoms in a formula ϕ of **PD**:

$$=\!(p_{j^1_0}\ldots,p_{j^1_{k_1}}),\ \ldots,\ =\!(p_{j^c_0},\ldots,p_{j^c_{k_c}}).$$

An *approximation sequence* $\Omega=\langle f_1,\ldots,f_c\rangle$ of ϕ is a sequence such that for each $1\leq\xi\leq c$, $f_\xi:\mathbf{2}^{\mathbf{k}_\xi}\to 2$ is a function from the maximal k_ξ-team on $\{j^\xi_0,\ldots,j^\xi_{k_\xi-1}\}$ into $2=\{0,1\}$. For each such sequence Ω, define a dependence atom-free (classical) formula ϕ^*_Ω, called an *approximation* of ϕ, by induction as follows:

[5] C.f. Theorem 5.2.11.

- $(p_i)^*_{\langle\rangle} := p_i$ and $(\neg p_i)^*_{\langle\rangle} := \neg p_i$;
- $(=\!(p_{j_0^\xi},\dots,p_{j_{k_\xi}^\xi}))^*_{\langle f_\xi\rangle} := \bigotimes_{s\in\mathbf{2}^{\mathbf{k}_\xi}} \left(p_{j_0^\xi}^{s(j_0^\xi)} \wedge\cdots\wedge p_{j_{k_\xi-1}^\xi}^{s(j_{k_\xi-1}^\xi)} \wedge p_{j_{k_\xi}^\xi}^{f_\xi(s)}\right)$;
- $(\psi\wedge\chi)^*_\Omega := \psi^*_{\Omega^0}\wedge\chi^*_{\Omega^1}$, where Ω^0 and Ω^1 are subsequences of Ω consisting of all the f_ξ's such that the dependence atoms $=\!(p_{j_0^\xi},\dots,p_{j_{k_\xi}^\xi})$ occur in ψ and χ, respectively;
- $(\psi\otimes\chi)^*_\Omega := \psi^*_{\Omega^0}\otimes\chi^*_{\Omega^1}$, where Ω^0 and Ω^1 are as above.

Next, we show that every **PD** formula is logically equivalent to the intuitionistic disjunction of all its approximations.

Lemma 4.4.3. *Let ϕ be a formula of* **PD** *and Λ the set of all its approximation sequences. Then*

$$\phi \equiv \bigvee_{\Omega\in\Lambda} \phi^*_\Omega.$$

Proof. We first show by induction on ϕ that for each $\Omega\in\Lambda$, $\phi^*_\Omega \models \phi$. Suppose all the occurrences of all dependence atoms in ϕ are as follows:

$$=\!(p_{j_0^1},\dots,p_{j_{k_1}^1}),\dots,=\!(p_{j_0^c},\dots,p_{j_{k_c}^c}).$$

Case $\phi := p_i$ or $\phi := \neg p_i$ is trivial as in this case, $\Omega = \langle\rangle$ and $\phi^*_{\langle\rangle} = p_i = \phi$.

Case $\phi := =\!(p_{j_0^\xi},\dots,p_{j_{k_\xi}^\xi})$ for $1\le\xi\le c$. Then $\Omega = \langle f_\xi\rangle$ for some $f_\xi : \mathbf{2}^{\mathbf{k}_\xi}\to 2$. Suppose for some team X,

$$X \models \bigotimes_{s\in\mathbf{2}^{\mathbf{k}_\xi}} \left(p_{j_0^\xi}^{s(j_0^\xi)} \wedge\cdots\wedge p_{j_{k_\xi-1}^\xi}^{s(j_{k_\xi-1}^\xi)} \wedge p_{j_{k_\xi}^\xi}^{f_\xi(s)}\right).$$

Then for each $s\in\mathbf{2}^{\mathbf{k}_\xi}$, there exists $X_s\subseteq X$ such that $X = \bigcup_{s\in\mathbf{2}^{\mathbf{k}_\xi}} X_s$ and

$$X_s \models p_{j_0^\xi}^{s(j_0^\xi)} \wedge\cdots\wedge p_{j_{k_\xi-1}^\xi}^{s(j_{k_\xi-1}^\xi)} \wedge p_{j_{k_\xi}^\xi}^{f_\xi(s)}.$$

For any $t,t'\in X$ such that

$$t\upharpoonright\{j_0^\xi,\dots,j_{k_\xi-1}^\xi\} = t'\upharpoonright\{j_0^\xi,\dots,j_{k_\xi-1}^\xi\} = s_0\in\mathbf{2}^{\mathbf{k}_\xi},$$

we must have that $t,t'\in X_{s_0}$. Thus

$$t(j_{k_\xi}^\xi) = f_\xi(s_0) = t'(j_{k_\xi}^\xi).$$

Hence $X \models =\!(p_{j_0^\xi},\dots,p_{j_{k_\xi}^\xi})$.

Case $\phi := \psi \otimes \chi$. By induction hypothesis, we have that $\psi^*_{\Omega^0} \models \psi$ and $\chi^*_{\Omega^1} \models \chi$, where Ω^0 and Ω^1 are subsequences of Ω consisting of all the f_ξ's with the dependence atoms $=\!(p_{j_0^\xi},\dots,p_{j^\xi_{k_\xi}})$ occurring in ψ and χ, respectively. It follows that $\psi^*_{\Omega^0} \otimes \chi^*_{\Omega^1} \models \psi \otimes \chi$, namely $(\psi \otimes \chi)^*_\Omega \models \psi \otimes \chi$.

Case $\phi := \psi \wedge \chi$. Similar to the above case.

Next, we show that $\phi \models \bigvee_{\Omega\in\Lambda} \phi^*_\Omega$ by induction on ϕ.

Case $\phi := p_i$ or $\neg p_i$ is trivial, as $\bigvee_{\Omega\in\Lambda}(p_i)^*_\Omega \equiv (p_i)^*_{\langle\rangle} \equiv p_i$ and $\bigvee_{\Omega\in\Lambda}(\neg p_i)^*_\Omega \equiv (\neg p_i)^*_{\langle\rangle} \equiv \neg p_i$.

Case $\phi := \;=\!(p_{j_0^\xi},\dots,p_{j^\xi_{k_\xi}})$ for $1 \le \xi \le c$. Suppose $X \models \;=\!(p_{j_0^\xi},\dots,p_{j^\xi_{k_\xi}})$ for some team X. Define a function $f_\xi : \mathbf{2^k} \to 2$ by taking

$$f_\xi(s) = \begin{cases} t(j^\xi_{k_\xi}) & \text{if } \exists t \in X(t \upharpoonright \{j_0,\dots,j_{k-1}\} = s); \\ 1 & \text{otherwise.} \end{cases}$$

By assumption, if for $t, t' \in X$,

$$t \upharpoonright \{j_0,\dots,j_{k-1}\} = t' \upharpoonright \{j_0,\dots,j_{k-1}\} = s,$$

then $t(j^\xi_{k_\xi}) = t'(j^\xi_{k_\xi})$, thus the function f_ξ is well-defined.

Clearly, for each $s \in \mathbf{2}^{\mathbf{k}_\xi}$,

$$X_s \models p^{s(j_0^\xi)}_{j_0^\xi} \wedge \cdots \wedge p^{s(j^\xi_{k_\xi-1})}_{j^\xi_{k_\xi-1}} \wedge p^{f_\xi(s)}_{j^\xi_{k_\xi}},$$

thus

$$X \models \bigotimes_{s\in\mathbf{2}^{\mathbf{k}_\xi}} \left(p^{s(j_0^\xi)}_{j_0^\xi} \wedge \cdots \wedge p^{s(j^\xi_{k_\xi-1})}_{j^\xi_{k_\xi-1}} \wedge p^{f_\xi(s)}_{j^\xi_{k_\xi}} \right).$$

Case $\phi := \psi \otimes \chi$. By induction hypothesis, we have that

$$\psi \models \bigvee_{\Omega^0\in\Lambda^0} \psi^*_{\Omega^0} \text{ and } \chi \models \bigvee_{\Omega^1\in\Lambda^1} \chi^*_{\Omega^1},$$

where Λ^0 is the set of all Ω^0's which are subsequences of some $\Omega \in \Lambda$ consisting of all the f_ξ's with the dependence atoms $=\!(p_{j_0^\xi},\dots,p_{j^\xi_{k_\xi}})$ occurring in ψ, and Ω^1 is obtained in the same way for χ.

Now, since $A \otimes (B \vee C) \models (A \otimes B) \vee (A \otimes C)$ for all formulas A, B, C, we obtain that

$$\psi \otimes \chi \models \left(\bigvee_{\Omega^0\in\Lambda^0} \psi^*_{\Omega^0}\right) \otimes \left(\bigvee_{\Omega^1\in\Lambda^1} \chi^*_{\Omega^1}\right) \models \bigvee_{\Omega^0\in\Lambda^0} \bigvee_{\Omega^1\in\Lambda^1} (\psi^*_{\Omega^0} \otimes \chi^*_{\Omega^1}),$$

where

$$\bigvee_{\Omega^0\in\Lambda^0} \bigvee_{\Omega^1\in\Lambda^1} (\psi^*_{\Omega^0} \otimes \chi^*_{\Omega^1}) \equiv \bigvee_{\Omega\in\Lambda} (\psi^*_{\Omega^0} \otimes \chi^*_{\Omega^1}) = \bigvee_{\Omega\in\Lambda} (\psi \otimes \chi)^*_\Omega,$$

as required.

Case $\phi := \psi \wedge \chi$ is proved similarly, using the fact that $A \wedge (B \vee C) \models (A \wedge B) \vee (A \wedge C)$ for all formulas A, B, C. ■

In the formula $\bigvee_{\Omega \in \Lambda} \phi^*_\Omega$, each ϕ^*_Ω is classical (does not contain any dependence atoms and intuitionistic disjunctions), but the whole formula is not in the language of **PD**. We shall view $\bigvee_{\Omega \in \Lambda} \phi^*_\Omega$ or the sequence

$$\langle \phi^*_\Omega \rangle_{\Omega \in \Lambda}$$

as a *weak normal form* for formulas of **PD**. We now define a natural deduction system for **PD** which will enable us to derive *in effect* the weak normal form $\bigvee_{\Omega \in \Lambda} \phi^*_\Omega$ for every formula ϕ.

Definition 4.4.4 (A natural deduction system for **PD**)**.** The rules are given as follows:

1. The rules ($\wedge$I), ($\wedge$E), ($\otimes$I), ($\otimes$WE), ($\otimes$Sub), (Com $\otimes$), (Ass $\otimes$), ($\bot$E), (EM$_0$), (Dstr $\wedge\otimes$) as in Definition 4.3.5 (see also Appendix).

2. Dependence Atom Strong Introduction: For any function $f : \mathbf{2^k} \to 2$ of the maximal k-team on $\{j_0, \ldots, j_{k-1}\}$ into $2 = \{0, 1\}$,

$$\cfrac{\bigotimes_{s \in \mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \cdots \wedge p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)} \right)}{=\!(p_{j_0}, \ldots, p_{j_k})} \ (\mathsf{DepSI})$$

3. Approximation Transition:

$$\cfrac{\begin{matrix} [\phi^*_{\Omega_0}] & & [\phi^*_{\Omega_m}] & \\ \vdots & \ldots & \vdots & \\ \theta & & \theta & \phi \end{matrix}}{\theta} \ (\mathsf{ApTr})$$

where $\{\Omega_0, \ldots, \Omega_m\}$ is the set of all approximation sequences of ϕ.

The Approximation Transition rule (ApTr) *has the same effect* as the combination of Intuitionistic Disjunction Elimination rule ($\vee$E) and following rule

$$\cfrac{\phi}{\bigvee_{i=1}^{m} \phi^*_{\Omega_i}} .$$

However, in the above deductive system for **PD**, we avoid the use of intuitionistic disjunction by taking the rule (ApTr) instead.

Next, we prove the Soundness Theorem for the above deductive system.

Theorem 4.4.5 (Soundness Theorem)**.** *For any formulas ϕ and ψ of* **PD***,*

$$\phi \vdash \psi \Longrightarrow \phi \models \psi.$$

Proof. It suffices to show that all of the deductive rules are valid. The validity of (DepSI) and (ApTr) follows from Lemma 4.4.3, and the validity of all the other rules follows from the proof of Theorem 4.3.7. ■

Corollary 4.4.6. *The following are derivable rules:*

1. *Rules (ex falso), (Dstr$\otimes\wedge$), (Dstr$^*\otimes\wedge\otimes$), (Dstr$^*\wedge\otimes\wedge$), (Com $\wedge$), (Ass $\wedge$) and ($\wedge$Sub) as in Corollary 4.3.6 (see also Appendix).*
2. *Dependence Atom Weak Elimination:*

$$\dfrac{\begin{array}{c}\forall f\\ [(=(p_{j_0},\dots,p_{j_k}))^*_{\langle f\rangle}]\\ \vdots\\ \theta\end{array} \qquad =(p_{j_0},\dots,p_{j_k})}{\theta}\,(\textsf{DepWE})$$

where $f : \mathbf{2^k} \to 2$ is any function from the maximal k-team $\mathbf{2^k}$ on $\{j_0,\dots,j_{k-1}\}$ into 2.

3. *Approximation Elimination:*

$$\dfrac{\phi^*_\Omega}{\phi}\,(\textsf{ApE})$$

where Ω is any approximation sequence of ϕ.

Proof. The rules in Item 1 are derived in the same way as in Corollary 4.3.6. Rule (DepWE) is a special case of the rule (ApTr).

We now proceed to derive the rule (ApE) by induction on ϕ. The case $\phi := p_i$ or $\neg p_i$ is trivial as in this case $\phi^*_\Omega = \phi^*_{\langle\rangle} = \phi$. The case $\phi = =(p_{j_0^\xi},\dots,p_{j_{k_\xi}^\xi})$ follows from (DepSI).

Case $\phi := \psi\otimes\chi$. By induction hypothesis, we have that $\psi^*_{\Omega^0}\vdash\psi$ and $\chi^*_{\Omega^1}\vdash\chi$, where Ω^0 and Ω^1 are as before. By ($\otimes$Sub), we derive that $\psi^*_{\Omega^0}\otimes\chi^*_{\Omega^1}\vdash\psi\otimes\chi$, namely $(\psi\otimes\chi)^*_\Omega\vdash\psi\otimes\chi$.

Case $\theta := \psi\wedge\chi$ is proved similarly using ($\wedge$Sub). ■

It is not hard to see that the rules (ApTr) and (ApE) imply *in effect* that

$$\bigvee_{\Omega\in\Lambda}\phi^*_\Omega \dashv\vdash \phi,$$

for any **PD** formula ϕ, where Λ is the set of all approximation sequences of ϕ. We now proceed to prove the completeness theorem for **PD** using the above weak normal form.

First, we show a completeness theorem for the dependence atom-free fragment of **PD** (which consists of all classical formulas of **PD**).

Proposition 4.4.7. *For any classical formulas ϕ, ψ of* **PD***,*

$$\phi\models\psi \Longrightarrow \phi\vdash\psi.$$

Proof. For classical formulas (which can be identified with formulas of classical propositional logic **CPL**), **PD** has the same deductive rules as **CPL**. Thus, it suffices to show that for classical formulas ϕ, ψ of **PD**,

$$\phi \models_{\mathbf{PD}} \psi \iff \phi \models_{\mathbf{CPL}} \psi.$$

"$\Longrightarrow$": Suppose $\phi \models_{\mathbf{PD}} \psi$. For any valuation s, we have that

$$\begin{aligned} s \models_{\mathbf{CPL}} \phi &\Longrightarrow \{s\} \models_{\mathbf{PD}} \phi \text{ (by Lemma 4.1.10)} \\ &\Longrightarrow \{s\} \models_{\mathbf{PD}} \psi \quad \text{(since } \phi \models_{\mathbf{PD}} \psi) \\ &\Longrightarrow s \models_{\mathbf{CPL}} \psi \text{ (by Lemma 4.1.10).} \end{aligned}$$

"$\Longleftarrow$": Suppose $\phi \models_{\mathbf{CPL}} \psi$. For any team X, we have that

$$\begin{aligned} X \models_{\mathbf{PD}} \phi &\Longrightarrow \forall s \in X,\ \{s\} \models_{\mathbf{PD}} \phi \quad \text{(by downwards closure)} \\ &\Longrightarrow \forall s \in X,\ s \models_{\mathbf{CPL}} \phi \text{ (by Lemma 4.1.10)} \\ &\Longrightarrow \forall s \in X,\ s \models_{\mathbf{CPL}} \psi \text{ (since } \phi \models_{\mathbf{CPL}} \psi) \\ &\Longrightarrow \forall s \in X,\ \{s\} \models_{\mathbf{PD}} \psi \text{ (by Lemma 4.1.10)} \\ &\Longrightarrow X \models_{\mathbf{PD}} \psi \text{ (}\psi \text{ is flat, by Lemma 4.1.9).} \end{aligned}$$

■

Theorem 4.4.8 (Completeness Theorem)**.** *For any formulas ϕ and ψ of* **PD***,*

$$\phi \models \psi \Longrightarrow \phi \vdash \psi.$$

Proof. Suppose $\phi \models \psi$. By Lemma 4.4.3, we have that

$$\phi \equiv \bigvee_{\Omega \in \Lambda} \phi^*_\Omega \text{ and } \psi \equiv \bigvee_{\Delta \in \Lambda'} \psi^*_\Delta,$$

where Λ and Λ' are the sets of all approximation sequences of ϕ and for ψ, respectively. For each $\Omega \in \Lambda$, we have that

$$\phi^*_\Omega \models \bigvee_{\Delta \in \Lambda'} \psi^*_\Delta.$$

By the Completeness Theorem of $\mathbf{PD}^{[\vee]}$, we have that

$$\phi^*_\Omega \vdash_{\mathbf{PD}^{[\vee]}} \bigvee_{\Delta \in \Lambda'} \psi^*_\Delta.$$

Thus, in $\mathbf{PID} = \mathbf{PD}^{[\vee]}$, we have that

$$\vdash_{\mathbf{PID}} \phi^*_\Omega \to \bigvee_{\Delta \in \Lambda'} \psi^*_\Delta.$$ [6]

By KP axiom of **PID**, we have that

$$\vdash_{\mathbf{PID}} (\neg\neg\phi^*_\Omega \to \bigvee_{\Delta \in \Lambda'} \psi^*_\Delta) \to \bigvee_{\Delta \in \Lambda'} (\neg\neg\phi^*_\Omega \to \psi^*_\Delta).$$

[6] The formulas ϕ^*_Ω and ψ^*_Δ may contain tensor disjunction $\otimes$, which is not in the language of **PID**, but as $\mathbf{PID} = \mathbf{PD}^{[\vee]}$, one can view them as shorthands for the equivalent formulas in **PID**.

Since the formula ϕ^*_Ω is classical, $\neg\neg\phi^*_\Omega \dashv\vdash \phi^*_\Omega$. Therefore

$$\vdash_{\mathbf{PID}} (\phi^*_\Omega \to \bigvee_{\Delta\in\Lambda'} \psi^*_\Delta) \to \bigvee_{\Delta\in\Lambda'} (\phi^*_\Omega \to \psi^*_\Delta),$$

which implies

$$\vdash_{\mathbf{PID}} \bigvee_{\Delta\in\Lambda'} (\phi^*_\Omega \to \psi^*_\Delta).$$

Now, by the disjunction property of **PID**, we have that for some $\Delta_\Omega \in \Lambda'$,

$$\vdash_{\mathbf{PID}} \phi^*_\Omega \to \psi^*_{\Delta_\Omega},$$

thus $\phi^*_\Omega \vdash_{\mathbf{PID}} \psi^*_{\Delta_\Omega}$. By the Soundness Theorem of **PID**, we obtain that

$$\phi^*_\Omega \models \psi^*_{\Delta_\Omega}.$$

Note that both ϕ^*_Ω and $\psi^*_{\Delta_\Omega}$ are classical, then by Proposition 4.4.7, we obtain that

$$\phi^*_\Omega \vdash_{\mathbf{PD}} \psi^*_{\Delta_\Omega}.$$

Finally, by (ApE), we derive $\phi^*_\Omega \vdash_{\mathbf{PD}} \psi$ for each $\Omega \in \Lambda$, therefore by (ApTr), we conclude that $\phi \vdash_{\mathbf{PD}} \psi$, as required. ■

Remark 4.4.9. *In the above proof, the use of* KP *axiom and disjunction property is not essential. Because each ϕ^*_Ω and ψ^*_Δ are classical, they can be turned into formulas in disjunctive normal form in the deductive system of* **PD** *(or* **CPL***). In this way, one obtains in* **PD** *that*

$$\phi^*_\Omega \dashv\vdash \Theta_{X_\Omega} \vdash \bigvee_{\Delta\in\Lambda'} \Theta_{Y_\Delta} \dashv\vdash \bigvee_{\Delta\in\Lambda'} \psi^*_\Delta,$$

for some teams X_Ω, Y_Δ. From this point on, one continues the proof with the same argument as in the proof of Theorem 4.3.10 for $\mathbf{PD}^{[\vee]}$.

Theorem 4.4.10 (Strong Completeness Theorem)**.** *Let Γ be a set of formulas and ϕ a formula of* **PD**. *Then*

$$\Gamma \vdash \phi \iff \Gamma \models \phi.$$

Proof. Since **PD** = **PID** (Corollary 4.4.2) and **PID** is compact (Theorem 4.2.19), we know that **PD** is also compact. Therefore the theorem follows. ■

We end this section with an application of the above given natural deduction system for **PD**. We will derive Armstrong axioms [2] mentioned in Section 1.1 in the system.

Example 4.4.11. The following Armstrong axioms are derivable in **PD**:

(i) $=\!(p_i, p_i)$

(ii) $=\!(p_{i_0}, p_{i_1}, p_{i_2}) \vdash =\!(p_{i_1}, p_{i_0}, p_{i_2})$

(iii) $=\!(p_{i_1}, p_{i_2}) \vdash =\!(p_{i_0}, p_{i_1}, p_{i_2})$

(iv) $=\!(p_{i_0}, p_{i_1}), =\!(p_{i_1}, p_{i_2}) \vdash =\!(p_{i_0}, p_{i_2})$

Proof. The derivations are as follows:

(i)

$$\cfrac{\cfrac{\cfrac{\cfrac{}{p_i\otimes\neg p_i}\;(\mathsf{EM}_0)}{(p_i\wedge p_i)\otimes(\neg p_i\wedge\neg p_i)}\;(\wedge\mathsf{I},\ \otimes\mathsf{Sub})}{\bigotimes_{s\in\mathbf{2^1}}(p_i^{s(i)}\wedge p_i^{f(s)})}}{=\!(p_i,p_i)}\;(\mathsf{DepSI})$$

where $\mathbf{2^1}$ is the maximal 1-team on $\{i\}$ and $f:\mathbf{2^1}\to 2$ is defined as

$$f(s)=s(i).$$

(ii)

$$\cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c}\forall f:\mathbf{2^2}\to 2\\ [(=\!(p_{i_0},p_{i_1},p_{i_2}))^*_{\langle f\rangle}]\end{array}}{\bigotimes_{s\in\mathbf{2^2}}(p_{i_0}^{s(i_0)}\wedge p_{i_1}^{s(i_1)}\wedge p_{i_2}^{f(s)})}}{\bigotimes_{s\in\mathbf{2^2}}(p_{i_1}^{s(i_1)}\wedge p_{i_0}^{s(i_0)}\wedge p_{i_2}^{f(s)})}\;(\mathsf{Com}\wedge,\ \otimes\mathsf{Sub})}{=\!(p_{i_1},p_{i_0},p_{i_2})}\;(\mathsf{DepSI})\qquad =\!(p_{i_0},p_{i_1},p_{i_2})}{=\!(p_{i_1},p_{i_0},p_{i_2})}\;(\mathsf{DepWE})$$

where $\mathbf{2^2}$ is the maximal 2-team on $\{i_0,i_1\}$.

(iii)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c}\forall f:\mathbf{2^1}\to 2\\ [(=\!(p_{i_1},p_{i_2}))^*_{\langle f\rangle}]\end{array}}{\bigotimes_{s\in\mathbf{2^1}}(p_{i_1}^{s(i_1)}\wedge p_{i_2}^{f(s)})}\qquad \cfrac{}{p_{i_1}\otimes\neg p_{i_1}}\;(\mathsf{EM}_0)}{\left(\bigotimes_{s\in\mathbf{2^1}}(p_{i_1}^{s(i_1)}\wedge p_{i_2}^{f(s)})\right)\wedge\left(p_{i_0}\otimes\neg p_{i_0}\right)}\;(\wedge\mathsf{I})}{\bigotimes_{s\in\mathbf{2^2}}(p_{i_0}^{s(i_0)}\wedge p_{i_1}^{s(i_1)}\wedge p_{i_2}^{g(s)})}\;(\mathsf{Dstr}\wedge\otimes)}{=\!(p_{i_0},p_{i_1},p_{i_2})}\;(\mathsf{DepSI})\qquad =\!(p_{i_1},p_{i_2})}{=\!(p_{i_0},p_{i_1},p_{i_2})}\;(\mathsf{DepWE})$$

where $\mathbf{2^1}$, $\mathbf{2^2}$ are the maximal 1-team on $\{i_1\}$ and 2-team on $\{i_0,i_1\}$, respectively, and $g:\mathbf{2^2}\to 2$ is defined as

$$g(s)=f(s\restriction\{i_1\}).$$

(iv) We first derive that for each $f_0:\mathbf{2^{1_0}}\to 2$ and $f_1:\mathbf{2^{1_1}}\to 2$,

$$(=\!(p_{i_0},p_{i_1})\wedge=\!(p_{i_1},p_{i_2}))^*_{\langle f_0,f_1\rangle}\vdash\,=\!(p_{i_0},p_{i_2}),\qquad (*)$$

where $\mathbf{2^{1_0}}$ and $\mathbf{2^{1_1}}$ are maximal 1-teams on $\{i_0\}$ and on $\{i_1\}$, respectively.

$$
\begin{array}{c}
(=\!(p_{i_0},p_{i_1})\wedge =\!(p_{i_1},p_{i_2}))^*_{\langle f_0,f_1\rangle}\\
\hline
\left(\bigotimes_{s_0\in\mathbf{2^{1_0}}}(p_{i_0}^{s_0(i_0)}\wedge p_{i_1}^{f_0(s_0)})\right)\wedge\left(\bigotimes_{s_1\in\mathbf{2^{1_1}}}(p_{i_1}^{s_1(i_1)}\wedge p_{i_2}^{f_1(s_1)})\right)\\
\hline
\bigotimes_{s_0\in\mathbf{2^{1_0}}}\left((p_{i_0}^{s_0(i_0)}\wedge p_{i_1}^{f_0(s_0)})\wedge\left(\bigotimes_{s_1\in\mathbf{2^{1_1}}}(p_{i_1}^{s_1(i_1)}\wedge p_{i_2}^{f_1(s_1)})\right)\right)\\
\hline
\bigotimes_{s_0\in\mathbf{2^{1_0}}}\bigotimes_{s_1\in\mathbf{2^{1_1}}}\left((p_{i_0}^{s_0(i_0)}\wedge p_{i_1}^{f_0(s_0)})\wedge(p_{i_1}^{s_1(i_1)}\wedge p_{i_2}^{f_1(s_1)})\right)\quad(\mathsf{Dstr}\wedge\otimes)\\
\hline
\bigotimes_{s_0\in\mathbf{2^{1_0}}}\bigotimes_{\substack{s_1\in\mathbf{2^{1_1}}\\ s_1(i_1)=f_0(s_0)}}(p_{i_0}^{s_0(i_0)}\wedge p_{i_1}^{f_0(s_0)}\wedge p_{i_1}^{s_1(i_1)}\wedge p_{i_2}^{f_1(s_1)})\quad(\bot\mathsf{E})\\
\hline
\bigotimes_{s_0\in\mathbf{2^{1_0}}}\bigotimes_{\substack{s_1\in\mathbf{2^{1_1}}\\ s_1(i_1)=f_0(s_0)}}(p_{i_0}^{s_0(i_0)}\wedge p_{i_2}^{f_1(s_1)})\quad(\wedge\mathsf{E},\ \otimes\mathsf{Sub})\\
\hline
\bigotimes_{s_0\in\mathbf{2^{1_0}}}(p_{i_0}^{s_0(i_0)}\wedge p_{i_2}^{f_1(s_1)})\quad(\text{for each } s_0\text{, such } s_1 \text{ is unqiue})\\
\hline
=\!(p_{i_0},p_{i_2})\quad(\mathsf{DepSI})
\end{array}
$$

Now, we derive $=\!(p_{i_0},p_{i_1}),=\!(p_{i_1},p_{i_2})\vdash=\!(p_{i_0},p_{i_2})$ as follows:

$$
\dfrac{\dfrac{\begin{array}{c}\forall f_0:\mathbf{2^{1_0}}\to 2,\ f_1:\mathbf{2^{1_1}}\to 2\\ \left[(=\!(p_{i_0},p_{i_1})\wedge=\!(p_{i_1},p_{i_2}))^*_{\langle f_0,f_1\rangle}\right]\end{array}}{=\!(p_{i_0},p_{i_2})}\,(\text{by }(*))\qquad (\wedge\mathsf{I})\,\dfrac{=\!(p_{i_0},p_{i_1})\qquad =\!(p_{i_1},p_{i_2})}{=\!(p_{i_0},p_{i_1})\wedge=\!(p_{i_1},p_{i_2})}}{=\!(p_{i_0},p_{i_2})}\,(\mathsf{ApTr})
$$

■

4.5 Axiomatizing propositional exclusion logic

It turns out that the method used in Section 4.4 can be generalized to obtain an axiomatization for the propositional variant of first-order exclusion logic, namely *propositional exclusion logic*. In this section, we will give this proof.

Let us first define the logic.

Definition 4.5.1. We call formulas of the form $p_{i_1}\cdots p_{i_k}\mid p_{j_1}\cdots p_{j_k}$ *exclusion atoms*. Well-formed formulas of *propositional exclusion logic* (**PExc**) are given by the following grammar:

$$\phi ::= p_i \;\Big|\; \neg p_i \;\Big|\; p_{i_1}\cdots p_{i_k}\,|\,p_{j_1}\cdots p_{j_k} \;\Big|\; \phi\wedge\phi \;\Big|\; \phi\otimes\phi,$$

where $p_i,p_{i_1},\dots,p_{i_k},p_{j_1},\dots,p_{j_k}$ are propositional variables and $k\geq 1$.

Definition 4.5.2. We inductively define the notion of a formula ϕ of **PExc** being *true* on a team X, denoted by $X \models \phi$. All the cases are the same as those of **PD** as defined in Definition 4.1.3 except the following:

- $X \models p_{i_1} \cdots p_{i_k} \mid p_{j_1} \cdots p_{j_k}$ iff for all $s, s' \in X$,
$$\langle s(i_1), \dots, s(i_k)\rangle \neq \langle s'(j_1), \dots, s'(j_k)\rangle.$$

It is easy to check that **PExc** has the *downwards closure property* and *the empty team property*, thus can be viewed as a sublogic of the maximal downwards closed logics.

Now, we proceed to axiomatize **PExc** using a similar method with that of **PD**. As in the case of **PD**, we start with analyzing approximation sequences of formulas of **PExc**.

Suppose the following are all the occurrences of all exclusion atoms in a formula ϕ of **PExc**:

$$p_{i_1^1} \cdots p_{i_{k_1}^1} \mid p_{j_1^1} \cdots p_{j_{k_1}^1}, \; \dots \dots, \; p_{i_1^c} \cdots p_{i_{k_c}^c} \mid p_{j_1^c} \cdots p_{j_{k_c}^c}.$$

An *approximation sequence* $\Omega = \langle o_1, \dots, o_c\rangle$ of ϕ is a sequence such that for each $1 \leq \xi \leq c$, $o_\xi \subseteq \mathbf{2^{2k_\xi}}$ with

$$\{\langle s(i_1^\xi), \dots, s(i_{k_\xi}^\xi)\rangle \mid s \in o_\xi\} \cap \{\langle s'(j_1^\xi), \dots, s'(j_{k_\xi}^\xi)\rangle \mid s' \in o_\xi\} = \emptyset, \tag{4.14}$$

where $\mathbf{2^{2k_\xi}}$ is the maximal $2k_\xi$-team on $\{i_1^\xi, \dots, i_{k_\xi}^\xi, j_1^\xi, \dots, j_{k_\xi}^\xi\}$. For any such sequence Ω, define an inclusion atom-free (classical) formula ϕ_Ω^*, called an *approximation* of ϕ, by induction the same way as in the case of **PD**, except the following case:

- $(p_{i_1^\xi} \dots p_{i_{k_\xi}^\xi} \mid p_{j_1^\xi} \dots p_{j_{k_\xi}^\xi})^*_{\langle o_\xi\rangle} := \bigotimes_{s \in o_\xi} \left(p_{i_1^\xi}^{s(i_1^\xi)} \wedge \dots \wedge p_{i_{k_\xi}^\xi}^{s(i_{k_\xi}^\xi)} \wedge p_{j_1^\xi}^{s(j_1^\xi)} \wedge \dots \wedge p_{j_{k_\xi}^\xi}^{s(j_{k_\xi}^\xi)} \right);$

Next, we show that every **PExc** formula is logically equivalent to the intuitionistic disjunction of all its approximations.

Lemma 4.5.3. *Let ϕ be a formula of* **PExc** *and Λ the set of all its approximation sequences. Then*

$$\phi \equiv \bigvee_{\Omega \in \Lambda} \phi_\Omega^*.$$

Proof. We prove the lemma by induction on ϕ. All the other cases are similar with those in the proof of Lemma 4.4.3 except the case that $\phi = p_{i_1^\xi} \dots p_{i_{k_\xi}^\xi} \mid p_{j_1^\xi} \dots p_{j_{k_\xi}^\xi}$.

In this case, we first show that for each $\langle o_\xi\rangle \in \Lambda$, $\phi^*_{\langle o_\xi\rangle} \models \phi$. Suppose

$$X \models \bigotimes_{s \in o_\xi} \left(p_{i_1^\xi}^{s(i_1^\xi)} \wedge \dots \wedge p_{i_{k_\xi}^\xi}^{s(i_{k_\xi}^\xi)} \wedge p_{j_1^\xi}^{s(j_1^\xi)} \wedge \dots \wedge p_{j_{k_\xi}^\xi}^{s(j_{k_\xi}^\xi)} \right).$$

Then for each $s \in o_\xi$, there exists $X_s \subseteq X$ such that $X = \bigcup_{s \in o_\xi} X_s$ and

$$X_s \models p_{i_1^\xi}^{s(i_1^\xi)} \wedge \dots \wedge p_{i_{k_\xi}^\xi}^{s(i_{k_\xi}^\xi)} \wedge p_{j_1^\xi}^{s(j_1^\xi)} \wedge \dots \wedge p_{j_{k_\xi}^\xi}^{s(j_{k_\xi}^\xi)}.$$

For any $t,t' \in X$, there exists $s,s' \in o_\xi$ such that $t \in X_s$ and $t' \in X_{s'}$. Since $\langle o_\xi \rangle$ is an approximation sequence of $p_{i_1^\xi} \dots p_{i_{k_\xi}^\xi} \mid p_{j_1^\xi} \dots p_{j_{k_\xi}^\xi}$,

$$\langle t(i_1^\xi),\dots,t(i_{k_\xi}^\xi)\rangle = \langle s(i_1^\xi),\dots,s(i_{k_\xi}^\xi)\rangle \neq \langle s'(j_1^\xi),\dots,s'(j_{k_\xi}^\xi)\rangle = \langle t'(j_1^\xi),\dots,t'(j_{k_\xi}^\xi)\rangle.$$

Hence $X \models p_{i_1^\xi} \dots p_{i_{k_\xi}^\xi} \mid p_{j_1^\xi} \dots p_{j_{k_\xi}^\xi}$.

Conversely, we show that $\phi \models \bigvee_{\Omega\in\Lambda} \phi_\Omega^*$. Suppose $X \models p_{i_1^\xi} \dots p_{i_{k_\xi}^\xi} \mid p_{j_1^\xi} \dots p_{j_{k_\xi}^\xi}$. Define

$$o_\xi = X \upharpoonright \{i_1^\xi,\dots,i_{k_\xi}^\xi, j_1^\xi,\dots,j_{k_\xi}^\xi\}.$$

Clearly, $X \models \bigotimes_{s\in o_\xi} \left(p_{i_1^\xi}^{s(i_1^\xi)} \wedge \dots \wedge p_{i_{k_\xi}^\xi}^{s(i_{k_\xi}^\xi)} \wedge p_{j_1^\xi}^{s(j_1^\xi)} \wedge \dots \wedge p_{j_{k_\xi}^\xi}^{s(j_{k_\xi}^\xi)} \right)$. It remains to show that $\langle o_\xi \rangle$ is an approximation sequence of $p_{i_1} \dots p_{i_k} \mid p_{j_1} \dots p_{j_k}$, namely to check that (4.14) is satisfied.

For any $s,s' \in o_\xi$, by definition, there are $t,t' \in X$ such that

$$t \upharpoonright \{i_1^\xi,\dots,i_{k_\xi}^\xi, j_1^\xi,\dots,j_{k_\xi}^\xi\} = s \upharpoonright \{i_1^\xi,\dots,i_{k_\xi}^\xi, j_1^\xi,\dots,j_{k_\xi}^\xi\}$$

and

$$t' \upharpoonright \{i_1^\xi,\dots,i_{k_\xi}^\xi, j_1^\xi,\dots,j_{k_\xi}^\xi\} = s' \upharpoonright \{i_1^\xi,\dots,i_{k_\xi}^\xi, j_1^\xi,\dots,j_{k_\xi}^\xi\}.$$

Since $X \models p_{i_1^\xi} \dots p_{i_{k_\xi}^\xi} \mid p_{j_1^\xi} \dots p_{j_{k_\xi}^\xi}$, we have that

$$\langle s(i_1^\xi),\dots,s(i_{k_\xi}^\xi)\rangle = \langle t(i_1^\xi),\dots,t(i_{k_\xi}^\xi)\rangle \neq \langle t'(j_1^\xi),\dots,t'(j_{k_\xi}^\xi)\rangle = \langle s'(j_1^\xi),\dots,s'(j_{k_\xi}^\xi)\rangle,$$

as required. ■

Next, we give a natural deduction system for **PExc** in which one can *in effect* show that

$$\bigvee_{\Omega\in\Lambda} \phi_\Omega^* \dashv\vdash \phi$$

for any **PExc** formula ϕ, where Λ is the set of all approximation sequences of ϕ.

Definition 4.5.4 (A natural deduction system for **PExc**). The rules are given as follows:

1. The rules ($\wedge$I), ($\wedge$E), ($\otimes$I), ($\otimes$WE), ($\otimes$Sub), (Com $\otimes$), (Ass $\otimes$), ($\bot$E), (EM$_0$), (Dstr $\wedge\otimes$) as in Definition 4.3.5 (see also Appendix).

2. Exclusion Atom Introduction:

   For any approximation sequence $\langle o \rangle$ of $p_{i_1} \dots p_{i_k} \mid p_{j_1} \dots p_{j_k}$,

$$\frac{\bigotimes_{s\in o} \left(p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_k}^{s(i_k)} \wedge p_{j_1}^{s(j_1)} \wedge \dots \wedge p_{j_k}^{s(j_k)} \right)}{p_{i_1} \dots p_{i_k} \mid p_{j_1} \dots p_{j_k}} \text{(Excl)}$$

3. Approximation Transition:

$$\cfrac{\cfrac{\begin{matrix}[\phi^*_{\Omega_0}]\\ \vdots\end{matrix}}{\theta}\ \cdots \quad \cdots\ \cfrac{\begin{matrix}[\phi^*_{\Omega_m}]\\ \vdots\end{matrix}}{\theta} \qquad \phi}{\theta}\ (\mathsf{ApTr})$$

where $\{\Omega_0,\ldots,\Omega_m\}$ is the set of all approximation sequences of ϕ.

Next, we prove the Soundness Theorem for the above system.

Theorem 4.5.5 (Soundness Theorem)**.** *For any* **PExc** *formulas ϕ and ψ,*

$$\phi \vdash \psi \Longrightarrow \phi \models \psi.$$

Proof. It suffices to show that all of the deductive rules are valid. The validity of (Excl) and (ApTr) follows from Lemma 4.5.3, and the validity of all the other rules follows from the proof of Theorem 4.3.7. ■

Theorem 4.5.6 (Completeness Theorem)**.** *For any formulas ϕ and ψ of* **PExc***,*

$$\phi \models \psi \Longrightarrow \phi \vdash \psi.$$

Proof. By a similar argument to that in the proof of Theorem 4.4.8. ■

Theorem 4.5.7 (Strong Completeness Theorem)**.** *For any set Γ of formulas and any formula ϕ of* **PExc***,*

$$\Gamma \models \phi \Longrightarrow \Gamma \vdash \phi.$$

Proof. Follows from Theorem 4.5.6 and the Compactness Theorem of **PID** (as **PExc** formulas are expressible in **PID**). ■

4.6 Axiomatizing propositional dependence logic with intuitionistic disjunction and non-empty atom

All of the logics we studied in sections 4.2-4.5, **PID**, **PD**$^{[\vee]}$, **PD** and **PExc**, are downwards closed. In this section, we study a naturally arisen non-downwards closed logic, namely propositional dependence logic with intuitionistic disjunction and non-empty atom. We show that this logic is maximal as it characterizes all n-teams, and we will generalize the method used in Section 4.3 to axiomatize this logic.

In Lemma 4.3.1 of the downwards closed logic **PD**$^{[\vee]}$, we were able to define an n-team *up to its subteams*. Observe that in the formula Θ_X in Lemma 4.3.1, if we could add an atom to each disjunct which says that the team described by the conjunction $(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)})$ is *non-empty*, then we would be able to define the team X *precisely*. Let us now introduce such an atom. We call this new atom *non-empty atom* and denote it by NE. Its team semantics is defined as follows.

Definition 4.6.1 (Non-empty Atom)**.** For any team X, define

- $X \models \mathsf{NE}$ iff $X \neq \emptyset$.

We stipulate that the non-empty atom NE is a 1-formula, written as $\mathsf{NE}(p_i)$. Clearly, the non-empty atom NE is not definable in any logic with the empty team property. For the logics that lack of the empty team property, stipulate

- $\bigotimes \emptyset := \bot$
- $\bigvee \emptyset := \bot \wedge \mathsf{NE}$

The formula $\bot$ is satisfied only by the empty team, whereas the formula $\bot \wedge \mathsf{NE}$ is satisfied by no teams. We call the former the *weak contradiction* and the latter the *strong contradiction*.

In a logic with the atom NE, a *classical formula* is a formula built from propositional variables, negated propositional variables and NE by conjunction $\wedge$ and tensor disjunction $\otimes$.

In this section, we consider the logic $\mathbf{PD}^{[\vee]}$ extended with NE, i.e., $\mathbf{PD}^{[\vee,\mathsf{NE}]}$. Now we prove that n-teams are definable *precisely* in $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ (c.f. Lemma 4.3.1).

Lemma 4.6.2. *Let X be an n-team on $N = \{i_1, \dots, i_n\}$. Define a formula Θ^*_X of $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ as*

$$\Theta^*_X := \bigotimes_{s \in X} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE}).$$

Then for any n-team Y on N,

$$Y \models \Theta_X \iff Y = X.$$

Proof. The direction "$\Longleftarrow$" is obvious. For "$\Longrightarrow$", suppose $Y \models \Theta^*_X$. If $X = \emptyset$, then $\Theta^*_X = \bot$, hence $Y = \emptyset = X$. Otherwise, for each $s \in X$, there exists a non-empty set Y_s such that

$$Y = \bigcup_{s \in X} Y_s \text{ and } Y_s \models p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE}.$$

Clearly, $Y_s = \{s\}$ implying $Y = X$. ∎

Next, we show that $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ is a maximal logic in the sense that it characterizes all n-teams for a fix n-element set of natural numbers (c.f. Definition 4.2.7 of maximal downwards closed logic).

Definition 4.6.3. A logic L with team semantics is called a *maximal logic* if for every n-element set $N = \{i_1, \dots, i_n\} \subseteq \mathbb{N}$,

$$\wp(\mathbf{2^n}) = \{[\![\phi]\!] : \phi(p_{i_1}, \dots, p_{i_n}) \text{ is an } n\text{-formula of } \mathsf{L}\}, \tag{4.15}$$

where $\mathbf{2^n}$ is the maximal n-team on N.

Theorem 4.6.4. $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ *is a maximal logic.*

Proof. It suffices to show the inclusion "$\subseteq$" of (4.15). For each $\mathcal{K} \in \wp(\mathbf{2^n})$, by Lemma 4.6.2, for any n-team Y on N,

$$Y \models \bigvee_{X \in \mathcal{K}} \Theta_X^* \iff \exists X \in \mathcal{K}(Y = X) \iff Y \in \mathcal{K},$$

thus $[\![\bigvee_{X\in\mathcal{K}} \Theta_X^*]\!] = \mathcal{K}$. Note that in particular, for the empty collection $\mathcal{K} = \emptyset$ of n-teams on N, we have that $[\![\bigvee_{X\in\emptyset} \Theta_X^*]\!] = [\![\bot \wedge \mathsf{NE}]\!] = \emptyset$. ■

Similar to the first-order logic case, the semantics of the *classical (contradictory) negation* $\sim \phi$ of a formula ϕ is define as:

$$X \models \sim \phi \iff X \not\models \phi$$

for all teams X. A consequence of the above theorem is, the classical negation $\sim \phi$ of every formula ϕ is definable in the logic $\mathbf{PD}^{[\vee,\mathsf{NE}]}$. Moreover, we have $\mathsf{NE} \equiv \sim \bot$. Therefore in fact, $\mathbf{PD}^{[\vee,\mathsf{NE}]} = \mathbf{PD}^{[\vee,\sim]}$, and $\mathbf{PD}^{[\vee,\sim]}$ is also a naturally arisen maximal logic. But in this chapter, we will restrict our attention to the logic $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ only, as the non-empty atom is considerably simpler than classical negation. Note also that from the equivalence of the two logics, we only derive that every instance of $\sim \phi$ is expressible in $\mathbf{PD}^{[\vee,\mathsf{NE}]}$, but the problem of whether classical negation is *uniformly definable*[7] in $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ is open.

As in the case of $\mathbf{PD}^{[\vee]}$, the proof of Theorem 4.6.4 shows that every $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ formula is logically equivalent to a formula of the normal form

$$\bigvee_{f\in F} \bigotimes_{s\in X_f} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE}),$$

where F is a finite set of indices and each X_f is an n-team on $\{i_1, \ldots, i_n\}$. Next, we give a natural deduction system for $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ for which the normal form will be syntactically derivable.

Definition 4.6.5 (A Natural Deduction System for $\mathbf{PD}^{[\vee,\mathsf{NE}]}$)**.** The rules are given as follows:

1. The rules ($\wedge$I), ($\wedge$E), ($\vee$I), ($\vee$E), ($\otimes$WE), ($\otimes$Sub), (Com $\otimes$), (Ass $\otimes$), ($\bot$E), (EM$_0$), (DepI), (DepE), (Dstr $\otimes\vee$), (Dstr $\otimes\vee\otimes$), as in Definition 4.3.5 (see also Appendix).

2. Weak Tensor Disjunction Introduction:

$$\frac{\phi}{\phi \otimes \psi}\,(*)\ (\otimes\mathsf{WI})$$

   ($*$) whenever ψ does not contain NE.

3. Tensor Disjunction Repetition:

$$\frac{\phi}{\phi \otimes \phi}\ (\otimes\mathsf{Rpt})$$

[7] See Definition 5.1.3 for the definition of *uniform definability*.

4. NE Introduction: $$\dfrac{}{(p_i \wedge \neg p_i) \vee \mathsf{NE}}\ (\mathsf{NE\,I})$$

5. Strong *ex falso*: $$\dfrac{(p_i \wedge \neg p_i) \wedge \mathsf{NE}}{\phi}\ (\mathsf{ex\ falso}^{+})$$

6. Strong Contradiction Introduction:

$$\dfrac{\Big(\bigotimes_{s\in X}(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \wedge \Big(\bigotimes_{s'\in Y}(p_{i_1}^{s'(i_1)} \wedge \cdots \wedge p_{i_n}^{s'(i_n)} \wedge \mathsf{NE})\Big)}{(p_{i_1} \wedge \neg p_{i_1}) \wedge \mathsf{NE}}\ (\mathbf{0}\mathsf{I})$$

where X and Y are n-teams on $\{i_1, \ldots, i_n\}$ with $X \neq Y$.

7. Strong Contradiction Contraction:

$$\dfrac{\phi \otimes ((p_i \wedge \neg p_i) \wedge \mathsf{NE})}{(p_i \wedge \neg p_i) \wedge \mathsf{NE}}\ (\mathbf{0}\mathsf{Ctr})$$

8. Distributive Laws:

$$\dfrac{\phi \wedge (\psi \otimes \chi)}{(\phi \wedge \psi) \otimes (\phi \wedge \chi)}\ (*)\ (\mathsf{Dstr}^{*} \wedge \otimes)$$

(∗) whenever ϕ does not contain NE.

$$\dfrac{\mathsf{NE} \wedge \bigotimes_{j\in J} \phi_j}{\bigvee_{\substack{f\in 2^J\\ f\neq \mathbf{0}}} \bigotimes_{\substack{j\in J\\ f(j)=1}} (\mathsf{NE} \wedge \phi_j)}\ (\mathsf{Dstr}\ \mathsf{NE} \wedge \otimes)$$

where $\mathbf{0} : J \to 2$ is defined as $\mathbf{0}(j) = 0$.

Corollary 4.6.6. *The following are derivable rules:*

1. *The usual commutative, associative, distributive and substitution laws for conjunction and intuitionistic disjunction.*

2. *Weak ex falso:*

$$\dfrac{p_i \wedge \neg p_i}{\phi}\ (\mathsf{ex\ falso}^{-})$$

whenever ϕ does not contain NE.

3. *Strong Contradiction Elimination:*

$$\dfrac{\phi \vee ((p_i \wedge \neg p_i) \wedge \mathsf{NE})}{\phi}\ (\mathbf{0}E)$$

4. *Distributive Laws (Dstr$\otimes\wedge$), (Dstr$\vee\otimes$) and (Dstr* $\wedge\otimes\wedge$) as in Corollary 4.3.6.*

5. *Tensor Disjunction Combination:*

$$\frac{\left(\bigotimes_{i\in I}\phi_i\right)\otimes\left(\bigotimes_{j\in J}\phi_j\right)}{\bigotimes_{k\in I\cup J}\phi_k}\,(\otimes\mathsf{Cmb})$$

whenever ϕ_i, ϕ_j are classical formulas.

6. *Tensor Disjunction Decomposition:*

$$\frac{\bigotimes_{k\in K}\phi_k}{\left(\bigotimes_{i\in I}\phi_i\right)\otimes\left(\bigotimes_{j\in J}\phi_j\right)}\,(\otimes\mathsf{Dcp})$$

where I,J,K are finite sets of indices with $I\cup J=K$.

Proof. The rules in Item 1 are derived by the standard argument, and the rules (Dstr $\otimes\wedge$) and (Dstr* $\wedge\otimes\wedge$) are proved by the same arguments as those in the proof of Corollary 4.3.6. The rules (Dstr $\vee\otimes$) and (Weak ex falso) are derived by a similar argument to that in the proof of Corollary 4.3.6, except that we use ($\otimes$Rpt) and ($\otimes$WI) instead of ($\otimes$I). It remains to derive other rules.

For (**0**E):

$$\frac{\phi\vee((p_i\wedge\neg p_i)\wedge\mathsf{NE})\qquad [\phi]\qquad \dfrac{[(p_i\wedge\neg p_i)\wedge\mathsf{NE}]}{\phi}\,(\text{ex falso}^+)}{\phi}\,(\vee\mathsf{E})$$

For ($\otimes$Cmb): Suppose ϕ_i and ϕ_j are classical formulas. If $I,J\neq\emptyset$, then the rule is derivable using ($\otimes$WE). It remains to derive the rule when $I=\emptyset$ or $J=\emptyset$. We only show the case that $I=\emptyset$. Noting that $\bigotimes\emptyset=\bot$, we have the following derivation:

$$\dfrac{\dfrac{\dfrac{\left(\bigotimes_{i\in\emptyset}\phi_i\right)\otimes\left(\bigotimes_{j\in J}\phi_j\right)}{\bot\otimes\left(\bigotimes_{j\in J}\phi_j\right)}}{\bigotimes_{j\in J}\phi_j}\,(\bot\mathsf{E})}{\bigotimes_{k\in\emptyset\cup J}\phi_k}$$

For ($\otimes$Dcp): If $I,J\neq\emptyset$, then the rule is derivable using ($\otimes$Rpt). It remains to derive the rule when $I=\emptyset$ or $J=\emptyset$. We only show the case that $I=\emptyset$. In this case $J=K$ and we have the following derivation:

$$\dfrac{\dfrac{\bigotimes_{k\in K}\phi_k}{\bot\otimes\Big(\bigotimes_{k\in K}\phi_k\Big)}\,(\otimes\mathsf{WI})}{\Big(\bigotimes_{i\in\emptyset}\phi_i\Big)\otimes\Big(\bigotimes_{j\in J}\phi_j\Big)}$$

■

Next, we check the Soundness Theorem of the above deductive system.

Theorem 4.6.7 (Soundness Theorem)**.** *For any* $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ *formulas* ϕ *and* ψ*,*

$$\phi\vdash\psi\Longrightarrow\phi\models\psi.$$

Proof. It suffices to show that all of the deductive rules are valid. The validity of the rules in Item 1 except for ($\otimes$WE) follows from Theorem 4.3.7, and the rules ($\otimes$Rpt), (NEI), (ex falso$^+$) and ($\mathbf{0}$Ctr) are easy to verified. It remains to verify the validity of the other rules.

The validity of the rule ($\otimes$WE) is checked by a similar argument to that in the proof of Theorem 4.3.7. Note that the crucial step can go through, since the classical formula χ of $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ is clearly closed under unions.

For ($\otimes$WI), assuming that ψ does not contain NE, it suffices to show that $\phi\models\phi\otimes\psi$. Since ψ does not contain NE (thus ψ has the empty team property), we have that $\emptyset\models\psi$. Thus, for any team X such that $X\models\phi$, we have that $\emptyset\cup X\models\phi\otimes\psi$.

For ($\mathbf{0}$I), by the locality property, it suffices to show that for all n-teams Z on $\{i_1,\dots,i_n\}$,

$$Z\not\models\Big(\bigotimes_{s\in X}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)\wedge\Big(\bigotimes_{s'\in Y}(p_{i_1}^{s'(i_1)}\wedge\cdots\wedge p_{i_n}^{s'(i_n)}\wedge\mathsf{NE})\Big),$$

where X and Y are two distinct n-teams on $\{i_1,\dots,i_n\}$. Note that the left and right disjuncts of the above formula are the formulas Θ_X^* and Θ_Y^*. By Lemma 4.6.2, if $Z\models\Theta_X^*\wedge\Theta_Y^*$, then $X=Z=Y$, which is a contradiction.

For (Dstr* $\wedge\otimes$), it suffices to show that if $X\models\phi\wedge(\psi\otimes\chi)$ for some team X, then $X\models(\phi\wedge\psi)\otimes(\phi\wedge\chi)$, whenever ϕ does not contain NE. Since $X\models\phi\wedge(\psi\otimes\chi)$, we have that $X\models\phi$, $Y\models\psi$ and $Z\models\chi$ for some teams $Y,Z\subseteq X$ with $Y\cup Z=X$. As ϕ does not contain NE, ϕ is downwards closed, which means that $Y\models\phi$ and $Z\models\phi$. It follows that $Y\models\phi\wedge\psi$ and $Z\models\phi\wedge\chi$, thus $X\models(\phi\wedge\psi)\otimes(\phi\wedge\chi)$.

For (Dstr NE $\wedge\otimes$), it suffices to show that if $X\models\mathsf{NE}\wedge\bigotimes_{j\in J}\phi_j$ for some team X, then $X\models\bigvee_{\substack{f\in 2^J\\ f\neq\mathbf{0}}}\bigotimes_{\substack{j\in J\\ f(j)=1}}(\mathsf{NE}\wedge\phi_j)$. By assumption, $X\neq\emptyset$ and there are teams $X_j\subseteq X$ for each $j\in J$ such that $\bigcup_{j\in J}X_j=X$ and $X_j\models\phi_j$. Define a function $f:J\to 2$ by taking

$$f(j)=\begin{cases}1 & \text{if } X_j\neq\emptyset;\\ 0 & \text{if } X_j=\emptyset.\end{cases}$$

Since $X \neq \emptyset$, there exists $j \in J$ such that $X_j \neq \emptyset$, thus $f \neq \mathbf{0}$. Clearly, for each $j \in J$ with $f(j) = 1$, we have that $X_j \models \mathsf{NE} \wedge \phi_j$. It follows that

$$X \models \bigotimes_{\substack{j \in J \\ f(j)=1}} (\mathsf{NE} \wedge \phi_j), \text{ thereby } X \models \bigvee_{\substack{f \in 2^J \\ f \neq \mathbf{0}}} \bigotimes_{\substack{j \in J \\ f(j)=1}} (\mathsf{NE} \wedge \phi_j).$$

■

Next, we show that every $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ formula is provably equivalent to a formula in the intended normal form.

Theorem 4.6.8. *Any n-formula $\phi(p_{i_1}, \dots, p_{i_n})$ of $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ is provably equivalent to a formula of the form*

$$\bigvee_{f \in F} \bigotimes_{s \in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE}),$$

where F is a finite set of indices, and each X_f is an n-team on $N = \{i_1, \dots, i_n\}$.

Proof. Let $\mathbf{2^n}$ be the maximal n-team on N. We prove the theorem by induction on $\phi(p_{i_1}, \dots, p_{i_n})$.

Case $\phi(p_{i_1}, \dots, p_{i_n}) = \mathsf{NE}(p_{i_1}, \dots, p_{i_n})$. Noting that $n \geq 1$ by stipulation, we prove that $\mathsf{NE} \dashv\vdash \theta$, where

$$\theta := \bigvee_{\substack{f \in 2^{\mathbf{2^n}} \\ f \neq \mathbf{0}}} \bigotimes_{\substack{s \in \mathbf{2^n} \\ f(s)=1}} (p_{i_1}^{s(i_1)} \wedge \dots \wedge \dots p_{i_n}^{s(i_n)} \wedge \mathsf{NE}),$$

where $\mathbf{0} : \mathbf{2^n} \to 2$ is the function defined as $\mathbf{0}(s) = 0$ for all $s \in \mathbf{2^n}$.

For $\mathsf{NE} \vdash \theta$, we have the following derivation:

(1) NE

(2) $(p_{i_1} \otimes \neg p_{i_1}) \wedge \dots \wedge (p_{i_n} \otimes \neg p_{i_n})$ (EM_0, $\wedge\mathsf{I}$)

(3) $\bigotimes_{s \in \mathbf{2^n}} (p_{i_1}^{s(i_1)} \wedge \dots \wedge \dots p_{i_n}^{s(i_n)})$ ($\mathsf{Dstr}^* \wedge \otimes$)

(4) $\mathsf{NE} \wedge \bigotimes_{s \in \mathbf{2^n}} (p_{i_1}^{s(i_1)} \wedge \dots \wedge \dots p_{i_n}^{s(i_n)})$ ((1), (3), $\wedge\mathsf{I}$)

(5) $\bigvee_{\substack{f \in 2^{\mathbf{2^n}} \\ f \neq \mathbf{0}}} \bigotimes_{\substack{s \in \mathbf{2^n} \\ f(s)=1}} (p_{i_1}^{s(i_1)} \wedge \dots \wedge \dots p_{i_n}^{s(i_n)} \wedge \mathsf{NE})$ (Dstr $\mathsf{NE} \wedge \otimes$)

For the other direction $\theta \vdash \mathsf{NE}$, we have the following derivation:

(1) θ

(2) $\bigvee_{\substack{f \in 2^{\mathbf{2^n}} \\ f \neq \mathbf{0}}} \Big(\mathsf{NE} \wedge \bigotimes_{\substack{s \in \mathbf{2^n} \\ f(s)=1}} (p_{i_1}^{s(i_1)} \wedge \dots \wedge \dots p_{i_n}^{s(i_n)})\Big)$ ($\mathsf{Dstr}^* \wedge \otimes \wedge$)

$$(3)\,\mathsf{NE}\wedge \bigvee_{\substack{f\in 2^{\mathbf{2^n}}\\ f\neq \mathbf{0}}} \bigotimes_{\substack{s\in \mathbf{2^n}\\ f(s)=1}} (p_{i_1}^{s(i_1)}\wedge\cdots\wedge\ldots p_{i_n}^{s(i_n)})\ \ (\mathsf{Dstr})$$

$$(4)\,\mathsf{NE}\ \ (\wedge\mathsf{E})$$

Case $\phi(p_{i_1},\ldots,p_{i_n}) = p_{i_k}$ for $1\leq k\leq n$. If $N\setminus\{i_k\}=\emptyset$, we show that

$$p_{i_k}(p_{i_k}) \dashv\vdash (p_{i_k}\wedge\neg p_{i_k})\vee(p_{i_k}\wedge\mathsf{NE}).$$

We have the following derivations:

$$(1)p_{i_k}$$
$$(2)(p_{i_k}\wedge\neg p_{i_k})\vee\mathsf{NE}\ \ (\mathsf{NE\,I})$$
$$(3)p_{i_k}\wedge\big((p_{i_k}\wedge\neg p_{i_k})\vee\mathsf{NE}\big)\ \ ((1),(2),\wedge\mathsf{I})$$
$$(4)\big(p_{i_k}\wedge(p_{i_k}\wedge\neg p_{i_k})\big)\vee(p_{i_k}\wedge\mathsf{NE})\ \ (\mathsf{Dstr})$$
$$(5)(p_{i_k}\wedge\neg p_{i_k})\vee(p_{i_k}\wedge\mathsf{NE})\ \ (\wedge\mathsf{E})$$

and

$$(1)(p_{i_k}\wedge\neg p_{i_k})\vee(p_{i_k}\wedge\mathsf{NE})$$
$$(2)p_{i_k}\wedge(\neg p_{i_k}\vee\mathsf{NE})\ \ (\mathsf{Dstr})$$
$$(3)p_{i_k}\ \ (\wedge\mathsf{E})$$

Now, assume $N\setminus\{i_k\}\neq\emptyset$. We prove that $p_{i_k}\dashv\vdash\theta$, where

$$\theta := \big(p_{i_k}\wedge\neg p_{i_k}\big)\vee \bigvee_{\substack{f\in 2^{\mathbf{2^{n-1}}}\\ f\neq\mathbf{0}}} \bigotimes_{\substack{s\in\mathbf{2^{n-1}}\\ f(s)=1}} (p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_{k-1}}^{s(i_{k-1})}\wedge p_{i_k}\wedge p_{i_{k+1}}^{s(i_{k+1})}\wedge\ldots p_{i_n}^{s(i_n)}\wedge\mathsf{NE}),$$ [8]

where $\mathbf{2^{n-1}}$ is the maximal $(n-1)$-team on $N\setminus\{i_k\}$ and $\mathbf{0}:\mathbf{2^{n-1}}\to 2$ is defined as $\mathbf{0}(s)=0$ for all $s\in\mathbf{2^{n-1}}$. For $p_{i_k}\vdash\theta$, we have the following derivation:

$$(1)p_{i_k}$$
$$(2)(p_{i_1}\otimes\neg p_{i_1})\wedge\cdots\wedge(p_{i_{k-1}}\otimes\neg p_{i_{k-1}})\wedge(p_{i_{k+1}}\otimes\neg p_{i_{k+1}})\wedge\cdots\wedge(p_{i_n}\otimes\neg p_{i_n})\ \ (\mathsf{EM}_0,\wedge\mathsf{I})$$
$$(3)\bigotimes_{s\in\mathbf{2^{n-1}}}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_{k-1}}^{s(i_{k-1})}\wedge p_{i_{k+1}}^{s(i_{k+1})}\wedge\cdots\wedge p_{i_n}^{s(i_n)})\ \ (\mathsf{Dstr}^*\wedge\otimes,\ \otimes\mathsf{Sub})$$
$$(4)p_{i_k}\wedge\bigotimes_{s\in\mathbf{2^{n-1}}}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_{k-1}}^{s(i_{k-1})}\wedge p_{i_{k+1}}^{s(i_{k+1})}\wedge\cdots\wedge p_{i_n}^{s(i_n)})\ \ ((1),(3),\wedge\mathsf{I})$$
$$(5)\bigotimes_{s\in\mathbf{2^{n-1}}}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_{k-1}}^{s(i_{k-1})}\wedge p_{i_k}\wedge p_{i_{k+1}}^{s(i_{k+1})}\wedge\cdots\wedge p_{i_n}^{s(i_n)})\ \ (\mathsf{Dstr}^*\wedge\otimes,\mathsf{Com}\wedge)$$

[8] Note that $p_{i_k}\wedge\neg p_{i_k} = \bigotimes\emptyset = \bigotimes_{\substack{s\in\mathbf{2^{n-1}}\\ \mathbf{0}(s)=1}}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_{k-1}}^{s(i_{k-1})}\wedge p_{i_k}\wedge p_{i_{k+1}}^{s(i_{k+1})}\wedge\ldots p_{i_n}^{s(i_n)}\wedge\mathsf{NE})$.

$(6)(p_{i_k} \wedge \neg p_{i_k}) \vee \mathsf{NE}$ $(\mathsf{NE\,I})$

$(7)\big((p_{i_k} \wedge \neg p_{i_k}) \vee \mathsf{NE}\big) \wedge \bigotimes_{s\in\mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_k} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \cdots \wedge p_{i_n}^{s(i_n)})$

((5), (6), $\wedge$I)

$(8)\Big((p_{i_k} \wedge \neg p_{i_k}) \wedge \bigotimes_{s\in\mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_k} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \cdots \wedge p_{i_n}^{s(i_n)})\Big)$

$\vee \Big(\mathsf{NE} \wedge \bigotimes_{s\in\mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_k} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \cdots \wedge p_{i_n}^{s(i_n)})\Big)$ (Dstr)

$(9)(p_{i_k} \wedge \neg p_{i_k}) \vee$

$\Big(\mathsf{NE} \wedge \bigotimes_{s\in\mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_k} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \cdots \wedge p_{i_n}^{s(i_n)})\Big)$ $(\wedge\mathsf{E})$

$(10)(p_{i_k} \wedge \neg p_{i_k}) \vee$

$\bigvee_{\substack{f\in 2^{\mathbf{2^{n-1}}}\\ f\neq \mathbf{0}}} \bigotimes_{\substack{s\in\mathbf{2^{n-1}}\\ f(s)=1}} \Big(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_k} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)} \wedge \mathsf{NE}\Big)$

(Dstr NE$\wedge\otimes$, Com)

For the other direction $\theta \vdash p_{i_k}$, we have the following derivation:

$(1)\theta$

$(2)\big(p_{i_k} \wedge \neg p_{i_k}\big) \vee$

$\Big(\bigvee_{\substack{f\in 2^{\mathbf{2^{n-1}}}\\ f\neq \mathbf{0}}} \bigotimes_{\substack{s\in\mathbf{2^{n-1}}\\ f(s)=1}} \big(p_{i_k} \wedge (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\big)\Big)$

(Com$\wedge$, $\otimes$Sub)

$(3)\big(p_{i_k} \wedge \neg p_{i_k}\big) \vee$

$\bigvee_{\substack{f\in 2^{\mathbf{2^{n-1}}}\\ f\neq \mathbf{0}}} \Big(p_{i_k} \wedge \bigotimes_{\substack{s\in\mathbf{2^{n-1}}\\ f(s)=1}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big)$

(Dstr$^*\wedge\otimes\wedge$)

$(4)p_{i_k} \wedge \Big(\neg p_{i_k} \vee \bigvee_{\substack{f\in 2^{\mathbf{2^{n-1}}}\\ f\neq \mathbf{0}}} \bigotimes_{\substack{s\in\mathbf{2^{n-1}}\\ f(s)=1}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{k-1}}^{s(i_{k-1})} \wedge p_{i_{k+1}}^{s(i_{k+1})} \wedge \ldots p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big)$

(Dstr)

$(5)p_{i_k}$ $(\wedge\mathsf{E})$

Case $\phi(p_{i_1}, \ldots, p_{i_n}) = \neg p_{i_k}$ for $1 \leq k \leq n$. Similar to the above case.

Case $\phi = {=\!}(p_{i_{j_0}}, \cdots, p_{i_{j_k}})(p_{i_1}, \ldots, p_{i_n})$. If $N \setminus \{i_{j_k}\} = \emptyset$, we show that

$$={\!}(p_{i_{j_k}})(p_{i_{j_k}}) \dashv\vdash \big(p_{i_{j_k}} \wedge \neg p_{i_{j_k}}\big) \vee \big(p_{i_{j_k}} \wedge \mathsf{NE}\big) \vee \big(\neg p_{i_{j_k}} \wedge \mathsf{NE}\big).^9$$

We have the following derivations:

$$\begin{array}{ll}
(1) & =\!(p_{i_{j_k}})\\
(2) & p_{i_{j_k}} \vee \neg p_{i_{j_k}} \quad (\mathsf{DepE})\\
(3) & (p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee \mathsf{NE} \quad (\mathsf{NE\,I})\\
(4) & (p_{i_{j_k}} \vee \neg p_{i_{j_k}}) \wedge \big((p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee \mathsf{NE}\big) \quad ((2),(3),\wedge\mathsf{I})\\
(5) & \big((p_{i_{j_k}} \vee \neg p_{i_{j_k}}) \wedge (p_{i_{j_k}} \wedge \neg p_{i_{j_k}})\big) \vee ((p_{i_{j_k}} \vee \neg p_{i_{j_k}}) \wedge \mathsf{NE}) \quad (\mathsf{Dstr})\\
(6) & (p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee ((p_{i_{j_k}} \vee \neg p_{i_{j_k}}) \wedge \mathsf{NE}) \quad (\wedge\mathsf{E})\\
(7) & (p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee (p_{i_{j_k}} \wedge \mathsf{NE}) \vee (\neg p_{i_{j_k}} \wedge \mathsf{NE}) \quad (\mathsf{Dstr})
\end{array}$$

and

$$\begin{array}{ll}
(1) & (p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee (p_{i_{j_k}} \wedge \mathsf{NE}) \vee (\neg p_{i_{j_k}} \wedge \mathsf{NE})\\
(2) & p_{i_{j_k}} \vee p_{i_{j_k}} \vee \neg p_{i_{j_k}} \quad (\wedge\mathsf{E})\\
(3) & p_{i_{j_k}} \vee \neg p_{i_{j_k}} \quad (\vee\mathsf{E})\\
(4) & =\!(p_{i_{j_k}}) \quad (\mathsf{DepI})
\end{array}$$

Now, assume $N \setminus \{i_{j_k}\} \neq \emptyset$. We show that $=\!(p_{i_{j_0}}, \cdots, p_{i_{j_k}}) \dashv\vdash \theta$, where

$$\theta := (p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee \bigvee_{f \in 2^{\mathbf{2^k}}} \bigvee_{\substack{u \in 2^{\mathbf{2^{n-1}}}\\ u \neq \mathbf{0}}} \bigotimes_{\substack{s \in \mathbf{2^{n-1}}\\ u(s)=1}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE}),$$

where $\mathbf{2^k}$ is the maximal k-team on $K = \{i_{j_0}, \ldots, i_{j_{k-1}}\}$ and $\mathbf{2^{n-1}}$ is the maximal $(n-1)$-team on $N \setminus \{i_{j_k}\}$.

For $=\!(p_{i_{j_0}}, \cdots, p_{i_{j_k}}) \vdash \theta$, we have the following derivation:

$$(1)\ =\!(p_{i_{j_0}}, \cdots, p_{i_{j_k}})$$

$$(2)\ \bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)})$$

(derived using (DepE), (EM_0), ($\mathsf{Dstr}^* \wedge\otimes$) by the same argument as that in the proof of Theorem 4.3.8 [the $\mathbf{PD}^{[\vee]}$ case])

$$(3)\ (p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee \mathsf{NE} \quad (\mathsf{NE\,I})$$

$$(4)\ (p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee \Big(\mathsf{NE} \wedge \bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)})\Big)$$

[9] Note that $(p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee (p_{i_{j_k}} \wedge \mathsf{NE}) \vee (\neg p_{i_{j_k}} \wedge \mathsf{NE}) = \bigotimes\emptyset \vee (p_{i_{j_k}} \wedge \mathsf{NE}) \vee (\neg p_{i_{j_k}} \wedge \mathsf{NE})$.

((2), (3), $\wedge$I, Dstr)

$$(5)(p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee \bigvee_{f \in 2^{\mathbf{2^k}}} \Big(\mathsf{NE} \wedge \bigotimes_{s \in \mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)})\Big) \quad \text{(Dstr)}$$

$$(6)(p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee \bigvee_{f \in 2^{\mathbf{2^k}}} \bigvee_{\substack{u \in 2^{\mathbf{2^{n-1}}} \\ u \neq \mathbf{0}}} \bigotimes_{\substack{s \in \mathbf{2^{n-1}} \\ u(s)=1}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})$$

(Dstr $\mathsf{NE} \wedge \otimes$)

For the other direction $\theta \vdash {=}(p_{i_{j_0}}, \cdots, p_{i_{j_k}})$, we have the following derivation:

$(1)\theta$

$$(2)(p_{i_{j_k}} \wedge \neg p_{i_{j_k}}) \vee \bigvee_{f \in 2^{\mathbf{2^k}}} \bigvee_{\substack{u \in 2^{\mathbf{2^{n-1}}} \\ u \neq \mathbf{0}}} \bigotimes_{\substack{s \in \mathbf{2^{n-1}} \\ u(s)=1}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)}) \quad (\wedge\mathsf{E}, \otimes\mathsf{Sub})$$

$$(3)\Big(\bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes \emptyset\Big) \vee \bigvee_{f \in 2^{\mathbf{2^k}}} \bigvee_{\substack{u \in 2^{\mathbf{2^{n-1}}} \\ u \neq \mathbf{0}}} \bigotimes_{\substack{s \in \mathbf{2^{n-1}} \\ u(s)=1}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)}) \quad (\vee\mathsf{I})$$

$$(4) \bigvee_{f \in 2^{\mathbf{2^k}}} \bigvee_{u \in 2^{\mathbf{2^{n-1}}}} \bigotimes_{\substack{s \in \mathbf{2^{n-1}} \\ u(s)=1}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)})$$

$(\mathsf{Com}\vee, \mathsf{Ass}\vee)$

$$(5) \bigvee_{f \in 2^{\mathbf{2^k}}} \bigvee_{u \in 2^{\mathbf{2^{n-1}}}} \bigotimes_{s \in \mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)})$$

($\otimes$WI, since for each $u \in 2^{\mathbf{2^{n-1}}}$, $\{s \in \mathbf{2^{n-1}} \mid u(s) = 1\} \subseteq \mathbf{2^{n-1}}$)

$$(6) \bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^{n-1}}} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_{j_{k-1}}}^{s(i_{j_{k-1}})} \wedge p_{i_{j_k}}^{f(s\upharpoonright K)} \wedge p_{i_{j_{k+1}}}^{s(i_{j_{k+1}})} \wedge \cdots \wedge p_{i_n}^{s(i_n)}) \quad (\vee\mathsf{E})$$

$(7)\,{=}(p_{i_{j_0}}, \cdots, p_{i_{j_k}})$ (derived using ($\otimes$WE), (DepI) by the same argument as that in the proof of Theorem 4.3.8 [the $\mathbf{PD}^{[\vee]}$ case])

Case $\phi(p_{i_1},\dots,p_{i_n})=\psi(p_{i_1},\dots,p_{i_n})\vee\chi(p_{i_1},\dots,p_{i_n})$. By induction hypothesis, we have that

$$\begin{aligned}\psi &\dashv\vdash \bigvee_{f\in F}\bigotimes_{s\in X_f}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE}),\\ \chi &\dashv\vdash \bigvee_{g\in G}\bigotimes_{s\in X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE}),\end{aligned}\tag{4.16}$$

where each $X_f, X_g\subseteq\mathbf{2^n}$. If $F,G\neq\emptyset$, then by ($\vee$E) and ($\vee$I), we derive that

$$\psi\vee\chi\dashv\vdash\bigvee_{h\in F\cup G}\bigotimes_{s\in X_h}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE}).$$

If $F=\emptyset$ or $G=\emptyset$, then $\psi\dashv\vdash(p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}$ or $\chi\dashv\vdash(p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}$. In the former case, by ($\mathbf{0}$E) and ($\vee$I), we derive that

$$\begin{aligned}\psi\vee\chi &\dashv\vdash((p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE})\vee\Big(\bigvee_{g\in G}\bigotimes_{s\in X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)\\ &\dashv\vdash\bigvee_{f\in F}\bigotimes_{s\in X_f}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\end{aligned}$$

Similarly, in the latter case, we derive $\psi\vee\chi\dashv\vdash\bigvee_{g\in G}\bigotimes_{s\in X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})$.

Case $\phi(p_{i_1},\dots,p_{i_n})=\psi(p_{i_1},\dots,p_{i_n})\otimes\chi(p_{i_1},\dots,p_{i_n})$. By induction hypothesis, we have that (4.16) holds. If $F=\emptyset$ or $G=\emptyset$, then $\psi\dashv\vdash(p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}$ or $\chi\dashv\vdash(p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}$. By ($\mathbf{0}$Ctr) and (ex falso$^+$), we derive $\psi\otimes\chi\dashv\vdash(p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}$, i.e., $\psi\otimes\chi\dashv\vdash\bigvee\emptyset$.

Now, assume $F,G\neq\emptyset$. We show that $\psi\otimes\chi\dashv\vdash\theta$, where

$$\theta:=\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{s\in X_f\cup X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE}).$$

For the direction $\psi\otimes\chi\vdash\theta$, we have the following derivation:

(1) $\psi\otimes\chi$

(2) $\Big(\bigvee_{f\in F}\bigotimes_{s\in X_f}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)\otimes\Big(\bigvee_{g\in G}\bigotimes_{s\in X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)$

(3) $\bigvee_{f\in F}\bigvee_{g\in G}\left(\Big(\bigotimes_{s\in X_f}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)\otimes\Big(\bigotimes_{s\in X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)\right)$
(Dstr $\otimes\vee$)

(4) $\bigvee_{f\in F}\bigvee_{g\in G}\bigotimes_{s\in X_f\cup X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})$ ($\otimes$Cmb)

The other direction $\theta\vdash\psi\otimes\chi$ is proved symmetrically using ($\otimes$Dcp) and (Dstr$\otimes\vee\otimes$).

Case $\phi(p_{i_1},\dots,p_{i_n}) = \psi(p_{i_1},\dots,p_{i_n}) \wedge \chi(p_{i_1},\dots,p_{i_n})$. By induction hypothesis, we have that (4.16) holds. If $F = \emptyset$ or $G = \emptyset$, then $\psi \dashv\vdash (p_{i_1} \wedge \neg p_{i_1}) \wedge \mathsf{NE}$ or $\chi \dashv\vdash (p_{i_1} \wedge \neg p_{i_1}) \wedge \mathsf{NE}$. In this case, we derive $\psi \wedge \chi \dashv\vdash (p_i \wedge \neg p_i) \wedge \mathsf{NE}$, i.e., $\psi \wedge \chi \dashv\vdash \bigvee \emptyset$ by ($\wedge$E) and (ex falso$^+$).

Now, assume $F, G \neq \emptyset$. We show that $\psi \wedge \chi \dashv\vdash \theta$, where

$$\theta := \bigvee_{h\in H} \bigotimes_{s\in X_h} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE}),$$

$$\{X_f \mid f \in F\} \cap \{X_g \mid g \in G\} = \{X_h \mid h \in H\}.$$

For $\psi \wedge \chi \vdash \theta$, we have the following derivation:

$$(1) \psi \wedge \chi$$

$$(2) \Big(\bigvee_{f\in F} \bigotimes_{s\in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \wedge \Big(\bigvee_{g\in G} \bigotimes_{s\in X_g} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big)$$

$$(3) \bigvee_{f\in F} \bigvee_{g\in G} \left(\Big(\bigotimes_{s\in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \wedge \Big(\bigotimes_{s\in X_g} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \right)$$

(Dstr)

$$(4) \bigvee_{\substack{(f,g)\in F\times G\\ X_f = X_g}} \left(\Big(\bigotimes_{s\in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \wedge \Big(\bigotimes_{s\in X_g} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \right)$$

$$\vee \bigvee_{\substack{(f,g)\in F\times G\\ X_f \neq X_g}} \left(\Big(\bigotimes_{s\in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \wedge \Big(\bigotimes_{s\in X_g} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \right)$$

$$(5) \bigvee_{\substack{(f,g)\in F\times G\\ X_f = X_g}} \left(\Big(\bigotimes_{s\in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \wedge \Big(\bigotimes_{s\in X_g} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \right)$$

$$\vee \big((p_i \wedge \neg p_i) \wedge \mathsf{NE}\big) \quad (\mathbf{0}\mathsf{I})$$

$$(6) \bigvee_{\substack{(f,g)\in F\times G\\ X_f = X_g}} \left(\Big(\bigotimes_{s\in X_f} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \wedge \Big(\bigotimes_{s\in X_g} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \right)$$

($\mathbf{0}$E)

$$(7) \bigvee_{h\in H} \left(\Big(\bigotimes_{s\in X_h} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \wedge \Big(\bigotimes_{s\in X_h} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \right) \quad (\vee\mathsf{E})$$

$$(8) \bigvee_{h\in H} \bigotimes_{s\in X_h} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE}) \quad (\wedge\mathsf{E})$$

For the other direction $\theta \vdash \psi \wedge \chi$, we have the following derivation:

$$(1) \bigvee_{h\in H} \bigotimes_{s\in X_h} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})$$

$$(2)\Big(\bigvee_{h\in H}\bigotimes_{s\in X_h}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)\wedge\Big(\bigvee_{h\in H}\bigotimes_{s\in X_h}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)$$

$(\wedge\mathsf{I})$

$$(3)\Big(\bigvee_{f\in F}\bigotimes_{s\in X_f}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)\wedge\Big(\bigvee_{g\in G}\bigotimes_{s\in X_g}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})\Big)$$

$(\vee\mathsf{I}, H\subseteq F,G)$

$(4)\psi\wedge\chi$

■

Similar with Lemma 4.3.9 in the case of $\mathbf{PD}^{[\vee]}$, the following lemma is crucial in the proof of the completeness theorem.

Lemma 4.6.9. *For any finite non-empty collections of n-teams $\{X_f \mid f\in F\}$, $\{Y_g \mid g\in G\}$ with the same domain, the following are equivalent:*

(a) $\bigvee_{f\in F}\Theta^*_{X_f}\models\bigvee_{g\in G}\Theta^*_{Y_g}$;

(b) *for each $f\in F$, we have that $X_f = Y_{g_f}$ for some $g\in G$.*

Proof. (a)⇒(b): For each $f\in F$, by Lemma 4.6.2, we have that

$$X_f\models\Theta^*_{X_f},\text{ thus } X_f\models\bigvee_{f\in F}\Theta^*_{X_f},$$

which by (a) implies that

$$X_f\models\bigvee_{g\in G}\Theta^*_{Y_g},$$

thus, there exists $g_f\in G$ such that $X_f\models\Theta^*_{Y_{g_f}}$. Hence we obtain by Lemma 4.6.2 that $X_f = Y_{g_f}$.

(b)⇒(a): Suppose X is any n-team satisfying

$$X\models\bigvee_{f\in F}\Theta^*_{X_f}.$$

Then $X\models\Theta^*_{X_f}$ for some $f\in F$, which by Lemma 4.6.2 and (b) means that $X = X_f = Y_{g_f}$ for some $g_f\in G$. By Lemma 4.6.2, $Y_{g_f}\models\Theta^*_{Y_{g_f}}$, thus

$$X\models\Theta^*_{Y_{g_f}},\text{ thereby } X\models\bigvee_{g\in G}\Theta^*_{Y_g},$$

as required. ■

Now, we prove the completeness theorem for $\mathbf{PD}^{[\vee,\mathsf{NE}]}$.

Theorem 4.6.10 (Completeness Theorem)**.** *For any $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ formulas ϕ and ψ,*

$$\phi\models\psi\Longrightarrow\phi\vdash\psi.$$

Proof. Suppose $\phi \models \psi$, where $\phi = \phi(p_{i_1},\dots,p_{i_n})$ and $\psi = \psi(p_{i_1},\dots,p_{i_n})$. By Theorem 4.6.8, we have that

$$\phi \dashv\vdash \bigvee_{f\in F} \Theta^*_{X_f}, \quad \psi \dashv\vdash \bigvee_{g\in G} \Theta^*_{Y_g}$$

for some finite sets $\{X_f \mid f \in F\}$ and $\{Y_g \mid g \in G\}$ of n-teams on $\{i_1,\dots,i_n\}$. Then, by the Soundness theorem, we have that

$$\bigvee_{f\in F} \Theta^*_{X_f} \models \bigvee_{g\in G} \Theta^*_{Y_g}.$$

If $F = \emptyset$, then $\phi \dashv\vdash \bot \wedge \mathsf{NE}$, thus, by (ex falso$^+$), we obtain $\phi \vdash \psi$. If $G = \emptyset$, then $\psi \dashv\vdash \bot \wedge \mathsf{NE}$, thus we must have that $\phi \dashv\vdash \bot \wedge \mathsf{NE}$, hence $\phi \vdash \psi$.

If $F, G \neq \emptyset$, then by Lemma 4.6.9, for each $f \in F$, we have that $X_f = Y_{g_f}$ for some $g_f \in G$. Thus, we have the following derivation:

$$\begin{array}{lll} (1) & \Theta^*_{X_f} & \\ (2) & \Theta^*_{Y_{g_f}} & \\ (3) & \bigvee_{g\in G} \Theta^*_{Y_g} & (\vee\mathsf{I}) \end{array}$$

Thus,

$$\Theta^*_{X_f} \vdash \bigvee_{g\in G} \Theta^*_{Y_g}$$

for each $f \in F$, which by ($\vee$E) implies that

$$\bigvee_{f\in F} \Theta^*_{X_f} \vdash \bigvee_{g\in G} \Theta^*_{Y_g},$$

namely $\phi \vdash \psi$. ■

Theorem 4.6.11 (Strong Completeness Theorem)**.** *For any set* Γ *of formulas and any formula* ϕ *of* $\mathbf{PD}^{[\vee,\mathsf{NE}]}$,

$$\Gamma \models \phi \Longrightarrow \Gamma \vdash \phi.$$

Proof. By a similar argument with that in the proof of Theorem 3.1.10 in [11], we can prove that $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ is compact. Then the theorem follows from Theorem 4.6.10. ■

4.7 Axiomatizing propositional independence logic with non-empty atom

The method used in the axiomatization of propositional dependence logic (Section 4.4) combined with that of $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ (Section 4.6) can be generalized to axiomatize propositional independence logic extended with the non-empty atom, i.e. $\mathbf{PInd}^{[\mathsf{NE}]}$. In this section, we will give such an axiomatization.

An independence atom

$$p_{i_1}\dots p_{i_k} \perp p_{j_1}\dots p_{j_m}$$

is satisfied by a team X if and only if for any two valuations $s, s' \in X$, there must exists a valuation $s'' \in X$ which witnesses the independence of $\langle p_{i_1}, \ldots, p_{i_k}\rangle$ and $\langle p_{j_1}, \ldots, p_{j_m}\rangle$. Here, the set of witnesses for s and s' must be non-empty, therefore, essentially, there is a non-empty atom in the underlying team semantics of the independence atom. In view of this, an axiomatization for the logic $\mathbf{PInd}^{[\mathsf{NE}]}$ gives some insight on the logic **PInd**. However, the problem of how to axiomatize propositional independence logic alone is still open.

We now proceed to axiomatize the logic $\mathbf{PInd}^{[\mathsf{NE}]}$. The main argument is a generalization of those in Section 4.4 and Section 4.6. Essentially, we make use of the disjunctive normal form of the maximal logic $\mathbf{PD}^{[\vee,\mathsf{NE}]}$, but as intuitionistic disjunction is not an eligible connective in $\mathbf{PInd}^{[\mathsf{NE}]}$, we will only use intuitionistic disjunction implicitly. This is achieved by taking approximations of each formula, which allows us to essentially push intuitionistic disjunction to the front of a formula in one step. Technically, the non-empty atom NE does not abey the usual distributive law, so the definition of approximations of formulas is more sophisticated in this case than in the case of **PD**.

Let $\phi(p_{i_1}, \ldots, p_{i_n})$ be an n-formula of $\mathbf{PInd}^{[\mathsf{NE}]}$. Suppose the following are all the occurrences of all atomic or negated atomic formulas in ϕ:

$$\alpha_1, \ldots, \alpha_c$$

where each α_ξ ($1 \le \xi \le c$) can be of the following forms:

$$p_{i_\xi}, \quad \neg p_{i_\xi}, \quad \mathsf{NE}, \quad p_{j_1^\xi} \ldots p_{j_{a_\xi}^\xi} \perp p_{k_1^\xi} \ldots p_{k_{b_\xi}^\xi},$$

where $\{i_\xi, j_1^\xi, \ldots j_{a_\xi}^\xi, k_1^\xi, \ldots k_{b_\xi}^\xi\} \subseteq \{i_1, \ldots, i_n\}$. A *strong approximation sequence* $\Upsilon = \langle u_1, \ldots, u_c\rangle$ of $\phi(p_{i_1}, \ldots, p_{i_n})$ is a sequence such that

- if $\alpha_\xi = p_{i_\xi}$, then
$$u_\xi \subseteq \{s \in \mathbf{2^n} \mid s(i_\xi) = 1\},$$
where $\mathbf{2^n}$ is the maximal n-team on $\{i_1, \ldots, i_n\}$;
- if $\alpha_\xi = \neg p_{i_\xi}$, then
$$u_\xi \subseteq \{s \in \mathbf{2^n} \mid s(i_\xi) = 0\};$$
- if $\alpha_\xi = \mathsf{NE}$, then $\emptyset \neq u_\xi \subseteq \mathbf{2^n}$;
- if $\alpha_\xi = p_{j_1^\xi} \ldots p_{j_{a_\xi}^\xi} \perp p_{k_1^\xi} \ldots p_{k_{b_\xi}^\xi}$, then $u_\xi \subseteq \mathbf{2^n}$ such that for $A = \{j_1^\xi, \ldots, j_{a_\xi}^\xi\}$, $B = \{k_1^\xi, \ldots, k_{b_\xi}^\xi\}$, we have that
$$\{(s \upharpoonright A, s' \upharpoonright B) \mid s, s' \in u_\xi\} = \{(s'' \upharpoonright A, s'' \upharpoonright B) \mid s'' \in u_\xi\}.$$

For each such sequence $\Upsilon = \langle u_1, \ldots, u_c\rangle$, define an independence atom-free (classical) formula $\phi^\star_\Upsilon$ of $\mathbf{PInd}^{[\mathsf{NE}]}$, called a *strong approximation* of ϕ, by induction as follows:

- for any atomic or negated atomic formula α_ξ ($1 \le \xi \le c$),
$$(\alpha_\xi)^\star_{\langle u_\xi\rangle} := \bigotimes_{s \in u_\xi} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE}) = \Theta^*_{u_\xi}$$

- $(\psi\otimes\chi)^\star_\Upsilon := \psi^\star_{\Upsilon_0}\otimes\chi^\star_{\Upsilon_1}$, where Υ_0 and Υ_1 are subsequences of Υ consisting of all the u_ξ's such that the atoms α_ξ occur in ψ and χ, respectively;
- $(\psi\wedge\chi)^\star_\Upsilon := \psi^\star_{\Upsilon_0}\wedge\chi^\star_{\Upsilon_1}$, where Υ_0 and Υ_1 are as above.

Next, we show that the every formula of $\mathbf{PInd}^{[\mathsf{NE}]}$ is logically equivalent to the intuitionistic disjunction of all its strong approximations.

Lemma 4.7.1. *Let $\phi(p_{i_1},\dots,p_{i_n})$ be an n-formula of $\mathbf{PInd}^{[\mathsf{NE}]}$ and Γ the set of all its approximation sequences. Then*

$$\phi\equiv\bigvee_{\Upsilon\in\Gamma}\phi^\star_\Upsilon.$$

Proof. We first show that for each $\Upsilon\in\Gamma$ with $\Upsilon=\langle u_1,\dots,u_c\rangle$, $\phi^*_\Upsilon\models\phi$. Put $N=\{i_1,\dots,i_n\}$. Assume X is an n-team on N such that $X\models\phi^\star_\Upsilon$. We show by induction on ϕ that $X\models\phi$.

The only interesting case is the case that $\phi=\alpha_\xi$ $(1\leq\xi\leq c)$ an atomic or negated atomic formula. In this case, $\Upsilon=\langle u_\xi\rangle$ and $(\alpha_\xi)^\star_{\langle u_\xi\rangle}=\Theta^*_{u_\xi}$. By Lemma 4.6.2, we know that $X\models\Theta^*_{u_\xi}$ implying $X=u_\xi$.

If $\alpha_\xi=p_{i_\xi}$, then as

$$u_\xi\subseteq\{s\in\mathbf{2^n}\mid s(i_\xi)=1\},$$

where $\mathbf{2^n}$ is the maximal n-team on N, we obtain that $u_\xi\models p_{i_\xi}$, i.e., $X\models p_{i_\xi}$. By a similar argument, we can prove that $X\models\neg p_{i_\xi}$ in case $\alpha_\xi=\neg p_{i_\xi}$.

If $\alpha_\xi=\mathsf{NE}$, then $X=u_\xi\neq\emptyset$, thus $X\models\mathsf{NE}$.

If $\phi:=p_{j^\xi_1}\dots p_{j^\xi_{a_\xi}}\perp p_{k^\xi_1}\dots p_{k^\xi_{b_\xi}}$, then by definition,

$$\{(s\upharpoonright A,s'\upharpoonright B)\mid s,s'\in u_\xi\}=\{(s''\upharpoonright A,s''\upharpoonright B)\mid s''\in u_\xi\},$$

where $A=\{j^\xi_1,\dots,j^\xi_{a_\xi}\}$, $B=\{k^\xi_1,\dots,k^\xi_{b_\xi}\}$. Thus, for any $s,s'\in X=u_\xi$, there exists $s''\in u_\xi=X$ such that

$$s''\upharpoonright A=s\upharpoonright A\text{ and }s''\upharpoonright B=s'\upharpoonright B.$$

This means that $X\models p_{j^\xi_1}\dots p_{j^\xi_{a_\xi}}\perp p_{k^\xi_1}\dots p_{k^\xi_{b_\xi}}$, as required.

Next, we show that $\phi\models\bigvee_{\Upsilon\in\Gamma}\phi^*_\Upsilon$. Assume X is an n-team on N such that $X\models\phi$. We show by induction on ϕ that $X\models\bigvee_{\Upsilon\in\Gamma}\phi^*_\Upsilon$.

If $\phi=\alpha_\xi$ $(1\leq\xi\leq c)$ an atomic or negated atomic formula, then for each $\Upsilon\in\Gamma$, $\Upsilon=\langle u_\xi\rangle$ and $(\alpha_\xi)^\star_{\langle u_\xi\rangle}=\Theta^*_{u_\xi}$. In view of Lemma 4.6.2, to show that $X\models\bigvee_{\langle u_\xi\rangle\in\Gamma}\Theta^*_{u_\xi}$ it suffices to show that $X=u_\xi$ for some $\langle u_\xi\rangle\in\Gamma$, namely to show that $\langle X\rangle$ is a strong approximation sequence of α_ξ.

If $\alpha_\xi=p_{i_\xi}$, then $X\models p_{i_\xi}$ implies that for any $s\in X\subseteq\mathbf{2^n}$, $s(i_\xi)=1$, thus

$$X\subseteq\{s\in\mathbf{2^n}\mid s(i_\xi)=1\}.$$

This means by definition that $\langle X\rangle$ is a strong approximation sequence of p_{i_ξ}, as required. The case $\alpha_\xi=\neg p_{i_\xi}$ is proved similarly.

If $\alpha_\xi = \mathsf{NE}$, then $X \models \mathsf{NE}$ implies that $\emptyset \neq X \subseteq \mathbf{2^n}$. This means by definition that $\langle X \rangle$ is a strong approximation sequence of NE, as required.

If $\alpha_\xi = p_{j_1^\xi} \dots p_{j_{a_\xi}^\xi} \perp p_{k_1^\xi} \dots p_{k_{b_\xi}^\xi}$, by definition, it suffices to show that

$$\{(s \upharpoonright A, s' \upharpoonright B) \mid s, s' \in X\} = \{(s'' \upharpoonright A, s'' \upharpoonright B) \mid s'' \in X\}.$$

The direction "$\supseteq$" is trivial, and the direction "$\subseteq$" follows easily from the fact that $X \models p_{j_1^\xi} \dots p_{j_{a_\xi}^\xi} \perp p_{k_1^\xi} \dots p_{k_{b_\xi}^\xi}$.

The induction cases are proved by a similar argument to that in the proof of Lemma 4.4.3. ■

As in the case of **PD**, we can view $\bigvee_{\Upsilon \in \Gamma} \phi^\star_\Upsilon$ or the sequence

$$\langle \phi^\star_\Upsilon \rangle_{\Upsilon \in \Lambda}$$

as a *weak normal form* for formulas of $\mathbf{PInd}^{[\mathsf{NE}]}$. We now define a natural deduction system for $\mathbf{PInd}^{[\mathsf{NE}]}$ which will enable us to derive *in effect* the weak normal form $\bigvee_{\Upsilon \in \Gamma} \phi^\star_\Upsilon$ for every formula ϕ.

Definition 4.7.2 (A natural deduction system for $\mathbf{PInd}^{[\mathsf{NE}]}$)**.** The rules are given as follows:

1. The rules ($\wedge$I), ($\wedge$E), ($\otimes$WI), ($\otimes$WE), ($\otimes$Rpt), ($\otimes$Sub), (Ass $\otimes$), (Com $\otimes$), (ex falso$^+$), ($\bot$E), (EM$_0$), (**0**I), (**0**Ctr) as in Definition 4.3.5 and Definition 4.6.5.

2. Independence Atom Strong Introduction: For any strong approximation sequence $\langle u \rangle$ of $p_{j_1} \dots p_{j_a} \perp p_{k_1} \dots p_{k_b} (p_{i_1}, \dots, p_{i_n})$,

$$\dfrac{\bigotimes_{s \in u} (p_{i_1}^{s(i_1)} \wedge \dots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})}{p_{j_1} \dots p_{j_a} \perp p_{k_1} \dots p_{k_b}} \ (\mathsf{IndSI})$$

3. Strong Approximation Transition:

$$\dfrac{\begin{matrix}[\phi^\star_{\Upsilon_0}] \\ \vdots \\ \theta\end{matrix} \quad \dots \quad \dots \quad \begin{matrix}[\phi^\star_{\Upsilon_m}] \\ \vdots \\ \theta\end{matrix} \quad \phi}{\theta} \ (\mathsf{SApTr})$$

where $\{\Upsilon_0, \dots, \Upsilon_m\}$ is the set of all strong approximation sequences of ϕ.

Next, we prove the Soundness Theorem for the above system.

Theorem 4.7.3 (Soundness Theorem)**.** *For any* $\mathbf{PInd}^{[\mathsf{NE}]}$ *formulas* ϕ *and* ψ*,*

$$\phi \vdash \psi \Longrightarrow \phi \models \psi.$$

Proof. It suffices to show that all of the deductive rules are valid. The validity of (IndSI) and (SApTr) follows from Lemma 4.7.1, and the validity of all the other rules follows from the proofs of Theorem 4.3.7 and Theorem 4.6.7. ■

Interesting derivable rules of the system are listed as follows.

Corollary 4.7.4. *The following are derivable rules:*

1. *Rules (Dstr$\otimes\wedge$), (Dstr$^*\wedge\otimes\wedge$), ($\otimes$Cmb), ($\otimes$Dcp), (Com $\wedge$), (Ass $\wedge$) and ($\wedge$Sub), as in Corollary 4.3.6 and Corollary 4.6.6 (see also Appendix).*
2. *Independence Atom Weak Elimination:*

$$\dfrac{\begin{array}{c}\forall\langle u\rangle\\ [(p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b})^{\star}_{\langle u\rangle}]\\ \vdots\\ \theta\end{array} \qquad p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b}}{\theta}\ \textsf{(IndWE)}$$

where $\langle u\rangle$ is a strong approximation sequences of $p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b}$.

3. *Strong Approximation Elimination:*

$$\dfrac{\phi^{\star}_{\Upsilon}}{\phi}\ \textsf{(SApE)}$$

where Υ is any strong approximation sequence of $\phi(p_{i_1},\dots,p_{i_n})$.

Proof. The rules in Item 1 are derived in the same way as in Corollary 4.6.6. Rule (IndWE) is a special case of rule (ApWE).

We now proceed to derive the rule (SApE) by induction on ϕ. The case that ϕ is an independence atom follows from (IndSI).

If $\phi = p_{i_j}$, then we have the following derivation:

$$\begin{array}{l}(1)(p_{i_j})^{\star}_{\langle u\rangle}\\ (2)\bigotimes_{s\in u}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}\wedge \textsf{NE})\\ (3)\bigotimes_{s\in u}p_{i_j}\quad(\wedge\textsf{E},\otimes\textsf{Sub})\\ (4)p_{i_j}\quad(\otimes\textsf{WE})\end{array}$$

The case $\phi = \neg p_{i_j}$ is proved similarly with the above case.

All the other cases are proved by a similar argument to that of the proof of Corollary 4.4.6. ■

As in the case of **PD**, rules (SApTr), (SApE) imply *in effect* that for any n-formula ϕ of $\mathbf{PInd}^{[\mathsf{NE}]}$,

$$\bigvee_{\Upsilon\in\Gamma} \phi^{\star}_{\Upsilon} \dashv\vdash \phi,$$

where Γ is the set of all strong approximation sequences of ϕ, although the formula $\bigvee_{\Upsilon\in\Gamma} \phi^{\star}_{\Upsilon}$ is not in the language of $\mathbf{PInd}^{[\mathsf{NE}]}$. The complicated rule (Dstr $\mathsf{NE}\wedge\otimes$) (in Definition 4.6.5 or see Appendix) for the NE atom indicates that there is little hope to manipulate the NE atom without having the intuitionistic disjunction in the logic. Therefore in proving the completeness theorem for $\mathbf{PInd}^{[\mathsf{NE}]}$, we will turn every $\phi^{\star}_{\Upsilon}$ formula into a formula $\Theta^{*}_{X_\Upsilon}$ in a better form. Now, we derive this in the deductive system. To simplify notations, we abbreviate $\Theta^{*}_{\mathbf{0}} := (p_{i_1} \wedge \neg p_{i_1}) \wedge \mathsf{NE}$.

Lemma 4.7.5. *Let $\phi(p_{i_1},\dots,p_{i_n})$ be an n-formula of $\mathbf{PInd}^{[\mathsf{NE}]}$. For any strong approximation sequence $\Upsilon = \langle u_1,\dots,u_c\rangle$ of ϕ, we have that*

$$\phi^{\star}_{\Upsilon} \dashv\vdash \Theta^{*}_{X_\Upsilon},$$

where X_Υ is an n-team on $\{i_1,\dots,i_n\}$ or $X_\Upsilon = \mathbf{0}$.

Proof. We prove the lemma by induction on ϕ.

In case $\phi = \alpha_\xi$ $(1 \le \xi \le c)$ an atomic or negated atomic formula, $\Upsilon = \langle u_\xi\rangle$ and $(\alpha_\xi)^{\star}_{\langle u_\xi\rangle} = \Theta^{*}_{u_\xi}$, thus $(\alpha_\xi)^{\star}_{\Upsilon} \dashv\vdash \Theta^{*}_{X_\Upsilon}$ holds trivially.

Case $\phi := \psi\otimes\chi$. By induction hypothesis, we have that

$$\psi^{\star}_{\Upsilon_0} \dashv\vdash \Theta^{*}_{X_{\Upsilon_0}} \text{ and } \chi^{\star}_{\Upsilon_0} \dashv\vdash \Theta^{*}_{X_{\Upsilon_0}}. \tag{4.17}$$

Since by ($\otimes$Sub),

$$\phi^{\star}_{\Upsilon} \dashv\vdash \psi^{\star}_{\Upsilon_0}\otimes\chi^{\star}_{\Upsilon_1} \dashv\vdash \Theta^{*}_{X_{\Upsilon_0}}\otimes\Theta^{*}_{X_{\Upsilon_1}},$$

it suffices to show that $\Theta^{*}_{X_{\Upsilon_0}}\otimes\Theta^{*}_{X_{\Upsilon_1}} \dashv\vdash \Theta^{*}_{X_\Upsilon}$ for some X_Υ.

If $X_{\Upsilon_0} = \mathbf{0}$, then taking $X_\Upsilon = \mathbf{0}$, we derive $\Theta^{*}_{\mathbf{0}} \vdash \Theta^{*}_{X_{\Upsilon_0}}\otimes\Theta^{*}_{X_{\Upsilon_1}}$ by (ex falso$^+$) and $\Theta^{*}_{X_{\Upsilon_0}}\otimes\Theta^{*}_{X_{\Upsilon_1}} \vdash \Theta^{*}_{\mathbf{0}}$ by ($\mathbf{0}$Ctr). The case $X_{\Upsilon_1} = \mathbf{0}$ is proved similarly. If $X_{\Upsilon_0}, X_{\Upsilon_1} \neq \mathbf{0}$, then by ($\otimes$Cmb) and ($\otimes$Dcp), we derive $\Theta^{*}_{X_{\Upsilon_0}}\otimes\Theta^{*}_{X_{\Upsilon_1}} \dashv\vdash \Theta^{*}_{X_{\Upsilon_0}\cup X_{\Upsilon_1}}$.

Case $\phi := \psi\wedge\chi$. By induction hypothesis, we have that (4.17) holds. Since by ($\wedge$Sub),

$$\phi^{\star}_{\Upsilon} \dashv\vdash \psi^{\star}_{\Upsilon_0}\wedge\chi^{\star}_{\Upsilon_1} \dashv\vdash \Theta^{*}_{X_{\Upsilon_0}}\wedge\Theta^{*}_{X_{\Upsilon_1}},$$

it suffices to show that $\Theta^{*}_{X_{\Upsilon_0}}\wedge\Theta^{*}_{X_{\Upsilon_1}} \dashv\vdash \Theta^{*}_{X_\Upsilon}$ for some X_Υ.

If $X_{\Upsilon_0} = \mathbf{0}$, then taking $X_\Upsilon = \mathbf{0}$, we derive $\Theta^{*}_{\mathbf{0}} \vdash \Theta^{*}_{X_{\Upsilon_0}}\wedge\Theta^{*}_{X_{\Upsilon_1}}$ by (ex falso$^+$) and $\Theta^{*}_{X_{\Upsilon_0}}\wedge\Theta^{*}_{X_{\Upsilon_1}} \vdash \Theta^{*}_{\mathbf{0}}$ by ($\wedge$E). The case that $X_{\Upsilon_1} = \mathbf{0}$ is proved similarly. If $X_{\Upsilon_0} = X_{\Upsilon_1} \neq \mathbf{0}$, then by ($\wedge$E) and ($\wedge$I), we derive $\Theta^{*}_{X_{\Upsilon_0}}\wedge\Theta^{*}_{X_{\Upsilon_1}} \dashv\vdash \Theta^{*}_{X_{\Upsilon_0}}$. If $X_{\Upsilon_0}, X_{\Upsilon_1} \neq \mathbf{0}$ and $X_{\Upsilon_0} \neq X_{\Upsilon_1}$, then we derive $\Theta^{*}_{\mathbf{0}} \vdash \Theta^{*}_{X_{\Upsilon_0}}\wedge\Theta^{*}_{X_{\Upsilon_1}}$ by (ex falso$^+$) and $\Theta^{*}_{X_{\Upsilon_0}}\wedge\Theta^{*}_{X_{\Upsilon_1}} \vdash \Theta^{*}_{\mathbf{0}}$ by ($\mathbf{0}$I). ■

Corollary 4.7.6. *Let Γ be the (non-empty) set of all strong approximation sequences of an n-formula $\phi(p_{i_1},\dots,p_{i_n})$ of $\mathbf{PInd}^{[\mathsf{NE}]}$. Then*

$$\phi \equiv \bigvee_{\Upsilon\in\Gamma} \Theta^{*}_{X_\Upsilon} \equiv \bigvee_{\Upsilon\in\Gamma_0} \Theta^{*}_{X_\Upsilon},$$

where each X_Υ is an n-team on $\{i_1,\dots,i_n\}$ or $X_\Upsilon=\mathbf{0}$, and

$$\Gamma_0=\{\Upsilon\in\Gamma\mid X_\Upsilon\neq\mathbf{0}\}.$$

Proof. It follows from Lemma 4.7.1, Lemma 4.7.5 and Soundness Theorem that

$$\phi\equiv\bigvee_{\Upsilon\in\Gamma}\phi^\star_\Upsilon\equiv\bigvee_{\Upsilon\in\Gamma}\Theta^*_{X_\Upsilon},$$

for some X_Υ such that $\phi^\star_\Upsilon\dashv\vdash\Theta^*_{X_\Upsilon}$.

Moreover, for each $\Upsilon\in\Gamma$ such that $X_\Upsilon=\mathbf{0}$, by definition,

$$\Theta^*_{X_\Upsilon}=\Theta^*_{\mathbf{0}}=(p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}.$$

Since for any formula ψ, $\big((p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}\big)\vee\psi\equiv\psi$, we obtain that $\bigvee_{\Upsilon\in\Gamma}\Theta^*_{X_\Upsilon}\equiv\bigvee_{\Upsilon\in\Gamma_0}\Theta^*_{X_\Upsilon}$, as required. ■

Now, we are in a position to prove the completeness theorem for $\mathbf{PInd}^{[\mathsf{NE}]}$. The proof is similar with that of Theorem 4.4.8.

Theorem 4.7.7 (Completeness Theorem)**.** *For any* $\mathbf{PInd}^{[\mathsf{NE}]}$ *formulas ϕ and ψ,*

$$\phi\models\psi\Longrightarrow\phi\vdash\psi.$$

Proof. Let $\phi(p_{i_1},\dots,p_{i_n})$ and $\psi(p_{i_1},\dots,p_{i_n})$. Suppose $\phi\models\psi$. By Corollary 4.7.6, we have that

$$\phi\equiv\bigvee_{\Upsilon\in\Gamma}\Theta^*_{X_\Upsilon}\equiv\bigvee_{\Upsilon\in\Gamma_0}\Theta^*_{X_\Upsilon}\text{ and }\psi\equiv\bigvee_{\Delta\in\Gamma'}\Theta^*_{X_\Delta}\equiv\bigvee_{\Delta\in\Gamma'_0}\Theta^*_{X_\Delta}.$$

where

(i) Γ, Γ' are the (non-empty) sets of all strong approximation sequences of ϕ and ψ, respectively;

(ii) $\phi^\star_\Upsilon\dashv\vdash\Theta^*_{X_\Upsilon}$ and $\psi^\star_\Delta\dashv\vdash\Theta^*_{X_\Delta}$ for all $\Upsilon\in\Gamma$ and $\Delta\in\Gamma'$;

(iii) $\Gamma_0=\{\Upsilon\in\Gamma\mid X_\Upsilon\neq\mathbf{0}\}$ and $\Gamma'_0=\{\Upsilon\in\Gamma'\mid X_\Upsilon\neq\mathbf{0}\}$.

To derive $\phi\vdash\psi$, by (SApTr) and (ii), it suffices to derive that for each $\Upsilon\in\Gamma$, $\Theta^*_{X_\Upsilon}\vdash\psi$. If $X_\Upsilon=\mathbf{0}$, then $\Theta^*_{X_\Upsilon}=\Theta^*_{\mathbf{0}}=(p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}$, thus we derive $\Theta^*_{\mathbf{0}}\vdash\psi$ by (ex falso$^+$).

If $X_\Upsilon\neq\mathbf{0}$, then $\Gamma_0\neq\emptyset$ and $\bigvee_{\Upsilon\in\Gamma_0}\Theta^*_{X_\Upsilon}\not\equiv(p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}$. Noting that by assumption,

$$\bigvee_{\Upsilon\in\Gamma_0}\Theta^*_{X_\Upsilon}\models\bigvee_{\Delta\in\Gamma'_0}\Theta^*_{X_\Delta},$$

we must have that $\bigvee_{\Delta\in\Gamma'_0}\Theta^*_{X_\Delta}\not\equiv(p_{i_1}\wedge\neg p_{i_1})\wedge\mathsf{NE}$, thereby $\Gamma'_0\neq\emptyset$. Now, by Lemma 4.6.9, there exists $\Delta\in\Gamma'$ such that $X_\Upsilon=X_\Delta$. Thus, we derive $\Theta^*_{X_\Upsilon}\vdash\psi$ as follows:

(1) $\Theta^*_{X_\Upsilon}$
(2) $\Theta^*_{X_\Delta}$ (since $X_\Upsilon=X_\Delta$)
(3) ψ^*_Δ (by Lemma 4.7.5)
(4) ψ (SApE).

This completes the proof. ■

We did not use Atomic Excluded Middle Rule (EM_0) in the proof of the above theorem, but this rule is required in the proof of the (Weak) Completeness Theorem: for any formula ϕ of $\mathbf{PInd}^{[\mathsf{NE}]}$,

$$\models \phi \implies \vdash \phi$$

(see Corollary 4.3.11).

Theorem 4.7.8 (Strong Completeness Theorem)**.** *For any set* Γ *of formulas and any formula* ϕ *of* $\mathbf{PInd}^{[\mathsf{NE}]}$*,*

$$\Gamma \models \phi \implies \Gamma \vdash \phi.$$

Proof. Follows from Theorem 4.7.7 and Compactness Theorem of $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ (as $\mathbf{PInd}^{[\mathsf{NE}]}$ is clearly a sublogic of the maximal logic $\mathbf{PD}^{[\vee,\mathsf{NE}]}$). ■

We end this section with an application of the natural deduction system of $\mathbf{PInd}^{[\mathsf{NE}]}$. In the following example, we derive the Geiger-Paz-Pearl's axioms [37] which axiomatized the implication problem for unconditional independence atoms (see Section 1.2).

Example 4.7.9. The following Geiger-Paz-Pearl's axioms are derivable in $\mathbf{PInd}^{[\mathsf{NE}]}$: Let $\bar{x} = p_{j_1} \cdots p_{j_a}$, $\bar{y} = p_{k_1} \cdots p_{k_b}$ and $\bar{z} = p_{i_1} \cdots p_{i_c}$.

(i) $\bar{x} \perp \bar{y} \vdash \bar{y} \perp \bar{x}$

(ii) $\bar{x} \perp \bar{y} \vdash \bar{z} \perp \bar{y}$, where $\bar{z}$ is a subsequence of $\bar{x}$.

(iii) $\bar{x} \perp \bar{y} \vdash \bar{u} \perp \bar{v}$, where $\bar{u} = p_{i_1} \cdots p_{i_a}$ is a permutation of $\bar{x}$ and $\bar{v} = p_{m_1} \cdots p_{m_b}$ is a permutation of $\bar{y}$.

(iv) $\{\bar{x} \perp \bar{y},\ \bar{x}\bar{y} \perp \bar{z}\} \vdash \bar{x} \perp \bar{y}\bar{z}$.

Proof. Put $A = \{j_1, \ldots, j_a\}$, $B = \{k_1, \ldots, k_b\}$ and $C = \{i_1, \ldots, i_c\}$. The derivations are as follows:

(i)

$$\dfrac{\dfrac{\overset{\forall\langle u\rangle}{\left[\bigotimes_{s\in u}(p_{j_1}^{s(j_1)} \wedge \cdots \wedge p_{j_a}^{s(j_a)} \wedge p_{k_1}^{s(k_1)} \wedge \cdots \wedge p_{k_b}^{s(k_b)} \wedge \mathsf{NE})\right]}}{p_{k_1} \ldots p_{k_b} \perp p_{j_1} \ldots p_{j_a}}\,(\mathsf{IndSI}) \qquad p_{j_1} \ldots p_{j_a} \perp p_{k_1} \ldots p_{k_b}}{p_{k_1} \ldots p_{k_b} \perp p_{j_1} \ldots p_{j_a}}\,(\mathsf{IndWE})$$

where (IndSI) is applicable because the strong approximation sequence $\langle u\rangle$ of the atom $p_{j_1} \ldots p_{j_a} \perp p_{k_1} \ldots p_{k_b}$ is clearly also a strong approximation sequence of the atom $p_{k_1} \ldots p_{k_b} \perp p_{j_1} \ldots p_{j_a}$.

(ii)

$$\dfrac{\dfrac{\dfrac{\overset{\forall\langle u\rangle}{\left[\bigotimes_{s\in u}(p_{j_1}^{s(j_1)} \wedge \cdots \wedge p_{j_a}^{s(j_a)} \wedge p_{k_1}^{s(k_1)} \wedge \cdots \wedge p_{k_b}^{s(k_b)} \wedge \mathsf{NE})\right]}}{\bigotimes_{s\in u}(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_c}^{s(i_c)} \wedge p_{k_1}^{s(k_1)} \wedge \cdots \wedge p_{k_b}^{s(k_b)} \wedge \mathsf{NE})}\,(\wedge\mathsf{E}, \otimes\mathsf{Sub})}{p_{i_1} \ldots p_{i_c} \perp p_{k_1} \ldots p_{k_b}}\,(\mathsf{IndSI}) \qquad p_{j_1} \ldots p_{j_a} \perp p_{k_1} \ldots p_{k_b}}{p_{i_1} \ldots p_{i_c} \perp p_{k_1} \ldots p_{k_b}}\,(\mathsf{IndWE})$$

where (IndSI) is applicable because $\langle u\rangle$ is a strong approximation sequence of $p_{i_1}\dots p_{i_c}\perp p_{k_1}\dots p_{k_b}$.

Indeed, since $\langle u\rangle$ is a strong approximation sequence of $p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b}$,

$$\{(s\upharpoonright A, s'\upharpoonright B)\mid s,s'\in u\}=\{(s''\upharpoonright A, s''\upharpoonright B)\mid s''\in u\}. \tag{4.18}$$

Since $C\subseteq A$, it follows that

$$\{(s\upharpoonright C, s'\upharpoonright B)\mid s,s'\in u\}=\{(s''\upharpoonright C, s''\upharpoonright B)\mid s''\in u\},$$

as required.

(iii)

$$\dfrac{\dfrac{\forall\langle u\rangle\quad \left[\bigotimes_{s\in u}(p_{j_1}^{s(j_1)}\wedge\dots\wedge p_{j_a}^{s(j_a)}\wedge p_{k_1}^{s(k_1)}\wedge\dots\wedge p_{k_b}^{s(k_b)}\wedge\mathsf{NE})\right]}{p_{i_1}\dots p_{i_a}\perp p_{m_1}\dots p_{m_b}}\,(\mathsf{IndSI})\qquad p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b}}{p_{i_1}\dots p_{i_a}\perp p_{m_1}\dots p_{m_b}}\,(\mathsf{IndWE})$$

where (IndSI) is applicable because $A=\{i_1,\dots,i_a\}$, $B=\{m_1,\dots,m_b\}$ and the strong approximation sequence $\langle u\rangle$ of $p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b}$ is obviously also a strong approximation sequence of $p_{i_1}\dots p_{i_a}\perp p_{m_1}\dots p_{m_b}$.

(iv) For each approximation sequence $\langle u_1,u_2\rangle$ of

$$(\alpha\wedge\beta)(p_{m_1},\dots,p_{m_d})=\\ (p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b})\wedge(p_{j_1}\dots p_{j_a}p_{k_1}\dots p_{k_b}\perp p_{i_1}\dots p_{i_c}),$$

we derive

$$(\alpha)^*_{\langle u_1\rangle}\wedge(\beta)^\star_{\langle u_2\rangle}\vdash p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b}p_{i_1}\dots p_{i_c} \tag{4.19}$$

as follows. If $u_1\neq u_2$, then we have the following derivation:

$$\dfrac{\dfrac{\dfrac{(\alpha)^*_{\langle u_1\rangle}\wedge(\beta)^\star_{\langle u_2\rangle}}{\big(\bigotimes_{s_1\in u_1}(p_{m_1}^{s_1(m_1)}\wedge\dots\wedge p_{m_d}^{s_1(m_d)}\wedge\mathsf{NE})\big)\wedge\big(\bigotimes_{s_2\in u_2}(p_{m_1}^{s_2(m_1)}\wedge\dots\wedge p_{m_d}^{s_2(m_d)}\wedge\mathsf{NE})\big)}}{(p_{m_1}\wedge\neg p_{m_1})\wedge\mathsf{NE}}\,(\mathbf{0}\mathsf{I})}{p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b}p_{i_1}\dots p_{i_c}}\,(\text{ex falso}^+)$$

If $u_1=u_2$, then we have the following derivation:

$$\dfrac{\dfrac{\dfrac{(\alpha)^*_{\langle u_1\rangle}\wedge(\beta)^\star_{\langle u_2\rangle}}{\big(\bigotimes_{s_1\in u_1}(p_{m_1}^{s_1(m_1)}\wedge\dots\wedge p_{m_d}^{s_1(m_d)}\wedge\mathsf{NE})\big)\wedge\big(\bigotimes_{s_2\in u_2}(p_{m_1}^{s_2(m_1)}\wedge\dots\wedge p_{m_d}^{s_2(m_d)}\wedge\mathsf{NE})\big)}}{\bigotimes_{s_1\in u_1}(p_{m_1}^{s_1(m_1)}\wedge\dots\wedge p_{m_d}^{s_1(m_d)}\wedge\mathsf{NE})}\,(\wedge\mathsf{E})}{p_{j_1}\dots p_{j_a}\perp p_{k_1}\dots p_{k_b}p_{i_1}\dots p_{i_c}}\,(\mathsf{IndSI})$$

where (IndSI) is applicable because $\langle u_1 \rangle$ is a strong approximation sequence of $p_{j_1} \dots p_{j_a} \perp p_{k_1} \dots p_{k_b} p_{i_1} \dots p_{i_c}$.
Indeed, since $\langle u_1, u_2 \rangle$ is an approximation sequence for $\alpha \wedge \beta$, we have (4.18) holds for u_1 and

$$\{(s \upharpoonright (A \cup B), s' \upharpoonright C) \mid s, s' \in u_2\} = \{(s'' \upharpoonright (A \cup B), s'' \upharpoonright C) \mid s'' \in u_2\}.$$

Thus, for any $s, s' \in u_1 = u_2$, there exists $s'' \in u_1 = u_2$ such that

$$(s \upharpoonright A, s' \upharpoonright B) = (s'' \upharpoonright A, s'' \upharpoonright B).$$

Moreover, for $s'', s' \in u_2 = u_1$, there exists $s''' \in u_2 = u_1$ such that

$$(s'' \upharpoonright (A \cup B), s' \upharpoonright C) = (s''' \upharpoonright (A \cup B), s''' \upharpoonright C).$$

It then follows that

$$\begin{aligned}(s \upharpoonright A, s' \upharpoonright (B \cup C)) =& (s'' \upharpoonright A, (s'' \upharpoonright B)^\frown (s' \upharpoonright C))\\ =& (s''' \upharpoonright A, (s''' \upharpoonright B)^\frown (s''' \upharpoonright C))\\ =& (s''' \upharpoonright A, s''' \upharpoonright (B \cup C)),\end{aligned}$$

which implies that

$$\{(s \upharpoonright A, s' \upharpoonright (B \cup C)) \mid s, s' \in u_1\} = \{(s''' \upharpoonright A, s''' \upharpoonright (B \cup C)) \mid s''' \in u_1\},$$

as required.
Now, we derive the axiom of item (iv) as follows:

$$\cfrac{\cfrac{\begin{array}{c}\forall \langle u_1, u_2 \rangle\\ (\alpha)^*_{\langle u_1 \rangle} \wedge (\beta)^\star_{\langle u_2 \rangle}\end{array}}{p_{j_1} \dots p_{j_a} \perp p_{k_1} \dots p_{k_b} p_{i_1} \dots p_{i_c}} \text{(by (4.19))} \quad \cfrac{\alpha \quad \beta}{\alpha \wedge \beta} (\wedge \mathsf{I})}{p_{j_1} \dots p_{j_a} \perp p_{k_1} \dots p_{k_b} p_{i_1} \dots p_{i_c}} (\mathsf{SApTr})$$

■

4.8 Axiomatizing propositional inclusion logic with non-empty atom

In this section, we generalize the method in Section 4.7 to axiomatize the propositional variant of first-order inclusion logic extended with the non-empty atom.

First, we define *propositional inclusion logic*.

Definition 4.8.1. We call formulas of the form $p_{i_1} \cdots p_{i_k} \subseteq p_{j_1} \cdots p_{j_k}$ *inclusion atoms*. Well-formed formulas of *propositional inclusion logic* (**PInc**) are given by the following grammar:

$$\phi ::= p_i \;\Big|\; \neg p_i \;\Big|\; p_{i_1} \cdots p_{i_k} \subseteq p_{j_1} \cdots p_{j_k} \;\Big|\; \phi \wedge \phi \;\Big|\; \phi \otimes \phi,$$

where $p_i, p_{i_1}, \dots, p_{i_k}, p_{j_1}, \dots, p_{j_k}$ are propositional variables.

Definition 4.8.2. We inductively define the notion of a **PInc** formula ϕ being *true* on a team X, denoted by $X \models \phi$. All the cases are the same as those of **PD** as defined in Definition 4.1.3 except the following:

- $X \models p_{i_1} \cdots p_{i_k} \subseteq p_{j_1} \cdots p_{j_k}$ iff for all $s \in X$, there exists $s' \in X$ such that

$$\langle s'(j_1), \ldots, s'(j_k) \rangle = \langle s(i_1), \ldots, s(i_k) \rangle.$$

It is easy to verify that **PInc** inherits most of the properties of first-order inclusion logic, in particular, **PInc** satisfies the *locality property* and the *union closure property* (c.f. Theorem 1.2.5): for any formula ϕ of **PInc**, and collection of teams $\{X_i\}_{i \in I}$,

$$\forall i \in I,\ X_i \models \phi \Longrightarrow \bigcup_{i \in I} X_i \models \phi.$$

In the rest of this section, we axiomatize **PInc** extended with the non-empty atom, namely **PInc**$^{[\mathsf{NE}]}$, using a similar method with that of **PInd**$^{[\mathsf{NE}]}$. Analogous to **PInd**, an inclusion atom

$$p_{i_1} \cdots p_{i_k} \subseteq p_{j_1} \cdots p_{j_k}$$

is satisfied by a team X if and only if for any valuation $s \in X$, there must exists a valuation $s' \in X$ which witnesses the values of the sequence $\langle p_{i_1}, \ldots, p_{i_k} \rangle$ being included in that of $\langle p_{j_1}, \ldots, p_{j_k} \rangle$. Here, the set of witnesses for s must be non-empty, therefore, essentially, there is a non-empty atom in the underlying team semantics of the inclusion atom, too. In view of this, we hope that an axiomatization for the logic **PInc** gives some insight on the logic **PInc**. However, the problem of how to axiomatize propositional inclusion logic alone is still open.

As in the case of **PInd**$^{[\mathsf{NE}]}$, we start with analyzing strong approximation sequences of **PInc**$^{[\mathsf{NE}]}$ formulas. Let $\phi(p_{i_1}, \ldots, p_{i_n})$ be an n-formula of **PInc**$^{[\mathsf{NE}]}$. Suppose the following are all the occurrences of all atomic or negated atomic formulas in ϕ:

$$\alpha_1, \ldots, \alpha_c$$

where each α_ξ ($1 \leq \xi \leq c$) can be of the following forms:

$$p_{i_\xi}, \quad \neg p_{i_\xi}, \quad \mathsf{NE}, \quad p_{j_1^\xi} \ldots p_{j_{m_\xi}^\xi} \subseteq p_{k_1^\xi} \ldots p_{k_{m_\xi}^\xi},$$

where $\{i_\xi, k_1^\xi, \ldots k_{b_\xi}^\xi, k_1^\xi, \ldots k_{m_\xi}^\xi\} \subseteq \{i_1, \ldots, i_n\}$. A *strong approximation sequence* $\Upsilon = \langle u_1, \ldots, u_c \rangle$ of $\phi(p_{i_1}, \ldots, p_{i_n})$ is a sequence such that for every α_ξ, the set u_ξ is defined as in the case of **PInd**$^{[\mathsf{NE}]}$, except

- if $\alpha_\xi = p_{j_1^\xi} \ldots p_{j_{m_\xi}^\xi} \subseteq p_{k_1^\xi} \ldots p_{k_{m_\xi}^\xi}$, then $u_\xi \subseteq \mathbf{2^n}$ such that

$$\{(s(j_1^\xi), \ldots, s(j_{m_\xi}^\xi)) \mid s \in u_\xi\} \subseteq \{(s'(k_1^\xi), \ldots, s'(k_{m_\xi}^\xi)) \mid s' \in u_\xi\}.$$

For any such sequence $\Upsilon = \langle u_1, \ldots, u_c \rangle$, define an inclusion atom-free formula $\phi_\Upsilon^\star$ of **PInc**$^{[\mathsf{NE}]}$, called a *strong approximation* of ϕ, the same way as in the case of **PInd**$^{[\mathsf{NE}]}$.

Next, we show that every **PInc**$^{[\mathsf{NE}]}$ formula is logically equivalent to the intuitionistic disjunction of all its strong approximations.

Lemma 4.8.3. *Let $\phi(p_{i_1},\dots,p_{i_n})$ be an n-formula of* **PInc**$^{[\mathsf{NE}]}$ *and Γ the set of all its strong approximation sequences. Then*

$$\phi \equiv \bigvee_{\Upsilon\in\Gamma} \phi^\star_\Upsilon.$$

Proof. We prove the lemma by induction on ϕ. All the other cases are similar with those in the proof of Lemma 4.7.1 except the case $\phi = p_{j_1^\xi}\dots p_{j_{m_\xi}^\xi} \subseteq p_{k_1^\xi}\dots p_{k_{m_\xi}^\xi}$.

In this case, we first show that for each $\langle u_\xi\rangle \in \Gamma$, $\phi^\star_{\langle u_\xi\rangle} \models \phi$. Suppose for some n-team X on $N = \{i_1,\dots,i_n\}$, $X \models \Theta^*_{u_\xi}$. Then by Lemma 4.6.2, $X = u_\xi$. Since

$$\{(s(j_1^\xi),\dots,s(j_{m_\xi}^\xi)) \mid s\in u_\xi\} \subseteq \{(s'(k_1^\xi),\dots,s'(k_{m_\xi}^\xi)) \mid s'\in u_\xi\}, \tag{4.20}$$

for any $s \in X = u_\xi$, there exists $s' \in u_\xi = X$ such that

$$\langle s'(k_1^\xi),\dots,s'(k_{m_\xi}^\xi)\rangle = \langle s(j_1^\xi),\dots,s(j_{m_\xi}^\xi)\rangle,$$

thus $X \models p_{j_1^\xi}\dots p_{j_{m_\xi}^\xi} \subseteq p_{k_1^\xi}\dots p_{k_{m_\xi}^\xi}$.

Conversely, we show that $\phi \models \bigvee_{\langle u_\xi\rangle\in\Gamma} \phi^\star_{\langle u_\xi\rangle}$. Suppose for some n-team X on N, $X \models p_{j_1^\xi}\dots p_{j_{m_\xi}^\xi} \subseteq p_{k_1^\xi}\dots p_{k_{m_\xi}^\xi}$. Taking $u_\xi = X$, clearly (4.20) holds, i.e. $u_\xi \in \Gamma$. By Lemma 4.6.2, $X \models \Theta^*_{u_\xi}$, i.e., $X \models \phi^\star_{\langle u_\xi\rangle}$, hence $X \models \bigvee_{\langle u_\xi\rangle\in\Gamma} \phi^\star_{\langle u_\xi\rangle}$. ■

Next, we define a natural deduction system for **PInc**$^{[\mathsf{NE}]}$ which will enable us to derive in effect the *weak normal form* $\bigvee_{\Upsilon\in\Gamma}\phi^\star_\Upsilon$ or $\langle\phi^\star_\Upsilon\rangle_{\Upsilon\in\Gamma}$ for every formula ϕ of **PInc**$^{[\mathsf{NE}]}$.

Definition 4.8.4 (A natural deduction system for **PInc**$^{[\mathsf{NE}]}$)**.** The rules are given as follows:

1. The rules ($\wedge$I), ($\wedge$E), ($\otimes$WI), ($\otimes$WE), ($\otimes$Rpt), ($\otimes$Sub), (Ass $\otimes$), (Com $\otimes$), (ex falso$^+$), ($\bot$E), (EM$_0$), (**0**I), (**0**Ctr) as in Definition 4.3.5 and Definition 4.6.5.
2. Inclusion Atom Strong Introduction: For any strong approximation sequence $\langle u\rangle$ of $p_{j_1}\dots p_{j_m} \subseteq p_{k_1}\dots p_{k_m}(p_{i_1},\dots,p_{i_n})$,

$$\dfrac{\bigotimes_{s\in u}(p_{i_1}^{s(i_1)}\wedge\dots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})}{p_{j_1}\dots p_{j_m}\subseteq p_{k_1}\dots p_{k_m}}\ (\mathsf{IncSI})$$

3. Strong Approximation Transition:

$$\dfrac{\begin{matrix}[\phi^\star_{\Upsilon_0}] \\ \vdots \\ \theta\end{matrix}\ \cdots\ \cdots\ \begin{matrix}[\phi^\star_{\Upsilon_m}] \\ \vdots \\ \theta\end{matrix}\qquad \phi}{\theta}\ (\mathsf{SApTr})$$

where $\{\Upsilon_0,\dots,\Upsilon_m\}$ is the set of all strong approximation sequences of ϕ.

Next, we prove the Soundness Theorem for the above system.

Theorem 4.8.5 (Soundness Theorem)**.** *For any formulas ϕ and ψ of* $\mathbf{PInc}^{[\mathsf{NE}]}$*,*

$$\phi \vdash \psi \Longrightarrow \phi \models \psi.$$

Proof. It suffices to show that all of the deductive rules are valid. The validity of (IncSI) and (SApTr) follows from Lemma 4.8.3, and the validity of all the other rules follows from the proofs of Theorem 4.3.7 and Theorem 4.6.7. ■

Lemma 4.8.6. *Let $\phi(p_{i_1},\ldots,p_{i_n})$ be an n-formula of* $\mathbf{PInc}^{[\mathsf{NE}]}$*. For any strong approximation sequence Υ of ϕ, we have that*

$$\phi_\Upsilon^\star \dashv\vdash \Theta^*_{X_\Upsilon},$$

where X_Υ is an n-team on $\{i_1,\ldots,i_n\}$ or $X_\Upsilon = \mathbf{0}$.

Proof. By a similar argument to that of the proof of Lemma 4.7.5. ■

Theorem 4.8.7 (Completeness Theorem)**.** *For any formulas ϕ and ψ of* $\mathbf{PInc}^{[\mathsf{NE}]}$*,*

$$\phi \models \psi \Longrightarrow \phi \vdash \psi.$$

Proof. By a similar argument to that in the proof of Theorem 4.7.7. ■

Note that as in the case of $\mathbf{PInd}^{[\mathsf{NE}]}$, Atomic Excluded Middle Rule (EM_0) is not needed in the proof of the above theorem, but this rule is required in the proof of the (Weak) Completeness Theorem: for any formula ϕ of $\mathbf{PInc}^{[\mathsf{NE}]}$,

$$\models \phi \Longrightarrow \vdash \phi.$$

Theorem 4.8.8 (Strong Completeness Theorem)**.** *For any set Γ of formulas and any formula ϕ of* $\mathbf{PInc}^{[\mathsf{NE}]}$*,*

$$\Gamma \models \phi \Longrightarrow \Gamma \vdash \phi.$$

Proof. Follows from Theorem 4.8.7 and the Compactness Theorem of $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ (as $\mathbf{PInc}^{[\mathsf{NE}]}$ is a sublogic of the maximal logic $\mathbf{PD}^{[\vee,\mathsf{NE}]}$). ■

4.9 Open problems

In this chapter, we defined the propositional variants of the logics of dependence and independence, and gave complete axiomatizations of these logics and their variants. Below we list the main open problems and future directions concerning the topics of this chapter (some of which are already mentioned in the corresponding sections).

1. We have shown in this chapter that **PID**, $\mathbf{PD}^{[\vee]}$ and **PD** are maximal downwards closed logics (Theorems 4.2.8, 4.3.2, 4.4.1), and $\mathbf{PD}^{[\vee,\mathsf{NE}]}$ is a maximal logic (Theorem 4.6.4). The relationship of the main logics discussed in this chapter in terms of expressive power is depicted in Figure 4.1 (c.f. Figure 3.2).

Classical propositional logic (**CPL**) is a proper sublogic of all the other logics, since all classical formulas are flat and all of the other logics have non-flat formulas. **PInc** and

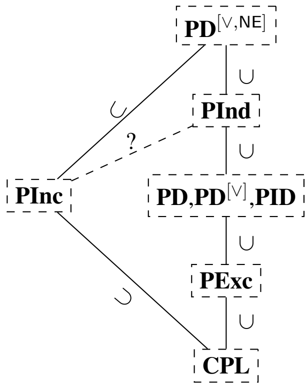


Figure 4.1: Expressive power of propositional logics

PInd are proper sublogics of the maximal logic $\mathbf{PD}^{[\vee,\mathsf{NE}]}$, since NE is not definable in neither of these logics. The downwards closed logic **PD** (or **PID**, $\mathbf{PD}^{[\vee]}$) is a proper sublogic of the non-downwards closed logic **PInd**, since dependence atoms are definable in **PInd** (Expression (4.5)). The logic **PExc** is downwards closed, thus it is a sublogic of the maximal downwards closed logics **PD**, $\mathbf{PD}^{[\vee]}$ and **PID**. Recall that first-order dependence logic is equivalent to first-order exclusion logic. However, propositional dependence logic is different from propositional exclusion logic as, e.g., with only one propositional variable, **PD** has in total 5 non-equivalent formulas:

$$p, \quad \neg p, \quad p \wedge \neg p, \quad p \otimes \neg p, \quad =\!(p),$$

whereas **PExc** has only 4 non-equivalent formulas, in particular, $p \mid p \equiv p \wedge \neg p$.

The logic **PInc** is different from the downwards closed logics **PD** and **PExc**, as it is not downwards closed. Recall that first-order inclusion logic is a proper sublogic of first-order independence logic, but the connection between **PInc** and **PInd** is unknown.

2. As discussed in Section 4.2, propositional intuitionistic dependence logic and inquisitive logic turn out to be essentially equivalent. This surprising connection certainly deserves to be further explored. In particular, the connection between first-order inquisitive logic ([11],[12]) and first-order (intuitionistic) dependence logic deserves investigation.

3. In Sections 4.7 and 4.8, we have axiomatized **PInd** extended with NE and **PInc** extended with NE. The problem of how to axiomatize **PInd** and **PInc** alone is open.

4. The deductive rule Approximation Transition (ApTr) in Definition 4.4.4 (see also Appendix) has a complex form. It is not known wether it is derivable by the simpler rule Dependence Atom Weak Elimination (DepWE). Similarly for Strong Approximation Transition (SApTr) rule and Independence Atom Weak Elimination (IndWE) rule.

5. It is proved in Corollary 4.4.2 that $\mathbf{PD}^{[\vee]} = \mathbf{PD}$. Given this fact, it is reasonable to conjecture that $\mathbf{PD}^{[\vee,\mathsf{NE}]} = \mathbf{PD}^{[\mathsf{NE}]}$, or even $\mathbf{PD}^{[\vee,\mathsf{NE}]} = \mathbf{PInd}^{[\mathsf{NE}]}$. This problem is open.

Chapter 5

Uniform definability in propositional dependence logic

We have proved in Theorem 4.4.1 that propositional dependence logic (**PD**) is a maximal downwards closed logic, therefore adding intuitionistic disjunction or intuitionistic implication into propositional dependence logic does not increase the expressive power of the logic. In particular, every formula with intuitionistic disjunction and intuitionistic implication can be translated equivalently into a formula of **PD** without these two connectives. In this chapter, we show that although such a non-uniform translation exists, neither of intuitionistic disjunction and intuitionistic implication is *uniformly definable* in propositional dependence logic. The work is inspired by [32], in which the weak universal quantifier $\forall^1$ (see Definition 2.2.6) is proved to be non-uniformly definable in first-order dependence logic. Also along this line, due to Ciardelli [11], in inquisitive logic or propositional intuitionistic dependence logic (**PID**), every instance of conjunction is expressible in terms of other connectives of the logic, but a uniform definition for conjunction does not exists. We adapt this result in this chapter in our framework.

In Section 5.1, we give formal definition of *uniform definability* of connectives, and make some remarks concerning the issues of definability and uniform definability in classical and intuitionistic propositional logic. In Section 5.2, we study the properties of contexts for **PD**, which is a crucial notion in the main proof of this chapter. Section 5.3 records the main results of this chapter. We prove that neither of intuitionistic implication and intuitionistic disjunction is uniformly definable in **PD**. We also include the result due to [11] that in the conjunction-free fragment of **PID**, every instance of $\wedge$ is definable, but $\wedge$ is not uniformly definable.

5.1 Contexts and Uniform Definability of Connectives

In this section, we define context and uniform definability of connectives for the following logics: classical propositional logic (**CPL**), intuitionistic propositional logic (**IPL**), propositional dependence logic (**PD**), and propositional intuitionistic dependence logic (**PID**). We also make some remarks concerning the issues of definability and uniformly definability of connectives in **CPL** and **IPL**.

Throughout this section, we use L to stand for any of the four propositional logics: **CPL**, **IPL**, **PD** and **PID**. For the purpose of this chapter, let us first recall basic definitions of L. The *language* of L consists of a set A of atoms and a set Ω of operator symbols. A *(well-formed) formula* in the language of L is any atom $\alpha \in \mathsf{A}$ or $\divideontimes(\alpha_1, \dots, \alpha_\gamma)$, where $\divideontimes \in \Omega$ is a γ-ary operator symbol, and $\alpha_1, \dots, \alpha_\gamma \in \mathsf{A}$. For the logics **CPL** and **IPL**, the corresponding set A of atoms consists of all propositional variables, that is, atoms are propositional variables. The set Ω of operator symbols for **CPL** contains classical negation $\neg$ and other classical connectives, the set Ω for **IPL** contains the nullary operator *falsum* $\bot$ and other intuitionistic connectives[1]. In this chapter, special attention needs to be paid to the syntax of **PD**, as well as that of **PID**. We stipulate (only in this chapter) that the language of **PD** consists of the set

$$\mathsf{A} = \{p_i, \neg p_i \mid i \in \omega\} \cup \{=\!\!(p_{i_1}, \dots, p_{i_k}) \mid i_1, \dots, i_k \in \omega\}$$

of atoms and the set $\Omega = \{\wedge, \otimes\}$ of operators. Both $\neg p_i$ and the dependence atom $=\!\!(p_{i_1}, \dots, p_{i_k})$ are considered as atoms that cannot be decomposed. Similarly, (only in this chapter) the language of **PID** consists of the set

$$\mathsf{A} = \{p_i, =\!\!(p_i) \mid i \in \omega\}$$

and the set $\Omega = \{\bot, \wedge, \vee, \to\}$.

For a formula ϕ of L, we call the class of all models of ϕ, denoted by $[\![\phi]\!]$, the *truth class* (or *truth set*) of ϕ. In particular:

- In **CPL**, the *semantic truth set* $[\![\phi]\!]$ of a formula ϕ is defined as

$$[\![\phi]\!] := \{s : \omega \to 2 \mid s \models \phi\}.$$

- In **IPL**, the *semantic truth class* $[\![\phi]\!]$ of a formula ϕ is defined as the class

$$[\![\phi]\!] := \{(\mathfrak{M}, w) \mid \mathfrak{M} \text{ is an intuitionistic Kripke model with a node } w \text{ and } \mathfrak{M}, w \models \phi\}.$$

- In **PD** or **PID**, the *semantic truth set* $[\![\phi]\!]$ of a formula ϕ is defined as

$$[\![\phi]\!] := \{X \subseteq 2^\omega : X \models \phi\}.$$

Let ∇^{L} be the set of all semantic truth sets (or classes) of all formulas of L, that is

$$\nabla^{\mathsf{L}} := \{[\![\phi]\!] \mid \phi \text{ is a formula of } \mathsf{L}\}.$$

In case the logic L is clear from the context, we simply write ∇ for ∇^{L}. In this chapter, by a γ-ary *connective* $\divideontimes$ of L we mean a γ-ary operator of L having a *compositional* meaning, that is, there is a γ-ary function $\boldsymbol{\divideontimes} : \nabla^\gamma \to \nabla$ such that for any formulas $\theta_1, \dots, \theta_\gamma$ of L,

$$[\![\divideontimes(\theta_1, \dots, \theta_\gamma)]\!] = \boldsymbol{\divideontimes}([\![\theta_1]\!], \dots, [\![\theta_\gamma]\!]). \tag{5.1}$$

For the logic **CPL**, a γ-ary connective is usually understood as a γ-ary Boolean function. We prove in the next lemma that our definition agrees with this usual understanding.

[1] Recall that intuitionistic negation is defined as: $\neg\phi := \phi \to \bot$.

Lemma 5.1.1. *In* **CPL**, *every function* $\circledast : \nabla^\gamma \to \nabla$ *induces a Boolean function* $f_\circledast : 2^\gamma \to 2$ *such that for all formulas* $\theta_1, \ldots, \theta_\gamma$, *all valuations* $s : \omega \to 2$,

$$s(\circledast(\theta_1, \ldots, \theta_\gamma)) = f_\circledast(s(\theta_1), \ldots, s(\theta_\gamma)), \tag{5.2}$$

and vice versa.

Proof. Let $\circledast : \nabla^\gamma \to \nabla$ be a function. Define

$$\widetilde{0} := \bot \text{ and } \widetilde{1} := \top.$$

Pick a valuation $s_0 : \omega \to 2$. Define the function $f_\circledast : 2^\gamma \to 2$ by taking for any $x_1, \ldots, x_\gamma \in 2$,

$$f_\circledast(x_1, \ldots, x_\gamma) = s_0(\circledast(\widetilde{x_1}, \ldots, \widetilde{x_\gamma})). \tag{5.3}$$

It remains to check that $f_\circledast$ satisfies Equation (5.2).

For some arbitrary formulas $\theta_1, \ldots, \theta_\gamma$, arbitrary valuations $s : \omega \to 2$, put

$$x_1 = s(\theta_1), \ldots, x_\gamma = s(\theta_\gamma).$$

Noting that

$$s(\circledast(\theta_1, \ldots, \theta_\gamma)) = s_0(\circledast(\widetilde{x_1}, \ldots, \widetilde{x_\gamma})),$$

Equation (5.2) follows from Equation (5.3).

Conversely, suppose $f : 2^\gamma \to 2$ is a function. We find a function $\circledast_f : \nabla^\gamma \to \nabla$ satisfying Equation (5.2). For any formulas $\theta_1, \ldots, \theta_\gamma$, define that

$$s \in \circledast_f([\![\theta_1]\!], \ldots, [\![\theta_\gamma]\!]) \iff f(s(\theta_1), \ldots, s(\theta_\gamma)) = 1,$$

for any valuation $s : \omega \to 2$. It remains to find a formula δ such that $[\![\delta]\!] = \circledast_f([\![\theta_1]\!], \ldots, [\![\theta_\gamma]\!])$.

Let $N = \{i_1, \ldots, i_n\}$ be the set of indices of all propositional variables occurring in the formulas $\theta_1, \ldots, \theta_\gamma$. Consider the set

$$S = \{s \restriction N \mid s \in \circledast_f([\![\theta_1]\!], \ldots, [\![\theta_\gamma]\!])\}.$$

Note that the set S is finite, since restricted to the n-element set N, there are in total 2^n possible distinct valuations. Now, consider the formula

$$\Theta_S = \bigvee_{s \in S} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)})$$

(c.f. the formula Θ_X of $\mathbf{PD}^{[\vee]}$ defined in Lemma 4.3.1). Since in **CPL**, the truth value of a formula depends only on the variables occurring in the formula (namely **CPL** is *local*, c.f. Lemma 4.1.6), it is not hard to see that $[\![\Theta_S]\!] = \circledast_f([\![\theta_1]\!], \ldots, [\![\theta_\gamma]\!])$. So we are done. ∎

It is well-known that in **IPL**, the intuitionistic connectives $\bot$, $\wedge$, $\vee$ and $\to$ are *independent* of each other, as none of them is *definable in terms of the other connectives* (see e.g. [80]). On the other hand, in **CPL**, the set $\{\neg, \vee\}$ of classical connectives is *functionally complete*, meaning that each Boolean function $f : 2^\gamma \to 2$ is *uniformly definable* by certain combination of the connectives in the set $\{\neg, \vee\}$. For example, for each formulas θ_1 and θ_2, their conjunction is uniformly defined as

$$(\theta_1 \wedge \theta_2) \equiv \neg(\neg\theta_1 \vee \neg\theta_2).$$

Other known functionally complete sets of connectives of **CPL** are

$$\{\neg, \wedge\},\ \{\neg, \rightarrow\},\ \{|\ ^2\},\ \text{etc.}$$

In this chapter, we study the uniform definability issue of **PD** and **PID**. To this end, we first define the notion of *uniform definability* formally for the logics L mentioned in this section. Basically, a connective is uniformly definable in L if and only if there is a context for L which defines the connective. A *context* for a logic L is a formula of L with distinguished atoms r_i ($i \in \omega$). Intuitively, these atoms r_i are understood as "holes" that are to be substituted uniformly by concrete instances of formulas.

Definition 5.1.2 (context)**.** A *context* for L is a formula of L with distinguished atoms r_i ($j \in \omega$). For the logic **CPL**, **IPL**, **PD** or **PID**, r_i can be understood as a distinguished propositional variable, and a context for these logics is a formula built from propositional variables r_i ($j \in \omega$) and other atoms using the connectives in L. We write $\phi[r_1, \ldots, r_\gamma]$ to mean that the distinguished atoms (distinguished propositional variables) occurring in the context ϕ are among $r_1, \ldots, r_\gamma$.

For example, the formula

$$\phi_0[r_1, r_2] := \neg(\neg r_1 \vee \neg r_2) \tag{5.4}$$

is a context for **CPL**, and the formula

$$\phi_1[r_1, r_2] := ((\neg p_1 \otimes r_1) \wedge (=\!(p_2, p_3) \otimes (r_1 \wedge r_2))) \tag{5.5}$$

is a context for **PD**.

As mentioned already, in a context $\phi[r_1, \ldots, r_\gamma]$, each distinguished atom r_i marks the places that are to be substituted uniformly by a formula of L. For any formulas $\theta_1, \ldots, \theta_\gamma$ of L, we write $\phi[\theta_1, \ldots, \theta_\gamma]$ for the formula $\phi(\theta_1/r_1, \ldots, \theta_\gamma/r_\gamma)$. Two contexts $\phi[r_1, \ldots, r_\gamma]$ and $\psi[r'_1, \ldots, r'_\gamma]$ for L are said to be *equivalent*, in symbols $\phi[r_1, \ldots, r_\gamma] \approx \psi[r'_1, \ldots, r'_\gamma]$, if and only if for any formulas $\theta_1, \ldots, \theta_\gamma$ of L,

$$[\![\phi[\theta_1, \ldots, \theta_\gamma]]\!] = [\![\psi[\theta_1, \ldots, \theta_\gamma]]\!].$$

Definition 5.1.3 (Uniform definability)**.** We say that a context $\phi[r_1, \ldots, r_\gamma]$ for L *uniformly defines* a γ-ary connective $\circledast$ if for all formulas $\theta_1, \ldots, \theta_\gamma$ of L,

$$[\![\phi[\theta_1, \ldots, \theta_\gamma]]\!] = [\![\circledast(\theta_1, \ldots, \theta_\gamma)]\!].$$

We say that a γ-ary connective $\circledast$ is *uniformly definable* in a propositional logic L, if there exists a context $\phi[r_1, \ldots, r_\gamma]$ for L which uniformly defines $\circledast$.

[2]"|" is called the *Sheffer stroke*, which is the binary connective defined by the following truth table:

A	B	$A \mid B$
0	0	1
0	1	1
1	0	1
1	1	0

For example, in **CPL**, the context $\phi_0[r_1,r_2]$ from (5.4) uniformly defines classical conjunction $\wedge$, since for any two formulas θ_1,θ_2 of **CPL**,

$$[\![\phi_0[\theta_1,\theta_2]]\!] = [\![\neg(\neg\theta_1 \vee \neg\theta_2)]\!] = [\![\theta_1 \wedge \theta_2]\!].$$

With our new terminology, a set C of connectives of a logic L is said to be *functionally complete* if and only if every γ-ary function $\divideontimes : \nabla^\gamma \to \nabla$ is uniformly definable by a context $\phi[r_1,\dots,r_\gamma]$ for L with connectives only from the set C. In particular, as already noted, the set $\{\neg,\vee\}$ is *functionally complete* for **CPL**, since every γ-ary function $\divideontimes : \nabla^\gamma \to \nabla$ (which corresponds to a Boolean function $f_{\divideontimes} : 2^\gamma \to 2$, by Theorem 5.1.1) is uniformly definable by a context $\phi[r_1,\dots,r_\gamma]$ with only $\neg$ and $\vee$ as connectives. On the other hand, none of the intuitionistic connectives $\circledast$ is uniformly definable in the fragment of **IPL** without the presence of $\circledast$. The notion of uniform definability is also related to *algebraic* (or *compositional*) *translation* (see e.g. [56]).

Most commonly used contexts do not contain extra atoms than the distinguished ones, for example classical conjunction $\wedge$ of **CPL** is defined by the context $\phi_0[r_1,r_2]$ from (5.4) of this kind. Contexts with extra atoms, e.g. $\phi_1[r_1,r_2]$ from (5.5) or

$$\phi_2[r_1] := r_1 \vee \mathsf{Rainy},$$

may intuitively make only little sense, but they are technically eligible,

Let $\circledast$ be a γ-ary connective of L. We say that *every instance of $\circledast$ is definable in* L if for every formulas $\theta_1,\dots,\theta_\gamma$ of L, there exists a formula ϕ of L such that

$$[\![\circledast(\theta_1,\dots,\theta_\gamma)]\!] = [\![\phi]\!],$$

In **CPL**, every instance of classical conjunction $\wedge$ is definable, as $\wedge$ is in fact uniformly definable. As mentioned, intuitionistic disjunction $\vee$ is not uniformly definable in the $\vee$-free fragment of **IPL** (i.e. **IPL**$[\bot,\wedge,\to]$); moreover, not every instance of $\vee$ is definable in **IPL**$[\bot,\wedge,\to]$, since given finitely many propositional variables, the full logic **IPL** has infinitely many non-equivalent formulas, whereas by Diego's Theorem, there are only finitely many non-equivalent $\vee$-free formulas (see e.g. Section 5.4 of [9]). For the logic **PD**, by Theorem 4.4.1, every instance of intuitionistic disjunction $\vee$ or intuitionistic implication $\to$ is definable in **PD**, however, we will show in this chapter that neither of $\vee$ and $\to$ is uniformly definable in **PD**. Moreover, we adapt the result due to [11] with our terminology that in the fragment of **PID** without conjunction $\wedge$, every instance of $\wedge$ is definable, but $\wedge$ is not uniformly definable.

5.2 Contexts for PD

In this section, we investigate the properties of contexts for propositional dependence logic.

In Definition 5.1.2, we defined contexts for the mentioned propositional logics L in general. In the case of **PD**, a context for **PD** is a formula ϕ with distinguished propositional variables r_i $(j \in \omega)$ built from the following grammar:

$$\phi ::= r_i \mid p_i \mid \neg p_i \mid {=}(p_{j_1},\dots,p_{j_k}) \mid (\phi \wedge \phi) \mid (\phi \otimes \phi),$$

where $p_i, p_{j_1}, \ldots, p_{j_k}$ are (non-distinguished) propositional variables. Note that for technical reasons that will become clear in Definition 5.2.7, we do not omit parentheses in a context. As emphasized in the previous section, we do not view negation as a connective, and dependence atoms cannot be decomposed, therefore by Definition 5.1.2, a context cannot have a subformula of the form $\neg r_i$ or $=\!(p_{j_1}, \ldots, p_{j_{m-1}}, r_i, p_{j_{m+1}} \ldots, p_{j_k})$. To make this idea clear, below we present the formal definition of subformulas of contexts for **PD**.

Definition 5.2.1 (Subformula)**.** Let ϕ be a context for **PD**. We define the set $\mathsf{Sub}(\phi)$ of subformulas of ϕ inductively as follows:

- $\mathsf{Sub}(r_i) = \{r_i\}$;
- $\mathsf{Sub}(p_i) = \{p_i\}$;
- $\mathsf{Sub}(\neg p_i) = \{\neg p_i\}$;
- $\mathsf{Sub}(=\!(p_{j_1}, \ldots, p_{j_k})) = \{=\!(p_{j_1}, \ldots, p_{j_k})\}$;
- $\mathsf{Sub}((\psi \wedge \chi)) = \mathsf{Sub}(\psi) \cup \mathsf{Sub}(\chi) \cup \{(\psi \wedge \chi)\}$;
- $\mathsf{Sub}((\psi \otimes \chi)) = \mathsf{Sub}(\psi) \cup \mathsf{Sub}(\chi) \cup \{(\psi \otimes \chi)\}$.

A context $\phi[r_1, \ldots, r_\gamma]$ is said to be *contradictory* if $\phi[r_1, \ldots, r_\gamma] \approx \bot$; it is said to be *tautological* if $\phi[r_1, \ldots, r_\gamma] \approx \top$. A contradictory context $\phi[r_1, \ldots, r_\gamma]$ defines uniformly a γ-ary connective that we call the *contradictory* connective. The following lemma shows that we may assume that a context is either contradictory or it does not contain a single contradictory subformula.

Lemma 5.2.2. *Let $\phi[r_1, \ldots, r_\gamma]$ be a context for* **PD***. If $\phi[r_1, \ldots, r_\gamma]$ is not contradictory, then there exists a context $\phi'[r_1, \ldots, r_\gamma]$ with no single contradictory subfromula such that $\phi'[r_1, \ldots, r_\gamma] \approx \phi[r_1, \ldots, r_\gamma]$.*

Proof. Assuming that $\phi[r_1, \ldots, r_\gamma]$ is not contradictory, we find the required formula ϕ' by induction on ϕ.

If $\phi[r_1, \ldots, r_\gamma]$ is an atom, then it clearly does not contain a single contradictory subformula.

If $\phi[r_1, \ldots, r_\gamma] = (\psi \wedge \chi)[r_1, \ldots, r_\gamma]$, which is not contradictory, then it is easy to see that none of $\psi[r_1, \ldots, r_\gamma]$ and $\chi[r_1, \ldots, r_\gamma]$ is contradictory. By induction hypothesis, there are $\psi'[r_1, \ldots, r_\gamma]$ and $\chi'[r_1, \ldots, r_\gamma]$ such that

$$\psi'[r_1, \ldots, r_\gamma] \approx \psi[r_1, \ldots, r_\gamma], \;\; \chi'[r_1, \ldots, r_\gamma] \approx \chi[r_1, \ldots, r_\gamma]$$

and none of ψ' and χ' contains a single contradictory formula. Let $\phi'[r_1, \ldots, r_\gamma] := (\psi' \wedge \chi')[r_1, \ldots, r_\gamma]$. Clearly,

$$(\psi \wedge \chi)[r_1, \ldots, r_\gamma] \approx (\psi' \wedge \chi')[r_1, \ldots, r_\gamma].$$

As we have assumed that $(\psi' \wedge \chi') \not\approx \bot$, by induction hypothesis, the set

$$\mathsf{Sub}((\psi' \wedge \chi')) = \mathsf{Sub}(\psi') \cup \mathsf{Sub}(\chi') \cup \{(\psi' \wedge \chi')\},$$

does not contain a single contradictory element.

If $\phi[r_1,\dots,r_\gamma] = (\psi \otimes \chi)[r_1,\dots,r_\gamma]$, which is not contradictory, then ψ and χ cannot be both contradictory. There are the following two cases:

Case 1: Only one of ψ and χ is contradictory. Without loss of generality, we may assume that $\psi[r_1,\dots,r_\gamma] \approx \bot$ and $\chi[r_1,\dots,r_\gamma] \approx \chi'[r_1,\dots,r_\gamma]$, where $\chi'[r_1,\dots,r_\gamma]$ is a context for **PD** which does not contain a single contradictory subformula. Clearly, for any formulas $\theta_1,\dots,\theta_\gamma$, any team X,

$$\begin{aligned} X \models (\psi \otimes \chi)[\theta_1,\dots,\theta_\gamma] &\iff X \models (\bot \otimes \chi')[\theta_1,\dots,\theta_\gamma] \\ &\iff X \models \chi'[\theta_1,\dots,\theta_\gamma], \end{aligned}$$

thus $(\psi \otimes \chi)[r_1,\dots,r_\gamma] \approx \chi'[r_1,\dots,r_\gamma]$. So we can take $\phi'[r_1,\dots,r_\gamma] := \chi'[r_1,\dots,r_\gamma]$.

Case 2: $\psi[r_1,\dots,r_\gamma] \approx \psi'[r_1,\dots,r_\gamma]$ and $\chi[r_1,\dots,r_\gamma] \approx \chi'[r_1,\dots,r_\gamma]$, where neither of $\psi'[r_1,\dots,r_\gamma]$ and $\chi'[r_1,\dots,r_\gamma]$ contains a single contradictory subformula. Let $\phi'[r_1,\dots,r_\gamma] := (\psi' \otimes \chi')[r_1,\dots,r_\gamma]$. Clearly,

$$(\psi \otimes \chi)[r_1,\dots,r_\gamma] \approx (\psi' \otimes \chi')[r_1,\dots,r_\gamma].$$

As we have assumed that $(\psi' \otimes \chi') \not\approx \bot$, by induction hypothesis, the set

$$\mathsf{Sub}((\psi' \otimes \chi')) = \mathsf{Sub}(\psi') \cup \mathsf{Sub}(\chi') \cup \{(\psi' \otimes \chi')\},$$

does not contain a single contradictory element.

■

Contexts for **PD** are *monotone* in the sense of the following lemma.

Lemma 5.2.3. *Let $\phi[r_1,\dots,r_\gamma]$ be a context for* **PD** *and $\theta_1,\dots,\theta_\gamma,\theta'_1,\dots,\theta'_\gamma$ formulas of* **PD**. *If $\theta_1 \models \theta'_1, \dots, \theta_\gamma \models \theta'_\gamma$, then $\phi[\theta_1,\dots,\theta_\gamma] \models \phi[\theta'_1,\dots,\theta'_\gamma]$.*

Proof. Suppose $\theta_1 \models \theta'_1, \dots, \theta_\gamma \models \theta'_\gamma$ and $X \models \phi[\theta_1,\dots,\theta_\gamma]$ for some suitable team X. We prove by induction on $\phi[r_1,\dots,r_\gamma]$ that $X \models \phi[\theta'_1,\dots,\theta'_\gamma]$.

The only interesting case is the case $\phi[r_1,\dots,r_\gamma] = r_i$ $(1 \le i \le \gamma)$. In this case, if $X \models r_i[\theta_1,\dots,\theta_\gamma]$, then $X \models \theta_i \models \theta'_i$, thus $X \models r_i[\theta'_1,\dots,\theta'_\gamma]$. ■

Corollary 5.2.4. *Let $\phi[r_1,\dots,r_\gamma]$ be a context for* **PD**. *If $\phi[r_1,\dots,r_\gamma] \not\approx \bot$, then there exists a non-empty team X such that $X \models \phi[\top,\dots,\top]$.*

Proof. Since $\phi[r_1,\dots,r_\gamma] \not\approx \bot$, there exist formulas $\theta_1,\dots,\theta_\gamma$ and a non-empty team X such that $X \models \phi[\theta_1,\dots,\theta_\gamma]$. As $\theta_i \models \top$ for all $1 \le i \le \gamma$, by Lemma 5.2.3, we obtain that $X \models \phi[\top,\dots,\top]$. ■

In the main proofs of this chapter, we will make use of syntax trees of contexts for **PD**. Now, we recall the notion of *labeled full binary tree*.

Definition 5.2.5 (Full Binary Tree)**.** A *full binary tree* is a triple $(T,\prec,r)$ which satisfies the following conditions:

(i) T is a non-empty set with $r \in T$. Elements of T are called *nodes* or *points*. The node r is called the *root* of $\mathfrak{T}$.

(ii) $\prec$ is a binary relation on T which satisfies the following conditions:

(a) $\prec$ is transitive, that is, for all $t_1, t_2, t_3 \in T$,

$$[t_1 \prec t_2 \text{ and } t_2 \prec t_3] \implies t_1 \prec t_3.$$

(b) $\prec$ is irreflexive, that is, for all $t \in T$, $t \not\prec t$.

(c) For all $t \in T \setminus \{r\}$, $r \prec t$.

(d) Each node $t \in T \setminus \{r\}$ has a unique immediate predecessor $t_0 \in T$. A node t_0 is called an *immediate predecessor* of a node t if $t_0 \prec t$ and there is no node t' with $t_0 \prec t' \prec t$. In this case, the node t_0 is called the *parent* of t, and t is called a *child* of t_0.

(e) Each parent has exactly two children. The nodes of T which have no children are called *leaves*.

A node $t_0 \in T$ is said to be an *ancestor* of a node $t_1 \in T$ if $t_0 \prec t_1$.The *depth* $\mathsf{d}(t)$ of a node t in a full binary tree $(T, \prec, r)$ is defined inductively as follows:

- $\mathsf{d}(r) = 0$;
- if t_1 is a child of t_0, then $\mathsf{d}(t_1) = \mathsf{d}(t_0) + 1$.

The *depth* $\mathsf{d}(T)$ of a tree $(T, \prec, r)$ is defined as $\mathsf{d}(T) = \sup\{\mathsf{d}(t) \mid t \in T\}$.

Definition 5.2.6 (Labeled Full Binary Tree)**.** A *labeled full binary tree* with root r is a quadruple $\mathfrak{T} = (T, \prec, r, \mathsf{f})$ such that $(T, \prec, r)$ is a full binary tree with root r and f is a labeling function from T into a non-empty set F.

In order to give a technical definition of syntax trees of contexts for **PD**, we will fix a specific *occurrence* of a subformula in a context. To this end, we count the number of parentheses in a context. For example, the context

$$\underset{1}{\Big(}\underset{2}{(}\, \neg p_1 \otimes r_1 \,\underset{3}{)} \wedge \underset{4}{\big(}{=}(p_2, p_3) \otimes \underset{5}{(}\, r_1 \wedge r_2 \,\underset{6}{)}\underset{7}{\big)}\underset{8}{\Big)} \tag{5.6}$$

has 8 parentheses (excluding the parentheses of the dependence atom). In the formula depicted above, we labeled each parenthesis by a natural number positioned right below the parenthesis. The parenthesis labeled with the natural number k is the k-th parenthesis of the formula (counting from the left). Let

$$\underset{k}{(}\, \phi \circledast \psi \,\underset{m}{)}$$

be a subformula of a context θ, where $\circledast \in \{\wedge, \otimes\}$ and the above two outermost parentheses are the k-th and the m-th parentheses in θ, respectively. The formula ϕ is said to be *bounded* by the k-th parenthesis, and every parenthesis in ϕ is said to be *inside the scope* of the k-th parenthesis. Similarly, the formula ψ is said to be *bounded* by the m-th parenthesis, and every parenthesis in ψ is said to be *inside the scope* of the m-th parenthesis. Our treatment of specific occurrences of subformulas is analogous to that in Section 5.2 of [78], one may compare (5.6) with Table 5.1 in [78].

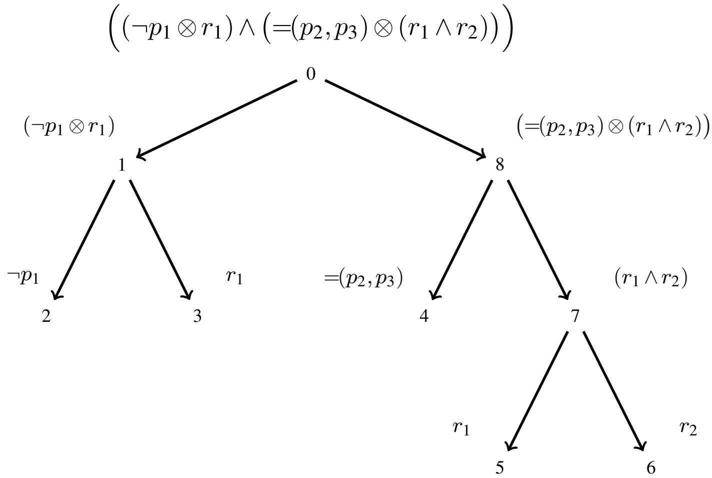


Figure 5.1: The syntax tree of $((\neg p_1 \otimes r_1) \wedge (=(p_2,p_3) \otimes (r_1 \wedge r_2)))$

Below we present the definition of syntax trees of contexts for **PD**. An example of a syntax tree is depicted in Figure 5.1.

Definition 5.2.7 (syntax tree)**.** The *syntax tree* of a context ϕ for **PD** is a labeled full binary tree $\mathfrak{T}_\phi = (T, \prec, r, \mathsf{f})$ satisfying

- $T := m+1$, where m is the number of all parentheses in ϕ;
- $r := 0$;
- $\prec := \{(0,k) \mid 0 < k \leq m\} \cup \{(k_1,k_2) \mid$ the k_2-th parenthesis is inside the scope of the k_1-th parenthesis$\}$;
- f is a function $\mathsf{f} : T \to \mathsf{Sub}(\phi)$ satisfying
  - $\mathsf{f}(0) = \phi$;
  - $\mathsf{f}(k) := \psi$, where ψ is the subformula of ϕ bounded by the k-th parenthesis.

  If $\mathsf{f}(k) = \psi$, we sometimes say that the node k is *labeled with* ψ or the formula ψ is *attached to* the node k.

Fact 5.2.8. *The leaf nodes of a syntax tree are always labeled with atoms.*

For a context $\phi[r_1,\dots,r_\gamma]$ for **PD**, if $X \models \phi[\theta_1,\dots,\theta_\gamma]$, then each occurrence of a subformula of $\phi[\theta_1,\dots,\theta_\gamma]$ is satisfied by a subteam of X. This can be described explicitly by a function σ which maps each node in the syntax tree $\mathfrak{T}_\phi$ of $\phi[\theta_1,\dots,\theta_\gamma]$ to a subteam of X satisfying the formula attached to the node. We now give the definition of such functions.

Definition 5.2.9 (Truth Function)**.** Let $\phi[r_1,\dots,r_\gamma]$ be a context for **PD** and $\theta_1,\dots,\theta_\gamma$ formulas of **PD**. Let N (with $|N| = n$) be the set of all indices of all propositional variables occurring in the formula $\phi[\theta_1,\dots,\theta_\gamma]$, and $\mathbf{2^n}$ the maximal n-team on N. Let $\mathfrak{T}_\phi = (T,\prec,r,\mathsf{f})$ be the syntax tree of ϕ. A function $\sigma : \mathfrak{T}_\phi \to \wp(\mathbf{2^n})$ is called a *truth function* for $\phi[\theta_1,\dots,\theta_\gamma]$ iff

(i) for all $k \in \mathfrak{T}_\phi$, $\sigma(k) \models \mathsf{f}(k)[\theta_1,\dots,\theta_\gamma]$;

(ii) if $\mathsf{f}(k) = (\psi \wedge \chi)$ and k_0, k_1 are the two children of k, then

$$\sigma(k) = \sigma(k_0) = \sigma(k_1);$$

(iii) if $\mathsf{f}(k) = (\psi \otimes \chi)$ and k_0, k_1 are the two children of k, then

$$\sigma(k) = \sigma(k_0) \cup \sigma(k_1).$$

A truth function σ is called a *truth function for* $\phi[\theta_1,\dots,\theta_\gamma]$ *over an* n*-team* X iff $\sigma(0) = X$.

Fact 5.2.10. *Let σ be a truth function for $\phi[\theta_1,\dots,\theta_\gamma]$. If k,k' are two nodes with $k \prec k'$, then $\sigma(k') \subseteq \sigma(k)$. In particular, if σ is a truth function for $\phi[\theta_1,\dots,\theta_\gamma]$ over an n-team X, then for all $k \in \mathfrak{T}_\phi$, $\sigma(k) \subseteq X$.*

Proof. Easy, by induction on $\mathsf{d}(k') - \mathsf{d}(k)$. ∎

First-order dependence logic has a game-theoretic semantics with perfect information games played with respect to teams (see Section 5.2 in [78]). With obvious adaptions, one can define a game-theoretic semantics for propositional dependence logic.[3] In fact, a truth function defined in Definition 5.2.9 corresponds to a *winning strategy* for Verifier in the game. An appropriate semantic game for **PD** has the property that $X \models \phi$ if and only if Verifier has a winning strategy in the corresponding game. In the next theorem, we show essentially the same property for truth functions. C.f. Lemma 5.12, Proposition 5.11 and Theorem 5.8 in [78].

Theorem 5.2.11. *Let $\phi[r_1,\dots,r_\gamma]$ be a context for* **PD** *and $\theta_1,\dots,\theta_\gamma$ formulas. Let N (with $|N| = n$) be the set of all indices of all propositional variables occurring in the formula $\phi[\theta_1,\dots,\theta_\gamma]$, and X an n-team on N. Then $X \models \phi[\theta_1,\dots,\theta_\gamma]$ iff there exists a truth function σ for $\phi[\theta_1,\dots,\theta_\gamma]$ over X.*

[3] In Definition 5.10 in [78], leave out game rules for quantifiers and make obvious modifications to game rules for atoms. We leave the details to the reader, as we will not go into this direction in this thesis.

Proof. The direction "$\Longleftarrow$" follows easily from the definition. For the other direction "$\Longrightarrow$", suppose $X \models \phi[\theta_1,\ldots,\theta_\gamma]$. Let $\mathfrak{T}_\phi = (T,\prec,r,\mathsf{f})$ be the syntax tree of ϕ. We define the value of σ on each node k of $\mathfrak{T}_\phi$ and check conditions (i)-(iii) of Definition 5.2.9 by induction on the depth of the nodes.

If $k = 0$ the root, then define $\sigma(0) = X$. Since $X \models \phi[\theta_1,\ldots,\theta_\gamma]$, condition (i) is satisfied for the node 0.

Suppose k is not a leaf node, $\sigma(k)$ has been defined already and conditions (i)-(iii) are satisfied for k. Let k_0, k_1 be the two children of k with $\mathsf{f}(k_0) = \psi$ and $\mathsf{f}(k_1) = \chi$ for some subformulas ψ, χ of ϕ. We distinguish two cases.

Case 1 $\mathsf{f}(k) = (\psi \wedge \chi)$. Define

$$\sigma(k_0) = \sigma(k_1) = \sigma(k).$$

Then condition (ii) for k_0, k_1 is satisfied. By induction hypothesis,

$$\sigma(k) \models (\psi \wedge \chi)[\theta_1,\ldots,\theta_\gamma],$$

thus

$$\sigma(k_0) \models \psi[\theta_1,\ldots,\theta_\gamma] \text{ and } \sigma(k_1) \models \chi[\theta_1,\ldots,\theta_\gamma],$$

namely condition (i) is satisfied for k_0, k_1.

Case 2 $\mathsf{f}(k) = (\psi \otimes \chi)$. By induction hypothesis,

$$\sigma(k) \models (\psi \otimes \chi)[\theta_1,\ldots,\theta_\gamma],$$

thus there exist n-teams $Y, Z \subseteq \sigma(k)$ such that $\sigma(k) = Y \cup Z$,

$$Y \models \psi[\theta_1,\ldots,\theta_\gamma] \text{ and } Z \models \chi[\theta_1,\ldots,\theta_\gamma].$$

Define $\sigma(k_0) = Y$ and $\sigma(k_1) = Z$. Then, conditions (i) and (ii) for k_0, k_1 are satisfied.

Hence σ is a truth function for $\phi[\theta_1,\ldots,\theta_\gamma]$ over X. ∎

The next lemma shows that a truth function is determined by its values on the leaves of the corresponding syntax tree.

Lemma 5.2.12. *Let $\phi[r_1,\ldots,r_\gamma]$ be a context for* **PD** *and $\theta_1,\ldots,\theta_\gamma$ formulas of* **PD**. *Let N (with $|N| = n$) be the set of all indices of all propositional variables occurring in the formula $\phi[\theta_1,\ldots,\theta_\gamma]$. Let $\mathfrak{T}_\phi = (T,\prec,r,\mathsf{f})$ be the syntax tree of ϕ. If $\sigma : \mathfrak{T}_\phi \to \wp(\mathbf{2^n})$ is a function satisfying conditions (ii),(iii) in Definition 5.2.9 and condition (i) with respect to $\theta_1,\ldots,\theta_\gamma$ for all leaf nodes, then σ is a truth function for $\phi[\theta_1,\ldots,\theta_\gamma]$.*

Proof. It suffices to prove that σ satisfies condition (i) with respect to $\theta_1,\ldots,\theta_\gamma$ for all nodes of $\mathfrak{T}_\phi$. We show this by induction on the depth of k.

Leaf nodes satisfy condition (i) by the assumption. Now, assume k is not a leaf. Then k has two children k_0, k_1 with $\mathsf{f}(k_0) = \psi$ and $\mathsf{f}(k_1) = \chi$ for some subformulas ψ, χ of ϕ. Since $\mathsf{d}(k_0), \mathsf{d}(k_1) > \mathsf{d}(k)$, by induction hypothesis, we have that

$$\sigma(k_0) \models \psi[\theta_1,\ldots,\theta_\gamma] \text{ and } \sigma(k_1) \models \chi[\theta_1,\ldots,\theta_\gamma]. \tag{5.7}$$

Now, we distinguish two cases.

Case 1: $\mathsf{f}(k) = (\psi \wedge \chi)$. Then, by condition (ii), $\sigma(k) = \sigma(k_0) = \sigma(k_1)$. It follows from (5.7) that $\sigma(k) \models (\psi \wedge \chi)[\theta_1, \ldots, \theta_\gamma]$.

Case 2: $\mathsf{f}(k) = (\psi \otimes \chi)$. Then, by condition (iii), $\sigma(k) = \sigma(k_0) \cup \sigma(k_1)$. It follows from (5.7) that $\sigma(k) \models (\psi \otimes \chi)[\theta_1, \ldots, \theta_\gamma]$.

■

5.3 Non-uniformly definable connectives in PD and PID

In this section, we prove that neither intuitionistic implication nor intuitionistic disjunction is uniformly definable in **PD**. At the end of the session, we include the result due to [11] that in the conjunction-free fragment of **PID**, every instance of $\wedge$ is definable, but $\wedge$ is not uniformly definable.

By Lemma 5.2.3, contexts for **PD** are monotone, thus **PD** cannot define uniformly non-monotone connectives. Below we show that intuitionistic implication is not monotone in the sense of Lemma 5.2.3, therefore not uniformly definable in **PD**.[4]

Theorem 5.3.1. *Intuitionistic implication is not uniformly definable in* **PD**.

Proof. Suppose there was a context $\phi[r_1, r_2]$ for **PD** which defines intuitionistic implication. Then for any formulas ψ, χ,

$$[\![\phi[\psi, \chi]]\!] = [\![\psi \to \chi]\!]. \tag{5.8}$$

For any non-empty team X, we have that

$$X \models \bot \to \bot \text{ and } X \not\models \top \to \bot.$$

Thus by (5.8),

$$X \models \phi[\bot, \bot] \text{ and } X \not\models \phi[\top, \bot].$$

But this contradicts Lemma 5.2.3 as $\bot \models \top$. ■

We now proceed to give another sufficient condition for connectives being not uniformly definable in **PD**. It will follow from this that intuitionistic disjunction is not uniformly definable in **PD**. To this end, we first make the following observations.

Fact 5.3.2. *Let $\phi[r_1, \ldots, r_\gamma]$ be a context for* **PD** *and $\theta_1, \ldots, \theta_\gamma$ formulas. Let σ be a truth function for $\phi[\theta_1, \ldots, \theta_\gamma]$ over a team X. In the syntax tree $\mathfrak{T}_\phi$ of ϕ, if a node k has no ancestor node with a label of the form $\psi \otimes \chi$, then $\sigma(k) = X$.*

Proof. Easy, by induction on the depth of k. ■

Lemma 5.3.3. *Let $\circledast$ be a γ-ary connective such that for every $1 \leq i \leq \gamma$, there are some formulas $\theta_1, \ldots, \theta_\gamma$ satisfying*

$$[\![\circledast(\theta_1, \ldots, \theta_\gamma)]\!] \not\subseteq [\![\theta_i]\!]. \tag{5.9}$$

If $\phi[r_1, \ldots, r_\gamma]$ is a context for **PD** *which uniformly defines $\circledast$, then in the syntax tree $\mathfrak{T}_\phi = (T, \prec, r, \mathsf{f})$, every leaf node labeled with r_i ($1 \leq i \leq \gamma$) has an ancestor node with a label of the form $\psi \otimes \chi$.*

[4] The author would like to thank Samson Abramsky for pointing out this fact.

Proof. Suppose there exists a leaf node k labeled with r_i which has no ancestor node with a label of the form $\psi \otimes \chi$. By assumption, for i, there exist formulas $\theta_1,\dots,\theta_\gamma$ satisfying (5.9). Let N (with $|N| = n$) be the set of all indices of all propositional variables occurring in the formula $\phi[\theta_1,\dots,\theta_\gamma]$. Take an n-team X such that

$$X \in [\![\circledast(\theta_1,\dots,\theta_\gamma)]\!] \text{ and } X \notin [\![\theta_i]\!].$$

Since $\phi[r_1,\dots,r_\gamma]$ uniformly defines $\circledast$,

$$X \in [\![\circledast(\theta_1,\dots,\theta_\gamma)]\!] = [\![\phi[\theta_1,\dots,\theta_\gamma]]\!],$$

thus $X \models \phi[\theta_1,\dots,\theta_\gamma]$. By Theorem 5.2.11, there is a truth function σ for $\phi[\theta_1,\dots,\theta_\gamma]$ over X. By the property of k and Fact 5.3.2, $\sigma(k) = X$. Thus

$$X \models r_i[\theta_1,\dots,\theta_\gamma], \text{ i.e., } X \in [\![\theta_i]\!],$$

which is a contradiction. ∎

The following elementary set theoretical lemma will be used in the proof of Lemma 5.3.5.

Lemma 5.3.4. *Let X,Y,Z be sets such that $|X| > 1$, $Y,Z \neq \emptyset$ and $X = Y \cup Z$. Then there exist $Y',Z' \subsetneq X$ such that $Y' \subseteq Y$, $Z' \subseteq Z$ and $X = Y' \cup Z'$.*

Proof. If $Y,Z \subsetneq X$, then taking $Y' = Y$ and $Z' = Z$, the lemma holds. Now, assume one of Y,Z equals X.

Case 1: $Y = Z = X$. Pick an arbitrary $a \in X$. Let $Y' = X \setminus \{a\} \subsetneq X$ and $Z' = \{a\}$. Since $|X| > 1$, we have that $Z' \subsetneq X$. Clearly, $X = (X \setminus \{a\}) \cup \{a\}$.

Case 2: Only one of Y and Z equals X. Without loss of generality, we assume that $Y = X$ and $Z \subsetneq X$. Let $Y' = X \setminus Z$ and $Z' = Z$. Clearly, $X = (X \setminus Z) \cup Z$ and $Y',Z' \subsetneq X$, as $\emptyset \neq Z \subsetneq X$.

∎

The next lemma is crucial to the proof of Theorem 5.3.6.

Lemma 5.3.5. *Let $\phi[r_1,\dots,r_\gamma]$ be a non-contradictory context for* **PD** *such that in the syntax tree $\mathfrak{T}_\phi = (T,\prec,r,\mathsf{f})$ of ϕ, every leaf node labeled with r_i ($1 \leq i \leq \gamma$) has an ancestor node labeled with a formula of the form $\psi \otimes \chi$. Let N (with $|N| = n$) be the set of all indices of all propositional variables occurring in the formula $\phi[\top,\dots,\top]$, and $\mathbf{2^n}$ the maximal n-team on N. If $\mathbf{2^n} \models \phi[\top,\dots,\top]$, then there exists a truth function σ for $\phi[\top,\dots,\top]$ over $\mathbf{2^n}$ such that $\sigma(x) \subsetneq \mathbf{2^n}$ for all leaf nodes x labeled with r_i ($1 \leq i \leq \gamma$).*

Proof. By Lemma 5.2.2, we may assume that $\phi[r_1,\dots,r_\gamma]$ does not contain a single contradictory subformula. Suppose $\mathbf{2^n} \models \phi[\top,\dots,\top]$. The required truth function σ over $\mathbf{2^n}$ is defined inductively on the depth of the nodes in the syntax tree $\mathfrak{T}_\phi$ in the same way as that in the proof of Theorem 5.2.11, except for the following case.

For each leaf node labeled with r_i, consider its ancestor node k with $\mathsf{f}(k) = (\psi \otimes \chi)$ of minimal depth, where $\psi,\chi \in \mathsf{Sub}(\phi)$ (the existence of such k is guaranteed by assumption). Let k_0, k_1 be the two children of k. Assuming that $\sigma(k)$ has been defined already, we now define $\sigma(k_0)$ and $\sigma(k_1)$.

By induction hypothesis,

$$\sigma(k) \models (\psi \otimes \chi)[\top, \ldots, \top].$$

The minimality of k implies that k has no ancestor node labeled with $\theta_0 \otimes \theta_1$, thus $\sigma(k) = \mathbf{2^n}$ by Fact 5.3.2. Then there exist teams $Y_0, Z_0 \subseteq \sigma(k) = \mathbf{2^n}$ such that $\mathbf{2^n} = Y_0 \cup Z_0$,

$$Y_0 \models \psi[\top, \ldots, \top] \text{ and } Z_0 \models \chi[\top, \ldots, \top].$$

Claim: There are non-empty teams Y, Z such that $\mathbf{2^n} = Y \cup Z$ and

$$Y \models \psi[\top, \ldots, \top] \text{ and } Z \models \chi[\top, \ldots, \top]. \tag{5.10}$$

Proof of Claim: If $Y_0, Z_0 \neq \emptyset$, then taking $Y = Y_0$ and $Z = Z_0$, the claim holds. Now, suppose one of Y_0, Z_0 is empty. Without loss of generality, we may assume that $Y_0 = \emptyset$. Then let $Z := Z_0 = \mathbf{2^n}$. Since $\psi[r_1, \ldots, r_\gamma] \not\approx \bot$, by Corollary 5.2.4 and locality of **PD**, there exists a non-empty n-team $Y \subseteq \mathbf{2^n}$ such that $Y \models \psi[\top, \ldots, \top]$. Thus Y and Z are as required. $\dashv$

Now, since $|\mathbf{2^n}| > 1$, by Lemma 5.3.4, there are teams $Y_0, Z_0 \subsetneq \mathbf{2^n}$ such that $Y_0 \subseteq Y$, $Z_0 \subseteq Z$ and $Y_0 \cup Z_0 = \mathbf{2^n}$. Define $\sigma(k_0) = Y_0$ and $\sigma(k_1) = Z_0$. Clearly, condition (iii) of Definition 5.2.9 for k_0, k_1 is satisfied. Moreover, by downwards closure, it follows from (5.10) that condition (i) for k_0, k_1 is also satisfied. Hence, such defined σ is a truth function for $\phi[\top, \ldots, \top]$ over $\mathbf{2^n}$.

It remains to check that $\sigma(x) \subsetneq \mathbf{2^n}$ for all leaf nodes x labeled with r_i ($1 \leq i \leq \gamma$). By assumption, there exists an ancestor k of x labeled with $(\psi \otimes \chi)$ of minimal depth. One of k's two children, say k_j, must be an ancestor of x or $k_j = x$. Thus, by Fact 5.2.10 and the construction of σ, we obtain that $\sigma(x) \subseteq \sigma(k_j) \subsetneq \mathbf{2^n}$. ■

Recall that in the proof of Theorem 4.4.1, for each n-element set $N \subseteq \mathbb{N}$, for each non-empty n-team X on N, we have constructed an n-formula $\Theta^\star_X$ of **PD** such that

$$Y \models \Theta^\star_X \iff X \nsubseteq Y$$

for any n-team Y on N.

Now, we give the intended sufficient condition for a non-contradictory connective being not uniformly definable in **PD**. In the proof, we will make use of the formula $\Theta^\star_X$.

Theorem 5.3.6. *Every non-contradictory γ-ary connective $\divideontimes$ satisfying the following conditions is not uniformly definable in* **PD***:*

(i) *For every $1 \leq i \leq \gamma$, there exist some formulas $\theta_1, \ldots, \theta_\gamma$ of* **PD** *satisfying*

$$[\![\divideontimes(\theta_1, \ldots, \theta_\gamma)]\!] \nsubseteq [\![\theta_i]\!]. \tag{5.11}$$

(ii) *There are formulas $\delta_1, \ldots, \delta_\gamma$ of* **PD** *such that $\models \divideontimes(\delta_1, \ldots, \delta_\gamma)$.*

(iii) *For any n-element set $N \subseteq \mathbb{N}$, there exist $1 \leq j_1 < \cdots < j_m \leq \gamma$ such that*

$$\mathbf{2^n} \not\models \circledast(\alpha_1, \ldots, \alpha_\gamma), \tag{5.12}$$

where $\mathbf{2^n}$ is the maximal n-team on N, and for each $1 \leq i \leq \gamma$,

$$\alpha_i = \begin{cases} \Theta^{\star}_{\mathbf{2^n}} & \textit{if } i = j_a \text{ , } 1 \leq a \leq m \\ \top & \textit{otherwise.} \end{cases} \tag{5.13}$$

Proof. Suppose $\circledast$ was uniformly definable in **PD**. Then there would exist a context $\phi[r_1, \ldots, r_\gamma]$ for **PD** such that for all **PD** formulas $\theta_1, \ldots, \theta_\gamma$,

$$[\![\phi[\theta_1, \ldots, \theta_\gamma]]\!] = [\![\circledast(\theta_1, \ldots, \theta_\gamma)]\!]. \tag{5.14}$$

Since $\circledast$ satisfies condition (i), by Lemma 5.3.3, in the syntax tree $\mathfrak{T}_\phi = (T, <, r, \mathsf{f})$ of $\phi[r_1, \ldots, r_\gamma]$, each node labeled with r_i ($1 \leq i \leq \gamma$) has an ancestor node labeled with a formula of the form $\psi \otimes \chi$.

By condition (ii), $\models \circledast(\delta_1, \ldots, \delta_\gamma)$ for some formulas $\delta_1, \ldots, \delta_\gamma$, thus by (5.14),

$$\models \phi[\delta_1, \ldots, \delta_\gamma].$$

As $\delta_i \models \top$ for all $1 \leq i \leq \gamma$, by Lemma 5.2.3 we have that

$$\models \phi[\top, \ldots, \top].$$

Let N (with $|N| = n$) be the set of all indices of all propositional variables occurring in $\phi[\top, \ldots, \top]$. Let $\mathbf{2^n}$ be the maximal n-team on N. We have that

$$\mathbf{2^n} \models \phi[\top, \ldots, \top].$$

Since $\phi[r_1, \ldots, r_\gamma] \not\approx \bot$, by Lemma 5.3.5 there exists a truth function σ for $\phi[\top, \ldots, \top]$ over $\mathbf{2^n}$ such that $\sigma(x) \subsetneq \mathbf{2^n}$ for all leaf nodes x labeled with r_i ($1 \leq i \leq \gamma$) in $\mathfrak{T}_\phi$.

By condition (iii), for the set N, there exist $1 \leq j_1 \leq \cdots \leq j_m \leq \gamma$ such that (5.12) holds. On the other hand, for each j_a ($1 \leq a \leq m$), as $\mathbf{2^n} \nsubseteq \sigma(x)$ holds for every leaf node x labeled with r_{j_a}, we have that $\sigma(x) \models \Theta^{\star}_{\mathbf{2^n}}$, i.e.,

$$\sigma(x) \models \mathsf{f}(x)[\alpha_1, \ldots, \alpha_\gamma],$$

where for each $1 \leq i \leq \gamma$, Equation (5.13) is the case.

Thus, by Lemma 5.2.12, σ is also a truth function for $\phi[\alpha_1, \ldots, \alpha_\gamma]$ over $\mathbf{2^n}$, where for each $1 \leq i \leq \gamma$, Equation (5.13) is the case, thereby

$$\mathbf{2^n} \models \phi[\alpha_1, \ldots, \alpha_\gamma].$$

Thus by (5.14), we obtain $\mathbf{2^n} \models \circledast(\alpha_1, \ldots, \alpha_\gamma)$. But this contradicts (5.12). ■

Theorem 5.3.7. *Intuitionistic disjunction is not uniformly definable in* **PD**.

Proof. It suffices to check that intuitionistic disjunction satisfies conditions (i)-(iii) in Theorem 5.3.6. Condition (i) is satisfied, since, e.g., $[\![\bot \vee \top]\!] \nsubseteq [\![\bot]\!]$ and $[\![\top \vee \bot]\!] \nsubseteq [\![\bot]\!]$. Condition (ii) is satisfied since, e.g., $\models \top \vee \top$. Lastly, for any n-element set $N \subseteq \mathbb{N}$, clearly $\mathbf{2^n} \not\models \Theta^{\star}_{\mathbf{2^n}} \vee \Theta^{\star}_{\mathbf{2^n}}$, thus condition (iii) is satisfied. ■

We have already proved that intuitionistic implication is not uniformly definable in **PD** in Theorem 5.3.1 by observing that intuitionistic implication is not monotone. In fact, the non-uniform definability of intuitionistic implication in **PD** also follows from Theorem 5.3.6, as intuitionistic implication also satisfies conditions (i)-(iii). Indeed, we have that (i) $[\![\bot \to \bot]\!] \nsubseteq [\![\bot]\!]$, (ii) $\models \top \to \top$ and (iii) $\mathbf{2^n} \not\models \top \to \Theta^{\star}_{\mathbf{2^n}}$, for any n-element set $N \subseteq \mathbb{N}$.

We end this section by including a result on the issue of definability and uniform definability of connectives in the logic **PID**. The proof of the next theorem is due to [11].

Theorem 5.3.8 ([11])**.** *In the fragment of* **PID** *without conjunction* $\wedge$ *(i..e. the logic* **PID**$[=\!(\cdot),\bot,\vee,\to]$*), every instance of* $\wedge$ *is definable but* $\wedge$ *is not uniformly definable.*

Proof. Proposition 3.5.5 in [11] shows that $\wedge$ is not uniformly definable in the fragment of inquisitive logic **InqL** without $\wedge$ (i.e. **InqL**$[\bot,\vee,\to]$). By Lemma 4.2.5, dependence atoms are finable in terms of $\vee$, thus **PID**$[=\!(\cdot),\bot,\vee,\to] =$ **PID**$[\bot,\vee,\to] =$ **InqL**$[\bot,\vee,\to]$.

Moreover, Proposition 2.5.2 in [11] shows that every non-empty downwards closed class $\mathcal{K}$ of n-teams is characterizable by a formula of **InqL**$[\bot,\vee,\to]$. This implies that **PID**$[=\!(\cdot),\bot,\vee,\to]$ is also a maximal downwards closed logic, therefore every instance of $\wedge$ is definable in the logic **PID**$[=\!(\cdot),\bot,\vee,\to]$. The basic idea of this proof is the following. By Theorem 4.2.8, $\mathcal{K} = [\![\bigvee_{X\in\mathcal{K}} \Psi_X]\!]$. Each Ψ_X is a (double) negated formula, thus flat (c.f. Corollary 2.1.2). Then, to evaluate the formula Ψ_X, it suffices to consider its satisfiability on singleton teams only. But on singleton teams, the formula Ψ_X behaves like a formula in **CPL** (c.f. Lemma 2.1.4 and Lemma 4.1.10). In **CPL**, the set $\{\neg,\vee\}$ of connectives is functionally complete, therefore Ψ_X viewed as a **CPL** formula is equivalent to a formula Ψ'_X of **CPL** with only $\neg$ and $\vee$ as connectives. Putting the argument together, we obtain that $\mathcal{K} = [\![\bigvee_{X\in\mathcal{K}} \Psi'_X]\!]$. ■

5.4 Concluding remarks

Team semantics was originally designed by Hodges ([50],[52]) for independence friendly logic in order to meet one of the fundamental needs of logic and language, namely "*compositionality*" (see e.g. [55],[53] for an overview). The idea of team semantics is a natural and powerful generalization of the usual semantics of classical logic. This new methodology provides a wide and solid framework for logics of dependence and independence.

On the other hand, the results of this chapter, as well as those in [11], [32] show that for logics L based on team semantics, even if every instance of a compositional connective $\circledast$ (i.e. Equation (5.1 holds for $\circledast$) is definable in L, a uniform (or compositional) definition for $\circledast$ does not necessarily exist in L. This phenomenon seems to indicate that the compositionality or uniformity in another level is lost in team semantics. In the author's opinion, this problem reflects some deep content of team semantics that surely deserves further investigation.

Chapter 6

Modal Intuitionistic Dependence Logic

In the preceding chapters, we studied first-order and propositional intuitionistic dependence logic. We devote the following two chapters to *modal intuitionistic dependence logic*.

Väänänen introduced in [79] *modal dependence logic* (**MD**), which incorporates the notion of "dependence" into modal logic. Loosely speaking, modal dependence logic can be understood as propositional dependence logic with modalities. A typical formula of **MD** can be of the following form:

$$\Box\Diamond =\!(p,q).$$

A corresponding practical statement can be as follows:

> *However the environment will be degraded in the next 100 years, it is possible that in 200 years from now, whether the earth will be destroyed depends only on whether there is another planet that crashes into the earth.*

The meaning of this formula is given on the usual Kripke models (of modal logic). As in the case of propositional logic, the dependence atom $=\!(p,q)$ only makes sense when it is evaluated on a set of possible worlds instead of a single world. These sets are called *teams* and the corresponding semantics is referred to as *team semantics*. Known results of model-theoretic properties of **MD** can be found in [76], and its computational issues are investigated in [76], [67], [17], [66], etc.

As discussed in Section 4.2, propositional dependence logic does not have a satisfactory treatment for conditional statements (especially conditionals of dependence facts). For the same reason, conditional modal statements such as the following one cannot have a reasonable interpretation in modal dependence logic:

> *However the environment will be degraded in the next 100 years, it is possible that in 200 years from now, if whether the earth will be destroyed depends only on whether there is another planet that crashes into the earth, then whether the human being will migrate to other planets depends only on whether the crash will occur.*

We propose to interpret the above conditional statement as

$$\Box\Diamond\big(=\!(p,q)\to=\!(p,r)\big),$$

where $\to$ is the intuitionistic implication studied in the preceding chapters.

In view of this, in this chapter, we introduce *modal intuitionistic dependence logic* (**MID**), which is the modal variant of propositional intuitionistic dependence logic studied in Section 4.2. We include in this chapter preliminary results on modal intuitionistic dependence logic. We give basic definitions in Section 6.1. In Section 6.2, we show that **MID** is a weak intermediate modal logic **K** with an axiom characterizing determinacy. We also give a translation from **MD** into **MID**. In Section 6.3, we reveal a connection between modal intuitionistic dependence logic and intuitionistic modal logic **IK**, defined independently in by Edwald [22], Fischer Servi [26] and Plotkin and Stirling [73], and show that model intuitionistic dependence logic is complete with respect to a certain set of finite bi-relation Kripke models. In Section 6.4, we give concluding remarks and open problems.

We, by no means, claim that the work in this chapter is complete in any sense for the investigation of modal intuitionistic dependence logic, however we present this chapter with the hope that these results will throw some light on the future research in this area.

6.1 Modal dependence logic and modal intuitionistic dependence logic

In this section, we give formal definition of modal dependence logic and modal intuitionistic dependence logic, and list their basic properties.

Definition 6.1.1. Let $p, p_1, \dots, p_k$ be propositional variables.

- Well-formed formulas of *modal dependence logic* (**MD**) are given by the following grammar:
$$\phi ::= p \mid \neg p \mid =\!(p_1,\dots,p_k) \mid \phi\wedge\phi \mid \phi\otimes\phi \mid \Box\phi \mid \Diamond\phi.$$
- Well-formed formulas of *modal intuitionistic dependence logic* (**MID**) are given by the following grammar:
$$\phi ::= p \mid \bot \mid =\!(p_1,\dots,p_k) \mid \phi\wedge\phi \mid \phi\vee\phi \mid \phi\to\phi \mid \Box\phi \mid \Diamond\phi.$$

In this chapter, we will mainly focus on **MID**. As before, to simplify notations, we apply Convention 1.1.2 to **MID**, in particular, $\phi\to\bot$ is abbreviated as $\neg\phi$ for any formula ϕ. In the usual modal logic, the modalities $\Box$ and $\Diamond$ are dual to each other, namely, e.g., $\Box\phi$ is equivalent to $\neg\Diamond\neg\phi$. However, as we will see from the semantics given below, this is not true for **MID**.

As for the usual modal logic, the semantics of **MD** and **MID** is defined with respect to the usual Kripke models of modal logic (not Kripke models of intuitionistic logic!).

Definition 6.1.2. A *(modal) Kripke frame* is an ordered pair $\mathfrak{F}=(W,R)$ consisting of a nonempty set W, a binary relation R on W. The set W is called the *domain* of $\mathfrak{F}$.

Elements of W are called *states*, *possible worlds*, *nodes* or *points*, while subsets of of W are called *teams* of $\mathfrak{F}$, i.e. a team is a set of possible worlds.

A *(modal) Kripke model* is a triple $\mathfrak{M} = (W, R, V)$, where (W, R) is a (modal) Kripke frame and $V : \text{Prop} \to \wp(W)$ is a valuation function from the set Prop of all propositional variables into the powerset of W.

If wRv, then v is called a *successor* of w, and w is called a *predecessor* of v. For any team X of a Kripke frame $\mathfrak{F}$, we define

$$R(X) = \{w \in W \mid \exists v \in X, \text{ s.t. } vRw\}.$$

Clearly, $R(\emptyset) = \emptyset$. In case $X = \{w\}$, we write $R(w)$ instead of $R(\{w\})$. If Y is a team such that

$$Y \subseteq R(X) \text{ and } \forall w \in X,\ R(w) \cap Y \neq \emptyset, \tag{6.1}$$

then Y is called a *successor team* of X, and we write XRY. Clearly, $\emptyset R \emptyset$ and the empty team is the unique successor team of the empty team itself.

A valuation $V : \text{Prop} \to \wp(W)$ of a Kripke model induces a function $\pi_V : W \to \wp(\text{Prop})$ defined by

$$\pi_V(w) = \{p \in \text{Prop} \mid w \in V(p)\}.$$

Conversely, a function $\pi_V : W \to \wp(\text{Prop})$ determines a valuation $V_\pi : \text{Prop} \to \wp(W)$ defined as

$$V_\pi(p) = \{w \in W \mid p \in \pi(w)\}.$$

In this thesis, we will use the terminologies V and π simultaneously.

The satisfaction relations of **MD** and **MID** are defined with respect to *teams*. A game theoretic semantics of **MD** based on *set game* is given in [79], such semantics can be generalized to **MID**, but in this thesis, we consider *team semantics* only. Below, we present the formal definition of team semantics of these logics.

Definition 6.1.3 (Team Semantics)**.** We inductively define the notion of a formula ϕ of **MD** or **MID** being *satisfied* in a Kripke model $\mathfrak{M} = (W, R, V)$ on a team $X \subseteq W$, denoted by $\mathfrak{M}, X \models \phi$, as follows:

- $\mathfrak{M}, X \models p$ iff $X \subseteq V(p)$;
- $\mathfrak{M}, X \models \neg p$ iff $X \cap V(p) = \emptyset$;
- $\mathfrak{M}, X \models \bot$ iff $X = \emptyset$;
- $\mathfrak{M}, X \models {=}(p_1, \cdots, p_k)$ iff for any $w, v \in X$, if

$$\pi_V(w) \cap \{p_1, \ldots, p_{k-1}\} = \pi_V(v) \cap \{p_1, \ldots, p_{k-1}\},$$

  then $\pi_V(w) \cap \{p\} = \pi_V(v) \cap \{p\}$;
- $\mathfrak{M}, X \models \phi \wedge \psi$ iff $\mathfrak{M}, X \models \phi$ and $\mathfrak{M}, X \models \psi$;
- $\mathfrak{M}, X \models \phi \otimes \psi$ iff there exist teams $Y, Z \subseteq X$ such that $X = Y \cup Z$,

$$\mathfrak{M}, Y \models \phi \text{ and } \mathfrak{M}, Z \models \psi;$$

- $\mathfrak{M}, X \models \phi \vee \psi$ iff $\mathfrak{M}, X \models \phi$ or $\mathfrak{M}, X \models \psi$;
- $\mathfrak{M}, X \models \phi \to \psi$ iff for any team $Y \subseteq X$,

$$\mathfrak{M}, Y \models \phi \Longrightarrow \mathfrak{M}, Y \models \psi;$$

- $\mathfrak{M}, X \models \Box\phi$ iff $\mathfrak{M}, R(X) \models \phi$;
- $\mathfrak{M}, X \models \Diamond\phi$ iff there exists a team Y such that XRY and $\mathfrak{M}, Y \models \phi$.[1]

Let L be any of the logics **MD** and **MID**. For any formula ϕ of L, if $\mathfrak{M}, X \models \phi$ holds for all teams X of $\mathfrak{M}$, then we say that ϕ is *true* in the Kripke model $\mathfrak{M}$, denoted by $\mathfrak{M} \models \phi$. If $\mathfrak{M} \models \phi$ holds for all Kripke models $\mathfrak{M}$, then we say that ϕ is *valid* in the logic L, denoted by $\models_{\mathsf{L}} \phi$ or simply $\models \phi$. The notions of *logical consequence* and *logical equivalence* are defined analogously to the first-order or propositional case. The *logic* **MID** is the set of all valid formulas of **MID**, namely

$$\mathbf{MID} = \{\phi : \models_{\mathbf{MID}} \phi\};$$

similarly for the *logic* **MD**.

Analogous to the first-order or propositional case, both of the logics **MD** and **MID** have the downwards closure property, the empty team property and the locality property. Moreover, Deduction Theorem holds in **MID**.

Theorem 6.1.4 (Downwards Closure)**.** *For any formula ϕ of* **MD** *or* **MID***, any Kripke model $\mathfrak{M}$, any teams X, Y of $\mathfrak{M}$,*

$$[\mathfrak{M}, X \models \phi \text{ and } Y \subseteq X] \Longrightarrow \mathfrak{M}, Y \models \phi.$$

Proof. Easy. ■

Lemma 6.1.5 (Empty Team Property)**.** **MD** *and* **MID** *have the* empty team property*, that is, every formula ϕ of any of the logics is satisfied on the empty team of any Kripke model $\mathfrak{M}$, i.e. $\mathfrak{M}, \emptyset \models \phi$.*

Proof. Easy. ■

Lemma 6.1.6 (Locality)**.** *Let $\phi(p_{i_1}, \dots, p_{i_n})$ be a formula of* **MD** *or* **MID***. For any Kripke model $\mathfrak{M} = (W, R, V)$, any team X of $\mathfrak{M}$, let $\mathfrak{M}^n = (W, R, V^n)$, where $V^n = V \upharpoonright \{p_{i_1}, \dots, p_{i_n}\}$. We have that*

$$\mathfrak{M}, X \models \phi \iff \mathfrak{M}^n, X \models \phi.$$

Proof. Easy. ■

[1] By the downwards closure property (see Theorem 6.1.4), it is equivalent to define this case as

"$\mathfrak{M}, X \models \Diamond\phi$ iff $\mathfrak{M}, Y \models \phi$ for some teams Y, Z such that XRZ and $Z \subseteq Y$.

However, we choose to use the strong version of the definition for reasons that will become clear in Definition 6.3.5 of powerset Kripke models in Section 6.3.

Lemma 6.1.7 (Deduction Theorem)**.** *For any formulas ϕ and ψ of* **MID***,*

$$\phi \models \psi \iff \models \phi \to \psi.$$

Proof. By the downwards closure property. ■

As in the first-order or propositional case, formulas ϕ satisfying

$$\mathfrak{M}, X \models \phi \iff \forall s \in X,\ \mathfrak{M}, \{s\} \models \phi$$

for all Kripke models $\mathfrak{M}$, all teams X of $\mathfrak{M}$ are called *flat* formulas. A formula built from propositional variables and negated propositional variables by conjunction $\wedge$, tensor disjunction $\otimes$ and the two modalities $\Box$, $\Diamond$ is called a *classical formula*. It is easy to show that formulas of **MID** or **MD** which do not contain dependence atoms or intuitionistic disjunction are flat. In particular, classical formulas are flat. Moreover, Sevenster showed [76] that on singleton teams, **MD** is equivalent to the usual modal logic.

The underlying propositional logics of **MD** and **MID** are **PD** and **PID**, therefore many of the relevant properties in Chapter 4 are true also for **MD** and **MID**. In particular, in **MID**, dependence atoms are eliminatable as

$$=\!(p_1,\dots,p_k) \equiv \big((p_1 \vee \neg p_1) \wedge \cdots \wedge (p_{k-1} \vee \neg p_{k-1})\big) \to (p_k \vee \neg p_k),$$

negated formulas are flat, all axiom schemas of the intermediate logic ND or KP, as well as the atomic double negation law $\neg\neg p \to p$ are valid, and **MID** is not closed under uniform substitution. Moreover, Lemma 2.1.3 holds for **MID**: a formula ϕ of **MID** is flat if and only if $\models_{\mathbf{MID}} \neg\neg\phi \leftrightarrow \phi$.

In **MD** and **MID**, we adopt the usual notions of disjoint unions of models, generated submodels, p-morphisms for the usual modal logic (see e.g. [4], [9]).

Definition 6.1.8. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be Kripke models.

- The *disjoint union* $\mathfrak{M} \uplus \mathfrak{M}' = (W_0, R_0, V_0)$ of $\mathfrak{M}$ and $\mathfrak{M}'$ is defined as

  $$W_0 = W \uplus W',\ R_0 = R \uplus R' \text{ and } V_0(p) = V(p) \uplus V'(p)$$

  for all $p \in$ Prop, where $\uplus$ takes the disjoint union of two sets.

- $\mathfrak{M}'$ is called a *submodel* of $\mathfrak{M}$ if

  $$W' \subseteq W,\ R' = R \cap (W' \times W') \text{ and } \pi_{V'} = \pi_V \restriction W'.$$

  A submodel $\mathfrak{M}'$ of $\mathfrak{M}$ is called a *generated submodel* of $\mathfrak{M}$ if $R(W') = W'$.

- A function f from W to W' is called a *p-morphism* or *bounded-morphism* of $\mathfrak{M}$ into $\mathfrak{M}$' if it satisfies the following conditions:

  (i) $\pi_V(w) = \pi_{V'}(f(w))$ for all $w \in W$;

  (ii) for any $w, v \in W$, wRv implies $f(w)R'f(v)$;

  (iii) $f(w)R'v'$ implies $\exists v \in W (wRv \wedge f(v) = v')$.

We leave it to the reader to check that truth of formulas of **MD** and **MID** are preserved under taking disjoint unions, generated submodels and p-morphic images.

Theorem 6.1.9. *Let* $\mathfrak{M} = (W, R, V)$*,* $\mathfrak{M}' = (W', R', V')$ *and* $\mathfrak{M}_i = (W_i, R_i, V_i)$ *(*$i \in I$*) be Kripke models.*

(i) *For every* $i \in I$ *and every* $X \subseteq W_i$*,*

$$\mathfrak{M}_i, X \models \phi \iff \biguplus_{i \in I} \mathfrak{M}_i, X \models \phi.$$

(ii) *If* $\mathfrak{M}$ *is a generated submodel of* $\mathfrak{M}'$*, then for all* $X \subseteq W$*,*

$$\mathfrak{M}, X \models \phi \iff \mathfrak{M}', X \models \phi.$$

(iii) *If* $f : \mathfrak{M} \to \mathfrak{M}'$ *is a p-morphism, then for all* $X \subseteq W$*,*

$$\mathfrak{M}, X \models \phi \iff \mathfrak{M}', f(X) \models \phi.$$

We end this section by verifying the disjunction property of **MID**.

Theorem 6.1.10 (Disjunction Property)**.** **MID** *has the disjunction property, that is, for any formulas* ϕ *and* ψ *of* **MID***, if* $\models \phi \vee \psi$*, then* $\models \phi$ *or* $\models \psi$*.*

Proof. Suppose $\mathfrak{M}_0, X_0 \not\models \phi$ and $\mathfrak{M}_1, X_1 \not\models \psi$. Let $\mathfrak{M} = \mathfrak{M}_0 \uplus \mathfrak{M}_1$, $X = X_0 \cup X_1$. Then we have that

$$\mathfrak{M}, X_0 \not\models \phi \text{ and } \mathfrak{M}, X_1 \not\models \psi,$$

thus as $X_0, X_1 \subseteq X$, by the downwards closure property,

$$\mathfrak{M}, X \not\models \phi \text{ and } \mathfrak{M}, X \not\models \psi.$$

Hence $\mathfrak{M}, X \not\models \phi \vee \psi$. ■

6.2 Modal intuitionistic dependence logic and weak intermediate modal logic

In this chapter, we investigate the connection between modal intuitionistic dependence logic and intuitionistic modal logic.

Intuitionistic modal logic has been studied extensively by philosophers, mathematicians and computer scientists since around 1950's. The basic idea is to combine intuitionistic logic and (classical) modal logic in order to obtain a logic which has both constructive and intensional content. As it turned out, this theoretically natural extension finds increasingly wide applications in the field of practical computer science. For a survey on intuitionistic modal logic, see e.g. [77], [86].

Modal intuitionistic dependence logic has a close connection with the intuitionistic modal logic **IK** defined independently by Edwald [22], Fischer Servi [26] and Plotkin and Stirling [73]. We prove in this section that **MID** is between **IK** with d(eterministic) axiom $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$ and the classical modal logic **K** with d axiom. We also prove that **MD** is a sublogic of **MID** in the sense that all formulas of **MD** are expressible in **MID**.

We first recall relevant definitions concerning intuitionistic modal logic **IK**. The (classical) modal logic **K** is defined as follows.

Definition 6.2.1. The modal logic **K** is the smallest set of formulas that contains the following axioms and is closed under the following rules:

1.	All axioms of **CPL**	4.	$(\Diamond\phi \to \Box\psi) \to \Box(\phi \to \psi)$
2.	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	5.	Modus Ponens (MP)
3.	$\Diamond\phi \leftrightarrow \neg\Box\neg\psi$	6.	Uniform Substituion (Sub)

All connectives and modalities in the above axioms have classical interpretations, whereas in intuitionistic modal logic, these logical constants have constructive meanings. In other words, the underlying propositional logic of classical modal logic is classical propositional logic, whereas intuitionistic modal logic is based on intuitionistic propositional logic. Among the independent intuitionistic modal logics considered in the literature, the most relevant one to our purpose is the following logic **IK** (also known as **FS**), introduced independently by Edwald [22], Fischer Servi [26][27] and Plotkin and Stirling in [73]. The axioms we listed below are given by Plotkin and Stirling in [73]. **IK** is a proper sublogic of **K**, i.e., **IK** $\subset$ **K**.

Definition 6.2.2 ([73])**.** The intuitionistic modal logic **IK** is the smallest set of formulas that contains the following axioms and is closed under the following rules:

1.	All axiom schemas of **IPL**	6.	$(\Diamond\phi \to \Box\psi) \to \Box(\phi \to \psi)$
2.	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	7.	Modus Ponens (MP)
3.	$\Box(\phi \to \psi) \to (\Diamond\phi \to \Diamond\psi)$	8.	Uniform Substituion (Sub)
4.	$\neg\Diamond\bot$	9.	Generalization (Gen): $\phi/\Box\phi$
5.	$\Diamond(\phi \vee \psi) \to (\Diamond\phi \vee \Diamond\psi)$		

In the next lemma we check that **IK** $\subseteq$ **MID**. However, **IK** $\neq$ **MID** as **MID** is not closed under Sub.

Lemma 6.2.3. *Let ϕ and ψ be formulas of* **MID**. *We have the following:*

1. *All axiom schemas of* **IPL** *are valid in* **MID**
2. *All axiom schemas of* **IK** *are valid in* **MID**
3. **MID** *is closed under* MP *and* Gen

Proof. Clearly, all axioms of **IPL** are valid in **MID**, and **MID** is closed under MP and Gen. It remains to check that all the other axiom schemas of **IK** are valid in **MID**. Axiom schemas 2-5 are verified straightforwardly. We will only check the validity of the axiom schema 6. By Deduction Theorem, it suffices to show that

$$\Diamond\phi \to \Box\psi \models \Box(\phi \to \psi).$$

Let $\mathfrak{M} = (W, R, V)$ be a Kripke model and $X \subseteq W$. Assume $\mathfrak{M}, X \models \Diamond\phi \to \Box\psi$. It suffices to show that $\mathfrak{M}, R(X) \models \phi \to \psi$. For any team $Y \subseteq R(X)$ such that $\mathfrak{M}, Y \models \phi$, consider the team

$$Z = \{w \in X \mid wRv \text{ for some } v \in Y\}.$$

Clearly ZRY, thus $\mathfrak{M}, Z \models \Diamond\phi$. Since $\mathfrak{M}, X \models \Diamond\phi \to \Box\psi$ and $Z \subseteq X$, we obtain that $\mathfrak{M}, Z \models \Box\psi$, thereby $\mathfrak{M}, R(Z) \models \psi$. As $Y \subseteq R(Z)$, by the downwards closure property, $\mathfrak{M}, Y \models \psi$. Hence $\mathfrak{M}, R(X) \models \phi \to \psi$, as required. ■

In **IK**, the two modalities $\Box$ and $\Diamond$ are independent of each other, in particular, $\neg\Box\phi \to \Diamond\neg\phi$ is not derivable and not a valid formula of **IK**. We invite the reader to check that this is also the case for **MID**.

In the next lemma we list without a proof the derivable formulas of **IK** that are most relevant for the rest of this chapter.

Lemma 6.2.4. *The following formulas are valid in* **MID** *as they are derivable in* **IK***:*

(i) $\vdash_{\mathbf{IK}} \neg\Diamond\phi \leftrightarrow \Box\neg\phi$;

(ii) $\vdash_{\mathbf{IK}} \Diamond(\phi \vee \psi) \leftrightarrow (\Diamond\phi \vee \Diamond\psi)$.

(iii) $\vdash_{\mathbf{IK}} (\Box\phi \vee \Box\psi) \to \Box(\phi \vee \psi)$;

(iv) $\vdash_{\mathbf{IK}} \Diamond\neg\phi \to \neg\Box\phi$.

In addition to all derivable formulas of **IK**, the formulas listed in the next lemma are also valid in **MID**. It will turn out later in this and the next sections that these formulas are of particular interests.

Lemma 6.2.5.

(i) $\models_{\mathbf{MID}} \neg\neg p \to p$;

(ii) $\models_{\mathbf{MID}} \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$;

(iii) $\models_{\mathbf{MID}} \neg\Box\phi \to \Diamond\neg\phi$, *whenever* ϕ *is flat; in particular* $\models_{\mathbf{MID}} \neg\Box\neg\phi \to \Diamond\neg\neg\phi$.

Proof. Item (i) follows from the fact that $\models_{\mathbf{PID}} \neg\neg p \to p$ (c.f. Lemma 2.1.3) and item (ii) is straightforward to verify. We only show (iii).

Let $\mathfrak{M} = (W, R, V)$ and $X \subseteq W$. We show $\neg\Box\phi \models \Diamond\neg\phi$, whenever ϕ is flat. Suppose $\mathfrak{M}, X \models \neg\Box\phi$. Then for any $w \in X$, we have that $\mathfrak{M}, \{w\} \not\models \Box\phi$, i.e. $\mathfrak{M}, R(w) \not\models \phi$. Since ϕ is flat, there exists $v_w \in R(w)$ such that $\mathfrak{M}, \{v_w\} \not\models \phi$. Define

$$Y = \{v_w \in R(X) \mid w \in X\}.$$

For any $v_w \in Y$, we have that $\mathfrak{M}, \{v_w\} \models \neg\phi$, thus $\mathfrak{M}, Y \models \neg\phi$ as $\neg\phi$ is flat. Clearly, XRY, thus $\mathfrak{M}, X \models \Diamond\neg\phi$, as required. ■

Let us call the formula

$$\Box(\phi \vee \psi) \to \Box\phi \vee \Box\psi$$

the d *axiom*. The d axiom is very strong, as it is known that the modal logic

$$\mathbf{K}\mathsf{d} := \mathbf{K} \oplus \mathsf{d}^2$$

is complete with respect to the class of deterministic frames (see [15]). A Kripke frame $\mathfrak{F} = \langle W, R \rangle$ is called *deterministic* if every node has at most one successor, i.e., for all $w, u, v \in W$,

$$wRu,\ wRv \Longrightarrow u = v.$$

We now show that over deterministic models (i.e. models whose underlying frames are deterministic), on singleton teams, **MID** is equivalent to **K**d.

Lemma 6.2.6. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a deterministic model and $w \in W$. Then for any modal formula ϕ,*

$$\mathfrak{M}, w \models_{\mathbf{Kd}} \phi \iff \mathfrak{M}, \{w\} \models_{\mathbf{MID}} \phi.$$

Proof. By induction on ϕ. The only interesting cases are the following ones.

If $\phi = \neg\psi$, then we have that

$$\begin{aligned} \mathfrak{M}, w \models_{\mathbf{Kd}} \neg\psi &\iff \mathfrak{M}, w \not\models_{\mathbf{Kd}} \psi \\ &\iff \mathfrak{M}, \{w\} \not\models_{\mathbf{MID}} \psi \text{ (by induction hypothesis)} \\ &\iff \mathfrak{M}, \{w\} \models_{\mathbf{MID}} \psi \to \bot. \end{aligned}$$

If $\phi = \Box\psi$, then we have that

$$\begin{aligned} \mathfrak{M}, w \models_{\mathbf{Kd}} \Box\psi &\iff \mathfrak{M}, v \models_{\mathbf{Kd}} \psi \text{ for the unique } v \text{ s.t. } wRv \text{ (if such exists)} \\ &\iff \mathfrak{M}, \{v\} \models_{\mathbf{MID}} \psi \text{ for the unique } v \text{ s.t. } wRv \text{ (if such exists)} \\ &\qquad \text{(by induction hypothesis)} \\ &\iff \mathfrak{M}, R(w) \models_{\mathbf{MID}} \psi \\ &\iff \mathfrak{M}, \{w\} \models_{\mathbf{MID}} \Box\psi. \end{aligned}$$

■

Next, we define intuitionistic **K**d, which is a proper sublogic of **K**d and will play an important role in this chapter.

Definition 6.2.7. The intuitionistic modal logic of **K**d, denoted by **IK**d, is the smallest set of formulas containing the following axioms and is closed under the following rules:

1. All **IK** axiom schemas
2. $\Box(\phi \vee \psi) \to \Box\phi \vee \Box\psi$
3. $\neg\Box\neg\phi \to \Diamond\neg\neg\phi$
4. All **IK** rules

Immediately from Lemma 6.2.5, we know that all axiom schemas of **IK**d are valid in **MID**, and **MID** is closed under all rules of **IK**d except for Sub. This means that **IK**d $\subsetneq$ **MID**. Next, we show that **MID** is a proper sublogic of **K**d, implying that **MID** is between **IK**d and **K**d.

Lemma 6.2.8. **IK**d $\subset$ **MID** $\subset$ **K**d.

Proof. We only check that **MID** $\subset$ **K**d. As law of excluded middle $p \vee \neg p$ fails in **MID**, we have that **MID** $\neq$ **K**d. Now, suppose $\phi \notin$ **K**d. Since **K**d is complete with respect to the class of deterministic frames, there exists a deterministic model $\mathfrak{M}$ and a point w in $\mathfrak{M}$ such that $\mathfrak{M}, w \not\models_{\mathbf{Kd}} \phi$. By Lemma 6.2.6, we have that $\mathfrak{M}, \{w\} \not\models_{\mathbf{MID}} \phi$, thereby $\phi \notin$ **MID**, as required. ■

[2] Denote by $\mathsf{L}_1 \oplus \mathsf{L}_2$ the smallest set of formulas containing all axioms of the two modal logics L_1 and L_2 and is closed under MP, Gen and Sub.

Modal logics with superintuitionistic logics as their underlying propositional logic are studied in the literature, especially the logics between intuitionistic **K4** and classical **K4** (see e.g. [85], [84], etc). In this chapter, for our purpose, we consider what we call *intermediate* **K**d *modal logic*.

Definition 6.2.9. An *intermediate* **K**d *modal logic* is a set L of formulas closed under MP, Sub, Gen such that $\mathbf{IK}\mathsf{d} \subseteq \mathsf{L} \subset \mathbf{K}\mathsf{d}$.

Recall from Theorem 4.2.10 that $\mathbf{PID} = \mathsf{KP}^{\neg}$, where **PID** is the underlying propositional logic of **MID**. Now, the logic

$$\mathsf{KP\_}\mathbf{K}\mathsf{d} := \mathbf{IK}\mathsf{d} \oplus \mathsf{KP}$$

is clearly an intermediate **K**d modal logic. In view of Lemma 6.2.8 and that $\mathbf{PID} = \mathsf{KP}^{\neg}$, we know that $\mathsf{KP\_}\mathbf{K}\mathsf{d} \subseteq \mathbf{MID}$. However, **MID** is not an intermediate **K**d modal logic, as it is not closed under Sub. Instead, **MID** is a *weak* intermediate **K**d modal logic defined as follows (the definition is inspired by [13] and [11]).

Definition 6.2.10. A *weak intermediate* **K**d *modal logic* is a set L of formulas closed under MP and Gen such that $\mathbf{IK}\mathsf{d} \subseteq \mathsf{L} \subset \mathbf{K}\mathsf{d}$.

Analogous to Proposition 3.31 in [13], we can prove the following lemma (recalling Definition 4.2.9 of the negative variant of a logic).

Lemma 6.2.11. *For any intermediate* **K**d *modal logic* L*, its negative variant* $\mathsf{L}^{\neg}$ *is the smallest weak intermediate* **K**d *modal logic including* L *and the atomic double negation axiom* $\neg\neg p \to p$ *for each propositional variable* p*. Moreover, if* L *has the disjunction property, then so does* $\mathsf{L}^{\neg}$.

Proof. By a similar proof with that of Proposition 3.31 in [13]. ■

Corollary 6.2.12. $\mathsf{KP\_}\mathbf{K}\mathsf{d} \subset \mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg} \subseteq \mathbf{MID}$.

Proof. By Lemma 6.2.11, and $\models_{\mathbf{MID}} \neg\neg p \to p$. ■

In summary, we have obtained so far the following inclusions:

$$\mathbf{IK}\mathsf{d} \subseteq \mathsf{KP\_}\mathbf{K}\mathsf{d} \subset \mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg} \subseteq \mathbf{MID} \subset \mathbf{K}\mathsf{d}. \tag{6.2}$$

However, we do not know whether

$$\mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg} = \mathbf{MID} \quad \text{or} \quad \vdash_{\mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg}} \phi \iff \models_{\mathbf{MID}} \phi,$$

i.e., whether **MID** is complete with respect to the deductive system of $\mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg}$.

In the remaining part of this section, we show that all formulas of **MD** are expressible in **MID**. This goal is achieved through a so-called *disjunctive-negative translation*, generalized from [13].

Definition 6.2.13 (Disjunctive-negative translation (c.f. [13]))**.** For any dependence atom-free formula ϕ of **MD** or **MID**, we define its *disjunctive-negative translation* $\mathsf{DN}(\phi)$ inductively as follows:

- $\mathsf{DN}(p) := \neg\neg p$
- $\mathsf{DN}(\bot) := \neg\neg\bot$

Assume that $\mathsf{DN}(\psi) = \neg\psi_1 \vee \cdots \vee \neg\psi_n$ and $\mathsf{DN}(\chi) = \neg\chi_1 \vee \cdots \vee \neg\chi_m$. Define

- $\mathsf{DN}(\psi \vee \chi) := \mathsf{DN}(\psi) \vee \mathsf{DN}(\chi)$
- $\mathsf{DN}(\psi \wedge \chi) := \bigvee\{\neg(\psi_i \vee \chi_j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$
- $\mathsf{DN}(\psi \otimes \chi) := \bigvee\{\neg\neg(\psi_j \to \neg\chi_i) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$
- $\mathsf{DN}(\psi \to \chi) := \bigvee\{\neg\neg\bigwedge_{1\leq i\leq m}(\chi_{j_i} \to \psi_i) \mid 1 \leq j_1, \ldots, j_n \leq m\}$
- $\mathsf{DN}(\Diamond\psi) := \bigvee\{\neg\Box\neg\neg\psi_i \mid 1 \leq i \leq n\}$
- $\mathsf{DN}(\Box\psi) := \bigvee\{\neg\Diamond\psi_i \mid 1 \leq i \leq n\}$

Next, we show that the disjunctive-negative translation is truth-preserving.

Lemma 6.2.14 (c.f. [13]). *For every dependence atom-free formula ϕ of* **MD** *or* **MID***, $\phi \equiv \mathsf{DN}(\phi)$.*

Proof. By induction on ϕ. The modality-free cases except for the case $\phi = \psi \otimes \chi$ of the inductive proof follow from Proposition 3.14 in [13], which makes essential use of the following clauses:

$$\models_{\mathbf{MID}} \neg\neg p \to p \text{ and } \models_{\mathbf{MID}} \mathrm{ND}_k \text{ for all } k \in \mathbb{N}^3.$$

If $\phi = \psi \otimes \chi$, then by induction hypothesis,

$$\psi \equiv \mathsf{DN}(\psi) = \neg\psi_1 \vee \cdots \vee \neg\psi_n \text{ and } \chi \equiv \mathsf{DN}(\chi) = \neg\chi_1 \vee \cdots \vee \neg\chi_m.$$

Thus

$$\begin{aligned}
\psi \otimes \chi &\equiv (\neg\psi_1 \vee \cdots \vee \neg\psi_n) \otimes (\neg\chi_1 \vee \cdots \vee \neg\chi_m) \\
&\equiv \bigvee_{1\leq i\leq n} \bigvee_{1\leq j\leq m} (\neg\psi_i \otimes \neg\chi_j) \\
&\quad \text{(c.f. the distributive laws } \mathsf{Dstr}\otimes\vee, \mathsf{Dstr}\otimes\vee\otimes \text{ of } \mathbf{PD}^{[\vee]}) \\
&\equiv \bigvee_{1\leq i\leq n} \bigvee_{1\leq j\leq m} (\neg\neg\psi_i \to \neg\chi_j) \quad (\neg\psi_i \text{ is flat, Lemma 2.1.5}) \\
&\equiv \bigvee_{1\leq i\leq n} \bigvee_{1\leq j\leq m} \neg\neg(\psi_j \to \neg\chi_i) \\
&\equiv \mathsf{DN}(\psi \otimes \chi)
\end{aligned}$$

If $\phi = \Diamond\psi$, then by induction hypothesis,

$$\Diamond\psi \equiv \Diamond(\neg\psi_1 \vee \cdots \vee \neg\psi_n)$$

[3] Note that ND_k is a special case of KP.

$$\begin{aligned}
&\equiv \Diamond\neg\psi_1 \vee \cdots \vee \Diamond\neg\psi_n \text{ (by Lemma 6.2.4 (ii))}\\
&\equiv \Diamond\neg\neg\neg\psi_1 \vee \cdots \vee \Diamond\neg\neg\neg\psi_n\\
&\equiv \neg\Box\neg\neg\psi_1 \vee \cdots \vee \neg\Box\neg\neg\psi_n \ \text{ (by Lemma 6.2.4 (iv) and Lemma 6.2.5 (iii))}\\
&\equiv \mathsf{DN}(\Box\psi).
\end{aligned}$$

If $\phi = \Box\psi$, then by induction hypothesis,

$$\begin{aligned}
\Box\psi &\equiv \Box(\neg\psi_1 \vee \cdots \vee \neg\psi_n)\\
&\equiv \Box\neg\psi_1 \vee \cdots \vee \Box\neg\psi_n \ \text{ (by Lemma 6.2.4 (iii) and Lemma 6.2.5 (ii))}\\
&\equiv \neg\Diamond\psi_1 \vee \cdots \vee \neg\Diamond\psi_n \ \text{ (by Lemma 6.2.4 (i))}\\
&\equiv \mathsf{DN}(\Box\psi).
\end{aligned}$$

■

Corollary 6.2.15.

- *Formulas of* **MD** *are expressible in* **MID***;*
- **MID**$^{[\otimes]}$ = **MID**.

Proof. Since

$$=\!\!(p_{i_1},\dots,p_{i_k}) \ \equiv \ \big((p_{i_1} \vee \neg p_{i_1}) \wedge \cdots \wedge (p_{i_{k-1}} \vee \neg p_{i_{k-1}})\big) \to (p_{i_k} \vee \neg p_{i_k}), \qquad (6.3)$$

and each formula $\mathsf{DN}(\phi)$ is a formula of **MID**. ■

The following corollary shows that in **MID**, although the two modalities $\Box$ and $\Diamond$ are not dual to each other (since $\neg\Box\phi \not\models \Diamond\neg\phi$), the two modalities are definable from each other via the disjunctive-negative translation.

Corollary 6.2.16. *Every formula of* **MID** *is logically equivalent to a* $\Box$*-free or* $\Diamond$*-free formula.*

Proof. By Lemma 6.2.4 (i), for any formula ψ of **MID**, we have that

$$\neg\Box\neg\neg\psi \equiv \neg\neg\Diamond\neg\psi \text{ and } \neg\Diamond\psi \equiv \neg\neg\neg\Diamond\psi \equiv \neg\neg\Box\neg\psi.$$

Thus, in Definition 6.2.13 of the disjunctive-negative translation, we can take equivalently

$$\mathsf{DN}(\Diamond\psi) := \bigvee\{\neg\neg\Diamond\neg\psi_i \mid 1 \leq i \leq n\}$$

or

$$\mathsf{DN}(\Box\psi) := \bigvee\{\neg\neg\Box\neg\psi_i \mid 1 \leq i \leq n\}.$$

This way, the resulting equivalent formula $\mathsf{DN}(\phi)$ will be $\Box$-free or $\Diamond$-free. ■

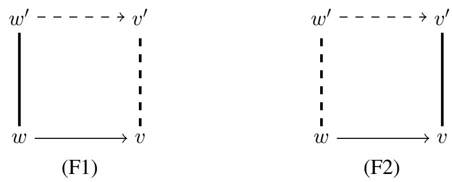


Figure 6.1

6.3 Model-theoretic properties of MID

Based on the results in the previous section, especially (6.2), in this section, we further reveal the relationship between **MID** and intuitionistic modal logic.

Intuitionistic modal logic **IK** is strongly complete with respect to *bi-relation Kripke models* [26],[27],[73]. Every modal Kripke model can be associated with a *powerset Kripke model*, which is a bi-relation Kripke model. For formulas of **MID**, the point-based satisfaction relation over these powerset Kripke models coincides with the set-based satisfaction relation given by team semantics. In the main part of this section, we will prove that **MID** is complete with respect to a class K of finite bi-relation Kripke models.

Let us start with recalling the Kripke semantics of **IK** ([26],[27],[73], see also [77]).

Definition 6.3.1 ([26],[73])**.** A *bi-relation Kripke frame* is a triple $\mathfrak{F} = (W, \geq, R)$, where

- W is a non-empty set;
- $\geq$ is a partial ordering and R is a binary relation on W;
- R and $\geq$ satisfy the following two conditions (F1) and (F2) (see Figure 6.1):

  (F1) If $w \geq w'$ and wRv, then there exists $v' \in W$ such that $v \geq v'$ and $w'Rv'$.

  (F2) If wRv and $v \geq v'$, then there exists $w' \in W$ such that $w \geq w'$ and $w'Rv'$.

A *bi-relation Kripke model* is a quadruple $\mathfrak{M} = (W, \geq, R, V)$ such that $(W, \geq, R)$ is a bi-relation Kripke frame and $V : \text{Prop} \to \wp(\text{W})$ is a function (a *valuation*) satisfying *monotonicity* with respect to $\geq$, that is,

$$[w \in V(p) \text{ and } w \geq v] \implies v \in V(p).$$

Definition 6.3.2 (satisfaction relation)**.** Let $\mathfrak{M} = (W, \geq, R, V)$ be a bi-relation Kripke model. We inductively define a satisfaction relation $\mathfrak{M}, w \Vdash \phi$ as follows:

- $\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$;
- $\mathfrak{M}, w \nVdash \bot$;
- $\mathfrak{M}, w \Vdash \psi \wedge \chi$ iff $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$;
- $\mathfrak{M}, w \Vdash \psi \vee \chi$ iff $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$;

- $\mathfrak{M}, w \Vdash \psi \to \chi$ iff for all v such that $w \geq v$,

$$\mathfrak{M}, v \Vdash \psi \Longrightarrow \mathfrak{M}, v \Vdash \chi;$$

- $\mathfrak{M}, w \Vdash \Diamond\psi$ iff there exists v such that wRv and $\mathfrak{M}, v \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \Box\psi$ iff for all u, v such that $w \geq u$ and uRv, it holds that $\mathfrak{M}, v \Vdash \psi$.

If $\mathfrak{M}, w \Vdash \phi$ for all nodes w in a model $\mathfrak{M}$, then we say that ϕ is *true* on the model $\mathfrak{M}$ and write $\mathfrak{M} \models \phi$. If $(\mathfrak{F}, V) \models \phi$ for any model $(\mathfrak{F}, V)$ on a frame $\mathfrak{F}$, then we say that ϕ is *valid* on the frame $\mathfrak{F}$ and write $\mathfrak{F} \models \phi$.

Lemma 6.3.3 (Monotonicity)**.** *For any formula ϕ of* **IK***, any bi-relation Kripke model $\mathfrak{M} = (W, \geq, R, V)$, any $w, v \in W$,*

$$\mathfrak{M}, w \Vdash \phi \text{ and } w \geq v \Longrightarrow \mathfrak{M}, v \Vdash \phi.$$

Proof. Easy. ■

Theorem 6.3.4 ([27], [73])**.** *The intuitionistic modal logic* **IK** *is strongly complete with respect to bi-relation Kripke frames.*

Proof. See [27], [73], and also [77]. ■

For propositional intuitionistic dependence logic, fixing a set of propositional variables $\{p_{i_1}, \dots, p_{i_n}\}$, it is proved essentially in [11] that there is a powerset intuitionistic Kripke model consisting of all non-empty teams on $\{i_1, \dots, i_n\}$, over which the usual Kripke semantics is equivalent to the team semantics (see [11] for details). Analogously, in the setting of **MID**, each modal Kripke model induces a so-called *powerset Kripke model*, which is a bi-relation Kripke mode.

Definition 6.3.5 (Powerset Kripke Models)**.** Let $\mathfrak{M} = (W, R, V)$ be a modal Kripke model. The *powerset Kripke model $\mathfrak{M}^\circ$ induced by* $\mathfrak{M}$ is a quadruple $\mathfrak{M}^\circ = (W^\circ, \supseteq, R^\circ, V^\circ)$, where

- $W^\circ = \wp(W) \setminus \{\emptyset\}$, i.e. W° consists of all nonempty teams on W;
- $\supseteq$ is the usual superset relation on W°;
- $XR^\circ Y$ iff XRY, for any $X, Y \in W^\circ$;
- $X \in V^\circ(p)$ iff $\mathfrak{M}, X \models p$, for any $X \in W^\circ$.

All powerset Kripke models are defined in the above way, in other words, each powerset Kripke models $\mathfrak{M}^\circ$ is induced by a unique modal Kripke model $\mathfrak{M}$.

As the reader may observe from Definition 6.3.5, in a powerset Kripke model $\mathfrak{M}^\circ$, elements of W° correspond to all non-empty teams of its associated modal Kripke model $\mathfrak{M}$. In an obvious way, the powerset Kripke model $\mathfrak{M}^\circ$ carries the information of teams of its associated modal Kripke model $\mathfrak{M}$. Note that the relation R° resembles the successor team relation R on W, but $R^\circ \neq R$ as R is viewed here as a relation between teams (sets of nodes), whereas R° is a relation between single nodes of W°!

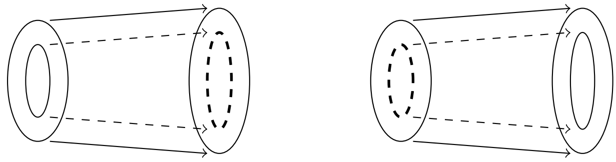


Figure 6.2

Fact 6.3.6. *Powerset Kripke models are bi-relation Kripke models.*

Proof. Clearly, the superset relation $\supseteq$ is a partial ordering, and the monotonicity of V° follows from the downwards closure property of the team semantics of propositional variables. It remains to check that any powerset Kripke model $\mathfrak{M}^\circ = (W^\circ, \supseteq, R^\circ, V^\circ)$ satisfies (F1) and (F2) (see also Figure 6.2).

For (F1), let $X, X', Y \in W^\circ$ be such that $X \supseteq X'$ and $XR^\circ Y$. Let

$$Y' = R(X') \cap Y.$$

Clearly, $Y \supseteq Y'$. We show that $X'R^\circ Y'$, i.e. (6.1) is satisfied for X', Y'. Clearly, $Y' = R(X') \cap Y \subseteq R(X')$. On the other hand, for any $w \in X' \subseteq XR^\circ Y$, there exists $v \in Y$ such that wRv, thus $v \in Y \cap R(X') = Y'$.

For (F2), let $X, Y, Y' \in W^\circ$ be such that $XR^\circ Y$ and $Y \supseteq Y'$. Let

$$X' = R^{-1}(Y') \cap X.$$

Clearly, $X \supseteq X'$. We show that $X'R^\circ Y'$, i.e. (6.1) is satisfied for X', Y'.

For any $v \in Y' \subseteq Y$, since $XR^\circ Y$, there exists $w \in X$ such that wRv, thus $w \in R^{-1}(Y') \cap X = X'$. It follows that $Y' \subseteq R(X')$.

On the other hand, for any $w \in X' = R^{-1}(Y') \cap X$, there exists $v \in Y'$ such that wRv. It follows that $v \in R(w) \cap Y'$. ■

We show in the next lemma that for formulas of **MID**, the team-based satisfaction relation with respect to the usual modal Kripke models is equivalent to the single-node-based satisfaction relation with respect to powerset Kripke models.

Lemma 6.3.7. *For any formula ϕ of* **MID***, any Kripke model $\mathfrak{M} = (W, R, V)$ and any non-empty team $X \subseteq W$, it holds that*

$$\mathfrak{M}, X \models \phi \iff \mathfrak{M}^\circ, X \Vdash \phi.$$[4]

Proof. By induction on ϕ. The only interesting case is the case $\phi = \Box\psi$. In this case, we have that

$$\mathfrak{M}^\circ, X \Vdash \Box\psi \Longrightarrow \mathfrak{M}^\circ, R(X) \Vdash \psi \text{ (since } X \subseteq X \text{ and } XR^\circ R(X))$$

[4]Note that the symbol "X" on the left-hand side stands for a team (a set of nodes), while the "X" on the right-hand side stands for a single node.

$$\Longrightarrow \mathfrak{M}, R(X) \models \psi \text{ (by induction hypothesis)}$$
$$\Longrightarrow \mathfrak{M}, X \models \Box\psi.$$

and that

$$\begin{aligned}\mathfrak{M}, X \models \Box\psi \Longrightarrow & \mathfrak{M}, R(X) \models \psi \\ \Longrightarrow & \text{for all non-empty } Y, Z \subseteq W \text{ s.t. } X \supseteq Y \text{ and } YRZ,\ \mathfrak{M}, Z \models \psi \\ & \quad \text{(since } Z \subseteq R(X) \text{ and } \models \text{ is downwards closed)} \\ \Longrightarrow & \text{for all } Y, Z \in W^{\circ} \text{ s.t. } X \supseteq Y \text{ and } YR^{\circ}Z,\ \mathfrak{M}^{\circ}, Z \Vdash \psi \\ & \quad \text{(by induction hypthesis)} \\ \Longrightarrow & \mathfrak{M}^{\circ}, X \Vdash \Box\psi.\end{aligned}$$

■

By a natural argument whose details will not be included in this thesis[5], one can show that **MID** has the *finite model property*, that is,

if $\not\models_{\mathbf{MID}} \phi$, then there exists a finite Kripke model $\mathfrak{M}$ such that $\mathfrak{M} \not\models \phi$.

Therefore, together with Lemma 6.3.7, we know that **MID** is complete with respect to finite powerset Kripke models.

In the rest of this section, we show that **MID** is complete with respect to a class K of certain finite bi-relation Kripke models. Recall that propositional intuitionistic dependence logic **PID** is complete with respect to negative saturated intuitionistic Kripke models (Theorem 4.2.17). These models are generalizations of the associated powerset intuitionistic Kripke models of **PID**. Here, for the logic **MID**, we will follow the same idea and define K as the class of finite bi-relation Kripke models having the properties abstracted from powerset Kripke models.

To give a definition of such class K, we study the KP axioms and the valid formulas of **MID** listed in Lemma 6.2.5, namely

$$\neg\neg p \to p, \quad \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi) \quad \text{and} \quad \neg\Box\neg\phi \to \Diamond\neg\neg\phi. \tag{6.4}$$

As pointed out already, these formulas are not valid in **IK** (we invite the reader to check it using Theorem 6.3.4, the completeness theorem of **IK**). We conjecture that **MID** is complete with respect to these axioms together with the axioms of **IK**, as well as the rules MP and Gen. Although this problem is open, and to axiomatize **MID** more axioms than these ones may be needed, it turns out that for the goal of this section, it is enough to consider these mentioned axioms only.

First of all, **PID** satisfies the KP axioms and atomic double negation law $\neg\neg p \to p$, and it is complete with respect to negative saturated intuitionistic Kripke models (Theorem 4.2.17). In view of this, we give the following definition.

[5]For each formula $\phi(p_1, \dots, p_n)$ of modal depth k, and a (possibly infinite) team X of a Kripke model $\mathfrak{M}$, *select* a finite submodel $\mathfrak{M}'$ and a finite subteam X' as follows: 1. Unravel the submodels of $\mathfrak{M}$ generated by each node $w \in X$, and take the disjoint union of all these unravelled tree-like models $\mathfrak{T}_w$. 2. Cut the forest $\biguplus_{w\in X} \mathfrak{T}_w$ up to depth k to form a new forest $\mathfrak{Fst}$. 3. Restrict attention to valuations on $\{p_1, \dots, p_n\}$ only. For each tree $\mathfrak{T}$ of $\mathfrak{Fst}$, starting from the deepest layer, layer by layer, identify isomorphic subtrees on each layer. For fixed k and n, the resulting new tree $\mathfrak{T}_0$ must be finite. 4. For the same reason, the resulting new forest $\mathfrak{Fst}_0$ contains at most finitely many non-isomorphic trees. Delete the isomorphic copies. The remaining forest is the required finite model $\mathfrak{M}'$, and the roots of the trees in the new forest $\mathfrak{M}'$ form the required team X'.

Definition 6.3.8. A bi-relation Kripke modal $\mathfrak{M} = (W, \geq, R, V)$ is said to be *negative* and *saturated* if V is negative and $(W, \geq)$ is saturated in the sense of Definition 4.2.16.

As in [11], and as we shall see in the sequel, in a saturated bi-relation Kripke model, $\geq$-endpoints (i.e. endpoints in the sense of Definition 4.2.15 with respect to $\geq$) behave as singleton teams of a usual modal Kripke model.

Secondly, in the following two lemmas we prove that each of the other two formulas of (6.4) characterizes a frame property under certain conditions.

We write $R_1 \circ R_2$ for the *composition* of the two relations R_1 and R_2, which is defined as

$$(x, y) \in R_1 \circ R_2 \text{ iff } \exists z(xR_1z \wedge zR_2y).$$

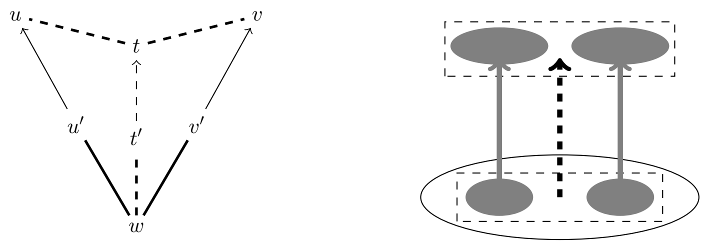
Lemma 6.3.9. *Let $\mathfrak{F} = (W, \geq, R)$ be a bi-relation Kripke frame. Then*

$$\mathfrak{F} \models \Box(p \vee q) \to (\Box p \vee \Box q) \iff \mathfrak{F} \text{ satisfies (G1')},$$

where (G1') is defined as follows:

(G1') *For all $w, u, v \in W$, if $u, v \in (\geq \circ R)(w)$, then there exists $t \in W$ such that $w \geq \circ Rt$, $t \geq u$ and $t \geq v$.*

Before we prove the lemma, we depict condition (G1') by the left figure as follows. This condition is abstracted from the corresponding property of powerset Kripke models (depicted by the right self-explanatory figure below).



Also, we point out that in case $\mathfrak{F}$ is finite, condition (G1') is equivalent to condition (G1) defined as follows:

(G1) For any $w \in W$ and any nonempty $X \subseteq (\geq \circ R)(w)$, there exists a node $u \in (\geq \circ R)(w)$ such that $u \geq v$ for all $v \in X$.

Proof of Lemma 6.3.9. Suppose $\mathfrak{F}$ satisfies (G1') and $(\mathfrak{F}, V), w \not\models \Box p \vee \Box q$ for some valuation V and some $w \in W$. Then there exists $u, v \in W$ such that $w \geq \circ Ru$, $w \geq \circ Rv$,

$$(\mathfrak{F}, V), u \not\models p \text{ and } (\mathfrak{F}, V), v \not\models q.$$

Let $t \in W$ be the point given by (G1'). Then by monotonicity, we have that $(\mathfrak{F}, V), t \not\models p \vee q$, which implies that $(\mathfrak{F}, V), w \not\models \Box(p \vee q)$.

Conversely, suppose $\mathfrak{F}$ does not satisfy (G1'). Then there exists $w, u, v \in W$ such that $w \geq \circ Ru$, $w \geq \circ Rv$ and for all $t \in W$ such that $w \geq \circ Rt$, either $t \not\geq u$ or $t \not\geq v$. Clearly, we can find a monotone valuation V such that

$$V(p) = W \setminus \geq^{-1}(v) \text{ and } V(q) = W \setminus \geq^{-1}(u).$$

For each $t \in W$ such that $w \geq \circ Rt$, by assumption, either $t \notin \geq^{-1}(v)$ or $t \notin \geq^{-1}(u)$, thus $(\mathfrak{F}, V), t \Vdash p \vee q$, thereby $(\mathfrak{F}, V), w \Vdash \Box(p \vee q)$. On the other hand, we have that $(\mathfrak{F}, V), u \nVdash q$ and $(\mathfrak{F}, V), v \nVdash p$, thus $(\mathfrak{F}, V), w \nVdash \Box p \vee \Box q$. ■

Lemma 6.3.10. *Let $\mathfrak{F} = (W, \geq, R)$ be a saturated bi-relation Kripke frame Then*

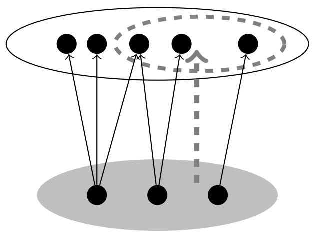
$$\mathfrak{F} \models \neg\Box\neg p \to \Diamond\neg\neg p \iff \mathfrak{F} \text{ satisfies (G2)},$$

where (G2) is defined as follows:

(G2) *Let $w \in W$ be an arbitrary point and E a set of $\geq$-endpoints such that $E \subseteq R(E_w)$ and for each $v \in E_w$, there exists $u_v \in E$ with vRu_v. Then there exists $t \in W$ such that*

$$wRt \text{ and } E_t \subseteq E.$$

Condition (G2) is abstracted from the corresponding property of powerset Kripke models (depicted by the self-explanatory figure below).



Proof of Lemma 6.3.9. Suppose $\mathfrak{F}$ satisfies (G2) and $(\mathfrak{F}, V), w \Vdash \neg\Box\neg p$ for some valuation V and some $w \in W$. Then, for each $v \in E_w$, $(\mathfrak{F}, V), v \nVdash \Box\neg p$. Since v is a $\geq$-endpoint and $\mathfrak{F}$ is saturated, there exists an endpoint u_v such that $vR\circ \geq u_v$ and $(\mathfrak{F}, V), u_v \Vdash p$. By (F2), there exists $v' \in W$ such that $v \geq v'$ and $v'Ru_v$. But as v is a $\geq$-endpoint, we must have that $v = v'$ and vRu_v, thus the set

$$E = \{u_v \mid v \in E_w\}$$

satisfies the condition in (G2). By (G2), there exists a point $t \in W$ such that

$$wRt \text{ and } E_t \subseteq E.$$

Hence $(\mathfrak{F}, V), w \Vdash \Diamond\neg\neg p$.

Conversely, suppose $\mathfrak{F}$ does not satisfy (G2). Then there exists $w \in W$ and a set E of $\geq$-endpoints satisfying $E \subseteq R(E_w)$ and for each $v \in E_w$, there exists $u_v \in E$ with vRu_v, and for all $t \in W$,

$$wRt \Longrightarrow E_t \nsubseteq E.$$

Clearly, we can find a monotone valuation V such that $V(p) = E$.

For each $v \in E_w$, since vRu_v, we have that $(\mathfrak{F}, V), v \nVdash \Box\neg p$, thus $(\mathfrak{F}, V), w \Vdash \neg\Box\neg p$. On the other hand, for each $t \in R(w)$, by assumption, there exists $s \in E_t$ such that $s \notin E$, thus $(\mathfrak{F}, V), s \nVdash p$. Hence $(\mathfrak{F}, V), w \nVdash \Diamond\neg\neg p$. ■

Now, we are ready to define the class K of generalizations of all powerset Kripke models.

Definition 6.3.11. Let K be the class of all finite negative $\geq$-saturated bi-relation Kripke models $\mathfrak{M} = (W, \geq, R, V)$ satisfying (G1) and (G2).

In the remaining part of this section, we show that **MID** is complete with respect to K, that is, we will prove the following theorem. The idea of the proof is inspired by that of Theorem 3.2.18 in [11].

Theorem 6.3.12. *For any formula ϕ of* **MID***, we have that*

$$\models_{\mathbf{MID}} \phi \iff \mathsf{K} \models \phi.$$

Proof of "$\Longleftarrow$". We leave it for the reader to check that each finite powerset Kripke model is indeed in K. Then, we have that

$$\begin{aligned} \mathsf{K} \models \phi \Longrightarrow\ & \mathfrak{M}^\circ \models \phi \text{ for all finite powerset Kripke models } \mathfrak{M}^\circ \\ \Longrightarrow\ & \mathfrak{M} \models \phi \text{ for all finite modal Kripke models } \mathfrak{M} \text{ (by Lemma 6.3.7)} \\ \Longrightarrow\ & \models_{\mathbf{MID}} \phi \text{ (by the finite model property of } \mathbf{MID}\text{).} \end{aligned}$$

■

To prove the other direction "$\Longrightarrow$" of Theorem 6.3.12, we first show that each model in K can be mapped p-morphically into a finite powerset Kripke model. As p-morphisms are truth-preserving, the required result will then follow. Now, we recall the definition of p-morphisms of bi-relation Kripke models given by Wolter and Zakharyaschev in [86].

Definition 6.3.13 ([86])**.** Let $\mathfrak{M}_1 = (W_1, \geq_1, R_1, V_1)$ and $\mathfrak{M}_2 = (W_2, \geq_2, R_2, V_2)$ be bi-relation Kripke models. A function $f : W_1 \to W_2$ is called a *p-morphism* iff

(P1) $w \in V_1(p) \Longleftrightarrow f(w) \in V_2(p)$ for all propositional variables p

(P2) $w \geq_1 v \Longrightarrow f(w) \geq_2 f(v)$

(P3) $wR_1v \Longrightarrow f(w)R_2 f(v)$

(P4) $f(w) \geq_2 v' \Longrightarrow \exists v \in W_1$ s.t. $f(v) = v'$ and $w \geq_1 v$

(P5) $f(w)R_2v' \Longrightarrow \exists v \in W_1$ s.t. $v' \geq_2 f(v)$ and wR_1v

(P6) $f(w)(\geq_2 \circ R_2)v' \Longrightarrow \exists v \in W_1$ s.t. $w \geq_1 \circ R_1 v$ and $f(v) \geq_2 v'$

Next, we prove the crucial lemma for the proof of the direction "$\Longrightarrow$" of Theorem 6.3.12.

Lemma 6.3.14. *For each finite bi-relation model $\mathfrak{M} = (W, \geq, R, V)$ in K, there exists a finite Kripke model $\mathfrak{N}$ such that there exists a p-morphism f of $\mathfrak{M}$ into the powerset Kripke model $\mathfrak{N}^\circ$ of $\mathfrak{N}$.*

Proof. Define a Kripke model $\mathfrak{N} = (W_0, R_0, V_0)$ as follows:

- W_0 is the set of all $\geq$-endpoints of W,
- $R_0 = R \restriction W_0$ and $V_0 = V \restriction W_0$.

Now, consider the powerset Kripke model $\mathfrak{N}^\circ = (W_0^\circ, \supseteq, R_0^\circ, V_0^\circ)$ of $\mathfrak{N}$. Define a function $f : W \to W_0^\circ$ by taking

$$f(w) = E_w \text{ for all } w \in W.$$

Since $\mathfrak{M}$ is saturated, $E_w \neq \emptyset$ for all $w \in W$, thus $E_w \in W_0^\circ$ and f is well-defined.

Before we continue the proof, let us ponder over the above construction. As defined, a $\geq$-endpoint e of $\mathfrak{M}$ is mapped through f into the singleton $\{e\} = E_e$. Intuitively, in the main argument of the proof, $\geq$-endpoints of $\mathfrak{M}$ are simulated by singletons of $\mathfrak{N}^\circ$. On the other hand, other points w of W are mapped into the sets E_w. Basically, during the proof, it is helpful for the reader to think of a node w of $\mathfrak{M}$ as a team formed by all $\geq$-endpoints seen from w, namely the set E_w.

Now, we proceed to show that f is a p-morphism, i.e., f satisfies (P1)-(P6).

(P1). It suffices to show that $\mathfrak{M}, w \Vdash p \iff \mathfrak{N}^\circ, E_w \Vdash p$. The direction "$\Longrightarrow$" follows from the monotonicity of V. For the direction "$\Longleftarrow$", if $\mathfrak{M}, w \nVdash p$, then since V is negative, $\mathfrak{M}, w \nVdash \neg\neg p$. Thus, there exists $v \in E_w$ such that $\mathfrak{M}, v \nVdash p$, which implies that $\mathfrak{N}, v \nVdash p$, thereby $\mathfrak{N}^\circ, E_w \nVdash p$.

(P2). Clearly, if $w \geq v$, then $E_w \supseteq E_v$, i.e. $f(w) \supseteq f(v)$.

(P3). Assume wRv, we show that $E_w R_0^\circ E_v$, namely $E_w R_0 E_v$. For any $s \in E_w$, by (F1) of $\mathfrak{M}$, there exists $t \in W$ such that

$$v \geq t \text{ and } sRt.$$

For each $t' \in E_v$ such that $t \geq t'$, by (F2), there exists $s' \in W$ such that

$$s \geq s' \text{ and } s'Rt'.$$

As s is a $\geq$-endpoint, we must have that $s = s'$ and sRt'.

On the other hand, for any $t \in E_v$, consecutively applying (F2) and (F1) of $\mathfrak{M}$, by a similar argument to the above, we can find an $s' \in E_w$ such that $s'Rt$. Hence we conclude that $E_w R_0 E_v$.

(P4). If $E_w \supseteq v'$, then as $\mathfrak{M}$ is $\geq$-saturated, there exists $v \in W$ such that $w \geq v$ and $E_v = v'$, as required.

(P5). If $E_w R_0^\circ v'$, then $E_w R_0 v'$. Clearly, v' is a set of $\geq$-endpoints such that $v' \subseteq R(E_w)$ and for each $s \in E_w$, there exists $t_s \in v'$ such that sRt_s. Thus, by (G2) of $\mathfrak{M}$, there exists $v \in W$ such that

$$wRv \text{ and } E_v \subseteq v',$$

as required.

(P6) Suppose $E_w \supseteq \circ R_0^\circ v'$. Then $v' \neq \emptyset$ and

$$v' \subseteq (\geq \circ R)(w).$$

By (G1) of the finite model $\mathfrak{M}$, there exists $v \in W$ such that

$$w \geq \circ Rv \text{ and } v \geq s \text{ for all } s \in v'.$$

Since v' is a set of $\geq$-endpoints, the latter of the above implies that $E_v \supseteq v'$. ■

Finally, we complete the proof of Theorem 6.3.12 as follows.

Proof of Theorem 6.3.12, the direction "$\Longrightarrow$". Suppose $\models_{\mathbf{MID}} \phi$. For each $\mathfrak{M} \in \mathsf{K}$, by Lemma 6.3.14, there is a Kripke model $\mathfrak{N}$ and a p-morphism $f : \mathfrak{M} \to \mathfrak{N}^{\circ}$. By assumption and Lemma 6.3.7, we know that $\mathfrak{N}^{\circ} \models \phi$. Since p-morphisms preserve truth, we conclude that $\mathfrak{M} \models \phi$, as required. ■

6.4 Concluding remarks and open problems

In this chapter, we defined syntactically a logic $\mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg}$ as the smallest set of formulas containing the following axioms and is closed under the following rules:

1. All **IPL** axiom schemas;
2. Axiom scheme of

$$(\mathsf{KP}) \qquad (\neg\phi \to (\psi \vee \chi)) \to ((\neg\phi \to \psi) \vee (\neg\phi \to \chi));$$

3. $\neg\neg p \to p$ for all propositional variables p;
4. All **IK** axiom schemas;
5. $\Box(\phi \vee \psi) \to \Box\phi \vee \Box\psi$;
6. $\neg\Box\neg\phi \to \Diamond\neg\neg\phi$;
7. Modus Ponens (MP);
8. Generalization (Gen): $\phi/\Box\phi$.

MID is sound with respect to the above system, as we showed in (6.2) in Section 6.2. Dependence atoms are eliminable in **MID** (see (6.3)), so for simplicity, we may identify **MID** with **MID** without dependence atoms. In this setting, the main open problem of this chapter is: is **MID** complete with respect to the deductive system of $\mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg}$, or equivalently does the following hold:

$$\vdash_{\mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg}} \phi \iff \models_{\mathbf{MID}} \phi \tag{6.5}$$

for all formulas ϕ of **MID**?

We proved in Theorem 6.3.12 that **MID** is complete with respect to the set K of finite bi-relation Kripke models, or

$$\models_{\mathbf{MID}} \phi \iff \mathsf{K} \models \phi. \tag{6.6}$$

Therefore, (6.5) reduces to whether the following is the case:

$$\vdash_{\mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg}} \phi \iff \mathsf{K} \models \phi. \tag{6.7}$$

In this direction, we know already the following:

- By Theorem 4.2.12, the underlying propositional logic of **MID**, i.e. **PID**, is complete with respect to the deductive system of $\mathsf{KP}^{\neg}$, namely axioms 1-3 and rules 7-8 of the above. Furthermore, by Theorem 4.2.17, **PID** (or $\mathsf{KP}^{\neg}$) is complete with respect to negative saturated intuitionistic Kripke models.

- By Theorem 6.3.4, **IK** (whose deductive system consists of axiom 4, rules 7-8 of the above and Sub) is complete with respect to bi-relation Kripke frames.

- By Lemma 6.3.9 and Lemma 6.3.10, over finite (saturated) bi-relation Kripke frames, axiom 5 and axiom 6 characterize (G1) and (G2), respectively.

The class K was defined as Kripke models having all of the properties mentioned above, and observe that in the proof of Theorem 6.3.12 or (6.6), we made essential use of all of the axioms of $\mathsf{KP\_}\mathbf{K}\mathsf{d}^{\neg}$. These seem to indicate that (6.7) should hold.

One related open problem (or, further and deeper problem) is: the logic

$$\mathsf{KP\_}\mathbf{K} := \mathbf{IK} \oplus \mathsf{KP}$$

is an intermediate **K** modal logics, as clearly $\mathbf{IK} \subseteq \mathsf{KP\_}\mathbf{K} \subseteq \mathbf{K}$. Is $\mathsf{KP\_}\mathbf{K}$ complete with respect to bi-relation Kripke frames $\mathfrak{F} = (W, \geq, R)$ such that $(W, \geq)$ is a KP-frame (see e.g. [9] for the definition of KP-frames)?

Chapter 7

Model Checking for Modal Intuitionistic Dependence Logic

In this chapter, we study the computational complexity of model checking problem for modal intuitionistic dependence logic. The model checking problem (MC) for a modal logic L of dependence and independence asks whether a given formula of L is satisfied by a given team of a given (modal) Kripke model.

The computational aspect of modal logics of dependence and independence deserves investigation for two reasons. *A priori*, the nature of team semantics gives such logics more complexity, as the successor search for formula evaluation has to be done for sets of states. On the other hand, in practice, particularly interesting properties involving dependence and independence are often supposed to be identified from a large amount of data (an example of such properties is given in Example 7.1.2).

For modal dependence logic (**MD**), the satisfiability problem (SAT) is showed by Sevenster [76] to be $\mathsf{NEXPTIME}$-complete, and a complete classification of the computational complexity of SAT for all operator fragments of **MD** is given in [67]. In [17], the computational complexity of MC for **MD** and some of its fragments (e.g. propositional dependence logic) are shown to be NP-complete. In this chapter, we investigate the computational complexity of MC for modal intuitionistic dependence logic.

By Corollary 6.2.15, we know that **MID** has the same expressive power as **MID** extended with tensor disjunction $\otimes$. In this chapter, we choose to identify these two logics, namely, formulas of **MID** in this chapter are built from the following grammar:

$$\phi ::= p \mid \neg p \mid \bot \mid =\!(p_1,\ldots,p_k) \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \Box\phi \mid \Diamond\phi,$$

where $p, p_1, \ldots, p_k$ are propositional variables.

Following [67] and [17], we will systematically analyze the complexity of MC for fragments of **MID** defined by restricting the set of modal operators ($\Diamond$, $\Box$), propositional operators ($\neg/\bot$, $\wedge$, $\otimes$, $\vee$, $\to$), as well as dependence operator ($=\!(\cdots)$) allowed in the logics (recall Definition 1.1.3). By the choice of the syntax of **MID**, modal dependence logic (**MD**), propositional dependence logic (**PD**) and propositional intuitionistic dependence logic (**PID**) are all viewed as operator fragments (sublogics) of **MID**. The method of systematically classifying the complexity of logic related problems by restricting the set of operators allowed in formulas was used by Lewis [65] for SAT for propositional logic, by Hemaspaandra et al. [44] [45] for SAT for modal logic, and by others. The motivation for this approach is twofold: theoretically, this systematic approach may lead to insights into the sources of hardness, i.e., the exact components of the logic that make SAT, MC and other problems hard; practically, by systematically examining all fragments of a logic, one might find useful fragments of the logic in practice with both efficient algorithms and high expressivity.

In Section 7.1, we give formal definition of the model checking problem for operator fragments of **MID**. Section 7.2 contains the main result of this chapter. We show that MC for **MID** in general is PSPACE-complete and that for **PID** is coNP-complete. In Section 7.3, we point out open problems.

The content of this chapter is based on the joint paper [18].

7.1 Model checking problem

Given a Kripke model $\mathfrak{M}$, a team X of $\mathfrak{M}$, and a formula ϕ of **MID**, the model checking problem for **MID** is the problem of deciding whether $\mathfrak{M}, X \models \phi$ holds.

Definition 7.1.1. Let L be a sublogic of **MID**. The *model checking problem* for L (denoted by L-MC) is defined as the decision problem of the set

$$\mathsf{L}\text{-MC} := \left\{ \langle \mathfrak{M}, X, \phi \rangle \;\middle|\; \begin{array}{l} \mathfrak{M} \text{ is a Kripke model, } X \text{ is a team of } \mathfrak{M}, \phi \text{ is a formula} \\ \text{of } \mathsf{L} \text{ and } \mathfrak{M}, X \models \phi \end{array} \right\}.$$

Note that in this chapter, we only consider the combined complexity of model checking problem for **MID**, i.e. the input consists of both a model and a formula. One can also consider the data complexity of model checking problem for **MID**, where the formula is fixed and the input consists of a model only. Usually, the data complexity of a model checking problem is lower than the combined complexity. In our case, for a given **MID** formula with finitely many propositional variables, there are even only finitely many irreducible models (with respect to p-morphisms, c.f. Footnote 5 on the proof of the finite model property of **MID** in Section 6.3), therefore the data complexity for **MID** model checking is not very interesting and is not the topic of this chapter.

A typical formula of **MID** expresses a modal property involving implications of dependence statements. Such properties are commonly found in many fields. Knowing whether these properties hold in certain sets of some system can be important in many cases. Below we present an example illustrating the applications of **MID**-MC in practice.

Example 7.1.2. Suppose the United Nations wants to build a model (represented as a Kripke model) of the imitation of the future of the earth and human race. Among all the candidate models, the United Nations wants to know which ones are optimistic models from the point of view of environmental degradation. One important criterion of being such an optimistic model is that in the model, the present world has to satisfy the following property ($*$):

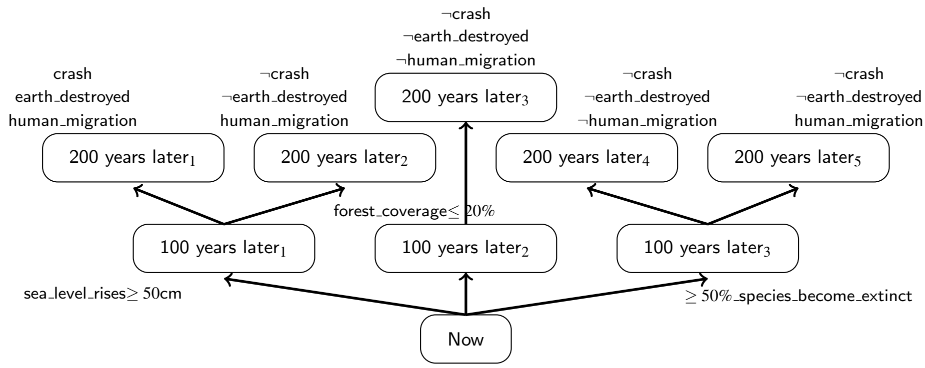
However the environment will be degraded in the next 100 years, it is possible that in 200 years from now, if whether the earth will be destroyed *depends only on whether there is another planet that crashes into the earth, then whether the human being will migrate to other planets depends only on whether the crash will occur.* *(∗)*

In the language of **MID**, the above criterion is interpreted by the formula

$$\Box\Diamond\big(=\!(\mathsf{crash},\mathsf{earth\_destroyed})\to =\!(\mathsf{crash},\mathsf{human\_migration})\big).$$

Selecting optimistic models with respect to this criterion is done by implementing an **MID**-MC on the candidate models.

For example, given the below depicted Kripke model $\mathfrak{M}$, where every symbol is self-explanatory, we achieve the above goal by checking whether



$$\mathfrak{M},\{\mathsf{now}\}\models\Box\Diamond\big(=\!(\mathsf{crash},\mathsf{earth\_destroyed})\to =\!(\mathsf{crash},\mathsf{human\_migration})\big)$$

holds. In this case, the above expression holds, as for the team

$$X=\{\mathsf{200\ years\ later}_1,\mathsf{200\ years\ later}_3,\mathsf{200\ years\ later}_4\},$$

it holds that

$$\mathfrak{M},X\models =\!(\mathsf{crash},\mathsf{earth\_destroyed})\to =\!(\mathsf{crash},\mathsf{human\_migration}),$$

therefore $\mathfrak{M}$ is an optimistic model with respect to criterion (∗). $\dashv$

In the next section, we will sometimes reduce one model checking problem to another in a complexity preserving way. Such reductions are defined as follows.

Definition 7.1.3. Let C be a countable set and $A,B\subseteq C$. Then A is *polynomial-time many-one reducible* to B, in symbols $A\leq^{\mathsf{P}}_{\mathrm{m}}B$, iff there is a *reduction* function $f:C\to C$ such that f is computable in polynomial time and for all $x\in C$,

$$x\in A\iff f(x)\in B.$$

If both $A\leq^{\mathsf{P}}_{\mathrm{m}}B$ and $B\leq^{\mathsf{P}}_{\mathrm{m}}A$, then we write $A\equiv^{\mathsf{P}}_{\mathrm{m}}B$.

Most complexity classes $\mathcal{C}$ with $\mathsf{P}\subseteq\mathcal{C}$ (e.g. PSPACE, coNP, etc.) are closed under the relation $\leq^{\mathsf{P}}_{\mathrm{m}}$, that is, if $A\leq^{\mathsf{P}}_{\mathrm{m}}B$ and $B\in\mathcal{C}$, then also $A\in\mathcal{C}$.

We end this section by pointing out that for any set Ω of **MID** operators, the complexity of **MID**[Ω]-MC is independent of the presence of $\bot$ and atomic negation $\neg$ in **MID**[Ω], that is, the following fact is ture.

Fact 7.1.4. **MID**[Ω]-MC $\equiv^{\mathsf{P}}_{\mathrm{m}}$ **MID**[$\Omega \setminus \{\bot, \neg\}$]-MC.

This is basically because, given an **MID**[Ω]-MC instance $\langle \mathfrak{M}, X, \phi \rangle$, in the formula ϕ, if one replaces all occurrences of $\bot$ by a fixed fresh propositional variable r, and all occurrences of every negated propositional variable $\neg p$ by a fresh propositional variable p', and modifies the valuation of $\mathfrak{M}$ in such a way that r is made to be true nowhere and p' is made to be true only on the states where p is false, then the resulting formula ϕ' and Kripke model $\mathfrak{M}'$ would satisfy

$$\mathfrak{M}, X \models \phi \iff \mathfrak{M}', X \models \phi'.$$

7.2 Complexity of model checking for fragments of MID

In this section we study the complexity of model checking problem for fragments of **MID** and obtain the results listed in Table 7.1. The results for the fragments where the intuitionistic implication $\to$ is not present have been obtained already in [17], so we will only consider the cases where $\to$ is involved. We start with giving a PSPACE algorithm for **MID**-MC.

Theorem 7.2.1. **MID**-MC *is in* PSPACE.

Proof. To prove the theorem, it suffices to give an algorithm for the problem that can be implemented on an alternating Turing machine running in polynomial time (AP Turing machines) [10]. An AP Turing machine uses an extension of ordinary non-deterministic guessing. Here the algorithm can switch between two guessing modes, namely universal and existential guessing. The existential guessing mode makes non-deterministic guessing in NP, whereas the universal guessing mode makes non-deterministic guessing in coNP. When the number of alternations is unbounded, AP Turing machines decide the PSPACE problems.

To prove the theorem, we consider an algorithm which has as input a Kripke model $\mathfrak{M}$, a formula ϕ of **MID**, and a team X of $\mathfrak{M}$. The output of the algorithm is "true" if and only if $\mathfrak{M}, X \models \phi$. By Fact 7.1.4, we may assume that ϕ does not contain $\bot$ or $\neg p$. In the cases

$$\phi \in \{p, =\!(p_1, \dots, p_n), \psi \wedge \chi, \psi \vee \chi\},$$

the algorithm checks whether $\mathfrak{M}, X \models \phi$ according to the team semantics in an obvious way. These cases are deterministic and can be done in PSPACE.

If $\phi = \psi \otimes \chi$ or $\phi = \Diamond \psi$ the algorithm guesses existentially the right fragmentation of the team and the right succeeding team, respectively. In case $\phi = \psi \to \chi$, the algorithm checks universally if for every team $Y \subseteq X$ (i.e. every computation path) with $\mathfrak{M}, Y \models \psi$, it also holds that $\mathfrak{M}, Y \models \chi$. Altogether the algorithm can be implemented on an alternating Turing machine running in polynomial time or – equivalently – on a deterministic machine using polynomial space.

Below we give the full algorithm Algorithm 7.1:

Algorithm 7.1: check($\mathfrak{M} = (W, R, \pi)$,$\phi$,$X$)

```
case φ
when φ = p
  foreach s ∈ X
    if not p ∈ π(s) then
      return false
  return true

when φ = =(p_1,...,p_n)
  foreach (s,s') ∈ X × X
    if π(s) ∩ {p_1,...,p_{n−1}} = π(s') ∩ {p_1,...,p_{n−1}} then
      if (q ∈ π(s) and not q ∈ π(s')) or (not q ∈ π(s) and q ∈ π(s')) then
        return false
  return true

when φ = ψ ∧ χ
  return (check(𝔐,X,ψ) and check(𝔐,X,χ))

when φ = ψ ⊗ χ
  existentially guess two sets of states Y,Z ⊆ W
  if not Y ∪ Z = X then
    return false
  return (check(𝔐,Y,ψ) and check(𝔐,Z,χ))

when φ = ψ ∨ χ
  return (check(𝔐,X,ψ) or check(𝔐,X,ψ))

when φ = ψ → χ
  universally guess a set of states Y ⊆ X
  if not check(𝔐,ψ,Y) or check(𝔐,χ,Y)
    return true
  return false

when φ = □ψ
  Y := ∅
  foreach s' ∈ W
    foreach s ∈ X
      if (s,s') ∈ R then
        Y := Y ∪ {s'}
            // Y is the set of all successors of all states in X, i.e. Y = R(X)
  return check(𝔐,Y,ψ)

when φ = ◇ψ
  existentially guess a set of states Y ⊆ W
    foreach s ∈ X
      if there is no s' ∈ Y with (s,s') ∈ R then
        return false
        // Y contains at least one successor for every state in X, i.e. XRZ for some Z ⊆ Y
  return check(𝔐,Y,ψ)
```

■

If we forbid tensor disjunction $\otimes$ and diamond $\Diamond$ in the sublogic of **MID** in question, the complexity of the above algorithm drops to coNP.

Corollary 7.2.2. **MID**$[\neg, =(\cdots), \wedge, \vee, \rightarrow, \Box]$-MC *is in* coNP. *In particular,* **PID**-MC *is in* coNP.

Proof. In Algorithm 7.1, existential guessing only applies to the cases $\phi = \psi \otimes \chi$ and $\phi = \Diamond\psi$. ■

If neither dependence atoms nor intuitionistic disjunction $\vee$ is allowed in the logic, the model checking problem can even be decided in deterministic polynomial time.

Theorem 7.2.3. $\mathbf{MID}[\neg,\wedge,\otimes,\to,\Box,\Diamond]$-MC *is in* P.

Proof. Formulas of the logic $\mathbf{MID}[\neg,\wedge,\otimes,\Box,\Diamond]$ (classical formulas) are flat, and can be identified with formulas of the usual modal logic (**M**) in negation normal form (identify $\otimes$ with the classical disjunction). It is not hard to show that

$$\mathbf{MID}[\neg,\wedge,\otimes,\Box,\Diamond]\text{-MC} \equiv^{\mathsf{P}}_{\mathrm{m}} \mathbf{M}\text{-MC}.$$

We know by [14] that **M**-MC is in P, so it suffices to show that

$$\mathbf{MID}[\neg,\wedge,\otimes,\to,\Box,\Diamond]\text{-MC} \equiv^{\mathsf{P}}_{\mathrm{m}} \mathbf{MID}[\neg,\wedge,\otimes,\Box,\Diamond]\text{-MC}. \tag{7.1}$$

The direction "$\geq^{\mathsf{P}}_{\mathrm{m}}$" holds trivially. We show the other direction "$\leq^{\mathsf{P}}_{\mathrm{m}}$".

For each formula ϕ of $\mathsf{L}_1 = \mathbf{MID}[\neg,\wedge,\otimes,\to,\Box,\Diamond]$, it suffices to find a logically equivalent formula ϕ^* in the language of $\mathsf{L}_2 = \mathbf{MID}[\neg,\wedge,\otimes,\Box,\Diamond]$ that can be obtained in polynomial time.

Consider two formulas ψ and χ of L_2. View ψ as a formula of the usual modal logic, the formula $\neg\psi$ has an equivalent formula ψ^- in negation normal form in the usual modal logic. The resulting formula ψ^- is in the language of L_2, as well as L_1. In L_1, it is not hard to prove by induction that

$$\psi \to \chi \ \equiv\ \psi^- \otimes \chi. \tag{7.2}$$

Now, for each formula ϕ of L_1, starting from the innermost intuitionistic implication $\to$, apply Equation (7.2) to eliminate all occurrences of the connective $\to$ in ϕ and obtain an equivalent formula ϕ^* in the language of L_2. Such a translation can clearly be done in polynomial time. So we are done. ■

In the remaining part of this section we provide hardness proofs for the model checking problems for various sublogics of **MID**. We first consider the sublogics without diamond $\Diamond$ and tensor disjunction $\otimes$.

Theorem 7.2.4. $\mathbf{PID}[\wedge,\vee,\to]$-MC *is* coNP-*hard.*

Proof. By Lemma 2.2.3, Lemma 4.2.5 and Fact 7.1.4,

$$\begin{aligned}\mathbf{PID}[\wedge,\vee,\to]\text{-MC} &\equiv^{\mathsf{P}}_{\mathrm{m}} \mathbf{PID}[\neg,=\!(\cdots),\wedge,\vee,\to]\text{-MC}\\ &\equiv^{\mathsf{P}}_{\mathrm{m}} \mathbf{PID}\text{-MC}.\end{aligned}$$

Thus it suffices to give a polynomial-time reduction from a known coNP-complete problem to **PID**-MC.

Consider the well-known coNP-complete problem

$$\textsc{Taut} = \{\phi \text{ is a tautology} \mid \phi \text{ is a formula of classical propositional logic}\}.$$

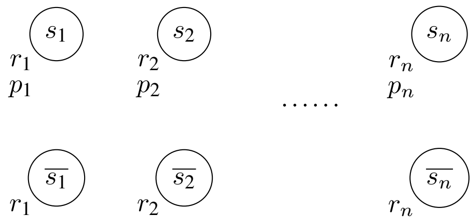


Figure 7.1: Kripke model $\mathfrak{M}$ in the proof of Theorem 7.2.4

Let $\phi(p_1,\dots,p_n)$ be an arbitrary formula of classical propositional logic in negation normal form. Let $r_1,\dots,r_n$ be new propositional variables. Let $\mathfrak{M} = (W,R,\pi)$ be a Kripke model defined as (see Figure 7.1)

$$\begin{aligned} W &:= \{s_1,\dots,s_n,\overline{s_1},\dots,\overline{s_n}\},\\ R &:= \emptyset,\\ \pi(s_i) &:= \{r_i,p_i\},\\ \pi(\overline{s_i}) &:= \{r_i\}. \end{aligned}$$

Define a formula $\phi^{\rightarrow}$ of **PID** inductively as follows:

$$\begin{aligned} p_i^{\rightarrow} &:= r_i \rightarrow p_i,\\ (\neg p_i)^{\rightarrow} &:= r_i \rightarrow \neg p_i,\\ (\theta_0 \wedge \theta_1)^{\rightarrow} &:= \theta_0^{\rightarrow} \wedge \theta_1^{\rightarrow},\\ (\theta_0 \vee \theta_1)^{\rightarrow} &:= \theta_0^{\rightarrow} \vee \theta_1^{\rightarrow}. \end{aligned}$$

Let

$$\alpha_n := \bigwedge_{i=1}^{n} \big(r_i \rightarrow {=}(p_i)\big),$$

and

$$\psi \;:=\; \alpha_n \rightarrow \phi^{\rightarrow}.$$

It suffices to show that $\phi \in \textsc{Taut}$ iff $\mathfrak{M}, W \models \psi$.

The general idea of the proof is as follows. By the construction, each team X of $\mathfrak{M}$ satisfying the formula α_n contains at most one of the states s_i and $\overline{s_i}$, for each i. In the Kripke model $\mathfrak{M}$, the state s_i simulates positive truth assignments for p_i (i.e. assignments σ such that $\sigma(p_i) = \top$), while $\overline{s_i}$ simulates negative truth assignments for p_i (i.e. assignments σ such that $\sigma(p_i) = \bot$). Thus, any maximal such team X simulates a truth assignment σ_X for $p_1,\dots,p_n$. Moreover, under the assignment σ_X, the formula ϕ of classical propositional logic will behave exactly as the formula $\phi^{\rightarrow}$ of **PID** under the team X, that is, $\sigma_X(\phi) = \top$ iff $\mathfrak{M}, X \models \phi^{\rightarrow}$. The required equivalence will then follow.

Formally, first suppose $\phi \in \textsc{Taut}$. By the downwards closure property, It suffices to show that for any maximal team $X \subseteq W$ such that $\mathfrak{M}, X \models \alpha_n$, it holds that $\mathfrak{M}, X \models \phi^{\rightarrow}$. Now, if Y is a maximal subteam of X such that $\mathfrak{M}, Y \models r_i$, then by the construction of $\mathfrak{M}$, we must have that $Y \subseteq \{s_i, \overline{s_i}\}$. Since $\mathfrak{M}, X \models r_i \rightarrow {=}(p_i)$, we have that $\mathfrak{M}, Y \models {=}(p_i)$, so p_i has a constant value in Y, which means that Y contains at most one state of s_i and $\overline{s_i}$. Therefore, the maximal team X contains exactly one of the states s_i and $\overline{s_i}$ for each $1 \leq i \leq n$.

Clearly, X induces a truth assignment σ_X for $p_1, \ldots, p_n$ defined as follows:

$$\sigma_X(p_i) := \begin{cases} \top & \text{if } s_i \in X, \\ \bot & \text{if } \overline{s_i} \in X. \end{cases}$$

Such team X and its *induced truth assignment* σ_X are in one-one correspondence; moreover, the assignment σ_X makes the classical formula ϕ true if and only if the team X satisfies the **PID** formula $\phi^{\rightarrow}$. We show this by showing a more general claim as follows:

Claim: For all subformulas χ of ϕ, it holds that $\sigma_X(\chi) = \top$ iff $\mathfrak{M}, X \models \chi^{\rightarrow}$.

Proof of Claim. An easy inductive proof. We only show the case that $\chi = \neg p_i$. First suppose $\sigma_X(\neg p_i) = \top$. Then $\overline{s_i} \in X$ and $s_i \notin X$, thus, for all non-empty team $Y \subseteq X$ such that $\mathfrak{M}, Y \models r_i$, we must have that $Y = \{\overline{s_i}\}$, hence $\mathfrak{M}, Y \models \neg p$, which implies that $\mathfrak{M}, X \models r_i \to \neg p_i$.

Conversely, suppose $\mathfrak{M}, X \models r_i \to \neg p_i$. Then we must have that $s_i \notin X$, thus by the maximality of X, $\overline{s_i} \in X$ and $\sigma_X(p_i) = \bot$, i.e. $\sigma_X(\neg p_i) = \top$. ⊣

Now, as $\phi \in$ TAUT, we have that $\sigma_X(\phi) = \top$. Hence we obtain by Claim that $\mathfrak{M}, X \models \phi^{\rightarrow}$, as required.

Conversely suppose that $\mathfrak{M}, W \models \psi$ and σ is an arbitrary truth assignment for $p_1, \ldots, p_n$. The truth assignment σ induces a team X_σ defined by

$$X_\sigma := \{s_i \mid \sigma(p_i) = \top\} \cup \{\overline{s_i} \mid \sigma(p_i) = \bot\}.$$

Clearly, $\mathfrak{M}, X_\sigma \models \alpha_n$, thus by assumption, $\mathfrak{M}, X_\sigma \models \phi^{\rightarrow}$. Hence by Claim we obtain that $\sigma(\phi) = \top$, thereby $\phi \in$ TAUT. ■

Theorem 7.2.5. *For all* $\{\wedge, \vee, \to\} \subseteq \Omega \subseteq \{\neg, =\!(\cdots), \wedge, \vee, \to, \Box\}$, **MID**[$\Omega$]-MC *is* coNP-*complete. In particular,* **PID**-MC *is* coNP-*complete.*

Proof. Follows from Theorem 7.2.2 and Theorem 7.2.4. ■

Next, we analyze the complexity of the model checking problem for fragments of **MID** containing tensor disjunction $\otimes$ and intuitionistic implication $\to$.

Theorem 7.2.6. *Let* $\Omega \supseteq \{=\!(\cdots), \wedge, \otimes, \to\}$. *Then* **MID**[$\Omega$]-MC *is* PSPACE-*complete.*

Proof. The upper bound follows from Theorem 7.2.1. For the lower bound we give a reduction to the problem from the well-known PSPACE-complete problem

$$\text{QBF} = \{\psi \text{ is a quantified Boolean formula} \mid \psi \text{ is true}\}.$$

Let $\psi = \forall x_1 \exists x_2 \ldots \forall x_{n-1} \exists x_n\, \phi$ be a quantified Boolean formula, where ϕ is quantifier-free. Assume without loss of generality that n is even and that $\phi = C_1 \wedge \cdots \wedge C_m$ is in 3CNF with

$$C_j = \alpha_{j0} \vee \alpha_{j1} \vee \alpha_{j2} \quad (1 \leq j \leq m)$$

for distinct propositional literals $\alpha_{j0}, \alpha_{j1}, \alpha_{j2}$. Let

$$r_1, \ldots, r_n, p_1, \ldots, p_n, c_1, \ldots, c_m, c_{10}, \ldots, c_{m0}, c_{11}, \ldots, c_{m1}, c_{12}, \ldots, c_{m2}$$

be distinct propositional variables. Define an instance $(\mathfrak{M} = (W, R, \pi), W, \theta)$ of **MID**$[\Omega]$-MC as follows

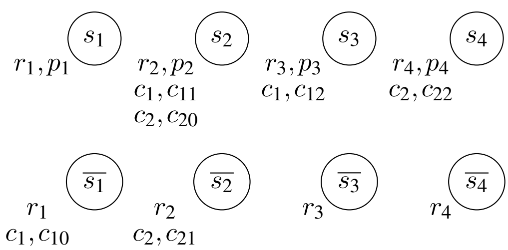
- $W := \{s_1, \ldots, s_n, \overline{s_1}, \ldots, \overline{s_n}\}$,
- $R := \emptyset$,
- $\pi(s_i) = \{r_i, p_i\} \cup \{c_j, c_{j0} \mid \alpha_{j0} = x_i,\ 1 \le j \le m\} \cup \{c_j, c_{j1} \mid \alpha_{j1} = x_i,\ 1 \le j \le m\} \cup \{c_j, c_{j2} \mid \alpha_{j2} = x_i,\ 1 \le j \le m\}$,
- $\pi(\overline{s_i}) = \{r_i\} \cup \{c_j, c_{j0} \mid \alpha_{j0} = \neg x_i,\ 1 \le j \le m\} \cup \{c_j, c_{j1} \mid \alpha_{j1} = \neg x_i,\ 1 \le j \le m\} \cup \{c_j, c_{j2} \mid \alpha_{j2} = \neg x_i,\ 1 \le j \le m\}$
  (see Figure 7.2 for an example of the construction of $\mathfrak{M}$),
- $\theta := \delta_1$, where

$$\delta_{2k-1} := (r_{2k-1} \rightarrow {=}(p_{2k-1})) \rightarrow \delta_{2k} \quad (1 \le k \le n/2),$$
$$\delta_{2k} := (r_{2k} \wedge {=}(p_{2k})) \otimes \delta_{2k+1} \quad (1 \le k < n/2),$$
$$\delta_n := (r_n \wedge {=}(p_n)) \otimes \phi',$$

and

$$\phi' := \bigwedge_{j=1}^{m} \Big(c_j \rightarrow \big(\ ({=}(c_{j0}) \wedge {=}(c_{j1}) \wedge {=}(c_{j2})) \otimes \big({=}(c_{j0}] \wedge {=}(c_{j1}) \wedge {=}(c_{j2}) \big) \ \big) \Big),$$

i. e. $$\theta = \big(r_1 \rightarrow {=}(p_1)\big) \rightarrow \Big(\big(r_2 \wedge {=}(p_2)\big) \otimes \cdots\cdots \cdots\cdots \otimes \Big(\big(r_{n-1} \rightarrow {=}(p_{n-1})\big) \rightarrow \Big(\big(r_n \wedge {=}(p_n)\big) \otimes \phi' \Big) \Big) \cdots \Big).$$



The model $\mathfrak{M}$ for $\phi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \neg x_2 \vee x_4)$

Figure 7.2: An example the construction of $\mathfrak{M}$ in the proof of Theorem 7.2.6

It suffices to show that $\psi \in \mathrm{QBF}$ iff $\mathfrak{M}, W \models \theta$.

The general idea of the proof is that the alternating operators $\to$ and $\otimes$ in the formula θ of **MID** simulate the alternating quantifiers $\forall$ and $\exists$ in the quantified Boolean formula ψ, respectively. In the formula ψ, for each $1 \leq k \leq n/2$, the universal quantifier $\forall x_{2k-1}$ corresponds to the formula δ_{2k-1} in θ , and we have that δ_{2k-1} is satisfied in a team X iff δ_{2k} is satisfied in all subteams $X_{2k-1} \subseteq X$ which satisfy $r_{2k-1} \to =\!(p_{2k-1})$, i. e. all subteams containing at most one of the states s_{2k-1} and $\overline{s_{2k-1}}$. Every existential quantifier $\exists x_{2k}$ in ψ corresponds to the subformula δ_{2k} of θ, and δ_{2k} is satisfied in a team X iff X can be split into X_{2k} and Y_{2k} such that $\mathfrak{M}, X_{2k} \models \delta_{2k+1}$ and $\mathfrak{M}, Y_{2k} \models r_{2k} \wedge =\!(p_{2k})$, i. e. δ_{2k+1} has to be satisfied in a team with exactly one of the states s_{2k} and $\overline{s_{2k}}$.

In the Kripke model $\mathfrak{M}$, starting from the team W of all states, for every nested level i of $\to$ or $\otimes$ drop exactly one of the states s_i and $\overline{s_i}$ according to the truth assignment of the Boolean variable x_i quantified by $\forall$ or $\exists$. Iterate this procedure until we arrive at a team X_n that contains exactly one of the states s_i and $\overline{s_i}$ for each $i \in \{1, \dots, n\}$. This team X_n is in fact the team induced by the complement of a truth assignment σ for $x_1, \dots, x_n$ (in a similar sense to that in the proof of Theorem 7.2.4) and $\mathfrak{M}, X_n \models \phi'$ iff $\sigma(\phi) = \top$. Then the required equivalence will follow.

Formally, first suppose that $\psi \in \mathrm{QBF}$. During the proof, we will construct a truth assignment σ for $x_1, \dots, x_n$ such that $\sigma(\phi) = \top$ by choosing values for

$$\sigma(x_1), \sigma(x_3), \dots, \sigma(x_{n-1}).$$

The assumption guarantees that an appropriate value for each $\sigma(x_{2k})$ that will satisfy the formula ϕ exists and they are determined by the values of $\sigma(x_1), \sigma(x_3), \dots, \sigma(x_{2k-1})$.

We have to show that

$$\mathfrak{M}, W \models (r_1 \to =\!(p_1)) \to \delta_2.$$

By the downward closure property , it suffices to show that for the maximal teams $X_1 \subseteq W$ such that $\mathfrak{M}, X_1 \models r_1 \to =\!(p_1)$, namely the teams $W \setminus \{s_1\}$ and $W \setminus \{\overline{s_1}\}$, it holds that

$$\mathfrak{M}, X_1 \models \delta_2, \text{ i. e. } \mathfrak{M}, X_1 \models (r_2 \wedge =\!(p_2)) \otimes \delta_3.$$

Choose the value of $\sigma(x_1)$ according to X_1 by letting

$$\sigma(x_1) := \begin{cases} \top & \text{if } X_1 = W \setminus \{s_1\}, \\ \bot & \text{if } X_1 = W \setminus \{\overline{s_1}\}. \end{cases}$$

Note that $\sigma \upharpoonright \{x_1\}$ is defined as the *complement* of the truth assignment induced by X_1 – which was used in the proof of Theorem 7.2.4. We will continue to define the truth assignment as the complement of the induced one. The reason for this will become clear when we show the connection between ϕ and ϕ' in the end.

Since $\psi \in \mathrm{QBF}$, by our discussion above, an appropriate value of $\sigma(x_2)$ that will satisfy the formula ϕ exists and is determined by $\sigma(x_1)$. Now we split the team X_1 into X_2 and Y_2 according to the value of $\sigma(x_2)$ by letting

$$Y_2 := \begin{cases} \{s_2\} & \text{if } \sigma(x_2) = \top, \\ \{\overline{s_2}\} & \text{if } \sigma(x_2) = \bot, \end{cases}$$

and $X_2 = X_1 \setminus Y_2$. Clearly, $\mathfrak{M}, Y_2 \models r_2 \wedge {=}(p_2)$ and it suffices to check that $\mathfrak{M}, X_2 \models \delta_3$.

As shown so far, to prove $\mathfrak{M}, W \models \delta_1$, it is enough to show that for every X_1 chosen as above, the above constructed X_2 satisfies $\mathfrak{M}, X_2 \models \delta_3$. Repeating the same arguments and constructions $n/2$ times, it remains to show that $\mathfrak{M}, X_n \models \phi'$.

Note that X_n and σ satisfy

$$\begin{array}{lll} s_i \in X_n \iff \sigma(x_i) & = \bot, \\ \overline{s_i} \in X_n \iff \sigma(\neg x_i) & = \bot. \end{array} \tag{7.3}$$

Moreover, since $\sigma(\phi) = \top$, for all $j \in \{1, \ldots, m\}$, it holds that $\sigma(\alpha_{j0}) = \top$ or $\sigma(\alpha_{j1}) = \top$ or $\sigma(\alpha_{j2}) = \top$.

Consider an arbitrary $j \in \{1, \ldots, m\}$. We illustrate the idea of the proof by an example. Let us suppose $\alpha_{j0} = \neg x_{i_0}$, $\alpha_{j1} = \neg x_{i_1}$, $\alpha_{j2} = x_{i_2}$ for some $i_0, i_1, i_2 \in \{1, \ldots, n\}$, and $\sigma(\neg x_{i_1}) = \top$. For any $X \subseteq X_n$ such that $\mathfrak{M}, X \models c_j$, by the construction of $\mathfrak{M}$, $X \subseteq \{\overline{s_{i_0}}, \overline{s_{i_1}}, s_{i_2}\}$. But in view of (7.3), we know that $\overline{s_{i_1}} \notin X$. Thus, $X \subseteq \{\overline{s_{i_0}}, s_{i_2}\}$, which implies that

$$\mathfrak{M}, X \models \big({=}(c_{j0}) \wedge {=}(c_{j1}) \wedge {=}(c_{j2})\big) \otimes \big({=}(c_{j0}) \wedge {=}(c_{j1}) \wedge {=}(c_{j2})\big),$$

as dependence atoms are always satisfied on singleton teams. This shows that $\mathfrak{M}, X_n \models \phi'$.

Conversely, suppose $\mathfrak{M}, W \models \theta$. Choose arbitrarily some values for

$$\sigma(x_1), \sigma(x_3), \ldots, \sigma(x_{2n-1})$$

and define the values for

$$\sigma(x_2), \sigma(x_4), \ldots, \sigma(x_{2n})$$

by reversing the above arguments and constructions, and repeat them $n/2$ times, we then arrive at (7.3) and $\mathfrak{M}, X_n \models \phi'$. The crucial observation is that when evaluating $(r_{2k-1} \to {=}(p_{2k-1})) \to \delta_{2k}$ we only need to consider the maximal teams satisfying $r_{2k-1} \to {=}(p_{2k-1})$ and there are exactly two of those, one without s_{2k-1} and the other one without $\overline{s_{2k-1}}$. And when evaluating $(r_{2k} \wedge {=}(p_{2k})) \otimes \delta_{2k+1}$ we have to consider only the complements of the maximal teams satisfying $r_{2k} \wedge {=}(p_{2k})$ and again there are exactly two, one without s_{2k} and the other one without $\overline{s_{2k}}$.

It remains to show that $\sigma(\phi) = \top$. That is to show that $\sigma(\alpha_{j0} \vee \alpha_{j1} \vee \alpha_{j2}) = \top$ for an arbitrarily chosen $j \in \{1, \ldots, m\}$. Again, we illustrate the idea of the proof by an example. Let us suppose

$$\alpha_{j0} = x_{i_0},\ \alpha_{j1} = x_{i_1} \text{ and } \alpha_{j2} = \neg x_{i_2}.$$

Now let $X \subseteq X_n$ be the maximal team such that $\mathfrak{M}, X \models c_j$. Then, by the construction of $\mathfrak{M}$, we know that $X \subseteq \{s_{i_0}, s_{i_1}, \overline{s_{i_2}}\}$. Since $\mathfrak{M}, X_n \models \phi'$,

$$\mathfrak{M}, X \models \big({=}(c_{j0}) \wedge {=}(c_{j1}) \wedge {=}(c_{j2})\big) \otimes \big({=}(c_{j0}) \wedge {=}(c_{j1}) \wedge {=}(c_{j2})\big).$$

Thus, by construction of $\mathfrak{M}$, we must have that$|X| \leq 2$. Say $s_{i_1} \notin X$, then, by maximality of X, we obtain that $s_{i_1} \notin X_n$ which means that $\sigma(x_{i_1}) = \top$ by (7.3), thereby $\sigma(\alpha_{j1}) = \top$. This way we obtain that $\sigma(\phi) = \top$. ■

Finally, we study the model checking problem for the sublogics of **MID** containing diamond $\Diamond$, intuitionistic disjunction $\vee$ and intuitionistic implication $\to$.

Theorem 7.2.7. *Let* $\Omega \supseteq \{\wedge, \vee, \to, \Diamond\}$. *Then* **MID**$[\Omega]$-MC *is* PSPACE-*complete.*

Proof. The upper bound again follows from Theorem 7.2.1. For the lower bound, noting that

$$\mathbf{MID}[\neg, =(\cdots), \wedge, \vee, \to, \Diamond]\text{-MC} \equiv^{\mathsf{P}}_{\mathsf{m}} \mathbf{MID}[\wedge, \vee, \to, \Diamond]\text{-MC},$$

it suffices to give a reduction from QBF to **MID**$[\neg, =(\cdots), \wedge, \vee, \to, \Diamond]$-MC in polynomial-time.

Let $\psi = \forall x_1 \exists x_2 \dots \forall x_{n-1} \exists x_n \, \phi$ be a quantified Boolean formula, where ϕ quantifier-free and without loss of generality we assume that n is even. Define an instance $(\mathfrak{M} = (W, R, \pi), X, \theta)$ of **MID**$[\neg, =(\cdots), \wedge, \vee, \to, \Diamond]$-MC as follows:

- $W := \bigcup_{1 \le i \le n} W_i, \quad R := \bigcup_{1 \le i \le n} R_i$ and for $1 \le i \le n/2$

$$\begin{aligned} W_{2i-1} &:= \{s_{2i-1}, \overline{s_{2i-1}}\} \\ W_{2i} &:= \{s_{2i}, \overline{s_{2i}}\} \cup \{t_i\} \cup \{t_{i1}, \cdots, t_{i(i-1)}\} \\ R_{2i-1} &:= \{(s_{2i-1}, s_{2i-1}), (\overline{s_{2i-1}}, \overline{s_{2i-1}})\} \\ R_{2i} &:= \{(t_i, t_{i1}), (t_{i1}, t_{i2}), \cdots, (t_{i(i-2)}, t_{i(i-1)})\} \\ &\quad \cup \{(t_{i(i-1)}, s_{2i}), (t_{i(i-1)}, \overline{s_{2i}})\} \\ &\quad \cup \{(s_{2i}, s_{2i}), (\overline{s_{2i}}, \overline{s_{2i}})\} \end{aligned}$$

- $\pi(s_j) := \{r_j, p_j\}$,
- $\pi(\overline{s_j}) := \{r_j\}$,
- $\pi(t) := \emptyset$, for $t \notin \{s_j, \overline{s_j} \mid 1 \le j \le n\}$
  (see Figure 7.3 for the construction of $\mathfrak{M}$),
- $X := \{s_i, \overline{s_i} \mid 1 \le i \le n,\ i \text{ odd}\} \cup \{t_i \mid 1 \le i \le n/2\}$;
- $\theta = \delta_1$, where

$$\begin{aligned} \delta_{2k-1} &:= (r_{2k-1} \to =(p_{2k-1})) \to \delta_{2k} \quad (1 \le k \le n/2), \\ \delta_{2k} &:= \Diamond \delta_{2k+1} \quad (1 \le k < n/2), \\ \delta_n &:= \Diamond \phi^{\to}, \end{aligned}$$

  and $\phi^{\to}$ is generated from ϕ by the same inductive translation as in the proof of Theorem 7.2.4,

$$\text{i.e. } \theta = (r_1 \to =(p_1)) \to \Diamond\Big((r_2 \to =(p_2)) \to \cdots\cdots$$
$$\cdots\cdots \to \Diamond\Big((r_{n-1} \to =(p_{n-1})) \to \Diamond\phi^{\to}\Big)\dots\Big).$$

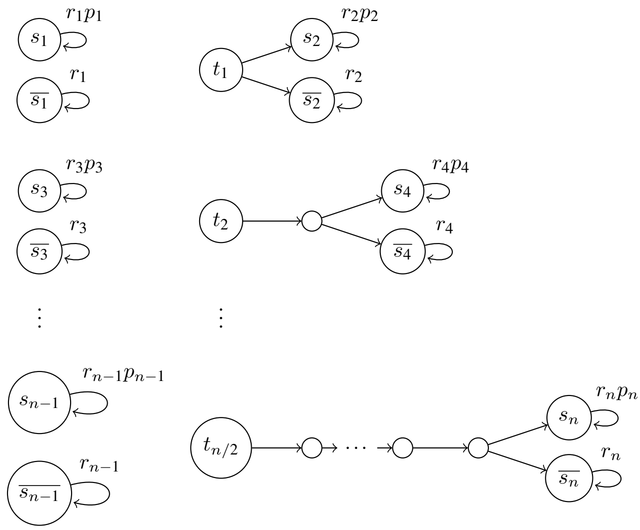


Figure 7.3: Kripke model $\mathfrak{M}$ in the proof of Theorem 7.2.7

We will show that $\psi \in \text{QBF}$ iff $\mathfrak{M}, X \models \theta$. The idea, analogous to the proof of Theorem 7.2.6, is that the alternating operators $\to$ and $\Diamond$ in the formula θ of **MID** simulate the quantifiers $\forall$ and $\exists$ in the quantified Boolean formula ψ, respectively. Note that the model $\mathfrak{M}$ can be viewed as the disjoint union of n submodels $\mathfrak{M}_i$ (as depicted in Figure 7.3), where each $\mathfrak{M}_i$ contains for p_i a positive state s_i with a loop and a negative state $\overline{s_i}$ with a loop, and for even $1 \leq i \leq n$, $\mathfrak{M}_i$ also contains a chain of length $(i/2-1)$ with a split leading to s_i and $\overline{s_i}$ at the end. The states in every $\mathfrak{M}_i$ with no proper predecessors (all s_{2k-1}, $\overline{s_{2k-1}}$, t_k's for $1 \leq k \leq n/2$, these form the team X) can be viewed as starting states and those with no proper successors (all s_i, $\overline{s_i}$'s for $1 \leq i \leq n$) can be viewed as final states. In the proof, we start with the team X of all starting states, and then for every nested level i of $\to$ we drop one of the states s_{2i+1} and $\overline{s_{2i+1}}$, while for every nested level i of $\Diamond$ we simultaneously move forward on the chains and thereby choose one of the states s_{2i} and $\overline{s_{2i}}$. Iterate this procedure until we arrive at a team X_n that contains exactly one of the final states s_i and $\overline{s_i}$ for each $i \in \{1, \ldots, n\}$. This team X_n induces a truth assignment σ for $x_1, \ldots, x_n$ as in the proof of Theorem 7.2.4, and $\mathfrak{M}, X_n \models \phi^{\to}$ iff $\sigma(\phi) = \top$.

Now, formally, suppose $\psi \in \text{QBF}$. We have to show that

$$\mathfrak{M}, X \models (r_1 \to =\!(p_1)) \to \delta_2.$$

By the downwards closure property, it suffices to show that for the maximal teams $X_1 \subseteq X$ such that $\mathfrak{M}, X_1 \models r_1 \to =\!(p_1)$, it holds that

$$\mathfrak{M}, X_1 \models \delta_2, \text{ i. e. } \mathfrak{M}, X_1 \models \Diamond \delta_3.$$

Analogous to the proof of Theorem 7.2.6, choose the value of $\sigma(x_1)$ according to X_1 by letting

$$\sigma(x_1) := \begin{cases} \bot & \text{if } X_1 = X \setminus \{s_1\}, \\ \top & \text{if } X_1 = X \setminus \{\overline{s_1}\} \end{cases}$$

(but here $\sigma \upharpoonright \{x_1\}$ is defined as the truth assignment induced by X_1 instead of the complementary one, as in the proof of Theorem 7.2.6). By a similar argument to that in the proof of Theorem 7.2.6, an appropriate value of $\sigma(x_2)$ that satisfies ϕ exists and is determined by $\sigma(x_1)$. We now choose X_2 such that $X_1 R X_2$ as follows:

$$X_2 := \begin{cases} R(X_1) \setminus \{\overline{s_2}\} & \text{if } \sigma(x_2) = \top, \\ R(X_1) \setminus \{s_2\} & \text{if } \sigma(x_2) = \bot, \end{cases}$$

It suffices to check that $\mathfrak{M}, X_2 \models \delta_3$. Again, analogous to the proof of Theorem 7.2.6, repeating the universal and the existential arguments $n/2$ times, it remains to show that $\mathfrak{M}, X_n \models \phi^{\rightarrow}$. And, analogous to (7.3), X_n and σ satisfy

$$\begin{aligned} s_i \in X_n &\iff \sigma(x_i) &= \top, \\ \overline{s_i} \in X_n &\iff \sigma(\neg x_i) &= \top, \end{aligned} \tag{7.4}$$

and moreover $\sigma(\phi) = \top$.

Noting that σ is the truth assignment induced by X_n, by the Claim in the proof of Theorem 7.2.4, we obtain that $\mathfrak{M}', X_n \models \phi^{\rightarrow}$, where $\mathfrak{M}' = (W', R', \pi')$ with

- $W' = \{s_i, \overline{s_i} \mid 1 \leq i \leq n\}$,
- $R' = \emptyset$,
- $\pi' = \pi \upharpoonright W'$

(i.e., $\mathfrak{M}'$ is the model constructed in the proof of Theorem 7.2.4, which can also be viewed as a submodel of $\mathfrak{M}$ consisting of all final states). Next, since $\phi^{\rightarrow}$ is modality-free, it follows that $\mathfrak{M}'', X_n \models \phi^{\rightarrow}$, where $\mathfrak{M}''$ is the submodel of $\mathfrak{M}$ generated by W' (namely $\mathfrak{M}'$ with all the loops). Finally, since the truth of **MID** formulas with respect to teams is invariant under taking generated submodels (Theorem 6.1.9), we conclude that $\mathfrak{M}, X_n \models \phi^{\rightarrow}$.

Conversely, suppose that $\mathfrak{M}, X \models \theta$. As in the proof of Theorem 7.2.6 we can reverse the above constructions and arrive at (7.4). The crucial point is that when evaluating $\Diamond\delta_{2k+1}$ we only need to consider minimal successor teams.

Now, by the construction of X_n, we have that $\mathfrak{M}, X_n \models \phi^{\rightarrow}$. Reversing the above argument, by the Claim in the proof of Theorem 7.2.4, we obtain that $\sigma(\phi) = \top$. ■

7.3 Concluding remarks and open problems

Table 7.1 contains all results we have obtained in this chapter. We have shown that model checking for **MID** in general is PSPACE-complete, and this still holds if we forbid $\Box$, $=(\cdots)$ and either $\Diamond$ or $\otimes$. If we forbid $\Diamond$ and $\otimes$, on the other hand, the complexity drops to coNP. In particular, **PID**-MC is coNP-complete.

Operators									Complexity	Reference
$\Box$	$\Diamond$	$\wedge$	$\otimes$	$\vee$	$\neg/\bot$	$\rightarrow$	$=(\cdots)$			
$*$	$*$	$+$	$+$	$*$	$*$	$+$	$+$		PSPACE	Theorem 7.2.6
$*$	$*$	$+$	$+$	$+$	$*$	$+$	$*$		PSPACE	Theorem 7.2.6, Lemma 4.2.5
$*$	$+$	$+$	$*$	$+$	$*$	$+$	$*$		PSPACE	Theorem 7.2.7
$*$	$-$	$+$	$-$	$+$	$*$	$+$	$*$		coNP	Theorem 7.2.5
$*$	$*$	$*$	$*$	$-$	$*$	$*$	$-$		P	Theorem 7.2.3
$*$	$*$	$*$	$*$	$*$	$*$	$-$	$*$		P / NP	[17]

$+$: operator present $-$: operator absent $*$: complexity independent of operator

Table 7.1: Classification of complexity for fragments of **MID**-MC
All results are completeness results except for the P cases which are upper bounds.

In Table 7.1, some of the cases are missing, e.g., the one where only conjunction is forbidden, the one where only both disjunctions are forbidden and the one from Theorem 7.2.7 but with dependence atoms allowed instead of intuitionistic disjunction.

One other main open problem is: the computational complexity of the satisfiability problem for **MID** is unknown.

Appendix: Rules of natural deduction systems for propositional logics of dependence and independence

Below we list all of the rules of natural deduction systems for the propositional logics considered in Chapter 4, and the main derivable rules used in the chapter.

The following table tabulates (in an obvious way) all of the natural deduction systems defined in Chapter 4. The listed logics are complete with respect to the corresponding deductive systems.

rules	$\mathbf{PD}^{[\vee]}$	**PD**	**PExc**	$\mathbf{PD}^{[\vee,\mathsf{NE}]}$	$\mathbf{PInd}^{[\mathsf{NE}]}$	$\mathbf{PInc}^{[\mathsf{NE}]}$
1-2	+	+	+	+	+	+
3-4	+			+		
5	+	+	+			
6-7				+	+	+
8-12	+	+	+	+	+	+
13-14	+			+		
(Dstr $\otimes\vee$)	+			+		
(Dstr $\otimes\vee\otimes$)	+			+		
(Dstr $\wedge\otimes$)	+	+	+			
(Dstr$^*\wedge\otimes$)				+		
(Dstr NE$\wedge\otimes$)				+		
16		+				
17					+	
18			+			
19						+
20		+	+			
21					+	+
22				+		
23-25				+	+	+

Rules:

1. Conjunction Introduction:

$$\frac{\phi \qquad \psi}{\phi\wedge\psi}\,(\wedge\mathsf{I})$$

2. Conjunction Elimination:

$$\frac{\phi\wedge\psi}{\phi}\,(\wedge\mathsf{E}) \qquad \frac{\phi\wedge\psi}{\psi}\,(\wedge\mathsf{E})$$

3. Intuitionistic Disjunction Introduction:

$$\frac{\phi}{\phi\vee\psi}\,(\vee\mathsf{I}) \qquad \frac{\psi}{\phi\vee\psi}\,(\vee\mathsf{I})$$

4. Intuitionistic Disjunction Elimination:

$$\frac{\phi\vee\psi \qquad \begin{matrix}[\phi]\\ \vdots\\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ \vdots\\ \chi\end{matrix}}{\chi}\,(\vee\mathsf{E})$$

5. Tensor Disjunction Introduction:

$$\frac{\phi}{\phi \otimes \psi}\ (\otimes\mathsf{I})$$

6. Weak Tensor Disjunction Introduction:

$$\frac{\phi}{\phi \otimes \psi}\ (*)\ (\otimes\mathsf{WI})$$

(∗) whenever ψ does not contain NE.

7. Tensor Disjunction Repetition:

$$\frac{\phi}{\phi \otimes \phi}\ (\otimes\mathsf{Rpt})$$

8. Weak Tensor Disjunction Elimination:

$$\frac{\phi \otimes \psi \qquad \begin{matrix}[\phi]\\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\chi}\ (\otimes\mathsf{WE})$$

whenever χ is a classical formula.

9. Tensor Disjunction Substitution:

$$\frac{\phi \otimes \psi \qquad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\phi \otimes \chi}\ (\otimes\mathsf{Sub})$$

10. Commutative and Associative Laws for Tensor Disjunction:

$$\frac{\phi \otimes \psi}{\psi \otimes \phi}\ (\mathsf{Com}\otimes) \qquad \frac{\phi \otimes (\psi \otimes \chi)}{(\phi \otimes \psi) \otimes \chi}\ (\mathsf{Ass}\otimes)$$

11. Contradiction Elimination:

$$\frac{\phi \otimes (p_i \wedge \neg p_i)}{\phi}\ (\bot\mathsf{E})$$

12. Atomic Excluded Middle:

$$\frac{}{p_i \otimes \neg p_i}\ (\mathsf{EM}_0)$$

13. Dependence Atom Introduction:

$$\frac{\bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \cdots \wedge \dots p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}\right)}{=\!(p_{j_0}, \dots, p_{j_{k-1}}, p_{j_k})}\ (\mathsf{DepI})$$

where $\mathbf{2^k}$ is the maximal k-team on the set $\{j_0, \dots, j_{k-1}\}$.

14. Dependence Atom Elimination:

$$\frac{=\!(p_{j_0}, \dots, p_{j_{k-1}}, p_{j_k})}{\bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \cdots \wedge \dots p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}\right)}\ (\mathsf{DepE})$$

where $\mathbf{2^k}$ is the maximal k-team on the set $\{j_0,\dots,j_{k-1}\}$.

15. Distributive Laws:

$$\frac{\phi\otimes(\psi\vee\chi)}{(\phi\otimes\psi)\vee(\phi\otimes\chi)}\,(\mathsf{Dstr}\otimes\vee)\qquad \frac{(\phi\otimes\psi)\vee(\phi\otimes\chi)}{\phi\otimes(\psi\vee\chi)}\,(\mathsf{Dstr}\otimes\vee\otimes)$$

$$\frac{\phi\wedge(\psi\otimes\chi)}{(\phi\wedge\psi)\otimes(\phi\wedge\chi)}\,(\mathsf{Dstr}\wedge\otimes)$$

$$\frac{\phi\wedge(\psi\otimes\chi)}{(\phi\wedge\psi)\otimes(\phi\wedge\chi)}\,(*)\,(\mathsf{Dstr}^*\wedge\otimes)$$

$(*)$ whenever ϕ does not contain NE.

$$\frac{\mathsf{NE}\wedge\bigotimes_{j\in J}\phi_j}{\bigvee_{\substack{f\in 2^J\\ f\neq\mathbf{0}}}\ \bigotimes_{\substack{j\in J\\ f(j)=1}}(\mathsf{NE}\wedge\phi_j)}\,(\mathsf{Dstr\ NE}\wedge\otimes)$$

where $\mathbf{0}:J\to 2$ is defined as $\mathbf{0}(j)=0$.

16. Dependence Atom Strong Introduction: For any function $f:\mathbf{2^k}\to 2$ of the maximal k-team on $\{j_0,\dots,j_{k-1}\}$ into $2=\{0,1\}$,

$$\frac{\bigotimes_{s\in\mathbf{2^k}}\left(p_{j_0}^{s(j_0)}\wedge\dots\wedge p_{j_{k-1}}^{s(j_{k-1})}\wedge p_{j_k}^{f(s)}\right)}{=\!(p_{j_0},\dots,p_{j_k})}\,(\mathsf{DepSI})$$

17. Independence Atom Strong Introduction: For any strong approximation sequence $\langle u\rangle$ of $p_{i_1}\dots p_{i_k}\perp p_{j_1}\dots p_{j_m}(p_{i_1},\dots,p_{i_n})$,

$$\frac{\bigotimes_{s\in u}(p_{i_1}^{s(i_1)}\wedge\dots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})}{p_{i_1}\dots p_{i_k}\perp p_{j_1}\dots p_{j_m}}\,(\mathsf{IndSI})$$

18. Exclusion Atom Introduction: For any approximation sequence $\langle o\rangle$ of $p_{i_1}\dots p_{i_k}\mid p_{j_1}\dots p_{j_k}$,

$$\frac{\bigotimes_{s\in o}\left(p_{i_1}^{s(i_1)}\wedge\dots\wedge p_{i_k}^{s(i_k)}\wedge p_{j_1}^{s(j_1)}\wedge\dots\wedge p_{j_k}^{s(j_k)}\right)}{p_{i_1}\dots p_{i_k}\mid p_{j_1}\dots p_{j_k}}\,(\mathsf{Excl})$$

19. Inclusion Atom Strong Introduction: For any strong approximation sequence $\langle u\rangle$ of the atom $p_{j_1}\dots p_{j_m}\subseteq p_{k_1}\dots p_{k_m}(p_{i_1},\dots,p_{i_n})$,

$$\frac{\bigotimes_{s\in u}(p_{i_1}^{s(i_1)}\wedge\dots\wedge p_{i_n}^{s(i_n)}\wedge\mathsf{NE})}{p_{j_1}\dots p_{j_m}\subseteq p_{k_1}\dots p_{k_m}}\,(\mathsf{IncSI})$$

20. Approximation Transition:

$$\frac{\begin{matrix}[\phi^*_{\Omega_0}] & & [\phi^*_{\Omega_m}] & \\ \vdots & \dots & \vdots & \\ \theta & & \theta & \phi\end{matrix}}{\theta}\,(\mathsf{ApTr})$$

where $\{\Omega_0, \ldots, \Omega_m\}$ is the set of all approximation sequences of ϕ.

21. Strong Approximation Transition:

$$\dfrac{\begin{matrix}[\phi^{\star}_{\Upsilon_0}] & & [\phi^{\star}_{\Upsilon_m}] & \\ \vdots & \ldots\ \ldots & \vdots & \\ \theta & & \theta & \phi\end{matrix}}{\theta}\ (\mathsf{SApTr})$$

where $\{\Upsilon_0, \ldots, \Upsilon_m\}$ is the set of all strong approximation sequences of ϕ.

22. NE Introduction:

$$\dfrac{}{(p_i \wedge \neg p_i) \vee \mathsf{NE}}\ (\mathsf{NEI})$$

23. Strong *ex falso*:

$$\dfrac{(p_i \wedge \neg p_i) \wedge \mathsf{NE}}{\phi}\ (\mathsf{ex\ falso}^+)$$

24. Strong Contradiction Introduction:

$$\dfrac{\Big(\bigotimes_{s \in X} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)} \wedge \mathsf{NE})\Big) \wedge \Big(\bigotimes_{s' \in Y} (p_{i_1}^{s'(i_1)} \wedge \cdots \wedge p_{i_n}^{s'(i_n)} \wedge \mathsf{NE})\Big)}{(p_i \wedge \neg p_i) \wedge \mathsf{NE}}\ (\mathbf{0}\mathsf{I})$$

where X and Y are n-teams on $\{i_1, \ldots, i_n\}$ with $X \neq Y$.

25. Strong Contradiction Contraction:

$$\dfrac{\phi \otimes ((p_i \wedge \neg p_i) \wedge \mathsf{NE})}{(p_i \wedge \neg p_i) \wedge \mathsf{NE}}\ (\mathbf{0}\mathsf{Ctr})$$

Main Derivable Rules:

1. Distributive Laws:

$$\dfrac{\phi \otimes (\psi \wedge \chi)}{(\phi \otimes \psi) \wedge (\phi \otimes \chi)}\ (\mathsf{Dstr} \otimes \wedge) \qquad \dfrac{\phi \vee (\psi \otimes \chi)}{(\phi \vee \psi) \otimes (\phi \vee \chi)}\ (\mathsf{Dstr} \vee \otimes)$$

$$\dfrac{(\phi \otimes \psi) \wedge (\phi \otimes \chi)}{\phi \otimes (\psi \wedge \chi)}\ (*)\ (\mathsf{Dstr}^* \otimes \wedge \otimes) \qquad \dfrac{(\phi \wedge \psi) \otimes (\phi \wedge \chi)}{\phi \wedge (\psi \otimes \chi)}\ (*)\ (\mathsf{Dstr}^* \wedge \otimes \wedge)$$

$(*)$ whenever ϕ is a classical formula.

2. *ex falso:*

$$\dfrac{p_i \wedge \neg p_i}{\phi}\ (\mathit{ex\ falso})$$

3. *Tensor Disjunction Combination:*

$$\dfrac{\Big(\bigotimes_{i \in I} \phi_i\Big) \otimes \Big(\bigotimes_{j \in J} \phi_j\Big)}{\bigotimes_{k \in I \cup J} \phi_k}\ (\otimes \mathit{Cmb})$$

whenever ϕ_i, ϕ_j are classical formulas.

4. *Tensor Disjunction Decomposition:*

$$\frac{\bigotimes\limits_{k\in K} \phi_k}{\left(\bigotimes\limits_{i\in I}\phi_i\right)\otimes\left(\bigotimes\limits_{j\in J}\phi_j\right)}\ (\otimes\mathsf{Dcp})$$

where I, J, K are finite sets of indices with $I \cup J = K$.

5. *Approximation Elimination:*

$$\frac{\phi^*_\Omega}{\phi}\ (\mathsf{ApE})$$

where Ω is any approximation sequence of ϕ.

6. *Strong Approximation Elimination:*

$$\frac{\phi^\star_\Upsilon}{\phi}\ (\mathsf{SApE})$$

where Υ is any strong approximation sequence of ϕ.

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