On The Primarity Of Some Block Intersection Graphs

by

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Declaration of Authorship

I declare that *On the primarity of some block intersection graphs* is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Sunday Vodah June 2018

Signed:..

Abstract

The thesis explores the interaction of factorizations of two structures: graphs and groups. We show that the simplicity of a group has a bearing on the primarity of vertex-transitive graphs. As a consequence, we show that factorizations in both structures in the existence of high symmetry are mutually interdependent.

It also explores the geometry of the projective spaces and the vector spaces in order to discuss some other interesting properties of a graph, $\Gamma_{[n,n-m]}$, defined on subgeometries satisfying some conditions. More specifically, the well known rank-nullity theorem of vector spaces is extended to projective spaces thereby showing that the block intersection graph of a set of subspaces of the same dimension is isomorphic to the block intersection graph of the set of their null spaces. In other words, a projective space of dimension n, the result holds whenever the union of the subspaces V^{n-m+1} and V^m is a direct sum.

One of the classes of graphs of major interest in this study are strongly regular. We explore some fundamental local structures of combinatorial design theory from which we construct strongly regular graphs as well as non-strongly regular graphs. This study identifies designs having the same combinatorial symmetry as the Steiner triple systems from projective geometry and in some cases also identify some graphs having the same combinatorial symmetry as the block intersection graphs of Steiner triple systems from projective geometry.

Goethals and Seidel [19] proved that any 2-design with just 2 intersection numbers has an inherent strongly regular graph.

In this study, we echo the result of Goethals and Seidel thereby showing that a block intersection graph of a tactical configuration on such a 2-design

with just 2 intersection numbers (quasi-symmetric design) produces an inherent strongly regular graph. Precisely, we show that the block intersection graph of tactical configurations on $1-\left(\frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7}, 7, (2^{n-1}-1)\right)$ designs are isomorphic to the block intersection graphs of Steiner triple systems obtained from projective geometry and are hence strongly regular.

June 2018.

In loving memories of

EMMANUEL OLUGBENGA VODAH,

HANNAH VODAH

and

FRANCIS VODAH

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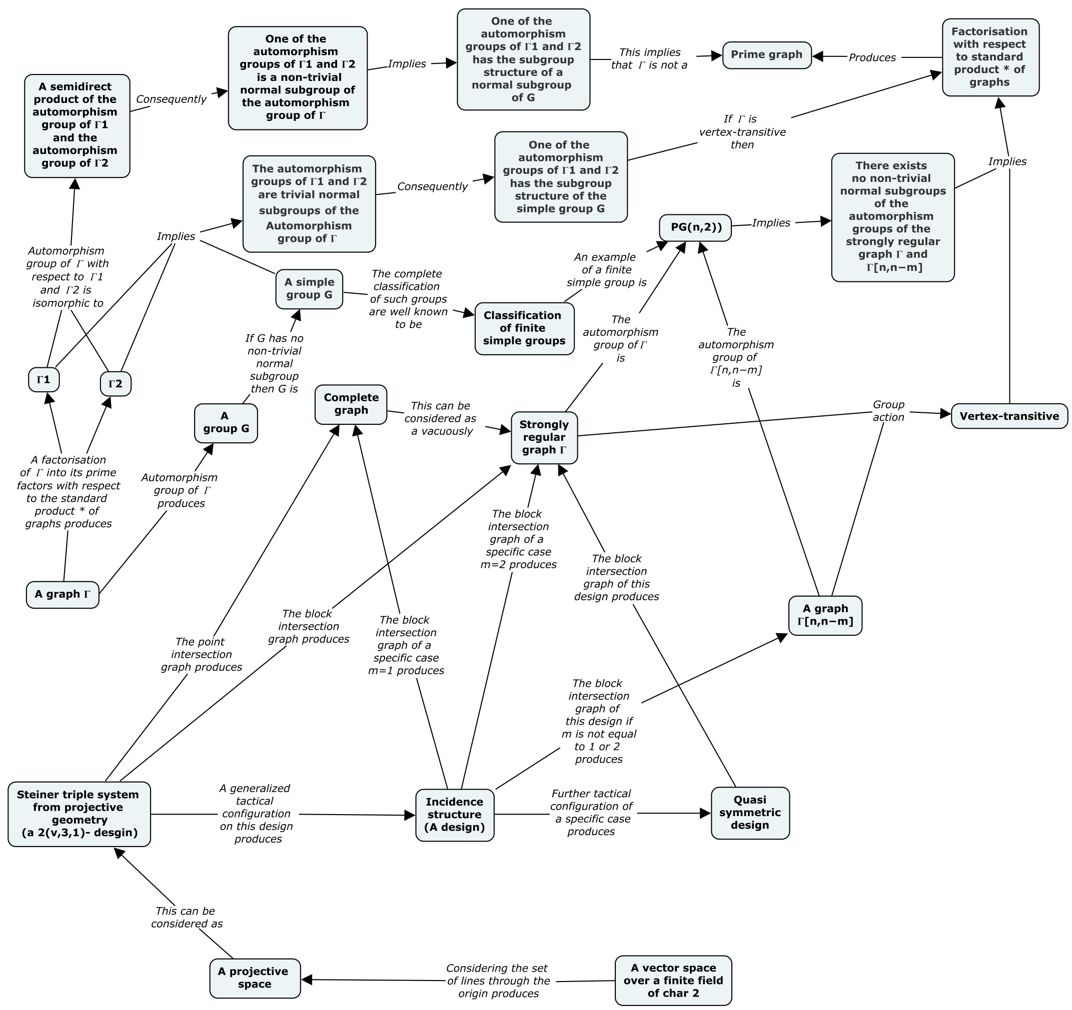
Concept map


One of the automorphism groups of Γ1 and Γ2 is a non-trivial normal subgroup of the automorphism group of Γ
Implies
One of the automorphism groups of Γ1 and Γ2 has the subgroup structure of a normal subgroup of G
This implies that Γ is not a
Prime graph
Produces
Factorisation with respect to standard product * of graphs
A semidirect product of the automorphism group of Γ1 and the automorphism group of Γ2
Consequently
If Γ is vertex-transitive then
The automorphism groups of Γ1 and Γ2 are trivial normal subgroups of the Automorphism group of Γ
Consequently
One of the automorphism groups of Γ1 and Γ2 has the subgroup structure of the simple group G
There exists no non-trivial normal subgroups of the automorphism groups of the strongly regular graph Γ and Γ[n,n−m]
Implies
Implies
PG(n,2))
Implies
An example of a finite simple group is
Automorphism group of Γ with respect to Γ1 and Γ2 is isomorphic to
A simple group G
The complete classification of such groups are well known to be
Classification of finite simple groups
The automorphism group of Γ is
The automorphism group of Γ[n,n−m] is
If G has no non-trivial normal subgroup then G is
Γ1
Γ2
A group G
Complete graph
This can be considered as a vacuously
Strongly regular graph Γ
Group action
Vertex-transitive
A factorisation of Γ into its prime factors with respect to the standard product * of graphs produces
Automorphism group of Γ produces
The block intersection graph of a specific case m=2 produces
The block intersection graph of this design produces
A graph Γ
The block intersection graph produces
The block intersection graph of a specific case m=1 produces
A graph Γ[n,n−m]
The point intersection graph produces
The block intersection graph of this design if m is not equal to 1 or 2 produces
Steiner triple system from projective geometry (a 2(v,3,1)- desgin)
A generalized tactical configuration on this design produces
Incidence structure (A design)
Further tactical configuration of a specific case produces
Quasi symmetric design
This can be considered as
A projective space
Considering the set of lines through the origin produces
A vector space over a finite field of char 2

Contents

Chapter 1

Introduction

1.1 Introduction and background

A tactical configuration consists of a finite set V of points, a finite set $\mathcal{B}$ of blocks and an incidence relation between them, so that all blocks are incident with the same number k points, and all points are incident with the same number r of blocks (See [14] for example). If $v := |V|$ and $b := |\mathcal{B}|$, then v, k, b, r are known as the parameters of the configuration. Counting incident point-block pairs, one sees that $vr = bk$.

In this thesis, we generalize tactical configurations on Steiner triple systems obtained from projective geometry. Our objects are subgeometries as blocks. These subgeometries are collected into systems and we study them as designs and graphs. Considered recursively is a further tactical configuration on some of the designs obtained and in what follows, we obtain similar structures as the Steiner triple systems from projective geometry. We also study these subgeometries as factorizations and examine the automorphism group of the new structures.

These tactical configurations at first sight do not form interesting structures. However, as will be shown, they offer some level of intriguing symmetries. It will be shown that they inherit the automorphism group of the parent geometry.

The block intersection graphs of the designs obtained are highly symmetrical graphs. They also inherit the automorphism group of the parent geometry.

A factor of a graph Γ is just a spanning subgraph and a graph factorization of Γ is a partition of the edges of Γ into factors. The process can also be reversed; that is, a product can be defined on the factors to produce the resultant graph Γ. One of the challenges in doing this is to determine the exact product.

A graph product is a relational structure built on the relations of the factor graphs. While there are $2^6 - 1$ artificial possibilities of products of graphs on two given graphs, howbeit, under reasonable and natural restrictions (such as associativity), the number of different products is actually quite limited. For instance, Imrich and Izbicki showed that there are exactly 20 associative graph products. [27]

There are three main products that have been studied in the literature: the Cartesian product, the direct product, and the strong product. Associativity allows for the easy extension of these products to arbitrarily many factors.

Specifically, a graph product is a relational structure that takes two graphs (factors) Γ_1 and Γ_2 to produce a graph Γ. The vertex set of the product is the Cartesian product $V(\Gamma_1) \times V(\Gamma_2)$. That is;

$$V(\Gamma_1 * \Gamma_2) = \{(u_1, v_1) | u_1 \in V(\Gamma_1) \text{ and } v_1 \in V(\Gamma_2)\}.$$

However, each product has different rules for adjacency of the vertices. The actions of automorphisms on some products of graphs have been well developed in [21].

We understudy these actions of automorphisms on the standard products of graphs as discussed in [21]. In addition, we investigate the automorphism groups of the block intersection graphs of the tactical configurations of this study and having found that the automorphism group is nontrivial and its only normal subgroups are the trivial group and the group itself, we show that any vertex-transitive graph Γ with such algebraic properties cannot be factorized with respect to the standard products of graphs. We therefore show that the block intersection graphs of the tactical configurations as well as that of the Steiner triple systems from projective geometry are primes.

We now provide a brief background to this study.

A Steiner system is an important type of block design that has been studied extensively. A Steiner triple system of order v is a collection of subsets of size three from a set of $v-$elements such that every pair of the elements of the set is contained in exactly one 3-subset. Many variations of block designs have been studied, but the most intensely studied are the balanced incomplete block designs (BIBDs or 2-designs) which are historically related to statistical issues in the design of experiments.

Steiner systems were defined through a posed problem for the first time by W.S.B. Woolhouse in 1844 [40]. The posed problem on Steiner triple systems was solved by Thomas Kirkman in 1847 [29]. In 1850, Kirkman introduced a variation of the problem now known as Kirkman's schoolgirl problem, which included an additional property called resolvability to triple systems. Oblivious of Kirkman's work, Jakob Steiner (1853) re-established triple systems. His extensive study on this subject matter made it more popular and it is not surprising that they are named after him.

Kirkman [29] established the fact that a Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$ in 1847. Bose [5] constructed such systems in 1939 for $v \equiv 3 \pmod 6$. Two decades later, Skolem [33] constructed triple systems for which $v \equiv 1 \pmod 6$.

Several studies have shown that Steiner triple systems exist in projective geometry and their automorphisms are well known ([8],[12],[26],[28]).

Strongly regular graphs are well known to be rooted in Steiner triple systems. Strongly regular graphs constitute one of the many highly characterized classes of graphs. As Cameron has noted, this important family of regular graphs appear between the highly structured and the apparently random. Strongly regular graphs have a lot of interesting properties for even a brief description here. See [9], [32] for a useful survey on strongly regular graphs.

In 1963, Bose [6] initiated the theory in the context of partial geometries and 2-class association schemes. This gave rise to the many combinatorial concepts of strongly regular graphs such as orthogonal arrays, conference matrices, Latin squares, geometric graphs and designs. A year later, Higman [23] introduced the study of the rank 3 permutation groups through strongly regular graphs. In 1975, Smith [34] considered the problem of permutation groups whose rank and subrank is 3. This result was generalized to strongly regular graphs satisfying local conditions such that the subconstituents with respect to some vertices are also strongly regular [10].

Over the years, the algebraic and the combinatorial aspects of strongly regular graphs have been well developed. In this study, we examine substructures of both the combinatorial and algebraic views of strongly regular graphs as well as the links between these two points of view.

In this study, we consider finite structures only.

1.2 Overview of the thesis

In the course of introducing the preliminaries of this study, we introduced our fundamental object of study. After giving the necessary preliminaries, we generalize tactical configurations from the fundamental object of study and in what follows, we consider the block intersections and intersection numbers of our tactical configurations. In addition, we introduce the concept of a recursive tactical configuration on our generalized configurations.

In Chapter 4 of this study, we construct block intersection graphs of the generalized tactical configurations as well as compare and contrast them to the block intersection graphs of their Steiner triple systems from projective geometries. Thereafter, we further investigate the properties of the new graphs thereby showing that in some cases, the graphs are isomorphic. We conclude the chapter by considering a specific graph from a recursive configuration of our tactical configuration.

In Chapter 5, having known that the automorphism group of the Steiner triple systems from projective geometry is isomorphic to the automorphism group of the block intersection graph of the Steiner triple systems, we investigate the automorphism groups of block intersection graphs of our generalized tactical configurations.

We also consider the actions of automorphisms on some products of graphs and having studied some algebraic properties of the block intersection graphs of our generalized tactical configurations, we conclude that the graphs are primes with respect to the standard product of graphs. As a result, we also conclude that factorizations in vertex-transitive graphs and groups are mutually interdependent.

In Chapter 6, we summarize the study and discuss various straddles of further studies.

Chapter 2

Preliminaries and notations

In undertaking this study, various algebraic structures, notations and definitions are called into action. We therefore introduce these concepts and present some basic definitions of the structures. We also present some fundamental results and mathematical concepts used in discussions throughout this study. As far as possible, we adhere to the commonly used notation and terminology.

We begin by introducing fundamental group theoretic concepts used in this study.

2.1 Group theoretic definitions

A very basic but pertinent notion of groups used in this study is that a nontrivial group G is *simple* if its only normal subgroups are the trivial group and the group itself. A group that is not simple can be broken into two smaller groups, a normal subgroup and the quotient group, and the process can be repeated.

Every action of a group decomposes the set into orbits. An action of a group on a set is called *transitive* when the set is nonempty and there is exactly one orbit.

The automorphism groups of the block intersection graphs of the tactical configurations of this study, the further tactical configurations as well as the

Steiner triple systems from projective geometrys are finite simple groups.

We now provide a brief background to simple groups. This is essential because the main result of this thesis has a direct application on the classification of finite simple groups to graphs. This is also important because it will serve as caveat to factorizations on graphs having these subgroup structures.

The classification of finite simple groups is a landmark result of modern mathematics. It was a monumental task. The project spans well over 110 years. The original proof is spread over scores of articles by dozens of researchers.

We now present the classification theorem.

2.1.1 The Classification Theorem

According to [38], the classification theorem for finite simple groups states that every finite simple group is isomorphic to one of the following:

1. a cyclic group C_p of prime order p;
2. an alternating group A_n, for $n \geq 5$;
3. a classical group:

   linear: $\mathrm{PSL}_n(q), n \geq 2$, except $\mathrm{PSL}_2(2)$ and $\mathrm{PSL}_2(3)$;
   unitary: $\mathrm{PSU}_n(q), n \geq 3$, except $\mathrm{PSU}_3(2)$;
   symplectic: $\mathrm{PSp}_{2n}(q), n \geq 2$, except $\mathrm{PSp}_4(2)$;
   orthogonal: $\mathrm{P\Omega}_{2n+1}(q), n \geq 3$, q odd;
   $\mathrm{P\Omega}^+_{2n}(q), n \geq 4$;
   $\mathrm{P\Omega}_{2n}(q), n \geq 4$
   where q is a power of prime p ;

4. an exceptional group of Lie type:

   $$G_2(q), q \geq 3; F_4(q); E_6(q); {}^2E_6(q); {}^3D_4(q); E_7(q); E_8(q)$$

   where q is a prime power, or

   $${}^2B_2(2^{2n+1}), n \geq 1; {}^2G_2(3^{2n+1}), n \geq 1; {}^2F_4(2^{2n+1}), n \geq 1$$

   or the Tits group ${}^2F_4(2)'$;

5. one of 26 sporadic simple groups:

 - the five Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$;
 - the seven Leech lattice groups Co_1, Co_2, Co_3, McL, HS, Suz, J_2;
 - the three Fischer groups Fi_{22}, Fi_{23}, Fi'_{24};
 - the five Monstrous groups $\mathbb{M}, \mathbb{B}$, Th, HN, He;
 - the six pariahs J_1, J_3, J_4, O'N, Ly, Ru.

Conversely, every group in this list is simple, and the only repetitions in this list are:

$$\begin{aligned} \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) &\cong \mathrm{A}_5; \\ \mathrm{PSL}_2(7) &\cong \mathrm{PSL}_3(2); \\ \mathrm{PSL}_2(9) &\cong \mathrm{A}_6; \\ \mathrm{PSL}_4(2) &\cong \mathrm{A}_8; \\ \mathrm{PSU}_4(2) &\cong \mathrm{PSp}_4(3). \end{aligned} \tag{2.1}$$

The actions of these groups on various natural geometrical or combinatorial objects to the point where much of the subgroup structure is revealed is well documented in [38].

One of the main results of this study considers factorizations of some block intersection graphs through their automorphism groups. The graphs in this regard are the block intersection graphs of 2-$(v, 3, 1)$ designs from projective geometry. It is well known that the automorphism group of these graphs are isomorphic to the projective general linear group ([26], [28]). Hence, we introduce the concept.

Definition 2.1. Let $\mathbb{F}$ be a field. Then the general linear group $\mathrm{GL}(n, \mathbb{F})$ is the group of invertible $n \times n$ matrices with entries in $\mathbb{F}$ under matrix multiplication.

If $\mathbb{F}$ is a finite field of order q, then it is written $\mathrm{GL}(n, q)$ instead of $\mathrm{GL}(n, \mathbb{F})$. It is a well known fact [37] that the order of the group $\mathrm{GL}(n, q)$ is

$$|\mathrm{GL}(n, q)| = \prod_{m=0}^{n-1} (q^n - q^m). \tag{2.2}$$

We now discuss the projective general linear group.

The quotient of $\mathrm{GL}(n, \mathbb{F})$ by its center $Z(\mathrm{GL}(n, \mathbb{F}))$ is called the projective linear group.

A Galois group is a group of field automorphisms under composition. It is denoted $\mathrm{Gal}(\mathbb{F})$. The Galois group acts on $\mathrm{GL}(n, \mathbb{F})$ by the Galois action on its entries.

A semilinear transformation is a transformation which is linear up to a field automorphism under scalar multiplication. The general semilinear group $\Gamma\mathrm{L}(n, \mathbb{F})$ is the group of all invertible semilinear transformations. The general semilinear group contains $\mathrm{GL}(n, \mathbb{F})$. It is conveniently written as a semidirect product:

$$\Gamma\mathrm{L}(n, \mathbb{F}) = \mathrm{Gal}(\mathbb{F}) \rtimes \mathrm{GL}(n, \mathbb{F})$$

where $\mathrm{Gal}(\mathbb{F})$ is the Galois group over its prime field $\mathbb{F}$.

The general semilinear group $\Gamma\mathrm{L}(n, \mathbb{F})$ is of important interest in this study because the associated projective semilinear group $\mathrm{P}\Gamma\mathrm{L}(n, \mathbb{F})$, which contains $\mathrm{PGL}(n, \mathbb{F})$ is the collineation group of projective space, for $n > 2$. This is as a result of the fundamental theorem of the projective geometry which is in the following terms.

Theorem 2.1. [1] *Any isomorphism between projective spaces of dimension at least 2 is induced by a semi-linear transformation between the underlying vector spaces, unique up to scalar multiplication.*

For more details on linear groups, see [13].

We now turn to a very fundamental concept crucial to this study.

2.2 Basics of design theory

We introduce designs in order to explore the subconstituents of strongly regular graphs which are very essential to the construction of the class of graphs of consideration in this study. Most of the concepts used here can be found in [2] and [35].

The most basic notion of the theory is that of an incidence structure. An incidence structure is a triple $\mathcal{D} = (V, \mathcal{B}, I)$ where V and $\mathcal{B}$ are any two disjoint sets and I is a binary relation between V and $\mathcal{B}$, that is $I \subseteq V \times \mathcal{B}$. The elements of V are called *points*, those of $\mathcal{B}$ *blocks* and those of I *flags*. It will always be clear from the context whether a given object is a point or a block. If the ordered pair $(p, B) \in I$, it is said that p is incident with B, B is said to contain the point p, or p is on B. If p is any point, (p) denotes the set of blocks incident with p, that is, $(p) := \{B \in \mathcal{B} : (p, B) \in I\}$. $|(p)|$ is said to be the *point degree* while $|\mathcal{B}|$ is said to be the *block degree*.

An incidence structure can also be considered as a *t-design* or a *block design*. This is made precise in the following manner.

An incidence structure $\mathcal{D} = (V, \mathcal{B}, I)$ in which a set V of v points and a family $\mathcal{B}$ of b subsets (blocks) containing k points each in such a way that any two points determine λ blocks, and each point is contained in r different blocks is called a block design with parameters (v, k, λ, b, r).

In this study, we consider *simple, regular and uniform designs*. A block design in which all the blocks have the same size is called uniform. Two identical blocks in a design are said to be *repeated* blocks. A design is said to be simple if it does not contain repeated blocks. A design is regular if every point occurs equally often in the design. A block design is said to be an *incomplete block design*, if $k < v$. It is called a *balanced* block design if every pair of distinct points in the design is contained in exactly λ blocks. A block design which is incomplete and balanced is called a *balanced incomplete* block design; simply, a BIBD.

We now discuss an incidence structure as a *t-design*.

Let V be a set with cardinality v and $\mathcal{B}$ is a collection of subsets (blocks) of size k selected from V. Let t, v and λ be positive integers such that $1 < k < v$, and any set of t points appears as a subset of exactly λ blocks. Then $(V, \mathcal{B})$ is called a $t-$design with parameters (v, k, λ).

A $t-$design with parameters (v, k, λ) or a $t-(v, k, \lambda)$ design is usually denoted $S_\lambda(t, k, v)$. Precisely, a block design or a $t-$design with $t \geq 2$ and $\lambda = 1$ is called a *Steiner system*. Should $k = 3, t = 2$ and $\lambda = 1$; then, we have a $2-$design with parameters $(v, 3, 1)$ denoted $S(2, 3, v)$ or a 2-$(v, 3, 1)$ design. These designs are called *Steiner triple systems*. As the name implies, we also refer to blocks of 2-$(v, 3, 1)$ designs in this study as *triples*. The substructures of these designs are of a major interest in the study.

The parameters b and r of a block design satisfy the following combinatorial identities.

Theorem 2.2. [2] *Let $\mathcal{D} = (V, \mathcal{B}, I)$ be an $S_\lambda(t, k; v)$. Then we have*

(a) $|(p)| = \lambda(v-1)/(k-1) := r$ *for all points* $p \in V$;

(b) $|\mathcal{B}| = \lambda v(v-1)/k(k-1) := b$.

Proof, see ([2], Theorem 2.10).

Corollary 2.1. *Let $\mathcal{D} = (V, \mathcal{B}, I)$ be an $S_\lambda(t, k; v)$ such that $v, k, \lambda \in \mathbb{N}$. Then the necessary conditions for the existence of an $S_\lambda(t, k; v)$ are*

(a) $\lambda(v-1) \equiv 0 \bmod (k-1)$;

(b) $\lambda v(v-1) \equiv 0 \bmod k(k-1)$.

We now consider the notion of an *induced substructure* of a design.

Let $\mathcal{D} = (V, \mathcal{B}, I)$ be an incidence structure and $V' \subseteq V$ and $\mathcal{B}' \subseteq \mathcal{B}$. Then the incidence structure induced by $\mathcal{D}$ on V' and $\mathcal{B}'$ is $\mathcal{D}' = (V', \mathcal{B}', I|V' \times \mathcal{B}')$, and $\mathcal{D}'$ is called an induced substructure of $\mathcal{D}$.

In order to consider induced substructures of 2-$(v, 3, 1)$ designs, we now introduce $1-$*designs*.

$1-$designs are simple structures where the only requirement other than constant block size is that each point appears in the same number of blocks. Generalized Quadrangles are examples of $1-$designs.

The fundamental object of this study in the following terms.

A 1-$(v, k, r = \lambda)$ design is called a *tactical configuration* (or simply a *configuration*) with parameters v, k, r, and $b = vr/k$.

Theorem 2.3. [4] *A 1-(v, k, r) design with b blocks exists if and only if $vr = bk$ and $b \leq \binom{v}{k}$.*

Theorem 2.4. [2] *Let $\mathcal{D}$ be a t-design and let $s < t$ be a positive integer. Then $\mathcal{D}$ is also an $s-$design. More specifically, if $\mathcal{D}$ has parameters v, k and λ_t where (λ_t is the number of blocks through a t-set), then the parameter λ_s (the number of blocks through an s-set) is given by*

$$\lambda_s = \lambda_t \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s}.$$

A special case of Theorem 2.4 is the fact that every $2-$design is a configuration.

In what follows, we consider a further tactical configuration of the tactical configuration discussed in this study. Hence, there is need to describe the concept.

In this consideration, the blocks of the parent 2-$(v, 3, 1)$ designs, $(V, \mathcal{B})$ are taken to be the set of points of a our tactical configuration and we use suggestive set-theoretic notation correspondingly. Starting from $(V, \mathcal{B})$, we define $\mathcal{B}$ to be a new set of points and the new blocks obtained to be $\overline{\mathcal{B}}$ such that $(\mathcal{B}, \overline{\mathcal{B}})$ is a tactical configuration of $(V, \mathcal{B})$. The tactical configuration of interest here excludes repeated blocks.

Another important concept in design theory considered in this study is *quasi-symmetric design*. First, we introduce the notion of a *symmetric design*.

A balanced incomplete block design, BIBD in which $b = v$ (or, equivalently, $r = k$ or $\lambda(v-1) = k^2 - k$) is called a *symmetric* BIBD. A symmetric $2-$design has $b = v$, and every two blocks intersect in exactly λ points.

A number x, for $0 \leq x < k$, is called an intersection number of a design, $\mathcal{D} = (V, \mathcal{B})$ if there exist blocks, $B, B' \in \mathcal{B}$ such that $|B \cap B'| = x$. Hence, for any block design, the intersection numbers are the cardinalities of the intersections of any two distinct blocks. k is not an intersection number in this study because we do not allow repeated blocks. It is well known that a 2-(v, k, λ) design, $\mathcal{D}$ is symmetric if and only if $\mathcal{D}$ has precisely one intersection number (λ).

Let x and y be non-negative integers with $x \leq y$. A design $\mathcal{D}$ is called quasi-symmetric with intersection numbers x and y if any two distinct blocks of $\mathcal{D}$ intersect in either x or y points, and both intersection numbers are realized. A quasi-symmetric design is called *proper* if $x \neq y$ and *improper* otherwise. Clearly, symmetric designs are improper quasi-symmetric designs and any 2-$(v, k, 1)$ design with $b > v$ is a proper quasi-symmetric design with $x = 0$ and $y = 1$. Thus linear spaces (2-$(v, k, 1)$ designs) provide examples of proper and improper quasi-symmetric designs. In general, the parameters $(v, b, r, k, \lambda; x, y)$ are called the standard parameters of a quasi-symmetric design.

2-$(v, k, 1)$ designs have only two intersection numbers since, no two blocks can meet in more than one point. An analogous design to this is a block design, $\mathcal{D} = (V, \mathcal{B})$ such that for any two $B, B' \in \mathcal{B}, |B \cap B'| \in \{0, y\}, 1 \leq x < \cdots < y < k, x, y \in \mathbb{N}$. That is, a design such that if any two blocks meet, they do so in more than one point. These designs mostly arise from designs as a result of subspaces of a vector space. In fact, this is the case of our tactical configrations in Chapter 3. We consider block intersection graphs from these designs in Chapter 4.

In Chapter 3 of this study, we consider a generalized tactical configurations of Steiner triple systems (2-$(v, 3, 1)$ designs) from projective geometry. We now introduce the design.

Proposition 2.1. [8] *Let V be the set of all non-zero vectors of $\mathbb{F}_2^{n+1}$, and let $\mathcal{B} = \{\{v_1, v_2, v_3\} : \{v_1, v_2, v_3\}$ is distinct, $v_1 + v_2 + v_3 = 0\}$. Then $(V, \mathcal{B})$ is a Steiner triple system of order $2^{(n+1)} - 1$.*

This design is from a *projective plane* of dimension n over a finite field of order 2, that is, $\mathrm{PG}(n, 2)$. In this study, the design above is known as the *projective Steiner triple system* of order $2^{(n+1)} - 1$ and it is denoted $(\mathrm{PG}(n, 2), \mathcal{B})$.

The case $n = 2$ is the well known *Fano plane*.

As indicated above, $\mathrm{PG}(n, 2)$ are realized from the non-zero vectors of a vector space, $V = \mathbb{F}_2^{n+1}$ of a dimension higher than the projective plane. The projective plane $\mathrm{PG}(n, 2)$ is of dimension n and not $(n + 1)$. This is because in projective geometry, lines through the origin consist of just two points of which the origin is one. Hence, lines through the origin are identified as points and any point of $\mathrm{PG}(n, 2)$ can be identified as the non-zero vector spanning the corresponding line.

A necessary condition for the existence of a Steiner triple system of order v is in the following:

Proposition 2.2. [29] *Let $(V, \mathcal{B})$ be a Steiner triple system of order v. Then $v \equiv 1$ or $3 \pmod 6$.*

Corollary 2.2. *Let $(V, \mathcal{B})$ be a Steiner triple system of order v. A point $x \in V$ is in exactly $\dfrac{v-1}{2}$ blocks.*

As a result of Proposition 2.2, we have that the total number of blocks in a projective Steiner triple system of order $2^{(n+1)}-1$ is

$$|\mathcal{B}| = \frac{(2^{n+1}-1)(2^n-1)}{3}.$$

As a result of our interest in tactical configurations from $(\mathrm{PG}(n,2),\mathcal{B})$, there is the need to discuss the subspaces of the underlying vector space.

For any $(n-m) = 0,1,2,\ldots,n-1$, a subspace of dimension $(n-m)$, or $(n-m)$-space, of a $\mathrm{PG}(n,2)$ is a set of points all of whose representing vectors form, together with the zero in $V = \mathbb{F}_2^{n+1}$, form a subspace of dimension $(n-m+1)$. The 1-dimensional (2-dimensional, 3-dimensional, n-dimensional) subspaces of V are called points (lines, planes, hyperplanes); in general, the $(n-m+1)$-dimensional subspaces are called $(n-m)$-flats. To avoid confusion, we denote $\mathrm{PG}((n-m),2)$ as an $(n-m)$-flat of $\mathrm{PG}(n,2)$.

The number of points of an $(n-m)$-flat in $\mathrm{PG}(n,2)$ is $(2^{n-m+1}-1)$ ([24], See [Theorem 3.1]) and the number of distinct $(n-m)$-flats in $\mathrm{PG}(n,2)$ is

$$\frac{\prod_{i=0}^{(n-m)}(2^{n-i+1}-1)}{\prod_{i=0}^{(n-m)}(2^{i+1}-1)}$$

([15], See [A.4.3]). In fact, the points of $\mathrm{PG}(n,2)$ together with $\mathrm{PG}((n-m),2)$ as blocks and incidence by natural containment form an incidence structure denoted by $\mathrm{PG}_{(n-m)}(n,2)$ ([2], [See Definition 2.15]). More so, $\mathrm{PG}_{(n-m)}(n,2)$ is a block design with parameters $v = 2^{n+1}-1$, $k = 2^{n-m+1}-1$,

$$r = \frac{\prod_{i=1}^{(n-m)}(2^{n-i+1}-1)}{\prod_{i=0}^{(n-m-1)}(2^{i+1}-1)}, \qquad \lambda = \frac{\prod_{i=2}^{(n-m+2)}(2^{n-i+1}-1)}{\prod_{i=0}^{(n-m-2)}(2^{i+1}-1)} \text{ and}$$

$$b = \frac{\prod_{i=0}^{(n-m)}(2^{n-i+1}-1)}{\prod_{i=0}^{(n-m)}(2^{i+1}-1)}$$

([2], [See Proposition 2.16]).

The designs obtained in this study are from tactical configurations of $(\mathrm{PG}(n,2),\mathcal{B})$ and differs from the design, $\mathrm{PG}_{(n-m)}(n,2)$ in terms of parameters and structure.

The automorphism groups of the structures at hand is another aspect of the discussion in consideration that recurs in this thesis.

In discrete structures generally, it is very pertinent to determine the group of automorphisms of a given structure in order to understand the symmetry of such a structure. We now present the notion of an automorphism group of a design.

Let $(V,\mathcal{B})$ and $(V',\mathcal{B}')$ be any two designs. An isomorphism is a map $\phi : V \to V'$ such that ϕ is a bijection and $\phi(B) \in \mathcal{B}'$ whenever $B \in \mathcal{B}$.

If two designs $(V,\mathcal{B})$ and $(V',\mathcal{B}')$ are isomorphic, it is denoted $(V,\mathcal{B}) \cong (V',\mathcal{B}')$. Should $(V,\mathcal{B})$ and $(V',\mathcal{B}')$ coincide then ϕ is said to be an automorphism.

Composing two automorphisms produces an automorphism. The identity is always an automorphism and the inverse of an automorphism is also an automorphism. Hence, the set of all automorphisms forms a permutation group under composition, which is called the automorphism group of the design.

The automorphism group of a design is always a subgroup of the symmetric group on v letters where v is the number of points of the design. For a design $\mathcal{D}$, the automorphism group is denoted Aut $\mathcal{D}$.

We now turn our attention to some fundamentals of graph theory discussed in this study.

2.3 Graph theoretic definitions

In this section, we briefly introduce some relevant concepts of graph theory. Furthermore, we present the fundamental class of graphs of importance to this study. Most of the concepts discussed here are found in [39].

The graphs of interest in this thesis are finite simple graphs.

Let V be a set and R a relation defined on V. Then $D = (V,R)$ is called a *digraph* if R is irreflexive, i.e., $(v, v) \notin R$ for all $v \in V$. The elements of V are called *vertices* and the elements of R are called *arcs*. The *out-degree* of a vertex x is the size of the set $\{y \in V : (x, y) \in R\}$. The *in-degree* of x is similarly defined as the size of the set $\{y \in V : (y, x) \in R\}$.

A graph $\Gamma = (V, E)$ is a digraph with the additional property that E is symmetric. In other words, a graph $\Gamma = (V, E)$ consists of a set of vertices V and a relation E which is irreflexive and symmetric. If it is not clear from the context, we will denote V by $V(\Gamma)$ and E by $E(\Gamma)$.

The arcs (x, y) and (y, x) are identified into a single *edge* and denoted $[x, y]$. Let V be a set and denote $V^{\{2\}}$ the family of all 2-subsets of V. Edges of the graph $\Gamma = (V, E)$ can be identified as subsets of $V^{\{2\}}$.

Two vertices x and y of a graph Γ are *adjacent* if there is an edge, $e = [x, y]$ joining them. The vertices x and y are said to be *incident* with e. If x and y are adjacent, it is said that they are *neighbours*. Similarly, two distinct edges e and e' are *adjacent* if they have a vertex in common.

Let Γ be a graph. The neighbourhood $N_\Gamma(v)$ of a vertex $v \in \mathrm{V}(\Gamma)$ is the set of vertices that are adjacent to v. The closed neighbourhood $N_\Gamma[v]$ of v is then the union $N_\Gamma[v] \cup \{v\}$. The *degree* $\deg(v)$ of a vertex, $v \in \mathrm{V}(\Gamma)$ is the number of edges incident on v, that is, the size of its neighbourhood, $|N_\Gamma(v)|$.

One of the foundational results commonly used in graph theory is the hand shaking lemma (See [3]), which is given in the following:

Let $\Gamma = (V, E)$ be a graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

A graph in which each vertex has the same degree is said to be *regular*. If each vertex has degree r, the graph is regular of degree r or r*-regular*. A *complete* graph is the one in which every two distinct vertices are adjacent. The complete graph on n vertices is denoted K_n. A graph having no edges is called a *null graph*.

A bipartite graph, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.

A sequence $x_0, x_1, \ldots, x_k$ of distinct vertices of a graph Γ is a path if

$[x_i, x_{i+1}] \in E(\Gamma)$ for all $i = 0, \ldots, k-1$. A graph Γ is said to be connected if for every pair of vertices, there is a path joining them.

Let u, v be vertices in a graph Γ. The *distance* from u to v is the length of a shortest path from u to v in Γ and is denoted $d(u, v)$. This is also known as the *geodesic distance*. The *eccentricity* of a vertex $u \in V(\Gamma)$ denoted $\mathrm{e}(u)$ is defined to be the maximum distance between u and any other vertex $v \in V(\Gamma)$; that is,

$$\mathrm{e}(u) = \max\{\mathrm{d}(u, v) | v \in V(\Gamma)\}.$$

The *diameter* of a graph Γ denoted $\mathrm{diam}(\Gamma)$ is defined to be maximum eccentricity of the graph, that is,

$$\mathrm{diam}(\Gamma) = \max\{e(y) | y \in V(\Gamma)\}.$$

A graph $\Gamma' = (V', E')$ is called a *subgraph* of $\Gamma = (V, E)$ if $V' \subset V$ and $E' \subset E$. Let $\Gamma = (V, E)$ be a graph and Γ' be a subgraph of Γ. Then Γ' is an *induced subgraph* on V' if $E' = E \cap V'^{\{2\}}$. A spanning subgraph of a graph Γ is a subgraph Γ' with $V(\Gamma) = V(\Gamma')$, that is, Γ' and Γ have exactly the same vertex set.

Let $\Gamma = (V, E)$ be a graph and $V' \subseteq V$. Then the induced subgraph $\Gamma[V']$ is a *clique* (*independent set*) if it is a complete graph (null graph).

The size of a clique is the number of vertices in the clique. A *maximal clique* (*maximal independent set*) is a clique (independent set) that cannot be extended by an additional vertex. That is, a maximal clique (maximal independent set) is one which is not contained in a larger clique (independent set). A *maximum clique* (*maximum independent set*) is the clique (independent set) of the largest possible size in a given graph Γ. The *clique number* $\omega(\Gamma)$ (*independence number* $\omega'(\Gamma)$) of a graph Γ is the number of vertices in the maximum clique (maximum independent set) of Γ.

In the following concepts, we now introduce some classes of graphs of interest in this study.

Definition 2.2. Given an incidence structure $\mathcal{D}$, the point graph of $\mathcal{D}$ is the graph Γ such that the vertex set of Γ has the same point set as $\mathcal{D}$. Any two vertices u and v of $\mathrm{V}(\Gamma)$ are adjacent whenever there is a block of $\mathcal{D}$ containing both u and v.

Definition 2.3. Given a combinatorial design, $\mathcal{D} = (V, \mathcal{B})$.

(a) The *block-intersection graph* of $\mathcal{D}$ is the graph having $\mathcal{B}$ as its vertex set, and in which two vertices B and B' are adjacent if and only if $B \cap B' \neq \emptyset$.

(b) The *i-block-intersection graph* of $\mathcal{D}$ is the graph having $\mathcal{B}$ as its vertex set, and in which two vertices B and B' are adjacent if and only if $|B \cap B'| = i$, for some $i \in \mathbb{N}$.

For a combinatorial design, $\mathcal{D} = (V, \mathcal{B})$ such that for any two $B, B' \in \mathcal{B}, |B \cap B'| \in \{0, y\}, 1 \leq x < \cdots < y < k$, that is, a design such that if any two blocks meet, they do so in more than one point. The *block-intersection graph* of $\mathcal{D}$ is the graph having $\mathcal{B}$ as its vertex set, and in which two vertices B and B' are adjacent if and only if $B \cap B' = y$. An example of such a block intersection graph is the Grassmann graph. This graph is considered in Chapter 4 of this study and hence we now define it.

Definition 2.4. Let $\mathbb{F}$ be a field, let V be a vector space of dimension $n \geq 2$ over $\mathbb{F}$, and let e be an integer satisfying $1 \leq e \leq n-1$. The Grassmann graph $\Gamma(n, e)$ is the graph whose vertices are the e-dimensional subspaces of V and two vertices are adjacent if and only if they meet in a subspace of dimension $e - 1$.

The block intersection graphs of the tactical configurations in this study have similar considerations in terms of edges.

It is well known that the block intersection graphs of 2-$(v, 3, 1)$ designs are strongly regular. These graphs are fundamental to this study. We now turn to them.

Definition 2.5. A *strongly regular graph* Γ with parameters (n, k, λ, μ) is a graph on n vertices which is regular with degree k and has the following properties:

(i) any two adjacent vertices have exactly λ common neighbors;

(ii) any two non-adjacent vertices have exactly μ common neighbors.

The parameters λ and μ respectively are undefined for complete and null graphs. The two classes are vacuously strongly regular. The parameters are not independent but a complete characterization of the parameter sets of strongly regular graphs is not known. The reader is referred to Cameron [9] and Seidel [32] for more basic properties of strongly regular graphs.

In Chapter 5 of this study, we discuss the factorizations of the block intersection graphs of our tactical configurations. Our major approach here is to consider the automorphism groups of the graphs. We now define the concept.

Definition 2.6. (a) Let Γ_1, Γ_2 be graphs. A *homomorphism* from Γ_1 to Γ_2 is a map $\alpha : V(\Gamma_1) \to V(\Gamma_2)$ that preserves adjacency; that is, $[\alpha(x), \alpha(y)] \in V(\Gamma_2)$ whenever $[x, y] \in V(\Gamma_1)$.

(b) A homomorphism $\alpha : V(\Gamma_1) \to V(\Gamma_2)$ is an *isomorphism* if

(i) α is a bijection;

(ii) α^{-1} is also a homomorphism.

In this case, it is said that Γ_1 is isomorphic to Γ_2 and written $\Gamma_1 \cong \Gamma_2$.

(c) A homomorphism $\alpha : \mathrm{V}(\Gamma_1) \to \mathrm{V}(\Gamma_2)$ for which Γ_1 and Γ_2 coincide is called an *endomorphism*. If in addition, α is a permutation, then it is an *automorphism*.

In other words, an *automorphism* α of a graph Γ is a permutation of the vertex set with the property that $\alpha(x)$ and $\alpha(y)$ are adjacent if and only if x and y are.

The set of all automorphisms forms a permutation group under composition and is denoted by Aut Γ.

Having discussed the automorphism group of a graph we now consider some actions of automorphisms on the vertex set of graphs under consideration in this study.

A graph Γ is said to be *vertex transitive* if for any two vertices $v_1, v_2 \in V(\Gamma)$, there exists an automorphism $\alpha : V(\Gamma) \to V(\Gamma)$ such that

$$\alpha(v_1) = v_2.$$

As alluded to in Section 2.1, every action of a group on a set decomposes the set into orbits. Hence, a graph Γ is said to be vertex transitive if Aut Γ has a single orbit on the vertex set of Γ.

A graph Γ is said to be *distance transitive* if Aut Γ has a single orbit on each of the sets $\{(v, w)|d(v, w) = k\}$ for $k = 0, 1, 2, \ldots, \mathrm{diam}(\Gamma)$.

In order to fully discuss one of the major results of this study, we consider the idea of graph products and their unique factorizations. We now discuss these concepts. Most of these concepts are found in [21].

2.3.1 Standard products of graphs and their unique factorizations

We begin with the introduction of the graph products which are important to this study.

Let Γ_1 and Γ_2 be graphs. Then a graph product $\Gamma_1 * \Gamma_2$ is any operation for which $V(\Gamma_1 * \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$ and the adjacency of two vertices in $\Gamma_1 * \Gamma_2$ depends only on the adjacencies of the corresponding vertices in the factors.

The four classical products are defined as follows.

(a) The *Cartesian product* of Γ_1 and Γ_2 is a graph denoted as $\Gamma_1 \Box \Gamma_2$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent precisely if $u_1 = u_2$ and $[v_1, v_2] \in E(\Gamma_2)$, or $v_1 = v_2$ and $[u_1, u_2] \in E(\Gamma_1)$.

(b) The *direct product* of Γ_1 and Γ_2 is the graph denoted as $\Gamma_1 \times \Gamma_2$, and for which vertices (u_1, v_1) and (u_2, v_2) are adjacent precisely if $[u_1, u_2] \in E(\Gamma_1)$ and $[v_1, v_2] \in E(\Gamma_2)$.

(c) The *strong product* of Γ_1 and Γ_2 is the graph denoted as $\Gamma_1 \boxtimes \Gamma_2$, and adjacency defined by $E(\Gamma_1 \boxtimes \Gamma_2) = E(\Gamma_1 \Box \Gamma_2) \cup E(\Gamma_1 \times \Gamma_2)$.

(d) The *lexicographic product* of Γ_1 and Γ_2 is the graph denoted as $\Gamma_1 \circ \Gamma_2$, two vertices (u_1, v_1) and (u_2, v_2) being adjacent whenever $[u_1, u_2] \in E(\Gamma_1)$ or $u_1 = u_2$ and $[v_1, v_2] \in E(\Gamma_2)$.

In all of the products above, Γ_1 and Γ_2 are considered as factors of the products.

According to the classification in [21], the cartesian product, direct product and the strong product are called the **fundamental products** of graphs while the classical products above constitute the **standard products** of graphs.

In Chapter 5 of this thesis, we show that the block intersection graphs of our tactical configurations are primes. We now introduce the concept.

The finite graph with one vertex and no edges, that is, a single point, is called the *trivial graph*.

Definition 2.7. A graph is *prime* with respect to a given graph product if it is nontrivial and cannot be represented as the product of two nontrivial graphs.

For instance, the cartesian product of a nontrivial graph Γ is prime if $\Gamma = \Gamma_1 \Box \Gamma_2$ implies that Γ_1 or Γ_2 is K_1.

Having introduced prime graphs, we now turn our attention to the automorphism groups of the prime factors of a graph. This is introduced in order to understand the role of the automorphisms of the factors of a graph on the graph itself.

2.3.2 Automorphism group of a graph with respect to its prime factors

The automorphism group of a graph is sometimes determined by the automorphism groups of its prime factors [21]. Hence the need to recall the following important results on prime factors.

Proposition 2.3. ([21], See Proposition 6.1) *Every nontrivial graph* Γ *has a prime factor decomposition with respect to the Cartesian product. The number of prime factors is at most* $\log_2|V(\Gamma)|$.

In fact, this argument holds for all standard products of graphs [21].

This implies that every nontrivial graph has a prime factorization with respect to any of the standard products, because a graph on n vertices cannot have more than $\log_2 n$ nontrivial factors, and so any factorization into a product of nontrivial graphs with a maximal number of factors is a prime factorization.

Theorem 2.5. (Sabidussi-Vizing, [31], [36]) *Every connected graph has a unique representation as a cartesian product of prime graphs, up to isomorphism and the order of the factors.*

There exists an analogous result for the strong product of graphs following the approach of Dörfler and Imrich (1970). ([21], See Chapter 7)

It is also well known that connected non-bipartite graphs factor uniquely into primes over the direct product. ([21], See Chapter 8).

In the case of the lexicographic product, prime factorization is not unique but there is a strong and predictable connection between different prime factorizations of the same graph. ([21], See Chapter 10).

We now discuss automorphisms of composites as a result of the cartesian product as well as their actions on Aut Γ.

Theorem 2.6. ([21], [Theorem 6.10]) *Suppose φ is an automorphism of a connected graph Γ with prime factor decomposition $\Gamma_1 \Box \Gamma_2 \Box \cdots \Box \Gamma_k$. Then there is a permutation π of $\{1, 2, \ldots, k\}$ and isomorphisms $\varphi_i : \Gamma_{\pi(i)} \to \Gamma_i$ for which $\varphi(x_1, x_2, \ldots, x_k) = \big(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \cdots, \varphi_k(x_{\pi(k)})\big)$. There are two possibilites in this regard:*

(a) *The permutation π is the identity. Then every φ_i is an automorphism of Γ_i. We say φ is generated by automorphisms of the factors Γ_i. If all factors are pairwise nonisomorphic, these automorphisms already generate the full automorphism group of Γ.*

(b) *At least two prime factors Γ_r and Γ_s are isomorphic. Let π be the transposition (r, s), and φ_r, φ_s a pair of isomorphisms from Γ_s onto Γ_r, respectively from Γ_r onto Γ_s. Furthermore, for indices i other than r or s, let φ_i be the identity on $V(\Gamma_i)$. Then the map φ corresponding to $\pi, \varphi_r, \varphi_s$ and the φ_i for $i \neq r, s$ is an automorphism. It is the transposition of two isomorphic prime factors of Γ.*

Theorem 2.7. ([21], [Corollary 6.11]) *The automorphism group of a connected graph with prime factor decomposition $\Gamma_1 \Box \Gamma_2 \Box \cdots \Box \Gamma_k$ is generated by automorphisms and transpositions of the prime factors.*

The following corollary on relatively prime graphs is as a result of the fact that transposition can only occur between isomorphic factors.

Corollary 2.3. ([21], [Corollary 6.12]) *Let* Γ *be the Cartesian product* $\Gamma_1 \Box \Gamma_2 \Box \cdots \Box \Gamma_k$ *of connected, relatively prime graphs. Then every automorphism* φ *of* Γ *preserves the layer structure of* Γ *with respect to the given product decomposition and can be written in the form* $\varphi(x_1, x_2, \ldots, x_k) = (\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_k(x_k))$, *where the* φ_i *are automorphisms of* Γ_i. *In this case,* Aut Γ *is the direct product of the automorphism groups of the factors.*

These results imply a simple theorem, which helps visualize the structure of the automorphism group of a product of prime graphs.

Theorem 2.8. ([21], [Theorem 6.13]) *The automorphism group of the Cartesian product of connected prime graphs is isomorphic to the automorphism group of the disjoint union of the factors.*

In order to discuss the automorphism group of the the strong, direct and the lexicographic products, the concepts of *relations* and *thin* graphs are used to discuss their prime factorisations. we now briefly introduce them.

Given a vertex x of Γ, the S-equivalence class containing x is denoted as

$$[x] = \{x' \in V(\Gamma) | N_\Gamma[x'] = N_\Gamma[x]\}.$$

Vertices x, y of a graph Γ are in the relation S, written xSy, provided that $N[x] = N[y]$. (We write S_Γ if there is a chance of ambiguity.)

The existence of vertices with identical neighborhoods complicates the discussion of prime factorizations over the direct product. To overcome this difficulty, a relation R is introduced on the vertices of a graph.

Given $x \in V(\Gamma)$, the set $[x] = \{x' \in V(\Gamma) | N_\Gamma(x') = N_\Gamma(x)\}$ denote the R-equivalence class containing x.

Two vertices x and x' of a graph Γ are in relation R, written xRx', precisely if $N_\Gamma(x) = N_\Gamma(x')$. (For clarity, we write R_Γ for R.)

To obtain an analogous of Theorem 2.6, the concept of S-prime graphs has been used by many authors (See [21]) to describe the unique prime factorisation of strong products of graphs. We now introduce it.

First, given a graph product $*$, it is natural to ask which graphs are nontrivial subgraphs of $*$-products.

Now, if Γ_1 and Γ_2 are graphs on at least two vertices, then a subgraph Γ of $\Gamma_1 * \Gamma_2$ is called nontrivial if each of the projections, ${}_{p\Gamma_1}(\Gamma)$ and ${}_{p\Gamma_2}(\Gamma)$ has at least two vertices.

A graph Γ is called $*$-S-prime if it cannot be represented as a nontrivial subgraph of a $*$-product graph. Graphs that are not $*$-S-prime are called $*$-S-composite. Clearly, $*$-S-prime graphs are prime with respect to $*$.

Another related concept here is the thinness of a graph.

A graph is called *R-thin* if all of its R-equivalence classes contain just one vertex. A graph is called *S-thin* if no two vertices are in the relation S.

In the case of the lexicographic and direct products, the concepts of *S-thin* and *R-thin* graphs are used in order to obtain an analogous of Theorem 2.6 and to express prime factorisations of the products.

We now present the automorphsm group of the strong product of graphs.

Theorem 2.9. ([21], [Theorem 7.18]) *The automorphism group of the strong product of connected, S-thin prime graphs is isomorphic to the automorphism group of the disjoint union of the factors.*

Analogously, the automorphism group of direct product of graphs is in the following terms.

Theorem 2.10. ([21], [Theorem 8.18]) *Suppose φ is an automorphism of a connected non-bipartite R-thin graph Γ that has a prime factorization $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$. Then there exists a permutation π of $\{1, 2, \ldots, k\}$, together with isomorphisms $\varphi_i : \Gamma_{\pi_{(i)}} \longrightarrow \Gamma_i$, such that $\varphi(x_1, x_2, \ldots, x_k) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)}))$.*

Thus Aut Γ is generated by the automorphisms of the prime factors and transpositions of isomorphic factors. Consequently, Aut Γ is isomorphic to the automorphism group of the disjoint union of the prime factors of Γ in this special case.

Theorem 2.11. ([21], [Theorem 10.13]) *Let $\Gamma_1 \circ \Gamma_2$ be the lexicographic product of simple nontrivial graphs. Then* Aut $(\Gamma_1 \circ \Gamma_2) =$ Aut $\Gamma_1 \circ$ Aut Γ_2 *if and only if Γ_2 is connected in case $R\Gamma_1$ is nontrivial and Γ_2 is connected in case $S\Gamma_1$ is nontrivial.*

In all of the standard graph products described in this study, it is very important to note that the automorphisms of a given graph with respect to any of the graph products are generated by the automorphisms of the prime factors.

Having given the necessary preliminaries, we now turn to the fundamental object of our study.

Chapter 3

Generalized tactical configurations from Steiner triple systems emanating from projective geometry

In this chapter we introduce systematically the fundamental concepts of this study and generalize them through an induction process. At the core, we present designs of interest and establish their intersection arrays as well as their intersection numbers.

Further, we explore the richness in symmetry and substructures of 2-designs from projective geometry and this leads us to the notion of a recursive tactical configurations.

Further tactical configurations obtained from the case $n - m = 2$ of our tactical configurations on Steiner triple systems from projective geometry yields an interesting result. This turns out to be quasi symmetric designs. We will study this separately in Subsection 3.3.1 of this Chapter. In Chapter 4, we will also consider its block intersection graphs. As alluded to, they turn out to be isomorphic to the block intersection graphs of Steiner triple systems from projective geometry.

We now present the fundamental object of this study.

3.1 Tactical configurations

In this section, we are particularly interested in configurations induced by $(n-m)$-flats on $(\mathrm{PG}(n,2),\mathcal{B})$ because we desire to identify substructures that inherit the combinatorial symmetry of the parent structure,$(\mathrm{PG}(n,2),\mathcal{B})$.

The configurations of interest in this study differs from that of ([2], [See Proposition 2.16]) in the sense that, we consider the point set of our configurations to be a family $\mathcal{B}$ of all triples from $(\mathrm{PG}(n,2),\mathcal{B})$ and blocks are defined to be a collection of triples induced by a given flat, $V^{(n-m)}$.

In order to avoid ambiguities, fix notation and more importantly because of our interest in uniform block designs in this study, we begin with the introduction of the following crucial concepts.

Definition 3.1. Let $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ such that $\mathcal{D} = (V, \mathcal{B})$ is a $2-(2^{n+1}-1, 3, 1)$ design. Let $V^{(n-m)}$ be an $(n-m)$-flat of V, $m \in \mathbb{N}$. We define the following:

(i) For any given $v \in V$, by $X(v)$ we mean

$$X(v) = \{B \in \mathcal{B} : v \in B\}.$$

That is, the set of triples of $\mathcal{B}$ containing v;

(ii) By $\mathcal{B}^{(n-m)}$, we mean

$$\mathcal{B}^{(n-m)} = \{B \in \mathcal{B} : B \cap V^{(n-m)} = B\}.$$

That is, the set of triples of $\mathcal{B}$ induced by $V^{(n-m)}$. This is also denoted $\langle V^{(n-m)} \rangle$;

(iii) For a given $v \in V^{(n-m)}$, by $\mathcal{B}_v^{(n-m)}$ we mean

$$\mathcal{B}_v^{(n-m)} = \{B \in \mathcal{B}^{(n-m)} : v \in B\}.$$

That is, a subset of the set of triples of $\mathcal{B}$ induced by $V^{(n-m)}$ containing v.

We now begin to explore the relationships of the sets described above in order to introduce some elementary properties of our configurations.

Proposition 3.1. *Let $m \in \mathbb{N}$ and let $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ such that $\mathcal{D} = (V, \mathcal{B})$ is a $2\text{–}(2^{n+1} - 1, 3, 1)$ design. For all $(n-m)$-flats, $V^{(n-m)}$ of V, let $\overline{\mathcal{B}}$ be the collection of all $\mathcal{B}^{(n-m)}$. Then*

(i) $|\mathcal{B}^{(n-m)}| = \dfrac{(2^{n-m+1} - 1)(2^{n-m} - 1)}{3}$;

(ii) $|\overline{\mathcal{B}}| = \dfrac{\prod_{i=0}^{(n-m)} (2^{n-i+1} - 1)}{\prod_{i=0}^{(n-m)} (2^{i+1} - 1)}$

Proof. (i) $V^{(n-m)}$ is a flat of V.

(ii) By [15](See Equation A.4.3), the number of distinct $(n-m)$-flats of PG$(n, 2)$ is $\dfrac{\prod_{i=0}^{(n-m)} (2^{n-i+1} - 1)}{\prod_{i=0}^{(n-m)} (2^{i+1} - 1)}$. □

Lemma 3.1. *Let $n, m \in \mathbb{N}, n > 2, m \geq 1$ and let $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ such that $\mathcal{D} = (V, \mathcal{B})$ is a $2\text{–}(2^{n+1} - 1, 3, 1)$ design. For any $v \in V$, consider $X(v)$ and let $X \subset X(v)$ such that $|X| = (2^{n-m} - 1)$. Then*

(i) $\left| \bigcup_{B \in X} B \right| = 2^{n-m+1} - 1$;

(ii) $\left(\bigcup_{B \in X} B \right)$ *defines an $(n-m)$-flat, $V^{(n-m)}$ of V and* $|\langle V^{(n-m)} \rangle| = \dfrac{(2^{n-m+1} - 1)(2^{(n-m)} - 1)}{3}$.

Proof. (i) Let $B, B' \in X$. If $B \neq B'$ then, $B \setminus \{v\} \cap B' \setminus \{v\} = \varnothing$, since $v \in B$ for all $B \in X$. The result therefore follows by inclusion-exclusion principle.

(ii) First, we show that $\left(\bigcup_{B\in X} B\right)$ together with $\{\mathbf{0}\} \in \mathbb{F}_2^{n+1}$ is closed under vector addition.

Let $x, y \in \left(\bigcup_{B\in X} B\right)$. By definition of $\mathcal{B}$, and since $X \subset X(v)$, we have that x, y is in exactly one block $B \in \mathcal{B}$. So, let $B = \{x, y, v\} \in X$. It follows that $v \in \left(\bigcup_{B\in X} B\right)$ and that $x + y + v = \mathbf{0}$ and hence; $x + y = v$.

In addition, $\left(\bigcup_{B\in X} B\right)$ is closed under scalar multiplication, since $\mathrm{Char}(\mathbb{F}_2)$ is 2.

Having shown that $\left(\bigcup_{B\in X} B\right)$ together with $\{\mathbf{0}\} \in \mathbb{F}_2^{n+1}$ is closed under vector addition and also closed under scalar multiplication, it follows that $\left(\bigcup_{B\in X} B\right)$ together with $\{\mathbf{0}\} \in \mathbb{F}_2^{n+1}$ defines a subspace of $\mathbb{F}_2^{n+1}$ and hence; $\left(\bigcup_{B\in X} B\right) = V^{(n-m)}$ defines an $(n-m)$-flat of V.

We now show that $|\langle V^{(n-m)}\rangle| = \dfrac{(2^{n-m+1}-1)(2^{(n-m)}-1)}{3}$.

$V^{(n-m)}$ together with $\{\mathbf{0}\} \in \mathbb{F}_2^{n+1}$ is an $(n-m+1)$-dimensional subspace of $\mathbb{F}_2^{n+1}$, since $V^{(n-m)}$ defines an $(n-m)$-flat of V. Hence, we have from (i) that

$$\left|V^{(n-m)}\right| = 2^{n-m+1} - 1.$$

Therefore, the result follows. $\square$

Lemma 3.2. *Let $V^{(n-m)}$ be an $(n-m)$-flat of V and let $\mathcal{B}^{(n-m)}$ be as defined in* Definition 3.1. *Then*

(i) *For every $v \in V^{(n-m)}$,*

$$|\mathcal{B}_v^{(n-m)}| = 2^{n-m} - 1,$$

and v partitions the set $V^{(n-m)} \setminus \{v\}$ into 2-element subsets;

(ii) *For each* $v_i \in V^{(n-m)}, i = 1, \cdots, |V^{(n-m)}|$,

$$\left(\bigcup_{B \in \mathcal{B}_{v_i}^{(n-m)}} B \right) = V^{(n-m)}.$$

Proof. (i) By Corollary 2.2, every $v \in V$ is in exactly $\dfrac{|V|-1}{2}$ triples of $\mathcal{B}$. By hypothesis, $V^{(n-m)}$ is an $(n-m)$-flat of V. It therefore follows that any $v \in V^{(n-m)}$ is also in exactly $\dfrac{2^{n-m+1}-2}{2}$ triples of $\left\langle V^{(n-m)} \right\rangle$, since $\left\langle V^{(n-m)} \right\rangle \subset \mathcal{B}$. Hence,

$$|\mathcal{B}_v^{(n-m)}| = 2^{n-m} - 1.$$

Now, we have that the set $V^{(n-m)} \setminus \{v\}$ is partitioned into 2-element subsets because $|B| = 3$ for any $B \in \mathcal{B}, v \in B$ for any $B \in \mathcal{B}_v^{(n-m)}$ and in addition, $|\mathcal{B}_v^{(n-m)}| = 2^{n-m} - 1$.

(ii) By the arguments of (i) above, we have that any $v_i \in V^{(n-m)}, i = 1, \cdots, |V^{(n-m)}|$, partitions the set $V^{(n-m)} \setminus \{v_i\}$ into 2-element subsets. The result immediately follows, since $\mathcal{B}_{v_i}^{(n-m)} = \{B \in \mathcal{B}^{(n-m)} : v_i \in B\}$. $\square$

Having discussed Lemma 3.1 and Lemma 3.2, we are now in a position to discuss the parameters of our tactical configurations. We will begin with the point degrees of the configurations, that is, the number of times a triple $B \in \mathcal{B}$ appears in the blocks of the tactical configurations.

As alluded to, a tactical configuration is simply a 1-$(v, k, r = \lambda)$ design with parameters v, k, r, and $b = vr/k$. In view of this, we have that $v = |\mathcal{B}|$, that is, the triples of $\mathrm{PG}(n, 2)$, and b is the total number of elements of the collection of all $\left\langle V^{(n-m)} \right\rangle$. The block degrees, k, is the content of Lemma 3.1(ii), while the point degrees, r, constitutes a greater part of the remainder of this section. The set of all blocks in a tactical configuration is denoted as $\overline{\mathcal{B}}$. Hence, we have the tactical configuration $(\mathcal{B}, \overline{\mathcal{B}})$.

We now focus on the point degrees.

In order to establish the degree of each point $B \in \mathcal{B}$, we employ mathematical induction. In the base case of this induction, each block of $\overline{\mathcal{B}}$ contains 7 triples as points. Each of the blocks of $\overline{\mathcal{B}}$ is also well known as a Fano plane.

The point degree of the base case is in the following terms.

Lemma 3.3. *For $n, m \in \mathbb{N}, n > 2, m \geq 1$, let $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ such that $\mathcal{D} = (V, \mathcal{B})$ is a $2-(2^{n+1} - 1, 3, 1)$ design. For all (2)-flats, $V^{(2)}$ of V, let $\overline{\mathcal{B}}$ be the collection of all $\mathcal{B}^{(2)}$. Then, every $B \in \mathcal{B}$ is in exactly $(2^{n-1} - 1)$ blocks of $\overline{\mathcal{B}}$.*

Proof. Let $B = \{v_1, v_2, v_3\}$ be any triple in $\mathcal{B}$. By Corollary 2.2, we have that each of the elements $v_1, v_2, v_3 \in V$ is in exactly $\dfrac{|V| - 1}{2} = 2^n - 1$ triples of $\mathcal{B}$. So, consider the sets $X(v_1), X(v_2)$ and $X(v_3)$. Clearly,

$$X(v_1) \cap X(v_2) \cap X(v_3) = B.$$

By Lemma 3.1 (i), we have that the number of points of an $(n - m)$-flat is $(2^{n-m+1} - 1)$. Hence for any 2-flat, $|V^{(2)}| = 7$ and by Lemma 3.1 (ii), $|\mathcal{B}^{(2)}| = 7$.

Now, let $B \in \mathcal{B}^{(2)}$ for some 2-flats of V. By Lemma 3.2, there exist $\mathcal{B}^{(2)}_{v_1}, \mathcal{B}^{(2)}_{v_2}, \mathcal{B}^{(2)}_{v_3} \subset \mathcal{B}^{(2)}$ such that

$$|\mathcal{B}^{(2)}_{v_1}| = |\mathcal{B}^{(2)}_{v_2}| = |\mathcal{B}^{(2)}_{v_3}| = 2^2 - 1$$

and $V^{(2)} = \left(\bigcup_{B \in \mathcal{B}^{(2)}_{v_i}} B \right)$, for each $i \in \{1, 2, 3\}$. In addition, $\mathcal{B}^{(2)}_{v_1} \cap \mathcal{B}^{(2)}_{v_2} \cap \mathcal{B}^{(2)}_{v_3} = B$.

Hence, it follows that any $\mathcal{B}^{(2)}$ containing B can be identified as $\left(\bigcup_{i=1,2,3} \mathcal{B}^{(2)}_{v_i} \right)$ such that $\mathcal{B}^{(2)}_{v_1} \cap \mathcal{B}^{(2)}_{v_2} \cap \mathcal{B}^{(2)}_{v_3} = B$.

Clearly, $\mathcal{B}^{(2)}_{v_1} \subset X(v_1), \mathcal{B}^{(2)}_{v_2} \subset X(v_2)$ and $\mathcal{B}^{(2)}_{v_3} \subset X(v_3)$. In addition, $X(v_1) \cap X(v_2) \cap X(v_3) = \mathcal{B}^{(2)}_{v_1} \cap \mathcal{B}^{(2)}_{v_2} \cap \mathcal{B}^{(2)}_{v_3} = B$, and

$$|\mathcal{B}^{(2)}_{v_1}| = |\mathcal{B}^{(2)}_{v_2}| = |\mathcal{B}^{(2)}_{v_3}| = 3.$$

Now, consider partitions of the $(2^n - 2)$ elements of $X(v_i) \setminus B$, for any $i \in \{1, 2, 3\}$ into $\dfrac{2^n - 2}{2}$ subsets, each of size 2.

Each of the partitions together with B defines a subset $\mathcal{B}^{(2)}$ of $\mathcal{B}$. Hence, the result. $\square$

In order to facilitate the discussion of the induction process, there is need to properly describe k-element subsets of $X(v)$ that define a flat of V. We now put this in the right context.

Definition 3.2. *Let $X(v) = \{B \in \mathcal{B} : v \in B\}$ and let X be a k-element subset of $X(v)$ such that $\left(\bigcup_{B\in X} B\right)$ defines a $(k-1)$-flat of V. By $X^k(v)$, we mean a collection of all such X.*

In view of the above definition and Lemma 3.3, we have the following.

Corollary 3.1. *For $n \geq 3$, and for a given $v \in V$, consider the set $X^3(v)$. Then*

(i) *for any $B \in X(v)$, every $B' \in X(v) \setminus B$ is in exactly one $X \in X^3(v)$;*

(ii) *every $B \in X(v)$ is in exactly $(2^{n-1} - 1)$ elements of $X^3(v)$.*

Proof. (i) For $X_i, X_j \in X^3(v), i \neq j, X_i \cap X_j = B$. Now, consider a partition of the $2^n - 2$ elements of $X(v) \setminus B$ into $\dfrac{2^n - 2}{2}$ subsets, each of size 2. The result therefore follows.

(ii) By Corollary 2.2, $|X(v)| = 2^n - 1$. Now, considering the partitions in (i) above, each of the partitions together with B defines an $X \in X^3(v)$. □

As a result of Corollary 3.1 and in order to establish a correspondence between the elements, $\mathcal{B}^{(2)}$ of $\overline{\mathcal{B}}$ and the elements, X of X^3_v, we have the following.

Lemma 3.4. *Let $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ such that $(V, \mathcal{B})$ is a 2–$(2^{n+1} - 1, 3, 1)$ design. For all (2)-flats, $V^{(2)}$ of V, let $\overline{\mathcal{B}}$ be a collection of all $\mathcal{B}^{(2)}$. Then, for any $v \in V$, each $X_i \in X^3_v$ determines a unique $\mathcal{B}^{(2)} \in \overline{\mathcal{B}}$.*

Proof. Let B_1, B_2, B_3 be distinct triples in X. Then, $B_1 \cap B_2 \cap B_3 = \{v\}$. Now consider $\bigcup_{B\in X} B$. By definition, $\bigcup_{B\in X} B$ defines a 2-flat of V and hence,

$$\left\langle \bigcup_{B\in X} B \right\rangle = \mathcal{B}^{(2)}.$$

By Lemma 3.1, we have that $\left|\bigcup_{B\in X} B\right| = 7$ and $\left|\left\langle \bigcup_{B\in X} B \right\rangle\right| = 7$.

We now consider the uniqueness of $\mathcal{B}^{(2)}$.

Let $B_1 = \{v_1, v_2, v_3\}, B_2 = \{v_1, v_4, v_5\}$, and $B_3 = \{v_1, v_6, v_7\}$. By definition, we have that

$$v_2 + v_3 = v_4 + v_5 = v_6 + v_7 = v_1.$$

Adding v_2 and v_3 to all sides, we have

$$0 = v_4 + v_5 + v_2 + v_3 = v_6 + v_7 + v_2 + v_3 = v_1 + v_2 + v_3.$$

Hence,

$$0 = v_4 + v_5 + v_2 + v_3 = v_6 + v_7 + v_2 + v_3 = 0, \tag{3.1}$$

since $B_1 = \{v_1, v_2, v_3\}$.

Eliminating v_3 from Equation (3.1), we obtain

$$0 = v_2 + v_4 + v_5 = v_2 + v_6 + v_7 = 0. \tag{3.2}$$

Eliminating v_2 from Equation (3.1), we also have

$$0 = v_3 + v_4 + v_5 = v_3 + v_6 + v_7 = 0. \tag{3.3}$$

Clearly, $v_4, v_5 \in B_2$ and $v_6, v_7 \in B_3$. Hence, it follows without loss of generality that Equations (3.2) and (3.3) can be re-arranged respectively as follows.

$$0 = v_2 + v_4 + v_6 = v_2 + v_5 + v_7 = 0;$$
$$0 = v_3 + v_4 + v_7 = v_3 + v_5 + v_6 = 0.$$

Hence, the triples $\{v_2, v_4, v_6\}, \{v_2, v_5, v_7\}, \{v_3, v_4, v_7\}$ and $\{v_3, v_5, v_6\}$ are identified as $B_4, \cdots, B_7$ respectively.

We now show that B_1, B_2, B_3 cannot be extended to a $\mathcal{B}^{(2)}$ distinct from that containing $B_4, \cdots, B_7$.

Now, suppose to the contrary that Equations (3.2) and (3.3) can be rearranged respectively as

$$0 = v_2 + v_4 + v_7 = v_2 + v_5 + v_6 = 0;$$
$$0 = v_3 + v_4 + v_6 = v_3 + v_5 + v_7 = 0.$$

Then, it follows that there exist

$$\{v_2, v_4, v_7\}, \{v_2, v_5, v_6\}, \{v_3, v_4, v_6\}, \{v_3, v_5, v_7\} \in \mathcal{B}$$

identified as $B_8, \cdots, B_{11}$ respectively such that

$$B_1 \cup B_2 \cup B_3 \bigcup_{i=8}^{11} B_i = \{v_1, \cdots, v_7\}.$$

Now consider $B_4 \cap B_8 = \{v_2, v_4\}, B_5 \cap B_9 = \{v_2, v_5\}, B_6 \cap B_{10} = \{v_3, v_4\}$ and $B_7 \cap B_{11} = \{v_3, v_5\}$. This contradicts the fact that every pair of elements of V is in exactly one triple in $\mathcal{B}$. $\square$

Consequently, 3 distinct triples define a unique $\mathcal{B}^{(2)}$ in the following.

Corollary 3.2. *Let* $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$. *Let* $V^{(2)}$ *be a 2-flat of* $V, \mathcal{B}^{(2)} = \langle V^{(2)} \rangle = \{B \in \mathcal{B} : B \cap V^{(2)} = B\}$, *and let* $\overline{\mathcal{B}}$ *be a collection of all* $\mathcal{B}^{(2)}$. *Then, any 3 triples* $\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3 \in \mathcal{B}$, *such that* $\mathrm{B}_1 \cap \mathrm{B}_2 \cap \mathrm{B}_3 \neq \varnothing$ *is in exactly one unique* $\mathcal{B}^{(2)} \in \overline{\mathcal{B}}$.

Having considered the base case, we now generalize point degrees of our tactical configurations by induction. Again as alluded to, in this general case, points of our tactical configurations just like in the base case are triples, while blocks are also defined to be sets of triples generated by flats.

We now extend the results of Lemma 3.3, Corollary 3.1, Lemma 3.4, Corollary 3.2, and Definition 3.2 to the point degrees of our configurations.

Theorem 3.1. *Let* $n, m \in \mathbb{N}$ *and* $(n-m) \in \{2, 3, \ldots, (n-2), (n-1)\}$. *Let* $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ *such that* $\mathcal{D} = (V, \mathcal{B})$ *is a* $2\text{–}(2^{n+1} - 1, 3, 1)$ *design. For all* $(n-m)$*-flats,* $V^{(n-m)}$ *of* V, *let* $\overline{\mathcal{B}}$ *be the collection of all* $\mathcal{B}^{(n-m)}$. *Then every* $B \in \mathcal{B}$ *is in exactly*

$$\frac{\prod_{i=1}^{(n-m-1)} (2^{n-i} - 1)}{\prod_{i=0}^{(n-m-2)} (2^{i+1} - 1)}$$

blocks of $\overline{\mathcal{B}}$.

Proof. We will prove by induction on $(n-m)$.

When $(n-m)=2$, by Lemma 3.3, the result holds.

Now, suppose the result is true for $(n-m)=k\geq 2$. That is, for $n\geq k+1$ and for any $v\in V$, every $B\in X(v)$ is in exactly

$$\frac{(2^{n-1}-1)(2^{n-2}-1)\cdots(2^{n-k+1}-1)}{1\times 3\times 7\times\cdots\times(2^{k-1}-1)},$$

(2^k-1)-element subsets, $X^{(2^k-1)}(v)$ of $X(v)$ such that $\left(\bigcup_{B\in X^{(2^k-1)}(v)} B\right)$ defines a k-flat. We need to show that it is also true for $(n-m)=k+1$.

Let $B=\{v_1,v_2,v_3\}\in\mathcal{B}$ such that $B\in\mathcal{B}^{(k+1)}$, for some $(k+1)$-flat of V. By Lemma 3.2, there exist $\mathcal{B}_{v_1}^{(k+1)},\mathcal{B}_{v_2}^{(k+1)},\mathcal{B}_{v_3}^{(k+1)}\subset\mathcal{B}^{(k+1)}$ such that $|\mathcal{B}_{v_1}^{(k+1)}|=|\mathcal{B}_{v_2}^{(k+1)}|=|\mathcal{B}_{v_3}^{(k+1)}|=2^{k+1}-1$ and $V^{(k+1)}=\left(\bigcup_{B\in\mathcal{B}_{v_i}^{(k+1)}} B\right)$, for any $i\in\{1,2,3\}$. In addition, $\mathcal{B}_{v_1}^{(k+1)}\cap\mathcal{B}_{v_2}^{(k+1)}\cap\mathcal{B}_{v_3}^{(k+1)}=B$.

Now, let $V^{(k)}=\{v_1,v_2,\cdots,v_{2^{k+1}-1}\}$ be a (k)-flat of V in $V^{(k+1)}$. It follows that $\mathcal{B}_{v_1}^{(k+1)},\mathcal{B}_{v_2}^{(k+1)},\cdots,\mathcal{B}_{v_{2^{k+1}-1}}^{(k+1)}\subset\mathcal{B}^{(k+1)}$ such that

$$|\mathcal{B}_{v_1}^{(k+1)}|=|\mathcal{B}_{v_2}^{(k+1)}|=\cdots=|\mathcal{B}_{2^{k+1}-1}^{(k+1)}|=2^{k+1}-1,$$

since the result is true for $(n-m)=k$.

In addition, $V^{(k+1)}=\left(\bigcup_{B\in\mathcal{B}_{v_i}^{(k+1)}} B\right)$, for any $i\in\{1,2,\cdots,2^{k+1}-1\}$ and

$$\mathcal{B}_{v_1}^{(k+1)}\cap\mathcal{B}_{v_2}^{(k+1)}\cap\cdots\cap\mathcal{B}_{2^{k+1}-1}^{(k+1)}=\left\langle V^{(k)}\right\rangle=\mathcal{B}^{(k)}.$$

In other words, any $\mathcal{B}^{(k+1)}\in\overline{\mathcal{B}}$ containing B can be identified as $\left(\bigcup_{i=1,2,\cdots,2^{k+1}-1}\mathcal{B}_{v_i}^{(k+1)}\right)$ such that

$$\mathcal{B}_{v_1}^{(k+1)}\cap\mathcal{B}_{v_2}^{(k+1)}\cap\cdots\cap\mathcal{B}_{2^{k+1}-1}^{(k+1)}=\mathcal{B}^{(k)}=\left\langle V^{(k)}\right\rangle.$$

Now, consider an incidence structure on the $(2^n - 2)$ elements of $X(v_i) \setminus B$, for any $i \in \{1, 2, 3\}$ into subsets of size, $|\mathcal{B}_{v_i}^{(k+1)} \setminus B| = 2^{k+1} - 2$, such that each $B' \in X(v_i) \setminus B$ is in exactly r' blocks and the total number of blocks from such an incidence structure is b', since $B = \{v_1, v_2, v_3\} \in \mathcal{B}^{(k+1)}$, for some $\mathcal{B}^{(k+1)} \in \overline{\mathcal{B}}$.

By Theorem 2.3, we have that $(2^n - 2)r' = b'(2^{k+1} - 2)$.

Furthermore, for each $\mathcal{B}_{v_i}^{(k+1)} \setminus B$, let $\mathcal{B}_{v_i}^{(k)}, \mathcal{B}'^{(k)}_{v_i}$ be partitions of $\mathcal{B}_{v_i}^{(k+1)} \setminus B$ into $(2^k - 1)$-element subsets, each of size $2^k - 1$.

Without loss of generality, and by Lemma 3.1, we have that

$$\left(\bigcup_{B \in \mathcal{B}_{v_i}^{(k)}} B \right) = V^{(k)} \text{ and thus } \left\langle V^{(k)} \right\rangle = \mathcal{B}^{(k)}.$$

Hence, we have that for each $\mathcal{B}_{v_i}^{(k+1)} \setminus B = \left(\bigcup_{B \in \left(\mathcal{B}_{v_i}^{(k)} \cup \mathcal{B}'^{(k)}_{v_i}\right)} B \right)$,

$$|\mathcal{B}_{v_i}^{(k)}| = |\mathcal{B}'^{(k)}_{v_i}| = 2^k - 1, \mathcal{B}_{v_i}^{(k)} \cap \mathcal{B}'^{(k)}_{v_i} = \emptyset$$

and $\mathcal{B}_{v_i}^{(k)}, \mathcal{B}'^{(k)}_{v_i} \subset \mathcal{B}_{v_i}^{(k+1)}$, since $|\mathcal{B}_{v_i}^{(k+1)} \setminus B| = 2^{k+1} - 2$.

Therefore, $|\mathcal{B}_{v_i}^{(k+1)} \setminus B| = 2|\mathcal{B}_{v_i}^{(k)}|$ or $2|\mathcal{B}'^{(k)}_{v_i}|$ and hence

$$(2^n - 2)r' = b'[2(2^k - 1)].$$

Now, consider a collection of all $\mathcal{B}_{v_i}^{(k)}$ for each $\mathcal{B}_{v_i}^{(k+1)}$. It follows that the collection of all $\mathcal{B}_{v_i}^{(k)}$ partitions the set $X(v_i) \setminus B$ into $(2^{n-1} - 1)$ element subsets, each of size $(2^{n-1} - 1)$, since the elements of $\mathcal{B}_{v_i}^{(k)}$ and $\mathcal{B}'^{(k)}_{v_i}$ are also elements of $X(v_i) \setminus B$. Hence, we have that

$$(2^{n-1} - 1)r' = b'|\mathcal{B}_{v_i}^{(k)}| \text{ or } (2^{n-1} - 1)r' = b'|\mathcal{B}'^{(k)}_{v_i}|.$$

In either case, we have $(2^{n-1} - 1)r' = b'(2^k - 1)$.

By induction hypothesis, we have that $n - m = k + 1$ and this implies that $n \geq (k + 2)$. In addition, the result holds for $n - m = k$.

Now, consider (2^k-1)-element subsets, $\mathcal{B}_{v_i}^{(k)}$ or $\mathcal{B'}_{v_i}^{(k)}$ from $(2^{n-1}-1)$ elements of $X(v_i) \setminus B$, for any $i \in \{1, 2, 3\}$ such that $\bigcup\limits_{B' \in \mathcal{B}_{v_i}^{(k)}} B'$ or $\bigcup\limits_{B' \in \mathcal{B'}_{v_i}^{(k)}} B'$ defines a k-flat.

We therefore have that

$$r' = \frac{(2^{n-2}-1)(2^{n-3}-1)\cdots(2^{n-k}-1)}{1 \times 3 \times 7 \times \cdots \times (2^{k-1}-1)}$$

and hence,

$$b' = \frac{(2^{n-1}-1)(2^{n-2}-1)\cdots(2^{n-k}-1)}{1 \times 3 \times 7 \times \cdots \times (2^{k-1}-1)(2^k-1)}.$$

Each of the $(2^{k+1}-2)$-element subsets together with B defines $\mathcal{B}_{v_i}^{(k+1)}$. This implies that $V^{(k+1)} = \left(\bigcup\limits_{B \in \mathcal{B}_{v_i}^{(k+1)}} B \right)$, for any $i \in \{1, 2, 3\}$.

By defiinition, each $V^{(k+1)}$ defines a $\mathcal{B}^{(k+1)} \subset \mathcal{B}$ and hence, the result.

□

Up to this point, we have only discussed the points set and its total number of points, the points degree, and the block size of our tactical configurations.

The case of the total number of blocks has been discussed in [15] (see Equation A.4.3) already. We only summarize in the following, since the blocks of our tactical configurations are generated by flats.

Lemma 3.5. [15] *The number of distinct* $(n-m)$*-flats of* $\mathrm{PG}(n, 2)$ *is*

$$\frac{\prod\limits_{i=0}^{(n-m)} (2^{n-i+1}-1)}{\prod\limits_{i=0}^{(n-m)} (2^{i+1}-1)}.$$

In view of Lemma 3.5, we have the following.

Corollary 3.3. *For all* $(n-m)$*-flats,* $V^{(n-m)}$ *of* V *such that* $\overline{\mathcal{B}}$ *is the collection of all* $\mathcal{B}^{(n-m)}$,

$$|\overline{\mathcal{B}}| = \frac{\prod_{i=0}^{(n-m)} (2^{n-i+1}-1)}{\prod_{i=0}^{(n-m)} (2^{i+1}-1)}.$$

The stage is now set for the introduction of the generalized tactical configurations of this study.

By the proof of Lemma 2.2, we have that $|\mathcal{B}| = \dfrac{(2^{n+1}-1)(2^{n+1}-2)}{6}$ for a $2-(2^{n+1}-1,3,1)$ design, $\mathcal{D} = (V, \mathcal{B})$. Hence, the size of the points set of our generalized tactical configuration is $\dfrac{(2^{n+1}-1)(2^n-1)}{3}$. Again by Lemma 3.1 (ii), we also have that $\left(\bigcup_{B \in X} B\right)$ defines an $(n-m)$-flat, $V^{(n-m)}$ of V and $|\left\langle V^{(n-m)}\right\rangle| = \dfrac{(2^{n-m+1}-1)(2^{(n-m)}-1)}{3}$. Hence the size of a block of our generalized tactical configuration is $\dfrac{(2^{n-m+1}-1)(2^{(n-m)}-1)}{3}$.

In view of the above together with Corollary 3.3, Theorem 2.3 and Theorem 3.1, we have the following.

Theorem 3.2. *Let* $n, m \in \mathbb{N}, (n-m) \in \{2, 3, \ldots, (n-2), (n-1)\}$*, and let* $\mathcal{D} = (V, \mathcal{B})$ *be the design* $(\mathrm{PG}(n,2), \mathcal{B})$*. For all* $(n-m)$*-flats,* $V^{(n-m)}$ *of* V*, let* $\overline{\mathcal{B}}$ *be the collection of all* $\mathcal{B}^{(n-m)}$*. Then,* $\left(\mathcal{B}, \overline{\mathcal{B}}\right)$ *is a*

$$1-\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, \frac{(2^{n-m+1}-1)(2^{n-m}-1)}{3}, \frac{\prod_{i=1}^{(n-m-1)} (2^{n-i}-1)}{\prod_{i=0}^{(n-m-2)} (2^{i+1}-1)}\right)$$

design with the number of blocks equal to

$$\frac{\prod_{i=0}^{(n-m)} (2^{n-i+1} - 1)}{\prod_{i=0}^{(n-m)} (2^{i+1} - 1)}.$$

Having discussed the parameters of our generalized tactical configurations, we now turn to block intersections and the numbers of triples in any intersection.

3.2 Block intersections and intersection numbers

In order to discuss block intersections and their intersection numbers, we employ some elementary linear algebra techniques in exploring the blocks of our configurations.

Due to the nature of blocks in the generalized tactical configurations discussed in the previous section, there are arrays of intersections between any two distinct blocks. We now discuss these intersection arrays beginning with the dimensions of the intersecting subflats.

Lemma 3.6. *The dimensions of the intersections of all two distinct* $(n-m)$*-flats of* $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}$ *are elements of* $\{(n-m-1), (n-m-2), \cdots, 0\}$.

Proof. To prove the result, we first observe the well-known dimensionality formular, $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$ for subspaces U_1 and U_2 of a vector space V.

Now, for all 2 distinct $(n-m)$-flats, V_1^{n-m}, V_2^{n-m} of V, we have that

$$\dim(V_1^{n-m} + V_2^{n-m}) = \dim(V_1^{n-m}) + \dim(V_2^{n-m}) - \dim(V_1^{n-m} \cap V_2^{n-m}) \tag{3.4}$$

We have essentially from Equation (3.4) that

(i) $\dim(V_1^{n-m} + V_2^{n-m}) \leq \dim(V_1^{n-m}) + \dim(V_2^{n-m})$ with equality when $V_1^{n-m} \cap V_2^{n-m} = \varnothing$;

(ii) $\dim(V_1^{n-m} + V_2^{n-m}) + \dim(V_1^{n-m} \cap V_2^{n-m}) - \dim(V_2^{n-m}) = \dim(V_1^{n-m})$.

This is by virtue of V_1^{n-m} and V_2^{n-m} being distinct $(n-m)$-flats of V.

From (ii) above, if $(V_1^{n-m} \cap V_2^{n-m}) \neq \varnothing$, we have that

$$\dim(V_1^{n-m} \cap V_2^{n-m}) < \dim(V_2^{n-m})$$

hence,

$$\dim(V_1^{n-m} \cap V_2^{n-m}) - \dim(V_2^{n-m}) < 0.$$

It therefore follows that

$$\dim(V_1^{n-m} + V_2^{n-m}) > \dim(V_1^{n-m}) = \dim(V_2^{n-m}).$$

This implies $\dim(V_1^{n-m} + V_2^{n-m}) \geq (n - m + 1)$.

Again, from (ii) above, consider the case $(V_1^{n-m} \cap V_2^{n-m}) = \varnothing$.

Clearly,

$$\dim(V_1^{n-m} + V_2^{n-m}) > \dim(V_1^{n-m}) = \dim(V_2^{n-m}).$$

This also implies that $\dim(V_1^{n-m} + V_2^{n-m}) \geq (n - m + 1)$.

It therefore follows that

$$\dim(V_2^{n-m}) < \dim(V_1^{n-m} + V_2^{n-m}) \leq \dim(V_1^{n-m}) + \dim(V_2^{n-m}).$$

This implies that

$$\dim(V_1^{n-m} + V_2^{n-m}) \in \{(n-m+1), (n-m+2), \cdots, 2(n-m)\}$$

since $\dim(V_1^{n-m}) = \dim(V_2^{n-m}) = (n - m)$.

Finally, Equation (3.4) can be re-arranged as follows:

$$\dim(V_1^{n-m} \cap V_2^{n-m}) = \dim(V_1^{n-m}) + \dim(V_2^{n-m}) - \dim(V_1^{n-m} + V_2^{n-m}).$$

Hence,

$$\begin{aligned}\dim(V_1^{n-m} \cap V_2^{n-m}) \in \{&2(n-m) - (n-m+1), 2(n-m) - (n-m+2), \cdots, \\ &2(n-m) - 2(n-m)\},\end{aligned}$$

since $\dim(V_1^{n-m} + V_2^{n-m}) = \{(n-m+1), (n-m+2), \cdots, 2(n-m)\}$. $\square$

As for the number of triples in the intersection of any two flats, that is, the intersection numbers of our configurations, we have the following.

Lemma 3.7. *Let $n, m \in \mathbb{N}, n > 2, m \geq 1$ and let $\mathcal{D} = (V, \mathcal{B})$ be the design $(\mathrm{PG}(n,2), \mathcal{B})$. For all $(n-m)$-flats, $V^{(n-m)}$ of V, let $\overline{\mathcal{B}}$ be the collection of all $\mathcal{B}^{(n-m)}$. Then, the intersection numbers of any 2 distinct blocks of $\overline{\mathcal{B}}$ are elements of*

$$\left\{\frac{(2^{n-m}-1)(2^{n-m-1}-1)}{3}, \frac{(2^{n-m-1}-1)(2^{n-m-2}-1)}{3}, \cdots, 1\right\}.$$

Proof. Let V_1^{n-m} and V_2^{n-m} be distinct $(n-m)$-flats of $V, \mathcal{B}^{(n-m)} = \left\langle V_1^{(n-m)} \right\rangle$, and $\mathcal{B}'^{(n-m)} = \left\langle V_2^{(n-m)} \right\rangle$. By Lemma 3.6, we have that $\dim(V_1^{n-m} \cap V_2^{n-m}) \in \{(n-m-1), (n-m-2), \cdots, 0\}$. Hence we have that

$$|(V_1^{n-m} \cap V_2^{n-m})| \in \{2^{(n-m)} - 1, 2^{(n-m-1)} - 1, \cdots, 2^{0+1} - 1\}$$

since, $(V_1^{n-m} \cap V_2^{n-m})$ is also a flat of $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}$. The result therefore follows since,

$$\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \left\langle (V_1^{n-m} \cap V_2^{n-m}) \right\rangle = \{B \in \mathcal{B} : B \cap (V_1^{n-m} \cap V_2^{n-m}) = B\}.$$

$\square$

The case for $n-m = 2$ is a special one. We obtain a configuration not with many intersections. As a result of Lemma 3.7, we have that the intersection of any two blocks when $n - m = 2$ is an element of the set $\{0, 1\}$. Thus, as a result of Lemma 3.7 and Theorem 3.2, we obtain the following.

Corollary 3.4. *Let $V^{(2)}$ be a 2-flat of $V, \mathcal{B}^{(2)} = \left\langle V^{(2)} \right\rangle = \{B \in \mathcal{B} : B \cap V^{(2)} = B\}$, and let $\overline{\mathcal{B}}$ be a collection of all $\mathcal{B}^{(2)}$. Then $\left(\mathcal{B}, \overline{\mathcal{B}}\right)$ is a quasi-symmetric design with the standard parameters,*

$$\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, \frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1 \times 3 \times 7}, (2^{n-1}-1), 7, (2^{n-1}-1);\right.$$

$$\left. 0, 1\right).$$

As alluded to in the introduction of this chapter, the design in Corollary 3.4 is of special attention in this study. In the next section, we shall consider a further tactical configuration on the design. In the interim, we consider the following further property of the design, since the intersection of any two blocks is an element of the set $\{0, 1\}$. This is required later in Chapter 4 to discuss isomorphism between the block intersection graphs from further tactical configurations of the design and the block intersection graph of Steiner triple systems from projective geometry.

Corollary 3.5. *Let $V^{(2)}$ be a 2-flat of $V, \mathcal{B}^{(2)} = \left\langle V^{(2)} \right\rangle = \{B \in \mathcal{B} : B \cap V^{(2)} = B\}$, and let $\overline{\mathcal{B}}$ be a collection of all $\mathcal{B}^{(2)}$. Then for any $\mathrm{B}_1, \mathrm{B}_2 \in \mathcal{B}$ such that $\mathrm{B}_1 \cap \mathrm{B}_2 \neq \varnothing$. The pair $(\mathrm{B}_1, \mathrm{B}_2)$ exists in a unique $\mathcal{B}^{(2)} \in \overline{\mathcal{B}}$.*

Proof. By Lemma 3.3, every $B \in \mathcal{B}$ is in exactly $(2^{n-1} - 1)$ blocks of $\overline{\mathcal{B}}$. In addition by Lemma 3.7, for all $\mathcal{B}^{(2)}, \mathcal{B}'^{(2)} \in \bar{\mathcal{B}}$, $|\mathcal{B}^{(2)} \cap \mathcal{B}'^{(2)}| = \{0, 1\}$, since $n - m = 2$. Hence for $\mathcal{B}^{(2)}, \mathcal{B}'^{(2)} \in \bar{\mathcal{B}}$ such that $\mathcal{B}^{(2)} \cap \mathcal{B}'^{(2)} \neq \varnothing$, $\mathcal{B}^{(2)} \cap \mathcal{B}'^{(2)} = B \in \mathcal{B}$.

Now, consider a partition of the $(2^{n-1} - 2)$ other triples containing B into $\dfrac{(2^{n-1} - 2)}{2}$ subsets, each of size 2. By Lemma 3.4, we have that B together with each partition defines a unique $\mathcal{B}^{(2)} \in \overline{\mathcal{B}}$.

Hence, the result follows. $\square$

Having considered intersections between any two blocks, we now count the exact number of intersections of all 2 distinct blocks in $\overline{\mathcal{B}}$.

Theorem 3.3. *Let $n, m \in \mathbb{N}, (n - m) \in \{2, 3, \ldots, (n - 2), (n - 1)\}$, and let $\mathcal{D} = (V, \mathcal{B})$ be the design $(\mathrm{PG}(n, 2), \mathcal{B})$. For all $(n - m)$-flats, $V^{(n-m)}$ of V, let $\overline{\mathcal{B}}$ be the collection of all $\mathcal{B}^{(n-m)}$. Then, there are exactly $(n - m)$ possible intersections between the blocks of $\overline{\mathcal{B}}$.*

Proof. LetV_1^{n-m} and V_2^{n-m} be distinct $(n-m)$-flats of $V, \mathcal{B}^{(n-m)} = \left\langle V_1^{(n-m)} \right\rangle$, and $\mathcal{B}'^{(n-m)} = \left\langle V_2^{(n-m)} \right\rangle$. By Lemma 3.6, we have that the dimensions of the intersection array of all distinct $(n - m)$-flats of $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}$ is $\{(n - m - 1), (n - m - 2), \cdots, 0\}$.

Now, consider the following, since $n = n - m + m$.

(i) if $n - m > m$ then, $n = 2m + k, k > 0$.

This implies $n - m = m + k$ and hence,

$$\begin{aligned}
&\{(n-m-1),(n-m-2),\cdots,0\}\\
&=\{(2m+k-m-1),(2m+k-m-2),\cdots,0\}\\
&=\{(m+k-1),(m+k-2),(m+k-3),\cdots,((m+k)-(m+k))\}\\
&=\{m+k-i\}, i=1,\cdots,m+k=n-m;
\end{aligned}$$

(ii) if $n - m < m$ then, $n = 2m - k, k > 0$.

This also implies $(n - m) = (m - k)$. Hence,

$$\begin{aligned}
&\{(n-m-1),(n-m-2),\cdots,0\}\\
&=\{(2m-k-m-1),(2m-k-m-2),\cdots,0\}\\
&=\{(m-k-1),(m-k-2),(m-k-3),\cdots,((m-k)-(m-k))\}\\
&=\{m-k-i\}, i=1,\cdots,(m-k)=(n-m).
\end{aligned}$$

(iii) if $n - m = m$ then, $n = 2m$. Hence,

$$\begin{aligned}
&\{(n-m-1),(n-m-2),\cdots,0\}\\
&=\{(2m-m-1),(2m-m-2),\cdots,0\}\\
&=\{(m-1),(m-2),(m-3),\cdots,(m-m)\}\\
&=\{m-i\}, i=1,\cdots,m=(n-m).
\end{aligned}$$

□

The exact intersection numbers of the configurations for any given n and m can then be obtained by Lemma 3.7, another important straddle of our study.

As alluded to, we now discuss further tactical configurations from Corollary 3.4.

3.3 Further tactical configurations

We present an example of a further tactical configuration from the original tactical configurations, $(\mathcal{B}, \overline{\mathcal{B}})$. We pay specific attention to the case $n - m =$

2 from the original tactical configurations. In Section 4.4 of the next chapter, we shall consider block intersection graphs of the given example as well as establish that the block intersection graphs of the given example is isomorphic to the block intersection graphs of Steiner triple systems from projective geometry.

As is obvious now, the configurations of Theorem 3.2, that is, the design $(\mathcal{B}, \overline{\mathcal{B}})$ has as its points set, the set $\mathcal{B}$ of triples from $\mathrm{PG}(n, 2)$, while blocks are the set of triples induced by an $(n-m)$-flat of $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}$.

Further, we extend the previous consideration in the sense that our points are now the blocks of the previous tactical configuration, that is, the set $\overline{\mathcal{B}}$, and a block is defined to be a collection of all the blocks in the previous tactical configuration with a common intersection.

This process over the original tactical configurations, $(\mathcal{B}, \overline{\mathcal{B}})$ could be recursive but that is not our focus here.

We now introduce the new tactical configurations on $(\mathcal{B}, \overline{\mathcal{B}})$ specifically when $n - m = 2$.

3.3.1 $1-\left(\dfrac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7}, (2^{n-1}-1), 7\right)$ design

Let $(\mathcal{B}, \overline{\mathcal{B}})$ be the design of Theorem 3.2, particularly when $n - m = 2$. We have that $(\mathcal{B}, \overline{\mathcal{B}})$ is a 1-$\left(\dfrac{(2^{n+1}-1)(2^n-1)}{3}, 7, 2^{n-1}-1\right)$ design with $\dfrac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7}$ blocks.

Now, let $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ such that $\mathcal{D} = (V, \mathcal{B})$ is a $2-(2^{n+1}-1, 3, 1)$ design. Let $V^{(2)}$ be a 2-flat of $V, \mathcal{B}^{(2)} = \langle V^{(2)} \rangle$, and let $\overline{\mathcal{B}}$ be a collection of all $\mathcal{B}^{(2)}$.

As alluded to, each $V^{(2)}$ is a Fano plane and hence, the total number of Fano planes is $\dfrac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7}$.

By Lemma 3.3, every $B \in \mathcal{B}$ is in exactly $(2^{n-1}-1)$ blocks of $\overline{\mathcal{B}}$.

Now, for each $B \in \mathcal{B}$, let $F_B = \{\mathcal{B}^{(2)} \in \overline{\mathcal{B}} : B \in \mathcal{B}^{(2)}\}$. Clearly, $|F_B| = (2^{n-1}-1)$. For clarity and easy identification, we will call such a structure

F_B, a *Fano block.*

Now, let $\mathcal{F}$ be a collection of all $F_B \subset \overline{\mathcal{B}}$ that is, the total collection of all Fano blocks from a 1-$\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, 7, (2^{n-1}-1)\right)$ design. By definition, $(\overline{\mathcal{B}}, \mathcal{F})$ is an induced substructure of the 1-$\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, 7, (2^{n-1}-1)\right)$ design since, each $\mathcal{B}^{(2)} \subset \mathcal{B}$ and each $F_B \subset \overline{\mathcal{B}}$.

In view of the consideration above, each of the $\frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7}$ blocks of the 1-$\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, 7, (2^{n-1}-1)\right)$ design is regarded as a point of the new tactical configurations, that is, a Fano plane is a point. Blocks are then described as the Fano blocks.

We now discuss other parameters of $(\overline{\mathcal{B}}, \mathcal{F})$ beginning with the point degree.

In order to facilitate this discussion, and bearing in mind that a Fano plane contains 7 triples of $\mathcal{B}$, there is the need to know the number of times a given Fano plane appears in the entire set, $\mathcal{F}$ of Fano blocks, that is, given a $\mathcal{B}^{(2)} \in \overline{\mathcal{B}}$, we are interested in the number of times $\mathcal{B}^{(2)}$ appears in $\mathcal{F}$. This we discuss below.

Lemma 3.8. *Let* $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ *such that* $\mathcal{D} = (V, \mathcal{B})$ *is a* $2\text{-}(2^{n+1}-1, 3, 1)$ *design.*

Let $V^{(2)}$ *be a 2-flat of* $V, \mathcal{B}^{(2)} := \langle V^{(2)} \rangle$, *and let* $\overline{\mathcal{B}}$ *be a collection of all* $\mathcal{B}^{(2)}$ *such that* $(\mathcal{B}, \overline{\mathcal{B}})$ *is a* 1-$\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, 7, (2^{n-1}-1)\right)$ *design, and let* $\mathcal{F}$ *be the total collection of all Fano blocks from* $(\mathcal{B}, \overline{\mathcal{B}})$. *Then, any given* $\mathcal{B}^{(2)}$ *is in exactly* 7 *Fano blocks.*

Proof. Let $V^{(2)}$ be a 2-flat of V such that $\mathcal{B}^{(2)} = \langle V^{(2)} \rangle$, and $|\mathcal{B}^{(2)}| = 7$.

Now, for each $B_i \in \mathcal{B}^{(2)}, i = 1, \cdots, 7$ consider the set $F_{B_i} = \{\mathcal{B}^{(2)} : B_i \in \mathcal{B}^{(2)}\}$. The result therefore follows since each F_{B_i} is generated by the elements of $\mathcal{B}^{(2)}$. $\square$

At this end, we have the points set and its total number of points, the block size, and the points degree of our tactical configurations. We will now consider the total number of blocks of $\mathcal{F}$.

Lemma 3.9. *Let* $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ *such that* $\mathcal{D} = (V, \mathcal{B})$ *is a* $2-(2^{n+1}-1, 3, 1)$ *design. Let* $V^{(2)}$ *be a 2-flat of* $V, \mathcal{B}^{(2)} := \langle V^{(2)} \rangle$, *and let* $\overline{\mathcal{B}}$ *be a collection of all* $\mathcal{B}^{(2)}$ *such that* $\left(\mathcal{B}, \overline{\mathcal{B}}\right)$ *is a* 1-$\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, 7, (2^{n-1}-1)\right)$ *design, and let* $\mathcal{F}$ *be the total collection of all Fano blocks from* $\left(\mathcal{B}, \overline{\mathcal{B}}\right)$. *Then,*

$$|\mathcal{F}| = \frac{(2^{n+1}-1)(2^n-1)}{3}.$$

Proof. By Theorem 3.2, we have that $|\overline{\mathcal{B}}| = \frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1 \times 3 \times 7}$.

By Lemma 3.8, any given $\langle V^{(2)} \rangle$ is in exactly 7 Fano blocks, and $|F_B| = (2^{n-1}-1)$.

In addition, we have that $(\overline{\mathcal{B}}, \mathcal{F})$ is an induced substructure of the 1-$\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, 7, (2^{n-1}-1)\right)$ design by definition, since each $\mathcal{B}^{(2)} \subset \mathcal{B}$ and each $F_B \subset \overline{\mathcal{B}}$. It therefore follows as an incidence structure that

$$\frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1 \times 3 \times 7} \times 7 = |\mathcal{F}| \times (2^{n-1}-1).$$

$\square$

Clearly, the number of blocks of Steiner triple systems from projective geometry is the same as the number of blocks of the design $(\overline{\mathcal{B}}, \mathcal{F})$. Hence, we are now in a position to generalize the parameters of further tactical configuration $(\overline{\mathcal{B}}, \mathcal{F})$.

In view of the proofs of Lemma 3.8 and Lemma 3.9, we have the following:

Theorem 3.4. *Let* $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ *such that* $\mathcal{D} = (V, \mathcal{B})$ *is a* $2-(2^{n+1}-1, 3, 1)$ *design. Let* $V^{(2)}$ *be a 2-flat of* $V, \mathcal{B}^{(2)} := \langle V^{(2)} \rangle$, *and let* $\overline{\mathcal{B}}$ *be a collection of all* $\mathcal{B}^{(2)}$ *such that* $\left(\mathcal{B}, \overline{\mathcal{B}}\right)$ *is a*
1-$\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, 7, (2^{n-1}-1)\right)$ *design, and let* $\mathcal{F}$ *be the total collection of all Fano blocks from* $\left(\mathcal{B}, \overline{\mathcal{B}}\right)$. *Then,* $\left(\overline{\mathcal{B}}, \mathcal{F}\right)$ *is a*

1-$\left(\dfrac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1 \times 3 \times 7}, (2^{n-1}-1), 7\right)$ *design with* $\dfrac{(2^{n+1}-1)(2^n-1)}{3}$ *blocks.*

The design $\left(\overline{\mathcal{B}}, \mathcal{F}\right)$ above differs from Steiner triple systems from projective geometry, that is, the 2-$(v,3,1)$ designs, in the sense that in 2-$(v,3,1)$ designs, any two meet in exactly one point. There exist no two triples that do not meet. In the design $\left(\overline{\mathcal{B}}, \mathcal{F}\right)$, we have from Lemma 3.7 that the intersection of any two blocks when $n-m=2$ is an element of the set, $\{0,1\}$.

However, it will be shown later that the block intersection graphs of these designs are isomorphic.

In order to conclude the results of this section, we summarize in the following:

Theorem 3.5. *Let* $(V,\mathcal{B})$ *be a 1-design such that* $(V,\mathcal{B})$ *is also a quasi-symmetric design with parameters* $(v,k,\lambda;0,1)$ *with* b *blocks. Then, the tactical configuration of* $(V,\mathcal{B})$ *is a* 1-(b,λ,k) *design with* v *blocks.*

Proof. Let $(V',\mathcal{B}')$ be a tactical configuration of $(V,\mathcal{B})$, and let k' be the block size of $(V',\mathcal{B}')$. Clearly, any block $B \in \mathcal{B}$ is a point of $(V',\mathcal{B}')$. Hence, the block size, k' of $(V',\mathcal{B}')$ is λ since, any $B' \in \mathcal{B}'$ can be considered to be the set of all λ blocks of b containing B.

Now, we need to show that any $B' \in \mathcal{B}'$ is in k blocks of $\mathcal{B}'$.

Let $X \in \mathcal{B}'$ be the set of all the λ blocks of b containing B. It follows that B $\notin \mathcal{B}' \setminus X$. Hence, for each $B' \in \mathcal{B}'$, we have k blocks of B' containing B, since $|B| = k$.

Hence as an incident structure, the result follows . $\square$

Chapter 4

Block intersection graphs of the generalized tactical configurations and their extensions

4.1 Introduction

In this chapter, we consider block intersection graphs of tactical configurations defined in Chapter 3 and examine their properties. These graphs are relations on the block intersection graphs from $2-(v,3,1)$ designs from projective geometry; hence, we will compare and contrast them.

As alluded to in Chapter 2 of this thesis, the block intersection graphs of the generalized tactical configurations are by definition similar to Grassmann graphs. The difference is that the vertex set of the block intersection graphs of these generalized tactical configurations are blocks generated by the projective subspace, $V^{(n-m)}$. That is, the set of triples generated by the subspace $V^{(n-m)}$ constitute a vertex. In addition, the geometry of a projective space differs from that of a vector space.

We further explore the relationship between the Grassmann graphs and the block intersection graphs of the tactical configurations of Chapter 3 in order to fully characterize the block intersection graphs of our tactical configurations. In some cases, it will be shown that we obtain similar results in terms of the Grassmann graphs of some specific subspaces.

Lastly, we consider the block intersection graphs of the tactical configurations discussed in Section 3.3.

We now define the graphs of our tactical configurations.

Definition 4.1. Let $n, m \in \mathbb{N}$ and $(n-m) \in \{2, 3, \ldots, (n-2), (n-1)\}$. Let $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$ such that $(V, \mathcal{B})$ is a $2-(2^{n+1}-1, 3, 1)$ design. For all $(n-m)$-flats, $V^{(n-m)}$ of V, let $\overline{\mathcal{B}}$ be the collection of all $\mathcal{B}^{(n-m)}$ such that $\mathcal{D} = \big(\mathcal{B}, \overline{\mathcal{B}}\big)$ is a tactical configuration of $(V, \mathcal{B})$.

The block intersection graph $\Gamma = (\overline{\mathcal{B}}, E)$ of $\mathcal{D}$ is the graph with

$$V(\Gamma) := \overline{\mathcal{B}};$$
$$E(\Gamma) := \{[\mathcal{B}_1^{(n-m)}, \mathcal{B}_2^{(n-m)}] : \mathcal{B}_1^{(n-m)} \cap \mathcal{B}_2^{(n-m)} = \mathcal{B}^{(n-m-1)}\}.$$

We denote such graphs by $\Gamma_{[n,n-m]}$.

In view of the definition above and as a result of Theorem 3.2, we have that

$$|V(\Gamma_{[n,n-m]})| = \frac{\prod_{i=0}^{(n-m)} (2^{n-i+1}-1)}{\prod_{i=0}^{(n-m)} (2^{i+1}-1)}.$$

We now begin to explore the properties of $\Gamma_{[n,n-m]}$.

4.2 Properties of $\Gamma_{[n,n-m]}$

In this section we consider the parameters of the block intersection graphs of our tactical configurations in Chapter 3 and explore the properties of the graphs in order to compare and contrast them with the block intersection graphs of Steiner triple systems from projective geometry.

We begin with the degree of $\Gamma_{[n,n-m]}$.

Lemma 4.1. *For each* $\mathcal{B}^{(n-m)} \in \overline{\mathcal{B}}$, *the degree of* $\mathcal{B}^{(n-m)}$,

$$\deg(\mathcal{B}^{(n-m)}) = (2^{n-m+1}-1)(2^{m+1}-2).$$

Proof. First we recall from Lemma 3.1 that every $\mathcal{B}^{(n-m)} \in V(\Gamma_{[n,n-m]})$ has a corresponding $(n-m)$-flat, $V^{(n-m)}$ with $(2^{n-m+1}-1)$ points of V, and by Corollary 3.3, $|\mathcal{B}^{(n-m)}| = \dfrac{(2^{n-m+1}-1)(2^{n-m}-1)}{3}$.

Now, let $\mathcal{B}_1^{(n-m)}, \mathcal{B}_2^{(n-m)} \in V(\Gamma_{[n,n-m]})$ such that $[\mathcal{B}_1^{(n-m)}, \mathcal{B}_2^{(n-m)}] \in E(\Gamma_{[n,n-m]})$. By definintion, $\mathcal{B}_1^{(n-m)} \cap \mathcal{B}_2^{(n-m)} = \mathcal{B}^{(n-m-1)}$. This again implies from Lemma 3.1 that $\left(\mathcal{B}_1^{(n-m)} \cap \mathcal{B}_2^{(n-m)}\right)$ belongs to a corresponding $(n-m-1)$-flat, $V^{(n-m-1)}$ with $(2^{n-m}-1)$ points of V. Hence, there is the need to consider the total number of $\mathcal{B}^{(n-m-1)}$ in a given $\mathcal{B}^{(n-m)} \in V(\Gamma_{[n,n-m]})$.

Now, consider an incidence structure on the $\dfrac{(2^{n-m+1}-1)(2^{n-m}-1)}{3}$ triples of $\mathcal{B}^{(n-m)}$ into subsets of size, $|\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)}| = \dfrac{(2^{n-m}-1)(2^{n-m-1}-1)}{3}$ such that each triple B is in exactly r blocks with a total of b blocks.

By Theorem 3.1, every triple B is in exactly $\dfrac{\prod_{i=1}^{(n-m-1)} (2^{n-i}-1)}{\prod_{i=0}^{(n-m-2)} (2^{i+1}-1)}$ blocks of $\overline{\mathcal{B}}$ for an $(n-m)$-flat of V. Hence, we have that every triple B is in exactly

$$r = \frac{\prod_{i=1}^{(n-m-2)} (2^{n-m-i}-1)}{\prod_{i=0}^{(n-m-3)} (2^{i+1}-1)}$$

blocks of size $\mathcal{B}^{(n-m-1)}$, since we consider an $(n-m-1)$-flat $V^{(n-m-1)}$ from an $(n-m)$-flat, $V^{(n-m)}$.

As an incidence structure, it follows that

$$
\begin{aligned}
b &= \frac{(2^{n-m+1}-1)}{(2^{n-m-1}-1)} \frac{\prod_{i=1}^{(n-m-2)} (2^{n-m-i}-1)}{\prod_{i=0}^{(n-m-3)} (2^{i+1}-1)} \\
&= \frac{(2^{n-m+1}-1)(2^{n-m-1}-1)(2^{n-m-2}-1)\cdots(2^2-1)}{(2^{n-m-1}-1)(1)(3)\cdots(2^{n-m-2}-1)} \\
&= (2^{n-m+1}-1).
\end{aligned}
$$

In other words, there are $(2^{n-m+1}-1)$ triples of size $|\mathcal{B}^{(n-m-1)}|$ to decide the edges of a given vertex $\mathcal{B}^{(n-m)} \in V(\Gamma_{[n,n-m]})$.

Next, for a given $\mathcal{B}^{(n-m-1)} \subset \mathcal{B}^{(n-m)}$, we consider the number of other vertices $V(\Gamma_{[n,n-m]}) \setminus \mathcal{B}^{(n-m)}$ containing $\mathcal{B}^{(n-m-1)}$.

For a given $V^{(n-m-1)} \subset V^{(n-m)}$, consider partitions of the set $V \setminus V^{(n-m-1)}$ into $\dfrac{|V \setminus V^{(n-m-1)}|}{|V^{(n-m)} \setminus V^{(n-m-1)}|}$ subsets, each of size, $|V^{(n-m)} \setminus V^{(n-m-1)}|$.

Each of the set $V^{(n-m)} \setminus V^{(n-m-1)}$ and $V^{(n-m-1)}$ defines a $V^{(n-m)}$ and hence, a $\mathcal{B}^{(n-m)} \in V(\Gamma_{[n,n-m]})$.

It therefore follows that there are $\dfrac{2^{n+1}-2^{n-m}}{2^{n-m}} = (2^{m+1}-1)$ vertices in $V(\Gamma_{[n,n-m]})$ containing $\mathcal{B}^{(n-m-1)}$.

Hence, we have that there are $(2^{n-m+1}-1)$ $\mathcal{B}^{(n-m-1)}$ in each $\mathcal{B}^{(n-m)} \in V(\Gamma_{[n,n-m]})$ and each $\mathcal{B}^{(n-m-1)}$ is in $(2^{m+1}-2)$ other vertices. The result therefore follows.

$\square$

In view of the arguments of the proof of Lemma 4.1, every vertex has equal degree. Therefore the block intersection graphs are regular with degree $(2^{n-m+1}-1)(2^{m+1}-2)$.

Now, considering the results of Theorem 3.2, and in addition, having shown that the block intersection graphs are regular with degree $(2^{n-m+1}-1)(2^{m+1}-2)$, it is imperative to point out at this instance, a special case of the considerations of graphs in this Chapter. This is summarized in the following.

Theorem 4.1. $\Gamma_{[n,n-1]}$ *is a complete graph on* $2^{n+1}-1$ *vertices.*

Proof. By hypothesis, $m=1$. In this case, it follows from Theorem 3.2 that

$$|V(\Gamma_{[n,n-1]})| = \frac{\prod_{i=0}^{(n-m)}(2^{n-i+1}-1)}{\prod_{i=0}^{(n-m)}(2^{i+1}-1)}.$$ Hence, we have that

$$|V(\Gamma_{[n,n-1]})| = \frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)\times\cdots\times 3}{1\times 3\times\cdots\times(2^{n-1}-1)(2^n-1)} = (2^{n+1}-1).$$

The result therefore follows from Lemma 4.1, since the degree of a given vertex $\mathcal{B}^{(n-1)} \in V(\Gamma_{[n,n-1]})$ is $(2^{n-m+1}-1)(2^{m+1}-2) = (2^n-1)(2^2-2) = (2^n-1)(2) = (2^{n+1}-2)$. $\square$

The total number of edges, $|E(\Gamma_{[n,n-m]})|$ is easily calculated by the hand shaking lemma. We have

$$2|E(\Gamma_{[n,n-m]})| = \sum_{\mathcal{B}^{(n-m)}\in V(\Gamma_{[n,n-m]})} \deg(\mathcal{B}^{(n-m)}).$$

Hence,

$$|E(\Gamma_{[n,n-m]})| = \frac{\prod_{i=0}^{(n-m)}(2^{n-i+1}-1)}{\prod_{i=0}^{(n-m)}(2^{i+1}-1)}(2^{n-m+1}-1)(2^m-1).$$

In order to compare and contrast these graphs with the strongly regular graphs of the block intersection graphs of $2-(2^{n+1}-1,3,1)$ designs, we now discuss some adjacency parameters of $\Gamma_{[n,n-m]}$.

Let $\Gamma_{[n,n-m]}$ be a block intersection graph of $\mathcal{D}$. Given any 3 distinct vertices $\mathcal{B}_1^{(n-m)}, \mathcal{B}_2^{(n-m)}, \mathcal{B}_3^{(n-m)} \in V(\Gamma_{[n,n-m]})$, we distinguish the following adjacency between any three.

Case 1: $([\mathcal{B}_1^{(n-m)},\mathcal{B}_2^{(n-m)}],[\mathcal{B}_1^{(n-m)},\mathcal{B}_3^{(n-m)}],[\mathcal{B}_2^{(n-m)},\mathcal{B}_3^{(n-m)}] \in E(\Gamma_{[n,n-m]})$ and $\mathcal{B}_1^{(n-m)}\cap\mathcal{B}_2^{(n-m)}\cap\mathcal{B}_3^{(n-m)} = \mathcal{B}^{(n-m-1)}$;

Case 2: $([\mathcal{B}_1^{(n-m)}, \mathcal{B}_2^{(n-m)}], [\mathcal{B}_1^{(n-m)}, \mathcal{B}_3^{(n-m)}], [\mathcal{B}_2^{(n-m)}, \mathcal{B}_3^{(n-m)}] \in E(\Gamma_{[n,n-m]})$ and $\mathcal{B}_1^{(n-m)} \cap \mathcal{B}_2^{(n-m)} \cap \mathcal{B}_3^{(n-m)} \neq \mathcal{B}^{(n-m-1)}$. As an illustration,

$$\begin{aligned} \mathcal{B}_1^{(n-m)} \cap \mathcal{B}_2^{(n-m)} &= \mathcal{B}^{(n-m-1)}, \\ \mathcal{B}_2^{(n-m)} \cap \mathcal{B}_3^{(n-m)} &= \mathcal{B}'^{(n-m-1)} \text{ and} \\ \mathcal{B}_1^{(n-m)} \cap \mathcal{B}_3^{(n-m)} &= \mathcal{B}''^{(n-m-1)}. \end{aligned}$$

In view of the two possibilities above, we classify adjacency in this study in the following:

Definition 4.2. For any 3 distinct vertices $\mathcal{B}_1^{(n-m)}, \mathcal{B}_2^{(n-m)}, \mathcal{B}_3^{(n-m)} \in V(\Gamma_{[n,n-m]})$,

(i) an adjacency of the form of **Case 1** above is called **Type I adjacency**, that is, the three vertices share a common $\mathcal{B}^{(n-m-1)}$;

(ii) an adjacency of the form of **Case 2** above is called **Type II adjacency**.

We now discuss the number of common neighbours of any two vertices of $\Gamma_{[n,n-m]}$ as a result of Type I adjacency.

Lemma 4.2. *Given* $\Gamma_{[n,n-m]} = (\overline{\mathcal{B}}, E)$ *such that* $m > 1$. *Any two adjacent vertices have* $(2^{m+1} - 3)$ *common neighbours of Type* I *adjacency.*

Proof. Let $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}$ be any two adjacent vertices of $\Gamma_{[n,n-m]}$. By definition, $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-1)}$. By the proof of Lemma 4.1, $\mathcal{B}^{(n-m-1)}$ is in $(2^{m+1} - 1)$ vertices of $V(\Gamma_{[n,n-m]})$. It therefore follows that $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ are incident to $(2^{m+1} - 3)$ other vertices of Type I adjacency. $\square$

Corollary 4.1. *For* $m > 1$, *any set of adjacent vertices of Type I adjacency forms a clique of size* $(2^{m+1} - 1)$.

It will be made evident shortly after the discussion of common neighbours as a result of Type II adjacencies that the cliques of Corollary 4.1 are not maximum cliques if $n \geq 2m$.

Before we consider the number of common neighbours of any two vertices of $\Gamma_{[n,n-m]}$ as a result of Type II adjacencies, more importantly, because $\Gamma_{[n,n-m]}$ is defined on flats, it may help to consider variations in $n = (n - m) + m$.

The following are possibilities of the right hand side of the equation $n = (n-m)+m$.

$$\text{Case 1: } n-m=m; \tag{4.1}$$
$$\text{Case 2: } n-m<m \Longrightarrow n-m=m-k, k>0 \Longrightarrow n=2m-k; \tag{4.2}$$
$$\text{Case 3: } n-m>m \Longrightarrow n-m=m+k, k>0 \Longrightarrow n=2m+k. \tag{4.3}$$

In view of the possibilities above, given a fixed $n = n-m+m$, there exists a fixed $k > 0$ such that the following holds.

$$\begin{aligned} n = (n-m)+(m) &= (n-m-k)+(m+k) \\ &= (m-k)+(n-m+k) \end{aligned} \tag{4.4}$$

Hence,

$$(n-m-k)+(m+k) = (m-k)+(n-m+k). \tag{4.5}$$

Now considering Equation 4.5, we have that $(n-m-k) = (m-k)$, since $(n-m-k) \neq (n-m+k)$ and $(m+k) \neq (m-k)$. Moreover, k is fixed. This implies that $n-m=m$.

Taking the above considerations into account, we have the following.

Lemma 4.3. *For a fixed n and $k > 0$ such that Equation* 4.4 *holds,*

$$n-m=m.$$

We are now in a position to discuss the number of common neighbours of any two vertices of $\Gamma_{[n,n-m]}$ as a result of Type II adjacencies.

Lemma 4.4. *For $m > 2$, let $\Gamma_{[n,n-m]} = (\overline{\mathcal{B}}, E)$ be a block intersection graph of $\mathcal{D} = (\mathcal{B}, \overline{\mathcal{B}})$. Any two adjacent vertices are mutually adjacent to $(2^{n-m+2}-4)$ vertices of Type* II *adjacency.*

Proof. Let $\mathcal{B}^{(n-m)} = \left\langle V^{(n-m)} \right\rangle, \mathcal{B}'^{(n-m)} = \left\langle V'^{(n-m)} \right\rangle$ and $\mathcal{B}''^{(n-m)} = \left\langle V''^{(n-m)} \right\rangle$ such that $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}, \mathcal{B}''^{(n-m)} \in V(\Gamma_{[n,n-m]})$ and $[\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}] \in E(\Gamma_{[n,n-m]})$. By definition, we have that $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-1)}$.

Now for a $\mathcal{B}^{(n-m-2)} \subset \mathcal{B}^{(n-m-1)}$, let $\mathcal{B}^{(n-m-2)} = \left\langle V^{(n-m-2)} \right\rangle$.

By Lemma 3.2(ii), we have that for any $v_i \in V^{(n-m-2)}, i = 1, \cdots, |V^{(n-m-2)}|$,
$\left(\bigcup_{B \in \mathcal{B}_{v_i}^{(n-m-2)}} B \right) = V^{(n-m-2)}$. Hence, $\mathcal{B}^{(n-m-2)}$ can be identified as $\mathcal{B}_{v_i}^{(n-m-2)}$.

By the same argument, $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ can equally be identified as $\mathcal{B}_{v_i}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}_{v_i}$ respectively since, $v_i \in V^{(n-m)}$ and v_i is also an element of $V'^{(n-m)}$.

In addition, $\mathcal{B}^{(n-m-1)}$ can as well be identified as $\mathcal{B}_{v_i}^{(n-m-1)}$, and finally, the set $\mathcal{B}$ of all triples is also identified as $X_{v_i} = \{B \in \mathcal{B} : v_i \in B\}$.

By Lemma 3.2(i), $|\mathcal{B}_{v_i}^{(n-m)}| = |\mathcal{B}'^{(n-m)}_{v_i}| = 2^{n-m} - 1$ and

$$|\mathcal{B}_{v_i}^{(n-m-2)}| = 2^{n-m-2} - 1.$$

Furthermore, we have that

$$\begin{aligned} |\mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-1)}| &= |\mathcal{B}'^{(n-m)}_{v_i} \setminus \mathcal{B}_{v_i}^{(n-m-1)}| \\ &= 2^{n-m} - 1 - (2^{n-m-1} - 1) \\ &= 2^{n-m-1}, \end{aligned}$$

and

$$|\mathcal{B}_{v_i}^{(n-m-1)} \setminus \mathcal{B}_{v_i}^{(n-m-2)}| = 2^{n-m-1} - 1 - (2^{n-m-2} - 1) = 2^{(n-m-2)}.$$

In addition,

$$\begin{aligned} |\mathcal{B}_{v_i}^{(n-m)} \cup \mathcal{B}'^{(n-m)}_{v_i}| &= |\mathcal{B}_{v_i}^{(n-m)}| + |\mathcal{B}'^{(n-m)}_{v_i}| - |\mathcal{B}_{v_i}^{(n-m)} \cap \mathcal{B}'^{(n-m)}_{v_i}| \\ &= 2(2^{n-m} - 1) - (2^{n-m-1} - 1) = 3(2^{n-m-1}) - 1. \end{aligned}$$

Hence, it follows that there are $2^n - 1 - [3(2^{n-m-1}) - 1] = 2^n - 3(2^{n-m-1})$ triples of the set, $X_{v_i} \setminus \left(\mathcal{B}_{v_i}^{(n-m)} \cup \mathcal{B}'^{(n-m)}_{v_i} \right)$.

We now consider other vertices of $\Gamma_{[n,n-m]}$ adjacent to $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ as a result of the set, $X_{v_i} \setminus \left(\mathcal{B}_{v_i}^{(n-m)} \cup \mathcal{B}'^{(n-m)}_{v_i} \right)$ and the subset, $\mathcal{B}_{v_i}^{(n-m-2)}$ of $\mathcal{B}_{v_i}^{(n-m-1)}$.

First, consider the sets, $\mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-1)}$ and $\mathcal{B}'^{(n-m)}_{v_i} \setminus \mathcal{B}_{v_i}^{(n-m-1)}$. We have that there are exactly

$$\frac{|\mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-1)}|}{2^{n-m-2}} = \frac{|\mathcal{B}'^{(n-m)}_{v_i} \setminus \mathcal{B}_{v_i}^{(n-m-1)}|}{2^{n-m-2}} = 2$$

distinct subsets, each of size, $2^{(n-m-2)}$ in each of the sets, $\mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-1)}$ and $\mathcal{B'}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-1)}$.

So, let $P_j^{(n-m-2)}, j = 1,2$ be the set of 2^{n-m-2} distinct subsets of $\mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-1)}$ and $P'^{(n-m-2)}_k, k = 1,2$ be the set of 2^{n-m-2} distinct subsets of $\mathcal{B'}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-1)}$. That is,

$$\bigcup_{j\in\{1,2\}} P_j^{(n-m-2)} = \mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-1)}$$

and

$$\bigcup_{k\in\{1,2\}} P'^{(n-m-2)}_k = \mathcal{B'}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-1)}.$$

Now consider the set, $X_{v_i} \setminus \left(\mathcal{B}_{v_i}^{(n-m)} \cup \mathcal{B'}_{v_i}^{(n-m)}\right)$.

We have that there are exactly

$$\frac{2^n - 3(2^{n-m-1})}{2^{n-m-2}} = \frac{2^n}{2^{n-m-2}} - \frac{3(2^{n-m-1})}{2^{n-m-2}}$$

distinct subsets, each of size 2^{n-m-2} in the set $X_{v_i} \setminus \left(\mathcal{B}_{v_i}^{(n-m)} \cup \mathcal{B'}_{v_i}^{(n-m)}\right)$.

Now, taking $\frac{2^n}{2^{n-m-2}}$ into account for a fixed n, we have from Equation 4.4 that

$$\begin{aligned} n = (n-m) + (m) &= (n-m-2) + (m+2) \\ &= (m-2) + (n-m+2). \end{aligned}$$

By Lemma 4.3, we have that $n-m$ must be equal to m. Hence, we have that $\frac{2^n}{2^{n-m-2}} = 2^{n-m+2}$.

It therefore follows that

$$\frac{2^n}{2^{n-m-2}} - \frac{3(2^{n-m-1})}{2^{n-m-2}} = 2^{n-m+2} - 3(2) = 2^{n-m+2} - 6.$$

Consequently, we have that there are exactly $2^{n-m+2} - 6$ distinct subsets, each of size 2^{n-m-2} in the set $X_{v_i} \setminus \left(\mathcal{B}_{v_i}^{(n-m)} \cup \mathcal{B'}_{v_i}^{(n-m)}\right)$.

So, let $P''^{(n-m-2)}_l, l \in \{1, 2, \cdots, (2^{n-m+2}-6)\}$, be the set of 2^{n-m-2} distinct subsets of $X_{v_i} \setminus \left(\mathcal{B}^{(n-m)}_{v_i} \cup \mathcal{B}'^{(n-m)}_{v_i}\right)$.

Without loss of generality, let

$$P = P^{(n-m-2)}_1 \cup P'^{(n-m-2)}_1.$$

In addition, let $P_l \in P''^{(n-m-2)}_l$. By Lemma 3.1, we have that the set,

$$P \cup P_l \cup \mathcal{B}^{(n-m-2)}_{v_i}$$

defines a $\mathcal{B}''^{(n-m)}_{v_i}$ for each $l \in \{1, 2, \cdots, (2^{n-m+2}-6)\}$ and hence a $\mathcal{B}''^{(n-m)} \in V(\Gamma_{[n,n-m]})$. Clearly, each $\mathcal{B}''^{(n-m)}$ is incident to $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$.

We now look at vertices adjacent to $\mathcal{B}^{(n-m)}_{v_i}$ and $\mathcal{B}'^{(n-m)}_{v_i}$ as a result of the set, $(\mathcal{B}^{(n-m)}_{v_i} \cup \mathcal{B}'^{(n-m)}_{v_i}) \setminus \mathcal{B}^{(n-m-1)}_{v_i}$ and $\mathcal{B}^{(n-m-2)}_{v_i}$.

Now, let

$$P^* = P^{(n-m-2)}_1 \cup P^{(n-m-2)}_2.$$

Again by Lemma 3.1 we have that each of the sets,

$$\mathcal{B}^{(n-m-2)}_{v_i} \cup P^* \cup P'^{(n-m-2)}_1$$

and

$$\mathcal{B}^{(n-m-2)}_{v_i} \cup P^* \cup P'^{(n-m-2)}_2$$

defines a $\mathcal{B}'''^{(n-m)}_{v_i}$ and hence a $\mathcal{B}'''^{(n-m)} \in V(\Gamma_{[n,n-m]})$. Clearly, each $\mathcal{B}'''^{(n-m)}$ is also incident to $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$.

In addition, the set

$$\mathcal{B}^* = \mathcal{B}^{(n-m-1)}_{v_i} \cup P^{(n-m-2)}_2 \cup P'^{(n-m-2)}_2$$

also defines a $\mathcal{B}''''^{(n-m)}_{v_i}$ and hence a $\mathcal{B}''''^{(n-m)} \in V(\Gamma_{[n,n-m]})$. Clearly, $\mathcal{B}''''^{(n-m)}$ is also incident to $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$.

It is very clear that $\mathcal{B}^*$ is a Type I adjacency. Hence, it follows that any two adjacent vertices are commonly adjacent to

$$(2^{n-m+2} - 6) + 2 = (2^{n-m+2} - 4)$$

vertices of Type II adjacency.

In order to complete the proof, we need to show that there is exactly one $\mathcal{B}^{(n-m-2)} \subset \mathcal{B}^{(n-m-1)}$ such that the result holds.

Suppose to the contrary that there exists a $\mathcal{B}'^{(n-m-2)} \subset \mathcal{B}^{(n-m-1)}$ such that $\mathcal{B}''^{(n-m)}, \mathcal{B}'''^{(n-m)}, \mathcal{B}''''^{(n-m)} \in V(\Gamma_{[n,n-m]})$ and are incident to both $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$.

Let $\mathcal{B}'^{(n-m-2)} = \langle V'^{(n-m-2)} \rangle$. For any $v_i' \in V'^{(n-m-2)}$, $i = 1, \cdots, |V'^{(n-m-2)}|$, similarly as in Lemma 3.2(ii), we have that $\left(\bigcup_{B \in \mathcal{B}_{v_i'}^{(n-m-2)}} B \right) = V'^{(n-m-2)}$.

Hence, $\mathcal{B}'^{(n-m-2)}$ can also be identified as $\mathcal{B}'^{(n-m-2)}_{v_i'}$.

Now, we have the following two possibilities, since $V^{(n-m-2)}, V'^{(n-m-2)} \subset V^{(n-m-1)}$.

(i) $V^{(n-m-2)} \cap V'^{(n-m-2)} \neq \emptyset$

This implies that there is at least a $v_i'' \in V^{(n-m-2)} \cap V'^{(n-m-2)}$ such that $\mathcal{B}^{(n-m-2)}$ and $\mathcal{B}'^{(n-m-2)}$ can be identified as $\mathcal{B}^{(n-m-2)}_{v_i''}$ and $\mathcal{B}'^{(n-m-2)}_{v_i''}$ respectively. By definition, we have that $\mathcal{B}^{(n-m-2)}_{v_i''} = \mathcal{B}'^{(n-m-2)}_{v_i''}$. Consequently, we have that $\mathcal{B}^{(n-m-2)} = \mathcal{B}'^{(n-m-2)}$. Hence we have a contradiction.

(ii) $V^{(n-m-2)} \cap V'^{(n-m-2)} = \emptyset$

In this case,

$$V^{(n-m-2)} \cup V'^{(n-m-2)} = V^{(n-m-1)},$$

since $\dim(V^{(n-m-1)}) = n - m - 1$. This implies that there exists a $v_i \in V^{(n-m-2)}$ and a $v_i' \in V'^{(n-m-2)}$ such that $\mathcal{B}^{(n-m-2)}$ and $\mathcal{B}'^{(n-m-2)}$ can be identified as $\mathcal{B}^{(n-m-2)}_{v_i}$ and $\mathcal{B}'^{(n-m-2)}_{v_i'}$ respectively.

We now show that this possibility cannot arise.

By the definition of $\mathcal{B}$, there exists a $B \in \mathcal{B}$, and a $v_i'' \in V$ such that $v_i + v_i^{'} + v_i^{''} = 0$, since any two elements of V are in exactly one block B of $\mathcal{B}$. Hence, it follows that $\{v_i, v_i^{'}, v_i^{''}\} \in (\mathcal{B}^{(n-m-2)}_{v_i} \cap \mathcal{B}'^{(n-m-2)}_{v_i'})$.

Now, it follows that $\{v_i, v_i^{'}, v_i^{''}\} \subset (V^{(n-m-2)} \cap V'^{(n-m-2)})$ since,

$$V^{(n-m-2)} = \left(\bigcup_{B \in \mathcal{B}_{v_i}^{(n-m-2)}} B \right), \text{ and } V'^{(n-m-2)} = \left(\bigcup_{B \in \mathcal{B}_{v_i^{'}}^{(n-m-2)}} B \right).$$

Hence we have a contradiction and the result therefore follows.

□

Corollary 4.2. *Any set of adjacent vertices of Type II adjacency forms a clique of size* $(2^{n-m+2} - 1)$.

Proof. By the arguments of the proof of Lemma 4.4, it is clear that $\mathcal{B}_{v_i}^{(n-m-1)} \cap P_l = \emptyset$, for any $P_l \in P_l^{''(n-m-2)}$. This implies that $\mathcal{B}_{v_i}^{(n-m-1)} \not\subset \mathcal{B}_l''$, for all $P_l \in P_l^{''(n-m-2)}$, where $\mathcal{B}_l'' = (P \cup P_l \cup \mathcal{B}_{v_i}^{(n-m-2)}), l \in \{1, 2, \cdots, (2^{n-m+2} - 6)\}$.

Hence any clique $\{\mathcal{B}_{v_i}^{(n-m)}, \mathcal{B'}_{v_i}^{(n-m)}, \mathcal{B}_l''\}, l = 1, 2, \cdots, (2^{n-m+2} - 6)$ has no other vertices common to $\mathcal{B}^{(n-m)}$, $\mathcal{B}'^{(n-m)}$ and $\mathcal{B}''$. □

Having discussed the clique sizes as a result of Type I and Type II adjacencies, we are now in a better position to discuss the nature of the cliques.

In consideration of Corollaries 4.1 and 4.2, we have that

$$(2^{m+1} - 1) - (2^{n-m+2} - 1) = 2^{m+1}(1 - 2^{n-2m+1})$$

and hence it follows that

$$2^{m+1}(1 - 2^{n-2m+1}) < 0, \text{ if } (n+1) \geq 2m,$$

and

$$2^{m+1}(1 - 2^{n-2m+1}) > 0, \text{ if } (n+1) < 2m.$$

In view of the above consideration and since we also have clearly from the proofs of Corollaries 4.1 and 4.2 that neither the cliques as a result of Type I adjacency nor the cliques as a result of Type II adjacency can be extended by an additional vertex of $\Gamma_{[n,n-m]}$, we therefore have the following corollary.

Corollary 4.3. $\Gamma_{[n,n-m]}$ *has a maximum clique of size*

$$\begin{cases} 2^{n-m+2} - 1, & \text{if } (n+1) \geq 2m; \\ 2^{m+1} - 1, & \text{if } (n+1) < 2m. \end{cases}$$

As a direct implication of Lemmas 4.2 and 4.4, we now give the total number of all common neighbours of any two adjacent vertices.

Theorem 4.2. *For $m > 1$, let $\Gamma_{[n,n-m]} = (\overline{\mathcal{B}}, E)$ be a block intersection graph of $\mathcal{D}$. The total number of all common neighbours of any two adjacent vertices, that is, of Type* I *and Type* II *adjacency is* $(2^{n-m+2} + 2^{m+1} - 7)$.

In view of Corollary 4.3, we also have the following.

Corollary 4.4. $\Gamma_{[n,n-m]}$ *is a non-bipartite graph.*

Proof. The size of the cliques of $\Gamma_{[n,n-m]}$ is greater than 2. $\square$

Having discussed the common neighbours and the total number of common neighbours of any two adjacent vertices,we now turn to the question of the common neighbours and the total number of common neighbours of any two non-adjacent vertices.

First, we recall from Lemma 3.7 that for any two blocks, $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)} \in \overline{\mathcal{B}}$,

$$\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}, \text{ where } |\mathcal{B}^{(n-m-j)}| \in \left\{ \frac{(2^{n-m}-1)(2^{n-m-1}-1)}{3}, \frac{(2^{n-m-1}-1)(2^{n-m-2}-1)}{3}, \cdots, 1 \right\}.$$

In view of the above, we now discuss the common neighbours and the total number of common neighbours of any two non-adjacent vertices.

Lemma 4.5. *Let $\Gamma_{[n,n-m]} = (\overline{\mathcal{B}}, E)$ be a block intersection graph of $\mathcal{D}$ and let $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ be any two non-adjacent vertices, such that $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}$, $j \in \{2, \cdots, (n-m)\}$.*

(i) *If $j = 2$ then, $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ have 9 common adjacent vertices.*

(ii) *If $j > 2$ then, there is no $\mathcal{B}''^{(n-m)} \in V(\Gamma_{[n,n-m]})$ such that*

$$[\mathcal{B}^{(n-m)}, \mathcal{B}''^{(n-m)}], [\mathcal{B}'^{(n-m)}, \mathcal{B}''^{(n-m)}] \in E(\Gamma_{[n,n-m]}).$$

Proof. (i) Let $\mathcal{B}^{(n-m)} = \left\langle V^{(n-m)} \right\rangle, \mathcal{B}'^{(n-m)} = \left\langle V'^{(n-m)} \right\rangle$ and $\mathcal{B}''^{(n-m)} = \left\langle V''^{(n-m)} \right\rangle$ such that $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}, \mathcal{B}''^{(n-m)} \in V(\Gamma_{[n,n-m]})$ and $[\mathcal{B}^{(n-m)}, \mathcal{B}''^{(n-m)}], [\mathcal{B}'^{(n-m)}, \mathcal{B}''^{(n-m)}] \in E(\Gamma_{[n,n-m]})$.

By definition, we have that there exists a $\mathcal{B}^{(n-m-1)} \subset \mathcal{B}^{(n-m)}$ and a $\mathcal{B}'^{(n-m-1)} \subset \mathcal{B}'^{(n-m)}$ such that $\mathcal{B}^{(n-m-1)}, \mathcal{B}'^{(n-m-1)} \subset \mathcal{B}''^{(n-m)}$.

By hypothesis, we also have that $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-2)}$, since $[\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}] \notin E(\Gamma_{[n,n-m]})$.

Now, let $\mathcal{B}^{(n-m-2)} = \left\langle V^{(n-m-2)} \right\rangle$.

By Lemma 3.2(ii), for any $v_i \in V^{(n-m-2)}, i = 1, \cdots, |V^{(n-m-2)}|$, $\left(\bigcup_{B \in \mathcal{B}_{v_i}^{(n-m-2)}} B \right) = V^{(n-m-2)}$. Hence, $\mathcal{B}^{(n-m-2)}$ can be identified as $\mathcal{B}_{v_i}^{(n-m-2)}$.

By the same argument, $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ can equally be identified as $\mathcal{B}_{v_i}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}_{v_i}$ respectively since, $v_i \in V^{(n-m)}$ and v_i is also an element of $V'^{(n-m)}$.

In addition the sets, $\mathcal{B}^{(n-m-1)}$ and $\mathcal{B}'^{(n-m-1)}$ can also be represented as $\mathcal{B}_{v_i}^{(n-m-1)}$ and $\mathcal{B}'^{(n-m-1)}_{v_i}$ respectively.

We now complete the proof of (i) by considering all other vertices of $V(\Gamma_{[n,n-m]})$ incident to both $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ due to $\mathcal{B}^{(n-m-2)}$.

Now consider the sets $\mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-2)}, \mathcal{B}'^{(n-m)}_{v_i} \setminus \mathcal{B}_{v_i}^{(n-m-2)}$, and the sets $\mathcal{B}_{v_i}^{(n-m-1)} \setminus \mathcal{B}_{v_i}^{(n-m-2)}, \mathcal{B}'^{(n-m-1)}_{v_i} \setminus \mathcal{B}_{v_i}^{(n-m-2)}$.

Clearly,
$$\begin{aligned} |\mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-2)}| &= |\mathcal{B}'^{(n-m)}_{v_i} \setminus \mathcal{B}_{v_i}^{(n-m-2)}| \\ &= (3)2^{(n-m-2)} \end{aligned}$$

and
$$\begin{aligned} |\mathcal{B}_{v_i}^{(n-m-1)} \setminus \mathcal{B}_{v_i}^{(n-m-2)}| &= |\mathcal{B}'^{(n-m-1)}_{v_i} \setminus \mathcal{B}_{v_i}^{(n-m-2)}| \\ &= 2^{(n-m-2)}. \end{aligned}$$

Hence, it follows that there are 3 distinct subsets, each of size, $2^{(n-m-2)}$ in each of the sets $\mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-2)}$ and $\mathcal{B}'^{(n-m)}_{v_i} \setminus \mathcal{B}_{v_i}^{(n-m-2)}$.

By Lemma 3.1, any pair of the subsets from $\mathcal{B}_{v_i}^{(n-m)} \setminus \mathcal{B}_{v_i}^{(n-m-2)}$ and $\mathcal{B}'^{(n-m)}_{v_i} \setminus \mathcal{B}_{v_i}^{(n-m-2)}$ together with $\mathcal{B}_{v_i}^{(n-m-2)}$ defines a $\mathcal{B}'''^{(n-m)} \in V(\Gamma_{[n,n-m]})$ which is incident to both $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$. Hence we have the result.

(ii) By hypothesis, $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}, j = 3, \cdots, (n-m)$.

Now, suppose to the contrary that there exists a $\mathcal{B}'''^{(n-m)} \in V(\Gamma_{[n,n-m]})$ incident to both $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$, whenever $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}$. That is, $[\mathcal{B}^{(n-m)}, \mathcal{B}'''^{(n-m)}], [\mathcal{B}'^{(n-m)}, \mathcal{B}'''^{(n-m)}] \in E(\Gamma_{[n,n-m]})$, whenever $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}$.

Let $\mathcal{B}^{(n-m-j)} = \left\langle V^{(n-m-j)} \right\rangle$. By a similar argument in (i) above, we have that there exists a $\mathcal{B}^{(n-m-1)} \subset \mathcal{B}^{(n-m)}$ and a $\mathcal{B}'^{(n-m-1)} \subset \mathcal{B}'^{(n-m)}$ such that $\mathcal{B}^{(n-m-1)}, \mathcal{B}'^{(n-m-1)} \subset \mathcal{B}'''^{(n-m)}$.

In addition, there exists a $v_i \in V^{(n-m-j)}$ such that

$$\begin{aligned}|\mathcal{B}^{(n-m)}_{v_i} \setminus \mathcal{B}^{(n-m-j)}_{v_i}| &= |\mathcal{B}'^{(n-m)}_{v_i} \setminus \mathcal{B}^{(n-m-j)}_{v_i}| \\ &= (2^j - 1)2^{(n-m-j)},\end{aligned}$$

and

$$\begin{aligned}|\mathcal{B}^{(n-m-1)}_{v_i} \setminus \mathcal{B}^{(n-m-j)}_{v_i}| &= |\mathcal{B}'^{(n-m-1)}_{v_i} \setminus \mathcal{B}^{(n-m-j)}_{v_i}| \\ &= 2^{(n-m-j)}(2^{j-1} - 1).\end{aligned}$$

Again by a similar argument as in (i) above, we have that there are $\dfrac{(2^j-1)}{(2^{j-1}-1)}$ distinct subsets, each of size, $2^{(n-m-j)}$ in each of the sets $\mathcal{B}^{(n-m)}_{v_i} \setminus \mathcal{B}^{(n-m-j)}_{v_i}$ and $\mathcal{B}'^{(n-m)}_{v_i} \setminus \mathcal{B}^{(n-m-j)}_{v_i}$.

We now complete the proof by showing that $\dfrac{2^j-1}{2^{j-1}-1} \notin \mathbb{N}$, for $j > 2$.

Now, suppose to the contrary that $2^{j-1} - 1$ divides $2^j - 1$, for $j > 2$.

It follows that $j = l + 2, l \in \mathbb{N}$, and $\dfrac{2^j-1}{2^{j-1}-1} = k, k \in \mathbb{N}$.

Hence,

$$\begin{aligned}(2^j - 1) &= k(2^{j-1} - 1) \\ 2^{l+2} - 1 &= k\left(2^{l+1} - 1\right) \text{ (substituting for j)} \\ k &= \frac{(2^{l+2}-1)}{(2^{l+1}-1)} \\ &= \frac{2(2^{l+1}-1)+1}{(2^{l+1}-1)}.\end{aligned}$$

This contradicts the fact that $k \in \mathbb{N}$, and hence we have the result. $\square$

From the proof of Lemma 4.5(i), it is clear that the path between each of the 9 vertices $\mathcal{B}^{(n-m)}, \mathcal{B}'''^{(n-m)}, \mathcal{B}'^{(n-m)}$ is a minimal path. As a result, we have the following.

Corollary 4.5. *Let $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)} \in V(\Gamma_{[n,n-m]})$, then $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ have distance 2 if $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-2)}$.*

Considering the result of Corollary 4.5, it is essential to generalize the discussion above to the distance between any two vertices $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)} \in V(\Gamma_{[n,n-m]})$ given that $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}, j \in \{2, \cdots, (n-m)\}$. A direct consequence of this is that it will lead us to the diameter of $\Gamma_{[n,n-m]}$.

First, there is the need to consider some properties of paths between any two vertices $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ of $V(\Gamma_{[n,n-m]})$ given that $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}$. We now discuss these properties in the following.

Lemma 4.6. *Let $\mathcal{B}_0^{(n-m)}, \mathcal{B}_1^{(n-m)}, \ldots, \mathcal{B}_j^{(n-m)}$ be a geodesic distance in $V(\Gamma_{[n,n-m]})$, then*

(i) $\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_j^{(n-m)} = \mathcal{B}_0^{(n-m)} \cap \mathcal{B}_j^{(n-m)}$;

(ii) $\dim(\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_j^{(n-m)}) = \dim(\mathcal{B}_0^{(n-m)}) - j$, *where* $j = \mathrm{d}(\mathcal{B}_0^{(n-m)}, \mathcal{B}_j^{(n-m)})$.

Proof. The proofs are by induction on j.

(i) Given that $j = 1$. By hypothesis, we have that $[\mathcal{B}_0^{(n-m)}, \mathcal{B}_1^{(n-m)}] \in E(\Gamma_{[n,n-m]})$. Hence, $\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} = \mathcal{B}^{(n-m-1)}$.

Now, assume that (i) above holds for $j = k$, that is,

$$\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)} = \mathcal{B}_0^{(n-m)} \cap \mathcal{B}_k^{(n-m)}.$$

We need to show that the result is also true for $j = k+1$.

$$\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_{k+1}^{(n-m)} =$$
$$(\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)}) \cap \mathcal{B}_{k+1}^{(n-m)}$$
$$= (\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_k^{(n-m)}) \cap \mathcal{B}_{k+1}^{(n-m)}$$
$$= (\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}) \cap (\mathcal{B}_k^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)})$$
$$= (\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}) \text{ since } [\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}] \notin E(\Gamma_{[n,n-m]}).$$

Hence we have the result.

(ii) Given that $j = 1$. By definition of $\Gamma_{[n,n-m]}$, we have that $[\mathcal{B}_0^{(n-m)}, \mathcal{B}_1^{(n-m)}] \in E(\Gamma_{[n,n-m]})$. Hence $\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} = \mathcal{B}^{(n-m-1)}$ which implies that $\dim(\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)}) = n-m-\mathrm{d}(\mathcal{B}_0^{(n-m)}, \mathcal{B}_1^{(n-m)})$.

Now assume that (ii) above holds for $j = k$, that is,

$$\dim(\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)}) = n-m-k.$$

We need to show that the result is also true for $j = k+1$.

Now, consider $\dim\left((\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)}) \cup \mathcal{B}_{k+1}^{(n-m)}\right)$. We have that

$$\dim\left((\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)}) \cup \mathcal{B}_{k+1}^{(n-m)}\right) =$$
$$\dim(\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)}) + \dim(\mathcal{B}_{k+1}^{(n-m)})$$
$$- \dim\left((\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)}) \cap \mathcal{B}_{k+1}^{(n-m)}\right).$$

By (i) above, we have that

$$\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)} = \mathcal{B}_0^{(n-m)} \cap \mathcal{B}_k^{(n-m)}.$$

Hence, we have that

$$\dim\left((\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)}) \cup \mathcal{B}_{k+1}^{(n-m)}\right)$$
$$= n-m-k+n-m-\dim\left((\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_k^{(n-m)}) \cap \mathcal{B}_{k+1}^{(n-m)}\right)$$
$$= 2(n-m)-k-\dim\left((\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}) \cap (\mathcal{B}_k^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)})\right)$$
$$= 2(n-m)-k-\dim(\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}) \text{ since}$$
$$[\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}] \notin E(\Gamma_{[n,n-m]}). \tag{4.6}$$

Again, consider $\dim\left((\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)}) \cup \mathcal{B}_{k+1}^{(n-m)}\right)$. We also have that

$$\begin{aligned}
&\dim\left((\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_1^{(n-m)} \cap \ldots \cap \mathcal{B}_k^{(n-m)}) \cup \mathcal{B}_{k+1}^{(n-m)}\right)\\
&= \dim\left((\mathcal{B}_0^{(n-m)} \cup \mathcal{B}_{k+1}^{(n-m)}) \cap \ldots \cap (\mathcal{B}_k^{(n-m)} \cup \mathcal{B}_{k+1}^{(n-m)})\right)\\
&= \dim(\mathcal{B}_k^{(n-m)} \cup \mathcal{B}_{k+1}^{(n-m)}) \text{ since } [\mathcal{B}_k^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}] \in E(\Gamma_{[n,n-m]})\\
&= \dim(\mathcal{B}_k^{(n-m)}) + \dim(\mathcal{B}_{k+1}^{(n-m)}) - \dim(\mathcal{B}_k^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)})\\
&= 2(n-m) - (n-m-1) \qquad ([\mathcal{B}_k^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}] \in E(\Gamma_{[n,n-m]})\\
&= n-m+1.
\end{aligned} \tag{4.7}$$

Combining Equations (4.6) and (4.7), we have that

$$n-m+1 = 2(n-m) - k - \dim(\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}).$$

Hence,

$$\begin{aligned}
\dim(\mathcal{B}_0^{(n-m)} \cap \mathcal{B}_{k+1}^{(n-m)}) &= 2(n-m) - k - (n-m+1)\\
&= n-m-(k+1).
\end{aligned}$$

Hence we also have this result. $\square$

An immediate implication of Lemma 4.6 is that we can find the distance between any two vertices of $\Gamma_{[n,n-m]}$ which we discuss in the following.

Corollary 4.6. *Let $j \in \{2, \cdots, (n-m)\}$, and let $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)} \in V(\Gamma_{[n,n-m]})$. Then $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ have distance j if $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}$.*

We are now in a position to discuss the diameter of $\Gamma_{[n,n-m]}$.

Theorem 4.3.

$$\operatorname{diam}(\Gamma_{[n,n-m]}) = \begin{cases} n-m, & \text{if } 2m \le n;\\ m, & \text{otherwise.}\end{cases}$$

Proof. (i) Let $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)} \in V(\Gamma_{[n,n-m]})$ such that $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}, j \in \{2, \cdots, (n-m)\}$. We have from Corollary 4.6 that

$d(\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}) = j$, since $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}$. Clearly j is at most $n-m$.

We now conclude the proof by considering the possible values of j since we have from Equations (4.1), (4.2) and (4.3) that for a fixed $n, n-m$ has the following possibilities:

(i) $n - m = m$;

(ii) $n - m > m \Longrightarrow n > 2m$;

(iii) $n - m < m \Longrightarrow n < 2m$.

It is clear from (i) and (ii) above that j is a maximum when $j = n - m$. Therefore, $n \geq 2m$.

In the case of (iii) above, we have that j is a maximum when $j = m$. Hence we have the result. □

At this point, we are now in a position to introduce one of the special cases considered in this study. As alluded to, the following corollary is one of the cases where the block intersection graphs of Steiner triple systems from projective geometry have the same parameters as compared to $\Gamma_{[n,n-m]}$. This is as a result of Corollary 3.3 and Lemmas 4.1, 4.2 and 4.5.

Isomorphism between these two graphs is a corollary of Theorem 4.6 in the next section. Hence in the interim, we discuss the parameters of this special case.

Theorem 4.4. *Let $n > 2, m = 2$ and $\Gamma_{[n,n-m]}$ be a block intersection graph of $\mathcal{D} = (\mathcal{B}, \overline{\mathcal{B}})$. Then, $\Gamma_{[n,n-m]}$ is a strongly regular graph with parameters*

$$\left(\frac{(2^{n+1}-1)(2^n-1)}{3}, 6(2^{n-1}-1), (2^n+1), 9\right).$$

Proof. $m = 2$ implies $n - m = n - 2$. Hence by Corollary 3.3,

$$\frac{\prod_{i=0}^{(n-2)} (2^{n-i+1}-1)}{\prod_{i=0}^{(n-2)} (2^{i+1}-1)} = \frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)(2^{n-2}-1)}{(1)(3)(2^{n-2}-1)(2^{n-1}-1)}$$

$$= \frac{(2^{n+1}-1)(2^n-1)}{3}.$$

Therefore, by Corollary 3.3 and Lemmas 4.1, 4.2 and 4.5, we have the result. □

The following corollary as a result of Lemma 4.5(ii), specifically for $m > 2$ is in contrast to the parameters of $\Gamma_{[n,n-m]}$ as compared to the block intersection graphs of Steiner triple systems from projective geometry.

Corollary 4.7. *Let $\Gamma_{[n,n-m]} = (\overline{\mathcal{B}}, E)$ be a block intersection graph of $\mathcal{D}$ such that $m > 2$. Then, the number of common adjacent vertices to any two non-adjacent vertices is an element of the set $\{0, 9\}$.*

As a direct consequence of Cororllary 4.7,Theorem 4.1 and Theorem 4.4, we have that the block intersection graphs of tactical configurations on Steiner triple systems from projective geometry are strongly regular if $m = 1, 2$. Otherwise, they not strongly regular.

In Chapter 2 of this study, we introduced the concepts of S-thin and R-thin graphs. We now consider the thinness of $\Gamma_{[n,n-m]}$.

Lemma 4.7. $\Gamma_{[n,n-m]}$ *is S-thin and R-thin if $m > 1$.*

Proof. By Theorem 4.1, $\Gamma_{[n,n-m]}$ is a complete graph if $m = 1$. Hence, we do not consider this case.

Now, let $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)} \in V(\Gamma_{[n,n-m]})$. It follows that

(i) $[\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}] \in E(\Gamma_{[n,n-m]})$;

(ii) $[\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}] \notin E(\Gamma_{[n,n-m]})$.

Now, if (i) above holds, it follows that $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}'^{(n-m-1)}$. By the arguments of the proof of Lemma 4.1, there are $(2^{n-m+1} - 2)$ other $\mathcal{B}^{(n-m-1)}$ in each of $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$. By Lemma 4.1 the degree of each of $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ is $(2^{n-m+1} - 1)(2^{m+1} - 2)$. Hence, it follows that

$$N_{\Gamma_{[n,n-m]}}[\mathcal{B}^{(n-m)}] \neq N_{\Gamma_{[n,n-m]}}[\mathcal{B}'^{(n-m)}]$$

and

$$N_{\Gamma_{[n,n-m]}}[\mathcal{B}^{(n-m)}] \cup \{\mathcal{B}^{(n-m)}\} \neq N_{\Gamma_{[n,n-m]}}[\mathcal{B}'^{(n-m)}] \cup \{\mathcal{B}'^{(n-m)}\}.$$

Hence, it follows that there are no two vertices in relations S and R.

We now consider (ii) above.

By Similar argument of (i) above, we have that each of the $(2^{n-m+1}-1)$ $\mathcal{B}^{(n-m-1)}$ in each of $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ have different adjacencies. Hence, we have that

$$N_{\Gamma_{[n,n-m]}}[\mathcal{B}^{(n-m)}] \neq N_{\Gamma_{[n,n-m]}}[\mathcal{B}'^{(n-m)}]$$

and

$$N_{\Gamma_{[n,n-m]}}[\mathcal{B}^{(n-m)}] \cup \{\mathcal{B}^{(n-m)}\} \neq N_{\Gamma_{[n,n-m]}}[\mathcal{B}'^{(n-m)}] \cup \{\mathcal{B}'^{(n-m)}\}$$

if $[\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}] \notin E(\Gamma_{[n,n-m]})$. Again, it follows that there are no two vertices in relations S and R. Therefore, we have the result. $\square$

4.3 Isomorphism in $\Gamma_{[n,n-m]}$.

In this section, we further explore the geometry of the projective space and the vector spaces in order to discuss some other interesting properties of $\Gamma_{[n,n-m]}$.

More specifically, we extend the well known rank-nullity theorem of vector spaces to the projective space thereby showing that the block intersection graph of a set of subspaces of the same dimension is isomorphic to the block intersection graph of the set of their null spaces. The result is the same for projective spaces.

We now introduce some elementary but pertinent properties of subspaces required in discussing this section.

It is well known that given a finite-dimensional vector space V and a subspace U of V, there exists a subspace W of V such that $V = U \oplus W$.

In the context of the notations of this study, it follows that given a subspace, V^m of $V = \mathbb{F}_2^n$, there is a subspace, V^{n-m} of V such that $V = V^{n-m} \oplus V^m$.

We now further explore relationships between V^{n-m} and V^m.

Theorem 4.5. *Let V^m, V^{n-m} be subspaces of $V = \mathbb{F}_2^n$ such that $V^{n-m} \oplus V^m = V$. Let f be a linear function on V^m, g a linear function on V^{n-m} and h a linear function on V. Then for a given $v_i \in V^m$, there exist a $v_{n-i} \in V^{n-m}$*

such that

$$f(v_i) = g(v_{n-i}), 1 \leq i \leq m.$$

Hence there is a 1-1 correspondence between the sets V^{n-m} and V^m.

Proof. V^m is finite-dimensional and hence there is a basis $v_1, v_2, \cdots, v_m$ of V^m. Of course, $v_1, v_2, \cdots, v_m$ is a linearly independent list of vectors in V. Hence this list can be extended to a basis $v_1, v_2, \cdots, v_m, v_{n-1}, v_{n-2}, \cdots, v_{n-m}$ of V, where $V^{n-m} = \text{span}(v_{n-1}, v_{n-2}, \cdots, v_{n-m})$.

Now,

$$V = V^{n-m} \oplus V^m$$

implies that

$$V = V^{n-m} + V^m$$

and

$$V^{n-m} \cap V^m = \{\mathbf{0}\}.$$

Consequently, we have that for a given $v \in V$, there exist $\alpha_i, \beta_{n-i} \in \mathbb{F}_2^n, i = 1, 2, \cdots, m$ such that

$$v = \sum_{i=1}^{m} \alpha_i v_i + \sum_{i=1}^{m} \beta_{n-i} v_{n-i},$$

since $v_1, v_2, \cdots, v_m, v_{n-1}, v_{n-2}, \cdots, v_{n-m}$ spans V.

Hence it follows that

$$h(v) = f(v_i) + g(v_{n-i}).$$

In addition, $V^{n-m} \cap V^m = \{\mathbf{0}\}$ implies that given a $v \in V^{n-m} \cap V^m, v = \mathbf{0}$ hence, there exist scalars $\alpha_i, \beta_{n-i} \in \mathbb{F}_2^n, i = 1, 2, \cdots, m$ such that $\alpha_i = \beta_{n-i} = 0, i = 1, 2, \cdots, m$, since $v_1, v_2, \cdots, v_m, v_{n-1}, v_{n-2}, \cdots, v_{n-m}$ is linearly independent.

This implies,

$$v = \sum_{i=1}^{m} \alpha_i v_i - \sum_{i=1}^{m} \beta_{n-i} v_{n-i} = 0;$$

thus,

$$v = \sum_{i=1}^{m} \alpha_i v_i = \sum_{i=1}^{m} \beta_{n-i} v_{n-i}$$

and hence we have the result.

□

Having seen that there exists a 1-1 correspondence between the sets V^{n-m} and V^m, we now extend the result to projective spaces.

We recall that a subspace of dimension $(n-m)$ or $(n-m)$-space of a $\mathrm{PG}(n,2)$ is a set of points all of whose representing vectors form, together with the zero in $V = \mathbb{F}_2^{n+1}$, a subspace of dimension $(n-m+1)$. In addition, $V^{(n-m)}$ is an $(n-m)$-flat of V, $m \in \mathbb{N}$.

Now given $n = (n-m)+m$, by Lemma 4.3, we have that for a fixed n and $k > 0$ such that Equation 4.4 holds, $n = (n-m)+m = m+(n-m)$, where $(n-m) = m$. Further, the following clarifications and definitions ensue, since the dimension of a projective space is one higher than that of a vector space.

Given $V = \mathbb{F}_2^{n+1}$, and $m > 0$. By $\mathbb{V}^{<n-m>}$ we mean the collection of all V^{n-m+1} subspaces of V, that is, the set

$$\mathbb{V}^{<n-m>} = \{V^{n-m+1} : V^{n-m+1} \subset V\}, \tag{4.8}$$

and hence by $\mathbb{V}^{<m-1>}$ we mean the collection of all V^m subspaces of V, that is, the set,

$$\mathbb{V}^{<m-1>} = \{V^m : V^m \subset V\}. \tag{4.9}$$

In the context of vertices of the block intersection graphs of consideration in this chapter, given a $V^{n-m+1} \in \mathbb{V}^{<n-m>}$,

$$\left\langle V^{n-m+1} \setminus \{\mathbf{0}\} \right\rangle = \mathcal{B}^{(n-m)},$$

that is, $V^{n-m+1} \setminus \{\mathbf{0}\}$ is the flat $V^{(n-m)}$ and for any $V^m \in \mathbb{V}^{<m-1>}$,

$$\langle V^m \setminus \{\mathbf{0}\} \rangle = \mathcal{B}^{(m-1)},$$

that is, $V^m \setminus \{\mathbf{0}\}$ is the flat $V^{(m-1)}$.

Hence, we have that

$$\overline{\mathcal{B}}^{(n-m)} = \{\mathcal{B}^{(n-m)} : \mathcal{B}^{(n-m)} = \left\langle V^{n-m+1} \setminus \{\mathbf{0}\} \right\rangle\}, \tag{4.10}$$

and

$$\overline{\mathcal{B}}^{(m-1)} = \{\mathcal{B}^{(m-1)} : \mathcal{B}^{(m-1)} = \langle V^m \setminus \{\mathbf{0}\}\rangle\}. \tag{4.11}$$

We now discuss isomorphisms between subspaces of Equations (4.8) and (4.9) and the flats of Equations (4.10) and (4.11). The flats essentially generate the vertex set of $\Gamma_{[n,n-m]}$.

In order to facilitate this discussion, we introduce an essential map as follows:

Consider the map $\phi : \mathbb{V}^{<n-m>} \longrightarrow \mathbb{V}^{<m-1>}$ defined by

$$\phi(V^{n-m+1}) = V^m, \text{ whenever } V^{n-m+1} \oplus V^m = V.$$

By Theorem 4.5, we have that there is a 1-1 correspondence between the sets V^{n-m+1} and V^m. Consequently, we have a 1-1 correspondence between the sets $\mathbb{V}^{<n-m>}$ and $\mathbb{V}^{<m-1>}$.

In addition, we have that the linear map ϕ induces a map $\phi^* : \overline{\mathcal{B}}^{(n-m)} \longrightarrow \overline{\mathcal{B}}^{(m-1)}$ defined as

$$\phi^*(\mathcal{B}^{(n-m)}) = \mathcal{B}^{(m-1)}.$$

That is,

$$\phi^*\left(\left\langle V^{n-m+1} \setminus \{\mathbf{0}\}\right\rangle\right) = (\langle V^m \setminus \{\mathbf{0}\}\rangle), \text{ whenever } n = (m-1) + (n-m+1).$$

As alluded to, this is as a result of the fact that the dimension of a vector space is one less than that of a projective space.

Consequently, ϕ^* is a bijection.

At this stage, we are now well-equipped to discuss one of the main results of this study. This result reveals isomorphic subgeometries of the geometry producing the designs as well as the block intersection graphs of this study.

Theorem 4.6. *For any* $m \in \mathbb{N}, \Gamma_{[n,n-m]} \cong \Gamma_{[n,m-1]}$.

Proof. Consider the map $\phi^* : V(\Gamma_{[n,n-m]}) \longrightarrow V(\Gamma_{[n,m-1]})$ defined as

$$\phi^*(\mathcal{B}^{(n-m)}) = \mathcal{B}^{(m-1)}.$$

Clearly, there is a 1-1 correspondence between the vertex sets $V(\Gamma_{[n,n-m]})$ and $V(\Gamma_{[n,m-1]})$. Hence, it is sufficient to show that ϕ^* preserves edges. That is, if $[{\mathcal{B}_1}^{(n-m)}, {\mathcal{B}_2}^{(n-m)}] \in E(\Gamma_{[n,n-m]})$, we need to show that $[{\mathcal{B}_1}^{(m-1)}, {\mathcal{B}_2}^{(m-1)}] \in E(\Gamma_{[n,m-1]})$.

First, we recall from Equation (4.4) of Lemma 4.3 that given $n = (n-m) + (m)$, we have that

$$\begin{aligned} n = (n-m) + (m) &= (n-m-k) + (m+k) \\ &= (m-k) + (n-m+k). \end{aligned}$$

Should $k = 1$, then we have

$$\begin{aligned} n = (n-m) + (m) &= (n-m-1) + (m+1) \\ &= (m-1) + (n-m+1). \end{aligned}$$

Now, let $[{\mathcal{B}_1}^{(n-m)}, {\mathcal{B}_2}^{(n-m)}] \in E(\Gamma_{[n,n-m]})$.

This implies that

$${\mathcal{B}_1}^{(n-m)} \cap {\mathcal{B}_2}^{(n-m)} = \mathcal{B}^{(n-m-1)} = \left\langle V^{n-m} \setminus \{\mathbf{0}\} \right\rangle.$$

Therefore,

$$\left\langle V_1^{n-m+1} \setminus \{\mathbf{0}\} \right\rangle \cap \left\langle {V_2}^{n-m+1} \setminus \{\mathbf{0}\} \right\rangle = \left\langle V^{n-m} \setminus \{\mathbf{0}\} \right\rangle.$$

By definition, we have that

$$V_1^{n-m+1} \setminus \{\mathbf{0}\} \cap {V_2}^{n-m+1} \setminus \{\mathbf{0}\} = V^{n-m} \setminus \{\mathbf{0}\}.$$

It therefore follows that

$$\begin{aligned} &V_1^{n-m+1} \cap {V_2}^{n-m+1} = V^{n-m}. \\ &\Longrightarrow (V \setminus V_1^m) \cap (V \setminus {V_2}^m) = V^{n-m} \quad (V^{n-m+1} \oplus V^m = V = \mathbb{F}_2^{n+1}). \\ &\Longrightarrow V \cap (V_1^m)^c \cap V \cap ({V_2}^m)^c = V^{n-m}. \\ &\Longrightarrow (V_1^m)^c \cap ({V_2}^m)^c = V^{n-m} \text{ (V is the whole space)}, \\ &\Longrightarrow (V_1^m \cup {V_2}^m)^c = V^{n-m} \text{ (De Morgan's Law)}. \end{aligned}$$

By complementation, we have $V_1^m \cup {V_2}^m = (V^{n-m})^c$.

$$\text{Hence, } V_1^m \cup {V_2}^m = V^{m+1} \qquad \text{(V is the whole space)}.$$

Now as for dimensions, we have that

$$\begin{aligned}\dim(V_1^m \cup V_2{}^m) &= \dim(V_1^m) + \dim(V_2{}^m) \\ &\quad - \dim(V_1^m \cap V_2{}^m) \\ &= \dim(V^{m+1}).\end{aligned}$$

$$\begin{aligned}&\implies \dim(V_1^m \cap V_2{}^m) = \dim(V_1^m) + \dim(V_2{}^m) - \dim(V^{m+1}), \\ &\implies \dim(V_1^m \cap V_2{}^m) = m + m - (m+1) = m - 1, \\ &\implies V_1^m \cap V_2{}^m = V^{m-1}.\end{aligned}$$

It follows that

$$V_1^m \setminus \{\mathbf{0}\} \cap V_2{}^m \setminus \{\mathbf{0}\} = V^{m-1} \setminus \{\mathbf{0}\}.$$

Again by definition, we have that

$$\langle V_1^m \setminus \{\mathbf{0}\}\rangle \cap \langle V_2{}^m \setminus \{\mathbf{0}\}\rangle = \left\langle V^{m-1} \setminus \{\mathbf{0}\}\right\rangle.$$

Hence, $\mathcal{B}^{(m-1)} \cap \mathcal{B}'^{(m-1)} = \mathcal{B}^{(m-2)}$.

Therefore, $[\mathcal{B}^{(m-1)}, \mathcal{B}'^{(m-1)}] \in E(\Gamma_{[n,m-1]})$, and hence the result. □

Although the considerations of the tactical configurations of this study does not include the case $n - m = 1$, by definition, this case considers the points of the set $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}$ as blocks. Nevertheless, the following corollary as a result of the proof of Theorem 4.6 further reveals isomorphisms between the point intersection graphs of Steiner triple systems from projective geometry and the block intersection graphs of one dimensional flats of V.

Corollary 4.8. $\Gamma_{[n,n-m]}$ *is isomorphic to the point graph of Steiner triple systems from projective geometry, if* $m = 1$.

Proof. By Theorem 4.6, $m = 1$ implies $\Gamma_{[n,n-1]} \cong \Gamma_{[n,0]}$.

By Theorem 4.1, $\Gamma_{[n,n-1]}$ is a complete graph on $2^{n+1} - 1$ vertices. Hence, we have that $|\overline{\mathcal{B}}^{(n-1)}| = |\overline{\mathcal{B}}^{(n)}| = |\overline{\mathcal{B}}^{(0)}| = 2^{n+1} - 1$.

By definition,

$$\overline{\mathcal{B}}^{(0)} = \{\mathcal{B}^{(0)} : \mathcal{B}^{(0)} = \left\langle V^1 \setminus \{\mathbf{0}\}\right\rangle\}.$$

Hence, $\overline{\mathcal{B}}^{(0)} = \{\langle v\rangle : v \in V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}\} = \{X(v) : v \in V\}$, where $X(v) = \{B \in \mathcal{B} : v \in B\}$.

By Definition 2.2, the point graph of Steiner triple systems $(V, \mathcal{B})$ from projective geometry is the graph $\Gamma = (V, E)$ with

$$\begin{aligned} V(\Gamma) &:= V; \\ E(\Gamma) &:= \{[u, v] : u, v \in \mathcal{B} \text{ for some } B \in \mathcal{B}\}. \end{aligned}$$

By Proposition 2.1, we have that $|V(\Gamma)| = 2^{(n+1)} - 1$.

Now, consider the map $\phi : V(\Gamma_{[n,0]}) \longrightarrow V(\Gamma)$ defined as

$$\phi(X(v)) = v.$$

Clearly there is a 1-1 correspondence between the set $\overline{\mathcal{B}}^{(0)}$ and the set V of the points set of Steiner triple systems from projective geometry. Hence, it is sufficient to show that ϕ preserves edges.

Let $[X(v), X(v')] \in E(\Gamma_{[n,0]})$. It follows that there exists a $B = \{v, v', v''\} \in X(v) \cap X(v')$, since we have from the definition of blocks (triples) of Steiner triple systems as alluded to in Section 2.2, that given any two $v, v' \in V$, there exists a $B \in \mathcal{B}$ such that v and v' are in exactly one block, $\{v, v', v''\} \in \mathcal{B}, v'' \in \mathcal{B}$. Hence, by definition of Γ, we have that $[v, v'] \in E(\Gamma)$. □

The following result identifies flats inheriting the same combinatorial symmetry of the fundamental object from which we construct tactical configurations.

Corollary 4.9. $\Gamma_{[n,n-m]}$ *is isomorphic to the block intersection graph of Steiner triple systems from projective geometry, if* $m = 2$.

Proof. By Theorem 4.6, $m = 2$ implies $\Gamma_{[n,n-2]} \cong \Gamma_{[n,1]}$.

By Theorem 4.4, $\Gamma_{[n,n-2]}$ is a strongly regular graph with parameters $\left(\dfrac{(2^{n+1}-1)(2^n-1)}{3}, 6(2^{n-1}-1), (2^n+1), 9\right)$.

The parameters above are clearly the same parameters of the block intersection graph of Steiner triple systems from projective geometry. Hence we need to show isomorphism between the block intersection graphs Γ of Steiner triple systems from projective geometry and $\Gamma_{[n,n-2]}$.

Now, $m = 2$ implies

$$\overline{\mathcal{B}}^{(1)} = \{\mathcal{B}^{(1)} : \mathcal{B}^{(1)} = \langle V^2 \setminus \{\mathbf{0}\}\rangle\}.$$

By definition, $V^2 \setminus \{\mathbf{0}\} = \{v, v', v''\}, v, v', v' \in V$, since $|V^2 \setminus \{\mathbf{0}\}| = 2^2 - 1$. Hence, $\langle V^2 \setminus \{\mathbf{0}\} \rangle = B$, where $B \in \mathcal{B}$, that is, a block of the set of triples of Steiner triple systems from projective geometry.

Now, consider the map $\phi : V(\Gamma_{[n,1]}) \longrightarrow V(\Gamma)$ defined as

$$\phi(\mathcal{B}^{(1)}) = B.$$

In other words, the map ϕ is an identity map, and clearly there is a 1-1 correspondence between the vertex sets $V(\Gamma_{[n,1]})$ and $V(\Gamma)$. Therefore, it is enough to show that ϕ preserves edges.

Let $[\mathcal{B}_1^{(1)}, \mathcal{B}_2^{(1)}] \in V(\Gamma)$. We have that

$$\begin{aligned} \mathcal{B}_1^{(1)} \cap \mathcal{B}_2^{(1)} &= v \in V \\ \phi(\mathcal{B}_1^{(1)}) \cap \phi(\mathcal{B}_2^{(1)}) &= B_1 \cap B_2 = v.(\text{ since } \mathcal{B}^{(1)} = \langle V^2 \setminus \{\mathbf{0}\} \rangle = B) \end{aligned}$$

Hence we have the result. $\square$

By similar argument of the proof of Theorem 4.6, we have the following on Grassmann graphs.

Theorem 4.7. *The Grassmann graph $Gr(n, m)$ of the set of all m-dimensional subspaces of a vector space V is isomorphic to the Grassmann graph $Gr(n, n-m)$ of the set of all $(n-m)$-dimensional subspaces of V.*

Given any two $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)} \in V(\Gamma_{[n,n-m]})$, we hope to further study i-block intersection graphs of our generalized tactical configurations of this study. That is, graphs on the set $\overline{\mathcal{B}}$ with edges defined to be $[\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)}] \in E(\Gamma_{[n,n-m]})$, whenever $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-i)}, 2 \leq i < (n-m-3)$.

We now discuss another important class of graphs having similar parameters as the block intersection graphs of Steiner triple systems alluded to in Section 3.3.

4.4 Graphs from a further configuration

As discussed earlier, we are interested in structures having similar properties to their parent structures.

In this section, we consider block intersection graphs of $1-\left(\frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7},(2^{n-1}-1),7\right)$ designs discussed in Chapter 3, examine their properties as well as compare them to the block intersection graphs from $2-(2^{n+1}-1,3,1)$ designs.

As indicated in Chapter 3, $1-\left(\frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7},(2^{n-1}-1),7\right)$ designs are quasi-symmetric.

Goethals and Seidel [19] proved that there is an inherent strongly regular graph in any 2-design with just two intersection numbers.

In this section, we echo the result of Goethals and Seidel [19], thereby showing that a block intersection graph of the tactical configuration on such a quasi-symmetric design produces a strongly regular graph. Precisely in this section, we show that block intersection graphs of our further tactical configurations on $1-\left(\frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7},7,(2^{n-1}-1)\right)$ designs are isomorphic to the block intersection graphs of their Steiner triple systems from projective geometry and hence, are strongly regular.

We now define the block intersection graphs of $1-\left(\frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7},(2^{n-1}-1),7\right)$ designs.

Definition 4.3. *Let $\left(\overline{\mathcal{B}},\mathcal{F}\right)$ be the design of Theorem* 3.4.

The block intersection graph $\Gamma=(\mathcal{F},E)$ of $\left(\overline{\mathcal{B}},\mathcal{F}\right)$ is the graph with

$$V(\Gamma):=\mathcal{F};$$
$$E(\Gamma):=\{[F_B,F_{B'}]:F_B\cap F_{B'}\neq\emptyset\},F_B\in\mathcal{F}.$$

We now explore the properties of block intersection graphs of $1-\left(\frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7},(2^{n-1}-1),7\right)$ designs.

By definition, for every $B\in\mathcal{B}$, $F_B=\{\mathcal{B}^{(2)}\in\overline{\mathcal{B}}:B\in\mathcal{B}^{(2)}\}$. In addition, the vertices of the block intersection graphs from $1-\left(\frac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7},(2^{n-1}-1),7\right)$ designs, just like the block intersection graphs from $2-(2^{n+1}-1,3,1)$ designs are defined by the blocks of their respective designs.

Hence, by Lemma 3.9 and the proof of Lemma 2.2, we have that

$$|V(\Gamma)| = \frac{(2^{n+1}-1)(2^n-1)}{3}.$$

In order to avoid unnecessary repetitions, it is better to consider isomorphism between the block intersection graphs of the $2-(2^{n+1}-1, 3, 1)$ and $1-\left(\dfrac{(2^{n+1}-1)(2^n-1)(2^{n-1}-1)}{1\times 3\times 7}, (2^{n-1}-1), 7\right)$ designs. This we discuss in the following.

Theorem 4.8. *Let $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}, \mathcal{B} = \{\{v_1, v_2, v_3\} : v_1 \neq v_2 \neq v_3, v_1 + v_2 + v_3 = 0\}$. Let $V^{(2)}$ be a 2-flat of $V, \mathcal{B}^{(2)} = \{B \in \mathcal{B} : B \cap V^{(2)} = B\}$, and let $\overline{\mathcal{B}}$ be a collection of all $\mathcal{B}^{(2)}$. For every $B \in \mathcal{B}$, let $F_B = \{\mathcal{B}^{(2)} \in \overline{\mathcal{B}} : B \in \mathcal{B}^{(2)}\}$, and let $\mathcal{F}$ be the total collection of all $F_B, B \in \mathcal{B}$ such that $(\overline{\mathcal{B}}, \mathcal{F})$ is a 1-design. Then the block intersection graph of $\Gamma = (\mathcal{B}, E)$ and $\Gamma' = (\mathcal{F}, E')$ are isomorphic.*

Proof. Consider $\sigma : \mathcal{B} \longrightarrow \mathcal{F}$ defined by

$$\sigma(B) = F_B.$$

By the proof of Theorem 4.8, it is clear that σ is a one-to-one correspondence. Hence, it is enough to show that σ preserves edges.

Let $[B, B'] \in E(\Gamma)$, we need to show that $[\sigma(B), \sigma(B')] \in E(\Gamma')$.

Now without loss of generality, let $B = \{v_1, v_2, v_3\}$ and $B' = \{v_1, v_4, v_5\}, v_i \neq v_j, i \neq j$.

By Corollary 3.5 (ii), (B, B') exists in a unique $\mathcal{B}^{(2)} \in \overline{\mathcal{B}}$. Therefore, it follows that $\mathcal{B}^{(2)} \in F_B \cap F_{B'}$ and hence we have the result. □

As alluded to at the end of Chapter 3, a further investigation on generalized recursive tactical configurations of our generalized tactical configurations is required. In what follows, the block intersection graphs of such generalized recursive tactical configurations are also of importance.

Chapter 5

Automorphism group and the primarity of $\Gamma_{[n,n-m]}$

In this chapter, we explore the symmetry of the block intersection graphs of our tactical configurations in order to discuss its full automorphism groups.

It is not surprising that the automorphism group of the underlying vector spaces of Steiner triple systems from projective geometry plays a major role as it does in the automorphism group of the block intersection graphs of Steiner triple systems from projective geometry.

It is well known that the projective general linear group acts on Steiner triple systems from projective geometry as well as their block intersection graphs [26].

In what follows, we explore the automorphism groups of $\Gamma_{[n,n-m]}$ as well as some actions of the automorphism on the vertex set of $\Gamma_{[n,n-m]}$. This automorphism group together with some properties of the standard product of graphs discussed in Chapter 2 is then used to show that any block intersection graph Γ having the subgroup structure of the class of groups of Aut $\Gamma_{[n,n-m]}$ and satisfying vertex transitivity is a prime graph.

As a result, we further establish that factorizations in both graphs and groups are mutually interdependent.

We begin with the automorphism group of $\Gamma_{[n,n-m]}$.

5.1 Automorphism groups of $\Gamma_{[n,n-m]}$

In this section, we use the fact that automorphism preserves the underlying vector spaces of our configurations to discuss the full automorphism groups of $\Gamma_{[n,n-m]}$.

The strategy employed here in determining the automorphism groups of $\Gamma_{[n,n-m]}$ is similar to that used in the automorphism groups of Grassmann graphs. Hence we begin by introducing the automorphism groups of Grassmann graphs.

Theorem 5.1. (Chow [11], cf. [[7] Theorem 9.31])

Let $\Gamma(n, e)$ *be a grassmann graph, and suppose that* $\Gamma(n, e)$ *is not complete* $(1 < e < n - 1)$. *Then*

$$\text{Aut } \Gamma \cong \begin{cases} \text{PGL}(n, q) & n \neq 2e \\ \text{PGL}(n, q) \times C_2 & n = 2e \end{cases}$$

In view of Theorems 4.1, 4.6 and Theorem 4.7, we have that Aut $\Gamma \cong \text{Sym}([n]_q)$, if $m \in \{1, n - 1\}$, where V is an n-dimensional vector space over a finite field of order q. Hence,

$$\text{Aut } \Gamma \cong \begin{cases} \text{PGL}(n, q) & n \neq 2e \\ \text{PGL}(n, q) \times C_2 & n = 2e \\ \text{Sym}([n]_q) & e \in \{1, n - 1\} \end{cases}$$

We are now fully equipped to discuss the full automorphism group of $\Gamma_{[n,n-m]}$.

Theorem 5.2.

$$\text{Aut } \Gamma_{[n,n-m]} \cong \begin{cases} \text{PGL}(n, 2) & n + 1 \neq 2m \\ \text{PGL}(n, 2) \times C_2 & n + 1 = 2m \\ \text{Sym}([n]_2) & m = 1. \end{cases}$$

Proof. First, we show that Aut $\Gamma_{[n,n-m]} \cong \text{PGL}(n, 2)$, if $n + 1 \neq 2m$. This will be done by induction on m.

The base case is $m = 2$ and by Theorem 4.6, $\Gamma_{[n,n-2]} \cong \Gamma_{[n,1]}$. In addition, by Corollary 4.9 $\Gamma_{[n,n-m]}$ is isomorphic to the block intersection graph of Steiner triple systems from projective geometry, if $m = 2$. Hence, the result holds.

Now, we assume that result holds for $m = k$ and show that the result holds for $m = k + 1$.

First, we observe that $n - m$ implies $n - m = n - (k + 1)$.

Hence, by Corollary 4.1 for $k > 2$, any set of adjacent vertices of Type I adjacency forms a clique of size $(2^{k+2} - 1)$. Now, let $\mathcal{C}'$ be the set of cliques as a result of Type I adjacency.

By Corollary 4.2, any set of adjacent vertices of Type II adjacency forms a clique of size $(2^{n-(k-1)} - 1)$. Again, let $\mathcal{C}''$ be the set of cliques as a result of Type II adjacency.

By Corollary 4.3, $\Gamma_{[n,n-(k+1)]}$ has a maximum clique of size

$$\begin{cases} 2^{n-k+1} - 1, & \text{if } (n+1) \geq 2(k+1); \\ 2^{k+2} - 1, & \text{if } (n+1) < 2(k+1). \end{cases}$$

Hence, it follows that $\Gamma_{[n,n-(k+1)]}$ has maximal cliques corresponding to the $(n-k)$-flats containing a given $(n-(k+1))$-flat and the $(k+1)$-flats contained in a given $(n-(k+1))$-flat, respectively.

Now, let $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ and define the $\Gamma_{\mathcal{C}}$ on $\mathcal{C}$ such that

$$\begin{aligned} \mathrm{V}(\Gamma_{\mathcal{C}}) &:= \mathcal{C}' \cup \mathcal{C}''; \\ \mathrm{E}(\Gamma_{\mathcal{C}}) &:= \{[C_1, C_2] : |C_1 \cap C_2| = 1, C_1, C_2 \in \mathcal{C}\}. \end{aligned}$$

It follows that $\Gamma_{\mathcal{C}}$ has two connected components Γ_1, and Γ_2 associated with the above partitioning of cliques into $\mathcal{C}'$ and $\mathcal{C}''$ and $\Gamma_1 \cong \Gamma_{[n,n-m+1]}$. In addition, $\Gamma_1 \cong \Gamma_{[n,n-k]}$, and $\Gamma_2 \cong \Gamma_{[n,k+1]}$.

Now, let G be the subgroup of Aut $\Gamma_{[n,n-(k+1)]}$ stabilizing Γ_1, and Γ_2 setwise. It follows that G is contained in the automorphism group of $\Gamma_{[n,n-2]}$, and G permutes the points of the projective space $V = \mathbb{F}_2^{n+1} \setminus \{\mathbf{0}\}$. By the Fundamental Theorem of Projective Geometry we have that $G \cong \mathrm{PGL}(n, 2)$ and hence Aut $\Gamma_{[n,n-(k+1)]} \cong \mathrm{PGL}(n, 2)$.

Now, we show that Aut $\Gamma_{[n,n-m]} \cong \mathrm{PGL}(n, 2) \times C_2$, if $n + 1 = 2m$.

Again, let $\mathcal{C}'$ be the set of cliques as a result of Type I adjacency and let $\mathcal{C}''$ be the set of cliques as a result of Type II adjacency and let $\Gamma_{\mathcal{C}}$ be as defined above.

In view of Corollaries 4.1 and 4.2, we have that

$$(2^{m+1}-1)-(2^{n-m+2}-1)=2^{m+1}(1-2^{n-2m+1})=0, \text{ if } n+1=2m.$$

It therefore follows that given a $C' \in \mathcal{C}'$ and a $C'' \in \mathcal{C}''$, $|C'| = |C''|$, if $n+1 = 2m$. Hence by similar argument above, $\Gamma_{\mathcal{C}}$ has two connected components Γ_1, and Γ_2 associated with the above partitioning of cliques into $\mathcal{C}'$ and $\mathcal{C}''$. In addition, $\Gamma_1 \cong \Gamma_{[n,n-m+1]}$, and $\Gamma_2 \cong \Gamma_{[n,m]}$.

By Theorem 4.6, we have that $\Gamma_{[n,m]} \cong \Gamma_{[n,n-m+1]}$, therefore $\Gamma_1 \cong \Gamma_2$. Hence by [16], we have the result.

We now complete the proof by showing that if $m = 1, \mathrm{Aut}\ \Gamma_{[n,n-m]} \cong \mathrm{Sym}([n]_2)$.

By Theorem 4.1, we have that $\Gamma_{[n,n-1]}$ is a complete graph on $2^{n+1}-1$ vertices. Hence, the result follows immediately.

□

Having determined that the automorphism groups of the graphs of the tactical configurations of this study inherit the automorphism groups of the block intersection graphs of Steiner triple systems from projective geometry, we now turn to discussing an important action of the automorphism group on the vertex set of $\Gamma_{[n,n-m]}$. This property will be used in the next subsection to finally characterize $\Gamma_{[n,n-m]}$.

Theorem 5.3. *$\Gamma_{[n,n-m]}$ is distance transitive.*

Proof. Let $\mathcal{B}^{(n-m)}, \mathcal{B}'^{(n-m)} \in V(\Gamma_{[n,n-m]})$.

By Corollary 4.6, $\mathcal{B}^{(n-m)}$ and $\mathcal{B}'^{(n-m)}$ have distance j if $\mathcal{B}^{(n-m)} \cap \mathcal{B}'^{(n-m)} = \mathcal{B}^{(n-m-j)}$ and by Theorem 4.3, we have that

$$\mathrm{diam}(\Gamma_{[n,n-m]}) = \begin{cases} n-m, & \text{if } 2m \leq n; \\ m, & \text{otherwise.} \end{cases}$$

By Theorem 5.2, we have that $\mathrm{PGL}(n,2)$ is a subgroup of $\mathrm{Aut}\ \Gamma_{[n,n-m]}$. Hence, it follows that $\Gamma_{[n,n-m]}$ is distance transitive, since $\mathrm{PGL}(n,2)$ is transitive on ordered bases. □

Having determined that the automorphism groups of the graphs of the tactical configurations of this study inherit the automorphism groups of the block intersection graphs of Steiner triple systems from projective geometry, it is now imperative to discuss the properties of the underlying graphs having such subgroup structures.

5.2 $\Gamma_{[n,n-m]}$ as prime graphs

In this section, we explore Aut $\Gamma_{[n,n-m]}$ as well as the properties of the standard products of graphs discussed in Chapter 2 to show that the block intersection graphs of our generalized tactical configurations in Chapter 3 are prime graphs.

As alluded to, $\mathrm{PGL}(n, 2)$ is a simple group ([26], [28]). Again, as discussed in the introduction of this section, this group is the automorphism group of the Steiner triple systems from projective geometry. By the results of the previous section, this group also acts on the block intersection graphs of our generalized tactical configurations of Chapter 3.

In addition, $\Gamma_{[n,n-m]}$ is vertex transitive since, $\mathrm{PGL}(n, 2)$ preserves the underlying vector subspaces of $V(\Gamma_{[n,n-m]})$.

Having discussed prime factorizations and the symmetry of graphs with respect to the standard products in Chapter 2, in addition, in view of the symmetry of $\Gamma_{[n,n-m]}$ discussed above, we are now better equipped to discuss the results of this study in a broader view of factorizations in both graphs and groups. This we summarize in the following.

Theorem 5.4. *Let Γ be a connected graph with vertex transitive automorphism group, which is simple. Then Γ is prime with respect to the standard products.*

Proof. Suppose to the contrary that $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$ is a prime factor decomposition of a graph Γ with respect to a standard product $*$. For each $i \in \{1, \ldots, k\}$, let Aut Γ_i be the automorphism group of Γ_i. Then consider

$$\{gsg^{-1} : g \in \mathrm{Aut}\ \Gamma, s \in \mathrm{Aut}\ \Gamma_i\} = \left\langle (\mathrm{Aut}\ \Gamma_i)^{\mathrm{Aut}\ \Gamma} \right\rangle.$$

By [22], $\left\langle (\mathrm{Aut}\ \Gamma_i)^{\mathrm{Aut}(\Gamma)} \right\rangle \lhd \mathrm{Aut}\ \Gamma$.

By hypothesis, Aut Γ is simple. Hence, it therefore follows that

$$\left\langle (\text{Aut } \Gamma_i)^{\text{Aut } \Gamma} \right\rangle = \{e, \text{Aut } \Gamma\},$$

where e is the identity in Aut Γ.

The case $\left\langle (\text{Aut } \Gamma_i)^{\text{Aut } \Gamma} \right\rangle = e$ is not a possibility since Aut Γ_i is contained in $\left\langle (\text{Aut } \Gamma_i)^{\text{Aut}(\Gamma)} \right\rangle$.

Now, we consider the case $\left\langle (\text{Aut } \Gamma_i)^{\text{Aut } \Gamma} \right\rangle = \text{Aut } \Gamma$.

By Theorems 2.8, 2.9, 2.10 and 2.11, Aut Γ is generated by the automorphisms of its prime factors and transpositions of isomorphic factors. Consequently, Aut Γ is isomorphic to the automorphism group of the disjoint union of isomorphic factors $\Gamma_i, i \in \{1, \ldots, k\}$.

In view of the fact that Γ is connected vertex-transitive, it therefore follows that all $\Gamma_i, i \in \{1, \ldots, k\}$ but one are K_1. □

The main result of this thesis is in the following terms.

Theorem 5.5. $\Gamma_{[n,n-m]}$ *is prime with respect to the standard products of graphs.*

Proof. We consider each of the standard products in succession.

(a) Cartesian product

By Theorem 2.8, we have that the automorphism group of the Cartesian product of connected prime graphs is isomorphic to the automorphism group of the disjoint union of the factors. Hence, in view of the fact that Aut $\Gamma_{[n,n-m]}$ is simple and that $\Gamma_{[n,n-m]}$ is vertex-transitive, the result therefore holds by Theorem 5.4.

(b) Strong product

By Theorem 2.9 we have that the automorphism group of the strong product of connected, S-thin prime graphs is isomorphic to the automorphism group of the disjoint union of the factors.

By Lemma 4.7, $\Gamma_{[n,n-m]}$ is S-thin. Hence, in view of the fact that Aut $\Gamma_{[n,n-m]}$ is simple and that $\Gamma_{[n,n-m]}$ is vertex-transitive, the result therefore holds by Theorem 5.4.

(c) Direct product

By Theorem 2.10, we have that if φ is an automorphism of a connected non-bipartite R-thin graph Γ that has a prime factorization $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$. Then there exists a permutation π of $\{1, 2, \ldots, k\}$, together with isomorphisms $\varphi_i : \Gamma_{\pi_{(i)}} \longrightarrow \Gamma_i$, such that

$$\varphi(x_1, x_2, \ldots, x_k) = \left(\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_k(x_{\pi(k)})\right).$$

It therefore follows that Aut Γ is generated by the automorphisms of the prime factors and transpositions of isomorphic factors. Consequently, Aut Γ is isomorphic to the automorphism group of the disjoint union of the prime factors of Γ.

By Corollary 4.4, $\Gamma_{[n,n-m]}$ is non-bipartite and by Lemma 4.7, $\Gamma_{[n,n-m]}$ is S-thin and R-thin if $m > 1$. Hence, in view of the fact that Aut $\Gamma_{[n,n-m]}$ is simple and that $\Gamma_{[n,n-m]}$ is vertex-transitive, the result therefore holds by Theorem 5.4.

(d) Lexicographic product

By Theorem 2.11, we have that if $\Gamma_1 \circ \Gamma_2$ is the lexicographic product of simple nontrivial graphs. Then Aut $(\Gamma_1 \circ \Gamma_2) =$ Aut $\Gamma_1 \circ$ Aut Γ_2 if and only if Γ_2 is connected in case $R\Gamma_1$ is nontrivial and Γ_2 is connected in case $S\Gamma_1$ is nontrivial.

Clearly, $\Gamma_{[n,n-m]}$ is nontrivial and again by Lemma 4.7, $\Gamma_{[n,n-m]}$ is S-thin and R-thin if $m > 1$. Hence, in view of the fact that Aut $\Gamma_{[n,n-m]}$ is simple and that $\Gamma_{[n,n-m]}$ is vertex-transitive, the result therefore holds by Theorem 5.4.

□

Clearly $\Gamma_{[n,n-m]}$ is a relation on the block intersection graph of Steiner triple systems from projective geometry which are well known to have simple and vertex-transitive automorphism groups [28]. Hence, by the same argument of Theorem 5.4, we have the following.

Corollary 5.1. *Let Γ be the block intersection graph of Steiner triple systems from projective geometry. Then Γ is a prime with respect to the standard product of graphs.*

Proof. By Corollary 4.9, we have that $\Gamma \cong \Gamma_{[n,n-m]}$, if $m = 2$. The result therefore follows.

$\square$

Chapter 6

Summary

In this chapter we give a brief summary of the main point of this study and suggests avenues of extensions of this work.

We now briefly summarize in the following.

The proof of Theorem 5.4 shows clearly that if the automorphism group of a vertex-transitive graph is simple, then it cannot be factorized with respect to the standard products of graphs. This is because automorphisms are generated by the automorphisms of the factors.

An implication of the above is that if the automorphism group of a graph is not simple then, the graph may also be factorized. This is because every connected graph has a unique representation as a product of prime graphs, up to isomorphism and the order of the factors, and that the automorphism group of the factors are normal subgroups of the automorphism of the graphs.

Hence, as alluded to in the abstract of this thesis, an immediate implication of the proof of Theorem 5.4 is that given a graph Γ whose subgroup structure is G, then factorizations in Γ are mutually interdependent on factorization in G. This is because automorphisms are generated by the automorphisms of the factors of Γ, and in addition, associativity allows for the easy extension of the fundamental graph products to arbitrarily many factors.

In view of the classification theorem for finite simple groups, it therefore follows that any vertex-transitive graph Γ with a subgroup structure of any of the finite simple group is prime.

We now suggests ways to improve on this study.

In Section 3.3, we explore the richness in symmetry and substructures of 2-designs from projective geometry and that leads us to more tactical configurations. Precisely, we produce an example of further tactical configurations and in the next chapter, we compare their block intersection graphs to the block intersection graphs of 2-$(v, 3, 1)$ designs from projective geometry.

Prior to that, the points of our configurations were considered as triples, while blocks are the set of triples induced by an $(n-m)$-flat of $V = \mathbb{F}_2^{n+1} \backslash \{\mathbf{0}\}$. In that section, we extended the previous consideration in the sense that points of the further tactical configurations were the blocks of the previous tactical configurations and blocks were defined to be collection of all the blocks in the previous tactical configurations with a common intersection.

Progressively in this manner, a recursive tactical configuration can be realized and also generalized.

The notion of a recursive tactical configurations described above could lead to so many questions. For instance, one may ask: (1) Does the process terminate? If it does, at what point does this happen? (2) If it does terminate, one may also ask how the final tactical configurations and its block intersection graphs compare or contrast to the original design and its block intersection graph respectively.

In addition, it will be interesting to see if the block intersection graphs of the recursive tactical configurations will also produce recursive prime graphs, since the block intersection graphs of Steiner triple systems from projective geometry as well as $\Gamma_{[n,n-m]}$ are prime graphs.

Another interesting further study we also hope to consider is the notion of i-block intersection graphs of our generalized tactical configurations with the following definitions.

$$V(\Gamma) := \overline{\mathcal{B}};$$
$$E(\Gamma) := \{[\mathcal{B}_1^{(n-m)}, \mathcal{B}_2^{(n-m)}] : \mathcal{B}_1^{(n-m)} \cap \mathcal{B}_2^{(n-m)} = \mathcal{B}^{(n-m-j)}\},$$
$$2 \leq j < (n-m-3).$$

Our take is that this might further reveal some interesting results.

Bibliography

[1] E. Artin, *Geometric Algebra*, Interscience, New York, 1957.

[2] T. Beth, D. Jungnickel and H. Lenz, *Design Theory,* Cambridge Univ. Press, Cambridge, 1993.

[3] N. L. Biggs, E. K. Lloyd, R. J. Wilson, *Graph Theory 1736 -1936*, Oxford Univ. Press, Oxford, 1976.

[4] D. Billington, A simple proof that all 1-design exists, *Discrete Math* **42** (1982) 321-322.

[5] R. C. Bose, On the construction of balanced incomplete designs, *Ann. Eugenics* **9** (1939) 353-399.

[6] R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.* **13** (1963) 389-419.

[7] Andries E. Brouwer, *Distance-Regular Graphs*, Springer, Berlin, 1989.

[8] P. J. Cameron, *Combinatorics,* Cambridge Univ. Press, Cambridge, 1994.

[9] P. J. Cameron, Strongly regular graphs, in *Selected topics in graph theory*, Academic Press, London, 1978.

[10] P. J. Cameron, J. M. Geothals and J. J. Seidel, Strongly regular graphs having strongly regular subconstituents, *J. Alg.* **55** (1978) 257-280.

[11] W. L. Chow, On the geometry of algebraic homogeneous spaces, *Ann. of Math.* (2) **50** (1949) 32-67.

[12] C. J. Colbourn, A. Rosa, *Triple Systems,* Oxford University, New York, 1999.

[13] J. D. Dixon,*The Structure of Linear Groups,* Van Nostrand Reinhold, London, 1971.

[14] P. Dembowski, *Finite Geometries*, Springer, Berlin, 1968.

[15] A. Dey, *Incomplete block designs*, World Scientific Publishing Company, Singapore, 2010.

[16] R. Frucht, On the groups of repeated graphs, *Bull. Amer. Math. Soc.* **55** (1949) 418420.

[17] C. Godsil, G. Royle, *Algebraic Graph theory*, Springer, New York, 2001.

[18] C. Godsil, G. F. Royle, *Cores of geometric graphs*. Preprint 2008.

[19] J. M. Goethals, J.J. Seidel, Strongly regular graphs derived from combinatorial designs, *Canad. J. Math* **22** (1970) 597-614.

[20] D. Gorenstein, *Finite Simple Groups: An Introduction to Their Classification*, Plenum Press, New York, 1982.

[21] R. Hammack, W. Imrich and S. Klavžar, *Handbook of product graphs*, CRC Press, Florida, 2011.

[22] Derek F. Holt, Bettina Eick, Eamonn A. O'Brien *Handbook of Computational Group Theory.* CRC Press, Florida. (2005).

[23] D. G. Higman, Finite permutation groups of rank 3, *Math Z.* **104** (1968) 417-450.

[24] J. W. P. Hirschfeld, *Projective geometries over finite fields*, Oxford Univ. Press, Oxford, 1979.

[25] D. A. Holton, J. Sheehan, *The Petersen graph*, Austral. Math. Soc. Lecture Series 7, Cambridge University, Cambridge, 1993.

[26] D. R. Hughes, F. C. Piper, *Projective Planes,* Springer, New York, 1973.

[27] W. Imrich, H. Izbicki, Associative products of graphs, *Monatsh. Math.* **80** (1975) 277-281.

[28] J. D. Key, E. E. Shult, Steiner triple systems with doubly transitive automorphism groups: A corollary to the classification theorem for finite simple groups, *J. Combin. Theory,* **3**6 (1984) 105-110.

[29] T. P. Kirkman, On a problem in combinations, *Camb.& Dublin Math. J.* **2** (1847) 191-204.

[30] B. Norman, *Algebraic Graph Theory,* Cambridge University Press, Cambridge, 1993.

[31] G. Sabidussi, Graph Multiplication, *Math Z.* **72** (1959/60), 446-457.

[32] J. J. Seidel, Strongly regular graphs, *Surveys in Combinatorics*, London Math. Soc. Lecture Note Series 38, Cambridge, 1979.

[33] T. Skolem, Some remarks on the triple system of Steiner, *Math. Scand.* **6** (1958) 273-280.

[34] M. S. Smith, On rank 3 permutation groups, *J. Alg.* **33** (1975) 22-42.

[35] D. R. Stinson, *Combinatorial Designs: Construction and Analysis,* Springer, New York, 2003.

[36] V.G. Vizing, Cartesian product of graphs, *Vychisl. Sistemy* **9** (1963), 30-43 (in Russian); *Comp. El. Syst.* **2** (1966) 352-365 (English transl.).

[37] A. J. Weir, Sylow p-subgroups of the general linear group over finite fields of characteristic p., *Proc. Amer. Math. Soc.* **6** (1955) 454-464.

[38] R. A. Wilson, *The finite simple groups*, Springer, New York, 2007.

[39] R. J. Wilson, *Introduction to Graph Theory*, Longman, England, 1996.

[40] W. S. B. Woolhouse, The Prize question, *Lady's and Gentlemen's Diary,* 1844.