The welfare effects of property tax classification in an urban area: A general equilibrium computational approach

Author: Joseph A. DiMasi

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THE WELFARE EFFECTS OF PROPERTY TAX CLASSIFICATION IN AN URBAN AREA: A GENERAL EQUILIBRIUM COMPUTATIONAL APPROACH

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Joseph A. DiMasi

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Joseph A. DiMasi
Boston College
Department of Economics
Chestnut Hill, MA 02167

ABSTRACT

Taxing different classes of property at different effective rates is a widespread occurrence in the United States, even though the practice violates many state constitutions. For purposes of tax discrimination, urban real property is commonly divided according to the use to which the property is applied. Typically, the major property categories considered are residential and business, or residential, commercial, and industrial.

This thesis investigates the structural and welfare effects of a change from a tax structure in an urban area that classifies property by use for tax purposes to one that does not discriminate in its treatment of property. To accomplish this, long run equilibrium models of urban spatial location are developed. In all models wage rates, and for one model output price of a composite commodity produced in the urban area, can vary in response to the change in tax policy. Conditions guaranteeing the existence of equilibrium for some of the models are developed, and proofs of the existence of equilibrium for those

models are provided.

Due to the analytical intractability of the models, the tax policy changes are simulated numerically through the use of a fixed point algorithm. The models are stylized, to the extent possible, to the Boston metropolitan area. In particular, the classification tax structure and parameterization of the functions of the model are chosen so that a resultant equilibrium resembles the Boston metropolitan area in or around 1980.

General equilibrium versions of compensating and equivalent variations in income are used as measures of welfare change. The qualitative welfare results obtained are quite robust. In all of the simulations conducted there is a welfare gain in moving from the particular classification tax structure used to one in which all property is taxed at the same effective rate.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

In the last two decades there have been significant theoretical advances in the field of urban economics. This work has provided a rigorous framework for analyzing and explaining important land use characteristics found in many urban areas. The analysis, for the most part, has consisted of micro-theoretic models of spatial location and has been structured so as to yield long-run static equilibrium solutions [see, for example, Henderson (1977), Mills (1980), and Muth (1969)]. Although most models have been fairly simple in structure, they have provided explanations for such widespread urban phenomena as population densities, structural densities, and land rent gradients that decline with distance from commercial and employment centers. Even though the basic framework of these models seems well suited to handling comparative static analyses of public policy changes, very little work utilizing a spatial structure to deal with local public finance policy problems has been done.[1] In fact, a public sector is totally absent from a good deal of work in the area.

The strong simplifying assumptions present in typical spatial location models have been made, generally speaking, to obtain analytic solutions. Usually, even slight generalizations and extensions render the models mathematically intractable. As a result, Mills and MacKinnon (1973) have argued in favor of the increased use of numerical computation to find solutions to otherwise intractable models of urban land use. A growing literature on the simulation of spatial models of urban economies has arisen in recent years. Much of this work has been accomplished through the use of a recently developed class of algorithms.[2,3,4] These "fixed point" or "simplicial search" methods of computing solutions to general equilibrium models have already proven to be useful tools in many areas of economic analysis.[5]

This study makes use of one of these computational techniques[6] to simulate an urban spatial location model in order to investigate the differential incidence of alternative property tax systems. In particular, the excess burden of a property tax system under which effective tax rates on real property differ according to the use to which the property is applied is compared to the excess burden of a system under which all effective property tax rates are required to be equal.[7] Differential property tax treatment, or as it is more commonly called, property tax classification, is a fairly widespread phenomenon in the United States. It is sometimes legally sanctioned but, for the most part, it has been practiced in an extra-legal manner.[8] An analysis which can ascertain whether or not a non-discriminatory property tax system is inferior, at least in terms of excess burden, to a currently existing classification scheme in an urban area has important policy implications.

There is a paucity of any such analyses in the literature. In one of only two theoretical studies that I have found which are closely related to this problem, Grieson (1974) presents a model in which property tax discrimination is allowed and suggests that some property tax assessors may classify property so as to minimize the deadweight loss of the tax. Besides a question as to whether assessors, in reality, have acted in such an optimal manner in classifying property, the analysis suffers from several shortcomings. A single town is considered in isolation. Thus, the effects of interaction with agents in other jurisdictions, at least in the particular urban area of which the town is a part, are ignored. In addition, restrictive functional forms are used. Demand and marginal cost are assumed to be linear.

In the other theoretical paper on the subject, Sonstelie (1979) develops a model of the incidence of a classified property tax. Results are first derived, as in the Grieson paper, in the context of a single jurisdiction model. The analysis is later extended, though, to a three jurisdiction model. The functional forms are also much more general than those found in the Grieson work. The Sonstelie framework differs, however, substantially from that found in this study. The spatial structure in the Sonstelie paper is, as acknowledged by the author, barely developed. In particular, the costs and nature of the transportation of people or goods is not modelled. The Sonstelie article treats the property tax as an ad valorem tax on the value of the production of two commodities, which he refers to as commercial and residential real estate. In our study the property tax, when levied on commercial or "business" property is an ad valorem tax on the capital and land inputs used in production. In the Sonstelie article, the tax policy changes, on which the incidence results are based, involve marginal increases in one or both of the tax rates on the two property types in a jurisdiction, with no regard for the amount of revenues raised in the jurisdiction before and after the change. Two policy changes are considered: an increase in the commercial real estate tax rate with no change in the residential tax rate, and an equal increase in both tax rates. In contrast, in our study an equilibrium in which the tax rates differ, but are in a given proportion to one another, is used as a base, and is compared to an equilibrium in which the tax rates are equal, but yield the same amount in tax revenues. In addition, the tax policy schemes analyzed in this study entail metropolitan wide, rather than jurisdiction specific, changes. Another important difference between Sonstelie's paper and this study involves the use of labor. Labor is not used as an input in production in the Sontelie article, whereas all non-housing production in this study can employ labor. Thus, the Sonstelie approach cannot capture the effects on wage income of changes in tax policy. Finally, the approach to measuring tax incidence in the Sontelie article differs fundamentally from the approach taken here. In the Sontelie paper incidence is described in terms of the effects that the tax policy changes have on the gross rents for commercial and residential real estate and on land rents. In this study the household utility maximization problem is explicitly considered and used to define incidence in terms of willingness-to-pay measures of welfare change.

From an empirical viewpoint, Cooper and Weinberg (1975) and

Wheaton (1975) have investigated the effect of a movement from a classification system, as has existed in Boston and Massachusetts generally, to a non-discriminatory system. In these studies, given full market values of existing residential, commercial, and industrial properties, effective tax rates on these properties, and the amounts of property tax revenues to be raised in each jurisdiction, the effect of a movement to an equal effective tax rate system is measured by changes in the tax liabilities of the residential and business sectors when either property values remain unchanged or change in accordance with some tax capitalization process. While this sort of analysis may have some relevance in the short-run, it clearly cannot suffice in the intermediate or long-run, for implicit in these studies is the assumption that no new structures are built, no old structures are demolished, and existing structures are maintained at the same level of quality and used for the same purposes.

The models developed in this study allow for the interaction of firms and residents in an entire urban area and for a reshaping of property values, structures, and uses in the region. These general models are stylized, somewhat, to fit the Boston metropolitan region. The treatment of the property tax given here is similar to its treatment in what may be called the classical view of property tax incidence. The key analogue, here, to the classical treatment is the assumption that the supply of structures, or capital, to the urban area is perfectly elastic. This assumption can be justified on the grounds that capital is sold in a national market and the demand for capital from the city under consideration is a small part of aggregate national demand. The controversy between the new view of the property tax [as represented in Aaron (1974, 1975) and Mieszkowski (1972)] and the classical view centers on the global incidence of the tax. Under the new view, however, the analysis of local changes is essentially the same as that which is found in the classical approach. Since we are considering, in this study, changes in the property tax system in one urban area, the analysis presented below cannot be said to show agreement with one approach as opposed to the other.

Evidence of the empirical significance of property tax rate discrimination is given in Table 1.1 taken from Aaron (1975, p. 60). Given the practice of imposing the same nominal tax rate on all property types in a city, the figures indicate clearly that housing and commercial-industrial property are treated differently. Commercial-industrial property is taxed at an effective rate that is higher than those for both types of housing in ten of the cities and that is higher than that for single-family housing in sixteen of the cities. This kind of disparity in effective rates is particularly notable in New York, San Diego, and Boston, whereas the practice of taxing commercial-industrial property at significantly lower rates is found in Memphis and St. Louis. Further evidence that the city of Boston, the metropolitan area of which is the object of consideration in this study, differentiates among property uses in levying taxes is found in Holland and Oldman (1974). As pointed out in Aaron (1975) it is estimated in that study that the average effective tax rate on commercial property in Boston was 8.2 percent, while the rate for residential property averaged 5.5 percent. Finally, Table 1.2 shows the differences in

TABLE 1.1. RELATIVE ASSESSMENT RATES ON SELECTED CLASSES OF PROPERTY IN THE TWENTY-FIVE LARGEST U.S. CITIES, 1971

PERCENT OF AVERAGE ASSESSMENT RATES FOR ALL PROPERTIES; CITY-WIDE AVERAGE = 100

City	Housing		
	Single-Family	Multifamily	Commercial/Industrial
New York	71	108	151
Chicago	95	111	145
Los Angeles	104	93	109
Philadelphia	94	114	110
Detroit	99	109	108
Houston	102	119	98
Baltimore	97	140	103
Dallas	89	136	100
Washington D.C.	98	112	92
Cleveland	97	106	128
Indianapolis	100	139	90
Milwaukee	99	104	121
San Francisco	106	91	108
San Diego	82	75	181
San Antonio	101	88	102
Boston	75	101	167
Honolulu	101	92	95
Memphis	100	109	67
St. Louis	105	135	76
New Orleans	101	101	110
Phoenix	93	117	104
Columbus	104	82	88
Seattle	94	140	NA
Pittsburgh	99	105	86
Denver	97	108	103

Source: Calculated from U.S. Bureau of the Census, 1972, Census of Governments, unpublished tabulations

NA: Not Available

effective tax rates in 1980 for a sample of ten cities and towns in the Boston metropolitan area. The jurisdictions vary in population and geographic location within the metropolitan area. Within a population group and geographic area, with the exception of Boston, the cities and towns were chosen randomly. Although the general level of tax rates appears to decline with population (and distance from the center of the region), the tendency of jurisdictions in this region, regardless of size and location, to tax business enterprises at rates in excess of those applied to residences is evident. The differences appear to be too large and too widespread to have occurred by chance.

The next part of this chapter reviews briefly the extant literature which uses a computational approach, specifically through implementation of a fixed point algorithm, to analyze issues in urban economics, usually in the context of traditional urban spatial location models. The nature of the models found in this work and the kinds of conclusions they engender is discussed. A brief survey of work in other fields of economics that use fixed point algorithms to compute equilibria and address policy questions is also provided. The final section of this chapter provides an overview of what the following chapters contain.

I. A Survey of the Literature on Economic Applications of Simplicial Search Algorithms

The first published application of a simplicial search algorithm to urban economics is a paper by MacKinnon (1974). It describes intuitively, but thoroughly, one type of simplicial search

TABLE 1.2. EFFECTIVE PROPERTY TAX RATES IN SELECTED CITIES AND TOWNS IN THE BOSTON SMSA, 1980

City/Town	Population Group	Effective Tax Rate (% of Full Market Value)		
		Single Family	Commercial	Industrial
Boston	> 50,000	4.60	10.52	13.59
Brookline	> 50,000	4.05	5.90	5.98
Cambridge	> 50,000	3.82	5.10	6.28
Newton	> 50,000	3.01	5.18	5.65
Belmont	25,000-50,000	2.82	3.80	4.31
Natick	25,000-50,000	3.24	4.52	4.26
Marblehead	10,000-25,000	2.31	2.67	5.25
Sictuate	10,000-25,000	3.50	4.20	4.20
Millis	5,000-10,000	2.95	3.53	3.96
Wrenham	< 5,000	2.28	2.63	2.70

Source: The Assessing Department of the City of Boston algorithm known as the Sandwich Method. The algorithm was used to solve several numerical models of urban land use. The purpose of doing so in the manner found in the paper was, as the author points out, to gain some computational experience, at modest cost, in investigating the difficulties in modelling discrete general equilibrium urban systems and to uncover fruitful directions for future research. The models followed along traditional lines in this area. The city is assumed to be circular, the central business district is taken to be a point at the center of the city, and residents commute to the central business district to work. In terms of functional forms, households maximize a Cobb Douglas utility function. The housing services production function (the only production function in the model) is also Cobb-Douglas. The algorithm searches over a set of prices for land, one for each ring in the urban area. A ring is the area between two concentric circles about the center of the city. In contrast, the models developed in this study exploit some duality tricks which, in the one household type model, allow us to search over just one housing service price instead of a whole set of such prices, one for each ring (or, alternatively, the set of land prices for the rings). This greatly improves on computational efficiency and allows for the handling, at reasonable cost, of richer models with additional endogenous variables. Sensitivity analysis, whereby the effects of changes in parameter values on the structure of the city and the welfare of its residents are determined, was carried out in the MacKinnon paper. One of the more interesting results obtained in this way was that, in the two household group model, increasing the size or income of one group reduced the welfare of the other, with a particularly strong effect occurring when the income of the richer group is raised.

Arnott and MacKinnon (1977a) then published a paper which, of all the simulation work published in this area, is closest in nature and intent to what is attempted here. The major questions asked in this study, however, are different and the modelling here is more general. The central business district in the Arnott and MacKinnon paper is again taken to be a point at the center of the region to which households commute to work. Cobb-Douglas functional forms are used. Unlike the previous paper, however, a property tax on housing is levied. Its value is fixed and there are no provisions in the model for determining the amount of property tax revenues that must be raised. In our study the tax rate is allowed to vary so that tax revenues raised equal an exogenously specified level. The model in Arnott and MacKinnon is parameterized to some extent to resemble metropolitan Toronto. The simplicial search algorithm that is utilized is one called the Vector Sandwich Method and is described rigorously in MacKinnon (1975). Key results obtained from their paper are that, depending on the model and welfare measure used, imposition of the property tax results in an excess burden that is 8.0% to 8.5% of net property tax revenues (revenues from the tax minus the reduction in land rents induced by the tax).

Another paper by Arnott and MacKinnon (1977b) uses essentially the same general equilibrium spatial location model as before (exclusive of a property tax), except that a choice of transportation mode is included. Four modes of travel are available to commuters:

walking, public transportation, inexpensive car and expensive car. Essentially the same computational procedure that was used in the previous paper is applied here. The effects of several types of government policy changes with respect to the various modes of travel are investigated. In particular, the policies involve changing prices and upgrading transportation facilities. The aggregate benefits and distributional impacts of these kinds of changes are quantified for a parameterization of the model that leads to a reasonably realistic city.

King (1977) presents a computational routine, based on the algorithm developed by Scarf (1973), for a spatial location model that is more general than that of MacKinnon (1974). An application of the algorithm to an urban economy, where utility and production are represented by CES functional forms, is presented. It is done, though, merely for illustrative purposes. The major contribution of the paper is the computational routine that is developed in it.

Arnott and MacKinnon (1978) simulate an urban residential location model that is similar to those found in their aforementioned papers. A major difference, though, is that the time costs of transportation congestion in the urban area are carefully modelled in this paper. They use a variant of the Vector Sandwich Method, mentioned above, to compute equilibria. The major results of this work are obtained when it is shown, by way of numerical counterexample, that, contrary to conventional wisdom, the shadow rent on land in residential use can be less than the market rent, and the shadow rent on land in transportation can be negative. The correct calculation of the shadow rents on land in residential and transportation use is important for cost-benefit analyses of government projects, such as road construction, that involve the acquisition of land. The authors argue in favor of future development of more sophisticated urban simulation models that can accurately calculate the shadow rents on land.

Richter (1978a) exhibits a use for these fixed point algorithms other than the obvious one--the numerical simulation of economic equilibrium models. In this paper the algorithm found in Scarf (1973) and the theorem underlying it are used to provide a constructive proof of the existence of equilibrium in a general function model. In particular, existence of a general equilibrium is proven for a model where local public goods are provided in different regions and consumers are free to move among regions to maximize utility on the basis of private goods prices and the public goods menus found in individual regions. There is a single tax authority that raises the revenues required to pay for the public goods provided in the region by imposing a proportional wealth tax. Production possibilities are described by an activity analysis matrix. Although the model might be applied to a set of suburbs in an urban area where different jurisdiction can provide different kinds and levels of local public goods, it is not in the tradition of urban spatial land use models since provision is not made for travel to an employment center. The set of suburbs application, however, does seem to be an appropriate vehicle to use to investigate the efficiency of local public goods economies of the type envisioned in Tiebout (1956). The Scarf algorithm can be used to calculate an equilibrium for the model in the paper once a set of functional forms and parameter values are specified. Such a numerical example is provided in the paper, but for illustrative purposes only.

Richter (1978b), however, presents a model that is of the traditional urban spatial location variety and shows how the Scarf algorithm can be used to compute equilibria for it. How the algorithm may be used to compute equilibria for urban land use models that include such realistic complications as various types of zoning, racial discrimination, and multiple work places is sketched. A small numerical example is also provided.

In a chapter on computational approaches to the study of neighborhood effects on urban land use, Richter (1979) explains intuitively how simplicial search algorithms can be used to simulate equilibria for urban spatial location models that include neighborhood externalities such as those generated by pollution, the level of which depends on proximity to the polluting activity, and the racial composition of neighborhoods in the presence of discriminatory attitudes. It is shown how both exogenous and endogenous neighborhood effects can be included in urban spatial location models that are amenable to simulation by fixed point algorithms. For illustrative purposes several computational examples are provided.

In a later paper, Richter (1980) provides a synthesis and generalization of much of the work on the use of simplicial search algorithms in computing solutions to urban spatial land use models. The centerpiece of this paper is a theorem (that is crucial to this dissertation) which, when applied to general equilibrium models, provides sufficient conditions for the existence of an equilibrium and indicates how a simplicial search algorithm can be used to compute that equilibrium. This synthesizes and generalizes the computational approaches of King (1977), Richter (1978b), and the papers of Arnott and MacKinnon described above. The procedures of King (1977) and Richter (1978b) utilize the Walras law and the property of homogeneity of degree zero of excess demands. As noted in the paper, though, it may be appropriate and useful in urban models to let some prices be exogenous. The price of capital is an obvious candidate since it may be bought and sold in a national market. In addition, part of household income may be exogenous and some of the income generated in the urban economy may flow out of the system as would be the case if absentee landowners existed. In these circumstances neither the Walras law nor homogeneity of degree zero for excess demands will be satisfied. Neither property, however, is required by the theorem. The Arnott and MacKinnon models do not require these properties but they are restricted in other ways. The theorem, however, can be applied to these models as well. General models which satisfy the hypotheses of the theorem are discussed. These include urban spatial models with endogenous externalities. So called dual or bid rent approaches to computation in urban models, whereby the equal utility for households of a given type and zero profits for competitive markets equilibrium conditions are exploited, are discussed. It is shown how they can be used to reduce the number of endogenous prices over which a fixed point algorithm must search, thereby increasing computational efficiency.

Finally, King (1980) shows how the Scarf algorithm can be used to compute equilibria for certain kinds of models with externalities.

Simple numerical examples are given for urban spatial location models with two different types of externalities. One simulation works with a model where discrimination by whites against blacks exists. The other involves a model that can be applied to a situation where a bakery, say, emits smoke that dirties the laundry wash and irritates the throats of nearby residents. The examples are meant merely to illustrate the procedure. In the general modelling in this paper four assumptions about externalities are made. Externalities are assumed to be non-depletable, producers and consumers act on the basis of anticipated levels of externalities, there is no possibility that recipients of externalities might bribe the producers, and the production of externalities is a linear homogeneous function of relevant inputs, outputs, or consumption.

While the above survey of that part of the urban economics literature that makes use of simplicial search algorithms is intended to be as complete as possible, we make no such claims for the following brief survey of work in other fields that also use these computational techniques. Our intent here is merely to convey something of the wide applicability and usefullness of these fixed point algorithms to economics.

In a series of related papers Shoven and Whalley (1973, 1974, 1977) discuss the issues of the existence and computation of a general equilibrium for an economy with ad valorem consumer and producer taxes. Shoven and Whalley (1973) describe a computational technique, based on Scarf's algorithm, that can be used to find a competitive equilibrium for a general Walrasian model with an arbitrary set of fixed ad valorem tax rates and where the revenue generated from the tax system is distributed among the consumers and/or is retained by government for the purchase of privately produced goods and services. The shares of the tax revenue distributed to the different consumers and government are fixed. It is also required that each recipient's allocation of tax revenues be a continuous function of total tax revenues. Production is described by an activity analysis matrix. In Shoven and Whalley (1974) this basic approach was extended to include a many country model with tariffs applying to trade flows. In these papers the computational routine essentially involves searching over a set of prices and tax revenues to find an equilibrium. In an interesting extension to the earlier papers, Shoven and Whalley (1977) consider the notion of equal yield tax alternatives and the problem of making the tax rates endogenous. In evaluating alternative tax systems, policymakers often compare proposals that will yield a given amount of tax revenue. It should be important, then, to develop computational procedures that can calculate equilibria for models constructed so that alternative tax regimes imposed within them have equal yields. The question of what are equal yields for different tax systems in a general equilibrium framework is an interesting one. In general, changes in the tax structure will alter relative prices. In such cases maintaining the same nominal amount of tax revenues may be inappropriate. In the paper different notions of tax yield equality are presented. It is shown how, using price indices applied to the amounts of revenue raised, it is possible to give every consumer the same real transfer payment with different tax schemes. It is also suggested that one might use a model where government is thought of as having a utility function defined on commodities purchased from the private sector in the case where tax revenues are retained by government and spent on goods and services. Computational routines, based on Scarf's algorithm, are constructed to handle all of these cases. To illustrate the use of the technique a model of the U.K. economy for the period 1968-1970 is simulated. In particular, the replacement of the distortionary system of taxation of income from capital in the U.K. by a single rate nondistortionary tax on capital income from all industries is considered. Although ad valorem tax rates are used in our study, the routines developed in the paper cannot be applied here since the models of the paper are restricted, for our purposes, in important ways. For example, in the paper production is characterized by an activity analysis matrix instead of continuous production functions and the framework is non-spatial.

As the Shoven and Whalley papers extended the range of models to which fixed point algorithms may be applied in one direction, MacKinnon (1979) extended their application in another--to the solution of models in which some or all industries exhibit increasing returns to scale. It is suggested that such equilibrium models may be useful in investigating policy changes where increasing returns to scale should not be assumed away. The paper provides an existence of equilibrium proof for a restricted class of models with increasing returns to scale industries and illustrates, with a numerical example, how fixed point algorithms can be used to calculate such equilibria.

In Mansur and Whalley (1982a) the fixed point algorithm approach to computing equilbria is applied to a general equilibrium multi-jurisdictional model where incomes of jurisdictions are interdependent. They have in mind, essentially, situations where production in one community has spillover effects in neighboring communities, as might occur with the production of some local public goods. The model is limited in this respect, though, in that no public good characteristics of the commodities which exhibit spillover effects are considered. The spillovers are expressed as a transfer of given fractions of production of outputs from one community to another. It is shown how Scarf's original algorithm [Scarf (1973)], or a recently developed refinement by Van der Laan and Talman (1979) that is much more computationally efficient, can be used to compute equilibria for the model. Application of the basic approach to an extension which models "brain drain" migration is suggested as an avenue for future research.

In another area of application, Imam and Whalley (1982) consider equilibrium with price regulation. Three formulations of equilibrium under price intervention policies are presented. The first considers product-specific legislated minimum and ceiling prices which are supported by a government marketing agency. The second involves legislated minimum prices with segmented markets of the type found in the urban-rural migration literature. The third formulation considers economy-wide minimum or ceiling prices supported through government market interventions similar to those that occur with agricultural price support programs. Some numerical examples obtained through the application of a fixed point algorithm are presented. It is suggested that it may be possible, with some modification, to apply the computational framework to analysis of financial market failures in less developed countries, where government monopolies of the types modelled in the paper as well as interest rate ceilings and floors often exist. Other possible areas of application mentioned include energy and transportation price regulation.

Finally, Fullerton, Shoven, and Whalley (1983) use a fixed point algorithm (the one used in this study) to investigate the welfare effects of alternative tax policies at the national level. In particular, they study the effects of a change from the existing United States income tax to a progressive consumption tax using a dynamic general equilibrium model of the United States economy. They calculate that the switch yields a stream of net gains the present discounted value of which is $650 billion in 1973 dollars. The effects of changing from the existing tax system to seven other tax plans, some of which involve the integration of corporate and personal income taxes, are calculated. They find that a combined policy of tax integration and savings deduction from the personal income tax yields the largest welfare gain, with the present discounted value of the stream of net gains in the neighborhood of $1 to $1.5 trillion.

As noted above, this last set of applications of simplicial search algorithms was given to illustrate the potential of these computational procedures. The applications, potential and actual, included topics in such areas as national public finance, international trade, labor economics, and price regulation.

II. Overview of the Study

Chapter 2 of this study will set out in general terms a base urban spatial location land use model that is to be used to investigate the efficiency aspects of property tax classification in an urban area. The model has many of the features of traditional urban land use models. It is a long-run equilibrium land use model where land is rented to the use that bids the most for it. The workers of households commute, from their residences, to work in a central business district, although unlike many, but not all, of these models the employment center here is more than just a point in space. The model differs, however, from all others of this type in that an endogenously determined property tax rate is included. The chapter concludes with a specification of functional forms for the functions of the base model and some derivations, obtained using those functional forms, that are needed to compute equilibria. The forms are not very restrictive. All production functions are of the CES type. Utility is CES in commodities and contains a leisure component. Chapter 3 begins with a discussion of the restrictions on the functions of the model needed to make them amenable to use on a computer and to ensure that they are consistent with economic theory. Conditions on the parameters that guarantee the existence of equilibrium for the model are then developed. An existence of equilibrium proof is provided which, by way of its dependence on a fixed point algorithm, also indicates how such an equilibrium can be computed. Any practical computation of this sort will, in general, result in only an approximation to an equilibrium and the last section of the chapter presents a discussion of how the error of the approximation can be shifted so that its interpretation is made economically meaningful. Chapter 4 develops two major extensions to the base model. First, a non-housing composite commodity which is bought and sold only in the urban area under consideration is added to the model. Second, multiple household types are included. The household groups can differ by preferences and/or income (exogenous, endogenous, or both). For the case where households differ in their endogenous income multiple labor types are added to the model. In the last section of this chapter the welfare measures that will be used in the sensitivity analysis are discussed. They are general equilibrium versions of compensating and equivalent variations in income. Chapter 5 begins with a specification of parameter values. Sensitivity analysis for the various models follows. Finally, Chapter 6 summarizes the results of the study and discusses prospects for future research.

CHAPTER 1

FOOTNOTES

[1] Richard Arnott (1979), however, has made a first step toward integrating residential location theory and optimal tax theory.

[2] A good treatment of the mathematical theory underlying these computational techniques and applications to standard Walrasian general equilibrium models, non-linear programming, and game theory can be found in the seminal work of Scarf (1973), and also in a more recent exposition, Scarf (1983).

[3] As mentioned below, Richter (1979, 1980) contain reasonably non-technical discussions of how these algorithms can be used to find solutions to urban spatial location models. For more technical dis-discussions and applications of these techniques to spatial models see Arnott and MacKinnon (1977a, 1977b, 1978), MacKinnon (1974), King (1977), and Richter (1978b).

[4] For examples of simulations of urban spatial location models which do not make use of these algorithms, see Mills (1972), Muth (1975), Steen (1982), and Sullivan (1983a, 1983b, 1983c).

[5] They have already found applications in such diverse fields as national public finance, international trade, and energy economics. The techniques seem to be particularly useful for analyzing public policy.

[6] The algorithm used in this study was developed by Merrill (1972) and is a variant of an algorithm developed by Scarf (1967) for computing fixed points.

[7] In practice, usually, residential, commercial, and industrial property are distinguished from one another for tax purposes. For the base model presented in Chapters 2 and 3, where there is one non-housing production sector, the differentiation will be between residential and "business" property. In the extensions to the base model presented in Chapter 4, however, where a local goods sector is added to the one traded good non-housing production sector, a distinction will be made between "commercial" and "industrial" business property.

[8] Most state constitutions prohibit classification of property for tax purposes. Currently, however, the constitutions of six states do permit property tax discrimination. In particular, the Massachusetts legislature, as a response to a court ruling, Town of Sudbury et al. v. Commissioner of Corporations and Taxation et al. Mass. 321, N.E. 2d 641 (1974), which found that classification violated the Massachusetts state constitution, has recently passed an amendment to the state constitution which sanctions classification. Property tax discrimination, however, had been widely practiced in Massachusetts and elsewhere for many years in spite of state law.

CHAPTER 2

A MODEL OF AN URBAN ECONOMY

Various models of urban spatial location and economic interaction will be employed to determine the welfare and structural implications of alternative means of financing local public services in an urban area, and to ascertain which economic parameters have important effects on the results, either normative or positive, or both. This set of models may be viewed as consisting of what may be called the base model and models that extend the basic framework in various directions. This chapter will set out the structure, in detail, of the base model. The model will first be expounded in terms that are as general as possible. That is, only general functional forms will be assumed. Then the specific functional forms chosen for the purpose of computer simulation and the derivations carried out on those functional forms, needed to perform the simulations, will be presented.

I. The Base Model

A. The General Setting

The analysis is carried on in the now familiar setting of a monocentric circular city.[1] The area of the city is divided into a set of distinct sections. These sections are concentric rings anchored about the geographic center of the urban area. The reason that concentric rings are chosen as the basic geographic unit is that different areas of the city will be differentiated one from another mainly by straight line distance to the city center. Concentric rings, if their widths are chosen small enough, possess the property that the distances from all points in the ring to the center of the region are all approximately the same. The first or innermost ring is taken to be relatively large and is meant to provide the area within which a central business district, or CBD, is located. The CBD is surrounded by much smaller rings which will house most or all of the residents of the city.[2] A fraction of the land in each ring is assumed to be devoted to what is to be considered exogenous uses. This fraction can be made to vary from ring to ring and may reflect, for example, the presence of land that is under water or otherwise unusable for housing and business purposes, land that is devoted to the transportation network, and land that is used by non-profit institutions such as hospitals, schools, and government agencies. Although these particular public, semi-public, and private users of land may indeed alter their location and the intensity with which they use land when relative prices in the urban area change, they all enjoy the benefit, at least in the urban area focused upon, of being exempt from the imposition of property taxes. Since this study is concerned primarily with the effects of changes in the property tax system, and since inclusion of the determinants of locational choice and quantity of land purchased by these institutions would be difficult and cumbersome, at best, we abstract from the effects on the urban area of changes in their decisions when property tax rates change. The portion of land in any ring that is left for what are considered to be endogenous uses is assumed to be homogeneous in quality.

There is assumed to be a given population of N identical households residing in the city. Each household contains one worker who makes a fixed number of work trips each year to the CBD. The analysis can be carried on in the same way with only cosmetic changes if it is assumed that households contain more than one worker, even a nonintegral amount, so long as the number is given exogenously. Just one work trip on any given work day is allowed, although here, also, no problems arise if more than one work trip is made provided that the number is given exogenously. The transportation network is assumed to be radial and dense, so that circumferential travel can be ignored. Commuting distance to the CBD, therefore, is just straight line distance. Every household residing in a given ring is treated, for commuting purposes, as though it lives at the midpoint of the ring. This is a very reasonable assumption if the widths of the rings are small. The work trips are assumed to be the only travel that the households undertake. These assumptions imply that all households residing in a given ring incur the same transportation costs.

B. Households

The analysis, of course, requires a more detailed characterization of the households residing in the urban area. Household labor supply is assumed to be constrained by an institutional work day. The household's worker must work eight hours, or however many are determined by convention, each day that is defined, also by institutional convention, to be a work day. Assuming full employment, then, each worker must supply a fixed amount of labor each year. Time, as will be seen below, will enter the analysis, so it is important to consider non-work hours. It is assumed that the time that a worker does not spend working is divided between leisure and commuting.

The households are utility maximizers. They are assumed, in the base model, to maximize utility over housing services, leisure, and what may be called a traded good. The traded good is a composite good which is sold in a national market of which the local market is a small part. The price of the traded good in the local market is therefore taken to be fixed. This composite commodity can be produced in the city, and can be exported from or imported to the city to eliminate surpluses and shortages in the local market.

A property tax is levied on residential housing. The tax is paid directly by households in the form of an ad valorem tax on housing services. The effective tax rate is uniform throughout the region. This assumption is made in full recognition of the fact that effective property tax rates do vary across and within jurisdictions in an urban area, even for property of the same type. Inclusion of the sorts of inadequate and uneven assessment practices and fiscal competition which may give rise to such variations would enormously complicate the analysis. In addition, it does not seem that these considerations bear directly on the issue of the efficiency of property tax classification for an entire urban area. This seems particularly true in a long-run analysis with one household type. A list of reasons why could include the following three. First, unfairly high or low assessment/sales ratios on individual structures or sets of structures in certain neighborhoods in a jurisdiction are likely to be randomly distributed about the average assessment/sales ratio for property of the same type in the jurisdiction if a lengthy enough time frame is used. The long-run average assessment/sales ratios, then, on individual structures and sites should be the same. Also, assessment techniques used in some jurisdictions which are clearly inferior to those used in other jurisdictions in the same urban area will almost certainly not persist for very long. Second, fiscal competition among local jurisdictions for residents, as characterized in the abundant Tiebout literature,[3] is probably relevant only for those internally homogeneous suburbs found in the outermost rings of metropolitan areas, and only for those regions which contain a large number of these communities. In this literature the form in which the fiscal competition is expressed consists of different jurisdictions offering different tax-expenditure packages to their residents to attract certain segments of the population. It is assumed that fiscal preferences can vary among individuals. With one household type, though, abstracting from differences in cost conditions and natural amenities among jurisdictions, there is no reason for different tax rates to arise in jurisdictions of roughly the same distance from the city center.[4] At different distances from the center, however, the residential tax base per capita can vary, the extent of which depends crucially on the price elasticity of housing demand. Then, tax rates would, in general, vary, even allowing for substitutions of consumption of other goods and services for some public services in low tax base jurisdictions. If the price elasticity of housing demand is inelastic, as some studies have indicated, then, in a one household type model, tax rates should be higher in communities with lower housing prices, restricting our attention of course, to jurisdictions that do not contain a significant amount of business property, or at least to those for which the business tax base per capita is roughly the same. In a monocentric circular city model of the type discussed here, housing prices decline with distance. Thus, if housing demand is inelastic, and per capita local public expenditures are constant throughout the region, tax rates would increase with distance, although perhaps not by much. In reality, though, we typically observe the reverse. That is, tax rates generally decline with distance from the central city, although the variation is probably much less among those distant suburbs which contain little business property than for the urban area as a whole. The difference can probably be accounted for by the observation that higher income groups tend to be more concentrated at locations further from the center. Thus, even if housing demand is inelastic, the effect of lower housing prices at greater distances on per capita residential tax bases can be offset and even somewhat more than offset by the effect of higher income on demand, if housing is a normal good, as it almost certainly is. Third, on the expenditure side, in the one household type model presented here, public service requirements would not vary, for example, for the commonly cited reason that some jurisdictions contain unusually large numbers of poor residents who have large demands for certain public services. In sum, then, with one household type, demands for local public services should not vary much, and per capita tax bases may not either, so that the use of an average tax rate for the entire region may not be a bad approximation to a more realistic model.

The households in this model are free to reside in any part of the region (i.e., to locate in any ring) subject only to their own financial and time constraints and the availability of housing. Instead of considering directly the full household choice problem (which includes the location decision), it will turn out to be very convenient, as will become apparent below, to consider the utility maximizing response of a household when it is required to locate in a given ring.

The choice problem facing a household, then, that is required to reside in a given ring, say ring j, can be expressed mathematically as follows:

$$\max_{x_T^j, x_H^j} \quad U(x_T^j, x_H^j, \ell^j) \tag{2.1}$$

subject to the constraints,

$$\bar{p}_T \cdot x_T^j + (1 + a_R \cdot t) \cdot p_H^j \cdot x_H^j = M + p_w \cdot \overline{W} - c \cdot u^j \tag{2.2}$$

$$\ell^j + \overline{W} + v \cdot u^j = T \tag{2.3}$$

where x_T^j = annual consumption of the traded good

x_H^j = annual consumption of housing services

ℓ^j = annual amount of leisure enjoyed by the household's worker

$\bar{p}_T$ = exogenous price of a unit of the traded good

p_H^j = price of a unit of housing services per year in ring j

a_R = assessment/sales ratio for residential property

t = nominal property tax rate

M = exogenous annual household income

p_w = hourly wage rate

$\overline{W}$ = fixed amount of man hours supplied each year by a worker (the product of the number of hours in the institutional work day and the fixed annual number of work trips made by a worker)

c = the product of twice the money cost of transporting one person one mile and the number of work trips made annually by a worker

u^j = distance in miles from the midpoint of ring j to the city center

v = the product of twice the amount of time spent by a worker each year commuting one mile and the number of work trips made annually by a worker

T = total amount of time available yearly for leisure, work, and commuting.

The houshold maximizes utility (2.1) over just its consumption of the traded good and housing services, subject to the budget constraint (2.2), which states that yearly expenditures on the traded good and housing services plus property tax payments must equal income net of the cost of transporting the worker to and from the CBD, and to the time constraint (2.3), which states that the time used for leisure, working, and commuting must add to a fixed amount. The unit time cost of travel term, v, is assumed to be constant. For a household residing in a given ring, then, the assumption of a fixed supply of labor implies that the amount of time available for leisure is fixed. The amount of time used for commuting and work is completely determined by location which leaves a given amount of time for leisure. Since the maximization is done subject to the household residing in ring j, the amount of leisure is not a choice variable. So, to carry out the maximization process for a given ring, we can substitute the time constraint (2.3), or in other words, the fixed amount of leisure, into the utility function U and then just maximize over the traded good and housing services subject to just the budget constraint (2.2). Finally, the unit money cost of travel term, c, is assumed to be constant. This, together with the assumption that v is constant, implies that we are ignoring transportation congestion.

C. Land Markets

Land is an integral component of this and any spatial location model and, as such, its ownership and use in production processes should be explicitly considered. Due to computational limitations engendered by the endogeneity of city size and the solution technique used (which, nonetheless, allows for a great deal of flexibility in modelling other aspects of the urban economy), it is necessary to treat residents and landowners as distinct groups.[5] The solution procedure will be described in detail below. We can assume either that the local government (since we are abstracting from inter-jurisdictional variations in tax and expenditure policies we may conceptualize local government as consisting of a single metropolitan-wide government or taxing authority) or a group of absentee landlords own all of the land in the city.

The uses to which land may be put are restricted by zoning regulations. The zoning takes the form of what may be called cumulative districting. In this form of zoning the different potential private uses of land are ranked. If an area is said to be zoned for one of these uses, it means that that type of land use and all uses that are ranked higher can locate there, but all land uses that are ranked lower are prohibited from locating there. In this model, there are three types of endogenous private land use. Land available for endogenous use can be used in the production of the traded good, in the production of housing services, and in an agricultural sector. We may refer to production of the traded good as an industrial activity. In the ranking of land uses for zoning purposes, then, industrial use is ranked lowest and residential use is ranked next. It is assumed that agricultural producers can locate anyplace in the region where landowners will rent to them. The first, or innermost ring, which is relatively large, is zoned industrial, which means that all three activities can locate in the CBD. All other rings are zoned residential, which means that only housing and agricultural production can locate there. The purpose of these zoning assumptions, aside from adding a characteristic of land markets actually observed, is to contain business activity within an area of definite size at the center of the city. This is necessitated by computational limitations. In any case, although suburbanization of employment has been an important phenomenon in urban development, business activity is still highly concentrated in many central cities.

Landowners in the region maximize their income by renting only to the highest bidder. If there should be a tie in the bid for land in a ring, it is assumed that any or all of the tied activities can locate there, landowners being indifferent as to which of the tied activities rents their land.

D. The Industrial Sector

A detailed specification of the behavioral assumptions made by the different kinds of firms that locate in the area, as well as the institutional settings, technological constraints, and market structures which they face is required. It is to that task that we now turn.

First, it is assumed that housing services in a ring are produced with a constant returns to scale technology using capital and land from that ring. Thus, the yearly aggregate supply of housing services in ring j is given by the production function:

$$s_H^j = s_H(K_H^j, L_H^j) \tag{2.4}$$

where K_H^j = amount of capital used yearly in the production of housing services in ring j

L_H^j = amount of land used yearly in the production of housing services in ring j

and s_H is homogeneous of degree 1 in capital and land. The producers of housing are assumed to be profit maximizers and the market for housing services is taken to be perfectly competitive. The capital and land markets are also presumed to be competitive, so that housing producers are price takers with respect to input as well as output prices.

The traded good is also produced with a constant returns to scale technology. Its production, however, is restricted to the CBD and it uses capital, labor, and land from the CBD as inputs. The yearly aggregate supply of the traded good from producers in the urban area is given by the production function:

$$s_T = s_T(K_T, L_T^1, W_T) \tag{2.5}$$

where K_T = amount of capital used yearly in local production of the traded good

L_T^1 = amount of land in the CBD used yearly in the production of the traded good

W_T = amount of labor used yearly in local production of the traded good

and s_T is homogeneous of degree 1 in capital, land, and labor. The producers of the traded good are profit maximizers and the market for this good is perfectly competitive. The market for labor is also assumed to be competitive so that traded good producers, like housing producers, are price takers with respect to all input and output prices.

Capital is sold in a national market so that we can take the price, inclusive of opportunity costs, of a unit of annual capital services, $\overline{p}_K$, to be fixed. We also make the assumption, which is fairly standard in these kinds of models, that capital can be transported, where needed, costlessly within the urban area.

As was mentioned above, the property tax levied on housing is applied to the value of housing services and is paid by residents.

Thus, profits in aggregate to producers of housing in ring j, if they are successful in bidding for land in the ring, may be expressed as follows:

$$\pi_H^j = p_H^j s_H^j - \overline{p}_K \cdot K_H^j - p_{LH}^j \cdot L_H^j \qquad (2.6)$$

where π_H^j = aggregate annual economic profits of housing producers from operations in ring j

p_{LH}^j= annual rental on land bid by housing producers for use of land in ring j

These assumptions will determine the behavior of housing producers. Producers of the traded good, however, are treated somewhat differently with respect to the tax system. In particular, the property tax levied on industrial property is treated as a factor value tax.[6] It is imposed on the values of capital and land used by the producers of the traded good. Thus, aggregate profits to these producers, if they are successful in bidding for land in the CBD, may be expressed as follows:

$$\pi_T = \overline{p}_T \cdot s_T - (1+a_I \cdot t) \cdot \overline{p}_K \cdot K_T - (1+a_I \cdot t) \cdot p_{LT}^1 \cdot L_T^1 - p_W \cdot W_T \qquad (2.7)$$

where π_T = aggregate annual economic profits of traded good producers from operations in the CBD

a_I = assessment/sales ratio for industrial property

p_{LT}^1= annual rental on land bid by traded good producers for use of land in the CBD

To close the production side of the model we assume that the rent offered for land anywhere in the region by the agricultural sector, $\overline{p}_A$, is given exogenously, and that the agricultural sector is untaxed. A fixed bid rent for land by the agricultural sector will allow for the determination of city size without taking conditions in agricultural markets explicitly into account. Such an assumption is common in circular city models with a variable border between the urban area and the agricultural hinterland. While this is certainly a simplification, making more realistic assumptions about a farm sector should not appreciably affect the results.

E. The Government Sector

The behavior of local government in this model has been specified to a large extent already. It has been stated that local government can be thought of as consisting of one regional government or taxing authority. Zoning regulations enacted by this government have been expressed, as well as the possibility of having the government own all the land in the region. It is also assumed that, under alternative tax schemes, the level of property tax revenues raised in the urban area is to be equal to a pre-specified amount. This requirement is made for the purpose of conducting differential incidence analysis. The intent is to examine the effects of different tax policies, all of which yield the same amount in revenues. Finally, the expenditure side of local public finance is abstracted from in that it is assumed that the government spends its tax revenues (also the land rents it collects if it owns the land) elsewhere.[7]

F. Solution Procedure

Solving this model involves finding a set of prices and a property tax rate which satisfy certain conditions. First, the requirements of market equilibrium must be met. In particular, for non-zero prices, the excess demands for housing services, land in each ring, and labor must be zero. Second, the amount of property taxes raised in the urban area at the equilibrium prices must be equal to the pre-specified amount mentioned above.

The fixed-point algorithm that is used essentially searches over a set of prices and property tax rates until a combination of prices and tax rate which yields an approximation to a solution, as just described, is found. There are many prices in the model, but the solution technique used allows for a reduction of the dimensionality of the search to a point where the algorithm need search over just three variables. Specifically, the algorithm searches over a set of vectors, $p = (p_W, p_H^2, t)$. The second component is the price of housing services in the second ring. All of the other prices in the model are either exogenous or, given the solution technique used, can be determined from a given vector p.

To see that this is so, let us first return to the utility maximization problem (2.1-2.3). Consider the choice problem facing a household required to reside in ring 2. Assuming the utility function is strictly quasi-concave, increasing in its arguments, and possesses the property of smoothness,[8] then, given a wage rate, p_W, a price of housing, p_H^2, and a tax rate, t, the maximization can be undertaken, in principle, and will result in unique consumption levels of housing services and the traded good. Since the maximization can be performed, we can form the indirect utility function of a household required to reside in ring 2 as,

$$V^2 = V^2(p_H^2, p_W, t) \quad , \tag{2.8}$$

suppressing the other parameters of the utility maximization problem including the price of the traded good, $\bar{p}_T$, and exogenous income, M. In other words, we can find a maximized level of utility, for households residing in ring 2, given any vector p.

It must be true, though, that in an equilibrium the level of utility achieved by all households, regardless of location, is the same.[9] We can exploit this equilibrium condition to find bid prices for housing services offered by households in all other rings.[10] To do this, consider the indirect utility function for a household residing in ring j,

$$V^j = V^j(p_H^j, p_W, t) \quad . \tag{2.9}$$

It must be true, though, in equilibrium that $V^j = V^2$, for all rings j in which households reside. If inversion of the indirect utility function with respect to the price of housing is possible, then knowledge of the value of V^2 will enable us to find a bid price for housing services in any ring j (i.e., a price such that the level of utility achieved by a resident of ring j is the same as that of a resident of ring 2). To ensure this we make the assumption that the direct utility function is such that households will wish to purchase a positive amount of housing services at every set of prices. Given this and the assumption that the direct utility function is increasing in each of its arguments, i.e., the property of local nonsatiation, the indirect utility function is decreasing in the price of housing.[11] Thus, V^j is a one-to-one function with respect to the price of housing in ring j.[12] Hence, we can indeed invert the indirect utility function, V^j, to obtain the bid price for housing services in ring j,

$$p_H^j = \Phi^j(p_w, t, V^2) \qquad (2.10)$$

where, again, other parameters of the maximization problem have been suppressed. We will, below, insure that, by construction, these housing service prices are equilibrium prices.

Before we can show that, though, we must turn to the production side of the economy and the allocation of land to different uses. In a competitive equilibrium with a constant returns to scale technology profits must be zero. We can impose a zero profits condition on producers in the model and exploit it to find bid rents for land by the housing and traded good industries, given only values for the components of the vector p and exogenous prices.

In particular, let us first turn to housing production. Assuming that the housing production function, s_H, possesses continuous second-order partial derivatives and is quasiconcave, the second-order condition for cost minimization will be satisfied (see Varian (1978), p. 12). Thus, in principle, we can find a cost function for housing services, c_H, and write costs in ring j as,

$$c_H^j = c_H(p_{LH}^j, \overline{p}_K, s_H^j) \ . \qquad (2.11)$$

It has been assumed, though, that the production function exhibits constant returns to scale. In such a case, minimized cost, for given input prices, is proportional to output. Specifically, in this situation, we may write minimized cost in ring j as,

$$c_H^j = s_H^j \cdot \hat{c}_H(p_{LH}^j, \overline{p}_K) \qquad (2.12)$$

where $\hat{c}_H(p_{LH}^j, \overline{p}_K) = c_H(p_{LH}^j, \overline{p}_K, 1)$

As noted above, in equilibrium the profits of housing producers must be zero. In other words, the price of housing services in a ring must be equal to the average cost of producing housing in that ring so that, at a solution, we may write,

$$p_H^j = \hat{c}_H(p_{LH}^j, \overline{p}_K) \qquad (2.13)$$

Given the price of housing services in ring j as determined above, then, a bid price for land in ring j by housing producers (i.e., a land rent such that profit-maximizing housing producers would be earning 0 profits) can be found if we can invert the average cost function, $\hat{c}_H$, with respect to p_{LH}^j. The average cost function, though, is increasing in the price of land.[13] Thus, the function, $\hat{c}_H$, is one-to-one with respect to the price of land. Therefore, we can invert it with respect to p_{LH}^j and so find the bid rent for land by housing producers as,

$$c_T = c_T(p_{LT}^1, \overline{p}_K, p_W, t, s_T) \tag{2.15}$$

The traded good production technology is also constant returns to sale, so that we may write costs as proportional to output,

$$c_T = s_T \, \hat{c}_T(p_{LT}^1, \overline{p}_K, p_W, t) \tag{2.16}$$

where

$$\hat{c}_T(p_{LT}^1, \overline{p}_K, p_W, t) = c_T(p_{LT}^1, \overline{p}_K, p_W, t, 1)$$

Profits must also be zero for traded good producers in long-run equilibrium so that we must have,

$$\overline{p}_T = \hat{c}_T(p_{LT}^1, \overline{p}_K, p_W, t) \tag{2.17}$$

The average cost function, $\hat{c}_T$, is increasing in the price of land in the CBD bid by traded good producers (see footnote 13), which indicates that it is one-to-one with respect to p_{LT}^1. Thus, we can invert the average cost function to find a bid rent for CBD land from traded good producers:

$$p_{LT}^1 = g(\overline{p}_K, p_W, t, \overline{p}_T) \tag{2.18}$$

Thus, given a vector p, land rents that are consistent with zero profits for profit-maximizing traded good and housing producers can be determined.

Do the housing bid rents actually prevail in the rings outside the CBD? What of rents in the CBD where both housing and traded good producers compete for land? To allocate land to alternative uses and so determine actual land rentals, we must refer to the assumption made earlier that land is rented to the highest bidder (or bidders). The actual land rentals, then, which prevail in the various rings, p_L^j, in equilibrium, are given by

$$p_L^1 = \max\{p_{LT}^1, p_{LH}^1, \overline{p}_A\} \tag{2.19}$$

$$p_L^j = \max\{p_{LH}^j, \overline{p}_A\} \quad j \geq 2 \tag{2.20}$$

In the case of a tie in bid rents, the land in the ring may be divided arbitrarily among the tied activities.[14] If we let L^j be the total amount of land in ring j available for endogenous uses, then the allocation of land to uses may be summarized as follows. For the housing industry,

$$\text{If } p_{LH}^j \begin{cases} < p_L^j \\ = p_L^j \text{ (tie for maximum bid)} \\ = p_L^j \text{ (no ties)} \end{cases} \text{ then } L_H^j \begin{cases} = 0 \\ \varepsilon\ [0, L^j] \\ = L^j \end{cases}$$

For the traded good industry,

$$\text{If } p_{LT}^{1} \begin{cases} < p_L^1 \\ = p_L^1 \quad \text{(tie for maximum bid)} \\ = p_L \quad \text{(no ties)} \end{cases} \quad \text{then } L_T^1 \begin{cases} = 0 \\ \varepsilon\ [0, L^j] \\ = L^1 \end{cases}$$

For the agricultural sector,

$$\text{If } \overline{p}_A \begin{cases} < p_L^j \\ = p_L^j \quad \text{(tie for maximum bid)} \\ = p_L^j \quad \text{(no ties)} \end{cases} \quad \text{then } L_A^j \begin{cases} = 0 \\ \varepsilon\ [0, L^j] \\ = L^j \end{cases}$$

where L_A^j is the amount of endogenous land in ring j allocated to agriculture. To ensure that the land allocations are consistent with available supplies we require that,

$$L_H^1 + L_T^1 + L_A^1 = L^1 \tag{2.21}$$

$$L_H^j + L_A^j = L^j \quad j \geq 2 \tag{2.22}$$

If we let these land allocations be the demands for land for these activities, and make the plausible assumption that the agricultural bid land rent, $\overline{p}_A$, is positive, then the land market will be in equilibrium. This can be done since the existence of constant returns to scale in production and the fact that bid rents are chosen so as to yield zero profits, implies that choosing the allocations to be consistent with the supply of land just amounts to specifying an otherwise indefinite scale of operation. Thus, for any vector p, by construction, the demand and supply of land everywhere in the city (i.e., in each ring) will be equated.

Having allocated land to uses we can now find supplies and other factor demands conditional on the vector p and the prices that have been derived from p. Given an allocation of land to the housing industry in a ring and the assumption that production is constant returns to scale, the supply of housing in a ring, obtained from profit maximization, will be determinate. In particular, the supply of housing in ring j is[15]

$$s_H^j = \kappa(p_K, \bar{p}_{LH}^{\,j}, L_H^j) \ . \tag{2.23}$$

Returning once again to the utility maximization problem (2.1-2.3), we see that, given a vector p, a unique optimizing level of demand for housing services from a household residing in ring j, $x_H^j(p_H^j, \bar{p}_T, p_W, t)$, can be found. In order to guarantee housing market equilibrium, a population level for ring j is selected so that the aggregate demand for housing in this ring is equal to the supply. In other words, we choose a population level, N_G^j, so that $N_G^j \cdot x_H^j = s_H^j$. The value that N_G^j takes, of course, will depend on the vector p. We therefore, in a sense, generate a population for each ring in which housing is present. It had better be, of course, that for a solution vector p^* the aggregate generated population is equal to the given urban population, N. The computational procedure that is used finds a vector p for which this condition is approximately satisfied. Given any vector p, though, the generated population for the entire city is given by

$$N_G = \sum_{j=1}^{\gamma} N_G^j \tag{2.24}$$

where γ is the largest index value for rings which contain a positive amount of housing. It must be true that at a solution, as mentioned above, $N_G = N$.

With the vector p, the traded good bid land rental generated from it, the exogenous prices of capital, and the allocation of CBD land, the profit maximizing levels of demand for labor and capital from the traded good industry can be found. The demand for labor, in general, is given by[16]

$$W_T = W_T(p_W, \overline{p}_K, p_{LT}^1, t, L_T^1) \ . \tag{2.25}$$

The demand for capital, in general, is given by

$$K_T = K_T(\overline{p}_K, p_W, p_{LT}^1, t, L_T^1) \ . \tag{2.26}$$

By construction both the housing and land markets will always be in equilibrium. Given an arbitrary vector p, however, the labor market may not be in equilibrium, the generated population may not be equal to the given population N, and the amount of property tax revenues generated in the urban area may not be equal to a pre-specified level, R.

To describe what a solution to our problem means, we can form what may loosely be called an excess demand correspondence. We associate with every non-negative vector p the set[17] E(p), where

$$E(p) = \begin{pmatrix} W_T - N \cdot \overline{W} \\ N - N_G \\ R - R_G \end{pmatrix}. \qquad (2.27)$$

The term R_G is the amount of property tax revenues generated in the urban area, given values for the components of the vector p. In other words,

$$R_G = t \cdot \left(\sum_{j=1}^{\gamma} a_R \cdot p_H^j \cdot x_H^j \cdot N_G^j + a_I \cdot \overline{p}_K \cdot K_T + a_I \cdot p_L^1 \cdot L_T^1 \right) \qquad (2.28)$$

To summarize, an outline of the basic elements of the foregoing procedure to derive a typical element of the excess demand correspondence E(p), for a given vector p, is illustrated in figures (2.1-2.4). Households and housing sector calculations are dealt with in figure 2.1. First, at a given vector p, the indirect utility of a household residing in ring 2, V^2, is found. This value is then used to derive bid housing service prices, p_H^j, for households in other rings. These bid housing prices, in turn, are used to calculate bid land rents for housing producers, p_{LH}^j, and, if housing succeeds in acquiring land in the ring, household demands for housing, x_H^j, for the associated rings. In addition, the bid land rents, assuming that housing is allocated land in the ring, determine housing service supplies, s_H^j, for the corresponding rings. Finally, the housing supplies and the household demands for housing are used to calculate generated population levels, N_G^j, for the rings, which are then summed to obtain the total generated population, N_G. Figure 2.2 symbolizes the progression of derivations for the traded good sector. Given a vector p, a bid land rent for CBD land for traded good producers is found. A comparison of this with the bid land rent for CBD land for housing producers and the exogenous agricultural land rent is used to determine the allocation of CBD land to the traded good industry, and this allocation, in turn, is used to derive the traded good industry's demands for labor and capital. Figure 2.3 illustrates, merely, that the bid land rents for the traded good and housing sectors, together with the agricultural land rent, are used to determine the actual land rents that prevail in the various rings and the allocation of land to the traded good and housing industries. Finally, figure 2.4 lists all of the endogenous prices and quantities that can be used to calculate generated property tax revenues, R_G. In principle, our goal is to find a vector $p^* \geqq 0$,[18] such that $0 \in E(p^*)$.[19] In practice, however, the computational procedure employed finds a vector p with an associated set E(p) such that E(p) contains an element which is at least approximately equal to a vector of zeroes.

II. Specific Functional Forms

In order to actually carry out the numerical calculations mentioned above we must first specify, precisely, the forms of the

FIGURE 2.1

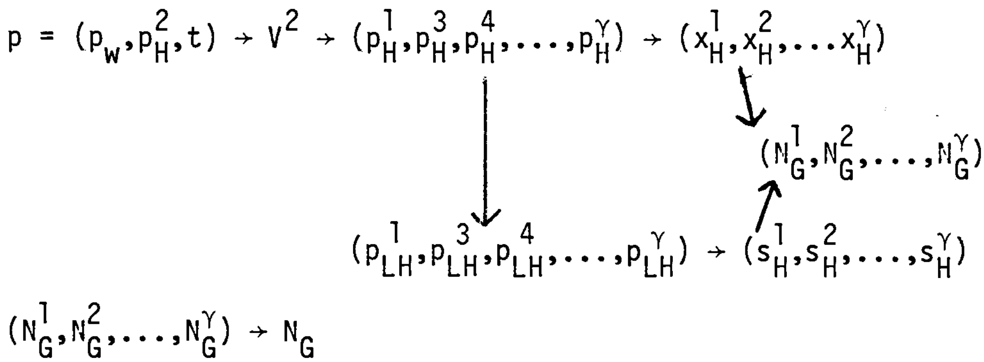

FIGURE 2.2

$$p = (p_W, p_H^2, t) \rightarrow p_{LT}^1 \rightarrow (L_T^1, W_T, K_T)$$

FIGURE 2.3

$$(p_{LT}^1, p_{LH}^1, p_{LH}^2, \ldots, p_{LH}^\gamma) \rightarrow (p_L^1, p_L^2, \ldots, p_L^\gamma, L_T^1, L_H^1, L_H^2, \ldots, L_H^\gamma)$$

FIGURE 2.4

$$(K_T, L_T^1, p_L^1, p_H^1, p_H^2, \ldots, p_H^\gamma, N_G^1, N_G^2, \ldots, N_G^\gamma, x_H^1, x_H^2, \ldots, x_H^\gamma, t) \rightarrow R_G$$

functions that we have been using, which up to now have been quite general. An advantage of the computational approach that we are using is that it allows us to specify fairly complex forms. We need not restrict our modelling, for example, to linear excess demands or Cobb-Douglas utility and production functions. The functional forms that we do choose, however, are constrained by the requirement that all of the manipulations of them proposed in the previous section can be carried out in closed-form or approximated in some way. It is to that specification that we now turn.

First, the utility function of a household residing in ring j is specified to be

$$U = A\cdot(\ell^j)^{\alpha_\ell}\cdot[\alpha_H(x_H^j)^{-\rho} + \alpha_T(x_T^j)^{-\rho}]^{\frac{-1}{\rho}} \tag{2.29}$$

where $A > 0$, $\alpha_\ell, \alpha_H, \alpha_T > 0$, $\rho > -1$ and $\rho \neq 0$. The utility function is in a constant elasticity of substitution, CES, form with regard to housing services and the traded good. Leisure, however, is combined with housing and the traded good in a multiplicatively separable manner in which the component containing leisure is given by a power function.[20]

The other functional forms to be specified are the production functions. The production of housing services in ring j is given by the CES function

$$s_H^j = B\cdot\left[\alpha_{KH}(K_H^j)^{-\rho_H} + \alpha_{LH}(L_H^j)^{-\rho_H}\right]^{\frac{-1}{\rho_H}} \tag{2.30}$$

where $B > 0, \alpha_{KH}, \alpha_{LH} > 0, \rho_H > -1, \rho_H \neq 0.$

Production of the traded good is also carried out by means of a CES production function

$$s_T = C \cdot \left[\alpha_{KT}(K_T)^{-\rho_T} + \alpha_{LT}(L_T^1)^{-\rho_T} + \alpha_{WT}(W_T)^{-\rho_T} \right]^{\frac{-1}{\rho_T}} \tag{2.31}$$

where

$$C > 0,\ \alpha_{KT}, \alpha_{LT}, \alpha_{WT} > 0, \rho_T > -1, \rho_T \neq 0 \ .$$

Given these three functions we can carry out all of the derivations mentioned in the previous section.

From the utility maximization problem (2.1-2.3) we will have to derive, given a vector p, the maximized level of utility attained by a household required to reside in ring 2. To do this we can first look at the dual problem of minimizing expenditures subject to a given level of utility, which determines the compensated demands for housing and the traded good, and so determines the expenditure function. Mathematically we may express this problem as

$$\min_{x_H^2, x_T^2} \ p_H^2(1+a_R \cdot t)x_H^2 + \bar{p}_T \cdot x_T^2 \tag{2.32}$$

$$\text{subject to } U(x_H^2, x_T^2, \ell^2) = U. \tag{2.33}$$

As indicated in the previous section, though, we will have to derive the cost functions associated with the production of both commodities. Since the cost minimization problem on the production side and the expenditure minimization problem on the consumer side are formally indentical, and since the utility function, taking leisure as fixed, is

CES in its other arguments, as the productions functions are, it will be useful to consider the problem (2.32-2.33) in a more general setting, with more general notation. This is done so that we may, with appropriate notational changes use the derived expenditure function in the general case to tell us not only what the expenditure function is in our specific instance, but also what the cost functions on the production side are for this model. Thus, to consider the problem in the most general way, let $x_1, x_2, \ldots, x_n$ be the commodities over which utility is maximized, and $p_1, p_2, \ldots, p_n$ be the associated prices. Let utility be given by the CES function,

$$U(x_1, \ldots, x_n) = A' \cdot \left(\sum_{i=1}^{n} \alpha_i x_i^{-\rho'} \right)^{\frac{-1}{\rho'}}$$

where $A' > 0$, $\alpha_i > 0$, $\rho' > -1$, $\rho' \neq 0$. The expenditure minimization problem can then be written as

$$\min_{x_i} \sum_{i=1}^{n} p_i x_i \tag{2.34}$$

$$\text{subject to } U(x_1, \ldots, x_n) = U \, . \tag{2.35}$$

The first-order conditions for this problem yield, for goods i and j,

$\dfrac{\frac{\partial U}{\partial x_i}}{\frac{\partial U}{\partial x_j}} = \dfrac{p_i}{p_j}$ or, since $\dfrac{\partial U}{\partial x_k} = \alpha_k \cdot x_k^{-(\rho'+1)} (A')^{-\rho'} U^{(\rho'+1)}$, for

$k = 1, 2, \ldots, n$ we may write

$$\frac{\alpha_i x_i^{-\rho'}}{\alpha_j x_j^{-\rho'}} = \frac{p_i \cdot x_i}{p_j \cdot x_j} \quad . \tag{2.36}$$

Since (2.36) holds for all i, we may sum both sides over i to obtain

$$\frac{\sum_{i=1}^{n} \alpha_i x_i^{-\rho'}}{\alpha_j x_j^{-\rho'}} = \frac{\sum_{i=1}^{n} p_i x_i}{p_j \cdot x_j} \tag{2.37}$$

Assuming the x_i's are solution values to the expenditure minimization problem, we may write the expenditure function as

$$c_U(p_1, p_2, \ldots, p_n, U) = \sum_{i=1}^{n} p_i x_i \ .$$

Substituting into (2.37), then, and rearranging to obtain an expression for the compensated demand function x_j, we get

$$x_j = \left(\frac{U}{A'}\right)^{\frac{\rho'}{1+\rho'}} (c_U)^{\frac{1}{1+\rho'}} \cdot \alpha_j^{\frac{1}{1+\rho'}} \cdot (p_j)^{\frac{-1}{1+\rho'}} \tag{2.38}$$

Since this must hold for all j, we can multiply expression (2.38) by p_j and sum over all j to get

$$c_U = \sum_{j=1}^{n} p_j x_j = \left(\frac{U}{A'}\right)^{\frac{\rho'}{1+\rho'}} (c_U)^{\frac{1}{1+\rho'}} \sum_{j=1}^{n} \alpha_j \left(\frac{p_j}{\alpha_j}\right)^{\frac{\rho'}{1+\rho'}} \quad . \tag{2.39}$$

Getting the expenditure function, c_u, on one side of the equation gives us, finally,

$$c_U(p_1,p_2,\ldots,p_n,U) = \left(\frac{U}{A'}\right)\cdot\left(\sum_{j=1}^{n}\alpha_j\left(\frac{p_j}{\alpha_j}\right)^{\frac{\rho'}{1+\rho'}}\right)^{\frac{1+\rho'}{\rho'}} \tag{2.40}$$

This general expenditure function can be useful, in finding expressions for the functions that we need to solve the model as discussed in the previous section, on both the consumption and production sides. To see this, first return to the optimization problem (2.32-2.33). The expenditure function for this problem is given by (2.40) if we let

$n = 2$, $x_1 = x_H^2$, $x_2 = x_T^2$, $p_1 = p_H^2\cdot(1+a_R\cdot t)$, $p_2 = \overline{p}_T$, $\rho' = \rho$, $\alpha_1 = \alpha_H$, $\alpha_2 = \alpha_T$, and $A' = A(\ell^2)^{\alpha_\ell}$. The expenditure function, then, for a household residing in ring 2 is

$$c_U(p_H^2(1+a_R\cdot t),\overline{p}_T,U) = \left(\frac{U}{A\cdot(\ell^2)^{\alpha_\ell}}\right)\left(\alpha_H\cdot\left(\frac{p_H^2(1+a_R\cdot t)}{\alpha_H}\right)^{\frac{\rho}{1+\rho}} + \alpha_T\left(\frac{\overline{p}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}\right)^{\frac{1+\rho}{\rho}} \tag{2.41}$$

If we define income net of transport costs for households in ring j to be Y^j (i.e., $Y^j = M + p_w\cdot\overline{W} - c\cdot u^j$), then we may invert the expenditure function (2.41) to find the maximized level of utility for a household in ring 2 as

$$V^2 = A\cdot Y^2\cdot(\ell^2)^{\alpha_\ell}\cdot\left(\alpha_H\left(\frac{p_H^2(1+a_R\cdot t)}{\alpha_H}\right)^{\frac{\rho}{1+\rho}} + \alpha_T\left(\frac{\overline{p}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}\right)^{\frac{-(1+\rho)}{\rho}} \tag{2.42}$$

The utility of a household residing in ring j, V^j, can, of course, be found by using Y^j, ℓ^j, and p_H^j instead of Y^2, ℓ^2 and p_H^2 in (2.42).

As indicated in the previous section, to solve the model we must invert the expression for V^j with respect to p_H^j, letting $V^j = V^2$ where V^2 is determined by (2.42). Doing this yields the bid price for housing services for households in ring j,

$$p_H^j = \frac{\alpha_H^{\frac{-1}{\rho}}}{(1+a_R\cdot t)} \cdot \left[\left(\frac{A\cdot Y^j\cdot(\ell^j)^{\alpha_\ell}}{V^2}\right)^{\frac{\rho}{1+\rho}} - \alpha_T\left(\frac{\bar{p}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}\right]^{\frac{1+\rho}{\rho}} \tag{2.43}$$

One other set of expressions on the demand side is required. They are the ordinary demand functions for housing services in each of the occupied rings, x_H^j, which will be needed to calculate the generated population levels, N_G^j, for each ring. We can find the x_H^j's by differentiating the indirect utility function, V^j, as shown in the equation, $x_H^j = \frac{\frac{-\partial V^j}{(p_H^j(1+a_R\cdot t))}}{\frac{\partial V^j}{\partial Y^j}}$, which is an application of what has often been referred to as Roy's identity. Performing both differentiations we obtain,

$$\frac{-\partial V^j}{(p_H^j(1+a_R\cdot t))} =$$

$$= A\cdot Y^j\cdot(\ell^j)^{\alpha_\ell}\cdot\left(\alpha_H\left(\frac{p_H^j(1+a_R\cdot t)}{\alpha_H}\right)^{\frac{\rho}{1+\rho}} + \alpha_T\left(\frac{\bar{p}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}\right)^{\frac{-(1+2\rho)}{\rho}}\cdot\left(\frac{p_H^j(1+a_R\cdot t)}{\alpha_H}\right)^{\frac{-1}{1+\rho}}$$

$$\frac{\partial V^j}{\partial Y^j} = A(\ell^j)^{\alpha_\ell}\cdot\left(\alpha_H\left(\frac{p_H^j(1+a_R\cdot t)}{\alpha_H}\right)^{\frac{\rho}{1+\rho}} + \alpha_T\left(\frac{\bar{p}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}\right)^{\frac{-(1+\rho)}{\rho}}$$

Thus, the demand for housing services in ring j is given by,

$$x_H^j = \frac{Y^j}{p_H^j\cdot(1+a_R\cdot t) + \alpha_T\cdot\left(\frac{\overline{p}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}\cdot\left(\frac{p_H^j(1+a_R\cdot t)}{\alpha_H}\right)^{\frac{1}{1+\rho}}} \tag{2.44}$$

On the production side, to solve the model, we need to be able to find the producers' bid prices for land in rings in which they are allowed to locate. Here, the general expenditure function, (2.40), will be of use, for if we interpret the prices there as input prices, U as an output level, and the utility function parameters in the expenditure function as corresponding parameters in the production functions, then (2.40) will give us producer cost functions. Specifically, if we let $n = 2, x_1 = K_H^j, x_2 = L_H^j, p_1 = \overline{p}_K, p_2 = p_{LH}^j, \alpha_1 = \alpha_{KH}, \alpha_2 = \alpha_{LH}, \rho' = \rho_H,$ $U = s_H^j$, and $A' = B$, we obtain the cost function for housing services in ring j,

$$c_H(\overline{p}_K, p_{LH}^j, s_H^j) = \left(\frac{s_H^j}{B}\right)\cdot\left[\alpha_{KH}\left(\frac{\overline{p}_K}{\alpha_{KH}}\right)^{\frac{\rho_H}{1+\rho_H}} + \alpha_{LH}\cdot\left(\frac{p_{LH}^j}{\alpha_{LH}}\right)^{\frac{\rho_H}{1+\rho_H}}\right]^{\frac{1+\rho_H}{\rho_H}} \tag{2.45}$$

Setting the price of housing services in ring j, p_H^j, equal to average cost, or $\frac{c_H}{s_H^j}$, and inverting with respect to p_{LH}^j gives us the bid rents for land in ring j for housing producers,

$$p_{LH}^j = \left[\left(\frac{(B\cdot p_H^j)^{\rho_H}}{\alpha_{LH}}\right)^{\frac{1}{1+\rho_H}} - \left(\frac{\alpha_{KH}(\overline{p}_K)^{\rho_H}}{\alpha_{LH}}\right)^{\frac{1}{1+\rho_H}}\right]^{\frac{1+\rho_H}{\rho_H}} \tag{2.46}$$

Similarly, for the traded good, if, in (2.40) we let $n = 3$, $x_1 = K_T$, $x_2 = L_T$, $x_3 = W_T$, $p_1 = \bar{p}_k\cdot(1+a_I\cdot t)$, $p_2 = p^1_{LT}\cdot(1+a_I\cdot t)$, $p_3 = p_w$, $\alpha_1 = \alpha_{KT}$, $\alpha_2 = \alpha_{LT}$, $\alpha_3 = \alpha_{wT}$, $\rho' = \rho_T$, $U = s_T$, and $A' = C$, then we obtain the cost function for traded good production in the CBD as

$$c_T(\bar{p}_K(1+a_I\cdot t),p^1_{LT}\cdot(1+a_I\cdot t),p_w,s_T) = \left(\frac{s_T}{C}\right)\cdot\left[\alpha_{KT}\left(\frac{\bar{p}_K(1+a_I\cdot t)}{\alpha_{KT}}\right)^{\frac{\rho_T}{1+\rho_T}} + \alpha_{LT}\left(\frac{p^1_{LT}(1+a_I\cdot t)}{\alpha_{LT}}\right)^{\frac{\rho_T}{1+\rho_T}} + \alpha_{wT}\left(\frac{p_w}{\alpha_{wT}}\right)^{\frac{\rho_T}{1+\rho_T}}\right]^{\frac{1+\rho_T}{\rho_T}}. \tag{2.47}$$

Setting the price of the traded good, $\bar{p}_T$, equal to average cost, or $\frac{c_T}{s_T}$, and inverting with respect to p^1_{LT} gives us the bid rent for CBD land for traded good producers,

$$p^1_{LT} = \left[\left(\frac{\left(\frac{c\cdot\bar{p}_T}{(1+a_I\cdot t)}\right)^{\rho_T}}{\alpha_{LT}}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{KT}\cdot(\bar{p}_K)^{\rho_T}}{\alpha_{LT}}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{wT}\left(\frac{p_w}{(1+a_I\cdot t)}\right)^{\rho_T}}{\alpha_{LT}}\right)^{\frac{1}{1+\rho_T}}\right]^{\frac{1+\rho_T}{\rho_T}} \tag{2.48}$$

Expressions for several more production side quantities are needed if the model is to be solved as proposed in the previous section. The supply of housing services in each occupied ring, s^j_H, is required to calculate the generated population levels in the various rings. We also need expressions for the demand for labor from traded good producers to find the excess demand for labor component of E(p), and the demand for capital from traded good producers so as to be able to calculate that part of generated property tax revenue that is collected from the business sector. Given, though, that all production is accomplished with a constant returns to scale technology, these outputs and factor demands are indeterminate unless the scale of production is somehow specified. It is the allocation of land, in each ring, to production according to bid rents, as described above, that will render these quantities determinate. To see how, let us consider the cost minimization condition (2.36).[21] Rearranging terms, we can express the factor x_i in terms of the factor x_j as follows:

$$x_i = \left(\frac{\alpha_i p_j}{\alpha_j p_i}\right)^{\frac{1}{1+\rho}} \cdot x_j \ . \qquad (2.49)$$

Thus, given a level of use for one input we can determine the profit-maximizing levels for all other inputs. In the case of housing we can, in (2.49), let $x_i = K_H^j$, $x_j = L_H^j$, $\alpha_i = \alpha_{KH}$, $\alpha_j = \alpha_{LH}$, $p_i = \overline{p}_K$, $p_j = p_{LH}^j$, and $\rho = \rho_H$ to obtain the demand for capital from housing producers in ring j,

$$K_H^j = \left(\frac{\alpha_{KH} \cdot p_{LH}^j}{\alpha_{LH} \cdot \overline{p}_K}\right)^{\frac{1}{1+\rho_H}} \cdot L_H^j \ . \qquad (2.50)$$

To find the supply of housing services in ring j we can substitute (2.50) into the production function (2.30) and obtain

$$s_H^j = B \cdot \left[\alpha_{KH} \left(\frac{\alpha_{LH} \cdot \bar{p}_K}{\alpha_{KH} \cdot p_{LH}^j} \right)^{\frac{\rho_H}{1+\rho_H}} + \alpha_{LH} \right]^{\frac{-1}{\rho_H}} \cdot L_H^j \, . \qquad (2.51)$$

Finally, using (2.49) we can, as desired, also find the demands for labor and capital by traded good producers. Letting $x_j = L_T^1$ and making the obvious substitutions for the other variables, and taking note of the fact that capital and land, but not labor, are taxed, we find that the demand for labor is given by

$$W_T = \left(\frac{\alpha_{wT} \cdot p_{LT}^1 \cdot (1+a_I \cdot t)}{\alpha_{LT} \cdot p_w} \right)^{\frac{1}{1+\rho_T}} \cdot L_T^1 \qquad (2.52)$$

and the demand for capital from traded good producers is

$$K_T = \left(\frac{\alpha_{KT} \cdot p_{LT}^1}{\alpha_{LT} \cdot \bar{p}_K} \right)^{\frac{1}{1+\rho_T}} \cdot L_T^1 \, . \qquad (2.53)$$

With (2.51-2.53) we now have all of the expressions needed to calculate an excess demand vector associated with a "price" vector, $p = (p_w, p_H^2, t)$. Actually, some problems in computation arise with some of these expressions when certain "price" vectors, p, are used in conjunction with a given parameterization. These problems and adjustments made to deal with them, along with the issue of the existence of equilibrium for the model are discussed in the next chapter.

CHAPTER 2

FOOTNOTES

[1] By city I do not mean any particular political jurisdiction, but rather an economically integrated urbanized region in which the labor demanded by employers located in the center is supplied, to an overwhelming extent, by the residents of the region. I have in mind, essentially, a metropolitan area.

[2] As will become clear below, it is possible to house some of the residents in the CBD alongside the firms which locate there.

[3] The seminal article for this literature is Tiebout (1956).

[4] The assumption of one household type, of course, is an analytical simplification. It will be relaxed below, although computations will still be done with a uniform tax rate posited. Conjectures on the implications of the tax policy changes considered on individual welfare in a model with multiple household types and the sort of fiscal competition present in the Tiebout context will be given below.

[5] The effect of changes in tax policies on land rents though, will be included in the welfare analysis to be discussed below. The purpose is to somehow capture the impact of tax policy on landowner welfare.

[6] This distinction is made primarily for analytical convenience, but it also reflects a difference in the way the tax is actually imposed. As a tax on residential property it is levied on the value of output (housing). As a tax on commercial or industrial property it must be levied on the value of some of the inputs.

[7] The assumption that the government spends its revenues elsewhere, of course, is not very realistic. Concentration on the revenue side of the government's budget, however, should suffice for differential incidence analysis. It is also possible, though, to easily incorporate a transfer of tax revenues in a lump sum fashion to the residents of the city.

[8] By smoothness we mean that continuous first and second order partial derivatives exist.

[9] The definition of equilibrium requires that no agent in the economy has an incentive to alter behavior. With no moving costs, if a household residing in a ring obtains a lower level of utility than households in some other ring, there is an incentive for that household to move, thus violating the definition of equilibrium.

[10] The duality process followed here is a slight generalization of a procedure found in Arnott and MacKinnon (1977a).

[11] The argument establishing this is fairly straightforward. Suppose we are given a price of housing $\hat{p}_H^j$, together with some wage and tax rate, and an associated level of indirect utility, $\hat{V}^j$. If the price of housing is lowered, then the household could now purchase at least as much of every commodity as it did before, and more of some or all of the commodities. Given local nonsatiation, then, the maximized level of utility in the new situation will be larger, as desired. The amount of time available for leisure need not concern us since the maximization problem considered has built into it a fixed amount of leisure. The result actually can also be established with the weaker assumption of local nonsatiation with respect to just one, any one, of the commodities. The level of utility must be higher with the lower price since the household can purchase more of the commodity with which it is nonsatiated and the same amount of all other commodities.

[12] V^j is one-to-one with respect to p_H^j if and only if $\bar{p}_H^j \neq \hat{p}_H^j$ implies $V^j(p_w,\bar{p}_H^j,t) \neq V^j(p_w,\hat{p}_H^j,t)$ for all $p_w,t \geqq 0$.

[13] An argument establishing this may be given as follows. Suppose $\bar{p}_{LH}^j$ is a price of land facing housing producers in ring j, and $\bar{\hat{c}}_H^j$ is the associated minimum cost of producing one unit of housing services. Now, let the price of land in ring j fall. Housing producers, then, could use the same amounts of capital and land to produce each unit of housing services as they did before, yet now the cost of doing so would be lower, providing that a positive amount of land had previously been employed. We may presume this to be true. The technology is constant returns to scale so that any amount of land usage is consistent with profit maximization. As shown below, we will fix the scale of operation by taking the amount of land used to be the amount in the ring available for endogenous use, when housing outbids all other uses for land in the ring. Since the objective here is merely to calculate a bid land rent, the fact that housing producers may be outbid for land by competing uses and thus employ no land is of no consequence. Thus, assuming that a positive amount of land is employed, average costs in the new situation must be lower.

[14] It happens to be the case that, for technical reasons relating to the way the algorithm approximates a solution, it does not matter how we allocate land in the case of a tie. Even though the computer program corresponding to the solution technique discussed in the text allocates land in some particular but arbitrary way in the case of ties among bid rents, when it searches over vectors p, the final output of the algorithm will indicate precisely how land is to be allocated among alternative uses in a ring, if the final solution approximation indicates ties in bid rents for land, so as to be consistent with an approximate equilibrium. This allocation, determined at an approximate equilibrium, may be quite different from the arbitrary rules of allocation adopted in the programming of the model described in the text.

[15] The output price, p_H^j, can be excluded from the argument list since, in the computational procedure used, it, and the bid land rent, p_{LH}^j, are not independent of one another.

[16] As in the previous footnote, the output price, $\bar{p}_T$, can be excluded from the argument lists for the two factor demands, since it, and the bid land rent, p_{LT}^1, are not determined independently of one another.

[17] The fact that land can be arbitrarily allocated to different uses in the case of ties in bid rents implies that E is a correspondence and not a function.

[18] By $p^* \geqq 0$ we mean that all components of the vector p^* are nonnegative.

[19] We are therefore not considering cases where the "supply" exceeds the "demand" at zero prices. For the population and tax revenues components of E(p) the reason is obvious. We could hardly consider a situation where the generated population exceeds the given population or where the tax revenues generated exceed the prespecified required amount to be a solution. In the case of the labor component, an excess supply at a zero price would imply that individuals are supplying labor services and incurring commuting costs for no return at all. An extension which makes the supply of labor endogenous will be developed in the next chapter.

[20] Because of a trade-off between generality in functional form and the number of parameters that must be specified, and a lack of much empirical evidence on the values of such parameters, we choose to stop at this level of generality. One may wonder, though, why the multiplicative power function of leisure form was chosen instead

of simply treating leisure in the same way as are all the commodities (i.e., add a third term to the "CES part" of the utility function). Since we are concerned with maximizing utility at a given location, however, leisure is treated as a constant in the optimization processes. If leisure is thought of as a constant it can be seen that either of these two utility functions is just a monotonic transformation of the other. Thus, they will yield the same demand and bid rent functions. It is a little more cumbersome to derive these functions, though, when utility is of the "full CES" type. For completeness in modelling, however, an option to refrain from work is given to households in the next chapter. The decision to work the institutional work day or not work at all can differ according to which of the two utility specifications is used. They will generally be in agreement, though, as to the labor-leisure choice. Finally, the analysis presented below (the proof of existence of equilibrium) is more easily conducted with the utility function given in the text. It should also be noted that, considering the labor-leisure choice as variable, neither of the two utility specifications is a generalization of the other.

[21] Alternatively, we could consider the problem of maximizing profits over the levels of use of the inputs. This would yield an identical first-order condition.

CHAPTER 3

COMPUTATION AND EXISTENCE OF EQUILIBRIUM IN THE BASE MODEL

Although we have elaborated fairly extensively on the method of computing an excess demand correspondence for a given "price" vector, we have not covered all of the fine points associated with actually calculating, on the computer, excess demands using the specific functional forms posited in the last chapter, and not much mention has been made of the search process, i.e., the algorithm that is used to find an equilibrium. In addition, the question of whether an equilibrium for the model exists and can be found by using the algorithm adopted has not been considered. All of these problems will be dealt with in this chapter. In section I, adjustments needed to ensure that some of the functions derived from the specification in section II of the last chapter can be calculated on a computer, and which restrict them so that they make economic sense for all possible price vectors, will be described. Restrictions on some of the derived functions and their parameterizations, which can be used to show that an equilibrium exists and can be obtained by using the algorithm, will also be given. Section II of this chapter contains a constructive proof of the existence of equilibrium for certain parameterizations of the model. The proof is dependent upon the algorithm that is chosen for use in this study. Thus, it not only establishes the existence of an equilibrium, but it also indicates a means by which an equilibrium can be found. Finally, a discussion of how the burden of the error of approximation can be shifted to make the interpretation of the error economically meaningful is given in section III.

I. Further Functional Restrictions

Even restricting our search to non-negative price-tax rate vectors, problems arise when one attempts to calculate some of the derived functions of the model, given the functional forms chosen for utility and production functions, at some price-tax rate combinations. They occur, in particular, with the producer bid land rent functional forms. To be specific let us review the bid land rent function (2.46) for housing producers. This, and any bid rent function, will make economic sense only if it yields a non-negative rent. Since the price of capital services, $\overline{p}_K$, is constant, though, and since the price of housing services in the second ring can, in the course of the search fall to zero, the expression in brackets,

$$\left[\left(\frac{(B\cdot p_H^j)^{\rho_H}}{\alpha_{LH}}\right)^{\frac{1}{1+\rho_H}} - \left(\frac{\alpha_{KH}(\overline{p}_K)^{\rho_H}}{\alpha_{LH}}\right)^{\frac{1}{1+\rho_H}}\right] \tag{3.1}$$

which is to be raised to a power, can be zero or negative. This problem and its remedy, though, differs with the sign of the substitution term, ρ_H.[1] Specifically, if $\rho_H > 0$ then (3.1) is non-positive when $p_H^j \leqq (\alpha_{KH})^{\frac{1}{\rho_H}} \cdot (\frac{\overline{p}_K}{B})$. What happens if this occurs, with the inequality being strict? If (3.1) is negative, the resulting bid land rent, using the form (2.46) as is, would be a negative, complex, or positive number depending on the value of ρ_H. Clearly, we can rule out, as economically meaningless, negative or complex land rents. What, though, if the expression (2.46) yields a positive value for p_{LH}^j?[2] Returning to the cost function (2.45) and dividing by s_H^j to obtain average cost, we can see that the problem can arise if we allow negative roots to be taken. If, for example, $p_{LH}^j = 25$ and $\rho_H = 1$, then $(p_{LH}^j)^{\frac{\rho_H}{1+\rho_H}} = \pm 5$. Allowing both roots, though, would render the cost function not a function at all, since the same land rent would be associated with two costs. To deal with this problem, then, besides restricting p_{LH}^j to non-negative values, we will allow only positive roots to be taken in expressions (2.45) and (2.46). In fact it will be presumed that wherever roots are encountered, whether in the utility function, the production functions, the bid rent functions or elsewhere, they will be positive. In relation to the bid land rent function for housing producers, these restrictions will ensure that (2.45) is a well-defined, one-to-one function with respect to p_{LH}^j, which can be unambiguously inverted. What is to be done, though, about the bid land rent in a ring when (3.1) is negative (i.e., when $p_H^j < (\alpha_{KH})^{\frac{1}{\rho_H}} \cdot (\frac{\bar{p}_K}{B})$)? Given the restrictions assumed, the value of the (constant) average cost function, $\frac{c_H}{s_H^j}$, cannot fall below a certain level. It's minimum is $(\alpha_{KH})^{\frac{1}{\rho_H}} \cdot (\frac{\bar{p}_K}{B})$, which it achieves at $p_{LH}^j = 0$. Thus, if the price of housing services is low enough so that (3.1) is negative, then profits must be negative. Intuitively, land rents must fall as the price of housing falls in order to keep profits zero. The rents cannot, however, fall below zero. Since, for this case, rents can fall to zero at a positive price of housing, profits must be negative at still lower housing prices. Housing producers would not want to locate in rings where this occurs. As a practical matter, though, in the programming a housing producer bid rent for land in a ring should be assigned when this problem is encountered. It can occur because the algorithm may indeed search in a region of price space where p_H^j is low enough relative to $\overline{p}_K$ to make profits negative in ring 2 and/or other rings considered (the algorithm is structured so that the potential region of search can include prices that are close to zero). The adjustment we shall make in this situation is to take the bid rent for land by housing producers in a ring to be zero whenever the price of housing services in that ring is less than or equal to

$(\alpha_{KH})^{\frac{1}{\rho_H}} \cdot (\frac{\overline{p}_K}{B})$. No harm is done by adopting this procedure since housing will not be allocated any land in a ring if the housing producer bid rent is zero. Housing producers would be outbid for land by agricultural producers.[3]

The situation is somewhat different if $\rho_H < 0$. In this case, even though the bid land rent falls with the bid price of housing, the expression (3.1) is non-positive if p_H^j is too large. In particular, this is true if $p_H^j \geq (\alpha_{KH})^{\frac{1}{\rho_H}} \cdot (\frac{\overline{p}_K}{B})$. Returning again to the cost function (2.46), it can be seen that average costs are bounded from above, although in this case they can approach zero. Substituting $p^j_{LH} = \infty$ into (2.46) and dividing by s^j_H reveals that the average cost of housing production in a ring cannot exceed $(\alpha_{KH})^{\frac{1}{\rho}} \cdot (\frac{\bar{p}_K}{B})$. Thus, if p^j_H is larger than this value, i.e., if (3.1) is negative, then profits are positive. A price of housing, therefore, which is large enough to make (3.1) non-positive is inconsistent with an equilibrium with zero profits and a positive finite land rent. We should not, however, adopt what was done for the case $\rho_H > 0$, which was to set the bid rent for land, in the programming, equal to zero. As (3.1) falls from positive levels to zero, with increases in the price of housing, the bid rent for land approaches infinity. Assigning infinity to the bid rent for land whenever $p^j_H \geq (\alpha_{KH})^{\frac{1}{\rho_H}} \cdot (\frac{\bar{p}_K}{B})$ cannot be done on the computer. We must, therefore, somehow place an upper bound on the bid rent function.[4] In addition, to complicate matters somewhat, there is an upper limit, albeit quite large, on numbers which can be stored by a computer. The upper bound on rents, therefore, cannot in practice be made arbitrarily large. The adjustment made in this situation is to choose an upper bound on housing bid land rents, say $\hat{p}_{LH}$, that is small enough to be handled by the computer, but so large that it can be safely ruled out as a land rent in a reasonable (i.e., realistic) equilibrium. Certainly, for a base case, which is meant to represent a real world urban economy, we can pick a value for $\hat{p}_{LH}$ that is significantly larger than any land rent that can be expected to prevail in reality. Although changing tax policies relative to those that prevail in a base case, as will be done below, will alter equilibrium land rents, it is highly unlikely that the system is so volatile as to lead to massive changes in land rents of the kind that would result in equilibrium land rents of $\hat{p}_{LH}$ or higher when the types of policies that will be considered are implemented. In terms of computation, once a value for $\hat{p}_{LH}$ is chosen we may assign a bid land rent of $\hat{p}_{LH}$ whenever the bid price of housing is high enough. In particular, dividing (2.46) by s_H^j to obtain average cost, a bid rent of p_{LH} is posited for housing producers in ring j whenever,[5]

$$p_H^j \geq \frac{1}{B} \cdot \left[\alpha_{KH}\left(\frac{\bar{p}_K}{\alpha_{KH}}\right)^{\frac{\rho_H}{1+\rho_H}} + \alpha_{LH}\left(\frac{\hat{p}_{LH}}{\alpha_H}\right)^{\frac{\rho_H}{1+\rho_H}}\right]^{\frac{1+\rho_H}{\rho_H}} \tag{3.2}$$

One more problem in this case ($\rho_H < 0$) remains. Inspection of (2.46) shows that p_H^j must be raised to a negative power. This is not something the computer will handle when $p_H^j = 0$ or some positive number it takes to be zero.[6] We can avoid this problem, though, by calculating the bid price of housing which would yield a bid land rent equal to the agricultural rent. Then, if the bid price of housing is less than that which would yield the agricultural rent, we may assign a value of zero to the bid land rent. Specifically, $p_{LH}^j = 0$ when,

$$p_H^j < \frac{1}{B} \cdot \left[\alpha_{KH}\left(\frac{\bar{p}_K}{\alpha_{KH}}\right)^{\frac{\rho_H}{1+\rho_H}} + \alpha_{LH}\left(\frac{\bar{p}_A}{\alpha_{LH}}\right)^{\frac{\rho_H}{1+\rho_A}}\right]^{\frac{1+\rho_H}{\rho_H}} \tag{3.3}$$

Again, this will do no harm since, in such situations, housing producers would not be allocated any land in the ring.

Turning now to bid land rents by traded good producers, it can be seen that the problems that can arise here are somewhat more complicated. Again it is assumed that only non-negative finite bid rents are admissible and that only positive roots are taken. Considering first the easier case of $\rho_T > 0$, an adjustment in our calculations of the bid land rent for CBD land for traded good producers, as given by (2.48), must be made. As was the case for housing, a problem arises when the bracketed expression in (2.48)

$$\left[\left(\frac{\left(\frac{C\cdot\bar{p}_T}{(1+a_I\cdot t)}\right)}{\alpha_{LT}}^{\rho_T}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{KT}(\bar{p}_K)^{\rho_T}}{\alpha_{LT}}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{WT}\left(\frac{p_W}{(1+a_I\cdot t)}\right)^{\rho_T}}{\alpha_{LT}}\right)^{\frac{1}{1+\rho_T}}\right] \quad (3.4)$$

becomes negative for some wage and tax rate combinations. To avoid, though, the possibility that (3.4) cannot be positive for any wage and tax rate, we adopt the following condition on the parameters of the model:

$$\left(\frac{(C\cdot\bar{p}_T)}{\alpha_{LT}}^{\rho_T}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{KT}(\bar{p}_K)}{\alpha_{LT}}^{\rho_T}\right)^{\frac{1}{1+\rho_T}} > 0 \qquad (c.1)$$

If this were not true, then, given that the tax rate must be non-negative, expression (3.4) would always be non-positive. In the case where $\rho_T < 0$, condition (c.1) also insures that the difference of the first two terms in (3.4) will always be positive. In any case, when

$\rho_T > 0$, the same sort of problem occurs when (3.4) is negative as when (3.2) is negative. With a little rearranging we can see that a negative (3.4) implies that,

$$\overline{p}_T < \frac{1}{C} \cdot \left[\alpha_{KT} \left(\frac{\overline{p}_K (1+a_I \cdot t)}{\alpha_{KT}} \right)^{\frac{\rho_T}{1+\rho_T}} - \alpha_{WT} \left(\frac{p_W}{\alpha_{WT}} \right)^{\frac{\rho_T}{1+\rho_T}} \right]^{\frac{1+\rho_T}{\rho_T}} \qquad (3.5)$$

Inspection of (2.47) then reveals that whenever the wage and tax rates are such that (3.4) is negative, the price of the traded good output will be less than the average cost of producing it, and so profits would be negative. To deal with this possibility we make the same sort of adjustment that was made for housing when $\rho_H > 0$. Specifically, whenever the wage and tax rates are such that (3.5) holds, the bid rent for CBD land by traded good producers is taken to be zero. This procedure serves the purpose of assigning a bid land rent for all price-tax rate combinations over which the algorithm searches in a natural and continuous manner. It also insures that the traded good industry is not allocated any land in the CBD when prices and the tax rate are such that profits to firms in the industry would be negative.

The situation, though, is much more complicated when $\rho_T < 0$. Analogous to the case $\rho_H < 0$ in housing production, computational problems arise when expression (3.4) is non-positive and when components of (3.4) involve attempts to raise zero, or a number that the computer takes to be zero, to a negative power. Since $\rho_T < 0$ the bid land rent, given by (2.48), rises to infinity as expression (3.4) falls to zero from positive levels. When (3.4) is negative, using (2.48) as is requires raising a negative number to a power. These problem situations (i.e., when (3.4) is non-positive) occur when the following inequality holds:

$$\bar{p}_T \geq \frac{1}{C} \cdot \left[\alpha_{KT} \left(\frac{\bar{p}_K (1 + a_I \cdot t)}{\alpha_{KT}} \right)^{\frac{\rho_T}{1+\rho_T}} + \alpha_{wT} \left(\frac{p_w}{\alpha_{wT}} \right)^{\frac{\rho_T}{1+\rho_T}} \right]^{\frac{1+\rho_T}{\rho_T}} \tag{3.6}$$

In other words, given that land rents should be real, finite, and non-negative, a non-positive (3.4), taking note of the cost function (2.47), implies that the price of the traded good exceeds the average cost of producing it. Profits, then, would be positive. A non-positive (3.4), therefore, is inconsistent with an equilibrium that possesses zero profits and sensible land rents.

What adjustments in assigning bid land rents should be made in this case? Here we take a somewhat different tack than was taken for housing production. The main purpose of the procedure which follows, though, is the same as that for the housing case. It has the effect of placing an upper bound on the bid rent for land from traded good producers and allows for the assignment of a non-negative bid land rent for all price-tax rate combinations.[7] To begin, we first calculate, given the tax rate component of the vector p used in the search process, the wage rate which would yield a bid rent for CBD land from traded good producers equal to the agricultural rent. Specifically, we obtain this "agricultural" wage rate, p_w^A, from the expression

$$p_w^A = \left[\left(\frac{(C\cdot\bar{p}_T)}{\alpha_{wT}}^{\rho_T}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{KT}\ (\bar{p}_K\cdot(1+a_I\cdot t))}{\alpha_{wT}}^{\rho_T}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{LT}\cdot(\bar{p}_A\cdot(1+a_I\cdot t))}{\alpha_{wT}}^{\rho_T}\right)^{\frac{1}{1+\rho_T}}\right]^{\frac{1+\rho_T}{\rho_T}} \tag{3.7}$$

Next we note that, in the model, the supply of labor is given exogenously. Let us take an amount of labor significantly in excess of this exogenous amount and use it in the demand for labor equation (2.52) along with the wage rate, (3.7), to obtain a rent for land. In particular, substituting, say $b\cdot N\cdot\bar{W}$ (where $b>>1$), for W_T and p_w^A for p_w in (2.52), making the assumption that all endogenous CBD land is allocated to the traded good sector, and solving for the bid land rent yields

$$\hat{p}_{LT}^1 = \left(\frac{b\cdot N\cdot\bar{W}}{L^1}\right)^{1+\rho_T} \cdot \left(\frac{\alpha_{LT}\cdot p_w^A}{\alpha_{wT}\cdot(1+a_I\cdot t)}\right) \tag{3.8}$$

This is the land rent which, given the tax rate and the wage rate that yields a bid land rent equal to the agricultural rent, results in a demand for labor that is b times the supply. In general the wage rate p_w^A and the land rent $\hat{p}_{LT}^1$ are not consistent in the sense that, given the tax rate, a wage rate of p_w^A would not yield a bid land rent of $\hat{p}_{LT}^1$. To account for this, we next invert the bid rent function (2.48) in terms of the wage rate to find that wage rate which would yield a bid rent of $\hat{p}_{LT}^1$. Specifically, we find that wage rate, which we will write as $\hat{p}_w$, according to the expression

$$\hat{p}_w = \left[\left(\frac{(C\cdot\bar{p}_T)^{\rho_T}}{\alpha_{wT}}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{KT}\ (\bar{p}_K\cdot(1+a_I\cdot t))^{\rho_T}}{\alpha_{wT}}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{LT}\cdot(\hat{p}^1_{LT}\cdot(1+a_I\cdot t))^{\rho_T}}{\alpha_{wT}}\right)^{\frac{1}{1+\rho_T}}\right]^{\frac{1+\rho_T}{\rho_T}} \quad (3.9)$$

We can use this value as a cutoff wage to bound the bid land rent and the demand for labor. If the actual wage rate, i.e., the wage rate component of the vector p, p_w, is less than $\hat{p}_w$ then $\hat{p}^1_{LT}$ is automatically assigned as the bid rent for CBD land for traded good producers. Thus, we avoid the problem of trying to raise expression (3.4) to a negative power when that expression is negative, zero, or even positive but very close to zero. This procedure, then, places, for a given tax rate, an upper bound on the bid land rent for traded good producers.[8] The model requires, though, that a demand for labor be assigned in all cases. The adjustment made here (i.e., when $p_w \leq \hat{p}_w$) is to assign, assuming that the traded good industry successfully bids for all of the endogenous CBD land, a demand determined by the demand equation (2.52) with p^1_{LT} substituted for $\hat{p}^1_{LT}$, p_w for $\hat{p}_w$, and L^1 for L^1_T. The procedure, therefore, also places an upper bound on the demand for labor. This demand, though, is not in general $b\cdot N\cdot\bar{W}$. It actually can be a somewhat larger value since $\hat{p}_w$ is likely to be smaller than p^A_w.[9] It is assumed that $\hat{p}^1_{LT}$ is chosen, by picking a large enough value for b, so that it is greater than the maximum bid land rent for the housing industry ($\hat{p}_{LH}$ if $\rho_H < 0$ and, if $\rho_H > 0$, the value for p^1_{LH} when the largest admissible value for p^1_H is substituted in (2.46). As a practical matter, an upper bound for p^1_H can be calculated, for a given size of the price-tax rate simplex, by substituting into (2.43) appropriate values for p^2_H, $\frac{\gamma^1(\ell^1)^{\alpha_\ell}}{\gamma^2(\ell^2)^{\alpha_\ell}}$, and t. What values we should choose for these terms, given the size of the simplex, can be ascertained by adapting part of the discussion below relating to condition (c.4) to this problem). This truncation of labor demand is appropriate since equilibrium requires that labor demand be equal to or less than the given labor supply. The somewhat artificial limit placed on demand, then, does not preclude a true equilibrium price vector from consideration if the limit on demand is chosen to be greater than the fixed supply.

II. The Existence of Equilibrium

In this section the problem of whether an equilibrium for the urban economy exists is explicitly considered. First, the algorithm that is used to compute an equilibrium is discussed briefly and a theorem that will be used to help establish the existence of equilibrium in this model will be stated. Next, a minor theoretical addition to the model, which allows us to depict, in an economically sensible manner, solutions to the model which involve zero wage rates, will be discussed. Then, several conditions on the parameters of the model which, together with condition (c.1), will be used to establish the existence of equilibrium are developed. Finally, a proof of the existence of equilibrium for the model is presented.

The algorithm that is used is a variant of a fixed-point computational procedure initially developed by Scarf (1967). The Scarf algorithm can be used to compute fixed points of certain kinds of functions and correspondences. We can describe the essence of the computational routine briefly. The procedure involves a search along a unit simplex, $S = \{p \geq 0 | \sum_{i=1}^{n} p_i = 1\}$. For economic models, points on the simplex usually represent price vectors. Each of the vectors, on the simplex, that are considered has associated with it what is referred to as a label. In many economic applications, including the ones done here, the labels are excess demands calculated on the basis of the associated price vector. At any one stage of the process, the search is conducted over a finite set of points selected in a regular fashion to serve as a grid for the simplex. Actually the search process considers collections of vectors on the gird, the number of them being equal to the number of components of a vector on the simplex, that are close together in the ordinary Euclidean sense. These collections are referred to as primitive sets. Geometrically, they possess the property that they form subsimplices with the same orientation as the unit simplex. The search continues along the simplex until a primitive set is found for which the associated labels satisfy a certain mathematical property. In terms of our economic application, a set of "price" vectors is found which are close together in the sense mentioned above (i.e., they form a primitive set) and for which a weighted average, with the weights determined by the algorithm, of the associated excess demands is approximately equal to a vector of zeroes. A weighted average of the "price" vectors in the primitive set, with the weights being the same as those for the excess demands, is then calculated. This is to serve as an approximate equilibrium "price" vector. Its associated excess demand vector may then be calculated as well. This is done for a given grid size. The grid size, however, may be increased, i.e., the grid is made denser so that the vectors in primitive sets are even closer together, in order to obtain a better approximation. A new approximate equilibrium price vector can then be found. The computational routine used keeps increasing the grid size until successive final "price" vectors (the ones obtained by taking the weighted average of the "price" vectors in the final primitive set) differ from each other, component by component, by an amount no larger than a level pre-specified by the user. In economic applications it can usually be shown that, in the limit, that is as the grid approaches infinite density, the vectors in the final primitive sets converge to an equilibrium price vector. Thus, in this sense a desired level of accuracy can be obtained by specifying the maximum amount by which the last two final price vectors can differ.

The algorithm that is actually used for this study is one developed by Merrill (1971) and is an extension of the Scarf algorithm. This routine improves upon the computational efficiency of the Scarf algorithm. In the latter algorithm the search is always initiated at a corner of the simplex. In the Merrill extension the addition of an artificial or "dummy" dimension to the simplex allows the search to be started at any point along the simplex. In particular, when moving from one grid size to a higher one the algorithm allows for the search along the denser grid to begin where the search along the previous coarser grid ended.

The algorithms mentioned above both work with the unit simplex. It will however, turn out to be necessary here to be able to use simplices of any size. This discrepancy, however, can be easily handled through an appropriate transformation of price vectors. There exists a one-to-one correspondence between simplices of any size and the unit simplex. Thus, searching over a unit simplex can be equivalent to searching over a simplex where the components of the price vectors sum to something larger than 1. We will indeed use such a mapping to translate prices on the unit simplex, over which the algorithm searches, into prices on the larger simplex, which are then used to calculate the excess demands that are to be the labels of the algorithm. A more troublesome aspect of these algorithms, though, is that the need to restrict the search to a simplex seemingly implies the necessity to normalize prices, i.e., to have the components of any price vector sum to 1 or some positive constant. Unfortunately, we cannot work with such a normalization in a model where some prices are given exogenously. The models considered here do contain exogenous prices. The price of capital is set in a national market. Furthermore, income has an exogenous component (presumably it can be interpreted as asset income derived from a fixed quantity of assets and the exogenous national rate of return on capital). Fortunately, though, a theorem found in Richter (1980), which will be presented below, can be invoked to show that the Scarf and Merrill algorithms can still be used to find equilibria for models with some exogenous prices. The search is still conducted over a simplex but now we require merely that the sum of the components of a vector of the endogenous prices over which we search be no larger than a positive constant. The difference between the constant and the sum of the components of the vector of endogenous prices is accounted for by an artificial variable (i.e., one which has no economic significance). The algorithm, therefore, is made to search over vectors which contain as components the endogenous variables mentioned plus this one artificial variable.

The manner in which the algorithm terminates, as described above, allows us to adequately handle the possibility of having to, in an equilibrium, allocate land in a given ring to more than one use because of ties in bid land rents. It was stated in footnote 14 of Chapter 2 that the final output of the algorithm will indicate how land in a ring is to be allocated to several uses if an equilibrium calls for this even though an arbitrary allocation is made in the programming when ties in bid land rents occur for given "price" vectors considered in the process of search. The part of the final output which does this consists of the weights mentioned above. The excess demands associated with the final primitive set are calculated using the particular land allocation rule adopted in the programming. The weighted average of these excess demands, however, yields a zero vector. The weights, then, which are part of the final output of the algorithm, can be used to allocate land in a given ring at a solution. For example, suppose the final primitive set consists of four "price" vectors.[10] Suppose also that the first two primitive set vectors call for all endogenous CBD land to be allocated to the traded good industry, while the remaining two call for all of that land to be used by the housing sector. Then, if the weights yielded at the termination of the algorithm were $\frac{3}{8}$, $\frac{2}{8}$, $\frac{2}{8}$, $\frac{1}{8}$ respectively, in the approximate equilibrium $\frac{5}{8}$ of the land available in the CBD would be allocated to the traded good industry while the remaining $\frac{3}{8}$ would be allocated to the housing sector. The final price vector, calculated using the weights, of course would not in general yield such an allocation. An excess demand calculated on the basis of it might call for all of the CBD land to be allocated to the traded good industry, or all of it to be allocated to the housing industry. The "price" vectors in the final primitive set, however, should all be close to one another as well as close to the final "price" vector. Thus, the final "price" vector can still be thought of as an approximation to one which yields a solution which allocates land as described above. It may well be the case, though, that all of the final primitive set vectors call for the same allocation of land to uses. In such cases, because of the continuity of the functions of the model, the final "price" vector should yield that same allocation and have an associated excess demand that is close to zero. It may also be the case that the final primitive set vectors yield essentially the same land allocations and the excess demands associated with the final "price" vector are small. In both situations we will use the final "price" vector and the allocations and quantities associated with it to represent an approximate solution.

Up to this point we have presumed that the algorithm will terminate at an approximate solution. It is not at all obvious, however, that a solution exists let alone that the algorithm employed will find an approximation to it if it does. As a first step towards dealing with the issues of the existence of equilibrium and a method for calculating approximations to an equilibrium, we state the following theorem due to Richter (1980).

Theorem 1. Let $E:P \rightarrow R^m$ be a correspondence, where R^m denotes Euclidean m-space, $P \equiv \{p \geq 0 | \sum_{i=1}^{m} p_i \leq d\}$ and $d > 0$. Suppose that:

(H.1) E is upper semicontinuous, bounded and convex;

(H.2) for each $p \in P$ with $\sum_{i=1}^{m} p_i = d$, there exists a non-zero, non-negative vector $\alpha \equiv (\alpha_1,\ldots,\alpha_m)$ such that $\alpha \cdot e \leq 0$ for all $e \in E(p)$, with $\alpha_i > 0$ only if $p_i > 0$.

Then there exists $e^* \leq 0$ and $p^* \in P$ with $e^* \in E(p^*)$ and $p^* \cdot e^* = 0$.

In applying this theorem to an economic model, the vectors p would usually be taken to be price vectors, and E would be taken to be an excess demand correspondence. Thus, given the hypotheses posited in the theorem, the conclusion would be interpreted to mean that an equilibrium exists. It states that there exists a price vector p^* such that its positive price components are associated with zero excess demands, and any zero price components are associated with zero or negative excess demands. The proof of the theorem uses the basic theorem underlying the Scarf algorithm (see Richter (1980)). As such, if the conditions of the theorem hold, we not only have a proof of the existence of equilibrium, but the proof itself indicates how that equilibrium can be computed. In particular, Scarf's algorithm (or Merrill's extension) can be used to find an approximate equilibrium.

For our purposes we may take the components of the vectors p in the theorem to be the wage rate, the price of housing services in ring 2, and a variable, to be explained below, that is related to the nominal property tax rate. If the vector p^* in the theorem turns out to be strictly positive, and the algorithm employed finds an approximation to it, then we will have found an approximation to a zero of the excess demand correspondence given in (2.27). This, of course, would mean that the labor market is in approximate equilibrium, the generated population is approximately equal to the given population, and generated tax revenues are approximately equal to the pre-specified level. Since, by construction, the housing and land markets are in equilibrium at any "price" vector, we would have found an approximate solution to the model.

Before we prove that, under certain restrictions on the parameters, our model satisfies the conditions of Theorem 1 and so possesses an equilibrium that can be approximated by the algorithm, let us extend the theoretical framework in the base model to allow households the option of choosing not to work. Implicit in the utility maximization problem (2.1-2.3) is the presumption that workers will commute to the CBD to work. Commuting money costs are subtracted from income, and the amount of time that must be spent in transit is a component of the time constraint (2.3). While this may be appropriate for most price-tax rate configurations, and certainly for reasonable equilibria, the algorithm can search over prices where it is not conceptually appropriate or, more importantly, it may even be the case that the parameterization calls for an equilibrium where the choice to work is not rational. If households are not literally forced to commute to the

CBD to work, then at very low wage rates it can be optimal for the household's worker to remain at home. This is feasible because of the exogenous income component of total income. The household, then, should have the option of avoiding the money and time costs of commuting and living off its exogenous income. If the wage rate is low enough, for given values of other prices, the parameters, and the tax rate, the household can achieve higher utility by refusing to work. This is obvious in the case of a zero wage rate, a value over which the algorithm, in the limit, can search, because working then returns nothing to the household in the form of additional income and yet it reduces its leisure time and requires it to pay the money costs of travel to and from the CBD on work trips. We can, without too much trouble, though, make adjustments to deal with such situations. To allow households the choice between working and not working we can extend the model to include a double utility maximization process. Here each household, required to reside in a ring, can be thought of as solving two utility maximization problems. The choice problem (2.1-2.3) is retained as one of the two. In addition, households will consider and solve the following optimization problem.

$$\max_{x_T^j, x_H^j} \quad U(x_T^j, x_H^j, T) \tag{3.10}$$

subject to the constraint,

$$\bar{p}_T \cdot x_T^j + (1+a_R \cdot t) \cdot p_H^j \cdot x_H^j = M \tag{3.11}$$

This is the choice problem for a household that decides not to work. The amount of time taken for leisure is now the total time available for work, leisure, and commuting. Thus, T is substituted in the utility function for leisure and the time constraint is eliminated. In addition, money costs of travel are deleted from the budget constraint so that expenditures can now be no larger than exogenous income. The maximized levels of utility for the two problems, say V_W^j and V_{NW}^j, respectively, are compared. The household chooses to work or not to work based on which level of indirect utility is higher. Thus, the level of utility actually achieved by a household residing in ring j is given by $V^j = \max\{V_W^j, V_{NW}^j\}$. Household demands, of course, are dependent on which option is taken. For the solution technique adopted, the relevant demand is the demand for housing services. Expression (2.44) shows us what that will be if, given the specific functional forms used, the household residing in ring j chooses to work. If, on the other hand, the household can achieve a higher level of utility by not working, then a solution of the problem (3.10-3.11) will give as the demand for housing services the following expression.

$$x_{HNW}^j = \frac{M}{p_H^j(1+a_R\cdot t) + \alpha_T\left(\frac{\bar{p}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}\cdot\left(\frac{p_H^j(1+a_R\cdot t)}{\alpha_H}\right)^{\frac{1}{1+\rho}}} \tag{3.12}$$

This demand differs from that in (2.44), which we may now write as x_{HW}^j, only in that total income (endogenous plus exogenous) net of the money costs of travel in the numerator is replaced by exogenous income. The solution procedure, however, involves finding just one level of utility for a given price-tax rate vector. The level of utility for households residing in the second ring is calculated and then used as the standard for the entire urban area, enabling bid prices for housing services to be determined. In this extension the same basic procedure is followed. The level of utility for households residing in ring two is calculated, as indicated above, as the maximum of V_W^2 and V_{Nw}^2. This level of utility is then used to find bid housing prices in other rings. Only now, given that households anywhere have the option of not working if it is in their best interest, there are two indirect utility functions which can be inverted with respect to the price of housing services. They will, in general, yield different bid prices. We choose, as the actual bid price of housing services, the larger of these two. Thus, if p_{Hw}^j, and p_{HNw}^j represent the bid prices for housing if the household decides to work and if it decides not to work, respectively, then the actual bid price in ring j is taken to be $p_H^j = \max\{p_{Hw}^j, p_{HNw}^j\}$. This is consistent with the assumption that households choose the labor option which yields the highest utility. Both indirect utility functions V_W^j and V_{Nw}^j, are decreasing functions of the price of housing services. Thus, given a price of housing services, p_H^j, as determined above, the labor option which yields a lower bid price of housing will also yield a lower level of utility.[11] The option, then, which yields the higher bid price in a ring is the one that is adopted by the households residing in that ring, and demands are then calculated accordingly. If there should be a tie in the bid prices for housing in a ring, or in the level of utility in the second ring, then the residents of the ring would be indifferent between working and not working. In such cases we can arbitrarily allocate the land in the ring available for housing, if the housing sector acquires land in the ring, between its use to provide housing for households that send members to work in the CBD and its use to provide housing for those households whose members remain at home. Given, then, the supply function for housing services in a ring, (2.51), and the household demands for housing by those that have members who work and those that do not, we can determine the number of households of each type to fit into the ring. This, in turn, allows us to find the generated population and housing property tax revenues for the ring. The fact that we allow the land, in a ring where households are indifferent between working and not working, to be allocated in any way implies that the upper semicontinuity of the excess demand correspondence is preserved. As was true with the allocation of land amony different production sectors in the case of ties in their bid land rents, it is also the case that the final output of the algorithm will indicate how land is to be allocated, in this situation, between its uses as housing for workers and housing for non-workers, at a solution for which households in some rings are as well off choosing to have all its members remain at home as they are having members work. The way in which the final output can be used to determine the allocation is the same. Given a rule for breaking ties in the bid housing prices, and in the utilities in ring two adopted in the programming, the number of households in a ring that send someone to work and the number that have all members remain at home are determined by taking a weighted average of the excess demand vectors associated with the final primitive set vectors, with the weights found by the algorithm at termination, as mentioned above.

Although the addition to the model of a labor choice corrects what may be considered a theoretical deficiency in one part of the model, it also creates a new problem in another part. The supply of labor in the model has been given exogenously. With an institutional work day, the presumption of a fixed number of workers implies that labor supply is constant. Since each of the given number of potential workers now have the option of not working, the supply of labor must be variable. What's more, although in an equilibrium the number of potential workers is fixed as a result of the exogeneity of population, the number of potential workers in the process of search can be thought of as varying as the price-tax rate vectors change, since the generated population varies. To account for this, we extend the model to make the supply of labor variable. At a given price-tax rate vector it is determined as that which is supplied by those potential workers among the generated population who find it optimal to work. The fact that labor supply is not fixed, however, necessitates that we posit another condition on the parameters, which will be used to establish the existence of equilibrium. Returning again to expression (2.48), as we did for condition (c.1), and using the first two terms in the bracketed expression, deleting the tax rate, we adopt the following condition on the parameters of the model.

$$\left[\left(\frac{(c \cdot \bar{p}_T)^{\rho_T}}{\alpha_{LT}} \right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{KT} (\bar{p}_K)^{\rho_T}}{\alpha_{LT}} \right)^{\frac{1}{1+\rho_T}} \right]^{\frac{1+\rho_T}{\rho_T}} < \bar{p}_A \text{ if } \rho_T < 0 \qquad \text{(c.2)}$$

The need for this condition will become apparent below in the existence proof. Inspection of equation (2.48) reveals the economic significance of the condition. When $\rho_T < 0$ and the wage rate is large enough relative to the tax rate, the condition implies that traded good producers will be outbid for land in the CBD by the agricultural sector. Thus, in such a situation the traded good industry would not acquire any land in the urban area. The importance of this, as will be seen below, is that in such situations the demand for labor in the urban area would have to be zero.

Now that we have introduced a labor choice in the model we can turn to one more computational loose end and investigate fully what can happen to the housing sector when the price of housing services over which we search, p_H^2, is zero, in order to form another condition which is to be used to establish the existence of equilibrium. With respect to the computation of the functions of the model, a problem can arise with expressions (2.42) and (2.43). In (2.42) p_H^2 is raised to a power. Two cases present themselves: $\rho < 0$ and $\rho > 0$. If $\rho < 0$ and p_H^2 is zero, or small enough that the computer takes it to be zero, then the level of utility, V^2, is infinite. This needn't, though, be a problem. When p_H^2 is zero, whether ρ_H is positive or negative, the bid price for land in the second ring from housing producers is taken to be zero. Housing, therefore, is not produced in the second ring. Furthermore, since the bid land rent function is nonincreasing in distance from the CBD, housing would not be allocated any land in rings further out. What, though, of the bid land rent in the first ring? Inspection of (2.43) reveals that if V^2 is infinite, and $\rho < 0$, the bid price for CBD land from housing producers must also be zero. Therefore, housing cannot locate anywhere in the urban area and the generated population must be zero. Thus, when $p_H^2 = 0$ and $\rho < 0$, we needn't bother with attempting to calculate (2.42) or (2.43). In such cases, when constructing a label for the given price-tax rate vector we may simply define the generated population, the generated housing property tax revenues, and the supply of labor to be zero. On the other hand, when $\rho > 0$ the indirect utility function, V^2, with respect to changes in p_H^2, is bounded from above. It achieves this upper bound when $p_H^2 = 0$. If we substitute this bound $A \cdot Y^2 \cdot (\ell^2)^{\alpha_\ell} \left(\alpha_T \left(\frac{\bar{p}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}\right)^{\frac{-(1+\rho)}{\rho}}$, into expression (2.43) for j = 1 we obtain

$$p_H^1 = \left(\frac{\bar{p}_T}{(1+a_R \cdot t)}\right) \cdot \left(\frac{\alpha_H}{\alpha_T}\right)^{\frac{1}{\rho}} \cdot \left[\left(\frac{Y^1(\ell^1)^{\alpha_\ell}}{Y^2(\ell^2)^{\alpha_\ell}}\right)^{\frac{\rho}{1+\rho}} - 1\right]^{\frac{1+\rho}{\rho}} \tag{3.13}$$

It will be useful to form a condition on the parameters which guarantees that the bid price of housing in the CBD, for any wage and tax rate, is low enough so that the housing bid land rent is less than the agricultural rent when $p_H^2 = 0$ and $\rho > 0$. If this condition holds then, noting the discussion above for the case $\rho < 0$, housing will not be allocated any land and the generated population will be zero, for any ρ, whenever $p_H^2 = 0$. The advantage of this will become apparent in the existence proof below. Such a condition would not seem to be very restrictive since, assuming, realistically, that the distance from the midpoint of the second ring to the midpoint of the CBD is not large, the bid price of housing in the CBD should not be very different from p_H^2, which in this case is 0. Given, then, that a price of housing close to zero would yield a low bid land rent, it does not seem unreasonable to suppose that the housing sector would be outbid for land, at least by the agricultural sector. In any case, the condition we adopt which guarantees this is given as

$$\overline{p}_T \left(\frac{\alpha_H}{\alpha_T}\right)^{\frac{1}{\rho}} \left[\left(\frac{(M-c\cdot u^1)}{(M-c\cdot u^2)} \frac{(\ell_w^1)^{\alpha_\ell}}{(\ell_w^2)^{\alpha_\ell}}\right)^{\frac{\rho}{1+\rho}} - 1\right]^{\frac{1+\rho}{\rho}} < \overline{p}_{HA} \text{ for } \rho > 0 \qquad \text{(c.3)}$$

where $\overline{p}_{HA}$ is the price of housing services which would yield a bid land rent equal to $\overline{p}_A$ from housing producers, which can be found by inverting (2.46) with respect to p_H^j and substituting $\overline{p}_A$ for p_{LH}^j, and ℓ_w^1, ℓ_w^2 are the amounts of leisure time available to residents of rings 1 and 2, respectively, if they work. One may wonder how (c.3) satisfies the requirement for the condition mentioned above. To see this, we note the similarities and differences with expression (3.13). The condition is the same as (3.13) except that t = 0 and the total incomes are replaced by exogenous income net of the money cost of commuting. The requirement will be satisfied if (3.13) can be shown to be no larger than the left-hand side of the inequality in (c.3). Since $t \geqq 0$, the component to the left of the brackets in (3.13) is less than or equal to the corresponding component in (c.3). Attention, then, must be focused on the bracketed expressions. Given the addition of a labor choice to the model, we need to consider three cases. We may ask what occurs, given that $p_H^2 = 0$ and $\rho > 0$, when the price-tax rate vector is such that it is optimal for potential residents in both rings 1 and 2 to work, when it is such that it is optimal for potential residents of ring 1 to work and for those of ring 2 to remain at home, and when it is such that it is optimal for potential residents of both rings not to work.[12] Note also that we ignored the case of potential residents of ring 1 choosing not to work, with those of ring 2 choosing instead to work. The reason is that it is not possible for residents in a given ring to find it best to remain at home while those in rings further out find it in their interest to work. This can be seen by noting that (2.43) is an increasing function of the product, $Y^j \cdot (\ell^j)^{\alpha_\ell}$. To find p_{Hw}^j, this product is given as $(M - p_w \cdot \overline{W} - c \cdot u^j) \cdot (T - \overline{W} - v \cdot u^j)^{\alpha_\ell}$. To find p_{HNw}^j, the appropriate product to use is $M \cdot T^{\alpha_\ell}$. Since the labor decision depends on which housing price is larger, it is optimal for residents of ring j to remain at home when

$$M \cdot T^{\alpha_\ell} > (M + p_w \cdot \overline{W} - c \cdot u^j)\ (T - \overline{W} - v \cdot u^j)^{\alpha_\ell}$$

For residents living in rings further out, however, both the total income earned and the amount of time for leisure associated with a decision to work will be lower (higher money and time costs of commuting) than those associated with the work decision for residents of ring j. Thus, the product of exogenous income and total time raised to the power α_ℓ must also exceed the product of total income and leisure time raised to the power α_ℓ if the residents work, in all rings at greater distances from the center than ring j. Hence, if it is optimal for potential residents of ring 1 not to work, it must also be optimal for those in ring 2 not to work. Returning, though, to the cases to be considered, suppose that residents in both rings 1 and 2 decide to work. In that case the ratio of incomes in (3.13) is $\frac{Y^1}{Y^2} = \frac{M+p_w \cdot \overline{W} - cu^1}{M+p_w \cdot \overline{W} - cu^2}$. This is a decreasing function of the wage rate.[13] Consequently, $\frac{Y^1}{Y^2} \leq \frac{M-c \cdot u^1}{M-c \cdot u^2}$. Thus, the bid price of housing in the CBD, p_H^1, is no larger than the left-hand side of the inequality in (c.3) and, as a result, the agricultural sector would outbid the housing sector for land everywhere in the urban area. Suppose, now, that potential residents of ring 1 find it optimal to work, but those of ring 2 do not. It must then be the case that $MT^{\alpha_\ell} > (M + p_w \cdot \overline{W} - c \cdot u^2) \cdot (T - \overline{W} - v \cdot u^2)^{\alpha_\ell}$. The ratio of products of income and leisure raised to the power α_ℓ in (3.13), however, in this situation becomes $\frac{Y^1 \cdot (\ell^1)^{\alpha_\ell}}{Y^2 (\ell^2)^{\alpha_\ell}} = \frac{(M+p_w \cdot \overline{W} - cu^1)}{M} \frac{(T-\overline{W}-v \cdot u^1)^{\alpha_\ell}}{T^{\alpha_\ell}}$. Given the inequality above, we then find that

$$\frac{Y^1 (\ell^1)^{\alpha_\ell}}{Y^2 (\ell^2)^{\alpha_\ell}} < \frac{(M+p_w \cdot \overline{W} - c \cdot u^1)}{(M+p_w \cdot \overline{W} - c \cdot u^2)} \frac{(T-\overline{W}-v \cdot u^1)^{\alpha_\ell}}{(T-\overline{W}-v \cdot u^2)^{\alpha_\ell}} \leq \frac{(M-c \cdot u^1)}{(M-c \cdot u^2)} \frac{(\ell^1_w)^{\alpha_\ell}}{(\ell^2_w)^{\alpha_\ell}} .$$

Thus, once again we have $p_H^1 < \overline{p}_{HA}$. Finally, consider the situation where residents of both rings find it best not to work. Then the ratio of products in (3.13) becomes

$$\frac{Y^1(\ell^1)^{\alpha_\ell}}{Y^2(\ell^2)^{\alpha_\ell}} = \frac{M}{M} \cdot \frac{T^{\alpha_\ell}}{T^{\alpha_\ell}} = 1 \ .$$

Since $u^1 < u^2$, however, $\frac{(M-c \cdot u^1)}{(M-c \cdot u^2)} \cdot \frac{(\ell_w^1)^{\alpha_\ell}}{(\ell_w^2)^{\alpha_\ell}} > 1$, and the desired result follows immediately. In sum, condition (c.3) will insure that housing is not allocated any land when $p_H^2 = 0$ and $\rho > 0$.

Before the last condition on the parameters of the model is expressed, a modification to the search process must be mentioned. Up to now the process has been characterized as a search over values of three endogenous variables--the wage rate, the price of housing services in ring 2, and the nominal property tax rate. This is still essentially what will be done. For the purposes of computation and proving the existence of equilibriums, however, the third component must be changed to a variable that involves, but is not merely, the tax rate. Let L_{Hw}^j and L_{HNw}^j be the amounts of land in ring j allocated to housing households that choose to work and households that choose to remain at home, respectively. The term L_H^2 in Chapter 2 represented the amount of land allocated to housing in ring 2, when all households were assumed to commute to work. We still retain the definition of L_H^2 as the amount of land allocated to housing in ring 2, only now it includes land allocated to housing either or both household types (those that work and those that do not). That is, $L_H^2 = L_{Hw}^2 + L_{HNw}^2$. A similar interpretation and formulation holds for L_H^j. Next, note that the housing supply expression, (2.51), is the product of a term that is constant, given a vector of prices for the model, and the amount of land in the ring allocated to housing. The supplies of housing for the two household types that can locate in the ring can be found as the product of this same constant term (for a given price vector) and the amounts of land allocated to the respective household types. Thus, since $L_H^j = L_{Hw}^j + L_{HNw}^j$, the total supply of housing in the ring can still be given by (2.51). We are interested, however, in the total value of housing services in ring 2. This can be expressed as $p_H^2 \cdot s_H^2$, where s_H^2 is defined by (2.51), using p_H^2 as the price of housing services facing producers. We are also interested in the total value of housing services in ring 2 when the price of housing services there is such that the bid rent for land by housing producers in the ring is just equal to the agricultural land rental. This can be expressed as $\overline{p}_{HA} \cdot \overline{s}_{HA}^2$, where $\overline{s}_{HA}^2$ is the supply of housing services in ring 2 that arises when the price of housing services in ring 2 is $\overline{p}_{HA}$ and all available land in the ring is allocated to housing (i.e., $L_H^2 = L^2$). It is a function only of the parameters and so can be calculated independently of the vector of variables over which we search, which we now write as $p = (p_w, p_H^2, \tau)$. The new third component of this vector is related to the tax rate. In particular, the nominal property tax rate is now defined to be

$$t = \frac{\tau}{a_R \cdot \max[p_H^2 \cdot s_H^2, \overline{p}_{HA} \cdot \overline{s}_{HA}^2]} \tag{3.14}$$

where $a_R > 0$. Looking at this differently, it can be seen that the new variable τ is the level of property tax revenues raised from housing in ring 2 when the price of housing services is high enough so that housing outbids agriculture for land in the ring or, if tied with agriculture, is allocated all of the available land in the ring, since $p_H^2 \cdot s_H^2$ is the total value of all housing services purchased in ring 2.

A redefinition of E(p) is now in order so as to include the modifications in the base model described above, and to accommodate the needs of computation and of the proof of the existence of equilibrium. The first component of E(p), representing the excess demand for labor, must now include a variable supply of labor, call it W^S, which was defined above. With regard to the second component, the generated population must reflect the fact that there are now two household types; those that choose to work and those that choose instead to be unemployed. Thus, letting N_G^{wj} and N_G^{Nwj} be the number of households in ring j that choose to work and remain at home respectively, the generated population can be expressed as

$N_G = \sum_{j=1}^{\Upsilon} (N_G^{wj} + N_G^{Nwj})$. The change needed in the last componet of E(p) is much less obvious. In fact, its purpose can be seen only in the context of the existence proofs. Thus, it will simply be presented at this point with no immediate explanation offered. In place of generated tax revenues, as given by (2.28), there is now the sum of τ, property tax revenues from housing for all rings except the second, and all business property tax revenues. Specifically, generated "property tax revenues" can now be defined as

$$R_G = \tau + t\cdot(a_R\cdot \sum_{\substack{j=1 \\ j\neq 2}}^{\gamma} p_H^j\cdot s_H^j + a_I\cdot(\bar{p}_K\cdot K_T + p_L^1\cdot L_T^1)) \qquad (3.15)$$

Note that R_G gives what may be called the actual level of property tax revenues generated at a given vector p, when p_H^2 is large enough so that, as mentioned above, τ represents the actual level of residential property tax revenues raised in ring 2.

One more concern must be addressed before we proceed to the last condition on the parameters and the existence proof. Some artificial, but nonetheless harmless, bounds must be placed on some of the correspondences of the model. Inspection of (2.43) reveals that the bid prices of housing services for households choosing to remain at home, p_{HNw}^j, are constant across all rings, for a given vector p. Intuitively, the reason that the bid housing price gradient for households that do not work is flat is that housing prices need not adjust to offset differences in the money and time costs of travel and the loss of leisure due to time spent working at different distances from the center, since no such costs and losses are incurred by these households anywhere in the region. Call this constant bid price for housing, p_{HNw}. The bid rents for land used in housing these household types, then, in light of (2.46), are also constant across rings, for a given p. Thus, if p_{HNw} is high enough, housing for households of this type will outbid agriculture for land in any ring and, as a result, the generated population becomes infinite. Since the housing bid price gradient for households that choose to work is downward sloping, it must eventually fall below the constant bid price, p_{HNw}, if it is not already below it at the center.[14] Thus, at least after a certain distance, housing for households that decide not to work outbids all other uses for land when p_{HNw} is high enough. This means that we would have to continue housing these households in rings ad infinitum as we move further and further away from the center. The generated population, in such circumstances, would have to be taken to be infinite.[15] Obviously we cannot allow this on the computer. It would also create problems for the existence proof. To avoid the problem, an upper bound is placed on the generated population of households who do not work, say $\hat{N}$, where $\hat{N}$ is well in excess of the given total population, N. We continue housing these household types in rings further and further out from the center until the number so housed equals $\hat{N}$. Using such a bound is innocuous since, in an equilibrium, the generated population must equal the given population. We are not, therefore, ruling out, with this procedure, any potential equilibria.

One more problem in this regard, though, remains to be resolved. Suppose that p_{HNw} is such that the bid land rent for housing of households who choose not to work is equal to the agricultural land rental, $\bar{p}_A$. In such cases none, some, or all of the available land in all rings, at least after a certain distance, can be allocated to housing these households. Thus, an infinite generated population is a possibility, but only one of many. In particular, we cannot rule out the possibility that an equilibrium involves just such a situation. Of course, if we do have an equilibrium, then the amount of land in a ring allocated to housing households who do not work must be less than the total amount available, except possibly for a finite number of rings. Furthermore, a given number of households of this type can be allocated to rings, at least beyond a certain point, in an infinite number of ways. It matters not in this model to households of this type if 1000 of them are housed in the 100'th ring or one of the 1000 is placed in each ring from the 101'th to the 1100'th. The convention adopted here to deal with these cases is to restrict the number of rings that will be considered. In particular, assuming that the bid price of land for producers of housing for households that do not work, say p_{LNw}, equals $\overline{p}_A$, we find the first ring, say $\hat{j}$, for which housing for these households is not outbid for land in the ring by any use. Next we find the ring, call it $\hat{\hat{j}}$, for which a generated population of households that choose to be unemployed of $\hat{N}$ arises when all of the available land in ring $\hat{j}$ that is not allocated to business or housing households that choose to work is allocated to housing households that choose to be unemployed, all of the available land in rings beyond $\hat{j}$ up to but not including $\hat{\hat{j}}$ is allocated to housing households that choose to be unemployed, at least some of the available land in ring $\hat{\hat{j}}$ is allocated to housing households that choose to be unemployed, and none of the land beyond ring $\hat{\hat{j}}$ is allocated to housing. The convention, then, is that no land is to be allocated to housing in rings beyond $\hat{\hat{j}}$. Restricting the number of rings to be considered in this way places an upper bound of $\hat{N}$ on the number of generated households that do not work. Note, however, that $\hat{\hat{j}}$ will, in general, vary with the vector p. To be more precise, though, about the process whenever $p_{LNw} = \overline{p}_A$, it is assumed that the generated population of households that do not work can take on any value in the interval $[0,\hat{N}]$. That generated population is housed in the following manner. All of the available land in ring $\hat{j}$ that is not used for business or housing households that choose to work, if any, is allocated to housing households that do not work and all available land in rings beyond $\hat{j}$ is allocated to housing households that do not work, until the generated population of these households is reached. Of course, in general, only part of the available land in the last ring considered, which may be $\hat{j}$ itself, will be allocated to housing. Any equilibrium where $p_{LNw} = \overline{p}_A$ can be characterized in this way. Clearly the entire population can be housed at such an equilibrium with the restricted city size since there is enough room to house as many as $\hat{N} > N$ households who choose to be unemployed. Within the context of the model, technically, there really is no need to justify this process of cramming the households who choose not to work, given their demand for housing, as close together as possible since, in a sense, an equilibrium where this occurs is as good as an alternative one in which a different pattern of location exists for those households within rings $\hat{j}$ to $\hat{\hat{j}}$, or where households who remain at home are housed in rings beyond $\hat{\hat{j}}$. The only difference among such equilibria is the location of households that do not work, in ring $\hat{j}$ and beyond, and households of this type are indifferent with respect to those locations.

Finally, we need to place an upper bound on labor demand when $\rho_T > 0$ (as shown above labor demand when $\rho_T < 0$ will be bounded given the restrictions placed on the model in this case). A level of labor demand, say $\hat{W}$, which is well in excess of the maximum labor supply,

$N \cdot \overline{W}$, is chosen. For $p_W > 0$, W_T may be calculated according to (2.52). If, however, the value obtained for W_T equals or exceeds $\hat{W}$, we use $\hat{W}$ as the traded good's demand for labor. If the value obtained is less than $\hat{W}$ then we use that value for labor demand. Since $\hat{W}$ exceeds the maximum labor supply we are not precluding any potential equilibria. Thus, although the bound is somewhat arbitrary, it is nonetheless harmless. A curious problem arises, however, when $\rho_T > 0$ and $p_W = 0$. Use of (2.52) to calculate labor demand would mean that, in these cases, infinity would have to be assigned to labor demand when any positive amount of the CBD land is allocated to the traded good industry, and zero would have to be assigned to labor demand when the traded good industry is not allocated any CBD land. In such situations ($\rho_T > 0$, $p_W = 0$), even if we assigned a value of $\hat{W}$ to labor demand when the amount of land allocated to the traded good industry is positive, a problem would still exist. The jump in labor demand from zero to $\hat{W}$ would destroy the upper semicontinuity of the excess demand for labor, a property we need to establish the existence of equilibrium. To remedy this we define the demand for labor when $\rho_T > 0$ and $p_W = 0$ in the following manner. If the traded good industry is not allocated any land in the CBD then, as always, the labor demand is taken to be zero. On the other hand, if the traded good industry is allocated some CBD land then labor demand can take on any positive value up to and including $\hat{W}$. This can be summarized symbolically as follows:

$$W_T \begin{cases} \varepsilon\ (0,\hat{W}] & \text{if} \quad L_T^1 > 0 \\ = \ 0 & \phantom{\text{if}} \quad L_T^1 = 0 \end{cases} \tag{3.16}$$

where W_T is labor demand when $\rho_T > 0$ and $p_W = 0$. This formulation forces the demand for labor to be upper semicontinuous. Of course, when $L_T^1 > 0$ this assignment of labor demand does not make economic sense. Given the functional forms used in the model, however, no genuine economic equilibrium would involve $L_T^1 > 0$ when $\rho_T > 0$ and $p_W = 0$ since then it could be said that the actual labor demand (infinite) would exceed the supply of labor. Furthermore, it will be seen below that the algorithm, in the limit, will not terminate at a point where $p_W = 0$ and $L_T^1 > 0$. On the other hand, when $L_T^1 = 0$ the assignment of labor demand by (3.16) would be in complete accord with the economics of the situation.

With the alteration in the search process and other modifications that have been made, the last condition on the parameters needed to establish existence can be developed. The objective of the condition is to ensure that it is not possible to house, in the city, the given number of households when the price of housing in ring 2 is so low that housing is outbid for land in ring 2 by agriculture or, if tied in its bid land rent in ring 2 with agriculture, is allocated less than the total amount of land in the ring available for endogenous use (in other words, when $p_H^2 \cdot s_H^2 < \overline{p}_{HA} \cdot \overline{s}_{HA}^2$, and the other conditions for a solution (in particular that R_G equals R) are satisfied). As will be seen below, the purpose of the condition is to guarantee that a solution to the model found by the search process represents a genuine economic equilibrium. Attention must be restricted to the generated population in the first two rings.[16] An upper bound for this population is to be found. The generated population of a household type in a ring is determined as the ratio of the supply of housing to that type and the demand for housing by such a household. Observation of expressions (2.46) and (2.51) reveals that the supply of housing in a ring is an increasing function of the bid price of housing services there. Furthermore, inspection of expressions (2.42) and (2.43) shows that the bid price of housing services in ring 1 is an increasing function of p_H^2. Finally, according to expression (2.44) the demand for housing in a ring is a decreasing function of the bid price for housing in that ring. Therefore, to find an expression which bounds from above the generated population in the situations considered, only the case of $p_H^2 = \bar{p}_{HA}$ is utilized. Given that the price of housing services in ring 2 is set, we seek to choose values of other variables that make p_H^1 as large as possible. Inspection of (2.42) and (2.43) reveals that p_H^1 is an increasing function of $\frac{Y^1 \cdot (\ell^1)^{\alpha_\ell}}{Y^2 \cdot (\ell^2)^{\alpha_\ell}}$. If households in both rings decide to work then this ratio can be expressed as the product of ratios, $\frac{(M+p_w \cdot \bar{W}-c \cdot u^1)(T-\bar{W}-v \cdot u^1)^{\alpha_\ell}}{(M+p_w \cdot \bar{W}-c \cdot u^2)(T-\bar{W}-v \cdot u^2)^{\alpha_\ell}}$. It was stated above (and proved in footnote 13), however, that the first ratio is a decreasing function of the wage rate. The second is constant, assuming that residents of both rings work. Households in any ring, j, will, as noted above, choose not to work when the wage rate is low enough so that

$$(M+p_w \cdot \bar{W}-c \cdot u^j) \cdot (T-\bar{W}-v \cdot u^j)^{\alpha_\ell} < M\, T^{\alpha_\ell} ,$$

and be indifferent between working and not working when there is equality of these two expressions. To establish an upper bound for

$\frac{Y^1(\ell^1)^{\alpha_\ell}}{Y^2(\ell^2)^{\alpha_\ell}}$, then, we find the wage rate, $\tilde{p}_w$, at which households residing

in ring 2 are indifferent between working and not working (i.e., find $\tilde{p}_w$ such that $(M+\tilde{p}_w \cdot \bar{W}-c \cdot u^2) \cdot (T-\bar{W}-v \cdot u^2)^{\alpha_\ell} = MT^{\alpha_\ell}$). Since $\frac{Y^1}{Y^2}$ is a decreasing function of p_w when households in both rings work, $\frac{Y^1(\ell^1)^{\alpha_\ell}}{Y^2(\ell^2)^{\alpha_\ell}}$ decreases as the wage rate increases above $\tilde{p}_w$. For wage rates

below $\tilde{p}_w$ households in ring 2 choose not to work. Thus, for $p_w \leq \tilde{p}_w$,

we have $\frac{Y^1(\ell^1)^{\alpha_\ell}}{Y^2(\ell^2)^{\alpha_\ell}} = \max\left[\frac{(M+p_w \cdot \bar{W}-c \cdot u^1) \cdot (T-\bar{W}-v \cdot u^1)^{\alpha_\ell}}{M \cdot T^{\alpha_\ell}}, 1\right].$ This function

decreases with decreases in p_w to the point where the wage rate is low enough so that households in ring 1 are indifferent between working and not working. At that point and for wage rates even lower,

$\frac{Y^1(\ell^1)^{\alpha_\ell}}{Y^2(\ell^2)^{\alpha_\ell}} = 1$. Therefore, $\frac{Y^1(\ell^1)^{\alpha_\ell}}{Y^2(\ell^2)^{\alpha_\ell}}$ is maximized when $p_w = \tilde{p}_w$. As a result, the value $\frac{Y^1(\ell^1)^{\alpha_\ell}}{Y^2(\ell^2)^{\alpha_\ell}} = \frac{(M+\tilde{p}_w \cdot \bar{W}-c \cdot u^1) \cdot (T-\bar{W}-v \cdot u^1)^{\alpha_\ell}}{MT^{\alpha_\ell}}$ will be used

to find an upper bound for the generated population.

There is one more variable which has an effect on the value of p_H^1--the nominal property tax rate. First, it should be noted that the tax rate must be bounded. It must be bounded from below by 0 and, if generated tax revenues are not to exceed the given level R, from above as well. In order to ascertain what value of t to choose to make p_H^1 as large as possible, the derivative of (2.43) for j=1, after (2.42) has been substituted in, with respect to the tax rate is taken. To simplify the exposition of this derivative the following substitutions for specific parameter expressions are made.

$$\delta = \frac{\rho}{1+\rho}$$

$$\delta_1 = \alpha_H^{\frac{-1}{\rho}}$$

$$\delta_2 = \left[\frac{(M+\tilde{p}_W\cdot\overline{W}-c\cdot u^1)(T-\overline{W}-v\cdot u^1)^{\alpha_\ell}}{MT^{\alpha_\ell}}\right]^{\delta}$$

$$\delta_3 = \alpha_H\cdot\left(\frac{\overline{P}_{HA}\cdot(1+a_R\cdot t)}{\alpha_H}\right)^{\delta}$$

$$\delta_4 = \alpha_T\cdot\left(\frac{\overline{P}_T}{\alpha_T}\right)^{\delta}$$

$$\delta_5 = (\delta_3 + \delta_4)^{\frac{-1}{\delta}}$$

$$\delta_6 = \delta_2\cdot((1+a_R\cdot t)\cdot\delta_5)^{-\delta} - \delta_4\cdot(1+a_R\cdot t)^{-\delta}$$

Let the bid price of housing services in ring 1, when $p_H^2 = \overline{P}_{HA}$ and $p_W = \tilde{P}_W$, be written as $\overline{P}_{HA}^1$. The derivative of this bid price with respect to the tax rate can now be expressed as follows:

$$\frac{d\overline{P}_{HA}^1}{dt} = a_R\cdot\delta_1\cdot\delta_4\cdot(\delta_6)^{\frac{1-\delta}{\delta}}\cdot(1+a_R\cdot t)^{-(\delta+1)}(1-\delta_2) \qquad (3.17)$$

All of the terms in the product on the right-hand side of (3.17) except the last are unambiguously positive. The sign of (3.17) therefore, varies with the sign of $(1-\delta_2)$. Since $(M+\tilde{p}_W \cdot \overline{W} - c \cdot u^1)(T-\overline{W}-v \cdot u^1)^{\alpha_\ell} > MT^{\alpha_\ell}$, δ_2 involves raising a number greater than 1 to a power the sign of which varies with the sign of ρ, the substitution term in the utility function. Thus, the following assertions can be made.

$$\rho > 0 = \delta_2 > 1 \Rightarrow \frac{d\overline{p}^1_{HA}}{dt} < 0$$
$$\rho < 0 = \delta_2 > 1 \Rightarrow \frac{d\overline{p}^1_{HA}}{dt} > 0 \qquad (3.18)$$

Hence, the bid price of housing services in ring 1, when $p^2_H = \overline{p}_{HA}$ and $p_W = \tilde{p}_W$, is maximized when $t = 0$ if $\rho > 0$, and when t is as large as possible if $\rho < 0$.

How can t be bounded for the case $\rho > 0$? Let R^1_G be the generated property tax base in ring 1. Then, assuming we have a zero of the third component of E(p) (i.e., $R_G = R$), we must have $\tau + t\, R^1_G = R$, because no housing exists beyond ring 2 in the situation considered here. Since $t = \frac{\tau}{a_R \cdot \overline{p}_{HA} \cdot \overline{s}^2_{HA}}$ in this case, we can solve for τ and obtain $\tau = \frac{a_R \cdot \overline{p}_{HA} \cdot \overline{s}^2_{HA} \cdot R}{a_R \cdot \overline{p}_{HA} \cdot \overline{s}^2_{HA} + R^1_G}$. This in turn allows us to write the tax rate as $t = \frac{R}{a_R \cdot \overline{p}_{HA} \cdot \overline{s}^2_{HA} + R^1_G}$. What is needed, though, is an upper bound for t that is a function of parameters only. The generated property tax base in ring 1, R_G^1, varies with P. For the purpose of expressing the upper bound that is sought, let $\overline{s}_{HA}^1 = s_H^1(\overline{p}_A, L^1)$ be the supply of housing in ring 1 when the bid land rental for CBD land by the housing industry is equal to the agricultural land rental, $\overline{p}_A$ (i.e., when $p_H^1 = \overline{p}_{HA}$), and all of the endogenous land in the CBD is allocated to housing. Also let $\overline{K}_T = K_T(\overline{p}_A, L^1)$ be capital demand by the traded good industry when its bid land rental is $\overline{p}_A$, and all of the endogenous CBD land is allocated to the traded good industry. Then, since the supply of housing to ring 1 and capital demand by the traded good industry are proportional, for a given p, to the respective amounts of CBD land allocated to these sectors, and the actual land rental that prevails in the CBD, in the case considered ($p_H^2 = \overline{p}_{HA}$), exceeds $\overline{P}_A$ (because $\overline{P}_{HA}^1 > \overline{P}_{HA}$), it must be true, even if CBD land is divided between housing and the traded good industry, that $\min[a_R \cdot \overline{p}_{HA} \cdot \overline{s}_{HA}^1 \, , \, a_I \cdot (\overline{p}_A \cdot L^1 + \overline{p}_R \cdot \overline{K}_T)] < R_G^1$. Replacing R_G^1 in the expression for t with the left-hand side of this inequality gives us an upper bound for t that depends only on the parameters of the model. Specifically, we define the bound $\hat{t}$ as follows:

$$t = \frac{R}{a_R \cdot \overline{p}_{HA} \cdot \overline{s}_{HA}^2 + \min[a_R \overline{p}_{HA} \cdot \overline{s}_{HA}^1, a_I \cdot (\overline{p}_A \cdot L^1 + \overline{p}_K \cdot \overline{K}_T)]} \qquad (3.19)$$

This value is to be used in forming a condition which satisfies our original objective. In particular, we want an upper bound for the generated population. When $\rho < 0$, since the supply of housing in ring 1, for a given allocation of land, increases and the household demands for housing in rings 1 and 2 decrease with increases in t, the value $\hat{t}$ will be substituted for t in calculating the upper bound on population. When $\rho > 0$, while the demands for housing still decrease with increases in t, the supply of housing in ring 1, for a given allocation of land now also decreases with increases in t. Since a solution to the problem of maximizing the ratio of the supply of housing in ring 1 to the household demand for housing in ring 1 with respect to the tax rate, in the situation considered, has not been found, we let the tax rate take on the value $\hat{t}$ when calculating household demand, but the value zero when calculating $\bar{p}^1_{HA}$, and so the supply of housing in ring 1.

Before the condition can be stated, something must be decided about the income term in the demand functions. Demand obviously decreases with decreases in net income here. Since $(T-\bar{W}-v \cdot u^j) < T$ for any j and households have the option of not working, net income will never fall below exogenous income, M. Therefore, we let $Y^1 = Y^2 = M$ in calculating housing demand in the context of the formation of the condition.

To ease exposition of the condition we suppress all variables except the price of housing in the ring and the amount of land in the ring allocated to housing in expressing the supply of housing services for the ring. Thus, we write $s^j_H = s^j_H(p^j_H, L^j_H)$. Similarly for housing demand we suppress all variables except the price of housing in the ring, the tax rate, and household net income in the ring. Thus, we write $x^j_H = x^j_H(p^j_H, t, Y^j)$. This now allows usto express upper bounds for the generated populations in the two rings. In particular, define $\hat{N}^1_G$ to be the upper bound for ring 1 as follows:

$$\hat{N}_G^1 = \begin{cases} \dfrac{s_H^1(\bar{p}_{HA}^{-1}(\bar{p}_{HA},\hat{t}),L^1)}{x_H^1(\bar{p}_{HA}^{-1}(\bar{p}_{HA},\hat{t}),\hat{t},M)} & \rho < 0 \\ & \text{if} \\ \dfrac{s_H^1(\bar{p}_{HA}^{-1}(\bar{p}_{HA},t=0),L^1)}{x_H^1(\bar{p}_{HA}^{-1}(\bar{p}_{HA},t=0),\hat{t},M)} & \rho > 0 \end{cases} \tag{3.20}$$

Next, we define $\hat{N}_G^2$ to be the upper bound for ring 2 as follows:

$$\hat{N}_G^2 = \frac{s_H^2(\bar{p}_{HA},L^2)}{x_H^2(\bar{p}_{HA},\hat{t},M)} \tag{3.21}$$

Note that all of the endogenous land in rings 1 and 2 is assumed to be allocated to housing. We can use these bounds to finally exhibit our last condition on the parameters of the model.

$$\hat{N}_G^1 + \hat{N}_G^2 < N \tag{c.4}$$

Condition (c.4) insures that a solution of the model (in the sense of a vector p^* such that $0 \varepsilon E(p^*)$) with $p_H^2 \cdot s_H^2 < \bar{p}_{HA} \cdot \bar{s}_{HA}^2$ does not exist.

Several things should be noted about this condition. First, satisfaction of the condition depends on the level of exogenous income relative to the required amount of property tax revenues. To be more precise, the condition is satisfied only if the actual level of exogenous income in the urban economy (based on the given number of households) exceeds the pre-specified level of property tax revenues to be raised. To see this, first note that from (3.19) and the fact that $\bar{p}^1_{HA} > \bar{p}_{HA}$ we can obtain

$R = \hat{t}\cdot(a_R\cdot\bar{p}_{HA}\cdot\bar{s}^2_{HA} + \min[a_R\cdot\bar{p}_{HA}\cdot\bar{s}^1_{HA}, a_I\cdot(\bar{p}_A\cdot L^1 + \bar{p}_K\cdot\bar{K}_T)]) \leq$

$a_R\cdot\hat{t}(\bar{p}_{HA}\cdot\bar{s}^1_{HA} + \bar{p}_{HA}\cdot\bar{s}^2_{HA}) < a_R\cdot\hat{t}\cdot(\bar{p}^1_{HA}\cdot s^1_H + \bar{p}_{HA}\cdot\bar{s}^2_{HA})$, where s^1_H is as defined in (3.20). Next, note that, from the definitions of N^1_G and N^2_G in (3.20) and (3.21), the fact that household net income must more than cover household tax payments, and condition (c.4), we must have $a_R\cdot\hat{t}\cdot(\bar{p}^1_{HA}\cdot s^1_H + \bar{p}_{HA}\cdot\bar{s}^2_{HA}) < M\cdot\hat{N}^1_G + M\cdot\hat{N}^2_G = M\cdot(\hat{N}^1_G + \hat{N}^2_G) < M\cdot N$. Thus, $R < M\cdot N$ and so total exogenous income for the given number of households must be larger than the property tax revenues to be raised, if the condition is to be satisfied. Second, the intent of the condition, to insure that it is not possible to house as many households as there actually are when $p^2_H\cdot s^2_H < \bar{p}_{HA}\cdot\bar{s}^2_{HA}$ and the non-population aspects of a solution are satisfied, will be achieved for some parameterizations which violate the condition. The condition was constructed on a worse than worst possible outcome basis. By this we mean that, in the situation considered, not all of the values of the variables used in finding the upper bounds for population in (3.20) and (3.21) can occur simultaneously, or even occur at all. For instance, in forming the condition it was assumed that $L^2_H = L^2$. Yet if all of the available land in ring 2 is allocated to housing, then $p^2_H\cdot s^2_H = \bar{p}_{HA}\cdot\bar{s}^2_{HA}$ where s^2_H is as defined in (3.21). The amount of land in ring 2 actually allocated to housing, in the cases considered, must be less than L^2. Thus, the generated population in ring 2 must really be less than $\hat{N}^2_G$. The tax rate used in the condition, $\hat{t}$, is an upper bound. The actual tax rate may be less than $\hat{t}$, with the result that the generated population in both rings would be lower. Furthermore, the tax rate in the case $\rho > 0$ involves an incongruity. One value is assumed in calculating $\bar{p}^1_{HA}$ and another is assumed to face households in ring 1. The two values were chosen to make $\hat{N}^1_G$ as large as possible. Thus, if the tax rate is the same for both situations, as it must really be, the generated population in ring 1 must be less than $\hat{N}^1_G$. A wage rate of $\tilde{p}_w$ was used in $\frac{Y^1(\ell^1)^{\alpha_\ell}}{Y^2(\ell^2)^{\alpha_\ell}}$ for calculating $\bar{p}^1_{HA}$, yet household income in ring 1 is taken to be M, even though households in that ring will choose to work, and so receive a higher net income, if the wage rate is $\tilde{p}_w$. Finally, the condition does not reflect the presumption that the labor market is in equilibrium. Indeed it is inconsistent with labor market equilibrium since the condition involves allocating all endogenous CBD land to housing, implying that labor demand is zero. Yet the wage rate $\tilde{p}_w > 0$ is used in part of (3.20) and, as noted, labor supply forthcoming from ring 1 will be positive if the wage rate is $\tilde{p}_w$ and housing is allocated some land there. In sum, then, condition (c.4) is sufficient but not necessary for our purposes. There are many parameterizations which give us what we want in spite of the fact that they do not satisfy (c.4). Finally, it should be noted that there are parameterizations that do satisfy (c.4). In fact, for any set of parameter values, exclusive of M, condition (c.4) will be satisfied if exogenous income is large enough. Proof of this assertion can be drawn from part of the proof of Corollary 2.5 below and the fact that household demand for housing in a ring, other things equal, goes to infinity as income tends toward infinity.

Given the conditions and restrictions mentioned above, we may finally establish that an equilibrium exists for the model. This will be accomplished by applying Theorem 1. We may state our result rigorously in the form of a second theorem.

Theorem 2. Let $P = \{p = (p_W, p_H^2, \tau) | p_W + p_H^2 + \tau \leqq d\}$ and $d > 0$. Given the model described above, with conditions (c.1)-(c.4), there exists $d > 0$ such that an economic equilibrium (with no excess supplies) exists. In particular, there is some $d > 0$ for which there exists $p^* \varepsilon P$ and $e^* \varepsilon E(p^*)$ such that $e^* = 0$ (with actual property tax revenue raised equal to the given amount).

Proof. To establish this we first show that the hypotheses of Theorem 1 are satisfied for some $d > 0$. This will prove that there exists $p^* \varepsilon P$ and $e^* \varepsilon E(p)$ such that $e^* \leqq 0$ and $p^* \cdot e^* = 0$. We then argue that we must have $e^* = 0$.

The upper semicontinuity of E is not difficult to see.[17] Each of the components of vectors in E(p) involves a sum of terms. With the exception of labor demand when $\rho_T > 0$, each of these terms is either a constant, one of the components of p (τ in the third component of vectors in E(p)), or an expression that is the product of a continuous function of p and the amount of land in a ring allocated to a particular endogenous use. If each of the correspondences (when they are defined on and map to sets of real numbers) in a sum of correspondences is upper semicontinuous, then the sum itself is an upper semicontinuous correspondence. Furthermore, if $E_i(p)$ is defined to be the set of ith components of vectors in E(p), we may consider E_i itself to be a correspondence. It can be shown that if E_i is upper semicontinuous, for all i, then E is upper semicontinuous. Thus, if we show that each of the terms in each component of vectors in E(p) is upper semicontinuous then it must be true that E is upper semicontinuous.

The constant terms and the term involving just a component of p can obviously be considered to be upper semicontinuous. As noted above, all other terms, with the exception of labor demand when $\rho_T > 0$, are, at a given p, proportional to the amount of land allocated to some use. We may cover all of these cases by investigating the upper semicontinuity of the correspondence $\hat{\phi}:P \rightarrow R$. This correspondence is to map "price" vectors $p \in P$ into sets of real numbers, elements of which, for a given p, take the form $\phi(p) \cdot L_i^j(p)$, where $\phi(p)$ is some continuous function of p and $L_i^j(p)$ is the amount of land in ring j allocated to use i when the price vector is p.

Recalling the model, we note that use i is allocated all of the land available for endogenous use in ring j, L^j, if it outbids all other uses; it is allocated none of the land if it is outbid by some other use; and it can be allocated any amount of land in the interval $[0,L^j]$ if it is not outbid for land in the ring but ties at least one other use with its bid rent for land. Suppose that $p^1,p^2,\ldots,p^k,\ldots$ is a sequence of points in P that converges to a point p^o in P. Assume that

$$e^1 = \phi(p^1)\cdot L_i^j(p^1), e^2 = \phi(p^2)\cdot L_i^j(p^2), \ldots, e^k = \phi(p^k)\cdot L_i^j(p^k), \ldots$$

is a sequence that converges to the point e. We must consider each of the three land allocation cases mentioned above when the price vector is p^o. Suppose that p^o is such that use i outbids all other uses for land in ring j. Then, $\phi(p^o)\cdot L_i^j(p^o) = \phi(p^o)\cdot L^j$. Given the continuity of the bid price functions it must be that, for price vectors p^k close enough to p^o (say $k \geq K$), use i obtains all of the available land in ring j. Thus, $e^k = \phi(p^k)\cdot L^j$ for k large enough ($k \geq K$). Hence, by the assumed continuity of ϕ we must have $e = \phi(p^o)\cdot L^j = \phi(p^o)\cdot L_i^j(p^o) \in \hat{\phi}(p^o)$. Suppose now that p^o is such that use i is outbid for land by some other use. Then, $\phi(p^o)\cdot L_i^j(p^o) = 0$. For price vectors, p^k, close enough to p^o, now, we have use i being outbid for land in ring j and so obtaining no land. Thus, $e^k = \phi(p^k)\cdot L_i^j(p^k) = 0$ for k large enough. Hence, once again we have $e = \phi(p^o)\cdot L_i^j(p^o) \in \hat{\phi}(p^o)$. Finally, suppose that p^o is such that use i is not outbid for land in ring j but is tied by some other use. In this case $\hat{\phi}(p^o)$ is a set of points since $L_i^j(p^o)$ can take on any value in $[0, L^j]$. Because ϕ is continuous, and use i cannot be allocated more than L^j units of land at any price vector, the sequence $e^1, e^2, \ldots e^k \ldots$, if it converges, must converge to $e = \phi(p^o)\cdot L$, where $L \in [0, L^j]$. Thus, since $L \in L_i^j(p^o)$, we have $e = \phi(p^o)\cdot L \in \hat{\phi}(p^o)$. Therefore ϕ is an upper semicontinuous correspondence.

Finally, consider labor demand when $\rho_T > 0$. In the region of price space where the above-mentioned upper bound, $\hat{W}$, on labor demand cannot be encountered, the actual demand for labor can be expressed as the product of a continuous function of p and the amount of land allocated to the traded good industry. The analysis just undertaken, then, can be used to show that the labor demand term in the first component of E is upper semicontinuous in this restricted region. Therefore, we may confine our attention to sequences of prices that converge to points where the upper bound may be binding. Again, we consider three possible land allocations at a point p^o, to which the sequence of prices $p^1, p^2, \ldots, p^k, \ldots$ converges. Let $W_T(p)$ represent labor demand at the price vector p, which will be a set of points if the traded good industry's bid land rent ties some other use in bidding for CBD land, but is not outbid by any use. If the wage rate component of p^o, p^o_w, is zero then, it should be recalled, labor demand is defined by (3.16). Assume that $e^1_w \varepsilon W_T(p^1), e^2_w \varepsilon W_T(p^2), \ldots, e^k_w \varepsilon W_T(p^k), \ldots$ is a sequence that converges to the point e_w. Suppose that p^o is such that the traded good industry outbids all other uses for CBD land. Then, $\hat{W} \varepsilon W_T(p^o)$ since the traded good industry is allocated all of the available CBD land ($L^1_T(p^o) = L^1$) and we are assuming that p^o is in a region of price space where (2.52) yields a value that exceeds $\hat{W}$ for more than one allocation of CBD land to the traded good industry.[18,19] For price vectors, p^k, close enough to p^o (say $k \geq K$), then, the upper bound will be binding for some CBD land allocations and the traded good industry is allocated L^1 units of land. Thus, $e^k_w = W_T(p^k) = \hat{W}$ for k large enough ($k \geq K$). Hence, we must have $e_w = \hat{W} \varepsilon W_T(p^o)$. Suppose now that p^o is such that the traded good industry is outbid for CBD land by some other use. Then, whether or not p^o_w is zero, $W_T(p^o) = 0$ since $L^1_T = 0$. For price vectors, p^k, close enough to p^0 now, the traded good industry is outbid for CBD land. Thus, $e_w^k = W_T(p^k) = 0$ for k large enough. So, since $e_w = 0$ we again have $e_w \in W_T(p^0)$. Finally, suppose that p^0 is such that the traded good industry is not outbid for CBD land, but its bid land rent is tied with that of some other use. Since $L_T^1(p^0)$ can now take on any value in $[0,L^1]$, labor demand at p^0, whether $p_w^0 = 0$ or not, can assume any value in $[0,\hat{W}]$. Because labor demand when $\rho_T > 0$ cannot exceed $\hat{W}$, the sequence $e_w^1, e_w^2, \ldots, e_w^k, \ldots$, if it converges, must converge to a point e_w, where $e_w \in [0,W]$. Thus, once again we have $e_w \in W_T(p^0)$, which shows finally that labor demand is upper semicontinuous. Therefore, every term in each component of E is upper semicontinuous. This establishes the upper semicontinuity of E.

The correspondence E is also easily seen to be bounded. The first component of E(p) is bounded from below since the demand for labor is nonnegative and the number of generated households that work, for a given d, is bounded from above.[20] The excess demand for labor is also bounded from above, because labor demand is bounded from above. When $\rho_T > 0$ labor demand is bounded from above by $\hat{W}$. When $\rho_T < 0$ as indicated in a discussion above, it is made to be bounded from above through the use of $\hat{p}_{LT}^1$ and $\hat{p}_w$.[21] The second component is bounded from below since the generated population of households that choose to be unemployed is bounded from above by $\hat{N}$, while, as noted, for a given d the generated population of households that work is also bounded from above. The second component is bounded from above by N. The third component is bounded from below because, for a given d, τ is bounded from above (in particular it can be no larger than d) and both the bid land rent and capital demand for the traded good industry are bounded from above.[22] The third component is bounded from above by R. Thus, E is a bounded correspondence.

The correspondence E is convex if and only if, for any $p \in P$ and $\lambda \in [0,1]$ with $e, e' \in E(p)$, we have $\lambda \cdot e + (1-\lambda) \cdot e' \in E(p)$. Let us examine the k'th components of these vectors. As has been noted, all of the components of vectors in E(p), with the exception of the first when $\rho_T > 0$, can be expressed as sums of terms where each is a constant, a component of p, or an expression that is the product of a function of p and the amount of land allocated to some endogenous use. The component of p and the functions of p just mentioned can be treated as constants here since the definition of convexity involves a single, although arbitrary, value of p. To aid in the analysis the following definitions of terms are made.

$\eta_i^j, \eta_i^{j'}$: the proportions of available land in ring j allocated to the endogenous, non-agricultural, use i associated with the "excess demands" e and e', respectively.

$\eta_A^j, \eta_A^{j'}$: the proportions of available land in ring j allocated to agriculture associated with the vectors e and e', respectively.

e_k : the k'th component of the vector e.

e'_k : the k'th component of the vector e'.

ϕ_i^{jk} : the non-land allocation term in the k'th component of e and e' that relates to use i in ring j.

We must have $\sum_{\text{all } i} \eta_i^j = 1 - \eta_A^j$ and $\sum_{\text{all } i} \eta_i^{j'} = 1-\eta_A^{j'}$, for all j.

Note that ϕ_i^{jk} is a function of p and is constant for a given p. Thus, it is invariant from one vector to another in the set E(p). The terms ϕ_i^{jk} are meant to represent those functions of p which, when multiplied by the amount of land in ring j allocated to use i, are the above-mentioned products that are particular items in the sums that comprise the "excess demands," or components of E(p). As such, we would define $\phi_i^{j\hat{k}}$ to be zero for all j if a product term relating to use i does not appear in the sum that represents the $\hat{k}$'th component of vectors in E(p) (as would be the situation for the traded good industry in the second component). Excluding the case of the first component of the excess demand vectors when $\rho_T > 0$, we can write the k'th components of e and e', respectively, as:

$$e_k = c_k + \sum_{j=1}^{\infty} \sum_{\text{all } i} \phi_i^{jk}(p) \cdot L_i^j(p) = c_k + \sum_{j=1}^{\infty} \sum_{\text{all } i} \eta_i^j \cdot \phi_i^{jk} \cdot L^j \tag{3.22}$$

$$e'_k = c_k + \sum_{j=1}^{\infty} \sum_{\text{all } i} \phi_i^{jk}(p) \cdot L_i^{j'}(p) = c_k + \sum_{j=1}^{\infty} \sum_{\text{all } i} \eta_i^{j'} \cdot \phi_i^{jk} \cdot L^j$$

where L_i^j and $L_i^{j'}$ are the amounts of land in ring j allocated to use i under e and e' respectively, and c_k is constant for a given p. We ask if the k'th component of the vector $\lambda e + (1-\lambda)e'$ can be the k'th "excess demand" of some vector in E(p). This component can be written using (3.22), as follows:

$$\lambda e_k + (1-\lambda)e'_k = c_k + \sum_{j=1}^{\infty} \sum_{\text{all } i} (\lambda \eta_i^j + (1-\lambda)\eta_i^{j'}) \cdot \phi_i^{jk} L^j \quad (3.23)$$

This will represent the k'th component of a vector in E(p) if, treating $(\lambda\eta_i^j + (1-\lambda)\eta_i^{j'})$ as the proportion of land in ring j allocated to use i, (3.23) is consistent with a feasible allocation of land in all rings, given the price vector p. If this is true then we must have $\sum_{\text{all } i} (\lambda\eta_i^j + (1-\lambda)\eta_i^{j'}) \leq 1$, for all j, with agriculture's allocated proportion of available land in ring j being the difference between 1 and the left-hand side of the inequality. Using the definitions it can be seen that $\sum_{\text{all } i} (\lambda\eta_i^j + (1-\lambda)\eta_i^{j'}) = \lambda \sum_{\text{all } i} \eta_i^j + (1-\lambda) \sum_{\text{all } i} \eta_i^{j'} = \lambda\cdot(1 - \eta_A^j) + (1-\lambda)\cdot(1-\eta_A^{j'}) = 1 - (\lambda\eta_A^j + (1-\lambda)\eta_A^{j'}) \leq 1$ since $\eta_A^j, \eta_A^{j'} \geq 0$. In what is really the natural approach to take, we assign $(\lambda\eta_A^j + (1-\lambda)\eta_A^{j'})$ to be the proportion of endogenous land in ring j allocated to agriculture. This allocation of land is consistent with the price vector p. If, under p, use i is outbid for land in ring j, then $\eta_i^j = \eta_i^{j'} = 0$ and so $(\lambda\eta_i^j + (1-\lambda)\eta_i^{j'}) = 0$ as well. If use i out-gids all other uses for land in ring j, then $\eta_i^j = \eta_i^{j'} = 1$ and so $(\lambda\eta_i^j + (1-\lambda)\eta_i^{j'}) = 1$ also. Finally, if use i is tied with at least one other use and not outbid by any use for land in ring j, then $0 \leq \eta_i^j, \eta_i^{j'} \leq 1$ and so $0 \leq (\lambda\eta_i^j + (1-\lambda)\eta_i^{j'}) \leq 1$. Therefore, the allocation of land defined by using $(\lambda\eta_i^j + (1-\lambda)\eta_i^{j'})$ is in line with the model in that, at the given vector p, a use would receive no land in a ring if it is outbid by some other use, all of the available land if it outbids all other uses, and anything from none to all of the available land if its bid land rent ties at least one other use and is not outbid by any use there. Combining all of this with the fact that the land allocation is consistent across k (i.e., use i receives the same proportion of land in ring j in each of the components of the "weighted average" vector) proves that $\lambda e + (1-\lambda)\cdot e' \in E(p)$.

This would be enough to prove the convexity of E except that we have excluded one possibility. Suppose that $\rho_T > 0$. If $p_W \neq 0$ there is no problem since then, labor demand can be expressed as the product of a function of p and the amount of CBD land allocated to the traded good industry. In these cases the first component of vectors in E(p) can be represented by (3.22). Thus, we need be concerned only with the case $p_W = 0$. In this case, labor demand is defined by (3.15). Since labor supply can be expressed in the form needed for (3.22), we need only be concerned with labor demand. In particular, if we let $W_T(p)$ and $W_T'(p)$ represent labor demand under e and e' respectively, then we must show that $\lambda W_T(p) + (1-\lambda)W_T'(p)$ is a potential labor demand, as we have defined it, when $\rho_T > 0$ and $p_W = 0$. Any such demand for labor must lie in the interval $[0,\hat{W}]$. If the traded good industry is outbid for CBD land, then $W_T(p) = W_T'(p) = 0$ and so $\lambda W_T(p) + (1-\lambda)W_T'(p) = 0$ as well. If the traded good industry outbids all other uses for CBD land then $0 < W_T(p), W_T'(p) \leq \hat{W}$ and so $0 < \lambda W_T(p) + (1-\lambda)\cdot W_T'(p) \leq \hat{W}$. Finally, if the traded good industry ties some other use in bidding for CBD land, but is not outbid for CBD land by any use, then $0 \leq W_T(p), W_T'(p) \leq \hat{W}$ and so $0 \leq \lambda W_T(p) + (1-\lambda)W_T'(p) \leq \hat{W}$ as well. Hence, $\lambda W_T(p) + (1-\lambda)W_T'(p)$ can serve as labor demand when the price vector is p. Therefore, E is a convex correspondence.

The upper semicontinuity, boundedness, and convexity of E imply that hypothesis (H.1) of Theorem 1 is satisfied. It remains to be shown that hypothesis (H.2) of that theorem is satisfied. To do this, we restrict our attention to price vectors p such that $p_W + p_H^2 + \tau = d$, for some $d > 0$. We must show that there is some $d > 0$ so that for every $p \varepsilon P$ with $p_W + p_H^2 + \tau = d$, there is a vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) \geqq 0$ such that $\alpha \cdot e \leqq 0$ for all $e \varepsilon E(p)$, with $\alpha_i > 0$ only if $p_i > 0$. To do this we apportion d among p_W, p_H^2, and τ by letting $p_W = a_1 d$, $p_H^2 = a_2 d$, and $\tau = a_3 d$, where $0 \leqq a_1, a_2, a_3 \leqq 1$ and $a_1 + a_2 + a_3 = 1$.

Before we discuss the various situations associated with different combinations of values for the a_i's, we must develop a certain constant term. There is a distance beyond which households that work cannot reside, no matter how high their earned income. Given an institutionally determined set of work days and length of time to be spent working on such days, along with the fixed per unit travel time, households residing in rings far enough from the CBD simply could not find enough time on work days to work and do the commuting needed to get to and from the workplace. Let j' be the index of the last ring for which residents of the ring who are employed would not have to spend more time working and commuting to and from work on a work day than the total amount of time available on such days.[23] Given this index and other parameters of the model the term, $\hat{a}$, can be defined.

$$\hat{a} = \frac{\dfrac{\overline{W} \cdot N}{B \cdot (\alpha_{LH})^{\frac{-1}{\rho_H}} \cdot \sum_{j=2}^{j'} L^j \cdot \left(\frac{\ell^j}{\ell^2}\right)^{\alpha_\ell}}}{1 + \dfrac{\overline{W} \cdot N}{B \cdot (\alpha_{LH})^{\frac{-1}{\rho_H}} \cdot \sum_{j=2}^{j'} L^j \cdot \left(\frac{\ell^j}{\ell^2}\right)^{\alpha_\ell}}} \qquad (3.24)$$

Note that $0 < \tilde{a} < 1$. Now define a new term, $\tilde{a}$, as follows: $\tilde{a} = \hat{a} + \varepsilon$, where ε is some positive constant such that $\varepsilon < 1 - \hat{a}$. Thus, we have $0 < \tilde{a} < 1$.

Given the definition of $\tilde{a}$ we can subdivide our problem of finding an appropriate d and vectors α , into the following three cases.

(i) $a_2 \geqq \tilde{a}$

(ii) $a_1 = \frac{1-\tilde{a}}{2}$, $a_2 < \tilde{a}$, $a_3 \leqq 1 - \frac{1-\tilde{a}}{2}$

(iii) $a_3 > \frac{1-\tilde{a}}{2}$, $a_1 < \frac{1-\tilde{a}}{2}$, $a_2 < \tilde{a}$

All of the a_i's, of course, are assumed in all cases to be non-negative and sum to 1. With respect to the values that p_W, p_H^2, and τ can assume when they sum to d, these cases are mutually exclusive and collectively exhaustive.

Let us first consider case (i). Here p_H^2 is always positive and increases with increases in d for any admissible a_2. If d, and so p_H^2 can be chosen high enough so that the generated population exceeds the given population, then we can choose $\alpha = (0,1,0)$ so as to satisfy (H.2) when the price vector is such that case (i) is in effect. To investigate this possibility, let us first consider the supply of housing in ring 2. As noted, the generated population of a household type in a ring is the ratio of the supply of housing to that type to the household demand for housing from a household of that type. We will concentrate first on the generated population of households in ring 2 that choose to work. If d is high enough, then p_H^2 will be large enough so that housing outbids agriculture for land in ring 2. Assume then that all of the available land in ring 2 is allocated to housing and, in particular, to housing households that work. Using the apportionment of d to the three "prices" via the a_i's, we study what happens to the supply of housing in ring 2 as d increases. In particular, using (2.46) and (2.51) we take the limit of s_H^2 as d goes to infinity.

$$\lim_{d\to\infty} s_H^2 = \lim_{d\to\infty} B\cdot\left[\alpha_{KH}\cdot\left(\frac{\alpha_{LH}\cdot\bar{p}_K}{\alpha_{KH}}\right)^{\frac{\rho_H}{1+\rho_H}}\cdot\left(\frac{1}{\left(\frac{(B\cdot a_2\cdot d)^{\rho_H}}{\alpha_{LH}}\right)^{\frac{1}{1+\rho_H}}-\left(\frac{\alpha_{KH}(\bar{p}_k)^{\rho_H}}{\alpha_{LH}}\right)^{\frac{1}{1+\rho_H}}}\right)+\alpha_{LH}\right]^{\frac{-1}{\rho_H}}\cdot L^2 \tag{3.25}$$

The value of this limit depends on the sign of ρ_H. In particular,

$$\lim_{d\to\infty} s_H^2 = \begin{cases} B\cdot\left[\alpha_{KH}\cdot\left(\dfrac{\alpha_{LH}\cdot\bar{p}_K}{\alpha_{KH}\cdot\hat{p}_{LH}}\right)^{\frac{\rho_H}{1+\rho_H}}+\alpha_{LH}\right]^{\frac{-1}{\rho_H}}\cdot L^2 & \rho_H < 0 \\ & \text{if} \\ B\cdot(\alpha_{LH})^{\frac{-1}{\rho_H}}\cdot L^2 & \rho_H > 0 \end{cases} \tag{3.26}$$

The limit when $\rho_H < 0$ is, in essence, infinite since in theory the upper bound, $\hat{p}_{LH}$, can be made as large as desired.

Turning now to household demand for housing services in ring 2 for those households that work, we again take the limit as d goes to infinity.

$$\lim_{d\to\infty} x_{HW}^2 = \frac{M + a_1 \cdot d \cdot \overline{W} - c \cdot u^2}{a_2 \cdot d\left[1 + \frac{a_3 \cdot d}{a_2 \cdot d \cdot s_H^2}\right] + \alpha_T\left(\frac{\overline{p}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}} \left[\frac{a_2 \cdot d \cdot \left(1 + \frac{a_3 \cdot d}{a_2 \cdot d \cdot s_H^2}\right)}{\alpha_H}\right]^{\frac{1}{1+\rho}}} \tag{3.27}$$

Note that the form $\frac{\tau}{a_R \cdot p_H^2 \cdot s_H^2}$ has been used for the tax rate, t. This is the form it must take for large values of d, since then $p_H^2 = a_2 \cdot d > \overline{p}_{HA}$. By dividing the numerator and denominator of the expression in (3.27) by d the limit can be evaluated. Again the value of the limit depends on the sign of a substitution parameter.

$$\lim_{d\to\infty} x_{Hw}^2 = \begin{cases} 0 & \text{if } \rho < 0 \\ \dfrac{a_1 \cdot \overline{W}}{a_2 + \dfrac{a_3}{\lim_{d\to\infty} s_H^2}} & \text{if } \rho > 0 \end{cases} \tag{3.28}$$

Since $a_2 \geqq \tilde{a} > 0$ the limit of x_{Hw}^2 when $\rho > 0$ is finite for admissible values of a_1, a_2, and a_3. In three out of the four sign combinations for ρ_H and ρ, the generated population of households in ring 2 that work, $N_G^{w2} = \frac{s_H^2}{x_{Hw}^2}$, goes to infinity or, with an appropriate choice of

$\hat{p}_{LH}$, as large a value as desired as d tends toward infinity. In these cases, then, the generated population can be made to exceed the given population by choosing a large enough value for d (and a large enough value for $\hat{p}_{LH}$ when $\rho_H < 0$ and $\rho > 0$). This result holds true even if we allow for the possibility that some or all of the land can be allocated to housing households that choose to be unemployed. For a given d, this will happen for small enough values of a_3. To be precise, if $a_3 > \frac{\tilde{p}_w}{d}$ then all households residing in ring 2 will decide to work, if $a_3 < \frac{\tilde{p}_w}{d}$ they all will choose to be be unemployed, and if $a_3 = \frac{\tilde{p}_w}{d}$ they will be indifferent between working and not working.[24] The limit of the supply of housing services provided for households in ring 2 that do not work would be given by (3.26) with L^2_{HNw} replacing L^2. The limit of household demand for housing by such households, on the other hand, will be zero no matter what sign ρ has, since household income for this group is M. Thus, d can be chosen large enough so that both N_G^{w2}, when L^2 units of land in ring 2 are allocated to housing households that work, and N_G^{Nw2}, when L^2 units of land in ring 2 are allocated to housing households that do not work, are greater than N. Therefore, for d large enough, the total generated population in ring 2 exceeds the given population for $a_3 > \frac{\tilde{p}_w}{d}$ and $a_3 < \frac{\tilde{p}_w}{d}$. The total generated population in ring 2 will also be greater than N when $a_3 = \frac{\tilde{p}_w}{d}$ since, for a given p, N_G^{w2} and N_G^{Nw2} are proportional to the amounts of land allocated to housing the respective household types.

When $\rho_H, \rho > 0$, however, the limit of the generated population of households that work, $\lim_{d\to\infty} N_G^{w2}$, is finite and may not exceed the given population. If the bid land rent gradient for housing of households that work could extend out from the center indefinitely as p_H^2 increased to infinity, then this case could be dispensed with quickly and easily. We would simply note that the supply and demand limits would be the same for all rings and the amount of land in a ring available for endogenous use increases to infinity as the distance of the ring from the center increases. As noted above, however, settlement of rings by households that work must be limited to rings 1 to j'. Thus, in this case, the limit of the total generated population of households that choose to work and reside in rings 2 and beyond, if all of the available land in these rings is allocated to housing these households, is

$$\lim_{d\to\infty} \sum_{j=2}^{j'} N_G^{wj} = \sum_{j=2}^{j'} \left(\frac{a_2 \cdot B \cdot (\alpha_{LH})^{\frac{-1}{\rho_H}} \cdot L^j + a_3}{a_1 \cdot \overline{W}} \right) \cdot \left(\frac{\ell^j}{\ell^2} \right)^{\alpha_\ell} \qquad (3.29)$$

The minimum value that this limit can assume for case (i) is obtained by letting $a_1 = 1-\tilde{a}$, $a_2 = \tilde{a}$, $a_3 = 0$. In this case (3.29) becomes

$\frac{\tilde{a}}{1-\tilde{a}} \cdot \frac{B}{\overline{W}} \cdot (\alpha_{LH})^{\frac{-1}{\rho_H}} \cdot \sum_{j=1}^{j'} L^j \cdot \left(\frac{\ell^j}{\ell^2}\right)^{\alpha_\ell}$. Given the definition of $\tilde{a}$ via (3.24) then, it can be seen that the limit (3.29), for any set of admissible values for the a_i's, exceeds N. Thus, for d large enough, the total generated population of households that work, when all of the available land in rings 2 to j' is allocated to housing these households, is greater than the given population. The part of the discussion for the other three sign combinations above on the possibility of having some or all of the available land allocated to housing households that do not work can be applied in total to this situation as well. Thus, a value can be found so that, if d equals or exceeds it, the total generated population of all households in the urban area will be larger than the given population, for any combination of signs for ρ_H and ρ. Therefore, for all price vectors satisfying case (i) we can choose $\alpha = (0,1,0)$. Hypothesis (H.2), for case (i), would then be satisfied.

Let us next consider price vectors associated with case (ii). Here we will concentrate on the excess supply of labor. In particular, we seek to find what happens to the demand for labor as d increases. To do so we analyze the bid land rent for CBD land by the traded good industry. Using the apportionment of d defined by the use of the terms, a_i, and doing a little rearranging, we may rewrite the bid land rent, (2.48), as

$$p_{LT}^{1} = \left[\left(\frac{1}{1 + \frac{a_I \cdot a_3 \cdot d}{a_R \cdot \max[a_2 \cdot d \cdot s_H^2, \bar{p}_{HA} \cdot \bar{s}_{HA}^2]}} \right)^{\frac{\rho_T}{1+\rho_T}} \cdot \left(\left(\frac{(C \cdot \bar{p}_T)^{\rho_T}}{\alpha_{LT}} \right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{WT}\ (a_1 \cdot d)^{\rho_T}}{\alpha_{LT}} \right)^{\frac{1}{1+\rho_T}} \right) - \left(\frac{\alpha_{KT} (\bar{p}_K)^{\rho_T}}{\alpha_{LT}} \right)^{\frac{1}{1+\rho_T}} \right]^{\frac{1+\rho_T}{\rho_T}} \tag{3.30}$$

where s_H^2 is a function of d (in particular, of $p_H^2 = a_2 \cdot d$).

When $\rho_T > 0$, given that $a_1 > 0$, it can be seen from (3.30) that the bid land rent falls to zero as d increases from low levels. It becomes zero at a finite value for d. For higher values, expression (3.5) will be operative and, by the convention adopted above, a bid land rent of zero would continue to be assigned. Thus, for d large enough the traded good industry will be outbid for CBD land. As a result, for d large enough, the demand for labor is zero. Suppose instead that $\rho_T < 0$. As d increases, since $a_1 > 0$, the bracketed expression in (3.30) approaches and can even become larger than the bracketed expression in condition (c.2). Thus, once again, for d large enough the traded good industry will be outbid for CBD land. Therefore, regardless of the sign of ρ_T, a finite value can be found such that if d equals or exceeds this value, the demand for labor must be zero. Since the supply of labor is always nonnegative, a value for d then can be chosen high enough so that the excess demand for labor is nonpositive. Thus, for such a value of d, hypothesis (H.2), for all price vectors associated with case (ii), can be satisfied by choosing $\alpha = (1,0,0)$.

Finally, consider price vectors associated with case (iii). Here the salient feature is that $a_3 > 0$. Generated property tax revenues, R_G, as defined in (3.15), clearly increase to infinity as τ does. Since τ is defined as $a_3 \cdot d$, generated property tax revenues will exceed the given amount of property tax revenues for large enough values of d. Thus, for such values of d, (H.2), for price vectors associated with case (iii), can be satisfied by letting $\alpha = (0,0,1)$.

Since all price vectors $p \in P$ with $p_W + p_H^2 + \tau = d$ are included in one of the three cases, we can now be assured that hypothesis (H.2) is satisfied for some $d > 0$. With both hypothesis of Theorem 1 satisfied, its conclusion is available to us. It tells us that there is a price vector $p^* \in P$ such that all of the components of some excess demand vector, $e^* \in E(p^*)$, associated with it are nonpositive and, for any component of e^* that is negative, the associated price is zero. It can be shown, however, that for our model all of the components of this excess demand vector must be zero.

Suppose that $p_W^* = 0$, where p_W^* is the wage rate component of p^*. With a zero wage rate no household residing anywhere in the city would choose to work. Thus, the supply of labor would be zero. Since $e^* \leqq 0$, then, the demand for labor under e^* must also be zero. Hence, the first component of e^*, the excess demand for labor, must be zero. Suppose now that $p_H^{2*} = 0$, where p_H^{2*} is the price of housing services in ring 2 component of p^*. As noted above, if $\rho < 0$ the bid price of housing (for households that work or remain at home) and so the housing bid land rent in all rings is zero. The generated population would then be zero. Condition (c.3), as noted above also, ensures that housing will be outbid by agriculture in all rings when $\rho > 0$ and the price of housing in ring 2 is zero. Thus, the generated population would be zero in this case as well. This would mean, however, that the given population exceeds the generated population, and so the second component of e^* would be positive. Since $e^* \leqq 0$, then, we must have $p_H^{2*} > 0$ and the generated population equal to the given population (i.e., the second component of e^* is zero). Finally suppose that $\tau^* = 0$, where τ^* is the third component of p^*. This implies that the nominal property tax rate is zero as well, and so that generated property tax revenues, as defined by (3.15), are zero. Thus, the given level of property tax revenues would exceed R_G under e^* and, as a result, the third component of e^* would be positive. Hence, we must have $\tau^* > 0$ with the third component of e^* equal to zero. Therefore, $e^* = 0$. All markets clear and the generated population equals the given population. The actual level of property tax revenues raised in this urban economy, however, would be less than the pre-specified amount of revenues to be raised if $p_H^{2*} \cdot s_H^{2*} < \bar{p}_{HA} \cdot \bar{s}_{HA}^2$, where s_H^{2*} is the supply of housing in ring 2 evaluated at p_H^{2*}. Then the actual level of property tax revenues raised in ring 2,

$a_R \cdot \left(\dfrac{\tau^*}{\bar{p}_{HA} \cdot \bar{s}_{HA}^2} \right) \cdot p_H^{2*} \cdot s_H^{2*}$, would be less than their surrogate in (3.15), τ^*. As discussed above, however, condition (c.4) guarantees that such a situation is not possible. Under (c.4) we cannot have $p_H^{2*} \cdot s_H^{2*} < \bar{p}_{HA} \cdot \bar{s}_{HA}^2$ with $e_1^* = e_3^* = 0$, since then the generated population would be less than the given population (i.e., $e_2^* > 0$).

Q.E.D.

This proof is valuable not merely because it demonstrates, under certain conditions on the parameters, the existence of equilibrium for the model but, as indicated above, it can be used to show that the Scarf algorithm (or the Merrill extension) can be employed to find approximations to the p^* and e^* of the theorem. This is precisely what we will do below in order to analyze the implications of certain tax policy changes. Thus, we get simultaneously, an existence proof and a method to compute an equilibrium (or at least an approximation to one).

There are several additional interesting results that can be obtained by using Theorem 2 and the analysis underlying it. Before we express and prove them, let us define three notions which will aid in the exposition. First, though, note that the term R_G^* in definition 2 is taken to be R_G as defined by (3.15), evaluated at p^* and the associated vector e^*.

Definition 1. A normal equilibrium is defined to be a price vector $p^* \varepsilon P$ and an excess demand vector $e^* \varepsilon E(p^*)$ such that $e^* = 0$ and $p_H^{2*} \cdot s_H^{2*} \geqq \bar{p}_{HA} \cdot \bar{s}_{HA}^2$.

Definition 2. A non-normal equilibrium is defined to be a price vector $p^* \varepsilon P$ and an excess demand vector $e^* \varepsilon E(p^*)$ such that $e_1^* = e_2^* = 0$, $R = R_G^* - \tau^* + a_R \cdot t^* \cdot p_H^{2*} \cdot s_H^{2*}$, and $p_H^{2*} \cdot s_H^{2*} < \bar{p}_{HA} \cdot \bar{s}_{HA}^2$.

Definition 3. A pseudo equilibrium is defined to be a price vector $p^* \varepsilon P$ and an excess demand vector $e^* \varepsilon E(p^*)$ such that $e^* = 0$ and $p_H^{2*} \cdot s_H^{2*} < \bar{p}_{HA} \cdot \bar{s}_{HA}^2$.

Non-normal equilibria are distinguished from normal equilibria by city size. Under a non-normal equilibrium no households locate beyond the first two rings and at least some of the endogenous land in ring 2 must be allocated to agriculture. Under a normal equilibrium households either locate beyond ring 2 or, if restricted to the first two rings, are allocated, for housing, all of the endogenous land in ring 2. The equilibrium of the conclusion of Theorem 2, by these definitions, is a normal equilibrium. Were it not for condition (c.4), though, the conclusion of the theorem would have to be modified to include the possibility that p^* and e^* could represent a pseudo equilibrium.

The existence of an equilibrium for the urban economy can be proven more easily, and with fewer conditions, in the absence of a government sector or, for the model above with the restriction that the amount of tax revenue to be raised is zero. In particular, we can state and prove the following corollary to Theorem 2.

<u>Corollary 2.1</u>. Given the model described above, exclusive of a government sector or with $R = 0$, and a parameterization that satisfies conditions(c.1)-(c.3), an equilibrium (with no excess supplies) exists.

<u>Proof</u>. Let $p = (p_W, p_H^2)$ and $E(p)$ be restricted to its first two components under the full model. The proof of Theorem 2, exclusive of any reference to taxes and tax rates, carries over completely to this situation. Thus, an equilibrium (with no excess supplies) exists for the model above with no government sector. If the full model is retained, with the exception that $R = 0$, then an equilibrium (with no excess supplies) also exists. If (p_W^*, p_H^{2*}) is the equilibrium price vector in the model with no government sector, then $(p_W^*, p_H^{2*}, 0)$ can serve as an equilibrium price vector for the full model with $R = 0$.

Q.E.D.

Note that the equilibrium in the conclusion of Corollary 2.1 may be normal or non-normal. Note also that condition (c.4) is no longer needed. Since, in the proof of the above corollary, the proof of Theorem 2 was used directly to show existence for the model with no government sector, we can be assured that the algorithm mentioned can be used to approximate the equilibrium for this model. The existence of equilibrium for the full model with R = 0, however, followed not from the proof of Theorem 2 directly, but rather from the existence of equilibrium for the model with no government sector. An alternative proof for the full model case, though, which uses the proof of Theorem 2 directly could have been presented almost as easily. The proof of Theorem 2 carries over to the full model with R = 0, except that now we must have $\tau^* = 0$ instead of $\tau^* > 0$. If $\tau^* > 0$, then $R_G^* > 0$ and so $R - R_G^* = -R_G^* < 0$. This would violate the conclusion of Theorem 1 as applied to our model. Of course, it would likely be more computationally efficient to use instead the model with no government sector. It should also be noted that the proof of Theorem 2 can be used, in essentially the same way as it was used in the proof of Corollary 2.1, to show that an equilibrium (normal or non-normal) must exist for the model, as described originally, but with a finite non-zero fixed property tax rate and no property tax revenue constraint. This can be stated as another corollary.

Corollary 2.2. Let the model be as described originally, except that the nominal property tax rate, $t \geq 0$, is fixed and there is no revenue requirement. Then, given a parameterization that satisfies conditions (c.1)-(c.3), an equilibrium (with no excess supplies) exists.

<u>Proof</u>. Let $p = (p_w, p_H^2)$ and E(p) be restricted to its first two components under the full model. The proof of Theorem 2, without reference to the third component of E in the full model but with reference to the fixed tax rate as it relates to the labor and housing components of E can clearly be applied here.

Q.E.D.

Existence of equilibrium is also obtained more easily if the model is weakened in another direction. Suppose now that production of the traded good is not modelled and, as in some urban residential location models, households receive all of their income exogenously but nonetheless commute to the point at the center of the region for employment. What was the original CBD can now be divided into a number of residential rings. Suppose also that there is still a property tax and a certain amount of revenues to be raised. For this restricted model we have the following result.

<u>Corollary 2.3</u>. Given the amended model just described, and a parameterization that satisfies conditions (c.3) and (c.4), a normal equilibrium exists.

<u>Proof</u>. Let $p = (p_H^2, \tau)$ and E(p) be restricted to the second and third components under the full model. Apply the proof of Theorem 2 without reference to production of the traded good and the wage rate.

Q.E.D.

Of course, this equilibrium can be approximated by the above-mentioned computational routine.

Weakening the model still further by ignoring both production in the (non-housing) business sector, and the government sector as it relates to revenue requirements, gives us what is essentially a generalization of the "Muth model" found in Arnott and MacKinnon (1977a). Such a model allows us to cut down even further on the number of conditions on the parameters required to establish the existence of equilibrium and a means of computing such an equilibrium. In particular, we have the following result.

Corollary 2.4. Let the model be that of Corollary 2.3, except that now there is no property tax revenue requirement and the nominal property tax rate, $t \geqq 0$, is fixed. Then, given a parameterization that satisfies condition (c.3), an equilibrium (with no excess supplies) exists.

Proof. Let $p = p_H^2$ and E(p) be restricted to the second component under the full model. The proof of Theorem 2 with no reference to the business sector and a property tax revenue constraint, but with reference to a fixed (possibly zero) nominal property tax rate, obviously can be applied here.

Q.E.D.

Note that the only condition on the parameters required here is the rather innocuous condition (c.3). In structure the Muth model in Arnott and MacKinnon (1977a) differs from this version of the base model in only one way. It differs in its treatment of the time cost of travel. Leisure does not enter the utility function there. Time costs, instead, are reflected in the budget constraint, as a constant per mile loss in money income for travel to and from the point CBD. While our way of treating the time problem seems superior, our model nonetheless could be amended to match the structure of the Arnott and MacKinnon model in this regard. None of our results would be altered. Indeed it would not even necessitate any changes in the proofs of our results. The model of Corollary 2.4 (amended by relegating time costs to the budget constraint) is a generalization of the Muth model in two ways. First, city size in Arnott and MacKinnon (1977a) is arbitrarily bounded from above by fixing the number of rings to be dealt with, although the actual radius of urban settlement can vary up to this bound. There is no bound on urban development in our model. Of course, no problem arises with parameterizations that lead to city sizes that are smaller than the bound. Second, Cobb-Douglas utility and production functions are used in the Muth model, whereas general CES functions are used here. The Cobb-Douglas function, of course, is just a special case (in the limit) of the general CES function. Our analysis would actually become easier with Cobb-Douglas forms. In fact, use of a Cobb-Douglas utility function eliminates the need for condition (c.3).[25] Arnott and MacKinnon (1977a) do not address the issue of existence in their paper. The analysis presented here, though, shows clearly than an equilibrium for their Muth model exists and can be approximated by a fixed point algorithm.

Returning to the full model, a few more results can be obtained.

It was stated above that satisfaction of condition (c.4) depends on the level of property tax revenues to be raised and/or exogenous income. Suppose we have a parameterization that satisfies condition (c.1)-(c.4) and so yields a normal equilibrium. We may ask how much we may presume about the existence of equilibrium for parameterizations with different levels of tax revenues to be raised and/or exogenous income. This leads us to the following result.

Corollary 2.5. Let the model be as described originally, with condition (c.1)-(c.4) satisfied for some parameterization with $R = \overline{R}$ and $M = \overline{M}$. Then a normal equilibrium exists for any parameterization with $0 \leqq R \leqq \overline{R}$ and $M \geqq \overline{M}$, and the same values that are given in the initial parameterization for all other parameters.

Proof. Conditions (c.1) and (c.2) are independent of R and M and so will be satisfied by all parameterizations alluded to in the conclusion. The term $\frac{M-c\cdot u^1}{M-c\cdot u^2}$ decreases with increases in M since $u^1 < u^2$. Thus, condition (c.3) is satisfied for $M > \overline{M}$ if it is satisfied at $M = \overline{M}$. Increases in M reduce $\hat{N}_G^1 + \hat{N}_G^2$ in two ways. First, increases in exogenous income, other things equal, increase household demands, x_H^1 and x_H^2. Second, changes in M affect $\hat{N}_G^1$ through changes in $\overline{p}_{HA}^1$.

The term $\frac{(M + \tilde{p}_w\cdot\overline{W}-c\cdot u^1)(T-\overline{W}-v\ u^1)^{\alpha_\ell}}{M\cdot T^{\alpha_\ell}}$ decreases with increases in M.

Decreases in this value, in turn, reduces $\overline{p}_{HA}^1$. The reduction in price reduces supply, s_H^1, and increases demand, x_H^1. Thus, $\hat{N}_G^1$ is lowered. Given values of the other parameters, then, if condition (c.4) is satisfied for some value of M, it will be satisfied for all higher values of M. The pre-specified level of tax revenues, R, has an impact on condition (c.4) only through the tax rate term, $\hat{t}$. Reductions in R reduce $\hat{t}$. A decrease in the tax rate used will increase household demands, x_H^1 and x_H^2, directly. The term $\hat{t}$ can also affect $\bar{p}_{HA}^1$. If $\rho < 0$, then reductions in $\hat{t}$ decrease $\bar{p}_{HA}^1$. The lower price, in turn, lowers supply, s_H^1, and increases demand, x_H^1, thereby lowering $\hat{N}_G^1$. If $\rho > 0$ then changes in $\hat{t}$ have no effect on $\bar{p}_{HA}^1$. In sum, reductions in R lower $\hat{N}_G^1 + \hat{N}_G^2$. Thus, given values for the other parameters, if (c.4) is satisfied for some value of R, it will also be satisfied for lower values of R. Thus, conditions (c.1)-(c.4) will be satisfied for $0 \leqq R \leqq \bar{R}$ and $M \geqq \bar{M}$. The proof of Theorem 2 can then be applied for the cases where $R > 0$. Corollary 2.1 shows that an equilibrium exists for $R = 0$. It must be normal since, here, (c.4) is satisfied at $R = 0$.

Q.E.D.

Thus, if we have a parameterization that satisfies the four conditions, we know not only that, for this parameterization, a normal equilibrium exists and can be approximated by the algorithm, but also that the same can be said if we lower the level of tax revenues to be raised and/or increase exogenous income.

The exogeneity of R suggests that genuine (i.e., normal and non-normal) equilibria may not exist for some parameterizations. There seems to be no a priori reason to suppose that, for a given population size, the wage and tax rates can always adjust to allow the urban economy to meet the revenue requirements of government, when those requirements are arbitrarily selected. It would seem that there may not exist a genuine equilibrium if R is chosen too high. This is certainly the case for a restricted version of the model. Suppose that (non-housing) business sector production is not modelled and households receive all of their income exogenously. Suppose also that the required amount of revenues, R, exceeds or equals aggregate household income. Then, a genuine equilibrium cannot exist. The city is simply too poor to support property tax revenue needs. In the full model, however, the issue becomes muddled by the fact that households can receive endogenous income and the business sector may contribute to property tax payments. Although we cannot prove it, it does not seem unreasonable to suppose that equilibrium may not exist for a parameterization that has, in some sense, tax revenue requirements that are too high.

Our last result concerns pseudo equilibria, which, as noted above, we cannot rule out as a possibility for a solution of the computational routine in the limit, if condition (c.4) is not satisfied. Even though we do not demonstrate the existence of a pseudo equilibrium for some parameterization, we may ask how might we change parameter values to get a genuine equilibrium should the algorithm, in the limit, yield a pseudo equilibrium for some parameterization. Of course, we may attempt to change parameter values so as to satisfy (c.4). This can certainly be done by simply raising exogenous income enough. Let us concentrate, though, on the tax revenues to be raised, R. We might satisfy (c.4) by lowering R enough. Then, assuming conditions (c.1)-(c.3) are satisfied, we could obtain a normal equilibrium for a particular value of R and all values lower than it. For some parameterizations, however, (c.4) will not be satisfied (just choose M low enough) even if R = 0. Can equilibria be found for some value of R in these cases? The problem with pseudo equilibria is that the tax revenues actually raised in the second ring in these cases are less than what is meant to represent them in R_G^*--the variable τ^*. Since the level of tax revenues to be raised is a parameter, we might think of lowering it to the level of the tax revenues actually raised in these cases. This leads to the following result.

Corollary 2.6. Let the model be as described originally, with conditions (c.1)-(c.3) satisfied for a particular parameterization. Let R be the given level of tax revenues to be raised under this parameterization. Suppose that, for this parameterization, the computational routine, in the limit, terminates at a price vector p^* and excess demand vector e^* that represents a pseudo equilibrium. Then a non-normal equilibrium exists if the level of tax revenues to be raised is changed to $\tilde{R} = \hat{R} - a_R \cdot t^* \cdot (\overline{p}_{HA} \cdot \overline{s}_{HA}^2 - p_H^{2*} \cdot s_H^{2*})$.

Proof. At the price vector $p^* = (p_W^*, p_H^{2*}, \tau^*)$ define R_G^{1*} to be R_G, as given by (3.15) evaluated at p^* and the associated vector e^*, minus τ^*. Then, at p^*, we have $\hat{R} = \tau^* + R_G^{1*}$. Actual tax revenues raised, however, are $a_R \cdot t^* \cdot p_H^{2*} \cdot s_H^{2*} + R_G^{1*}$. In this situation, by the definition of t given in (3.14), we have $\tau^* = a_R \cdot t^* \cdot \overline{p}_{HA} \cdot \overline{s}_{HA}^2$. Thus, at the price vector p^*, actual tax revenues raised will be equal to the required level if that amount is $\tilde{R}$. Since p^* and the associated vector e^*, with $R = \hat{R}$, represents a pseudo equilibrium, then, at p^*, even if $R = \hat{R}$, there is an excess demand vector, $e^* \varepsilon E(p^*)$, with the generated population equal to the given population and labor demand equal to labor supply.

Q.E.D.

Thus, by lowering revenues to be raised by just the right amount we can show the existence of a non-normal equilibrium. Note that $\tilde{R} > 0$. This can be seen by observing that

$$\tilde{R} = \hat{R} - a_R \cdot t^* (\bar{p}_{HA} \cdot \bar{s}_{HA}^2 - p_H^{2*} \cdot s_H^{2*}) =$$

$$= \tau^* + R_G^{1*} - \tau^* + \frac{\tau^*}{\bar{p}_{HA} \, \bar{s}_{HA}^2} \cdot p_H^{2*} \cdot s_H^{2*} = R_G^{1*} + \frac{\tau^*}{\bar{p}_{HA} \cdot \bar{s}_{HA}^2} \cdot p_H^{2*} \cdot s_H^{2*} .$$

This last expression must be positive since $\tau^* > 0$ and, if no land is allocated to housing in ring 2 (i.e., $s_H^{2*} = 0$), $R_G^{1*} > 0$ because then, all of the given population must be housed in ring 1. Unfortunately, though, the equilibrium we are discussing is not a solution that the computational routine would find. At p^*, generated tax revenues are $\hat{R} > \tilde{R}$. If the algorithm were run with $R = \tilde{R}$, then it would have to terminate, in the limit, at a different price vector. In fact, assuming that (c.1)-(c.3) are satisfied, the computational routine used will never terminate, in the limit, at a non-normal equilibrium. If (c.1)-(c.3) are not violated, then the algorithm, in the limit, will terminate at a price vector and associated excess demand vector where $R_G^* = R$. Suppose that the terminal price vector

p^* yields a non-normal equilibrium. At a non-normal equilibrium, however, $p_H^{2*} \cdot s_H^{2*} < \bar{p}_{HA} \cdot \bar{s}_{HA}^2$. Hence, the actual tax revenues generated at p^* would be less than the given level of revenues thereby violating the definition of a non-normal equilibrium. This weakness of the framework, however, is greatly mitigated by the fact that non-normal equilibria are exceedingly uninteresting. At such an equilibrium all economic activity in the urban area occurs in the CBD, or in the CBD and the first, rather small, residential ring surrounding it. Clearly, no realistic equilibrium for the urban area would be characterized in this way. What happens, though, in the situation considered in Corollary 2.6 when the computational routine is run with the lower level of revenues, $\tilde{R}$? Given the assumption that (c.1)-(c.3) are satisfied and the proof of Theorem 2, the outcome, in the limit, must be either a normal equilibrium or another pseudo equilibrium. The possibility that a normal equilibrium could result raises the spectre of multiple equilibria. Unfortunately, we cannot rule out the possibility of multiple equilibria of any kind (multiple non-normal equilibria, multiple normal equilibria, and combinations of normal and non-normal equilibria). As has been done in some studies that use these fixed point algorithms, to test for multiple equilibria we will start the algorithm, for a given parameterization, at widely differing points on the price simplex. Although this does not prove uniqueness of a normal equilibrium that has been found by the algorithm, it does seem to provide some evidence that other normal equilibria do not exist for the given parameterization. It does not, however, provide any evidence that non-normal equilibria do not exist for the given parameterization since, as noted, the algorithm cannot terminate, in the limit, at such equilibria. If running the algorithm with $R = \tilde{R}$ results in another pseudo equilibrium then the process can be repeated. The level of given tax revenues can be lowered again and the existence of another non-normal equilibrium established. This could, in theory, continue indefinitely. Thus, it can be stated if a pseudo equilibrium is found by the computational routine for some parameterization, then multiple equilibria exist and/or an infinite number of non-normal equilibria exist for the set of parameterizations that differ from the given parameterization only by virtue of having a lower value of R.

III. The Burden of the Error of Approximation

The solution price vector p^* to which we have been referring will be found, it has been stated, at termination of the algorithm in the limit. By this it is meant that the price vectors at which the algorithm terminates for given grid sizes tend toward p^* as the grid becomes infinitely dense. In practice, the grid size in use when the routine terminates must be finite. Thus, in general, the price vector found at termination will be only an approximation to the true equilibrium price vector. Of course, the approximate equilibrium price vector can be made arbitrarily close to the true equilibrium price vector by choosing a dense enough grid. In general, though, the components of the excess demand vector associated with the terminal price vector will be only approximately zero. In particular, the difference between the final generated population and the given population may be non-zero.

One may object on aesthetic grounds to a characterization of a solution to the model in which not all of the given number of households or more than the given number of households are actually housed in the urban area. It might be better at termination to house exactly the number of households given in the parameterization and, as a result, shift the error of approximation for the second component of E(p) to one or some of the markets in the model. Errors of approximation could then be expressed in terms of excess demands or suboptimal behavior on the part of agents in markets.

There are several ways, that we can imagine, to do this. We might, for instance, if too few households are generated, somehow fit the difference uniformly across rings by placing an equal number in each ring. Although this seems, at first glance, to be a symmetric approach, in a meaningful sense it really is not. It amounts to treating different rings differently since population densities vary from ring to ring. A better approach to take, it seems, would be to change the population of each ring by the same percentage of the population generated by the algorithm. In particular, let N_G^* represent the generated population at termination of the algorithm. Regardless of whether the final generated population exceeds or is less than the given population, we define the percentage change in the population of each ring in the set of rings occupied in the final approximation to be

$$\eta_G^* = \frac{N-N_G^*}{N_G^*} \tag{3.31}$$

The population of an occupied ring now becomes $(1 + n_G^*) \cdot N_G^{j*}$, where N_G^{j*} is the generated population of ring j at the final approximation. In this way the population of each ring is changed by the same proportion, and the number of households finally housed in the urban area equals the given number.

We must ask, though, just how we are to accomplish this addition or subtraction of households from rings. In this regard we must also consider what effects the process will have on the markets of the model. We could shift the population error to the housing markets by increasing the aggregate demand for housing in each ring by the same percentage referred to above, while keeping household demands the same. This would preserve the optimization processes of households. It suffers, however, from the same sort of interpretational weakness as does the final approximation. Since nothing was done to alter supply in a ring, it is not possible to house more households than is done in the final approximation, when too few households are generated there. The best approach to take seems to be the following. Change the number of households in a ring, not by changing aggregate demand for housing in a ring, but rather by changing housing supply in each ring. The supplies in each ring are to increase or decrease by the same proportion as does the population. This places the burden of the error associated with the generated population on housing producers. We can still preserve land market equilibrium by having the increased (decreased) supply arise not from housing producers using more (less) land than is allocated to housing in the final approximation, but rather by housing producers using more (less) capital than is optimal at the final price vector. Housing producers, of course, would then be earning negative economic profits. The story that would have to be told here to justify an approximation of this type is that actual rates of return to housing producers in an acceptable approximate solution are less than a normal rate of return, by an amount small enough that housing producers would not bother to alter their production plans.

In the results to be presented below, population densities and capital/land ratios will be reported. They should be adjusted, though, to account for the changes mentioned. It is easy to calculate the new population densities. They are simply the densities under the final approximation, increased or decreased by the proportion given in (3.31). We can obtain the new capital/land ratios via the housing production function using the amount of land allocated to housing in a ring under the final approximation, and the new supply of housing for that ring. This process of adding and subtracting households also has effects elsewhere in the model. In particular, labor supply and tax revenues generated would change. Thus, appropriate adjustments to the errors in the first and third components of E should be made. It is a simple matter to do this for tax revenues generated. Housing property tax revenues in ring j can be expressed as $a_R \cdot t^* \cdot p_H^{j*} \cdot s_H^{j*}$, and the adjustment that is to be made to housing supply has already been noted. Matters are a little more complicated with labor supply because of the option that households have to refrain from work. If all households in a ring, under the final approximation, choose to work, then all of the households added to or subtracted from the ring, as the case may be, are used for the recalculation of labor supply. If all of the households in a ring, under the final approximation, choose to be unemployed, then the change in population in that ring is to have no effect on labor supply. Finally, if, under the final approximation, some households in a ring choose to work while others in the ring do not, then labor supply is adjusted in accordance with the proportion of the total population of the ring, under the final approximation, that choose to work. That is, the potential labor services of a proportion of the households added to (subtracted from) the ring is added to (subtracted from) aggregate labor supply, with this proportion being the proportion of the population of the ring, under the final approximation, that work.

As has been noted, the burden of the population error in the final approximation can be shifted entirely. For the most part, it is shifted onto the housing producer profit maximization process. But just what happens to profits when this is done, and is it measurable? We may define a rate of return to housing producers as follows. Let p_K be the purchase price of a physical unit of capital and r be the annual interest cost of (or normal rate of return on) capital. The relationship of these new variables with the annual rental price of a unit of capital, $\overline{p}_K$, given in the model above is $\overline{p}_K = r \cdot p_K$. We may then define the actual rate of return received by housing producers for housing in ring j on a dollar's worth of capital expenditures to be

$$r_a^j = \frac{p_H^j \cdot s_H^j(K_H^j, L_H^j) - p_{LH}^j \cdot L_H^j}{p_K \cdot K_H^j} \qquad (3.32)$$

If the optimal amount of capital used in ring j by housing producers for the final approximate price vector, say K_H^{j*}, is used, then the normal rate of return, r, will be earned by producers. Using more or less capital will result in a value of r_a that is less that r. To measure the error due to using non-optimal amounts of capital in housing production, which would be done to house exactly the given number of households, we need not specify values for the new variables, p_K and r. We express the error as the percentage decline in the actual rate of return from the level of the normal rate of return as follows:

$$1-\frac{r_a^j}{r} = \frac{p_H^j[s_H^j(K_H^{j*},L_H^j)\cdot\hat{K}_H^j - s_H^j(\hat{K}_H^j,L_H^j)\cdot K_H^{j*}] - p_{LH}^j\cdot L_H^j\cdot[\hat{K}_H^j - K_H^{j*}]}{[p_H^j\cdot s_H^j(K_H^{j*},L_H^j) - p_{LH}^j\cdot L_H^j]\cdot\hat{K}_H^j} \tag{3.33}$$

where $\hat{K}_H^j$ is the amount of capital that must be used in housing production in ring j, to yield the supply of housing in that ring required to house the given number of households in the urban area under the procedure to shift errors mentioned above, and all other variables assume the values given for them at the solution obtained at termination of the algorithm.

Acceptable errors can then be defined in terms of this percentage decline in the rate of return. Note finally that the sub-optimal rates of return earned by housing producers will, in general, vary from ring to ring.

CHAPTER 3

FOOTNOTES

[1] The elasticity of factor substitution in housing production is $\sigma_H = \frac{1}{1+\rho_H}$. A positive ρ_H implies that the elasticity of substitution is less than 1, while a negative ρ_H implies that it is greater than 1.

[2] It can be positive if, for instance, $\rho_H = 1$. Then if, say,

$$\left(\frac{(Bp_{LH}^{j})^{\rho_H}}{\alpha_{LH}}\right)^{\frac{1}{1+\rho_H}} = 5 \text{ and } \left(\frac{\alpha_{KH}(\overline{p}_K)^{\rho_H}}{\alpha_{LH}}\right)^{\frac{1}{1+\rho_H}} = 10, \text{ expression (2.46) would}$$

yield $p_{LH}^{j} = 25$.

[3] Inspection of (2.43) reveals that the bid price of housing services declines with distance from the CBD. Thus, if the housing price in a ring is low enough to yield a zero bid land rent in the ring, then we know that this must also be true for all rings farther out. Land is not allocated to housing, though, unless the bid rent by housing producers is at least as large as the agricultural rent. We should not, therefore, continue to calculate bid prices for rings farther out after one is found which yields a land rent which is equal to or less than the agricultural rent.

[4] As will be indicated below, in any attempt to use the algorithm, the size of the price-tax rate simplex over which the program searches must be pre-specified. This can be set at will but, as described below, it must be large enough to allow equilibrium prices to be found. Thus, given a specification of the size of the simplex, the bid price of housing services cannot exceed a certain amount. It may be that a size can be chosen which is large enough to yield an approximate equilibrium and yet is small enough so that (3.1) is never non-positive. In this case the bid rent function need not be bounded. In general, though, an adjustment should be made.

[5] As can be seen from the existence proof presented below, choosing an upper bound on housing bid land rents will not prevent an approximate equilibrium from being found if the bound can be as large as

desired. In principle we could keep increasing the bound until the equilibrium is found, but, as mentioned above, there is a limit to how high the bound can be when the computer is used. The existence proof provided below presumes that this machine constraint is not binding.

[6] Just as the computer is limited in its ability to deal with extremely large, in absolute value, numbers it is also limited in dealing with very small, in absolute value, non-zero numbers. If the value of a variable is positive but sufficiently small the computer will treat it as though it were zero.

[7] It has the additional effect of placing an upper bound on the demand for labor. The importance of this, aside from allowing us to avoid the problem of dealing with infinity on the computer, will become apparent in the next section when the issue of the existence of equilibrium is raised.

[8] The bound varies with the tax rate as can be seen by noting that it appears directly as an argument in expression (3.8) and indirectly through its influence on p_w^A. Actually, we can state unambiguously the direction of change in $\hat{p}_{LH}^1$ as the tax rate changes. Inspection of (2.48), with the substitution of $\bar{p}_A$ for p_{LT}^1, reveals that $\frac{p_w^A}{(1+a_I \cdot t)}$ appears separately as an argument. The only other place where it has an effect is in the first component of the bracketed expression in (2.48). An increase in t will raise that component. To offset this effect, so as to keep the bid rent constant at $\bar{p}_A$, the term $\frac{p_w^A}{(1+a_I \cdot t)}$ must be decreased. Taking note of this, an inspection of (3.8) shows that $\hat{p}_{LT}^1$ must decrease with increases in the tax rate. There is a limit, though, to how low $\hat{p}_{LT}^1$ can be, for a given size of the price-tax rate simplex, since an upper bound on the tax rate exists. In any case, the level of b can be raised to make $\hat{p}_{LH}^1$, for any given size of the simplex, as large as desired.

[9] Arguing heuristically, the agricultural rent should be relatively low and, since the bid land rent is inversely related in (2.48) to the wage rate, the wage rate which yields the agricultural rent, for a given tax rate, should be relatively large. For reasonable parameterizations, i.e., those that yield equilibrium CBD land rents for traded good producers well in excess of the agricultural rent, it would seem that the labor demand derived using the agricultural land rent and the wage rate associated with it is not likely to be very large for many

values for the tax rate. In such cases p^1_{LT} would generally exceed $\overline{p}_A$, and p_w would be less than p^A_w. Whenever it should happen, though, that $p_w > p^A_w$ for some tax rate, no CBD land would be allocated to the traded good industry. To be more precise, though, the whole issue can be avoided with certainty by choosing a large enough value for b. As noted above, $\frac{p^A_w}{(1+a_I \cdot t)}$ decreases as t increases. Given a size of the simplex, however, t cannot exceed a certain level; say $\overline{t}$. We can choose b large enough, then, so that $\hat{p}^1_{LT}$ evaluated at $\overline{t}$ is greater than $\overline{p}_A$. If this is done then $\hat{p}^1_{LT}$ will exceed $\overline{p}_A$, and $\hat{p}_w$ will be less than p^A_w for any admissible tax rate.

[10] For the basic model this will be true even though $p = (p^2_w, p^2_H, t)$ contains three components. Use of the procedure to find a solution that is indicated by the proof of Theorem 1, to be stated below, requires that a fourth dimension be added. This extra "price" is a dummy variable with no economic interpretation. It also does not correspond to the artificial dimension introduced as part of the Merrill extension mentioned above.

[11] More precisely, the bid prices p^j_{Hw} and p^j_{HNw} are chosen so that the households would attain the same level of utility, V^2, under either option. Then, if the actual price of housing services is the higher of the two bid prices, the option with the lower bid price would yield a level of utility less than V^2.

[12] For simplicity of argument, we may presume that cases of indifference between working and not working are treated as though the optimal decision is to work. The result that (c.3) implies that the housing sector is outbid for land in the CBD when $p^2_H = 0$ and $\rho > 0$ is not changed by explicitly considering cases of indifference.

[13] Differentiating $\frac{Y^1}{Y^2}$ with respect to p_w, we obtain $\frac{d\left(\frac{Y^1}{Y^2}\right)}{dp_w} = \frac{c\overline{W}(u^1 - u^2)}{(Y^2)^2}$. This is negative since $u^1 < u^2$.

[14] At greater distances net income falls to zero. Noting (2.43), we can see that the bid price for housing, and so the bid land rent by the housing industry, falls to zero as income does. Thus, the housing bid land rent gradient for households that work must eventually fall below any positive value.

[15] One may wonder if this conclusion need be true since it might seem that the infinite series $\sum_{j=\tilde{j}}^{\infty} N_G^{Nwj}$, where N_G^{Nwj} was defined above and $\tilde{j}$ is the index of the first ring for which housing for these household types outbids all other uses for land, can be convergent. This, however, does not occur. The term N_G^{Nwj} can be calculated as the ratio of the supply of housing in ring j allocated to households who do not work divided by the demand for housing in the ring by such a household. This supply of housing, however, is, for a given price vector, proportional to the amount of land in the ring allocated to housing for this group. In the situation considered, all of the endogenous land, L^j, is allocated to housing this household type. Thus, our sum can be written, for a given p, as the product of a constant term and the sum of the amounts of endogenous land for all rings from $\tilde{j}$ on, $\sum_{j=\tilde{j}}^{\infty} L^j$. It can be easily shown, however, that the areas of concentric rings of a given width increase to infinity as the radius of the rings goes to infinity. In particular, in our context, let the radius of some ring be u and the width of any ring be ω. Then, the land area available for endogenous use in the next ring, assuming say that one-third of all land in a ring is available for endogenous use, is $\frac{\pi}{3}\cdot((u+\omega)^2 - u^2) = \frac{\pi}{3}(2u\omega + \omega^2)$. Clearly this area goes to infinity as u does. Therefore, $\sum_{j=\tilde{j}}^{\infty} L^j = \infty$, and so

$$\sum_{j=\tilde{j}}^{\infty} N_G^{Nwj} = \infty.$$

[16] One may question whether this is valid when $p_{HNw} = \overline{p}_{HA} = p_H^2$. In such cases, households who choose to be unemployed can be generated and housed in rings beyond ring 2. Only housing and agriculture, however, can use endogenous land in ring 2 and beyond. Given then the convention referred to above, by which the total generated population of households who do not work are housed, given their demand for housing, as close together as possible starting at ring $\hat{j}$ (ring 2 in this case) if some of the available land in ring $\hat{j}$ is not allocated

to other non-agricultural uses, or at $\hat{j}+1$ if all of the available land in ring $\hat{j}$ is allocated to other non-agricultural uses, housing extends beyond ring 2 only if all of the endogenous land in ring 2 is used for housing. But then $p_H^2 \cdot s_H^2 = \bar{p}_{HA} \cdot \bar{s}_{HA}^2$.

[17] A correspondence $E:P \to V$ can be defined to be upper semi-continuous in the following manner. Suppose $p^1, p^2, \ldots, p^k, \ldots$ is a sequence of points in P which converges to a point p^o in $P(p^k \to p^o)$. Let $e' \in E(p^1), e^2 \in E(p^2), \ldots, e^k \in E(p^k), \ldots$ be a sequence that converges to a point e $(e^k \to e)$. Then E is upper semicontinuous if $e \in E(p)$.

[18] If $p_W^o \neq 0$ then $\hat{W} = W_T(p^o)$. If, however, $p_W^o = 0$ then by (3.15), $\hat{W} \in W_T(p^o)$ (i.e., labor demand is a set of points, one of which is $\hat{W}$).

[19] The situation where p^o is such that (2.52) yields W when $L_T^1 = L^1$ will not be explicitly considered, in this case, since the upper semicontinuity result would then follow straightforwardly from combining the two approaches (i.e., the analyses for the two distinct regions of price space).

[20] For a given d, p_H^2 and τ (and so t) are bounded from above. Given the functional forms posited, this implies that the generated population of households that work in a ring is bounded from above. The number of rings with households that work must be finite since the bid land rent gradient for housing such households must fall below the agricultural land rental at some finite distance. Thus, the generated population of households that work is bounded.

[21] Both $\hat{p}_{LT}^1$ and $\hat{p}_W$ are defined and produce an upper bound for labor for a given finite t. Since τ is bounded for a given d, the tax rate will be bounded as well. Thus, the upper bound on labor would be the largest of the upper bounds for the set of admissible values of t.

[22] When $\rho_T < 0$ the bid land rent is bounded by the highest value that can be obtained for $\hat{p}_{LT}^1$, for the given value of d. When $\rho_T > 0$ the bid land rent is bounded by the value of p_{LT}^1 obtained after substituting zero for p_W and t in (2.48). From (2.53) it can be seen that capital demand is bounded from above if the bid land rent for the traded good industry is bounded from above.

[23] Households residing in a ring, where commuting time plus work time equals the total amount of time available for commuting, working, and (say, non-sleep) leisure on work days, might still be willing to work. So long as the number of work days in a year is less than the total number of days in the year, households would find it optimal to work if the wage rate were high enough. They could enjoy positive amounts of leisure (and so non-zero utility) from their non-work days.

[24] This statement is valid, of course, only if we presume that d is large enough so that $\frac{\tilde{p}_W}{d} < 1$.

25. If the utility of a household residing in ring j is given by $U(x_H^j, x_T^j) = A \cdot (x_H^j)^{\alpha} (x_T^j)^{\beta}$ with $\alpha + \beta = 1$, then the bid price of housing services in ring j would be, $p_H^j = p_H^2 \left(\frac{Y^j}{Y^2}\right)^{\frac{1}{\alpha}}$. Thus, if $p_H^2 = 0$ then we must have $p_H^1 = 0$.

CHAPTER 4

EXTENSIONS TO THE BASE MODEL

Taxing business property in the urban area at a higher effective tax rate, holding revenues constant, appears to give obvious direct benefits to residents in the model. The residential effective tax rate, it would seem, should fall. There are, however, in reality opposing effects on the welfare of households, one of which has found expression in the base model. Higher effective tax rates on commercial-industrial property should lead to reduced demands for labor and so lower wage rates. Production for a business sector (the traded good industry) that uses labor as an input has been modelled. Thus, the effect of changes in tax structure on labor income has in some way been captured. One other potential major effect on household welfare, however, has not. Changes in relative effective tax rates might cause price changes for some non-housing consumer commodities. To allow for this, in section I we extend the base model so that it includes what may be called a local good. It is a commodity that may be consumed by households and produced in the urban area. Although this same commodity[1] may be produced elsewhere, it is not imported to the urban area under consideration. Neither is any of the local good that is produced in the urban area exported to other regions. Thus, the relevant market is the urban area, and so, unlike the traded good, its price can change in response to changes in business tax rates in the given urban area. The modifications that must be made to accomodate incorporation of this new commodity in the extended model are presented in section I. Section II considers an additional important extension. Neither the base model nor the model which includes a local good can tell us anything about distributional issues. To address such concerns, in section II multiple household groups are added to the framework. The household groups may differ by preferences and/or income.[2] The differences in income may arise because of differences in exogenous household income and/or differences in endogenous labor income. To yield labor incomes that vary by household group, multiple labor types are incorporated in the production technologies of the traded and local goods. Finally, section III introduces and discusses the welfare measures that are to be used in the numerical simulations.

I. A Model with a Local Good

To extend the base model in a desired direction we add a commodity which can be consumed and produced in the urban area under consideration and has a price that is endogenous. To ensure this we assume that the commodity cannot be imported to or exported from the city. Thus, market clearing for the commodity involves equating local demand and local supply. The price of the local good will therefore be sensitive to changes in local tax policy.

A. Structure of the New Model

Production of the local good is to be thought of as a commercial activity, whereas production of the traded good is now considered to be an industrial activity. This allows us to make a three-way classification of property in the model which mirrors the three major categories found in practice - commercial industrial, and residential. This classification is made operative in the model by adding an assessment/sales ratio for commercial property, a_C, which may differ from the assessment/sales ratios already introduced (i.e. a_I and a_R). As with the traded good, we restrict production of the local good to a CBD. We now, however, divide that CBD into two rings. To the cumulative districting scheme mentioned above we now add to the ranking of land uses, commercial (i.e. local good) production. It is ranked above industrial (i.e. traded good) use of land but below housing production. The first ring is zoned industrial so that all uses, including local good production, may locate there. The second ring, however, is zoned commercial so that all land uses, with the exception of traded good production, are allowed there. The third ring and all rings further out are zoned residential so that only housing and agriculture can bid for land in these rings.

The production technology for the local good is constant returns to scale and uses capital, labor, and land from the first two rings as inputs. In particular, output of the local good in a ring is given by the following CES production function.

$$s_C^j = D\,[\alpha_{KC}\,(K_C^j)^{-\rho_C} + \alpha_{WC}(W_C^j)^{-\rho_C} + \alpha_{LC}(L_C^j)^{-\rho_C}]^{-\frac{1}{\rho_C}} \qquad (4.1)$$

for $j = 1,2$ and with $D>0$, α_{KC}, α_{LC}, $\alpha_{WC}>0$, $\rho_C>-1$, $\rho_C\neq 0$

The term K_C^j, W_C^j, and L_C^j represent capital, labor, and land usage, respectively, by the local good industry in ring j. Setting price equal to average cost for the local good industry and inverting yields the industry's bid land rent.

$$P_{LC} = \left[\left(\frac{\left(\frac{D \cdot P_C}{(1+a_C \cdot t)} \right)^{\rho_C}}{\alpha_{LC}} \right)^{\frac{1}{1+\rho_C}} - \left(\frac{\alpha_{KC} (\overline{P}_K)^{\rho_C}}{\alpha_{LC}} \right)^{\frac{1}{1+\rho_C}} - \left(\frac{\alpha_{WC} \left(\frac{P_W}{(1+a_C t)} \right)^{\rho_C}}{\alpha_{LC}} \right)^{\frac{1}{1+\rho_C}} \right]^{\frac{1+\rho_C}{\rho_C}} \quad (4.2)$$

where P_C is the endogenous price of local good output.

Households may purchase the commodity at the same price regardless of the location of their residence. Note that the bid land rent is not indexed by ring even though the local good industry may locate in either or both of the first two rings. No cost of transporting the good to a marketplace is assumed and labor can be hired at the same rate anywhere in the CBD, so that the industry's bid land rent gradient is flat. This bid land rent and the first order conditions for cost minimization can be used to find expressions for the local good industry's demands for labor and capital.

$$W_C^j = \left(\frac{\alpha_{WC} \cdot P_{LC} \cdot (1+a_C t)}{\alpha_{LC} \; P_W} \right)^{\frac{1}{1+\rho_C}} \cdot L_C^j \qquad j = 1,2 \qquad (4.3)$$

$$K_C^j = \left(\frac{\alpha_{KC} \cdot P_{LC}}{\alpha_{LC} \; \overline{P}_K} \right)^{\frac{1}{1+\rho_C}} \cdot L_C^j \qquad j = 1,2 \qquad (4.4)$$

where L_C^j is the amount of land in ring j available for endogenous use that is allocated to the local good industry.

These demand relations can be used to express local good supply from a ring as a function of input prices and the amount of land in the ring allocated to local good production. In particular, we may write

$$s_C^j = D\cdot[\alpha_{KC}\cdot\left(\frac{\alpha_{LC}\cdot\overline{P}_K}{\alpha_{KC}\cdot P_{LC}}\right)^{\frac{\rho_C}{1+\rho_C}} + \alpha_{WC}\left(\frac{\alpha_{LC}\cdot P_W}{\alpha_{WC}\cdot P_{LC}\cdot(1+a_C\cdot t)}\right)^{\frac{\rho_C}{1+\rho_C}} + \alpha_{LC}]^{\frac{-1}{\rho_C}}\cdot L_C^j \quad (4.5)$$

Turning to the consumption side, we add the local good to the household's list of commodities for potential consumption by forming the following household utility function for ring j.

$$U^j = A\cdot(\ell^j)^{\alpha_\ell}\cdot[\alpha_H\cdot(x_H^j)^{-\rho} + \alpha_T\cdot(x_T^j)^{-\rho} + \alpha_C(x_C^j)^{-\rho}]^{\frac{-1}{\rho}} \quad (4.6)$$

where $A>0$, α_H, α_T, $\alpha_C > 0$, $\rho > -1$, $\rho \neq 0$

The term x_C^j is consumption of the local good by a household residing in ring j.

The price of housing services over which the algorithm will search is again taken to be the one for the first ring that is zoned residential; in this case, ring 3. The benchmark utility level is now given by the indirect utility function for households residing in ring 3. It can be expressed as follows:

$$V^3 = A\cdot Y^3\cdot(\ell^3)^{\alpha_\ell}\left[\alpha_H\left(\frac{P_H^3(1+a_R\cdot t)}{\alpha_H}\right)^{\frac{\rho}{1+\rho}} + \alpha_T\cdot\left(\frac{\overline{P}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}} + \alpha_C\cdot\left(\frac{P_C}{\alpha_C}\right)^{\frac{\rho}{1+\rho}}\right]^{\frac{-(1+\rho)}{\rho}} \quad (4.7)$$

The housing bid price gradient is then found by inverting expression (4.7). In particular, the bid price for housing services in ring j is given as

$$P_H^j = \left(\frac{(\alpha_H)^{\frac{-1}{\rho}}}{(1+a_R\cdot t)}\right)\cdot\left[\left(\frac{A\cdot Y^j\cdot(\ell^j)^{\alpha_\ell}}{V^3}\right)^{\frac{\rho}{1+\rho}} - \alpha_T\left(\frac{\overline{P}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}} - \alpha_C\left(\frac{P_C}{\alpha_C}\right)^{\frac{\rho}{1+\rho}}\right]^{\frac{(1+\rho)}{\rho}} \quad (4.8)$$

Given an expression for indirect utility, use of Roy's identity will yield the following demands for housing services and the local good for households residing in ring j.

$$x_H^j = \frac{Y^j}{P_H^j(1+a_R.t)+\alpha_T\left(\frac{\overline{P}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}.\left(\frac{P_H^j(1+a_R.t)}{\alpha_H}\right)^{\frac{1}{1+\rho}}+\alpha_C\left(\frac{P_C}{\alpha_C}\right)^{\frac{\rho}{1+\rho}}.\left(\frac{P_H^j(1+a_R.t)}{\alpha_H}\right)^{\frac{1}{1+\rho}}} \quad (4.9)$$

$$x_C^j = \frac{Y^j}{P_C+\alpha_H\left(\frac{P_H^j(1+a_R.t)}{\alpha_H}\right)^{\frac{\rho}{1+\rho}}.\left(\frac{P_C}{\alpha_C}\right)^{\frac{1}{1+\rho}}+\alpha_T\left(\frac{\overline{P}_T}{\alpha_T}\right)^{\frac{\rho}{1+\rho}}.\left(\frac{P_C}{\alpha_C}\right)^{\frac{1}{1+\rho}}} \quad (4.10)$$

We may again distinguish between the demands of households in a ring that work and of those in a ring that do not work. They will differ only in the income terms in (4.9) and (4.10). For the local good, the demands for households that choose to work and households that choose to be unemployed will be represented by x_{CW}^j and x_{CNW}^j, respectively.

The excess demand correspondence, E(p), defined on the price vector, p, is to be represented, in part, by the three components given for it above - one for the excess demand for labor, one for the difference between the given population and the generated population, and one for the difference between a pre-specified level of tax revenues and generated tax revenues. Some adjustments must be made, however, to the expressions for the demand for labor and the generated property tax revenues. Labor demand can now arise from two industries. Thus, aggregate labor demand, W^d, is given as

$$W^d = W_T + W_C^1 + W_C^2 \quad (4.11)$$

Business property tax revenues can now also arise from two industries. Thus, we let generated property tax revenues be expressed as

$$R_C=\tau+t\cdot(a_R\cdot\sum_{\substack{j=1\\j\neq 3}}^{\Upsilon} p_H^j\cdot s_H^j + a_I\cdot(\bar{p}_K\cdot K_T + p_L^1\cdot L_T^1)+a_C\cdot(\bar{p}_K\cdot(K_C^1+K_C^2)+p_L^1\cdot L_C^1+p_L^2\cdot L_C^2)) \quad (4.12)$$

The property tax rate, t, is now defined in terms of ring 3. In particular, it is now expressed as

$$t = \frac{\tau}{a_R\cdot\max\,[p_H^3\cdot s_H^3,\ \bar{p}_{HA}\cdot\bar{s}_{HA}{}^3]} \quad (4.13)$$

where $a_R > 0$ and $\bar{s}_{HA}{}^3$ is the supply of housing services in ring 3 forthcoming when the price of housing services there is $\bar{p}_{HA}$ and all available land in the ring is allocated to housing production (i.e. $L_H^3 = L^3$).

The price vector, p, and the excess demand correspondence, E(p), must, however, be augmented in the extended model by a fourth component. For the price vector the new component is the price of the local good. Thus, search is now to be conducted over vectors $p=(p_W,\ p_H^3,\tau,p_C)$. The corresponding component of E(p) is the excess demand for the local good. Aggregate demand for the local good is dependent on household demand and on the size and distribution of the generated population. Aggregate demand for the local good, x_C^d, may be determined as follows:

$$X_C^d = \sum_{j=1}^{\Upsilon} (x_{CW}^j\cdot N_G^{Wj} + x_{CNW}^j\cdot N_G^{NWj}) \quad (4.14)$$

The aggregate supply of the local good, s_C, is just the sum of the supplies from the two CBD rings. Thus, we may write

$$s_C = s_C^1 + s_C^2 \quad (4.15)$$

The excess demand correspondence, E, defined on price vectors, p, can now be represented as follows:

$$E(p) = \begin{pmatrix} W^d - W^s \\ N - N_G \\ R - R_G \\ X_C^d - s_C \end{pmatrix} \qquad (4.16)$$

Our goal is to find a price vector $p^* = (p_W^*, p_H^{3*}, \tau^*, p_C^*)$ such that there exists $e^* \varepsilon E(p^*)$ with $e_1^* = e_2^* = e_3^* = 0$ and $p_C^* \cdot (X_C^{d*} - s_C^*) \leqq 0$, where X_C^{d*} and s_C^* are aggregate demand and supply for the local good, respectively, evaluated at p^*.[3]

B. Restrictions on the New Model

In addition to the restrictions on the functional forms and the conditions on the parameters delineated in the previous chapter for the base model, similar restrictions and conditions relating to the local good must be imposed here. As was true for the traded good industry's bid land rent, problems can arise when the bracketed expression in (4.2) becomes negative. If $\rho_C > 0$, for a wage rate-tax rate-price of local good combination at which the bracketed expression is negative, the price of the local good will be less than the average cost of producing it, and so profits would be negative.[4] Proceeding as we did with the traded good, then, a bid land rent of zero is assumed for the local good industry when $\rho_C > 0$ and the bracketed expression in (4.2) is negative.

On the other hand, the problems are again more complicated when $\rho_C < 0$. When $\rho_C < 0$ and the price vector is such that the bracketed expression in (4.2) is negative, the price of the local good will exceed the average cost of producing it, and so profits would be positive. Dealing with this sort of problem, in relation to the local good is, however, more complicated than it was for the traded good since output price can now vary. Once again, however, we choose some level of labor demand, say $b' \cdot N \cdot \overline{W}$, well in excess of the maximum potential supply, $N \cdot \overline{W}$. Working, as we are, with a given simplex, the wage rate must be bounded from above. In particular, it cannot exceed d. Our approach is to first substitute d for the wage rate and the large level of labor services just mentioned ($b' \cdot \overline{N} \cdot W$) for labor demand in expression (4.3). Then, using the minimum of the total amounts of land available for endogenous use in the two CBD rings and inverting (4.3) with respect to p_{LC}, we obtain, at a given tax rate, an upper bound, say $\hat{p}_{LC}$, for the local good industry's bid land rent.

$$\hat{p}_{LC} = \left(\frac{b' \cdot N \cdot \overline{W}}{\min[L^1, L^2]}\right)^{1+\rho_C} \left(\frac{\alpha_{LC} \cdot d}{\alpha_{WC}(1+a_C \cdot t)}\right) \tag{4.17}$$

Next, we find the local good output price that is consistent with this bid land rent and a wage rate of d. That is, we invert (4.2) with respect to p_C and substitute $\hat{p}_{LC}$ for p_{LC} and d for p_W to obtain

$$\hat{p}_C = \frac{1}{D} \cdot [(\alpha_{LC}[(1+a_C \cdot t) \cdot \hat{p}_{LC}]^{\rho_C})^{\frac{1}{1+\rho_C}} + (\alpha_{KC}[(1+a_C \cdot t) \cdot \overline{p}_K]^{\rho_C})^{\frac{1}{1+\rho_C}}$$
$$+ (\alpha_{WC}(d)^{\rho_C})^{\frac{1}{1+\rho_C}}]^{\frac{1+\rho_C}{\rho_C}} \tag{4.18}$$

A convention that we adopt to bound the bid land rent, then, is to assign a value of $\hat{p}_{LC}$ for the local good industry's bid land rent whenever $p_C \geqq \hat{p}_C$.[5] More must be done, however, if the bid land rent is to be bounded in all situations. If $p_C < \hat{p}_C$, then we must look to the wage rate. In particular, we invert (4.2) with respect to the wage rate and substitute $\hat{p}_{LC}$ for p_{LC} to obtain

$$\hat{p}_{WC} = \left[\left(\frac{[D \cdot p_C]^{\rho_C}}{\alpha_{WC}} \right)^{\frac{1}{1+\rho_C}} - \left(\frac{\alpha_{KC}[\overline{p}_K(1+a_C t)]^{\rho_C}}{\alpha_{WC}} \right)^{\frac{1}{1+\rho_C}} - \left(\frac{\alpha_{LC} \cdot [\hat{p}_{LC} \cdot (1+a_C \cdot t)]^{\rho_C}}{\alpha_{WC}} \right)^{\frac{1}{1+\rho_C}} \right]^{\frac{\rho_C}{1+\rho_C}} \quad (4.19)$$

This is the wage rate consistent with the actual output price, p_C, a bid land rent of $\hat{p}_{LC}$, and the given tax rate. To have the bid land rent bounded by $\hat{p}_{LC}$, we assign a value of $\hat{p}_{LC}$ for the local good industry's bid land rent whenever $p_W \leqq \hat{p}_{WC}$ and $p_C < \hat{p}_C$.[6] In all other cases (with $\rho_C < 0$), the bid land rent is assigned by calculating the right-hand side of (4.2). The result is that, for $\rho_C < 0$, the bid land rent for the local good industry will never exceed $\hat{p}_{LC}$.

It is assumed that the minimum upper bound on the local good's bid land rent ($\hat{p}_{LC}$ when t is as high as it can be, given d) is chosen so that it is greater than the upper bound on housing bid land rents. It is also assumed that the maximum upper bound on the local good's bid land rent ($\hat{p}_{LC}$ when $t = 0$) is less than the minimum upper bound on the traded good industry bid land rent when $\rho_T < 0$.[7] This will ensure that the local good industry will outbid housing and agriculture for land in ring 2 when the constraint $p_{LC} \leqq \hat{p}_{LC}$ is binding, and that the traded good industry will outbid housing, agriculture, and the local good industry for land in ring 1 when the traded good bid land rent bound is binding. The implication of this is that labor demand will exceed the maximum potential supply of labor when either the local or traded good's upper bound on bid land rents is binding.

We wish, as we did in the base model, to make E(p) a bounded correspondence. In particular, we must have bounded demands for labor. Assumptions made in the previous chapter serve to bound the traded good industry's demand for labor. Bounding the local good industry demand for labor when $\rho_C < 0$, however, calls for a somewhat different analysis. Setting $p_W = \hat{p}_{WC}$ in (4.3) to calculate an upper bound for labor demand is not satisfactory since $\hat{p}_{WC}$ falls to zero as p_C does. Instead, we assign as the local good industry demand for labor in a ring, assuming that the industry successfully bids for some land in the ring, the minimum of $b' \cdot N \cdot \overline{W}$ and the demand that would be found by using (4.3) and the actual wage rate. No potential equilibrium is sacrificed in this process since $b' \cdot N \cdot \overline{W}$ exceeds the maximum labor supply. In practice, we may assign $b' \cdot N \cdot \overline{W}$ as the local good labor demand in a ring when the bound on bid land rents, $\hat{p}_{LC}$, is binding and the local good industry outbids other uses for land in the ring. A practical problem can arise, though, when $\hat{p}_{LC}$ is not in effect. The value for $\hat{p}_{WC}$ can be very small. If the actual wage rate is very close to a very small $\hat{p}_{WC}$, then calculation of labor demand by (4.3) can lead to a very high value - perhaps too high for the computer. To avoid this potential problem we circumvent the calculation of labor demand via (4.3) when it would yield a value higher than $b' \cdot N \cdot \overline{W}$, in the following way.

First calculate the bid land rent, p_{LC}, for the given price vector (we are assuming that $p_C < \hat{p}_C$ and $p_W > \hat{p}_{WC}$). Using this value and the given tax rate in expression (4.3) and setting labor demand in (4.3) equal to $b'\cdot N\cdot \overline{W}$, we then solve for the wage rate. If the actual wage rate for the given vector p is less than this value, then actual labor demand, as determined by (4.3), would exceed $b'\cdot N\cdot \overline{W}$. Thus, if the actual wage rate is less than or equal to the value of p_W that has been calculated in the manner mentioned above, we automatically assign a labor demand of $b'\cdot N\cdot \overline{W}$ (assuming the local good industry outbids other uses for land in the ring), without first calculating the actual demand for labor. If the actual wage rate is greater than the wage rate value mentioned above, then labor demand is calculated and assigned according to (4.3), with the actual wage rate used as the value of p_W.

If $\rho_C > 0$, then the problem of bounding the local good labor demands is completely analagous to the problem of bounding the traded good demand for labor in the base model. A level of labor demand, say $\hat{W}$, which is well in excess of the maximum labor supply is chosen.[8] If $p_W > 0$, W_C^j may be calculated, in principle, according to (4.3). If, however, the value that would be obtained for W_C^j equals or exceeds $\hat{W}$, then $\hat{W}$ is assigned as the local good demand for labor in ring j. On the other hand, as before, the case $p_W = 0$ presents problems for both boundedness and upper semi-continuity. They are resolved here as they were for the traded good in the base model. If the local good industry is not allocated any land in a CBD ring then local good labor demand for that ring is taken to be zero. If, however, the local good industry is allocated some land in the ring then local good labor demand in the ring can take on any positive value up to and including $\hat{W}$. Symbolically, the assignment of local good labor demand when $\rho_C > 0$ and $p_W = 0$ is given as follows:

$$W_C^j \begin{cases} \varepsilon\ (0,\hat{W}] & \text{if} \quad L_C^j > 0 \\ = 0 & \quad\quad L_C^j = 0 \end{cases} \qquad j \equiv 1,2 \qquad (4.20)$$

One last bounding problem remains. The new (fourth) component of E(p), the excess demand for the local good, can be infinite. Inspection of (4.10) reveals that household demand for the local good (and so aggregate demand when a positive number of households is generated) becomes infinite when the price of the local good is zero. We have no choice, then, but to place a lower bound on the local good price. In particular, we choose some low positive value, $\overline{p}_C$. If, in the search process, the price vector is such that the actual value for p_C is less than $\overline{p}_C$, then we let $\overline{p}_C$ be the value assumed for the local good price in all calculations. In principle, this needn't be a problem since, for any parameterization, it can be shown that an equilibrium, with a zero local good price, cannot exist. To see this, first note that, at an equilibrium with a zero local good price, aggregate demand for the local good must be infinite. This follows from the fact that a positive number of households must be generated at an equilibrium. Supply of the local good, on the other hand, must be finite since land, labor, and capital used in production are all bounded from above.[9] Therefore, it is impossible to have a zero or negative excess demand for the local good at an equilibrium with a zero local good price. Hence, no such equilibrium exists.

The bound, $\bar{p}_C$, can be chosen low enough, then, so that the effective region of search on the price-tax rate simplex contains the "price" vector associated with any particular equilibrium.[10]

Now that the special structure for this new model has been delineated, it is natural to ask if an equilibrium for the model, at least under certain conditions on the parameters, exists. Unfortunately, unlike what was done in the previous chapter for the base model, a rigorous existence proof for the extended model was not obtained. It is possible, however, to present the following argument which is, in part, heuristic.

We again attempt to use Theorem 1 as the basis for the existence argument. In particular, the reasoning again proceeds along lines needed to show that the two hypotheses of that theorem are satisfied. In order to preserve results obtained for the base model we retain conditions (c.1) - (c.3). Condition (c.4) must, however, be altered somewhat in structure, but not in nature, to account for changes in the specification of the model. Instead of relating to the first two rings, the condition must now refer to the first three rings, with the third ring taking the place of the second ring in the base model condition. Furthermore, the upper bound on the tax rate, $\hat{t}$, must be re-defined. In the situation relevant to the condition (which has been discussed in chapter 3), the actual tax rate can now be written as $t = \dfrac{R}{a_R \bar{p}_{HA} \bar{s}_{HA}^{3} + (R_G^1 + R_G^2)}$, where $\bar{s}_{HA}^3$ is the supply of housing services in ring 3 when the land rent facing the housing industry for land in the ring is $\bar{p}_A$ and all of the land in the ring available for endogenous use is allocated to the housing industry, and R_G^2 is the generated property tax base in ring 2. Suppose we let $\overline{K}_C^j = K_C^j(\overline{p}_A, L^j)$ be capital demand by the local good industry for CBD ring j when its bid land rent is $\overline{p}_A$ and all available land in ring j is allocated to the local good industry, and define the following two variables.

$$R^1 = \min[a_R \cdot \overline{p}_{HA} \cdot \overline{s}_{HA}^{\,1},\ a_I \cdot (\overline{p}_A \cdot L^1 + \overline{p}_K \cdot \overline{K}_T),\ a_C \cdot (\overline{p}_A \cdot L^1 + \overline{p}_K \cdot \overline{K}_C^1)] \qquad (4.21)$$

$$R^2 = \min[a_R \cdot \overline{p}_{HA} \cdot \overline{s}_{HA}^{\,2},\ a_C \cdot (\overline{p}_A \cdot L^2 + \overline{p}_K \cdot \overline{K}_C^2)] \qquad (4.22)$$

Then, by reasoning analogous to that given for the base model in a similar situation, it must be true that $R^1 < R_G^1$ and $R^2 < R_G^2$. Using these facts, we define the upper bound on the tax rate as follows:

$$\hat{t} = \frac{R}{a_R \cdot \overline{p}_{HA} \cdot \overline{s}_{HA}^{\,3} + (R^1 + R^2)} \qquad (4.23)$$

It is also necessary to re-define $\tilde{p}_W$ so that it refers to ring 3. Finally, to form the new condition, say (c.4)´, equation (3.20), which defines $\hat{N}_G^1$, is retained and equation (3.21), which defines $\hat{N}_G^2$, is indexed to refer to ring 3 instead of ring 2. This leaves the term $\hat{N}_G^2$ to be specified. It is defined exactly as is $\hat{N}_G^1$ in (3.20), with all ring indices referring to ring 2. The condition, appropriate to the extended model, that is the analogue of (c.4) can now be given as follows:

$$\hat{N}_G^1 + \hat{N}_G^2 + \hat{N}_G^3 < N \qquad (c.4)´$$

Satisfaction of hypothesis (H.1) of Theorem 1 for the extended model can be established rigorously. To do this, we must first argue that E is upper semicontinuous. It is clear that the arguments establishing this for the base model in the proof of Theorem 2 can be used here. The supplies of the local good, the demands by the local good industry for labor (when $\rho_C < 0$) and capital, and aggregate demand for the local good in a ring can be expressed as products of a continuous function of p and the amount of land in a ring allocated to some endogenous use. In addition, the argument in the proof of Theorem 2 establishing the upper semicontinuity of traded good labor demand when $\rho_T < 0$ can be invoked to show the upper semicontinuity of local good labor demand, using (4.20) in place of (3.16), when $\rho_C < 0$. Thus, E, for the extended model, is upper semicontinuous.

Second, it must be shown that E(p) is a bounded set for any admissible p. Clearly, from the analysis for the base model, the population component of E(p) is bounded. The labor and tax revenue components are also bounded. This follows from the analysis in the previous chapter and the fact that labor and capital demands for the local good industry are bounded, for a given d. Since labor and capital demands and the amount of land used by the local good industry are bounded, the supply of the local good must also be bounded. Household demand for the local good is bounded from above since the local good price is bounded from below by $\overline{p}_C$. From the analysis in the previous chapter, for a given d, the generated population is bounded from above. Thus, aggregate demand for the local good is bounded, and so the component of E(p) that represents the excess demand for the local good must be bounded. Hence, E(p) is a bounded set.

Finally, the convexity of E(p), for any p, must be verified. It is clear that, with the exception of cases where $p_W = 0$ with $\rho_T > 0$ and/or $\rho_C > 0$, all of the components of excess demand vectors for this model can once again be put in the form (3.22). As noted above, the terms in the excess demand components relating to the local good, in these situations, can be expressed as the products of a function of p and the amount of land in a ring allocated to some endogenous use. Thus, convexity follows for these cases. When $p_W = 0$ and $\rho_C > 0$, the argument in the proof of Theorem 2 proving the convexity of the traded good demand for labor when $p_W = 0$ and $\rho_T > 0$ can be adapted in total to show the convexity of the local good labor demand. Thus, E is a convex correspondence and so hypothesis (H.1) of Theorem 1 must be satisfied.

In turning to hypothesis (H.2) of Theorem 1, however, the argument must lose some rigor. The complexity added by considering this local good, that competes for land with other endogenous uses in the two CBD rings and for which its aggregate demand and supply must be equated, is sufficiently great to make it necessary that our argument, at this point, be merely heuristic. Hypothesis (H.2) will be satisfied if, for a given p and large enough value of d, we can find a vector α such that $\alpha \cdot e \leqq 0$ for all $e \in E(p)$, with α_i positive only if the corresponding "price", p_i, is positive. To find such a vector, we make the following assumption [which can also be found in Richter (1980)]:

<u>Assumption 1</u>. There exists d>0 such that for all $p \geqq 0$ with $\sum_{\text{all } i} p_i \geqq d$, $p \cdot e < 0$ for all $e \in E(p)$.

This assumption, at least for many parameterizations, seems plausible. It implies that if the sum of the components of the price vector p is sufficiently large, then the value of the "excess demands" for all vectors $e \in E(p)$ is negative. It seems reasonable to expect this to be true for many parameterizations if high prices are associated with negative excess demands for the corresponding components of e; while low prices are associated with positive excess demands for the corresponding components of e. The value of the excess demands, p·e, can be viewed as a weighted average of the excess demand components, with the highest weights attached to the excess demands that are likely to be negative, and the lowest weights attached to those excess demands that are likely to be positive. Thus, in such situations, we may expect p·e to be negative if prices can be large enough.

In the case of a component of e that actually does represent an excess demand for some good or service, this association between the magnitude of the price and the sign of the corresponding excess demand certainly seems likely and natural. The two components of vectors in E(p) that are of this type are the excess demands for labor services and the local good. Inspection of the labor demands for the traded good and local good industries particular to our specification shows that they behave normally with respect to the wage rate; increases in the wage rate, ceteris paribus, lowers labor demand in a ring (assuming that the particular industry outbids all other uses for land in the CBD ring). In addition, the supply of labor tends to exhibit a positive relationship to the wage rate, since higher wage rates can lead some households, that would otherwise choose to be unemployed, to send a member to work.[11] Thus, there seems to be a tendency for high wage rates to be associated with negative excess demands for labor. With respect to the excess demand for the local good, the relevant price is p_C. It is easily seen that the household demand for the local good, (4.10), is negatively related to the price of the local good, while the supply function, (4.5), is positively related to p_C. Thus, there appears to be a natural tendency for high local good prices to be associated with negative excess demands for the local good.[12]

In our model, however, two components of "excess demand" vectors are not excess demands for some good or service: the population and tax revenue components. The price associated with the population component is the price of housing services in ring 3. It can be seen from the definition of the generated population in a ring that an increase in p_H^3, provided that housing is able to obtain land in ring 3, will lead directly to an increase in the aggregate generated population via an increase in the generated population for ring 3. Equations (2.51) and (4.9) show that a higher price for housing in ring 3 leads to an increased supply and a decreased household demand there. An increase in p_H^3 also tends to increase the total generated population through a shifting up of the entire housing price gradient. At least for increases in p_H^3 above the level at which no land is allocated to housing, the total generated population must increase because the higher housing prices in previously occupied rings leads to increases in the generated populations in those rings, for the same reasons mentioned above in relation to ring 3, and the higher housing price gradient increases the number of rings in which land is allocated to housing. Thus, high housing prices for ring 3 tend to be associated with large total generated populations and so, negative population components. Finally, the "price" associated with the tax revenue component is the variable, τ. Clearly, if τ is large enough, then the tax revenue component will be negative.

Thus, there seems to be a tendency for high "price" components of the vector, p, to be associated with negative excess demands in the corresponding elements of vectors in E(p). So, Assumption 1 may be satisfied for some parameterizations of the model. In such cases, hypothesis (H.2) will be satisfied and a solution of the type given in the conclusion of Theorem 1 exists. Condition (c.4)´ guarantees that such a solution represents a normal equilibrium.[13]

II. Models with Multiple Household Types

All of the models considered to this point have used the assumption that all households in the urban area are identical. Homogeneity was posited, in particular, for preferences and exogenous income, and, to the extent that all households had the option of working an institutional work day at the same rate of pay, for endogenous income as well. It would be interesting, however, to investigate the effects that alternative property tax schemes have on different groups; particularly different income groups. In the following sections, the changes and additional restrictions that must be made to model different household groups, as well as the question of the existence of equilibrium, will be explored.

A. Preference and Income Differentials

To capture the possibility that different residents have different tastes, the household utility functions must be indexed by group. Retaining the same functional form for utility, then, the level of utility for a household of type i residing in ring j is given as follows:

$$U^{ij} = A^i \cdot (\ell^j)^{\alpha_{\ell i}} \cdot [\alpha_{Hi} \cdot (x_H^{ij})^{-\rho_i} + \alpha_{Ti} \cdot (x_T^{ij})^{-\rho_i} + \alpha_{Ci} \cdot (x_C^{ij})^{-\rho_i}]^{\frac{-1}{\rho_i}} \qquad (4.24)$$

where $A^i > 0$, $\alpha_{Hi}, \alpha_{Ti}, \alpha_{Ci} > 0$, $\rho_i > -1$, $\rho_i \neq 0$

Note that the amount of leisure time enjoyed by a household residing in ring j is not indexed by household group. This reflects an assumption that commuting time for households residing at a given distance from the center of the region does not differ by household type. This, in turn, is consistent with the implicit assumption of one mode of transport. Household demands for any of the goods can differ because of differences in the preference parameters and/or differences in income. With one type of labor, variation in income net of the money cost of travel can arise only from differences in exogenous income. The amount of exogenous income received by most households, however, is likely to be, in realistic terms, only a small part of total income. For this reason, it may be useful to allow for variation in the endogenously determined component of income. To do this, multiple labor types must be introduced along with the multiple household types. Households, then, may differ not only by preferences, but also by the type of labor skills that they possess. Producers are assumed to treat the services of different labor types as different inputs in production. Thus, the wages of different labor types are determined in different markets. The supply sides of these labor markets are assumed to be independent in the sense that workers cannot acquire skills that they do not have or lose skills that they do possess, and so move from one market to another, even if relative wage rates change. The market wage rate facing household group i is defined to be p_{Wi}. If we let M_i represent the exogenous income of household group i, then income net of the money cost of commuting for a household of this type residing in ring j can be written as follows:

$$Y^{ij} = M_i + p_{Wi} \cdot \overline{W} - c \cdot u^j \tag{4.25}$$

The demands for the goods are defined as they were before, but with Y^{ij} replacing Y^j and the utility function parameters now indexed by i. This also applies to the indirect utility and bid housing price functions.

Now that there are multiple household types in the model, the question of which group or groups are allowed to reside in a given ring arises. To answer this question, bid housing prices for the first ring zoned residential for all of the household types are introduced to the set of prices over which the algorithm is to search. In a model with a local good the first residential ring is the third. Let p_H^{i3} be the bid price of housing services in the third ring for the i'th household group. The actual price of housing for the third ring is chosen to be the maximum of the bid prices for the different groups. Thus, assuming that there are n household types, the price of housing in ring 3 is $p_H^3 = \max\{p_H^{13}, p_H^{23}, \ldots, p_H^{n3}\}$. Whether or not a household group manages to locate in ring 3, its bid housing price there can be used to establish its benchmark utility level. Once again, the option of not working can be provided to households in the model. Thus, the level of utility achieved by a household of type i anywhere in the region, at a given vector of prices over which the algorithm searches, is given by $V^{i3} = \max\{V_W^{i3}, V_{NW}^{i3}\}$, where V_W^{i3} and V_{NW}^{i3} are the utility levels of a household of type i residing in ring 3 when the household chooses to send and not to send, respectively, a member to work. The utilities are calculated on the basis of

p_H^{i3} and the other non-housing "price" components of the search vector. The utility levels can then be used to determine the bid housing prices, p_{HW}^{ij} and p_{HNW}^{ij}, for households of a given type, residing in rings other than the third, that have some or no members, respectively, that work. The price of housing that actually prevails in one of these rings is taken to be the maximum of these bid prices for the ring for all household groups. Thus, the price of housing in ring j is given as $p_H^j = \max\{p_{HW}^{1j}, \ldots, p_{HW}^{nj}, p_{HNW}^{1j}, \ldots, p_{HNW}^{nj}\}$. The maximum, if it is unique, determines which group resides in the ring and whether or not such households work. If there is a tie for the maximum, then households of different types and/or households that make different decisions about labor supply may reside in the same ring, with the available land in the ring arbitrarily allocated to housing these households.

Turning to a depiction of the production side, the multiple labor types are considered to be separate inputs in the production functions. The number of labor types need not be equal to the number of household groups, but it cannot exceed that number. Assuming that the number of labor types is $m \leqq n$, output of the traded and local goods can be written as follows:

$$s_T = C\cdot[\alpha_{KT}\cdot(K_T)^{-\rho_T} + \alpha_{LT}\cdot(L_T)^{-\rho_T} + \sum_{i=1}^{m} \alpha_{WTi}\cdot(W_{Ti})^{-\rho_T}]^{\frac{-1}{\rho_T}} \quad (4.26)$$

$$s_C^j = D\cdot[\alpha_{KC}\cdot(K_C^j)^{-\rho_C} + \alpha_{LC}\cdot(L_C^j)^{-\rho_C} + \sum_{i=1}^{m} \alpha_{WCi}\cdot(W_{Ci}^j)^{-\rho_C}]^{\frac{-1}{\rho_C}} \quad j = 1,2 \quad (4.27)$$

where $C, D, \alpha_{KT}, \alpha_{KC}, \alpha_{LT}, \alpha_{LC}, \alpha_{WTi}, \alpha_{WCi} > 0$ and $\rho_T, \rho_C > -1$, $\rho_T, \rho_C \neq 0$

The terms W_{Ti} and W_{Ci}^j represent labor demands for the traded and local good industries, respectively. The bid land rents for the two industries can then be expressed and calculated as follows:

$$p_{LT} = \left[\left(\frac{(C \cdot \bar{p}_T/(1+a_I \cdot t))^{\rho_T}}{\alpha_{LT}} \right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{KT} \cdot (\bar{p}_K)^{\rho_T}}{\alpha_{LT}} \right)^{\frac{1}{1+\rho_T}} - \sum_{i=1}^{m} \left(\frac{\alpha_{WTi} \cdot (p_{Wi}/(1+a_I \cdot t))^{\rho_T}}{\alpha_{LT}} \right)^{\frac{1}{1+\rho_T}} \right]^{\frac{1+\rho_T}{\rho_T}} \tag{4.28}$$

$$p_{LC} = \left[\left(\frac{(D \cdot p_C/(1+a_C \cdot t))^{\rho_C}}{\alpha_{LC}} \right)^{\frac{1}{1+\rho_C}} - \left(\frac{\alpha_{KC} \cdot (\bar{p}_K)^{\rho_C}}{\alpha_{LC}} \right)^{\frac{1}{1+\rho_C}} - \sum_{i=1}^{m} \left(\frac{\alpha_{WCi} \cdot (p_{Wi}/(1+a_C \cdot t))^{\rho_C}}{\alpha_{LC}} \right)^{\frac{1}{1+\rho_C}} \right]^{\frac{1+\rho_C}{\rho_C}} \tag{4.29}$$

Each industry now has multiple demands for labor. They each, of course, are dependent on the wage rate set in the market for that type of labor. Given the production functions, (4.26) and (4.27), labor demands can be determined as follows:

$$W_{Ti} = \left(\frac{\alpha_{WTi} \cdot p_{LT} \cdot (1+a_I \cdot t)}{\alpha_{LT} \cdot p_{Wi}} \right)^{\frac{1}{1+\rho_T}} \cdot L_T^1 \tag{4.30}$$

$$W_{Ci}^j = \left(\frac{\alpha_{WCi} \cdot p_{LC} \cdot (1+a_C \cdot t)}{\alpha_{LC} \cdot p_{Wi}} \right)^{\frac{1}{1+\rho_C}} \cdot L_C^j \qquad j = 1,2 \tag{4.31}$$

for $i = 1,2,\ldots,m$

With multiple labor types more markets must be cleared in equilibrium. Because of this, the number of elements of vectors in the range of the excess demand correspondence E must be expanded to include the excess demands for the different kinds of labor. The associated prices are the wage rates, p_{Wi}. The excess demand vectors must also be expanded in another direction. It is assumed that the numbers of households of each type, N_i, to be housed in the urban area are given exogenously. Analagous to what was done in the one group model, the number of households of a given type and labor supply decision generated in a ring is the ratio of the supply of housing to the group to the demand for housing by a household from this group. In an equilibrium we must have the total number of generated households of each type equal to the exogenously given number for that type. If we let N_{Gi}^{Wj} and N_{Gi}^{NWj} be the numbers of generated households of type i residing in ring j that do and do not send a member to work, respectively, then our new population equilibrium conditions can be written as follows:

$$N_i - \sum_{j=1}^{\gamma} (N_{Gi}^{Wj} + N_{Gi}^{NWj}) = 0 \qquad i = 1,2,\ldots,n \qquad (4.32)$$

As a result, we add as components to the excess demand vectors, the differentials between the given and generated populations for the various household types. The prices that are to be associated with these population differentials are the bid housing prices, p_H^{i3}, for the third ring. Thus, the price vector over which the algorithm is to search, in general, for this model is given as $p = (p_{W1},\ldots,p_{Wm},p_H^{13},\ldots,p_H^{n3}, \tau, p_C)$. The dimension of the simplex over which the algorithm is to search, therefore, is increased by n+m-2 over that for the one household type models.

B. Restrictions on the Model

Although most of the bounding procedures for the base and local good models can be retained, keeping the excess demand correspondence bounded in the presence of multiple household and labor types introduces a few additional considerations. Care must be taken to guarantee the boundedness of the generated populations of households of each type, the demands for the various kinds of labor services, and the industry bid land rents. The following exposition will be confined, as much as possible, to the new aspects of the bounding problem. Previous discussions on the rationale for certain bounding procedures that remain relevant for the new model will not be repeated here.

In the one household group model a bound of $\hat{N}$, greater than the total given population N, on the generated number of households that do not send any members to work was imposed. The same thing can be done in this model for each household type. For simplicity, we let $N = \sum_{i=1}^{n} N_i$ and choose $\hat{N}$ well in excess of N. The value $\hat{N}$ can then be taken as an upper bound for all household groups on the number of generated households that supply no labor. Since housing prices are bounded from above for a given value of d, the number of generated households, for any group, that supply labor is also bounded from above. Thus, the total generated population of each household group is bounded. This also clearly indicates that the aggregate demand for the local good is bounded from above, provided that the lower bound on the local good price, $\overline{p}_C$, is retained.

Matters are a bit more complicated when it comes to bounding and calculating industry bid land rents and labor demands. As was true before, however, no particularly thorny problems arise here in calculating bid land rents for the traded good industry when $\rho_T > 0$ or the local good industry when $\rho_C > 0$. What was done in this regard in the earlier models is still valid. Extra care must be taken, however, in other situations.

Suppose that $\rho_T < 0$. We again choose some level of labor demand, $b \cdot N \cdot \overline{W}$, that is well in excess of the maximum potential labor supply for any labor type.[14] Next, we find the minimum of the labor coefficients, α_{WTi}, in the production function (4.26). Let k be the index for the minimum.[15] Thus, $\alpha_{WTk} = \min_i \{\alpha_{WTi}\}$. An upper bound, at a given tax rate, on the traded good bid land rent, $\hat{p}_{LT}$, is then found by inverting the labor demand function (4.30) for the k'th labor type with respect to the land rent, using $b \cdot N \cdot \overline{W}$ for labor demand, L^1 for the amount of land used by the traded good industry, and d for the wage rate. Thus, this upper bound can be written as follows:

$$\hat{p}_{LT} = \left(\frac{b \cdot N \cdot \overline{W}}{L^1}\right)^{1+\rho_T} \cdot \left(\frac{\alpha_{LT} \cdot d}{\alpha_{WTk} \cdot (1+a_I \cdot t)}\right) \tag{4.33}$$

For practical reasons, though, consider what values of the endogenous variables lead to a binding bid land rent constraint.[16] The constraint will be binding for low values of one or more of the wage rates. To place lower bounds on the individual wage rates such that actual wage rates below these values must result in a binding bid land rent constraint, the following terms, derived by inverting (4.28) with respect to the sum found therein and substituting $\hat{p}_{LT}$ for p_{LT}, is defined.

$$\hat{p}_{WTs} = \left(\frac{(C \cdot \overline{p}_T/(1+a_I \cdot t))^{\rho_T}}{\alpha_{LT}}\right)^{\frac{1}{1+\rho_T}} - \left(\frac{\alpha_{kT} \cdot (\overline{p}_k)^{\rho_T}}{\alpha_{LT}}\right)^{\frac{1}{1+\rho_T}} - (\hat{p}_{LT})^{\frac{\rho_T}{1+\rho_T}} \tag{4.34}$$

If the i'th labor type were the only labor group, then (4.34) can be used to define the wage rate, $\hat{p}_{WTi}$, that would yield a bid land rent of $\hat{p}_{LT}$.

In particular, this term is given as follows:

$$\hat{p}_{WTi} = (\hat{p}_{WTs})^{\frac{1+\rho_T}{\rho_T}} \cdot (\alpha_{LT}/\alpha_{WTi})^{\frac{1}{\rho_T}} \cdot (1+a_I \cdot t) \tag{4.35}$$

Clearly, if $p_{Wi} \leqq \hat{p}_{WTi}$, for any i, then the land rent constraint must be binding, and $\hat{p}_{LT}$ would be assigned as the traded good industry bid land rent. If none of the wage rates are this low, then attention is turned to the sum in (4.28). If the value of the sum is greater than or equal to $\hat{p}_{WTs}$, then the constraint must be binding, and so the assigned bid land rent would be $\hat{p}_{LT}$.

Essentially the same procedure is undertaken for the local good industry when $\rho_C < 0$, although the analysis is more involved due to the fact that the local good price is endogenous. As was done in the local good model, a large level of labor demand, $b' \cdot N \cdot \overline{W}$, the minimum of the amounts of land available in the two CBD rings, and the size of the simplex term, d, are used to define an upper bound, $\hat{p}_{LC}$, on the local good bid land rent. Analagous to what was done for the traded good industry, the minimum of the labor coefficients in the production function (4.27) is used. Thus, assuming that the index of the minimum coefficient is h, the upper bound on the bid land rent, for a given tax rate, can be written as follows:

$$\hat{p}_{LC} = \left(\frac{b' \cdot N \cdot \overline{W}}{\min[L^1, L^2]}\right)^{1+\rho_C} \cdot \left(\frac{\alpha_{LC} \cdot d}{\alpha_{WCh} \cdot (1+a_C \cdot t)}\right) \tag{4.36}$$

This time, finding which price vectors result in a binding bid land rent constraint must involve considering variation in output price. If the local good price is high enough, then the bid land rent constraint will be binding. To find a value such that the constraint must be binding if the local good price exceeds it, we merely re-define $\hat{p}_C$ by substituting the minimum labor coefficient, α_{WCh}, for α_{WC} in (4.18). Then, if $p_C \geqq \hat{p}_C$ the constraint will be binding and a bid land rent of $\hat{p}_{LC}$ can be assigned automatically. When $p_C < \hat{p}_C$ the analysis proceeds exactly as it did for the traded good industry. Expressions analogous to (4.34) and (4.35) can be defined as follows:

$$\hat{p}_{WCs} = \left(\frac{(D\cdot p_C/(1+a_C\cdot t))^{\rho_C}}{\alpha_{LC}}\right)^{\frac{1}{1+\rho_C}} - \left(\frac{\alpha_{KC}\cdot(\overline{p}_k)^{\rho_C}}{\alpha_{LC}}\right)^{\frac{1}{1+\rho_C}} - (\hat{p}_{LC})^{\frac{\rho_C}{1+\rho_C}} \tag{4.37}$$

$$\hat{p}_{WCi} = (\hat{p}_{WCs})^{\frac{1+\rho_C}{\rho_C}} \cdot (\alpha_{LC}/\alpha_{WCi})^{\frac{1}{\rho_C}} \cdot (1+a_C\cdot t) \tag{4.38}$$

If $p_{Wi} \leqq \hat{p}_{WCi}$, for any i, then the land rent constraint is binding and a bid land rent of p_{LC} can be assigned automatically. If $p_{Wi} > \hat{p}_{WCi}$, for all i, then the sum in (4.29) should be calculated. If it is greater than or equal to $\hat{p}_{WCs}$, then a bid land rent of $\hat{p}_{LC}$ is automatically assigned. Otherwise, the bid land rent is calculated according to (4.29).

The procedure for bounding labor demand when $\rho_C < 0$ in the one household group model mentioned above can also be applied here for the local good industry when $\rho_C < 0$ and for the traded good industry when $\rho_T < 0$. Upper bounds of $b\cdot N\cdot\overline{W}$ for each of the traded good labor demands and $b'\cdot N\cdot\overline{W}$ for each of the local good labor demands are chosen. Thus, in assigning labor demand for the i'th labor type the minimum of $b\cdot N\cdot\overline{W}$ and the value calculated according to (4.30) is taken to be the traded good's demand, and the minimum of $b'\cdot N\cdot\overline{W}$ and the value calculated according to (4.31) is taken to be the local good demand. When the bid land rent constraint for one of the industries is binding, and the industry outbids all other uses for land in a ring, the labor demand constraints for all of the labor types used by the industry in the ring must also be binding, and so the appropriate upper bound can be assigned as labor demand, without first using (4.30) or (4.31) to calculate any values. If a bid land rent constraint is not binding, then, as was done for the one group local good model, the problem of attempting to calculate demand according to (4.30) or (4.31), when that would yield a very high value, can be circumvented by substituting the appropriate upper bound on labor demand in the equation and then solving for the wage rate. If the actual wage rate is less than or equal to this value, and the industry outbids other uses for land in the ring, then the upper bound is automatically assigned as labor demand. Otherwise, demand is assigned according to (4.30) or (4.31).

Finally, for the cases $\rho_T > 0$ and $\rho_C > 0$ an upper bound on labor demand of $\hat{W}$, well in excess of the maximum labor supply for any labor type, is set as was done in the previous model. The process of assigning labor demand in these situations is identical to what has been done above for the two industries. If $p_{Wi} = 0$, then labor demand is assigned as follows:

$$W_{Ti} \begin{cases} \varepsilon(0,\hat{W}] & \text{if } L_T^1 > 0 \\ = 0 & \phantom{\text{if }} L_T^1 = 0 \end{cases} \tag{4.39}$$

$$W_{Ci} \begin{cases} \varepsilon(0,\hat{W}] & \text{if } L_C^j > 0 \\ = 0 & \phantom{\text{if }} L_C^j = 0 \end{cases} \qquad j = 1,2 \tag{4.40}$$

C. Existence of Equilibrium

In the most general multiple group model (one which contains a local good) a discussion of the existence of equilibrium must, as it was in the local good model with one group, be, at least in part, heuristic. We shall discuss briefly the elements of the argument that are peculiar to the multiple group model, taking the existence of equilibrium discussion for the local good model mentioned above as given.

First, the conditions (c.3) and (c.4)´ should be replaced in a straightforward manner by sets of similar conditions, {(c.31),...,(c.3n),(c.41)´, ...,(c.4n)´}, one condition of each type for each household group. The demands and supplies in the conditions (c.4i)´, and the utility function and budget constraint parameters of conditions (c.3i) should refer to individual household groups.[17] Second, the restrictions made in the previous section imply that the generated population levels for the various household groups and the labor demands for the different labor types are bounded. As a result, the excess demand correspondence for this model, E, is bounded. Upper semicontinuity of the labor demands when $\rho_C > 0$ and/or $\rho_T > 0$ follows from (4.39) and (4.40). Clearly, all of the components of other "excess demands", labor supplies, and labor demands for other cases are upper semicontinuous. Convexity is also obviously satisfied in the new model. Thus, hypothesis (H.1) of Theorem 1 must hold.

Hypothesis (H.2) will also be satisfied if, once again, Assumption 1 can be made. Its plausibility in terms of the one group local good model has already been mentioned. That analysis carries over here. The only difference in structure between the models as far as the argument is concerned is that there simply are more population differentials and excess demand for labor components to consider. The prices, however, associated with the population differentials are the corresponding housing prices in ring 3, and those associated with the excess demands for labor are the wage rates for the corresponding labor types. Thus, the arguments for the one population differential and the one excess demand for labor, given above, can clearly be applied here to cover the multiple group cases. With Assumption 1 satisfied, a solution as given in the conclusion of Theorem 1 exists. Conditions (c.41)′- (c.4n)′ guarantee that such a solution will be a normal equilibrium.

Although we are forced to rely on this heuristic argument for the most general multiple group model, a rigorous proof of the existence of equilibrium can be obtained for a somewhat more restrictive model. The problems involved in trying to prove existence for the full multiple group and the one group local good models arise, essentially from the complications caused by the presence of the local good. Eliminating the local good from the model, then, might allow us to prove the existence of equilibrium, at least under certain conditions on the parameters. This is indeed the case. An existence proof for a multiple group model with a traded, but not a local, good can be presented. While such a model is, of course, not as general as the multiple group model with a local good, it does constitute a generalization of the base model.

Structurally, eliminating the local good from the multiple group model set forth in the previous two subsections involves merely striking any references to the local good sector in the functional forms of the model, and the price and excess demand vectors. For simplicity of notation and clarity in highlighting the important adjustments that must be made to the base model proof in order to prove existence for the multiple group model, the following theorem is presented in terms of a two household and one labor group model. The proof, however, can easily be generalized to cover a model involving any number of household and labor groups.

<u>Theorem 3</u>. Let $P = \{p = (p_W, p_H^{12}, p_H^{22}, \tau) | p_W + p_H^{12} + p_H^{22} + \tau \leqq d\}$ and $d > 0$. Given the model described above with two household groups and one labor type, and conditions (c.1), (c.2), (c.31), (c.32) and at least one of conditions (c.41)´ and (c.42)´,[18] there exists $d > 0$ such that an economic equilibrium (with zero excess supplies) exists. In particular, there is some $d > 0$ for which there exists $p^* \varepsilon P$ and $e^* \varepsilon E(p^*)$ such that $e^* = 0$. (with actual property tax revenues raised equal to the pre-specified amount).

<u>Proof</u>. From the discussion given above it is clear that the excess demand correspondence, E, for this model is upper semicontinuous, bounded, and convex. Thus, hypothesis (H.1) of Theorem 1 must be satisfied. It remains to be shown that hypothesis (H.2) of that theorem is also satisfied.

As was done in the proof of Theorem 2, to show that (H.2) holds we restrict our attention to price vectors where the price components sum to d (i.e., $p_W + p_H^{12} + p_H^{22} + \tau = d$). To keep the notation consistent, as much as possible, with that of the proof of Theorem 2, we apportion d among p_W, p_H^{12}, p_H^{22}, and τ by letting $p_W = a_1 \cdot d, p_H^{12} = a_2 \cdot d$, $\tau = a_3 \cdot d$, and $p_H^{22} = a_4 \cdot d$. The a_i's are defined so that $0 \leqq a_1, a_2, a_3, a_4 \leqq 1$ and $a_1 + a_2 + a_3 + a_4 = 1$. The constant term $\hat{a}$, expressed in (3.24) in the proof of Theorem 2, must be re-defined to serve a similar purpose in this proof. Taking note of the fact that there are two household groups, $\hat{a}$ is defined as follows:

$$\hat{a} = \frac{(\overline{W}\ N)/(B\cdot(\alpha_{LH})^{\frac{-1}{\rho_H}} \cdot \sum_{j=2}^{j'} (L^j/2)\cdot \min\ [(\ell_W^j/\ell_W^2)^{\alpha_\ell^1},\ (\ell_W^j/\ell_W^2)^{\alpha_\ell^2}]\cdot)}{1+\ (\overline{W}\cdot N)/(B\cdot(\alpha_{LH})^{\frac{-1}{\rho_H}} \cdot \sum_{j=2}^{j'} (L^j/2)\cdot \min\ [(\ell_W^j/\ell_W^2)^{\alpha_\ell^1},\ (\ell_W^j/\ell_W^2)^{\alpha_\ell^2}])} \tag{4.41}$$

As was done in the proof for the base model, a positive constant, ε, is used to define the term $\tilde{a} = \hat{a} + \varepsilon$, where $\varepsilon < 1 - \hat{a}$. Thus, we again have $0 < \tilde{a} < 1$.

The term, $\tilde{a}$, is used to partition the problem of finding a value of d high enough so that (H.2) is satisfied by considering the following three cases:

(i) $a_2 + a_4 \geqq \tilde{a}$

(ii) $a_1 \geqq \frac{1-\tilde{a}}{2}$, $a_2 + a_4 < \tilde{a}$, $a_3 \leqq 1 - \frac{1-\tilde{a}}{2}$

(iii) $a_3 > \frac{1-\tilde{a}}{2}$, $a_1 < \frac{1-\tilde{a}}{2}$, $a_2 + a_4 < \tilde{a}$

The analysis for cases (ii) and (iii) here is identical to that given for cases (ii) and (iii) in the proof of Theorem 2, and so will not be repeated. For case (ii) the vector $\alpha = (1,0,0,0)$, and for case (iii) the vector $\alpha = (0,0,0,1)$, can be chosen to satisfy (H.2). Thus, we need only be concerned with case (i). Without loss of generality, we may assume that $a_2 \geqq a_4$ for price vectors satisfying case (i).

For all cases other than ρ_1, $\rho_H > 0$, the proof of Theorem 2 may be applied, and so $\alpha = (0,1,0,0)$ chosen to satisfy (H.2). Furthermore, if $\rho_2 < 0$ and the second household group is allocated all of the available land in some ring for a high enough value of d, then, from (3.28), it can be seen that the generated population of households of this type will exceed the given number, and so we may choose $\alpha = (0,0,0,1)$. If the second group is not allocated all of the land in a ring in the limit when $\rho_2 < 0$, then it can be seen that the analysis to be presented below for the case $\rho_1, \rho_2, \rho_H > 0$ will suffice since, for a high enough value of d, the second group's household demand for housing in a ring when $\rho_2 < 0$ will be less than it is when $\rho_2 > 0$. Thus, we restrict our attention to situations where $\rho_1, \rho_2, \rho_H > 0$.

We again seek to show that the limit as d goes to infinity of the total generated population of households of all types that choose to work and reside in rings 2 and beyond, when all of the available land in these rings is allocated to housing households that send a member to work, is greater than the total number of households, $N = N_1 + N_2$. There are now two household types competing for land in rings 2 to j'. Which group, in the limit, is allocated land in a ring, j, can be determined, using expression (2.43) indexed by household type, by noting the value of the following limit:

$$\lim_{d \to \infty} \frac{p_H^{1j}}{p_H^{2j}} = \frac{a_2 \cdot (\ell_W^j / \ell_W^2)^{\alpha_\ell^1}}{a_4 \cdot (\ell_W^j / \ell_W^2)^{\alpha_\ell^2}} \tag{4.42}$$

If the limit (4.42) is greater than 1, then household group 1 is allocated, in the limit, all of the available land in ring j. If the limit is less than 1, then household group 2 obtains, in the limit, all of the land in the ring. Finally, if the value of the limit is 1, then the land, in the limit, can be arbitrarily allocated to the two household groups. Taking note now, of (4.42), the limits (3.26) and (3.28), and the definition of the generated population in a ring of households of a given type that send a member to work, it can be seen that the following inequality, for the limit of the total generated population of households referred to above, must hold.

$$\lim_{d\to\infty}\sum_{j=2}^{j'}(N_{G1}^{Wj}+N_{G2}^{Wj}) \geqq \sum_{j=2}^{j'} \max\,[a_2\cdot(\ell_W^j/\ell_W^2)^{\alpha_\ell^1},\ a_4\cdot(\ell_W^j/\ell_W^2)^{\alpha_\ell^2}]\cdot(B\cdot(\alpha_{LH})^{\frac{-1}{\rho_H}}\cdot L^j/(a_1\overline{W})) \tag{4.43}$$

Note that the right hand side will equal the limit on the left hand side if $a_3 = 0$. It is also important to see the validity of the following:

$$\max[a_2\cdot(\ell_W^j/\ell_W^2)^{\alpha_\ell^1},\ a_4\cdot(\ell_W^j/\ell_W^2)^{\alpha_\ell^2}] \geqq \frac{\tilde{a}}{2}\cdot\min[(\ell_W^j/\ell_W^2)^{\alpha_\ell^1},\ (\ell_W^j/\ell_W^2)^{\alpha_\ell^2}] \tag{4.44}$$

This inequality follows from the fact that $a_2 \geqq \frac{\tilde{a}}{2}$. Finally, given the definitions of $\hat{a}$ and $\tilde{a}$, the following inequality also holds.

$$\frac{\tilde{a}}{1-\tilde{a}}\cdot(B/\overline{W})\cdot(\alpha_{LH})^{\frac{-1}{\rho_H}}\cdot\sum_{j=2}^{j'}(L^j/2)\cdot\min[(\ell_W^j/\ell_W^2)^{\alpha_\ell^1},\ (\ell_W^j/\ell_W^2)^{\alpha_\ell^2}] > N \tag{4.45}$$

Given the restrictions on the a_i's implicit in case (i) and the assumption that $a_2 \geqq a_4$, and inequality (4.44), the right hand side of (4.43) must be greater than or equal to the left hand side of (4.45). Thus, for d large enough, the total generated population of households that work, when all of the available land in rings 2 to j' is allocated to housing these households, is greater than the sum of the given population levels for the two groups. Therefore, in such situations, hypothesis (H.2) will be satisfied when $\alpha = (0,1,0,1)$ is chosen. The discussion in the proof of Theorem 2 on the possibility of having some or all of the available land allocated to housing households that do not work can clearly be adapted here to cover all of the possible sign combinations for ρ_1, ρ_2, and ρ_H. Hence, hypothesis (H.2) must hold for this model.

Thus, the conclusion of Theorem 1 is valid here. That is, there is a price vector $p^* \in P$ and $e^* \in E(p^*)$ such that $e^* \leqq 0$ and $p^*\, e^* = 0$. The discussion in the proof of Theorem 2 showing that $e^* = 0$ for the base model can also be used to show that $e^* = 0$ for this model. In the context of this model, it is condition (c.31) that ensures that $p_H^{12*} > 0$, while condition (c.32) ensures that $p_H^{22*} > 0$. Finally, satisfaction of at least one of the conditions (c.41)′ and (c.42)′ ensures that the solution, (p^*,e^*), is a normal economic equilibrium.

Q.E.D.

III. Welfare Measures

There are several types of welfare measures that might be applied to the models to determine the normative effects of tax system changes. The ones used here take account of household preferences and are, essentially, general equilibrium versions of the partial equilibrium concepts of Hicks compensating and equivalent variation in income. The computational procedure used in this study can easily deal with such measures.

To describe how these measures can be formed and computed, we consider a solution for the model under a property tax classification scheme for different property uses that mirrors, at least on average, a real world classification system to be a base case. The change contemplated for the tax system is a movement to a tax structure in which there is no discrimination in the treatment of property (ie. all users are taxed at the same effective rate). We shall refer to the former solution type as the classification case, and the latter as the equal rate case.

A compensating variation measure can be formed as follows. First, a solution for the classification case is found, and the level of utility attained by households in that solution is noted. Now, consider a situation in which the government can levy a lump sum tax or subsidy on households. Next, find a solution to the equal rate version of the model with the government collecting the lump sum tax or providing the lump sum subsidy so as to constrain the level of utility attained in equilibrium to be equal to the level that was achieved in the classification case. This tax or subsidy provides a willingness-to-pay measure for the households. Thus, it can be used as a measure of welfare change for residents. Landowners, whether they be absentee private owners or the government, however, are affected by the tax system changes through changes in land rents. Differential land rents for a solution can be defined as the difference between aggregate land rents in the urban area on land used for endogenous purposes and what rents for the urban area would be if all endogenous land were rented at the agricultural rental, $\overline{p}_A$.[19] The change in differential land rents in moving from the classification case to the equal rate constrained utility case can be used as a measure of welfare change for landowners and/or simply added to the sum of the compensating variations for the households to obtain an aggregate measure of welfare change.[20]

Something very similar to this can be done to find an equivalent variation measure. To do this we find a solution for the equal rate case, and note the level of utility obtained by households in that solution. Next, we find a solution to the classification model with the government levying a lump sum tax or subsidy so as to constrain the level of utility to be what it was in the equal rate case. This tax or subsidy constitutes another measure of welfare change for the residents of the urban area. A change in differential land rents can again be used to capture the welfare effects of the change in the tax system on landowners.

One other means of distinguishing between the two tax systems involves comparing what may be called the excess burdens of the alternative tax regimes. We consider a no-tax system here to be a base case and compare what are, essentially, equivalent variation measures of the two systems relative to the base case.[21] The model is run under the assumption that there are no property taxes, with the government levying a lump sum tax or subsidy so as to ensure that the level of utility attained in equilibrium is, first, equal to what it is in the classification case, and then, to what it is in the equal rate case. Taking into account the lost property tax revenues and changes in differential land rents, we can find measures of the deadweight loss for each tax system. To keep computation costs reasonable, this last means of measuring welfare change was not employed in the simulations to be presented in the next chapter.

Finally, it should be noted that for a multiple household group model, the compensating and equivalent variation measures can be applied to each group. Thus, we could obtain information on how, and the extent to which different household groups are affected by a given change in the tax system.

CHAPTER 4

FOOTNOTES

[1] The commodity is, like the traded good, a composite commodity. In view of the nature of contemporary urban production it may be thought of as being heavily weighted in favor of services.

[2] Given a concern with distributional issues and limited information on differences in preferences, the emphasis here should be on differences in income.

[3] Although in principle, for a general function model, a zero local good price equilibrium is a possibility, it is not one that we can find since, for technical reasons, we will have to bound the local good price from below at a positive level. As indicated below, however, given the functional forms used, a zero local good price equilibrium is not possible.

[4] Note that a condition analagous to (c.1) is not required here since output price can vary.

[5] If this were not done then, no matter what the wage rate, the local good industry bid land rent, as calculated by (4.2), would exceed $\hat{p}_{LC}$ and increase to infinity as p_C rose above $\hat{p}_C$ up to the point where the bracketed expression in (4.2) became zero. Higher values of p_C would be inconsistent with zero profits.

[6] It can be seen from (4.2) that, when $p_C < \hat{p}_C$, wage rates below $\hat{p}_{WC}$ would yield bid land rents in excess of $\hat{p}_{LC}$. The bid land rents increase to infinity as the wage rate falls to the point where the bracketed expression in (4.2) becomes zero. Still lower values of p_W would be inconsistent with zero profits.

[7] This can always be done if appropriate choices of the values of b and b′ are made.

[8] For simplicity, we use the same notation, and so implicitly the same value, for the bound on the local good labor demands as we do for the traded good labor demand. The analysis is not altered if the bounds differ from one another.

[9] From (4.4) it can be seen that capital demand for the local good industry is bounded from above if the local good industry bid land rent is bounded from above.

[10] Of course, if multiple equilibria exist, then the practice of bounding the local good price could exclude some equilibrium price vectors from the search. If, however, the local good price components of all the equilibrium price vectors are bounded from below at a positive level, then $\underline{p}_C$ can be chosen low enough, in theory, so that the region of search contains all possible equilibrium price vectors.

[11] Since this is not the only factor at work in the model with respect to labor supply, however, the stated relationship is tentative for at least one reason. Although an analysis of (4.8) shows that an increase in the wage rate will flatten the housing price gradient about the price of housing in ring 3 and so extend the boundary of the urban area, the effect on aggregate labor supply is actually unclear. The supply of workers may diminish because the lower housing bid land rents in the CBD rings may mean that housing is outbid for land there, when it may not have been prior to the increase in the wage rate, and because household demands for housing in the residential rings initially occupied rise with the higher incomes, thus lowering the generated population in those rings.

[12] Once again the question of what happens to the size of the generated population is relevant. Aggregate demand for the local good is dependent on the number of households generated, as well as on household demand. If the generated population rises with an increase in p_C, then aggregate demand for the local good may increase. Furthermore, the issue of what happens to aggregate supply depends on whether the local good industry is able to obtain some CBD land. If the industry is initially outbid by other uses for land, then a given increase in p_C may not lead to the industry outbidding other uses for land, and so may not lead to an increase in supply.

[13]The definition of a normal equilibrium should, of course, now be extended to include the excess demand for the local good. A normal equilibrium would be one for which the excess demand for the local good is zero. It has been argued above that a zero local good price equilibrium is not possible given the functional forms used in this model.

[14] If there is more than one labor type, then $N \cdot \overline{W}$ must exceed the maximum potential labor supply for any labor type.

[15] If there is a tie for the minimum, then we arbitrarily choose the index of one of the tied coefficients.

[16] As noted previously, the computer cannot raise zero, positive numbers it takes to be zero, or negative numbers to a power.

[17] Note that if the local good sector is included in the model, a value for p_C must be chosen in conditions (c.3i) and (c.4i)´. In both cases the value to choose would have to be d, thus making satisfaction of the conditions dependent on the size of the simplex.

[18] Conditions (c.41)´ and (c.42)´ for this model, of course, are defined for the first two, not the first three, rings and make no reference to a local good sector.

[19] In order to make meaningful comparisons of aggregate land rents for different solutions it is necessary to subtract agricultural rents because the number of rings in urban use is endogenous and no limit is set, in the model, on the number of rings that can be considered for some use.

[20] Implicit in summing the dollar measures for the different agents to obtain a measure of the total gain or loss to society is the assumption, commonly made for practical reasons in many welfare analyses, that a dollar given to one individual is valued the same as a dollar given to any other individual; or at least that this is a suitable approximation.

[21] An equivalent variation measure, as opposed to a compensating variation measure, should be used here since three different equilibrium cases are involved in the comparisons. It is well known that an equivalent variation measure is transitive, while a compensating variation measure is not.

CHAPTER 5

PARAMETERIZATION AND SENSITIVITY ANALYSIS

The models presented in the previous chapters are sufficiently complex to render them analytically intractable. Thus, to obtain any results about the effects of changes in the property tax system, qualitative or quantitative, numerical simulations must be done. To accomplish this, values must be chosen for the parameters of the models. It is our intent to parameterize the models so that a base case equilibrium for each of them corresponds roughly to the reality for the Boston metropolitan area in or around the year 1980. This chapter presents a detailed discussion of the analysis and data used to parameterize these models and the results of sensitivity analysis applied to the base case parameterization.

There does not seem to be good data available on some of the parameters of the model. On the other hand, realistic data are available on what should be reasonable values for some of the endogenous prices in the model. Through exploitation of some equilibrium conditions, and budget share and elasticity definitions, these "desirable" solution prices are used to find values for the otherwise difficult to determine parameters. Thus, our base case parameterizations are conditional on equilibrium values of some of the endogenous prices. As a result, there are basically two approaches to the sensitivity analysis that can be taken. After finding a base case parameterization and solution, and welfare results for a change in tax structure for that parameterization, sensitivity analysis can be conducted, in a rather simplistic way, by just varying parameter values from what they are in the base case. New equilibria can then be computed for the classification tax structure, and also for changes in that system. Resultant changes in urban structure and welfare from that of the base case can be noted. Since some of the parameters in the base case are calculated conditional on certain endogenous prices and other parameters, varying a parameter value without re-calibrating, in accordance with the relationships among prices and parameters used to find the original parameterization, the whole set of parameter values can lead to an equilibrium that is fundamentally different than the base case equilibrium. Thus, if the base case equilibrium is thought to represent reality, then this approach should be viewed as one which can tell us how the equilibrium under one tax system or another will change if there is a change in tastes and/or technology. It can also tell us how the welfare results will be altered by such changes in tastes and technology. In such an approach, then, the original parameterization is taken to be the "correct" one. Alternatively, for any change in a parameter value, the entire system of parameters can be re-calculated in accordance with the equilibrium and other relations that were exploited to complete the parameterization for the base case. If this approach is taken, then we should not expect much of a change in the nature of the equilibrium for a given tax system, but the welfare effects for a given system may be quite different. This approach, then, is consistent with the notion that the parameters, or the raw data used to calculate some of them, may be subject to estimation or measurement error. As such, the sensitivity analysis can therefore be interpreted as an attempt to take these potential errors into account when presenting results; particularly the welfare effects. With a few exceptions, to be noted below, the latter approach is the one taken here.

Due to computational problems and costs, results for a multiple group model are not given. Since exogenous household income should represent only a small portion of total income, a multiple group model of any significance should allow for multiple labor types. Attempts to find solutions for a three household type - three labor type model (in particular one with low, middle and high income groups) were unsuccessful. Accumulated numerical round-off errors, it seemed, prevented convergence of the algorithm. The errors, in turn, may have been caused by the substantial amount of bounding of the functions of the model that must be done. In any case, the lack of success and costs of computation argued against further attempts at this time. Computational costs tend to increase dramatically with the dimensions of the simplex over which algorithms of this type search. Experience with the base and local good models showed that solutions obtained for the local good model generally took three times as long to find as solutions for the base model. The increase in the dimensionality of search over the base model, however, was just one. The multiple group model, in general, involves an increase in dimensionality over that for the local good model of $n + m - 2$. For the three household type - three labor type model, in particular, the number of endogenous prices over which the algorithm searches is double that for the local good model.[1] As a result, attempts to find solutions by further use of the algorithm employed in this study, or by using more efficient algorithms, will be relegated to future research.

The parameterization of the base case for the base model is discussed in detail in section I. The results of the sensitivity analysis for the base model are presented in section II. The methods by which the base case for the local good model is parameterized, and the values of model parameters, are delineated in section III, while the sensitivity analysis results for this model are presented in section IV, the final section of the chapter.

II. Base Model Parameterization

As noted above, we seek to parameterize a base case so that the resultant equilibrium yields an urban area that resembles, in some ways, the Boston metropolitan area in 1980. Thus, the data used to parameterize the base case for the base model, to the extent possible, are selected from published reports relating to the Boston region for this year. In all, for the base model there are twenty-seven parameters to choose.[2] The values of those parameters chosen for the base case are presented in table 5.1. The rest of this section is devoted to discussing the methodology and data used to select those parameter values.

Direct estimates of the coefficients of the household utility function are not available. Housing price elasticity of demand estimates and budget share data, however, can be found. This information and the definitions of the own-price elasticity of housing demand and the housing budget share for typical households can then be used to implicitly define the household utility function parameters - α_T, α_H, and ρ.[3] This can be done in the following manner. From the housing demand function (2.44), it is easy to see that the own-full price[4] elasticity of housing demand, η^j, for a household residing in ring j is given as follows:

$$\eta^j = \frac{-[p_H^j\cdot(1+a_R\cdot t) + (1/1+\rho)\cdot\alpha_T\cdot(\bar{p}_T/\alpha_T)^{\frac{\rho}{1+\rho}}\cdot(p_H^j\cdot(1+a_R\cdot t)/\alpha_H)^{\frac{1}{1+\rho}}]}{p_H^j\cdot(1+a_R\cdot t) + \alpha_T\cdot(\bar{p}_T/\alpha_T)^{\frac{\rho}{1+\rho}}\cdot(p_H^j\cdot(1+a_R\cdot t)/\alpha_H)^{\frac{1}{1+\rho}}} \tag{5.1}$$

This elasticity varies with the housing services price, p_H^j, and so, in general, will vary by ring. We focus attention, though, on the elasticity for a typical, or average, household living somewhere in the interior of the urban area. The unit price of housing facing such a household in a realistic equilibrium is to be expressed as p_H, while t is to represent the nominal property tax rate facing all households in an equilibrium that is to mirror, on average, the actual situation for metropolitan Boston in or around 1980. For ease of exposition, the following definitions are made.

$$a = p_H\cdot(1+a_R\cdot t) \tag{5.2}$$

$$b = \alpha_T\cdot(\bar{p}_T/\alpha_T)^{\frac{\rho}{1+\rho}}\cdot(a/\alpha_H)^{\frac{1}{1+\rho}} \tag{5.3}$$

Thus, the housing price elasticity of demand, η, for our average household can be written as follows:

$$\eta = \frac{-(a+(1/(1+\rho))\cdot b)}{a + b} \tag{5.4}$$

This expression can then be inverted to express the elasticity of substitution term, ρ, in terms of a, b, and η.

$$\rho = \frac{-b}{(a+b)\cdot\eta+a} \; -1 \tag{5.5}$$

Thus, to choose a value for ρ, we must find values for these three terms. Since property tax discrimination was operative in the Boston area during the period in question, a classification scheme is assumed for the base case. Figures used for the assessment/sales ratios for different property classes in the classification amendment to the Massachusetts state constitution that was passed (which were thought to represent the status quo, on average, for the state) were 40% for residential property, 50% for commercial property, and 55 % for industrial property. The average figures for the Boston area may actually show a greater divergence between residential and business ratios, and so the higher business property figure, 55%, was used for industrial property in the base model, even though no business property in that model is defined as commercial. In particular, we let $a_R = .40$ and $a_I = .55$. To find a value for a, something must be assumed about what the nominal tax rate is likely to be in a base case equilibrium. It is commonly noted that urban residential property tax rates, expressed as a percent of annual housing rentals, generally average about 20 to 25 percent (see e.g. Mills (1980), p. 128). Since Massachusetts, and the Boston region in particular, is thought to be a high property tax area, a residential tax rate of 25% was targeted for the base case. Given that $a_R = .40$, this implies a value of t = .625. Finally, an assumption about the value of the unit price of housing services facing the average household in the base case equilibrium must be made. The value chosen will be that derived from data on FHA-insured existing single-family homes in Massachusetts in 1981.[5] A discussion of how it is derived must be deferred until we take up the parameterization of the housing production function. For now, we simply note that it can be derived from published data, and the value found is p_H = \$4752.42. Thus, the value for a is \$5940.525.

A value for b can be found by looking at budget share data. The gross of tax housing budget share of income net of commuting costs for a typical household, h, can be expressed as $h = a/(a+b)$. This can be inverted to find $b = [(1-h)\cdot a]/h$. Two sources were used to determine a housing budget share net of property taxes for before-tax income. Data from U.S. Department of Housing and Urban Development (1982) on FHA-insured homes for Massachusetts in 1981 on borrower income and housing costs[6] indicated a housing budget share net of property taxes for before-tax income of approximately 18.6%. On the other hand, data from U.S. Department of Labor (1978) on average shelter expenditures and before-tax income for families in the Boston SMSA during 1972-73 yielded a figure of slightly more than 15.1% for the above-mentioned budget share. As a compromise, the approximate average of 16.8% was used for this budget share in the base case parameterization. Income for the model, though, should be viewed as after-tax income not saved (i.e. consumption expenditures). To obtain the budget share, h, we must also net out commuting costs from consumption expenditures. The consumer expenditure survey data in U.S. Department of Labor (1978) include an average annual family consumption expenditure level, inclusive of transportation costs, of $9302.11 for the Boston SMSA in 1972-73. This figure can be scaled up, in a manner described below, so that it is expressed in 1981 dollars. An annual per-mile money cost of commuting of $32.4375, also discussed below, is assumed. Since commuting costs vary by residential location, a location must be assumed for the average household. It is assumed that the typical household lives 5.5 miles from the employment center for the region.[7] The consumption expenditure figure mentioned can then be adjusted to account for the commuting costs of a household living 5.5 miles from the center. This gives a figure that allows us to convert the budget share of 16.8% found above into one appropriate to income net of transportation costs for the model. Doing so yields a housing budget share, net of property taxes, of 20.87%. Using a residential property tax rate of 25%, a gross of property tax budget share of h = 26.1% is found. Given this value and that for the term a found above, we find that b = 16820.107.

The price elasticity of housing demand for samples of FHA data for owner-occupants across different cities and time periods has been estimated in Muth (1971) and Polinsky and Ellwood (1979). These studies suggest a price elasticity of -.7. Thus, we let $\eta = -.7$. The values for a,b, and η can then be used to determine, through (5.5), the value $\rho = .6834$. The utility function coefficients, α_T and α_H, can be set by using (5.3). Inverting (5.3) with respect to the ratio of these coefficients yields:

$$\frac{\alpha_T}{\alpha_H} = (b)^{1+\rho}\cdot(\overline{p}_T)^{-\rho}\cdot(a)^{-1} \tag{5.6}$$

Since the units in which we measure the traded good are arbitrary, what matters for the analysis is the product, $C\cdot\overline{p}_T$, rather than $\overline{p}_T$ and C considered independently of one another. Thus, one of the two terms may be chosen arbitrarily. We choose $\overline{p}_T = 100$. Substituting this into (5.6) then gives a ratio for the utility coefficients of 93.975781. Normalizing so that the coefficients add to 1, then yields the values $\alpha_T = .989471$ and $\alpha_H = .010529$.

Parameters in the household budget and time constraints yet to be determined are v, $\overline{W}$, T, c, and M. An average rush-hour travel speed for urban travel of 20 m.p.h. seems reasonable. It is assumed that each worker works 250 days a year, while the institutionally determined length of a work day is 8 hours. Thus, we choose v = 25 and $\overline{W}$ = 2000. The total amount of time per day available for work, leisure, and commuting is taken to be 16 hours. Thus, using a 365 day year, we assume that T = 5840. Data on the money cost of travel were obtained from U.S. Department of Transportation (1980). The cost per vehicle-mile was calculated from data contained in the study to be 10.38¢/mi..[8] A vehicle occupancy rate of 1.6 persons/vehicle was assumed. This is a common highway engineer's estimate and also agrees precisely with data on the numbers of vehicles and passengers per day crossing a boundary of the downtown Boston business district in a Boston region transportation study [Boston Redevelopment Authority (1967, p.44)]. Thus, the value chosen for the money travel cost parameter is c = $32.4375. The data in U.S. Department of Labor (1978) on total family income and wages and salaries indicate that the proportion of income that has wages and salaries as its source fluctuates in a very narrow range about 75% for Boston, other regions in the Northeast, and the country as a whole. As a result, we take exogenous income to be 25% of total income. As noted above, however, the relevant income measure for the model is consumption expenditure. We have that value for Boston for the 1972-73 period. We need the corresponding figure, though, for the 1980-81 period. Personal outlays in the U.S. between 1972 and 1981 increased from $747.2 billion to $1,898.9 billion, while the number of households increased from 66.7 million to 82.4 million.[9] Thus, we can multiply 1972 consumption expenditures for the Boston SMSA by a factor of (1898.9/82.4)/(747.2/66.7) to express income in 1981 dollars. Doing so, we obtain approximately $19,136 for income. Thus, we choose M = $4784.

The housing production function is parameterized in the manner given in Muth (1975). From previous work [Muth (1969)], Muth assumes a value for the housing elasticity of substitution of $\sigma_H = .75$. This implies that $\rho_H = .3333$. Koenker (1972) has also estimated σ_H, and found a value of .7. For the base case, the Muth figure is used. The procedure employed by Muth to estimate the coefficients of the CES production function involves solving a set of two simultaneous equations, for these coefficients, that were parameterized using 1966 FHA data for a number of cities, and the assumed value for σ_H. For this study, we use the 1981 FHA data for Massachusetts. The equations, which result from profit maximization and the production technology, are given as follows:

$$(\alpha_{LH}/\alpha_{KH}) = (p_{LH}\cdot L_H/(\bar{p}_K\cdot K_H))^{\frac{1}{\sigma_H}}\cdot(p_{LH}/\bar{p}_K)^{\frac{-(1-\sigma_H)}{\sigma_H}} \qquad (5.7)$$

$$\alpha_{KH} = [(\alpha_{LH}/\alpha_{KH})\cdot(L_H)^{\frac{-(1-\sigma_H)}{\sigma_H}} + (K_H)^{\frac{-(1-\sigma_H)}{\sigma_H}}]^{-1}\cdot(s_H)^{\frac{-(1-\sigma_H)}{\sigma_H}} \qquad (5.8)$$

Note that the equations are not indexed by ring. Thus, terms like p_{LH}, L_H, and K_H should be thought of as referring to housing for an average household. Equation (5.7) is derived by taking the ratio of the two marginal productivity conditions for profit maximization, while equation (5.8) is simply a re-statement of the production function. A unit of housing is taken to be the amount supplied by a typical FHA-insured existing home in Massachusetts for 1981. As a result, we let $s_H = 1$, and use the FHA data for such a typical house in choosing values in (5.7) and (5.8). The data alluded to reveals a median lot size for existing homes of 7200 sq. ft., or L_H = .1652893 acres. A median site price of \$.87/ sq. ft. therefore yields site expenditures for the median lot of \$6264. Since such a price embodies expenditures for non-building capital invested in the land, we follow Muth's practice of reducing this figure by one-half to obtain a price for raw land. Doing so, yields a per-acre price of land of \$18949. Assuming an interest rate of 8.5% per year[10], gives us p_{LH} = \$1610.63. We define a unit of structure (capital) to be that obtained with an expenditure of \$1000. The FHA data reveals a median house price for existing homes in Massachusetts of \$36,363. Subtracting the derived price of raw land for the median lot, \$3132, from the median house price gives us structural expenditures of \$33,231. Thus, we take K_H=33.231. Assuming depreciation, maintenance, and repair costs of 3.5% per year, and mortgage insurance, hazard insurance, and miscellaneous costs of 1.5% per year, together with the interest rate of 8.5%, the annual rental for a unit of structure is $\overline{p}_K$ = \$135. Substituting these values into (5.7) and (5.8) will then yield α_{LH} = .0307426 and α_{KH} = 3.0349022. Finally, these coefficients can be used to find a value for the unit price of annual housing services facing the typical household in the base case. Setting price equal to average cost for housing for the typical household yields:

$$p_H = [(\alpha_{LH})^{\sigma_H} \cdot (p_{LH})^{1-\sigma_H} + (\alpha_{KH})^{\sigma_H} \cdot (\overline{p}_K)^{1-\sigma_H}]^{\frac{1}{1-\sigma_H}} \qquad (5.9)$$

Using the values obtained, for the terms on the right hand side of (5.9), gives a price of housing services, mentioned above, of p_H = \$4752.42.

The CBD size in the model is fixed. We wish the radius of the employment node to be roughly consistent with the reality for the Boston area. Employment in the model, however, is completely centralized. In reality, much employment is suburbanized. The real world Boston CBD, therefore, will employ fewer workers than will the model CBD (assuming all workers in the model choose to work). Thus, it is reasonable to choose the size of the CBD in the model to be larger than that of the actual CBD. Some data which helps in this regard can be found in a transportation study for the city of Boston [Boston Redevelopment Authority (1967)]. A geographical area, referred to as "Boston Proper" and defined in the study, can be used as a notion of a CBD (for the city of Boston). Data is also available in the study on the number of persons crossing the boundary of this area during the workday peak travel periods. A value for the number of workers traveling to the Boston CBD during the morning peak can be approximated using this data. One approach to take to determine the CBD size for the model, would be to first estimate the radius of the Boston Proper area, and find the total amount of land in an annulus with that radius. The employment total can then be used to establish a figure for number of workers per unit of land. This and the number of workers specified in the model can be used to determine the amount of land in the first ring, and so a CBD radius, required to yield the same value for workers per unit of land. One problem with this approach, though, is that, given jurisdictional boundaries in the region, the actual CBD for the region as a whole surely extends beyond the city of Boston's border. Thus, the figure for employment probably under-estimates the actual number of CBD workers and so the number of workers per unit of land. It seems that it would be better, instead, to estimate the actual amount of land, used for production, in arriving at a figure for the number of workers per unit of land. Our estimate for the Boston Proper land area is 2.64 square miles, or 1687 acres. Not all of this land, though, is used for production. The study indicates that 34.7% of the land in what is called the Central Area of Boston (a subset of the Boston Proper Area) is devoted to roads and parking. In addition, some of the land in the Boston Proper area is used for parks, churches, and the like. As a result, we assume that 60% of the Boston Proper area (i.e. 1012 acres) is allocated to production. Data on the number of people crossing the Boston Proper boundary during peak periods by mode is given in Table 12 of the study. Excluding those crossing the boundary in steamships and trucks, we arrive at an estimate of the number of workers in the Boston Proper area after adjusting the table figures, which give totals for the two peak periods combined, to account for the fact that some drivers enter and exit the boundary during the same peak period, and so that the data refer to just the morning peak period.[11] The figure arrived at for the number of workers is 165,798. This gives a worker per acre figure of 163.83. The data on total population and the number of people per household in U.S. Bureau of the Census (1982b) show that the number of households in both the Boston SMSA and the Boston Urbanized Area for 1980 is very close to 1 million. Thus, we choose N = 1 million. We also make the assumption of 1 worker per household, so that there are 1 million workers in the model (assuming that all households choose to send a member to work). Finally, it is also assumed that one-third of the land in every ring is available for endogenous use. This is consistent with what is assumed in other studies of this type (see e.g. Muth (1975) and

Richter (1979)]. Thus, the model CBD should contain 18,312 acres in total (6104 acres available for endogenous use). This implies a CBD radius of 3.02 miles. This was rounded down to 3 miles. All other rings are assumed to have a width of one-eighth of a mile. This particular figure was chosen because it is the length of an average city block. There is not much of a problem, though, with choosing an even smaller width. Computational costs are not very sensitive to this term.

To parameterize the traded good production function, the value of certain endogenous prices must be targeted. That is, values which some endogenous prices should hold, at least approximately, in the base case equilibrium must be chosen. In particular, we seek solution values for p_W and p^1_{LT} that are consistent with a realistic base case equilibrium. To determine P_W we use the annual total income (exogenous plus endogenous) figure of $19,136 for a typical household, expressed in 1981 dollars, given above. Assuming that wages constitute 75% of income, we obtain a target hourly wage rate of $7.18. For the CBD traded good land rental, a value is sought that is high enough so that we can be assured that the traded good industry will be allocated all of the available CBD land in the base case equilibrium, but low enough so that it is reasonably close to the housing bid land rent for the first residential ring.[12] Thus, given the values chosen for the housing production function parameters, we choose a target CBD annual land rent of p^1_{LT} = $12,000. A value of the elasticity of substitution term, ρ_T, can now be determined conditional on a ratio of production function coefficients. Assuming full employment of the potential labor force in equilibrium and that the traded good industry outbids other uses for CBD land, the labor demand equation (2.52) can be inverted with respect to ρ_T to obtain

$$\rho_T = \frac{\ell n\ ((\alpha_{WT}/\alpha_{LT})\cdot p^1_{LT}\cdot(1+a_I\cdot t)/p_W)}{\ell n\cdot(N\cdot\overline{W}/L^1)} \tag{5.10}$$

If we determine the value of ρ_T in this way, and the scale parameter, C, in a manner to be mentioned below, it can be seen that our target prices will be approximately realized in equilibrium, no matter what value is chosen for the ratio α_{WT}/α_{LT} (provided, of course, that it is not chosen so that (5.10) yields a value that lies outside of the range of acceptible values for ρ_T). Given this degree of arbitrariness, then, we let $\alpha_{WT} = .20$ and $\alpha_{LT} = .01$. Substituting these values into (5.10), we obtain $\rho_T = -.1572$. To parameterize α_{KT} and C, we must look to the amount of tax revenues that are to be raised in equilibrium. Data provided to me by the Assessing Department of the city of Boston shows that the cities and towns in the Boston SMSA raised approximately \$1.7 billion in property tax revenues for 1980. Thus, we let R = \$1.7 billion. It is possible to establish, approximately, what the amount of property tax revenues raised from the residential sector should be in the base case equilibrium, and so, given R, approximately what the level of business tax revenues will be. It was stated above that the typical household in the base case equilibrium is to have an income of \$19,136, live 5.5 miles from the employment center, and spend 20.87% of its income net of commuting costs on housing. Given the value chosen for c and a residential tax rate of 25%, then, the typical household should pay about \$989 per year in property taxes. Thus, residential property taxes for the entire urban area should approximate \$989 million. Consequently, business property taxes should approximate \$711 million. The business property tax rate is implied by the assumed value of $t = .625$. Therefore, with the value taken for p^1_{LT} in equilibrium and assuming again that the traded good industry obtains all of the available CBD land, revenues arising from the business tax on land can be subtracted from total business tax revenues to obtain revenues from the tax on the traded good's use of capital. Given the business property tax rate and the rental rate on capital, $\overline{p}_K$, we can then determine the amount of capital, K_T, that the traded good industry should be using in the base case equilibrium. Hence, we can finally determine a value for α_{KT} by inverting (2.53) as follows:

$$\alpha_{KT} = (K_T/L^1)^{1+\rho_T} \cdot (\alpha_{LT} \cdot \overline{p}_K/p^1_{LT}) \tag{5.11}$$

Substituting in the values found for the terms on the right hand side of (5.11) yields $\alpha_{KT} = .0808035$. The scale parameter, C, can now be found by inverting the bid land rent function (2.48) with respect to C. All other terms in (2.48) have been specified. Using these values in the inverted expression yields C = 777.11.

Just two parameters remain to be specified-- $\overline{p}_A$ and α_ℓ. Their values were simply chosen so that, with the other parameter values selected, a reasonable population density gradient and city size would result. The effects of changes in these values are realized mainly in the steepness of the housing price gradient and in city size, with increases in α_ℓ and $\overline{p}_A$ both leading to steeper gradients and smaller city sizes. The values chosen for the base case parameterization were $\overline{p}_A$ = \$450/acre and $\alpha_\ell = .31$.

TABLE 5.1

BASE CASE PARAMETERIZATION

Utility Function:	$A = 1$ $\alpha_H = .010529$ $\alpha_T = .989471$ $\alpha_\ell = .31$ $\rho = .6834$
Budget and Time Constraints:	$v=25$ $\overline{W}=2000$ $c=32.4375$ $T=5840$ $M=4784$
Housing Production Function:	$B=1$ $\alpha_{LH}=.0307426$ $\alpha_{KH}=3.0349022$ $\rho_H=.3333$
Traded Good Production Function:	$C=777.11$ $\alpha_{LT}=.01$ $\alpha_{WT}=.20$ $\alpha_{KT}=.0808035$ $\rho_T=-.1572$
Miscellaneous:	$\overline{p}_T=100$ $\overline{p}_A=450$ $a_R=.40$ $a_I=.55$ $\overline{p}_K=135$ $N=1,000,000$ $R=1,700,000$ CBD radius=3

II. Base Model Results

In this section computed solutions (i.e. economic equilibria) for the base model are presented and discussed. The solutions were found with the aid of a fixed point algorithm, FIXPOINT, developed at the Computer Research Center of the National Bureau of Economic Research. To make use of this program users must write a subroutine peculiar to their particular applications. To measure the computation costs for these solutions, it is perhaps better to note the number of excess demand vectors calculated, and so the number of price vectors passed by the algorithm to the user written subroutine, before an equilibrium is found, rather than actual CPU time. All of the base model solutions presented were found after 100 to 200 price vectors were examined by the subroutine.

The parameterization in table 5.1 was used to yield the status quo solution, i.e. the one which is to reflect the reality for the Boston SMSA, and serve as the basis of comparison for a move to an equal rate tax system and changes in the technological and other parameters of the model. Tables 5.2 to 5.16 present detailed results for various solutions. Data is provided for the first, or CBD, ring, the second ring, every tenth ring thereafter, and the last, or boundary, ring. The population density gradients are expressed in terms of households per acre of land used for residential development, the housing capital/land ratios are measured in units of capital per acre of residential land, and the bid housing price and land rent gradients are expressed in dollars per acre. Other data given for individual solutions are the equilibrium hourly wage rate, the effective property tax rates, the residential and business tax bases, the traded good bid land rent for CBD land, and aggregate land rents for the urban area. With the exceptions of table 5.3, which characterizes the equilibrium for the base case parameterization with an equal tax rate system, and table 5.14, which relates to an equal rate tax system and a change in the population parameter, these tables apply to solutions for the classification tax scheme (i.e. a_R = .40 and a_I = .55).

As noted in the previous section, there are basically two approaches that we can take in conducting the sensitivity analysis; one being to recalibrate the entire parameterization when considering a change in one parameter value so that the resultant equilibrium, under tax classification, is essentially the same as the base case classification equilibrium. We may then investigate whether the welfare results are affected, even though the classification solutions are the same. For most of the reported parameter changes this is what was done. However, the second approach mentioned above (simply change one parameter value with no concern for duplicating the base case solution) was tried on a number of parameters. It is the solutions for these changes that are presented in tables 5.4 to 5.16. Since all parameter changes of the first type are designed to yield the important aspects of a given solution, the structural characteristics of solutions for these changes are not presented. Welfare results for these changes, and those of the second type as well, are, however, depicted in tables 5.17 and 5.18. The zeroes for population density and the capital/land ratio for the CBD ring in all of the tables reflect the fact that the traded good industry outbids housing for CBD land in all solutions. With tables 5.2 and 5.3, analysis of the important structural effects of a change from a status-quo classification tax system to a non-discriminatory one can be given. It can be seen that the gradients (density, capital/land ratio, housing price, housing land rent) become flatter with the move to an equal rate system. City size, however, remains constant. As might be expected, the more favorable treatment of business property in the equal rate case relative to the classification case leads, in equilibrium, to greater demands for labor and land, and so a higher wage rate and CBD land rent. The effective tax rate, for all property types, in the equal rate solution is between the rates on residential and commercial property for the classification case. Thus, the traded good industry faces a lower tax rate, and the residents a higher tax rate, in the new equilibrium. As a result, the urban area's residents spend less, net of property taxes, in aggregate on housing and the value of the property (capital and land) used by the traded good industry increases. Finally, aggregate land rents in the urban area increase. Welfare results for this and other situations will be discussed below.

Since the FHA data used to parameterize the housing production function was thought to be fairly reliable, re-calibration to duplicate the base case solution for cases where one of the housing production parameters is altered was not done. Instead, we consider, in tables 5.4 to 5.9, changes (increases and decreases) in individual housing production function parameters, holding all other parameters constant. Thus, we may view comparisons of the solutions for these changes with that for the base case as an indication of what would happen if the housing production technology changes.

Since it was thought that the solutions may be much more sensitive to changes in the elasticity of factor substitution than to changes in the land and capital coefficients, the latter were increased and decreased by 20%, while the elasticity of substitution was increased and decreased by only 10%.[13] Table 5.4 characterizes the equilibrium when the elasticity of substitution is lowered from .75 to .675 (i.e. ρ_H = .4815). Table 5.5 characterizes the solution when the elasticity of substitution is increased to .825. Results here are perhaps surprising. A comparison of equilibrium utilities for the two changes and the base case reveal that, at least over the range of parameter values considered, the welfare of the residents is increased with decreases in the elasticity of substitution. Decreased substitution possibilities led to higher welfare in spite of the fact that wages fell ($7.30 to $7.10) and the residential tax rate nearly doubled (16.6% to 31.3%). An explanation of the welfare result seems to lie in the fact that housing prices increased substantially with increases in the elasticity of substitution. Housing prices in the first residential ring increased by a factor of nearly 16 for the high elasticity relative to the low elasticity case. One last effect that is quite noticeable is that city size appears to be sensitive to the elasticity, but only on the low elasticity side. There is virtually no difference in city size between the base case and the increased elasticity case, but the urban area increases by 12 rings (1.5 miles) for the decreased elasticity case.

Tables 5.6 and 5.7 show the effects of a 20% decrease and a 20% increase, respectively, for the land coefficient, α_{LH}. Here the effects on wages and the tax rates are minimal. There is no change in the wage rate and small increases in the tax rates, with a lower coefficient. Examination of the utilities again reveals that decreases in the parameter lead to increases in welfare. This also, it seems, can be attributed to lower housing prices in the case where the parameter is decreased. In this case, though, just what it is that causes this effect seems clear. A decrease in α_{LH} can be shown to imply technological progress. Given that $\rho_H > 0$ in this case, a decrease in α_{LH} means that more housing services can be produced with the same amounts of capital and land. Thus, it is not surprising that equilibrium housing land rents fall and welfare increases. Finally, city size is sensitive to the coefficient, with decreases in α_{LH} leading to smaller city sizes. This time the effect is symmetric, with a change in size of 10 rings for both cases. Tables 5.9 and 5.10 reveal that the effects of increases and decreases in α_{KH} are qualitatively the same as those for α_{LH}, except that there is now a measurable change in the wage rate. As might be expected, though, the changes in α_{LH} have a greater impact on the population density gradient.

Tables 5.10 and 5.11 show the effects of 50% changes in the agricultural land rent. For the most part, the effects are minor. Changes in the parameter seem to have greatest impact on city size and the steepness of the population density gradient. Decreases in the agricultural land rent appear, not surprisingly, to lead to larger city sizes and flatter density gradients. Also not surprisingly, a smaller $\overline{p}_A$ seems to lead to higher utility for the area residents. Since there is no change in the wage rate and even a slight increase in the residential tax rate with the lower agricultural rent, the effect on welfare again seems to be fueled by lower housing prices.

The results for changes in α_ℓ are similar. Tables 5.12 and 5.13 depict solutions for a 20% decrease and a 20% increase, respectively, for this parameter. Again, city size is smaller and the population density gradient flatter for a decrease in the value of the parameter. Welfare comparisons do not make sense here since it is a utility function parameter that is being varied.

As will be discussed below, a move to an equal rate tax system for the base case parameterization leads to an increase in household welfare. If the change is made, then, it would seem possible that some migration to this urban area from other regions would occur. The model, however, depicts a closed region. There can be intra-regional movement, but provision is not made for inter-regional migration. In reality, if the welfare gain is large enough (it must at least cover moving costs) residents in other regions would be attracted to the area in which the tax change is made. To examine the effects on the area of such migration, and to bound the extent of the movement, the population parameter was increased by 20% to 1.2 million. If the population is increased, however, so should the tax revenues that have to be raised. The simple assumption made here is that per capita tax revenues should be constant. Thus, we also increase tax revenues by 20% so that R = \$2.04 billion. The model was then run under the equal rate system. The results are presented in table 5.14. As was to be expected, the gradients are shifted up, the city expands, and the tax bases increase. The wage and tax rates, however, are virtually unaffected. Examination of equilibrium utilities reveals that household welfare is lower with the larger population than it is for the base case under the classification tax scheme. Thus, even with zero moving costs, migration to the area because of the tax policy change would be less than the 200,000 household increase set for this case.

Finally, the impact of decisions to decrease or increase the size of the area that is zoned industrial (i.e. the CBD) can be studied. Tables 5.15 and 5.16 show the effects of changes in zoning policy that would decrease and increase, respectively, by 10% the radius of the CBD ring. There appear to be tendencies for wage rates to increase, and the traded good industry bid CBD land rent to decrease with increases in CBD size. Both of these effects can be attributed, quite understandably, to an increase in the supply of land to the traded good industry. There also seems to be a weak tendency for the rates to decrease with CBD size. The size of the urban area, after accounting for a change in the radius of the first ring, increases with increases in CBD size. The equilibrium utilities also show that household welfare tends to increase with the radius of the CBD.

Tables 5.17 and 5.18 present welfare results for various parameter changes of movements from a classification tax system to an equal rate system. For the base case such a move results in a compensating variation welfare gain of about $36.5 million per year for residents of the area. Taking into account changes in differential land rents, the aggregate welfare change is $37.1 million. Results using an equivalent varia-measure (not shown) agree quite closely with those compensating variation based figures. The household equivalent variation welfare gain is $33.9 million per year and the aggregate welfare gain is $34.8 million per year. To keep compensation costs low, it was decided to use just one of the two measures for the parameter changes. Since a classification solution was calculated in each case, the compensating variation measure was chosen. The compensating variation approach requires the computation of two equilibria, while using the equivalent variation approach would result in the calculation of three solutions for each parameter change.

The second to tenth sets of results in table 5.17 refer to changes considered for individual parameters or data values used to find model parameter values, that are accompanied by whatever changes in the whole set of parameters are necessary to maintain the major aspects of the base case classification solution. The second set listed in the table correspond to the two budget share extremes gleaned from two different sources - the Consumer Expenditure Survey and the FHA data. The housing budget shares, net of property taxes, on before-tax income for these two sources were 15.1% and 18.6%, respectively. These figures give rise to the values of h that are listed. The third set reflects a decrease and an increase in the assumed value for the housing price elasticity of demand which, it can be recalled, was used to parameterize the utility function. The value for ρ_H in case 4 of the table refers to an alternative estimate of the housing elasticity of factor substitution (σ_H=.7) found by Koenker (1972). Cases 5 and 6 refer to changes in income levels. The assumed percentage of total income, for the typical household, that is exogenous was altered to 15% and then to 35%. The base case figure for total income was decreased and then increased by 20%. Case 7 presents results for 20% changes in the per mile money cost of commuting. Case 8 refers to 10% changes in the CBD radius. A figure of $12,000 for the CBD land rent in equilibrium was assumed in the base case parameterization. Case 9 of the table shows the effects of varying this land rent by 50%. Finally, case 10 relates to 10% changes in the assumed residential tax rate of 25% (t=.625). Cases 11 to 16 give the welfare results for the parameter changes referred to in tables 5.4 to 5.16. It should be recalled that the ratio α_{WT}/α_{LT} was chosen arbitrarily, within certain limits, for the base case parameterization. The elasticity of substitution term was calculated conditional on this ratio. Altering the ratio, which is equivalent to altering the value of the elasticity of substitution for traded good production, will not change the classification solution, but could change the welfare results. Such results for three very different ratios are presented in case 17.

For a given α_{WT}/α_{LT} ratio, the results in table 17 reveal a notable consistency, both qualitatively and quantitatively. In all cases the residents benefit from a change to an equal rate tax system. There is an aggregate welfare gain in all cases as well. Changes in differential land rents are all small relative to the compensating variation measure.

Among the parameter changes for which the parameterization is re-calibrated, the largest variations in welfare are for the budget share, price elasticity, and total income changes. The changes that result in the greatest variation in welfare - 20% changes in total income - also seem to yield the most interesting conclusion. The gain in household welfare from the change in tax policy is greater for lower total household income levels. This suggests that perhaps, in a multiple group model, the lowest income groups will benefit the most, in terms of a dollar measure of willingness-to-pay, from a shift to a non-discriminatory tax system, thus strengthening the argument for such a change.

Among the parameter changes for which there is no re-calibration, the largest welfare variation results from the 20% changes in the capital coefficient, α_{KH}. All changes, though, agree quite closely with the welfare results for the base case. More significant divergences from the base case welfare results are found for the variations in α_{WT}/α_{LT}. Given the method of parameterization, the ratio is positively related to ρ_T, and so negatively related to σ_T, the traded good elasticity of factor substitution. The result obtained, stated in terms of this elasticity, is that the welfare gains are greater for larger assumed values of σ_T. Thus, we may state that our welfare results are conditional on the value of one arbitrarily specified parameter, σ_T. If this parameter can be estimated, though, we can then be quite confident in the magnitude of the welfare results for the base case. Unfortunately, it would seem to be a difficult parameter to estimate. It is a production function parameter for a rather general composite commodity - one which is a composite of all non-housing consumer goods and services for an urban area. It should be strongly noted, though, that the qualitative results are robust. In all cases, including those for variation in the assumed value of σ_T, there is a welfare gain for households, and in aggregate, in adopting an equal rate tax system.

All of the results considered above are based, in essence, on changes in one parameter or data value at a time. It is possible that the results could be quite different for multiple changes. It is not feasible, though, to investigate the thousands of parameter change combinations that could be considered. To gather some evidence on the effects that multiple parameter changes would have and on bounds for the welfare changes, the following was done. For each of the re-calibration single parameter cases (2-10) of table 5.17, which change resulted in the higher compensating variation value and which resulted in the higher aggregate welfare gain was noted. Solutions were then found for parameterizations where all of the kinds of parameter or data value changes for cases 2 to 10 occur simultaneously. Welfare results are given in table 5.18. Cases 1 and 2 of the table give the results for parameterizations that use all of the changes for cases 2 to 10 of table 5.17 that lead to the higher welfare gain. Case 1 uses the changes that led to the higher compensating variation, and case 2 uses the changes that led to the higher aggregate welfare gain. Cases 3 and 4 present results for parameterizations that use all of the changes for cases 2 to 10 of table 5.17 that lead to the smaller welfare gain. Since, for some of the cases in table 5.17, both of the compensating variation and aggregate welfare measures were lower than the corresponding measures for the base case solution, cases 5 and 6 of table 5.18 refer to parameterizations for which the base case parameter values for those cases in table 5.17 where this occurs were used along with the changes that led to the higher welfare gain in the other cases. Although we cannot be certain of it, the lowest and highest welfare gains in table 5.18 can serve as lower and upper bounds on the welfare gain for any combination of the parameter changes given in cases 2 to 10 of table 5.17. It seems likely that these values should at least be close to the true bounds. The results in table 5.18 show that the aggregate welfare gain varies from \$9.5 million to \$57.8 million per year. The qualitative result that a change to an equal rate tax system leads to an improvement in welfare, however, still holds.

TABLE 5.2 - BASE CLASSIFICATION

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5232	6798
2	20.94	587.18	5228	5343
12	17.68	494.23	5137	5342
22	14.75	411.15	5047	4180
32	12.14	337.51	4958	3213
42	9.85	272.85	4869	2420
52	7.84	216.68	4781	1779
62	6.12	168.51	4693	1273
72	4.66	127.83	4606	880
82	3.44	94.08	4520	585
87 (City Limit)	2.91	79.62	4477	468

Wage rate: $7.18

Effective tax rate (residential property): 24.9%

Effective tax rate (industrial property): 34.2%

Residential tax base: \$3.9419 x 10^9

Business tax base: \$2.0669 x 10^9

Traded good industry bid land rent (CBD): $11,996

Aggregate land rents: \$3.6081 x 10^8

TABLE 5.3 - BASE CASE: EQUAL RATE

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5226	6686
2	20.84	579.98	5221	6613
12	17.61	488.63	5132	5262
22	14.71	406.93	5042	4123
32	12.12	334.45	4954	3174
42	9.84	270.75	4866	2395
52	7.85	215.37	4779	1765
62	6.14	167.82	4692	1266
72	4.68	127.60	4606	878
82	3.47	94.18	4520	586
87 (City Limit)	2.94	79.84	4478	470

Wage rate: $7.25

Effective tax rate (residential property): 27.5%

Effective tax rate (industrial property): 27.5%

Residential tax base: $\$3.9130 \times 10^9$

Business tax base: $\$2.2214 \times 10^9$

Traded good industry bid land rent (CBD): $12,752

Aggregate land rents: $\$3.6213 \times 10^8$

TABLE 5.4 - ρ_H = .4815

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	2099	5821
2	14.71	279.93	2097	5771
12	13.07	248.26	2049	4830
22	11.52	218.40	2002	3955
32	10.07	190.37	1955	3259
42	8.70	164.19	1909	2618
52	7.43	139.87	1863	2065
62	6.25	117.45	1818	1594
72	5.17	96.94	1774	1199
82	4.19	78.36	1729	875
92	3.31	61.73	1686	615
99 (City Limit)	2.75	51.26	1655	467

Wage rate: $7.10

Effective tax rate (residential property): 31.3%

Effective tax rate (industrial property): 43.0%

Residential tax base: $\$2.7912 \times 10^9$

Business tax base: $\$1.8588 \times 10^9$

Traded good industry bill land rent (CBD): $10.964

Aggregate land rents: $\$3.8390 \times 10^8$

TABLE 5.5 - ρ_H = .2121

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	32687	7477
2	23.83	1200.30	32669	7385
12	19.36	972.72	32316	5724
22	15.56	779.51	31963	4376
32	12.35	616.99	31612	3296
42	9.67	481.66	31262	2442
52	7.45	370.26	30913	1775
62	5.64	279.71	30564	1264
72	4.19	207.19	30217	878
82	3.05	150.04	29871	594
88 (City Limit)	2.48	122.12	29664	463

Wage rate: $7.30

Effective tax rate (residential property): 16.6%

Effective tax rate (industrial property): 22.9%

Residential tax base: $6.9004 x 10^9

Business tax base: $2.3287 x 10^9

Traded good industry bid land rent (CBD): $13,272

Aggregate land rents: $3.8130 x 10^8

TABLE 5.6 - α_{LH} = .0245941

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5006	7261
2	26.06	728.37	5002	7168
12	21.36	595.25	4914	5477
22	17.23	478.74	4827	4096
32	13.65	377.93	4740	2989
42	10.57	291.85	4654	2118
52	7.98	219.52	4569	1449
62	5.83	159.90	4484	949
72	4.09	111.90	4400	590
77 (City Limit)	3.37	91.91	4358	454

Wage rate: $7.18

Effective tax rate (residential property): 25.0%

Effective tax rate (industrial property): 34.4%

Residential tax base: $3.8970 x 10^9

Business tax base: $2.0615 x 10^9

Traded good industry bid land rent (CBD): $11,970

Aggregate land rents: $3.2865 x 10^8

TABLE 5.7 - α_{LH} = .0368911

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5448	6403
2	17.43	490.16	5444	6341
12	15.02	421.14	5350	5179
22	12.82	358.56	5258	4179
32	10.84	302.17	5166	3327
42	9.06	251.71	5074	2608
52	7.47	206.95	4983	2008
62	6.07	167.59	4893	1516
72	4.84	133.36	4804	1118
82	3.79	103.95	4715	802
92	2.89	79.07	4627	557
97 (City Limit)	2.50	68.23	4583	457

Wage rate: \$7.18

Effective tax rate (residential property): 24.7%

Effective tax rate (industrial property): 34.0%

Residential tax base: \$3.9899 x 10^9

Business tax base: \$2.0707 x 10^9

Traded good industry bid land rent (CBD): \$12,015

Aggregate land rents: \$3.9090 x 10^8

TABLE 5.8 - $\alpha_{KH} = 2.4279218$

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	2831	6677
2	21.97	489.96	2828	6601
12	18.45	410.01	2769	5206
22	15.30	338.81	2711	4037
32	12.51	275.97	2653	3071
42	10.06	221.06	2596	2284
52	7.93	173.66	2539	1656
62	6.11	133.31	2483	1164
72	4.58	99.52	2428	788
82	3.32	71.81	2373	510
84 (City Limit)	3.09	66.95	2362	465

Wage rate: $7.13

Effective tax rate (residential property): 28.8%

Effective tax rate (industrial property): 39.6%

Residential tax base: $\$3.1549 \times 10^9$

Business tax base: $\$1.9484 \times 10^9$

Traded good industry bid land rent (CBD): $11,411

Aggregate land rents: $\$3.4014 \times 10^8$

TABLE 5.9 - α_{KH}= 3.6418826

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	8693	6887
2	20.33	679.94	8687	6812
12	17.21	574.18	8556	5437
22	14.41	479.47	8426	4276
32	11.91	395.32	8296	3306
42	9.70	321.22	8168	2506
52	7.77	256.64	8040	1858
62	6.11	201.04	7913	1342
72	4.68	153.84	7786	939
82	3.49	114.45	7660	633
89 (City Limit)	2.79	91.19	7573	468

Wage rate: $7.22

Effective tax rate (residential property): 22.1%

Effective tax rate (industrial property): 30.4%

Residential tax base: $4.6769 x 10^9

Business tax base: $2.1571 x 10^9

Traded good industry bid land rent (CBD): $12,439

Aggregate land rents: $3.7505 x 10^8

TABLE 5.10 - $\overline{p}_A$ = 225

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5208	6392
2	20.01	560.64	5203	6320
12	16.84	470.58	5113	5004
22	14.01	390.25	5023	3899
32	11.49	319.20	4934	2983
42	9.28	256.97	4845	2234
52	7.36	203.08	4757	1632
62	5.71	157.03	4670	1158
72	4.31	118.29	4584	794
82	3.16	86.32	4498	522
92	2.22	60.55	4412	325
98 (City Limit)	1.76	47.81	4361	237

Wage rate: $7.18

Effective tax rate (residential property): 24.9%

Effective tax rate (industrial property): 34.3%

Residential tax base: \$3.9205 x 10^9

Business tax base: \$2.0619 x 10^9

Traded good industry bid land rent (CBD): $11,972

Aggregate land rents: \$3.4764 x 10^8

TABLE 5.11 - $\bar{p}_A$ = 675

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5254	7165
2	21.77	610.86	5250	7086
12	18.42	515.37	5159	5649
22	15.41	429.88	5068	4436
32	12.73	353.95	4979	3423
42	10.36	287.14	4890	2590
52	8.28	228.96	4801	1915
62	6.49	178.92	4713	1378
72	4.97	136.50	4626	961
80 (City Limit)	3.93	107.70	4557	701

Wage rate: $7.18

Effective tax rate (residential property): 24.8%

Effective tax rate (industrial property): 34.1%

Residential tax base: $\$3.9488 \times 10^9$

Business tax base: $\$2.0678 \times 10^9$

Traded good industry bid land rent (CBD): $12,001

Aggregate land rents: $\$3.7111 \times 10^8$

TABLE 5.12 - α_ℓ = .248

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5176	5903
2	18.87	528.58	5172	5843
12	16.14	451.02	5092	4729
22	13.68	381.06	5012	3777
32	11.46	318.38	4933	2972
42	9.49	262.67	4854	2300
52	7.74	213.60	4776	1746
62	6.20	170.80	4698	1296
72	4.88	133.94	4620	937
82	3.75	102.62	4544	657
91 (City Limit)	2.89	78.85	4475	462

Wage rate: $7.18

Effective tax rate (residential property): 24.9%

Effective tax rate (industrial property): 34.3%

Residential tax base: $\$3.9346 \times 10^9$

Business tax base: $\$2.0655 \times 10^9$

Traded good industry bid land rent (CBD): $11,990

Aggregate land rents: $\$3.4760 \times 10^8$

TABLE 5.13 - α_ℓ = .372

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5285	7701
2	22.96	644.35	5280	7609
12	19.13	535.16	5178	5940
22	15.72	438.46	5078	4554
32	12.72	353.64	4978	3419
42	10.10	280.06	4880	2505
52	7.86	217.08	4782	1784
62	5.95	163.99	4684	1227
72	4.37	120.07	4588	810
82	3.09	84.56	4492	507
84 (City Limit)	2.87	78.40	4473	459

Wage rate: $7.18

Effective tax rate (residential property): 24.8%

Effective tax rate (industrial property): 34.1%

Residential tax base: \$3.9517 x 10^9

Business tax base: \$2.0737 x 10^9

Traded good industry bid land rent (CBD): $12,030

Aggregate land rents: \$3.7357 x 10^8

TABLE 5.14 - EQUAL RATE: $N=1.2x10^9$, $R=2.04x10^9$

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5285	7701
2	23.18	645.01	5280	7260
12	19.71	546.78	5190	6113
22	16.58	458.54	5100	4834
32	13.77	379.87	5011	3761
42	11.29	310.34	4922	2873
52	9.10	249.50	4834	2148
62	7.20	196.85	4746	1566
72	5.58	151.92	4660	1108
82	4.20	114.17	4573	757
92	3.07	83.07	4488	496
94 (City Limit)	2.87	77.60	4471	453

Wage rate: $7.24

Effective tax rate (residential property): 27.5%

Effective tax rate (industrial property): 27.5%

Residential tax base: $4.7161 x 10^9

Business tax base: $2.6555 x 10^9

Traded good industry bid land rent (CBD): $14,834

Aggregate land rents: $4.3808 x 10^8

TABLE 5.15 - CBD RADIUS = 2.7 MI. (NO RE-CALIBRATION)

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5246	7016
2	21.46	601.31	5241	6939
12	18.15	506.88	5150	5526
22	15.17	422.38	5060	4333
32	12.51	347.40	4970	3339
42	10.17	281.46	4882	2522
52	8.12	224.10	4793	1861
62	6.35	174.82	4706	1337
72	4.85	133.10	4619	929
82	3.60	98.39	4532	621
89 (City Limit)	2.86	77.98	4472	455

Wage rate: $7.17

Effective tax rate (residential property): 24.9%

Effective tax rate (industrial property): 34.2%

Residential tax base: $\$3.9448 \times 10^9$

Business tax base: $\$2.0609 \times 10^9$

Traded good industry bid land rent (CBD): $14,306

Aggregate land rents: $\$3.5199 \times 10^8$

TABLE 5.16 - CBD RADIUS = 3.3 MI. (NO RE-CALIBRATION)

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5218	6564
2	20.38	571.92	5214	6491
12	17.18	480.59	5123	5147
22	14.31	399.06	5033	4017
32	11.75	326.89	4944	3079
42	9.51	263.60	4855	2311
52	7.55	208.73	4767	1693
62	5.87	161.78	4680	1205
72	4.45	122.21	4593	829
82	3.27	89.48	4507	547
86 (City Limit)	2.86	78.19	4473	457

Wage rate: \$7.19

Effective tax rate (residential property): 24.8%

Effective tax rate (industrial property): 34.1%

Residential tax base: \$3.9446 x 10^9

Business tax base: \$2.0780 x 10^9

Traded good industry bid land rent (CBD): \$10,253

Aggregate land rents: \$3.6936 x 10^8

TABLE 5.17 - WELFARE CHANGE
(classification to equal rate)

Parameter Δ	Compensating Variation	Δ in Differential Rents	Aggregate Welfare Δ
1. Base Case	36.477×10^6	6.652×10^5	37.142×10^6
2. h = .234	40.667×10^6	-6.912×10^5	39.976×10^6
h = .289	32.037×10^6	1.054×10^6	33.091×10^6
3. $\eta = -0.5$	40.759×10^6	5.384×10^5	41.298×10^6
$\eta = -0.9$	33.083×10^6	-9.835×10^5	32.100×10^6
4. $\rho_H = .4286$	34.410×10^6	1.503×10^6	35.913×10^6
5. M = 2870	34.710×10^6	6.041×10^5	35.314×10^6
M = 6697	34.293×10^6	2.970×10^6	37.263×10^6
6. Y = 15309	44.754×10^6	-4.720×10^4	44.706×10^6
Y = 22962	24.918×10^6	1.652×10^6	26.570×10^6
7. c = 25.95	34.984×10^6	9.399×10^6	35.925×10^6
c = 38.925	34.908×10^6	7.942×10^6	35.703×10^6
8. CBD: 2.7 mi. (re-calibrated)	37.362×10^6	-2.251×10^5	37.137×10^6
CBD: 3.3 mi. (re-calibrated)	33.156×10^6	1.173×10^6	34.328×10^6
9. $p^1_{LT} = 8000$	39.602×10^6	-9.642×10^5	38.638×10^6
$p^1_{LT} = 16000$	31.882×10^6	1.592×10^6	33.474×10^6

TABLE 5.17 (continued)

Parameter Δ	Compensating Variation	Δ in Differential Rents	Aggregate Welfare Δ
10. $t = .5625$	37.439×10^6	-4.758×10^5	36.963×10^6
$t = .6875$	30.597×10^6	2.617×10^6	33.214×10^6
11. $\rho_H = .4815$ (not re-calibrated)	47.081×10^6	-7.166×10^5	36.963×10^6
$\rho_H = .2121$ (not re-calibrated)	19.725×10^6	2.387×10^6	33.214×10^6
12. $\alpha_{LH}=.0245941$	34.974×10^6	3.583×10^5	35.333×10^6
$\alpha_{LH}=.0368911$	35.196×10^6	3.301×10^5	35.526×10^6
13. $\alpha_{KH}=2.4279218$	41.468×10^6	1.145×10^6	42.613×10^6
$\alpha_{KH}=3.6418826$	30.153×10^6	1.015×10^6	31.168×10^6
14. $\overline{p}_A = 225$	35.109×10^6	7.247×10^5	35.834×10^6
$\overline{p}_A = 675$	34.887×10^6	8.476×10^5	35.735×10^6
15. $\alpha_\ell = .248$	34.974×10^6	8.479×10^5	35.849×10^6
$\alpha_\ell = .372$	35.398×10^6	7.540×10^4	35.473×10^6
16. CBD: 2.7 mi. (not re-calibrated)	35.485×10^6	4.183×10^5	35.903×10^6
CBD: 3.3 mi. (not re-calibrated)	35.120×10^6	4.364×10^5	35.557×10^6

TABLE 5.17 (continued)

Parameter Δ	Compensating Variation	in Δ Differential Rents	Aggregate Welfare Δ
17. $\alpha_{LT}=\alpha_{wT}$ = .10	57.160×10^6	2.004×10^6	59.165×10^6
α_{LT} = .005, α_{wT} = .40	28.162×10^6	1.278×10^6	29.440×10^6
α_{LT} = .02, α_{wT} = .10	43.278×10^6	1.035×10^6	44.313×10^6

TABLE 5.18 - WELFARE RESULTS
(multiple parameter changes)

Parameter Δ	Compensating Variation	Δ in Differential Rents	Aggregate Welfare Δ
1. Max welfare Δ (resident)	54.208×10^6	-1.512×10^6	52.697×10^6
2. Max welfare Δ (aggregate)	58.724×10^6	-9.187×10^5	57.805×10^6
3. Min welfare Δ (resident)	2.726×10^6	6.786×10^6	9.511×10^6
4. Min welfare Δ (aggregate)	4.146×10^6	5.542×10^6	9.688×10^6
5. Max welfare Δ (resident; base)	55.748×10^6	-1.345×10^6	54.027×10^6
6. Max welfare Δ (aggregate; base)	48.055×10^6	3.352×10^6	51.407×10^6

III. Local Good Model Parameterization

In comparison to what was done for the base model, full parameterization of a base case for the local good model is much more difficult. This is so not merely because there is an additional production function to parameterize. The problem of parameterization is also made much more difficult because there is an additional price in the demand functions and the local good market must clear at an equilibrium price vector. As a result, we will find it necessary to make assumptions about the local good industry's shares of the labor force and tax revenues raised from the business sector. On the other hand, we are fortunate in that many of the parameter values chosen for the base case of the base model can be retained for the parameterization of the local good model. Thus, the analysis and data underlying the choice of such values need not be mentioned again.

The CBD in the local good model consists of two rings and something explicit should now be stated about commuting to an employment center. The simulations reported for the base model were run on the assumption that the employment node was located at the midpoint of the one CBD ring. All workers were assumed to commute along a straight line to this midpoint. For workers residing in the first ring, the terminal points of the work trip were taken to be the midpoint of the CBD and the boundary between the CBD and the next ring along any ray from the mathematical center of all the annuli. For lack of evidence that something else is more appropriate, in the parameterization of the local good model the radius of each of the two CBD rings is taken to be half of the CBD radius in the base case parameterization of the base model. In particular, the radius of each of the first two rings is taken to be 1.5 miles. For purposes of establishing commuting distance, all workers are assumed to travel along a ray to an employment node located at the midpoint of the CBD. Since the radius of the two CBD rings is the same, the workers travel to the border between the two rings. Workers residing in either of the two CBD rings are assumed to commute, along a ray, from the midpoint of the ring in which they reside to the boundary between the two rings. Because the radius of the two rings is the same, this commuting assumption implies that work trip distance is the same for households residing in the first two rings, and thus that bid housing service prices and land rents will be the same in the two rings. Aside from having to distinguish between the two CBD rings in the local good model, all of the miscellaneous parameter values listed in Table 5.17 for the base model can be retained for the local good model. The parameter values listed in Table 5.17 for the budget and time constraints, and the housing production function also remain the same in our parameterization of the local good model. In addition, an extra assessment/sales ratio must be added to the parameter list. Following what was specified in the Massachusetts classification amendment, we let $a_c = .50$.

Finding values for the utility function, traded good production function, and local good production function parameters, though, requires some analysis. Once again, the process of parameterization is conducted in terms of the choices of a typical household in long run equilibrium. The basic approach to selecting values for the utility function substitution term, ρ, that was used for the base model parameterization can also be used here. Formally, the only difference is that the term b must be redefined to take account of the extra term in the denominator of the household demand function in the local good model. In particular, b is defined as follows:

$$b = [\alpha_T \cdot (\bar{p}_T/\alpha_T)^{\frac{\rho}{1+\rho}} + \alpha_C \cdot (p_C/\alpha_C)^{\frac{\rho}{1+\rho}}] \cdot (a/\alpha_H)^{\frac{1}{1+\rho}} \qquad (5.12)$$

Given this definition, it is once again true that h = a/(a+b), where h and a are as defined above. Since the values used for h and a should be the same as they were in the base model parameterization, the value of b must remain the same. The term ρ can again be defined by (5.5). Thus, the value chosen for ρ in the local good model is identical to the value used in the base case parameterization of the base model. The process of finding the coefficients of the goods in the utility function, however, does differ from what was done before. To find these values we first note that the local good budget share on income net of commuting costs for a typical household, q, can be written as follows:

$$q = \frac{p_C}{p_C + [\alpha_H \cdot (a/\alpha_H)^{\frac{\rho}{1+\rho}} + \alpha_T \cdot (\bar{p}_T/\alpha_T)^{\frac{\rho}{1+\rho}}] \cdot (p_C/\alpha_C)^{\frac{1}{1+\rho}}} \qquad (5.13)$$

We denote the denominator of (5.13), which also happens to be the denominator of the demand function for the local good, x_C, for a typical household, by using the term, $\bar{s}$. Thus, we have $q = p_C/\bar{s}$.

We also note that aggregate supply and demand for the local good must be equal in equilibrium. Aggregate demand in the base case equilibrium will be approximately equal to $N \cdot x_C$. Let the variable $\hat{s}_C^2$ be defined as $\hat{s}_C^2 = N \cdot x_C$. This term should be approximately equal to aggregate supply of the local good in the base case equilibrium. The superscript of 2 indicates that our target (base case) equilibrium is to be one in which the local good is produced only in the second CBD ring. The first ring, in the target equilibrium, is to be reserved for traded good production. With the definition of $\hat{s}_C^2$, and the local good demand, x_C, we may write $\bar{s} = N \cdot Y/\hat{s}_C^2$, where Y is once again taken to be income net of commuting costs for the typical household. Given the value of Y used in the base case parameterization of the base model, and using the target supply of the local good forthcoming from production in the second ring in equilibrium (which will be established below) as the value of $\hat{s}_C^2$, a value for $\bar{s}$ can be determined. This value, together with a target equilibrium price, p_C, for the local good, then allows us to find the budget share q. The target price may be chosen arbitrarily. Doing so, we let $p_C = 100$.

Manipulation of the definition of $\bar{s}$ yields the following equality:

$$\frac{\bar{s} - p_C}{(p_C)^{\frac{1}{1+\rho}}} = (a)^{\frac{\rho}{1+\rho}} \cdot (\alpha_H/\alpha_C)^{\frac{1}{1+\rho}} + (\bar{p}_T)^{\frac{\rho}{1+\rho}} \cdot (\alpha_T/\alpha_C)^{\frac{1}{1+\rho}} \qquad (5.14)$$

Given the definition of q, the left hand side of (5.14) can be expressed as $[(1-q)/q] \cdot (p_C)^{\frac{\rho}{1+\rho}}$. This expression is determinate given the values of ρ, q, and the target price, p_C, mentioned above. We may also now rewrite equation (5.12) as follows:

$$b/(a)^{\frac{1}{1+\rho}} = (\bar{p}_T)^{\frac{\rho}{1+\rho}} \cdot (\alpha_T/\alpha_H)^{\frac{1}{1+\rho}} + (p_C)^{\frac{\rho}{1+\rho}} \cdot (\alpha_C/\alpha_H)^{\frac{1}{1+\rho}} \tag{5.15}$$

The left hand side of (5.15) is determinate, given the data. Equations (5.14) and (5.15) constitute a system of two equations, in the three unknown utility function coefficients, that must hold in equilibrium. To aid in the exposition of the method of solution for the utility function coefficients, we define the following terms:

$$\bar{c}_1 = b/(a)^{\frac{1}{1+\rho}}$$

$$\bar{c}_2 = [(1-q)/q] \cdot (p_C)^{\frac{\rho}{1+\rho}}$$

$$c_{11} = c_{22} = (\bar{p}_T)^{\frac{\rho}{1+\rho}}$$

$$c_{12} = (p_C)^{\frac{\rho}{1+\rho}}$$

$$c_{21} = (a)^{\frac{\rho}{1+\rho}}$$

This allows us to write the system of two simultaneous equations more succinctly as follows:

$$c_{11} \cdot (\alpha_T/\alpha_H)^{\frac{1}{1+\rho}} + c_{12} \cdot (\alpha_C/\alpha_H)^{\frac{1}{1+\rho}} = \bar{c}_1 \tag{5.16}$$

$$c_{21}\cdot(\alpha_H/\alpha_C)^{\frac{1}{1+\rho}} + c_{22}\cdot(\alpha_T/\alpha_C)^{\frac{1}{1+\rho}} = \bar{c}_2 \qquad (5.17)$$

We seek to find values for α_H,α_T, and α_C. To do this, first solve equation (5.17) for α_C. This yields the following:

$$\alpha_C = \left[\frac{c_{21}\cdot(\alpha_H)^{\frac{1}{1+\rho}} + c_{22}\cdot(\alpha_T)^{\frac{1}{1+\rho}}}{\bar{c}_2}\right]^{1+\rho} \qquad (5.18)$$

Next, substitute the right hand side of (5.18) for α_C in equation (5.16). After some rearranging, we obtain an expression for the ratio α_T/α_H as follows:

$$\frac{\alpha_T}{\alpha_H} = \left(\frac{[\bar{c}_1 - (c_{12}\cdot c_{21}/\bar{c}_2)]}{[c_{11} + (c_{12}\cdot c_{22}/\bar{c}_2)]}\right)^{1+\rho} \qquad (5.19)$$

The value of this ratio is determinate, given the data and target values mentioned above. For ease in exposition, define $\bar{c}$ to be equal to the term inside the outer parenthesis on the right hand side of (5.19). Next, substitute this value in equation (5.17) for $(\alpha_T/\alpha_H)^{\frac{1}{1+\rho}}$. After a little rearranging, an expression for α_C/α_H can be found. In particular, we obtain

$$\frac{\alpha_C}{\alpha_H} = \left(\frac{\bar{c}_1 - c_{11}\cdot\bar{c}}{c_{12}}\right)^{1+\rho} \qquad (5.20)$$

The individual utility coefficient values can now be found if we normalize so that the coefficients add to some constant. In particular, we normalize so that the coefficients add to 1. Thus, we may find α_H in terms of the ratios in (5.19) and (5.20) as follows:

$$\alpha_H = \frac{1}{(\alpha_T/\alpha_H) + (\alpha_C/\alpha_H) + 1} \qquad (5.21)$$

The values of α_T and α_C can then be determined by using the value of α_H obtained, and the values of the right hand sides of (5.19) and (5.20). All of this is conditional, of course, on the local good budget share, and that, in turn, is dependent upon $\hat{s}^2_C$. Thus, we must establish, approximately, the amount of the local good that is to be supplied in equilibrium.

The amount of the local good supplied in a base case equilibrium depends on the target local good price, which has already been specified, and the local good production function parameters. Given p_C, we specify the local good production function parameters in the same manner used to parameterize the traded good production function for the base model. To do so, however, assumptions must be made about the number of workers employed by the local good industry and the amount of tax revenues to be raised from the local good sector. We may, for example, assume that 50 percent of the given labor force is to be employed in the local good sector, and 50 percent of the total amount of tax revenues to be raised from taxes on business property[14] is to arise from the local good sector. Analogous to what was done for the traded good in the base model, then, the labor demand function for the local good industry for production in ring 2 can be inverted to find the substitution term, ρ_C, once a target land rent for ring 2, and the coefficient ratio α_{wC}/α_{LC}, have been chosen. For the base case parameterization, to ensure that the local good industry outbids housing for land in ring 2, we choose p_{LC} = \$10,000/acre. The values of the coefficients, α_{wC} and α_{LC} were chosen, somewhat arbitrarily, to be $\alpha_{wC} = .30$ and $\alpha_{LC} = .01$. An analogue to equation (5.10) for the local good can then be used to find ρ_C. Given the data and the target prices used,[15] we find a value of $\rho_C = -.1133$. The amount of business tax revenues assumed to be raised from taxes on local good property can be used to find the amount of capital, K_C, used by the local good industry in equilibrium. A local good analogue (for ring 2) to equation (5.11) is then used to determine the coefficient α_{KC}. Given the value found for ρ_C and values chosen for other parameters and the target local good bid land rent, a value of $\alpha_{KT} = .09096017$ is obtained. The bid land rent function (4.2) can now be inverted with respect to the scale parameter, D. All other terms in (4.2) have been specified. As a result, we obtain D = 676.07.

With the values of the parameters of the local good production function chosen, and the amounts of labor, capital, and land used by local good producers in the base case equilibrium known, the aggregate amount of the local good that will be supplied in equilibrium can be determined. It is this value that is used for $\hat{s}_C^2$. Given $\hat{s}_C^2$, the local good budget share for a typical household, and so the utility function coefficients, can be chosen. For the data used to parameterize the base case, the local good budget share is 45.75% and the utility function coefficients are α_H = .01627393, α_T = .3013052, and α_C = .6824208.

Finally, values of the parameters of the traded good production function can be found in the manner given in the discussion of the parameterization of the base model. What is different now is that the number of workers employed by the traded good industry and the tax revenues raised from taxes on traded good property are less than what they were for the base model parameterization. Values for the traded good production function and all other parameters, when the traded and local good industries each employ 50 percent of the labor force and each contribute 50 percent of total tax revenues raised from the business sector, are listed in Table 5.19.

TABLE 5.19

Base Case Parameterization

Utility Function:	$A = 1$ α_H=.01627393 α_T=.3013052 α_C=.6824208
	ρ =.6834
Budget and Time Constraints:	$v = 25$ $\overline{W}$=2000 c=32.4375 T=5840 M=4784
Housing Production Function:	$B = 1$ α_{LH}=.0307426 α_{KH}=3.0349022 ρ_H=.3333
Traded Good Production Function:	$C = 251.94$ α_{LT}=.01 α_{wT}=.20 α_{KT}=.09096017
	ρ_T = -.1794
Local Good Production Function:	$D = 676.07$ α_{LC}=.01 α_{wC}=.30 α_{KC}=.1033591
	ρ_C = -.1133
Miscellaneous:	$\overline{p}_T$ = 100 $\overline{p}_A$=450 a_R=.40 a_C=.50 a_I=.55
	$\overline{p}_K$ = 135 N=1,000,000 R=1,700,000
	CBD ring 1 radius = 1.5 CBD ring 2 radius = 1.5

IV. Local Good Model Results

This section presents the more important results obtained for equilibria of the model with a local good. These solutions were, generally, much more difficult and costly to find than those for the base model. As noted above, the local good solutions were often three times as costly, in terms of the number of price vectors examined by the user written subroutine, as the base case solutions. The typical cost increase may be thought of as being even larger than this since, in general, the computational time involved in calculating excess demand vectors for price vectors used in the search is greater for the local good model due to its increased complexity. Variation in the cost of the local good model solutions, however, was quite substantial. The number of price vectors examined before a solution was found varied from about 160 to just over 1000.

In the absence of strong evidence to the contrary, the equilibrium obtained under classification with labor and tax revenue shares of 50% was, somewhat arbitrarily, taken, for illustrative purposes, to be a base case solution. As can be seen in Table 5.25, the welfare results for the case agree fairly closely with those of the base case for the base model. The aggregate welfare change for a move to an equal rate tax system using a compensating variation measure is about $31.7 million per year, while the same change in tax structure yields, using an equivalent variation measure (not shown), an aggregate welfare gain of about $33.2 million. The major characteristics of urban area structure for the base case parameterization under classification and equal rate tax systems are given in Tables 5.20 and 5.21, respectively. As was true for the base model, the gradients reported became somewhat flatter with a move to a non-discriminatory tax system. Results for both models are also qualitatively the same for the wage rate, the tax rates, and the residential tax base. The more favorable treatment of business property and less favorable treatment of residential property results in an increased demand for labor and so a higher wage rate, a higher effective tax rate on residential property as well as lower rates for business property, and lower household expenditures on housing net of property taxes. The lower tax rates on business property also, as expected, yield higher land rents for both CBD rings, and higher business tax bases in equilibrium. Unlike the base model, though, the move to an equal rate system for the local good base case parameterization results in slightly lower aggregate land rents. What is, however, perhaps most surprising about the results is that the local good price rises after the change to an equal rate system. One might expect that the lower effective tax rate facing the local good industry would result in a lower local good price. It appears, though, that in the new general equilibrium the supply side effects of the tax rate reduction are outweighed by income effects on the demand side. Wage rates, and so household incomes, rise under the equal rate system. Apparently, the increase in demand for the local good induced by the higher incomes is large enough to raise the local good price in the new equilibrium.

To investigate how different assumptions about the local good industry's employment and tax revenue shares affect the results, the shares were each varied to values of 25% and 75%, as well as 50%.

This yields nine different combinations of employment and tax revenue shares. The results for a combination of a 50% employment and a 50% tax revenue share were discussed above. Important characteristics of urban structure under an equal rate tax system for the other eight combinations are presented in Tables 5.22 to 5.24. The parameterizations for each of these combinations were chosen so that they yielded solutions under a classification tax scheme that duplicated the important aspects of a certain solution--one which is meant to represent, as best as possible, the reality for the Boston SMSA in or around 1980. Thus, the structural results for the solutions obtained for these combinations under the classification tax scheme are not presented.

Inspection of Tables 5.22 to 5.24 reveals that some of the equilibrium values of the variables presented tend to vary monotonically with changes in the employment and tax shares. The wage rate clearly varies positively with the local good's employment share and negatively with the tax share. No such monotonicity can be shown for the effective tax rate on local good property. On the other hand, the local good price unequivocably shows a tendency to increase with increases in the employment share and decrease with increases in the tax share. The price of housing services in the first residential ring (i.e., ring 3) tends to vary negatively with the tax share. With one exception (tax share = 75%), it increases or remains constant with increases in the employment share, holding constant the tax share. Also, with one exception each, the residential tax base tends to increase with increases in the employment share and decrease with increases in the tax share. The local good tax base in the equal rate equilibria varies positively with assumed values of each of the shares. On the other hand, the traded good tax base exhibits a relationship with the employment share that is not monotonic. It is, however, negatively related to the local good tax share. The local good bid land rent for CBD land in the equal rate equilibria varies positively with the employment share and negatively with the tax share. While the traded good bid land rent for CBD land also varies positively with the local good employment share, its relationship with the tax share is not monotonic. Finally, aggregate land rents for the urban area, with one exception (employment share = 25%), appears to decrease with increases in the local good tax share, while its relationship with the employment share does not exhibit monotonicity.

Welfare results for the nine combinations of employment and tax share for a move from a classification to an equal rate tax system are presented in Table 5.25. Compensating variation and aggregate welfare change measures for the different combinations are given. The range of the welfare gains is larger than that shown for the base model. Compensating variation ranges from \$17.1 million to \$75.8 million, while the aggregate welfare change measure varies from \$14.8 million to \$78.6 million. The combinations which yield welfare gains closest to those found for the base case of the base model are 50% for both employment and tax share, and 25% for both shares. Whatever the variation in magnitudes, the robustness of the qualitative result that there is a welfare gain in the long run, both for households considered separately and in conjunction with landowners, in moving from a status-quo classification tax structure to one in which all property is taxed at the same effective rate is once again clearly demonstrated. As was the case for the base model, there is a welfare gain from such a change in tax structure in all cases considered for the local good model.

TABLE 5.20 - BASE CASE: CLASSIFICATION

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5276	7553
2 (CBD)	0	0	5276	7553
3	20.54	575.76	5217	6549
13	17.32	484.03	5127	5196
23	14.43	402.21	5037	4058
33	11.86	329.58	4948	3113
43	9.60	265.96	4859	2338
53	7.63	210.77	4771	1715
63	5.94	163.51	4683	1222
73	4.51	123.66	4597	842
83	3.31	90.67	4510	557
87(city limit)	2.90	79.28	4476	466

Wage rate: $7.18

Effective tax rate (residential property): 24.9%

Effective tax rate (commercial property): 31.1%

Effective tax rate (industrial property): 34.2%

Local good price: $100

Residential tax base: $3.9820 x 10^9

Local good tax base: $1.1310 x 10^9

Traded good tax base: $1.0241 x 10^9

Local good CBD land rent: $9946

Traded good CBD land rent: $15,369

Aggregate land rents: $3.0435 x 10^8

TABLE 5.21 - BASE CASE: EQUAL RATE

Ring	Population Density	Capital/Land Ratios:Housing	Housing Service Prices	Housing Land Rents
1 (CBD)	0	0	5271	7452
2 (CBD)	0	0	5271	7452
3	20.45	570.24	5212	6465
13	17.25	479.81	5122	5136
23	14.39	399.00	5033	4016
33	11.84	327.39	4945	3085
43	9.60	264.53	4857	2322
53	7.64	209.95	4769	1706
63	5.96	163.17	4683	1219
73	4.53	123.67	4597	842
83	3.34	90.92	4511	559
87(city limit)	2.93	79.59	4477	468

Wage rate: $7.26

Effective tax rate (residential property): 27.1%

Effective tax rate (commercial property): 27.1%

Effective tax rate (industrial property): 27.1%

Local good price: $100.34

Residential tax base: $\$3.9609 \times 10^9$

Local good tax base: $\$1.1803 \times 10^9$

Traded good tax base: $\$1.1234 \times 10^9$

Local good CBD land rent: $10,332

Traded good CBD land rent: $17,131

Aggregate land rents: $\$3.0405 \times 10^8$

TABLE 5.22 - EQUAL RATE

Employment Share	Tax Share	Wage Rate	Tax Rate	Local Good Price	Housing Service Price (ring 3)
25%	50%	$7.24	27.1%	$99.63	$5212
75%	50%	7.32	26.8	101.08	5213
25%	25%	7.26	27.2	100.41	5212
50%	25%	7.31	27.0	101.09	5213
75%	25%	7.41	26.6	102.55	5216
25%	75%	7.21	27.1	99.04	5212
50%	75%	7.23	27.0	99.55	5211
75%	75%	7.24	26.9	99.89	5213

TABLE 5.23 - EQUAL RATE

Employment Share	Tax Share	Tax Base: Residential	Tax Base: Local Good	Tax Base: Traded Good
25%	50%	$\$3.9647 \times 10^9$	$\$1.1756 \times 10^9$	$\$1.1215 \times 10^9$
75%	50%	3.9713×10^9	1.1841×10^9	1.1590×10^9
25%	25%	3.9586×10^9	5.9050×10^8	1.6464×10^9
50%	25%	3.9696×10^9	6.0266×10^8	1.6434×10^9
75%	25%	4.0226×10^9	6.0811×10^8	1.7685×10^9
25%	75%	3.9581×10^9	1.7558×10^9	5.5852×10^8
50%	75%	3.9550×10^9	1.7604×10^9	5.5776×10^8
75%	75%	3.9681×10^9	1.7649×10^9	5.7013×10^8

TABLE 5.24 - EQUAL RATE

Employment Share	Tax Share	Land Rents Local Good (CBD)	Land Rents Traded Good (CBD)	Aggregate
25%	50%	$10312	$17065	$\$3.0426 \times 10^8$
75%	50%	10347	17665	3.0496×10^8
25%	25%	10358	16772	3.0394×10^8
50%	25%	10527	16772	3.0560×10^8
75%	25%	10593	17927	3.0923×10^8
25%	75%	10270	17016	3.0355×10^8
50%	75%	10277	17029	3.0332×10^8
75%	75%	10289	17421	3.0446×10^8

TABLE 5.25 - WELFARE CHANGE
(classification to equal rate)

Employment Share	Tax Share	Compensating Variation	Δ in Differential Rents	Aggregate Welfare Δ
50%	50%	32.045×10^6	-4.058×10^5	31.639×10^6
25%	50%	27.252×10^6	-2.051×10^6	25.200×10^6
75%	50%	50.603×10^6	1.934×10^6	52.537×10^6
25%	25%	37.674×10^6	-2.120×10^6	35.553×10^6
50%	25%	45.501×10^6	-7.973×10^5	44.704×10^6
75%	25%	75.769×10^6	2.801×10^6	78.570×10^6
25%	75%	17.098×10^6	-2.328×10^6	14.771×10^6
50%	75%	20.964×10^6	-1.954×10^6	19.011×10^6
75%	75%	29.281×10^6	-1.118×10^6	28.163×10^6

CHAPTER 5

FOOTNOTES

[1] The dimension of the simplex over which the algorithm searches actually less than doubles. As noted above, the set of search variables for any case must include one artificial, i.e. non-economic, variable.

[2] The parameters to which we refer are those found in the economic model that is used. In addition to these, there are some, what may be called, system parameters to choose, such as the initial grid size, the number of significant digits desired for accuracy purposes, and so on that relate to the mathematical operations carried on by the algorithm for any application. Values for these parameters will not be discussed.

[3] Since only ordinal utility matters, the scale term, A, in the utility function can be chosen arbitrarily. For all parameterizations it will be taken to be 1.

[4] By full price we mean the housing price gross of property tax payments. In particular, we are referring to $p_H^j \cdot (1+a_R \cdot t)$.

[5] Although the FHA data is provided for selected housing areas as well as for states, Boston is not one of the selected areas. Data for existing, rather than new, homes had to be used since a small sample size precluded reporting of data on new homes for Massachusetts.

[6] Housing costs used were reported average monthly payments to principal and interest, mortgage insurance premium, debt service, hazard insurance, and maintenance and repairs.

[7] Although this assumption must be made somewhat arbitrarily, the distance chosen seems reasonable a priori, and, in fact, corresponds closely to what is obtained in the computed solutions. In any case, the results are not very sensitive to this assumption.

[8] Costs such as maintenance and repairs, replacement tires, gasoline, oil, and state and federal taxes on gasoline and oil were used. The data applies to a compact size, 2-door sedan, purchased for $5,215, operated 100,000 miles over a 10-year period, and then scrapped for $40. Prices used were for the Baltimore area.

[9] Data on personal outlays can be found in U.S. Department of Commerce (1973, 1982), while that on the number of households is found in U.S. Bureau of the Census (1973, 1982a).

[10] The appropriate interest rate, it seems, should be a nominal one and reflect, to some extent, experience relative to the period for which the data is provided. Since interest rates rose in the 70's and 80's over traditional levels, a value somewhat higher than the 6% used by Muth was chosen. The interest rate chosen also yields a housing rental rate, net of property taxes, that corresponds closely with that which would be obtained from the data on median house price and assumed percentages to account for costs of depreciation, maintenance, insurance, and so forth, mentioned below.

[11] It is likely that many drivers are simply passing through the Boston CBD on their way to work during the morning peak. Table 22 of the study shows that, for what is probably only a part of the true morning peak period, on average 36,130 vehicles entered and 22,260 left the Boston Proper area. To somehow account for the phenomenon of drivers simply passing through the area, it was assumed that one-half of the 22,260 vehicles that exited the area were also vehicles that had entered the area during the morning peak. Thus, we assume that 42.8% of the auto (we also make the same assumption for the relatively small number of pedestrians in the data) travellers crossing the area boundary during the morning peak are area workers. This kind of adjustment was not made for public transit passengers. Finally, since 58,390 vehicles crossed the boundary during the limited (7:30 - 8:30 am) morning peak period mentioned and 64,212 vehicles crossed the boundary during a similarly limited (4:30 - 5:30 pm) afternoon peak, the data in Table 12, which give totals for the morning (7:30 - 9:00 am) and afternoon (4:30 - 6:00 pm) peak periods combined, were decreased by 53.4%.

[12] The CBD land rental needn't be very close to the housing land rental in ring 2 since it can be thought of as an average land rent for a relatively large annulus.

[13] Care should be taken to limit the extent of changes in parameters because this can lead to problems in finding system parameters that lead to searches for equilibria that are reasonably short in duration, and because equilibria may not exist for large enough parameter changes.

[14] The total amount of property tax revenues to be raised, and the amount arising from the tax on residential property, take on the same values here that were given in the base case parameterization of the base model. Thus, the total amount of tax revenues to be raised from taxes on business property is given.

[15] Since income for the typical household and the effective residential tax rate paid by households should be what they were for the base model, the values chosen for p_W and t do not change.

CHAPTER 6

CONCLUSIONS AND PROSPECTS FOR FUTURE RESEARCH

This study has presented a framework for analyzing the structural and normative effects of replacing a property tax system which classifies property according to use for tax purposes with one which does not discriminate in its treatment of real property. The analysis was conducted within a long run equilibrium model of urban land use. Results were obtained by numerical simulation which was conducted with the use of a fixed point algorithm. Section I of this chapter briefly reviews and summarizes the simulation results, while section II outlines some extensions and variants of the models developed above that may be pursued in the future.

I. Summary of Results

Equilibria for monocentric models of urban spatial location were calculated under classification and equal rate property tax regimes. A model in which all labor is employed in an industry that produced a good that could be exported to or imported from the urban area, so that its price is set in a national market, served as a base model for the simulations. An extension to the model with a traded good sector was also simulated. The extension added a local good sector to the urban area. In particular, in addition to the traded good industry, labor could find employment in an industry which produced a good which cannot be imported to or exported from the urban area. Base case parameterizations for the two models were chosen so that the resultant equilibria produced urban areas that are stylized after the Boston metropolitan area in or around 1980. Sensitivity analysis was conducted by varying parameter values from their base case levels. Structurally, the effects of a change from a classification tax scheme that is meant to represent, on average, the reality for the Boston metropolitan area to an equal rate tax system for the region were qualitatively the same for all parameterizations considered. The change resulted in lower effective tax rates on business property, and higher effective rates on residential property, in equilibrium. This, in turn, led to increased demands for labor and land from the business sector, and so higher wage rates and CBD land rents in equilibrium. Households spent less on housing net of property taxes, but more on housing gross of property taxes, in the new (equal rate) equilibria. The higher residential tax rates induced households to live further out, on average, from the center of the region to locations where net of tax housing prices are lower, in order to offset, somewhat, the effect of the higher tax rate on gross of tax housing costs. This is evidenced by the flatter population density gradients for the equal rate solutions. The welfare results showed some degree of variability for different parameterizations. Multiple parameter changes from base case values yielded a range of $9.5 million to $54 million annually for the aggregate welfare gain in moving to an equal rate system. Results for different assumptions about the local good industry's shares of employment and business tax revenues showed aggregate welfare gains varying from $14.8 million to $78.6 million annually. The qualitative result that there is a welfare gain in moving from a classification scheme as has existed in the Boston metropolitan area to an equal rate system, however, was seen to hold for all of the simulations.

II. Future Research

It is possible, using the basic computational framework presented in this study to extend the analysis in several directions. Modelling for multiple household and labor types has already been developed and exposited in Chapter 4. It is hoped that the severe computational problems encountered when attempts were made to find solutions for multiple group specifications that were sufficiently complex to warrant investigation, can be overcome in the future through further experimentation with the algorithm used for this study or through the use of more efficient algorithms.

One extension not mentioned above is the incorporation of costs of transport for the business sectors. We may envision producers of the traded or local good having to incur a cost to transport their product to a certain point for distribution. It would be a simple matter to include such an extension. The one CBD ring in the base model and the two CBD rings in the local good model can be divided into a number of smaller rings equal in width, say, to that of the residential rings. It can be assumed, then, that traded and local good producers in the various rings must ship their product to a distribution center. If the per-mile cost of transporting a unit of a particular good is taken to be constant, then all that need be done to account for business sector transportation costs is to establish bid land rent, labor, and capital demand functions for the various CBD rings in which the traded and local good industries can locate, and use output price net of the cost of transporting a unit of the good from the ring to the distribution center in place of output price in these functions. Although attempts may be made in the future to incorporate such an extension, it does not seem likely that this sort of change would appreciably affect the results.

It may prove more interesting, however, to consider adapting the computational framework to handle somewhat different tax policy questions. We may, for example, attempt to examine the impact of replacing the property tax in the model partially or fully with a sales tax. The sales tax could be imposed on sales of the traded and/or local good in the urban area. It is also possible to investigate another, but much less widely practiced, form of property tax classification; that of classification by type of property. It is a fairly simple matter to allow for different effective property tax rates on land and capital. The nature of the way the tax is imposed on residential property, however, would have to be altered. Property taxes would be levied on the capital and land used in the production of housing and paid by housing producers. The effects on urban structure, and the welfare of residents and landowners, of changing the ratio of effective tax rates for capital and land can then be studied.

An extension to the analysis that will be attempted is to search, within the context of the model already developed, for an optimal classification of property for tax purposes. Both the particular classification scheme used in this study, and the equal rate system are merely special cases of general property tax classification. Thus, if it is assumed that property taxes are to be imposed, it is natural to ask which classification of residential and business property, out of the infinite number possible, leads to the largest level of welfare for households, or landowners and households combined. A search process for the base model that may be used can be outlined as follows. To keep matters simple, we consider only household welfare. Different classification schemes can be posited by making appropriate assumptions about the values of the assessment/sales ratios, a_R and a_I. There is no loss of generality in normalizing so that the ratios sum to one. Thus, we search over the set $G = \{(a_R,a_I)|a_R,a_I \geqq 0,\ a_R + a_I = 1\}$ for values of a_R and a_I that maximize household equilibrium utility. Calculation of equilibrium utility for different combinations of a_R and a_I requires separate simulations. Thus, to keep computation and time costs down to a reasonable level the search process should economize on the number of combinations considered. In that spirit, the following is proposed. Solutions of the model are calculated for vectors in G where a_R is some multiple of .05. Noting equation (3.14), it can be seen that the case $(a_R,a_I) = (0,1)$ cannot be considered because of the way that the nominal property tax rate is defined. Of the solutions calculated, the one yielding the highest household utility in equilibrium is noted. Suppose that the vector in G that is associated with this solution has $a_R = m\cdot(.05)$, where m is some positive integer less than 20. Solutions are then calculated for vectors in G with components that are multiples of .01 and have $(m-1)\cdot(.05) < a_R < (m+1)\cdot(.05)$. If m = 20, then the additional solutions calculated are only those for vectors in G with components that are multiples of .01 and have $.95 < a_R < 1$. The equilibrium utility levels for this second set of solutions are compared to one another and to the utility level for the case $a_R = m\cdot(.05)$. Of the vectors in G that are associated with these equilibrium utility levels, the one that yields the highest household utility is taken to be the optimal classification scheme. The welfare gain in moving from the classification scheme that represents the reality for the urban area to the optimal classification scheme can then be calculated by using a compensating or equivalent variation measure in the same manner as was done in considering a change to an equal rate system. Assuming that m < 20, the search for the optimal classification scheme described above entails finding 27 solutions in addition to those for the status quo classification and equal rate systems [i.e., in addition to those for $(a_R,a_I) = (.40,.55)$ and $(a_R,a_I) = (.50,.50)$].

Something quite similar can be done in searching for an optimal classification scheme for the local good model. Search must now be conducted over vectors in the set $H = \{(a_R,a_I,a_C)|a_R,a_I,a_C \geqq 0, a_R + a_I + a_C = 1\}$ to find assessment/sales ratios, a_R, a_I, and a_C, that maximize equilibrium household utility. To keep search costs reasonable, the following is suggested. Instead of using multiples of .05 at the first stage, we consider values of a_R that are positive multiples of .10. For each such value of a_R, solutions are computed for all vectors in H for which $a_I + a_C = 1 - a_R$ and the ratios a_I and a_C take on values that are multiples of .10. There are fifty-five of these solutions to calculate. From the group of vectors in H associated with these solutions, the one which yields the largest equilibrium household utility is noted. The next stage in the search process involves finding solutions for some vectors in H that are close to the one that is noted at the end of the first stage. In particular, we use vectors in H for which one of the three components is larger by the amount .05 and another of the three is smaller by the amount .05 than the corresponding components for the vector found at the end of the first stage of the process. There are six vectors of this sort, and so six more solutions must be computed at this stage. The equilibrium household utilities for each of these cases are compared to one another and to the utility level for the vector that yielded maximum utility at the first stage. Of the vectors associated with these solutions, the one which yields the highest household utility is noted. The third stage then involves finding solutions for vectors in H that are very close to the one found at the end of the second stage. In particular, the vectors vary by .02 in two of their components from the values of the corresponding components for the vector found at the end of the second stage. This again produces six vectors and so requires an additional six solutions. Utility levels are noted and compared as in the second stage, and this produces a vector that yields maximum utility for the group considered. Finally, the fourth stage entails finding solutions for vectors in H which have two components that differ by .01 from the corresponding components of the vector found at the end of the third stage. The vector yielding highest household utility at the end of the fourth stage is taken to be the optimal classification. When the vectors found at the end of each stage are all interior to the simplex H, this search process requires the calculation of 73 solutions.

One last prospect for future research involves extending the models to allow for regional migration; that is, a system of cities model may be constructed. The models developed above are models of a closed region. It is possible that tax policy changes for a metropolitan area may induce some migration from other areas. To capture this, we may develop a model of a system of metropolitan areas in which the population level of each urban area is endogenous. As is customary in the system of cities literature, the model would have, as an equilibrium condition, households of the same type achieving the same level of utility at a solution, regardless of the urban area in which they locate.

The computational problems that may arise when attempting to solve a full multi-region model could, however, be quite severe. For such a model, in the list of variables over which the algorithm searches, we should have variables of the type used in the one region model specified for each urban area. In addition, one other variable would probably have to be listed. It is one that should be associated with an expression relating to the allocation of a fixed economy-wide population to the different urban areas. As noted above, computational costs tend to increase substantially (usually exponentially) with the number of variables over which the algorithm searches. Mansur and Whalley (1982b) and

Richter (1981), however, decompose large models, if they contain enough special structure to allow this, into a number of partially independent subsystems. The basic approach is to find a solution to the large system by combining solutions to the smaller subsystems. Solutions for the subsystems would involve search over a number of variables that is substantially less than the total number of search variables for the entire system. The cost of doing this is that the number of solutions that have to be computed increases; only one solution is required for the original model. Given the relationship between computed costs and the number of variables over which the fixed point algorithm searches, however, the benefits of using a decomposed structure can outweigh the costs of doing so.

Richter (1981) presents an outline of a decomposition of a general competitive equilibrium model for possible computation by fixed point algorithms. Computing a solution to the full model entails finding a price vector that yields zero excess demands for all the goods in the economy. If, however, the model can be decomposed appropriately, instead of searching over a simplex constructed by using all of the prices in the economy, to increase computational efficiency basically the following is suggested. Let $\{p_1,p_2,\ldots,p_n\}$ be the set of prices for all commodities in the economy. Suppose that a proper subset of this set, say $\{p_1,p_2,\ldots,p_{n_0}\}$, appear as arguments in all of the excess demand functions. It is assumed that the commodities, and so their excess demand functions, can be divided into a number of distinct groups. In one group the excess demands are functions of all the prices. In each of the other groups the excess demands are functions only of $p^o=(p_1,p_2,\ldots,p_{n_o})$ and a proper subset of the remaining prices, none of which appear as arguments, with the exception of the first group mentioned, in the excess demand functions for other groups. The computational strategy involves two levels of calculation. At the lower level solutions are found for each of the commodity groups, with the exception of the first, conditional on the vector of prices p^o. By a solution for a group we mean values for the prices on which the excess demand functions for the group depend that yield zeroes for all excess demands in the group. Computation of these solutions may be accomplished with the use of a fixed point algorithm. The solutions generate values for all prices in the economy other than those in p^o. These values are then passed to the first level where, together with p^o, they are substituted into the excess demand functions for the first group. If this results in zeroes for all of the excess demand functions in the group, then we have found an equilibrium for the full model. If not, then the prices in p^o are adjusted in some manner and the new vector is then passed to the second level where the process outlined above is repeated. Search continues in this way until a solution on the first level is found. The adjustment of p^o to find a solution for the system of excess demands on the first level may be conducted according to a fixed point algorithm.

As mentioned in Richter (1981), an example of an economic system that can be decomposed in this way would be a competitive equilibrium model of a regional economy with traded and non-traded goods. The traded good markets are national, while the non-traded good markets are regional.

Thus, the excess demands on the first level of the decomposition mentioned above can be taken to be those for the traded goods. Each group on the second level would consist of the excess demands for the non-traded goods marketed in a given region.

It is our intent here, though, to discuss how a bi-level decomposition can be applied to a generalization of the models developed and simulated for this study that includes multiple urban areas and household inter-regional migration. A heuristic proof of the existence of equilibrium that utilizes Theorem 1 will be presented. Computation of solutions to such a model may be done in the future with the use of a fixed point algorithm adjusted to deal with a decomposed model.

For simplicity, we assume that there are just two urban areas of the type modelled above and one household group. Let N_i and N now be defined as the population in region i and the total fixed economy-wide population, respectively. The regional populations are allowed to vary, but in equilibrium we must have $N_1+N_2=N$. It is assumed that movement from one urban area to another by households is costless. The property tax revenues to be raised in an urban area should, in some way, depend on the population in the area. The simplest assumption to make is that per capita tax revenues raised in an urban area are constant with respect to population changes. Thus, we let $R_i = \hat{R}_i \cdot N_i$ be the amount of property tax revenues to be raised in region i, where $\hat{R}_i$ is the fixed per capita revenue level for the region. Per capita tax revenues may vary across regions. Now, let $V_i^* (N_i)$, i=1,2, be the equilibrium level of utility in region i when the population for the area is constrained to be equal to N_i. It is assumed that V_i^* is strictly decreasing in N_i.

An intuitive explanation of why this last assumption may be valid might go something like this. An increase in population will tend to depress wage rates because of the increased supply of labor, and to raise housing service prices because of increased demand for housing. Both of these factors tend to decrease equilibrium household utility. Urban area agglomeration economies, however, have thus far been ignored. This may be appropriate for the models developed above since the urban area's population size is constant. The absence of explicit treatment of urban area agglomeration economies in a system of cities model is, however, less tenable. Incorporation of agglomeration economies might render the equilibrium utility assumption invalid at low population levels. Increases in population may lead to economies that outweigh the negative effects of size mentioned above and so result in higher equilibrium utility. In neighborhoods of reasonable equilibrium population levels for the regions, however, scale economies may be exhausted or more than offset by the negative price effects of size. In any case, in order to keep the discussion at this preliminary stage simple, we ignore the scale economies issue, and so are able to preserve the equilibrium utility assumption and the existence of equilibrium arguments given above for the one region models.

Setting up the full model to be solved at one stage using a fixed point algorithm would involve search over two sets of the variables given above for a one region model plus at least one of the population levels for the two regions. This may be quite costly and troublesome. Thus, in the hope of achieving greater computational efficiency, the following decomposition of the model is proposed. We set up a bi-level process where the lower level would consist of equilibrium problems for the two regions, conditional on population levels for each of the regions. The upper level passes population levels, N_1 and N_2, to the lower level. A fixed point algorithm is then used to find regional equilibria. This results in the equilibrium utility levels, $V_1^*(N_1)$ and $V_2^*(N_2)$. These utility values are then passed to the upper level, where they are checked to see whether or not they are equal. If they are equal, and the population levels used to generate them add to the fixed total population, N, then an equilibrium for the full model would have been found. If either of the utilities are not equal or the regional populations do not sum to N, then the regional population levels are adjusted according to a fixed point algorithm, and the new values are passed to the lower level where the process is repeated.

The process of search and the existence of equilibrium for the regional economies on the lower level has already been discussed. Thus, we confine our attention to the upper level. At this level a fixed point algorithm searches over the simplex $S_u = \{(N_1,N_2) \geqq 0 \mid N_1+N_2 \leqq d\}$ where $d > 0$. There are two "excess demand functions" associated with vectors in S_u. In particular, we define the vector-valued function, E_u, as follows:

$$E_u(N_1,N_2) = \begin{pmatrix} V_1^*(N_1) - V_2^*(N_2) \\ N - (N_1+N_2) \end{pmatrix}$$

Values for the regional population levels that yield a zero vector for the value of the function E_u would represent part of a solution to the full model. To prove that such values exist and will be found by the algorithm, we seek to show that the conditions of Theorem 1 are satisfied. Before this can be done, though, another assumption must be made. To ensure the boundedness of E_u, we must truncate the equilibrium utility functions should they rise to infinity at low population levels. To accomplish this and to satisfy hypothesis (H.2) of Theorem 1, the following is assumed about the nature of the two urban areas. It is assumed that small enough population levels, n_1 and n_2, can be found so that $V_1^*(n_1) > V_2^*(N)$ and $V_2^*(n_2) > V_1^*(N)$. Given this, the equilibrium utility functions can be truncated as follows:

$$V_1^*(N_1) = V_1^*(n_1) \qquad N_1 \leq n_1$$
$$\text{if}$$
$$V_2^*(N_2) = V_2^*(n_2) \qquad N_2 \leq n_2$$

The boundedness, continuity, and convexity assumptions of hypothesis (H.1) are now easily seen to be satisfied for E_u. Hypothesis (H.2) is also satisfied. Suppose that $N_1+N_2=d$ for a large value of d; in particular, we should have $d>N$. For $N_2 > 0$, we choose $\alpha=(0,1)$. Then, $N-(N_1+N_2)=N-d<0$. If $N_2=0$, then $\alpha=(1,0)$ is chosen. In that case, $V_1^*(N_1)-V_2^*(N_2)=V_1^*(d)-V_2^*(0)<V_1^*(N)-V_2^*(n_2)<0$. Thus, both hypotheses of the theorem are satisfied, and so the conclusion, expressed in terms of regional population levels, N_1^* and N_2^*, and E_u, is available to us. We cannot have $N_1^* = 0$, for then $V_1^*(N_1^*)-V_2^*(N_2^*)=V_1^*(0)-V_2^*(N) = V_1^*(n_1)-V_2^*(N) > 0$, since $N_1^* = N_2^* = 0$ would imply that $N-(N_1^*+N_2^*)>0$. We also cannot have $N_2^*=0$, for then $V_1^*(N_1^*)-V_2^*(N_2^*) \leq V_1^*(N) - V_2^*(0) = V_1^*(N)-V_2^*(n_2)< 0$, since $N_1^* \geq N$ if $N_2^* = 0$. Hence, provided that it turns out that $N_1^* \geq n_1$ and $N_2^* \geq n_2$, the regional population levels, N_1^* and N_2^*, represent a solution to the upper level problem, and so we would have found an equilibrium for the full model. Further analysis and computation of such a model, and/or a similar one which incorporates agglomeration economies, may be attempted in the future.

REFERENCES

Aaron, H.J. (1974). "A New View of Property Tax Incidence." American Economic Review, 64, no. 2: 212-221.

Aaron, H.J. (1975). Who Pays the Property Tax? Brookings Institution, Washington, D.C.

Arnott, R.J. (1979). "Optimal Taxation in a Spatial Economy with Transport Costs." Journal of Public Economics, 11: 307-334.

Arnott, R.J. and J.G. MacKinnon. (1977a). "The Effects of the Property Tax: A General Equilibrium Simulation." Journal of Urban Economics, 4: 389-407.

Arnott, R.J. and J.G. MacKinnon. (1977b). "The Effects of Urban Transportation Changes: A General Equilibrium Simulation." Journal of Public Economics, 8: 19-36.

Arnott, R.J. and J.G. MacKinnon. (1978). "Market and Shadow Land Rents with Congestion." American Economic Review, 68: 588-600.

Boston Redevelopment Authority. (1967). Transportation Facts for the Boston Region. City of Boston, Massachusetts.

Cooper, K.G. and R.M. Weinberg. (1975). Simulating the Economic Impacts of Revaluation in Boston. Federal Reserve Bank of Boston, Massachusetts.

Fullerton, D., J.B. Shoven and J. Whalley. (1983). "Replacing the U.S. Income Tax with a Progressive Consumption Tax: A Sequenced General Equilibrium Approach." Journal of Public Economics, 20: 3-23.

Grieson, R.E. (1974). "The Economics of Property Taxes and Land Values: The Elasticity of Supply of Structures." Journal of Urban Economics, 1: 367-381.

Henderson, J.V. (1977). Economic Theory and the Cities. Academic Press, New York.

Holland, D.M. and O. Oldman. (1974). Estimating the Impact of 100% of Market Value Property Tax Assessments of Boston Real Estate. Boston Urban Observatory.

Imam, H. and J. Whalley. (1982). "General Equilibrium with Price Intervention Policies: A Computational Approach." Journal of Public Economics, 18: 105-119.

King, A.T. (1977). "Computing General Equilibrium Prices for Spatial Economies." Review of Economics and Statistics, 59: 340-350.

King, A.T. (1980). "General Equilibrium with Externalities: A Computational Method and Urban Applications." Journal of Urban Economics, 7: 84-101.

Koenker, R. (1972). "An Empirical Note on the Elasticity of Substitution Between Land and Capital in a Monocentric Housing Market." Journal of Regional Science, 12: 299-306.

MacKinnon, J.G. (1974). "Urban General Equilibrium Models and Simplicial Search Algorithms." Journal of Urban Economics, 1: 161-183.

MacKinnon, J.G. (1975). "An Algorithm for the Generalized Transportation Problem." Regional Science and Urban Economics, 5: 445-464.

MacKinnon, J.G. (1979). "Computing Equilibria with Increasing Returns." European Economic Review, 12: 1-16.

Mansur, A. and J. Whalley. (1982a). "General Equilibrium in Multijuristictional Models with Income Interdependence." Journal of Economic Theory, 26: 183-190.

Mansur, A. and J. Whalley. (1982b). "A Decomposition Algorithm for General Equilibrium Computation with Application to International Trade Models." Econometrica, 50, no. 6: 1547-1557.

Merill, O.H. (1972). "Applications and Extensions of an Algorithm that Computes Fixed Points of Certain Upper Semi-continuous Point to Set Mappings." unpublished doctoral dissertation, University of Michigan.

Mieszkowski, P. (1972). "The Property Tax: An Excise or a Profits Tax?" Journal of Public Economics, 1: 73-96.

Mills, E.S. (1972). Studies in the Structure of the Urban Economy. Johns Hopkins Press, Baltimore.

Mills, E.S. (1980). Urban Economics. 2nd Edition Scott-Foresman and Company, Glenview, Illinois.

Mills, E.S. and J.G. MacKinnon. (1973). "Notes on the New Urban Economics." Bell Journal of Economics and Management Science, 4: 593-601.

Muth, R.F. (1969). Cities and Housing. University of Chicago Press, Chicago, Illinois.

Muth, R.F. (1971). "The Derived Demand for Urban Residential Land." Urban Studies, 8: 243-254.

Muth, R.F. (1975). "Numerical Solution of Urban Residential Land-Use Models." Journal of Urban Economics, 2: 307-332.

Polinsky, A. and D. Ellwood. (1979). "An Empirical Reconciliation of Micro and Grouped Estimates of the Demand for Housing." Review of Economics and Statistics, 61: 199-205.

Richter, D.K. (1978a). "Existence and Computation of a Tiebout General Equilibrium." Econometrica, 46: 779-805.

Richter, D.K. (1978b). "The Computation of Urban Land Use Equilibria." Journal of Economic Theory, 19: 1-27.

Richter, D.K. (1979). "A Computational Approach to the Study of Neighborhood Effects in General Equilibrium Urban Land Use Models." In The Economics of Neighborhood. Ed. David Segal. Academic Press, New York, N.Y.

Richter, D.K. (1980). "A Computational Approach to Resource Allocation in Spatial Urban Models." Regional Science and Urban Economics, 10: 17-42.

Richter, D.K. (1981). "Decomposition Procedures for Large Scale Optimization and Economic Equilibrium Problems." National Science Foundation Proposal #SES-8105839.

Scarf, H.E. (1967). "The Approximation of Fixed Points of a Continuous Mapping." Siam Journal of Applied Mathematics, 15: 1328-1343.

Scarf, H.E. (with the collaboration of Terje Hansen). (1973). The Computation of Economic Equilibria. Yale University Press, New Haven, Connecticut.

Scarf, H.E. (1981). "The Computation of Equilibrium Prices." In Handbook of Mathematical Economics. Eds. Kenneth J. Arrow and Michael D. Intrilligator. North-Holland, New York, N.Y.

Shoven, J. and J. Whalley. (1973). "General Equilibrium with Taxes: A Computational Procedure and an Existence Proof." Review of Economic Studies, 40, no. 4: 475-489.

Shoven, J. and J. Whalley. (1974). "On the Computation of Competitive Equilibrium on International Markets with Tariffs." Journal of International Economics, 4: 341-354.

Shoven, J. and J. Whalley. (1977). "Equal Yield Tax Alternatives: General Equilibrium Computational Techniques." Journal of Public Economics, 8: 211-224.

Sonstelie, J. (1979). "The Incidence of a Classified Property Tax." Journal of Public Economics, 12: 75-85.

Steen, R.C. (1982). "Effects of the Current and Alternative Systems of Urban Public Finance." unpublished doctoral dissertation, Princeton University.

Sullivan, A.M. (1983a). "A General Equilibrium Model with External Scale Economies in Production." Journal of Urban Economics, 13: 235-255.

Sullivan, A.M. (1983b). "The General Equilibrium Effects of Congestion Externalities." Journal of Urban Economics, 14: 80-104.

Sullivan, A.M. (1983c). "Second-Best Policies for Congestion Externalities." Journal of Urban Economics, 14: 105-123.

Tiebout, C.M. (1956). "A Pure Theory of Local Expenditures." Journal of Political Economy, 64: 416-424.

U.S. Bureau of the Census (1973). Current Population Reports, Series P-20, No. 246, "Household and Family Characteristics: March 1972." U.S. Government Printing Office, Washington, D.C.

U.S. Bureau of the Census (1982a). Current Population Reports, Series P-20, No. 374, "Population Profile of the United States: 1981." U.S. Government Printing Office, Washington, D.C.

U.S. Bureau of the Census (1982b). U.S. Census of Population, 1980. U.S. Government Printing Office, Washington, D.C.

U.S. Department of Commerce, Bureau of Economic Analysis (1973). Survey of Current Business. 53, no. 7.

U.S. Department of Commerce, Bureau of Economic Analysis (1982). Survey of Current Business. 62, no. 7.

U.S. Department of Housing and Urban Development (1982). FHA Homes 1981: Data for States and Selected Areas on Characteristics of FHA Operations under Section 203. U.S. Government Printing Office, Washington, D.C.

U.S. Department of Labor, Bureau of Labor Statistics (1978). Consumer Expenditure Survey: Integrated Diary and Interview Survey Data, 1972-73. Bulletin 1992, U.S. Government Printing Office, Washington, D.C.

U.S. Department of Transportation, Federal Highway Administration (1980). Cost of Owning and Operating Automobiles and Vans, 1979. U.S. Government Printing Office, Washington, D.C.

Van der Laan, G. and A.J.J. Talman (1979). "A Restart Algorithm for Computing Fixed Points without an Extra Dimension." Mathematical Programming, 17: 74-84.

Varian, H.R. (1978). Microeconomic Analysis. W.W. Norton and Company, New York, N.Y.

Wheaton, W.C. (1975). The Statewide Impact of Full Property Revaluation in Massachusetts. Federal Reserve Bank of Boston, Massachusetts.